Published for SISSA by Springer

Received: July 3, 2023
Accepted: July 7, 2023
Published: July 25, 2023

Partition functions of non-Lagrangian theories from the holomorphic anomaly

Francesco Fucito,[a] **Alba Grassi,**[b,c] **Jose Francisco Morales**[a] **and Raffaele Savelli**[a]

[a] *I.N.F.N – sezione di Roma Tor Vergata and Dipartimento di Fisica, Università di Roma Tor Vergata, Via della Ricerca Scientifica, I-00133 Roma, Italy*

[b] *Section de Mathématique, Université de Géneve, 1211 Geneva 4, Switzerland*

[c] *CERN, Theory Division, 1211 Geneva 23, Switzerland*

E-mail: fucito@roma2.infn.it, alba.grassi@cern.ch, morales@roma2.infn.it, savelli@roma2.infn.it

Abstract: The computation of the partition function in certain quantum field theories, such as those of the Argyres-Douglas or Minahan-Nemeschansky type, is problematic due to the lack of a Lagrangian description. In this paper, we use the holomorphic anomaly equation to derive the gravitational corrections to the prepotential of such theories at rank one by deforming them from the conformal point. In the conformal limit, we find a general formula for the partition function as a sum of hypergeometric functions. We show explicit results for the round sphere and the Nekrasov-Shatashvili phases of the Ω background. The first case is relevant for the derivation of extremal correlators in flat space, whereas the second one has interesting applications for the study of anharmonic oscillators.

Keywords: Conformal and W Symmetry, Nonperturbative Effects, Supersymmetric Gauge Theory

ArXiv ePrint: 2306.05141

https://doi.org/10.1007/JHEP07(2023)195

Contents

1 Introduction

In [1, 2] it was shown that in a suitable limit, the Argyres-Douglas (AD) limit, the moduli space of a massive $\mathcal{N} = 2$ supersymmetric gauge theory of the Yang-Mills type leads to isolated superconformal field theories (SCFT). A first attempt to classify such theories appeared in [3, 4] while more recent results were obtained in [5–14]. In this paper we will be concerned with the rank-one version of the AD SCFT's as well as of those of the Minahan-Nemeschansky (MN) type [15]. Being such SCFT's isolated and strongly coupled, their analytic treatment is troublesome given that a Lagrangian description is not available. To circumvent these difficulties, at least five different strategies have appeared in the literature: the conformal bootstrap, the AGT duality, the matrix-model methodology, the large-charge expansion, and the geometric approach based on the Ω-background technology. The numerical conformal bootstrap is a method that exploits the constraints coming from the symmetries of the theory to give numerical estimates of the parameters of interest [16–20]. The AGT duality, in its original formulation, relates the partition function of a four-dimensional SQCD with four massive flavors with a four-point correlator of a two-dimensional conformal field theory [21, 22]. Making some (or all) of these points collide leads to rank-one SCFT's [23–28]. Similar ideas have also been used to provide matrix-model representations for the partition function in a class of AD theories [29–36]. Both the AGT and the matrix-model technology have been useful to study AD theories with large deformation parameters (see also [37]). Another original perspective has been explored in the context of the large-charge expansion [38–46], where it was suggested that one can have an approximate description of such strongly coupled SCFT's in terms of an universal effective field theory. Finally, using the genus expansion of the Ω background, as well as ideas coming from localization [47–61], it has been possible to study chiral/anti-chiral correlators of non-Lagrangian theories [45, 62]. Such analytic results, even if based on the first two leading terms in the expansion of the prepotential for small curvatures, show surprisingly good agreement with the numerical bootstrap method [18] as well as with the large-charge expansion [43, 44]. To improve the analytic estimate of [45, 62] and get an exact result, one should incorporate higher curvature terms in the prepotential. This is the main motivation of this paper.

To accomplish this task we use the recursion equations following from the refined holomorphic anomaly. The latter was originally investigated in topological field theories [63], and then revisited in [64–75] after the introduction of the Ω background [47–52]. We want to emphasize that, when approaching the AD point, it is essential to employ the holomorphic anomaly equation. Indeed this technique, while providing expressions which are perturbative in the Ω-background parameters, is exact in all the other parameters of the theory. This is an important difference with respect to localization techniques à la Nekrasov, which instead cannot be used in the context of strongly coupled field theories.

All rank-one SCFT's of the AD and MN type are characterized by the dimension of their Coulomb-branch parameter and they can be treated in a uniform way. First, we specialize the holomorphic anomaly equation to a specific one-parameter family of deformations of these SCFTs. This allows us to compute the free energy exactly in the deformation parameters and order by order in the Ω background parameters $\epsilon_{1,2}$. When we turn off the deformation and go to the conformal point, we discover significant simplifications. More precisely, we find that their partition function can be expressed as an infinite sum of confluent U-hypergeometric functions

$$\mathcal{Z}(a,\epsilon_1,\epsilon_2) = \mathrm{e}^{\frac{\mathcal{F}_0(a)}{\epsilon_1\epsilon_2}}\, E_2^{\gamma/2} \sum_{n=0}^{\infty} \left(\frac{E_{2\delta}}{E_2^{\delta}}\right)^n c_n\, \mathrm{U}\left(-\frac{\gamma}{2}+n\delta, \frac{1}{2}, -\frac{6a^2}{E_2\epsilon_1\epsilon_2}\right) \qquad \epsilon_1\epsilon_2 \neq 0\,, \tag{1.1}$$

where a is the local coordinate on the Coulomb branch, $\delta = 2\,,3$ depending on the SCFT,

E_I are the Eisenstein functions evaluated at the fixed value of the modular parameter $\tau_* = \mathrm{i}\,, e^{\frac{\pi \mathrm{i}}{3}}$, and γ a constant determined by the conformal dimension of the Coulomb-branch operator (see section 3.1 for more details). Finally, the coefficients c_n are pure rational numbers depending only on the phase of the Ω background. They are determined by the gap conditions [66, 73], that ensure consistency of the expansion near singular monopole points. In order to compute them, deforming away from the conformal point is essential. Nevertheless, we check that their value is independent of the particular deformation we choose. We will also show that in the so-called Nekrasov-Shatashvili limit (NS) [76], i.e. $\epsilon_1 \to 0$, the summation in (1.1) undergoes a non-trivial re-organization in terms of a different set of functions. This limit is relevant for the study of quantum-mechanical anharmonic oscillators.

This paper is organized as follows. In section 2 we review the holomorphic anomaly equation and explain how to solve it recursively. In section 3 we specialize this algorithm to the isolated rank-one conformal field theories and show that important simplifications occur leading to (1.1). In section 4 we focus on the example of the sphere ($\epsilon_1 = \epsilon_2$) which is relevant for the computation of the extremal correlators of these SCFTs. In section 5 we discuss the NS limit. We conclude in section 6 with a few hints for further investigations. Several technical details as well as conventions are relegated to five appendices.

2 The Ω-background prepotential

2.1 Holomorphic anomaly equation

We consider rank-one $\mathcal{N} = 2$ supersymmetric (in general non-Lagrangian) theories living on an Ω-background specified by the parameters ϵ_1, ϵ_2 and by a Seiberg-Witten (SW) geometry. We denote by (a, a_D) the SW periods, by u the Coulomb-branch parameter and omit the dependence on all remaining parameters: couplings and masses. The partition function on the Ω-background can be written as

$$\mathcal{Z}(a, \epsilon_1, \epsilon_2) = e^{\frac{\mathcal{F}(a,\epsilon_1,\epsilon_2)}{\epsilon_1 \epsilon_2}} \tag{2.1}$$

with $\mathcal{F}$ the prepotential. The theory prepotential is regular in the limit $\epsilon_1, \epsilon_2 \to 0$ so it can be expanded as

$$\mathcal{F}(a, \epsilon_1, \epsilon_2) = \sum_{g=0} (\epsilon_1 \epsilon_2)^g \mathcal{F}_g(a, \beta) = \sum_{h,s \geq 0} (\epsilon_1 + \epsilon_2)^{2h} (\epsilon_1 \epsilon_2)^s \mathcal{F}_{s,h}(a) \tag{2.2}$$

with

$$\mathcal{F}_g(a, \beta) = \sum_{h=0}^{g} (\beta + \beta^{-1})^{2h} \mathcal{F}_{g-h,h}(a)\,, \qquad \beta = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\,. \tag{2.3}$$

The $\mathcal{F}_0(a)$ term represents the theory prepotential in flat space which can be determined out of the SW geometry. Higher derivative terms are given by the reduced partition function

$$\widehat{\mathcal{Z}}(a, \epsilon_1, \epsilon_2) = e^{-\frac{\mathcal{F}_0(a)}{\epsilon_1 \epsilon_2}} \mathcal{Z}(a, \epsilon_1, \epsilon_2) \tag{2.4}$$

JHEP07(2023)195

that unlike $\mathcal{Z}$ has a regular limit when the Ω-background is turned off. This function will be the main object of our study. We introduce the IR coupling

$$q(a) = e^{\pi i \tau(a)} = e^{\pi i \frac{\partial a_D}{\partial a}} = e^{-\frac{\partial^2 \mathcal{F}_0(a)}{2\partial a^2}} \ . \tag{2.5}$$

The partition function $\widehat{\mathcal{Z}}$ can be alternatively viewed as a function of q or as a function of a. In particular, one can express $\widehat{\mathcal{Z}}(q)$ in terms of the Eisenstein's series $E_2(q), E_4(q), E_6(q)$ that form a basis of quasi-modular functions, see appendix B for all the relevant definitions.[1] All $\mathcal{F}_g(q)$'s have weight zero and $a(q)$ has weight one. S-duality covariance constrains the dependence of the partition function on E_2. Indeed the full dependence on this form is determined by the anomaly equation [66, 69, 73, 77][2]

$$\partial_{E_2} \widehat{\mathcal{Z}}(q) = \frac{\epsilon_1 \epsilon_2}{24}\, \partial_a^2 \widehat{\mathcal{Z}}(q) \ . \tag{2.6}$$

In (2.6) the derivatives on the l.h.s. is carried out keeping E_4, E_6 constant. In the r.h.s. of (2.6) the partition functions is meant as a function of q and a. Writing

$$\widehat{\mathcal{Z}}(q, \epsilon_1, \epsilon_2) = \sum_{g=0}^{\infty} (\epsilon_1 \epsilon_2)^g \widehat{\mathcal{Z}}_g(q, \beta) \tag{2.7}$$

one finds the recursive equation

$$\partial_{E_2} \widehat{\mathcal{Z}}_g = \frac{1}{24}\, \partial_a^2 \widehat{\mathcal{Z}}_{g-1} \, . \tag{2.8}$$

For the "reduced" prepotential $\widehat{\mathcal{F}}(q,\beta) = \mathcal{F}(q,\beta) - \mathcal{F}_0(q)$ one finds

$$\partial_{E_2} \widehat{\mathcal{F}} = \frac{1}{24} \left[\epsilon_1 \epsilon_2 \partial_a^2 \widehat{\mathcal{F}} + \left(\partial_a \widehat{\mathcal{F}} \right)^2 \right] , \tag{2.9}$$

or equivalently, using (2.2),

$$\partial_{E_2} \mathcal{F}_g = \frac{1}{24} \left[\partial_a^2 \mathcal{F}_{g-1} + \sum_{g'=1}^{g-1} \partial_a \mathcal{F}_{g'} \partial_a \mathcal{F}_{g-g'} \right] . \tag{2.10}$$

Equations (2.10) allows to compute $\mathcal{F}_g$ recursively starting from $\mathcal{F}_1(q,\beta)$ up to E_2-independent terms. On the other hand $\mathcal{F}_1(q,\beta)$ is determined in terms of $\frac{\partial a}{\partial u}(q)$ and the discriminant $\Delta(q)$ characterising the dynamics in flat space via the formula

$$\mathcal{F}_1(q,\beta) = -\frac{1}{2} \log \frac{\partial a}{\partial u}(q) + \frac{\beta^2 + \beta^{-2}}{24} \log \Delta(q) \ . \tag{2.11}$$

[1] As we will see later, in specific cases it is convenient to replace $E_4(q), E_6(q)$ with different modular functions.

[2] Throughout this paper we consider the holomorphic version of the anomaly equation, obtained by replacing $\widehat{E}_2(\tau,\bar{\tau}) = E_2(\tau) - \frac{3}{\pi \mathrm{Im}(\tau)} \to E_2(\tau)$.

2.2 Seiberg-Witten elliptic curve

The functions $\frac{\partial a}{\partial u}(q)$ and $\Delta(q)$ entering $\mathcal{F}_1$ are described by the SW elliptic geometry. For a recent discussion see also [78]. We write the Seiberg-Witten curve in the Weierstrass form[3]

$$y^2 = 4z^3 - g_2(u)z - g_3(u)\,, \tag{2.12}$$

with discriminant

$$\Delta(u) = 16[g_2^3(u) - 27g_3^2(u)]\,. \tag{2.13}$$

The SW periods are given by

$$\omega_1 = \frac{\partial a}{\partial u} = \frac{1}{\pi}\oint_\alpha \frac{dz}{y(z)}\,, \qquad w_2 = \frac{\partial a_D}{\partial u} = \frac{1}{\pi}\oint_\beta \frac{dz}{y(z)} \tag{2.14}$$

and the complex coupling parameter q introduced in (2.5) can also be given in terms of

$$q = e^{\pi \mathrm{i} w_2/\omega_1}\,. \tag{2.15}$$

The functional dependence $u(q)$ and $\omega_1(q)$ is determined by solving the elliptic geometry formulae

$$g_2(u) = \frac{4E_4(q)}{3\omega_1(q)^4}\,, \qquad g_3(u) = \frac{8E_6(q)}{27\omega_1(q)^6} \tag{2.16}$$

for $u(q)$ and $\omega_1(q)$ in terms of $E_4(q)$ and $E_6(q)$. Once this is done, all functions of u can be viewed as functions of q. For example, the discriminant is given by

$$\Delta(q) = 16(g_2^3 - 27g_3^2) = \frac{1024(E_4(q)^3 - E_6(q)^2)}{27\omega_1(q)^{12}} \tag{2.17}$$

and the first gravitational correction becomes

$$\mathcal{F}_1(q) = -\frac{1}{2}\log\omega_1(q) + \frac{\beta^2+\beta^{-2}}{24}\log\Delta(q)\,. \tag{2.18}$$

To compute higher derivative terms, we need derivatives with respect to a, that can be translated into derivatives with respect to q using the chain rule

$$\partial_a \mathcal{F}_g(q,\beta) = \xi D_\tau \mathcal{F}_g(q,\beta) \tag{2.19}$$

with

$$D_\tau = \frac{\partial_\tau}{\pi \mathrm{i}} = q\partial_q \tag{2.20}$$

and

$$\xi = q^{-1}\frac{dq}{da} = \frac{1}{\omega_1 D_\tau u} = \frac{\nu'(u)\, E_4\, E_6}{2\omega_1(E_6^2 - E_4^3)} \tag{2.21}$$

where ξ is a modular form of weight -3 and

$$\nu(u) = \log\frac{27g_3(u)^2}{g_2(u)^3} = \log\frac{E_6(q)^2}{E_4(q)^3}\,. \tag{2.22}$$

[3] In appendix A, we collect some results which are useful to bring to this standard Weierstrass form the different expressions used in the literature for the elliptic geometry of rank-one theories.

JHEP07(2023)195

Here we have used (2.16) and computed the derivatives with respect to τ using

$$D_\tau E_2 = \frac{1}{6}(E_2^2 - E_4)\,, \quad D_\tau E_4 = \frac{2}{3}(E_2\, E_4 - E_6)\,, \quad D_\tau E_6 = E_2\, E_6 - E_4^2\,. \tag{2.23}$$

Plugging (2.19) into (2.10) one can compute $\mathcal{F}_g$ order by order in g up to E_2 invariant terms. The general form of $\mathcal{F}_g$ is

$$\mathcal{F}_g(q,\beta) = \xi^{2g-2}\left(\sum_{\ell=1}^{3g-3} c_{g,\ell}(\beta,E_4,E_6)\, E_2^\ell + h_g(\beta,E_4,E_6)\right), \quad g \geq 2\,, \tag{2.24}$$

where $c_{g,\ell}(\beta,E_4,E_6)$ is a modular form of weight $6g-6-2\ell$ and $h_g(\beta,E_4,E_6)$ is a modular function of weight $6g-6$, known as holomorphic ambiguity, which cannot be determined using (2.10).

The holomorphic ambiguities are fixed by imposing the so-called "gap conditions" [65, 66, 73, 79], that determine the behavior of the prepotential near the points where the elliptic curve degenerates. Rank-one SCFT's can be deformed in such a way that the discriminant of their SW curve takes the particularly simple form

$$\Delta(u) \sim \prod_{i=1}^{n}(u - u_0 e^{\frac{2\pi \mathrm{i}}{n}}) = u^n - u_0^n \tag{2.25}$$

leading to n equivalent singularities in the u-plane.

According to (2.17), the zeroes of the discriminant $u \sim u_0$ correspond to the point where $q = 0$. This limit can be studied, expanding

$$a(q) = \int^q \frac{dq'}{q'\xi(q')} \tag{2.26}$$

for small q, and inverting the series to get $q(a)$ for small a. Plugging this into the $\mathcal{F}_g$, the holomorphic ambiguities h_g are determined by requiring the gap conditions [73, 75]

$$\mathcal{F}_g(a) \underset{q\to 0}{\approx} (2g-3)! \sum_{k=0}^{g} \widehat{B}_{2k}\widehat{B}_{2g-2k}\frac{\beta^{2g-2k}}{a^{2g-2}} + O(a^0) \tag{2.27}$$

where

$$\widehat{B}_m = \frac{\left(\frac{1}{2^{m-1}} - 1\right) B_m}{m!} \tag{2.28}$$

and B_k are the Bernoulli numbers. We will work out two different choices of Ω background, i.e. $\beta = 1$ and $\beta = 0$. In these cases (2.27) becomes:

$$\beta = 1: \quad \mathcal{F}_g(a) \underset{q\to 0}{\approx} -\frac{B_{2g}}{2g(2g-2)\, a^{2g-2}} + O(a^0)\,, \tag{2.29}$$

$$\beta = 0: \quad \mathcal{F}_g(a) \underset{q\to 0}{\approx} -\frac{(1-2^{1-2g})\,(2g-3)!B_{2g}}{(2g)!\, a^{2g-2}} + O(a^0)\,. \tag{2.30}$$

It is important to stress that in more complicated setups,[4] equations (2.16) or (2.22) are hard to solve or admit several inequivalent solutions, often related to each other by modular

[4]For instance where (2.25) does not hold.

JHEP07(2023)195

SW	$\mathcal{H}_0$	$\mathcal{H}_1$	$\mathcal{H}_2$	E_6	E_7	E_8
N_7	2	3	4	8	9	10
d	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{3}{2}$	3	4	6
g_2	0	u	0	0	u^3	0
g_3	u	0	u^2	u^4	0	u^5
τ	$e^{\frac{\pi \mathrm{i}}{3}}$	i	$e^{\frac{\pi \mathrm{i}}{3}}$	$e^{\frac{\pi \mathrm{i}}{3}}$	i	$e^{\frac{\pi \mathrm{i}}{3}}$

Table 1. SW data for isolated rank-1 $\mathcal{N} = 2$ SCFTs.

transformations. In such cases (see appendix C for details), the $\mathcal{F}_g$'s transform non-trivially under modular transformations and a different set of gap conditions on their modular transformed $\mathcal{F}_g^D$ is required at $q_D \to 0$. The basis of modular functions to use is also adapted according to such situations.

3 Isolated rank-1 conformal field theories

3.1 The partition function

Rank-one conformal field theories can be realized in F-theory as a single D3-brane, probing a singularity built out of a certain number N_7 of coinciding mutually non-perturbative 7-branes [80]. The low-energy dynamics on the D3-brane is described by a SW elliptic curve specified by a single Coulomb-branch parameter u, a u-independent modular parameter τ and a discriminant

$$\Delta(u) \sim g_2^3 - 27 g_3^2 \sim u^{N_7} \ . \tag{3.1}$$

Prototypical examples are the AD theories $\mathcal{H}_0$, $\mathcal{H}_1$, $\mathcal{H}_2$, and the Minahan-Nemeschansky theories E_6, E_7, E_8. These are all isolated, non-Lagrangian field theories and are the focus of the present work. They can be split into two classes depending on the value of the modular parameter

$$\begin{array}{lllll} \mathbf{A}: & \tau = e^{\frac{\pi \mathrm{i}}{3}}, & y^2 = 4x^3 - u^{b_3}, & b_3 = 1,2,4,5, & \mathcal{H}_0, \mathcal{H}_2, \mathrm{E}_6, \mathrm{E}_8 \\ \mathbf{B}: & \tau = \mathrm{i}, & y^2 = 4x^3 - u^{b_2} x, & b_2 = 1,3, & \mathcal{H}_1, \mathrm{E}_7 \end{array} \tag{3.2}$$

The conformal dimension d of the Coulomb-branch parameter is given by

$$d = \frac{12}{12 - N_7} \tag{3.3}$$

that follows from the requirement that the SW period $\frac{\partial a}{\partial u}$ be of dimension $1-d$ and therefore the conformal dimension of the holomorphic differential be $[dx/y] = 1 - d$. From (3.1) and (3.3) it follows that N_7 is an integer, multiple of 2 or 3, and smaller than 12. In table 1 we collect the SCFT data for all possible choices of N_7.[5] In the theories of type $\mathbf{A}$ the

[5]The case of $N_7 = 6$ is special because both g_2 and g_3 are generically non-vanishing, with the ratio g_2^3/g_3^2 an arbitrary complex number. The associated SCFT is therefore not isolated and it corresponds to the $SU(2)$ gauge theory with four massless fundamental hypermultiplets.

JHEP07(2023)195

modular form E_4 vanishes, whereas E_2, E_6 are constants. Similarly in the theories of type **B** the modular form E_6 vanishes, whereas E_2, E_4 are constants. Therefore, in all these cases, the free energy is a function of β and of the following dimensionless quantities

$$x = \frac{E_2 \epsilon_1 \epsilon_2}{6a^2} \qquad\qquad \kappa = \frac{E_{2\delta}}{E_2^\delta}\,, \tag{3.4}$$

where

$$\delta = \begin{cases} 3 & \quad \mathbf{A} \\ 2 & \quad \mathbf{B} \end{cases}. \tag{3.5}$$

For these SCFTs one finds

$$u \sim a^d\,, \qquad \Delta(u) = a^{12(d-1)}\,, \tag{3.6}$$

where d is the conformal dimension of the Coulomb-branch operator (see table 1). The first correction to the SW prepotential takes the general form[6]

$$\mathcal{F}_1(a,\beta) = \gamma \log\left(\frac{a}{\sqrt{\epsilon_1\epsilon_2}}\right) \tag{3.7}$$

with

$$\gamma = \frac{d-1}{2}(1+\beta^2+\beta^{-2})\,. \tag{3.8}$$

By dimensional analysis, the higher corrections take the form

$$\mathcal{F}_g(a,\beta) = \frac{\mathfrak{f}_g(\beta)}{a^{2g-2}}\,, \tag{3.9}$$

where $\mathfrak{f}_g(\beta)$ are numbers. The latter can be computed recursively using the holomorphic anomaly equation with boundary conditions fixed by an E_2-independent function. We can make the following Ansatz

$$\widehat{\mathcal{Z}}(a,\beta) = E_2^{\frac{\gamma}{2}} \sum_{n=0}^{\infty} \kappa^n c_n f_n(x,\beta)\,, \tag{3.10}$$

where c_n are numerical coefficients encoding the holomorphic ambiguities and depend on the phase of the Ω background. Plugging (3.10) into (2.6) leads to the confluent hypergeometric equation[7]

$$2x^3 f_n''(x) + x\,(3x-2)\, f_n'(x) + (2n\delta - \gamma)\; f_n(x) = 0\,, \tag{3.11}$$

where the boundary conditions are chosen such that (3.10) has a power-like behavior for $x \to 0$. The final solution is

$$\widehat{\mathcal{Z}}(a,\beta) = E_2^{\gamma/2} \sum_{n=0}^{\infty} \kappa^n c_n\, \mathrm{U}\left(-\frac{\gamma}{2} + n\delta, \frac{1}{2}, -\frac{1}{x}\right)\,, \tag{3.12}$$

with $\mathrm{U}(a,b,z)$ the confluent U hypergeometric function[8] and c_0 is an overall normalization which can be set to $c_0 = 1$ without loss of generality. The coefficients $\{c_n\}_{n\geq 1}$ are β-dependent coefficients encoded in the E_2-independent part of $\widehat{\mathcal{Z}}$. In the next section we will

[6]Throughout the paper we will omit any additive constant to $\mathcal{F}_1$.

[7]We remark that in a SCFT τ is independent of a, and thus E_2 and a are independent variables.

[8]Our conventions are the same as in `Mathematica`.

derive the first few coefficients c_n for the theories in table 1, and show that they are rational numbers. The strategy will be to first turn on suitable mass or coupling deformations for such theories, in order to isolate a monopole point where the gap condition can be imposed. The coefficients c_n will then be derived by turning off the deformation.[9] We check explicitly that the final result is independent of the deformation. We also note that (3.12) as it is written holds for $\epsilon_i \neq 0$, i.e. all phases of the Ω background except the NS phase. Indeed if we consider the NS limit there is a non-trivial re-organization of (3.12) which we discuss in section 5.

3.2 Deformations

Conformal invariance can be broken by turning on masses or couplings. Here we consider the simplest deformation splitting democratically the discriminant into its N_7 roots

$$\Delta(u) \sim u^{N_7} - m^{dN_7} \tag{3.13}$$

We will refer to m generically as a mass deformation, although for the case of $\mathcal{H}_0$, where masses are not available, the dimension-one parameter m is related to the IR-relevant coupling c via $m = c^{\frac{5}{4}}$. The deformed SW curves look like

$$\begin{aligned} &\mathbf{A}: \qquad y^2 = 4x^3 - m^{\frac{4b_3}{6-b_3}} x - u^{b_3}\,, \qquad b_3 = 1,2,4,5 \\ &\mathbf{B}: \qquad y^2 = 4x^3 - u^{b_2} x - m^{\frac{6b_2}{4-b_2}}\,, \qquad b_2 = 1,3 \end{aligned} \tag{3.14}$$

In all these examples, we will derive q-exact formulae for the first few $\mathcal{F}_g$'s. An important ingredient in our procedure will be to parametrize the holomorphic ambiguities for **A** and **B** theories respectively in the following form ($g \geq 2$)

$$\begin{aligned} h_g^A(\beta, q) &= \frac{E_4^{3g-3}}{E_6^{g-1}} \sum_{i=0}^{\left[\frac{5g-5}{3}\right]} \left(\frac{E_6^2}{E_4^3}\right)^i h_{g,i}(\beta)\,, \\ h_g^B(\beta, q) &= E_6^{g-1} \sum_{i=0}^{\left[\frac{3g-3}{2}\right]} \left(\frac{E_4^3}{E_6^2}\right)^i h_{g,i}(\beta)\,, \end{aligned} \tag{3.15}$$

where $h_{g,i}(\beta)$ are q-independent coefficients to be determined. The above expressions are dictated by the requirement that h_g has modular weight $6g-6$, allowing only integer powers of E_4 and E_6, such that h_g does not grow faster than its corresponding non-ambiguous part when $E_4 \to 0$ and $E_6 \to 0$.

In appendix C, we will consider an alternative deformation of the AD theory $\mathcal{H}_1$ described by the SW curve

$$y^2 = 4x^3 - ux - cu + 4c^3 \tag{3.16}$$

where c is the IR-relevant coupling. In particular we will show that the results for the $\mathcal{F}_g$'s in the conformal limit are the same, independently of the deformation used to compute them. An analogous match for the theory $\mathcal{H}_2$ is obtained in appendix D, where we consider the $N_f = 3$-SQCD description of this AD theory.

[9] In [81] it was also observed that, to determine the partition function of topologically twisted $\mathcal{H}_0$, one has to first perturb the theory away from the conformal point.

4 Examples: $\beta = 1$

In this section, we consider the Ω background given by $\epsilon_1 = \epsilon_2 = \epsilon$, i.e. $\beta = 1$. This choice enters for example the computation of the round-sphere partition function [54, 82] and of extremal correlators [45, 46, 57–60, 62]. Despite such a large interest, the holomorphic anomaly techniques have not been explored so far for this particular phase of the Ω background.[10] In the following we will compute (2.24) stopping at the first order in g in which the holomorphic ambiguity contributes in the conformal limit. This is dictated by a reason of simplicity given that the formulae become very large. In appendix E we will give results up to $g = 18, 7, 15$ for $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2$ respectively.

4.1 $\mathcal{H}_0$ theory

The SW curve for the deformed $\mathcal{H}_0$ theory is

$$y^2 = 4x^3 - cx - u\,. \tag{4.1}$$

Plugging $g_2 = c$, $g_3 = u$ into (2.16) and (2.21) gives

$$\begin{aligned}
&u = \frac{c^{\frac{3}{2}} E_6(q)}{3\sqrt{3} E_4(q)^{\frac{3}{2}}}, \qquad \omega_1 = \left(\frac{4E_4}{3c}\right)^{\frac{1}{4}}, \qquad \xi = \frac{3^{\frac{7}{4}}\, E_4(q)^{\frac{9}{4}}}{2^{\frac{1}{2}} c^{\frac{5}{4}} (E_6^2 - E_4^3)} \\
&\mathcal{F}_1 = \frac{1}{12} \log\left(c^{\frac{9}{2}} \frac{E_4(q)^3 - E_6(q)^2}{E_4(q)^{\frac{9}{2}}}\right), \qquad \Delta = 16(c^3 - 27u^2)\,.
\end{aligned} \tag{4.2}$$

In this case the holomorphic ambiguity takes the form of the first expression in (3.15). Solving recursively the holomorphic anomaly equation (2.10), one finds the first few terms:

$$\begin{aligned}
\mathcal{F}_2 =& \frac{\xi^2}{24\,12^2}\left[\frac{5}{3}E_2^3 + \frac{3E_6}{E_4}E_2^2 - \frac{\left(34E_4^3+21E_6^2\right)}{E_4^2}E_2 + h_2(q)\right] \\
\mathcal{F}_3 =& \frac{\xi^4}{24\,12^4}\left[\frac{5}{6}E_2^6 + \frac{10E_6}{E_4}E_2^5 + \frac{\left(16E_4^3+67E_6^2\right)}{2E_4^2}E_2^4 - \frac{\left(1465E_6E_4^3+147E_6^3\right)}{9E_4^3}E_2^3\right. \\
&\left. - \frac{\left(11897E_4^6+59376E_6^2E_4^3+6300E_6^4\right)}{30E_4^4}E_2^2 + \frac{\left(104257E_6E_4^3+95565E_6^3\right)}{15E_4^2}E_2 + h_3(q)\right] \\
\mathcal{F}_4 =& \frac{\xi^6}{24\,12^6}\left[\frac{1105}{1296}E_2^9 + \frac{865E_6}{48E_4}E_2^8 + \left(\frac{3589E_6^2}{24E_4^2} + \frac{2039E_4}{72}\right)E_2^7 + \left(\frac{41491E_6^3}{72E_4^3} + \frac{69869E_6}{216}\right)E_2^6\right. \\
&+\left(\frac{175987E_6^4}{240E_4^4} - \frac{43813E_6^2}{60E_4} - \frac{149791E_4^2}{720}\right)E_2^5 - \left(\frac{76559E_6^5}{48E_4^5} + \frac{399439E_6^3}{15E_4^2} + \frac{10250789E_4E_6}{720}\right)E_2^4 \\
&-\left(\frac{20125E_6^6}{4E_4^6} + \frac{11223703E_6^4}{120E_4^3} + \frac{92285669E_6^2}{1080} + \frac{1372051E_4^3}{270}\right)E_2^3 + \left(\frac{154401743E_6^5}{360E_4^4} + \frac{576047063E_6^3}{360E_4}\right. \\
&\left.+\frac{328463299E_4^2E_6}{630}\right)E_2^2 - \left(\frac{14652664E_6^6}{45E_4^5} + \frac{13723519199E_6^4}{3600E_4^2} + \frac{11480517509E_4E_6^2}{3150}\right)E_2 \\
&\left.-\left(\frac{753433829E_4^4}{3150}\right)E_2 + h_4(q)\right]
\end{aligned} \tag{4.3}$$

[10] We also note that the holomorphic anomaly equation for the sphere and the standard topological string phase ($\epsilon_1 = -\epsilon_2$) is actually the same. What changes are the initial data, i.e. $\mathcal{F}_1$, and the gap conditions.

JHEP07(2023)195

The ambiguous part is given by

$$\begin{aligned}
h_2 &= \frac{1619}{15} E_6 \\
h_3 &= -\frac{140891 E_6^4}{45 E_4^3} - \frac{1206371 E_6^2}{90} - \frac{124319 E_4^3}{63} \\
h_4 &= \frac{26737369 E_6^7}{540 E_4^6} + \frac{7883698699 E_6^5}{3600 E_4^3} + \frac{21429183673 E_6^3}{4050} + \frac{25632734639 E_4^3 E_6}{18900}
\end{aligned} \tag{4.4}$$

which has been determined by imposing the gap conditions (2.29).

The conformal limit. The theory becomes conformal in the limit $c \to 0$ and fits into the class **A** according to (3.2). In this limit $\tau \to e^{\pi i/3}$. Therefore E_2, E_6 become constants[11] and E_4 vanishes. More precisely, using (4.2), we find

$$\begin{aligned}
u &\approx \left(\frac{5^6}{2^9 3^3 E_6}\right)^{\frac{1}{5}} a^{\frac{6}{5}}, \\
E_4 &\approx \left(\frac{2^6 E_6^4}{3^3\, 5^4}\right)^{\frac{1}{5}} c\, a^{-\frac{4}{5}}, \\
\xi &\approx \left(\frac{2^{11} 3^2}{E_6 5^9}\right)^{\frac{1}{5}} c\, a^{-\frac{9}{5}}\,.
\end{aligned} \tag{4.5}$$

From the above formulae we notice that while both E_4 and ξ go to zero in the limit $c \to 0$, their ratio stays finite and goes like

$$\frac{\xi}{E_4} \approx \frac{6}{5\, E_6\, a}\,. \tag{4.6}$$

Keeping only the leading terms in (4.3) and (4.4), and using (4.6), one finds

$$\begin{aligned}
\mathcal{F}_2 &\approx -\frac{7\, E_2}{800\, a^2}, \\
\mathcal{F}_3 &\approx -\frac{7\, E_2^2}{8000\, a^4}, \\
\mathcal{F}_4 &\approx -\frac{161\, E_2^3}{768000\, a^6} + \frac{26737369\, E_6}{12960000000\, a^6}\,.
\end{aligned} \tag{4.7}$$

The above formulae reproduce the result (3.12), with

$$\beta = 1\,, \qquad \delta = 3\,, \qquad \gamma = \frac{3}{10}\,, \qquad \kappa = \frac{E_6}{E_2^3}\,, \qquad x = \frac{E_2 \epsilon^2}{6a^2}\,, \tag{4.8}$$

and

$$c_0 = 1, \qquad c_1 = -\frac{26737369}{2^8\; 3\; 5^7}. \tag{4.9}$$

Higher-genus prepotentials $\mathcal{F}_g$ can also be computed. Results for the ambiguity coefficients c_n are listed in (E.2). As we can see, the growth of c_n is relatively fast. It is likely that the sum over hypergeometric is divergent. However, a more detailed analysis is needed.

[11] Their numerical values are $E_2 \approx 1.103$, $E_6 \approx 2.881$.

4.2 $\mathcal{H}_1$ theory

The SW curve for the $\mathcal{H}_1$ theory deformed by the second-order mass Casimir is

$$y^2 = 4x^3 - ux - m^2\,. \tag{4.10}$$

In this case $g_2 = u$ and $g_3 = m^2$ leading to

$$\begin{aligned} u &= \frac{3E_4 m^{\frac{4}{3}}}{E_6^{\frac{2}{3}}}\,, \qquad \omega_1 = \left(\frac{8E_6}{27m^2}\right)^{\frac{1}{6}}\,, \qquad \xi = \frac{\sqrt{\frac{3}{2}}E_6^{\frac{3}{2}}}{2m(E_4^3 - E_6^2)} \\ \mathcal{F}_1 &= \frac{1}{12}\log\left(m^6\frac{E_4^3 - E_6^2}{E_6^3}\right)\,, \qquad \Delta = 16(u^3 - 27m^4) \end{aligned} \tag{4.11}$$

Here the holomorphic ambiguity takes the form of the second expression in (3.15). Solving recursively the holomorphic anomaly equation (2.10), one finds the first few terms

$$\mathcal{F}_2 = \frac{\xi^2}{24\,12^2}\left[\frac{5}{3}E_2^3 + \frac{3E_4^2}{E_6}E_2^2 + \left(-\frac{9E_4^4}{E_6^2} - 46E_4\right)E_2 + h_2(q)\right] \tag{4.12}$$

$$\begin{aligned} \mathcal{F}_3 = \frac{\xi^4}{24\,12^4}&\left[\frac{5}{6}E_2^6 + \frac{10E_4^2}{E_6}E_2^5 + \left(\frac{63E_4^4}{2E_6^2} + 10E_4\right)E_2^4 + \left(\frac{9E_4^6}{E_6^3} - \frac{461E_4^3}{3E_6} - \frac{310E_6}{9}\right)E_2^3\right. \\ &\left. + \left(-\frac{27E_4^8}{E_6^4} - \frac{7068E_4^5}{5E_6^2} - \frac{6871E_4^2}{6}\right)E_2^2 + \left(\frac{8289E_4^7}{5E_6^3} + \frac{9425E_4^4}{E_6} + \frac{6716E_6E_4}{3}\right)E_2 + h_3(q)\right]\,. \end{aligned}$$

The ambiguous part is given by

$$\begin{aligned} h_2 &= \frac{351E_4^3}{5E_6} + \frac{566E_6}{15} \\ h_3 &= -\frac{1112E_4^9}{9E_6^4} - \frac{4842049E_4^6}{630E_6^2} - \frac{3186886E_4^3}{315} - \frac{12220E_6^2}{21} \end{aligned} \tag{4.13}$$

which has been determined by imposing the gap conditions (2.29).

The conformal limit. The theory becomes conformal in the limit $m \to 0$ and fits into the class **B** according to (3.2). In this limit $\tau \to \mathrm{i}$. Therefore E_2, E_4 become constants[12] and E_6 vanishes. More precisely, using (4.11), we find

$$\begin{aligned} u &\approx \left(\frac{3^5}{2^{10}E_4}\right)^{\frac{1}{3}} a^{\frac{4}{3}} \\ E_6 &\approx \frac{32m^2E_4^2}{3\,a^2} \\ \xi &\approx \frac{64m^2}{3a^3} \end{aligned} \tag{4.14}$$

[12]Their numerical values are $E_2 \approx 0.955$, $E_4 \approx 1.456$.

From the above formulae we notice that while both E_6 and ξ go to zero in the limit $m \to 0$, their ratio stays finite and goes like

$$\frac{\xi}{E_6} \approx \frac{2}{a\,E_4^2}\,. \tag{4.15}$$

Keeping only the leading terms in (4.12) and (4.13), and using (4.15), one finds

$$\begin{aligned}\mathcal{F}_2 &\approx -\frac{E_2}{96\,a^2}\\ \mathcal{F}_3 &\approx -\frac{243\,E_2^2+1112E_4}{279936\,a^4}\end{aligned} \tag{4.16}$$

The above formulae reproduce the result (3.12), with

$$\beta=1\,,\qquad \delta=2\,,\qquad \gamma=\frac{1}{2}\,,\qquad \kappa=\frac{E_4}{E_2^2}\,,\qquad x=\frac{E_2\epsilon^2}{6a^2}\,, \tag{4.17}$$

and

$$c_0=1\,,\qquad c_1=-\frac{139}{972}\,. \tag{4.18}$$

Higher-genus prepotentials $\mathcal{F}_g$ can also be computed. Results for the ambiguity coefficients c_n are listed in (E.3).

4.3 $\mathcal{H}_2$ theory

The SW curve for the $\mathcal{H}_2$ theory deformed by the second-order mass Casimir is

$$y^2=4x^3-m^2x-u^2\,. \tag{4.19}$$

In this case $g_2=m^2$ and $g_3=u^2$ leading to

$$\begin{aligned}&u=\frac{\sqrt{E_6}m^{\frac{3}{2}}}{3^{\frac{3}{4}}E_4^{\frac{3}{4}}}\,,\qquad \omega_1=\left(\frac{4E_4}{3m^2}\right)^{\frac{1}{4}}\,,\qquad \xi=\frac{3\sqrt{2}E_4^{\frac{3}{2}}\sqrt{E_6}}{(E_6^2-E_4^3)\,m}\\ &\mathcal{F}_1=\frac{1}{12}\log\left(\frac{(E_4^3-E_6^2)\,m^9}{E_4^{\frac{9}{2}}}\right)+\text{const}\,,\qquad \Delta=16(m^6-27u^4)\end{aligned} \tag{4.20}$$

Here the holomorphic ambiguity takes again the form of the first expression in (3.15). Solving recursively (2.10) one finds the first few terms

$$\mathcal{F}_2=\frac{\xi^2}{24\,12^2}\left[\frac{5}{3}E_2^3+\frac{3E_4^2}{E_6}E_2^2+\left(-\frac{3E_6^2}{E_4^2}-52E_4\right)E_2+h_2(q)\right] \tag{4.21}$$

$$\begin{aligned}\mathcal{F}_3=\frac{\xi^4}{24\,12^4}&\left[\frac{5}{6}E_2^6+\frac{5\left(5E_4^3+7E_6^2\right)}{6E_4E_6}E_2^5+\left(\frac{2E_4^4}{E_6^2}+\frac{185E_4}{6}+\frac{26E_6^2}{3E_4^2}\right)E_2^4-\left(\frac{251E_4^3}{6E_6}+\frac{1207E_6}{9}+\frac{19E_6^3}{6E_4^3}\right)E_2^3\right.\\ &\left.-\left(\frac{153E_4^5}{2E_6^2}+\frac{9167E_4^2}{6}+\frac{29353E_6^2}{30E_4}+\frac{3E_6^4}{E_4^4}\right)E_2^2+\left(\frac{2343E_4^4}{E_6}+\frac{277747E_6E_4}{30}+\frac{51607E_6^3}{30E_4^2}\right)E_2+h_3(q)\right]\end{aligned}$$

$$\begin{aligned}\mathcal{F}_4=\frac{\xi^6}{24\,12^6}&\left[\frac{1105E_2^9}{1296}+\left(\frac{985E_4^2}{144E_6}+\frac{805E_6}{72E_4}\right)E_2^8+\left(\frac{445E_4^4}{36E_6^2}+\frac{8135E_4}{72}+\frac{3781E_6^2}{72E_4^2}\right)E_2^7\right.\\ &+\left(\frac{11E_4^6}{4E_6^3}+\frac{54395E_4^3}{216E_6}+\frac{117511E_6}{216}+\frac{10921E_6^3}{108E_4^3}\right)E_2^6+\left(\frac{59E_4^5}{2E_6^2}-\frac{8509E_4^2}{48}-\frac{4097E_6^2}{36E_4}+\frac{13583E_6^4}{240E_4^4}\right)E_2^5\end{aligned}$$

JHEP07(2023)195

$$
\begin{aligned}
&-\left(\frac{99E_4^7}{2E_6^3}+\frac{1176895E_4^4}{144E_6}+\frac{9441703E_6E_4}{360}+\frac{1150283E_6^3}{144E_4^2}+\frac{577E_6^5}{24E_4^5}\right)E_2^4\\
&-\left(\frac{7273E_4^6}{E_6^2}+\frac{775503E_4^3}{10}+\frac{97040443E_6^2}{1080}+\frac{15561623E_6^4}{1080E_4^3}-\frac{9E_6^6}{E_4^6}\right)E_2^3\\
&+\left(\frac{13905E_4^8}{4E_6^3}+\frac{6407761E_4^5}{18E_6}+\frac{512201711}{360}E_6E_4^2+\frac{29391479E_6^3}{40E_4}+\frac{42036497E_6^5}{1260E_4^4}\right)E_2^2\\
&-\left(\frac{160687E_4^7}{E_6^2}+\frac{10391931E_4^4}{4}+\frac{385527557}{90}E_6^2E_4+\frac{8137162319E_6^4}{8400E_4^2}+\frac{3300704E_6^6}{315E_4^5}\right)E_2+h_4(q)\Bigg]
\end{aligned}
\tag{4.22}
$$

The ambiguous part is given by

$$
\begin{aligned}
h_2 &= \frac{147E_4^3}{5E_6}+\frac{1178E_6}{15}\\
h_3 &= -\frac{3529E_4^6}{14E_6^2}-\frac{1038589E_4^3}{126}-\frac{6008447E_6^2}{630}-\frac{150032E_6^4}{315E_4^3}\\
h_4 &= \frac{63691E_4^9}{10E_6^3}+\frac{25347539E_4^6}{30E_6}+\frac{132133663}{30}E_6E_4^3+\frac{150291551071E_6^3}{45360}\\
&\quad+\frac{11994210803E_6^5}{37800E_4^3}+\frac{12428E_6^7}{27E_4^6}
\end{aligned}
\tag{4.23}
$$

which has been determined by imposing the gap conditions (2.29).

The conformal limit. The theory becomes conformal in the limit $m\to 0$ and fits into the class **A** according to (3.2). In this limit $\tau\to e^{\pi i/3}$. Therefore E_2, E_6 become constants[13] and E_4 vanishes. More precisely, using (4.20), we find

$$
\begin{aligned}
u &\approx \left(\frac{8}{27E_6}\right)^{\frac14} a^{\frac32}\\
E_4 &\approx \frac{m^2E_6}{2a^2}\\
\xi &\approx \frac{3m^2}{2a^3}
\end{aligned}
\tag{4.24}
$$

From the above formulae we notice that while both E_4 and ξ go to zero in the limit $m\to 0$, their ratio stays finite and goes like

$$
\frac{\xi}{E_4}\approx\frac{3}{E_6\,a}\,. \tag{4.25}
$$

Keeping only the leading terms in (4.21) and (4.23), and using (4.25), one finds

$$
\begin{aligned}
\mathcal{F}_2 &\approx -\frac{E_2}{128\,a^2}\\
\mathcal{F}_3 &\approx -\frac{E_2^2}{2048\,a^4}\\
\mathcal{F}_4 &\approx -\frac{243\,E_2^3-12428\,E_6}{2654208\,a^6}
\end{aligned}
\tag{4.26}
$$

[13] Their values are clearly the same as in the $\mathcal{H}_0$ theory.

The above formulae reproduce the result (3.12), with

$$\beta = 1\,, \qquad \delta = 3\,, \qquad \gamma = \frac{3}{4}\,, \qquad \kappa = \frac{E_6}{E_2^3}\,, \qquad x = \frac{E_2 \epsilon^2}{6a^2}\,, \tag{4.27}$$

and

$$c_0 = 1\,, \quad c_1 = -\frac{3107}{3072}\,. \tag{4.28}$$

Higher-genus prepotentials $\mathcal{F}_g$ can also be computed. Results for the ambiguity coefficients c_n are listed in (E.4).

4.4 $\mathbf{E_6}$ theory

The SW curve for the E_6 theory deformed by the eighth-order mass Casimir is

$$y^2 = 4x^3 - m^8 x - u^4\,. \tag{4.29}$$

In this case $g_2 = m^8$ and $g_3 = u^4$ leading to

$$\begin{gathered} u = \frac{E_6^{\frac{1}{4}} m^3}{3^{\frac{3}{8}} E_4^{\frac{3}{8}}}\,, \qquad \omega_1 = \left(\frac{4E_4}{3m^8}\right)^{\frac{1}{4}}\,, \qquad \xi = \frac{2\sqrt{2}\,3^{\frac{5}{8}} E_4^{\frac{9}{8}} E_6^{\frac{3}{4}}}{(E_6^2 - E_4^3)\, m} \\ \mathcal{F}_1 = \frac{1}{12} \log\left(\frac{(E_4^3 - E_6^2)\, m^{36}}{E_4^{\frac{9}{2}}}\right) + \text{const}\,, \qquad \Delta = 16(m^{24} - 27u^8) \end{gathered} \tag{4.30}$$

Here the holomorphic ambiguity takes again the form of the first expression in (3.15). Solving recursively (2.10) one finds the first few terms

$$\begin{aligned} \mathcal{F}_2 =& \frac{\xi^2}{24\,12^2}\left[\frac{5E_2^3}{3} + \left(\frac{9E_4^2}{2E_6} - \frac{3E_6}{2E_4}\right) E_2^2 + \left(\frac{6E_6^2}{E_4^2} - 61E_4\right) E_2 + h_2(q)\right] \\ \mathcal{F}_3 =& \frac{\xi^4}{24\,12^4}\left[\frac{5E_2^6}{6} + \frac{5\left(5E_4^3 + 3E_6^2\right) E_2^5}{4E_4 E_6} + \left(\frac{39E_4^4}{4E_6^2} + \frac{115E_4}{4} + \frac{3E_6^2}{E_4^2}\right) E_2^4\right. \\ &+ \frac{\left(81E_4^6 - 4869E_6^2 E_4^3 - 8165E_6^4 + \frac{57E_6^6}{E_4^3}\right) E_2^3}{72E_6^3} + \left(-\frac{1566E_4^5}{5E_6^2} - \frac{107869E_4^2}{60} - \frac{474E_6^2}{E_4} - \frac{3E_6^4}{4E_4^4}\right) E_2^2 \\ &\left. + \frac{\left(3969E_4^9 + 562005E_6^2 E_4^6 + 957017E_6^4 E_4^3 + 75585E_6^6\right) E_2}{120E_4^2 E_6^3} + h_3(q)\right] \\ \mathcal{F}_4 =& \frac{\xi^6}{24\,12^6}\left[\frac{1105E_2^9}{1296} + \left(\frac{985E_4^2}{96E_6} + \frac{745E_6}{96E_4}\right) E_2^8 + \left(\frac{303E_4^4}{8E_6^2} + \frac{8399E_4}{72} + \frac{70E_6^2}{3E_4^2}\right) E_2^7\right. \\ &+ \frac{\left(19143E_4^9 + 198654E_6^2 E_4^6 + 159643E_6^4 E_4^3 + 11244E_6^6\right) E_2^6}{432E_6^3 E_4^3} \\ &+ \frac{\left(27459E_4^{12} + 638676E_6^2 E_4^9 - 734194E_6^4 E_4^6 - 547740E_6^6 E_4^3 + 25455E_6^8\right) E_2^5}{2880E_4^4 E_6^4} \\ &- \frac{\left(2599047E_4^{12} + 47027130E_6^2 E_4^9 + 64395032E_6^4 E_4^6 + 8268330E_6^6 E_4^3 - 555E_6^8\right) E_2^4}{2880E_4^5 E_6^3} \\ &- \frac{\left(1954449E_4^{15} + 135004860E_4^{12} E_6^2 + 452446390E_4^9 E_6^4 + 215240996E_4^6 E_6^6 + 12237645E_6^8 E_4^3 - 540E_6^{10}\right) E_2^3}{4320E_6^4 E_4^6} \end{aligned} \tag{4.31}$$

$$
\begin{aligned}
&+\frac{\left(484499421E_4^{12}+8105065554E_6^2E_4^9+13840109482E_6^4E_4^6+3239239130E_6^6E_4^3+39065765E_6^8\right)E_2^2}{10080E_4^4E_6^3}\\
&-\frac{\left(482143347E_4^{15}+70093473948E_6^2E_4^{12}+394187429578E_6^4E_4^9+312678218260E_6^6E_4^6\right)E_2}{100800E_4^5E_6^4}\\
&-\frac{\left(31095995615E_6^8E_4^3+29687000E_6^{10}\right)E_2}{100800E_4^5E_6^4}+h_4(q)\Bigg]
\end{aligned}
\tag{4.32}
$$

The ambiguous part is given by

$$
\begin{aligned}
h_2 &= \frac{441E_4^3}{10E_6}+\frac{383E_6}{6}\\
h_3 &= -\frac{21909E_4^6}{20E_6^2}-\frac{3426562E_4^3}{315}-\frac{4063991E_6^2}{630}-\frac{21205E_6^4}{252E_4^3}\\
h_4 &= \frac{150204789E_4^9}{1600E_6^3}+\frac{2879128369E_4^6}{1400E_6}+\frac{731025235537E_6E_4^3}{151200}\\
&\quad+\frac{166127444801E_6^3}{90720}+\frac{22522691E_6^5}{320E_4^3}+\frac{8E_6^7}{27E_4^6}
\end{aligned}
\tag{4.33}
$$

which has been determined by imposing the gap conditions (2.29).

The conformal limit. The theory becomes conformal in the limit $m \to 0$ and fits into the class **A** according to (3.2). In this limit $\tau \to e^{\pi i/3}$. Therefore E_2, E_6 become constants[14] and E_4 vanishes. More precisely, using (4.30), we find

$$
\begin{aligned}
u &\approx \frac{a^3}{6\sqrt{6}E_6^{\frac{1}{2}}}\\
E_4 &\approx \frac{2^4\,3^3\,E_6^2\,m^8}{a^8}\\
\xi &\approx \frac{2^6\,3^4\,E_6\,m^8}{a^9}
\end{aligned}
\tag{4.34}
$$

From the above formulae we notice that while both E_4 and ξ go to zero in the limit $m \to 0$, their ratio stays finite and goes like

$$
\frac{\xi}{E_4} \approx \frac{12}{E_6\,a}\,. \tag{4.35}
$$

Keeping only the leading terms in (4.31) and (4.33), and using (4.35), one finds

$$
\begin{aligned}
\mathcal{F}_2 &\approx \frac{E_2}{4a^2}\\
\mathcal{F}_3 &\approx -\frac{E_2^2}{32a^4}\\
\mathcal{F}_4 &\approx \left(\frac{E_2^3}{192}+\frac{E_6}{81}\right)\frac{1}{a^6}
\end{aligned}
\tag{4.36}
$$

[14] Their values are clearly the same as in the $\mathcal{H}_0$ and $\mathcal{H}_2$ theories.

The above formulae reproduce the result (3.12), with

$$\beta = 1\,, \quad \delta = 3\,, \quad \gamma = 3\,, \quad \kappa = \frac{E_6}{E_2^3}\,, \quad x = \frac{E_2\epsilon^2}{6a^2}\,, \tag{4.37}$$

and

$$c_0 = 1 \quad , \quad c_1 = -\frac{8}{3}\,. \tag{4.38}$$

Higher-genus prepotentials $\mathcal{F}_g$ can also be computed in exactly the same manner as in the previous examples.

4.5 E_7 theory

The SW curve for the E_7 theory deformed by the eighteenth-order mass Casimir is

$$y^2 = 4x^3 - u^3x - m^{18}\,. \tag{4.39}$$

In this case $g_2 = u^3$ and $g_3 = m^{18}$ leading to

$$\begin{aligned} u &= \frac{3^{\frac{1}{3}}m^4E_4^{\frac{1}{3}}}{E_6^{\frac{2}{9}}}\,, \qquad \omega_1 = \sqrt{\frac{2}{3}}\left(\frac{E_6^{\frac{1}{6}}}{m^3}\right)\,, \qquad \xi = \frac{3^{\frac{13}{6}}E_4^{\frac{2}{3}}E_6^{\frac{19}{18}}}{2^{\frac{3}{2}}m(E_4^3-E_6^2)} \\ \mathcal{F}_1 &= \frac{1}{12}\log\left(\frac{(E_4^3-E_6^2)m^{54}}{E_6^3}\right) + \text{const}\,, \quad \Delta = 16(u^9 - 27m^{36}) \end{aligned} \tag{4.40}$$

Here the holomorphic ambiguity takes again the form of the second expression in (3.15). Solving recursively (2.10) one finds the first few terms

$$\mathcal{F}_2 = \frac{\xi^2}{24\,12^2}\left[\frac{5E_2^3}{3} + \left(\frac{E_4^2}{3E_6} + \frac{8E_6}{3E_4}\right)E_2^2 + \left(\frac{7E_4^4}{E_6^2} - 62E_4\right)E_2 + h_2(q)\right] \tag{4.41}$$

$$\begin{aligned} \mathcal{F}_3 = \frac{\xi^4}{24\,12^4}\Bigg[&\frac{5E_2^6}{6} + \frac{10E_2^5(17E_4^3+10E_6^2)}{27E_4E_6} + E_2^4\Big(\frac{191E_4^4}{18E_6^2} + \frac{80E_6^2}{27E_4^2} + \frac{754E_4}{27}\Big) \\ &+ \frac{E_2^3(229E_4^6 - 7487E_4^3E_6^2 - 7250E_6^4)}{81E_6^3} - \frac{E_2^2(210E_4^9 + 142680E_4^6E_6^2 + 577955E_4^3E_6^4 + 101728E_6^6)}{270E_4E_6^4} \\ &- \frac{E_2(6905E_4^9 + 128405E_4^6E_6^2 + 59572E_4^3E_6^4 - 2624E_6^6)}{135E_4^2E_6^3} + h_3(q)\Bigg] \end{aligned} \tag{4.42}$$

The ambiguous part is given by

$$\begin{aligned} h_2 &= -\frac{191E_4^3}{15E_6} + \frac{82E_6}{15} \\ h_3 &= -\frac{302161E_6^4}{5184E_4^3} - \frac{2294657E_4^3}{2592} + \frac{230203E_6^2}{3240} - \frac{1544857E_4^9}{25920E_6^4} \end{aligned} \tag{4.43}$$

$$\tag{4.44}$$

which has been determined by imposing the gap conditions (2.29).

The conformal limit. The theory becomes conformal in the limit $m \to 0$ and fits into the class **B** according to (3.2). In this limit $\tau \to i$. Therefore E_2, E_4 become constant[15] and E_6 vanishes. More precisely, using (4.40), we find

$$
\begin{aligned}
u &\approx \frac{3\,a^4}{2^{10}\,E_4} \\
E_6 &\approx \frac{2^{45}\,E_4^6\,m^{18}}{3^3\,a^{18}} \\
\xi &\approx \frac{2^{46}\,E_4^4\,m^{18}}{3\,a^{19}}
\end{aligned}
\tag{4.45}
$$

From the above formulae we notice that while both E_6 and ξ go to zero in the limit $m \to 0$, their ratio stays finite and goes like

$$
\frac{\xi}{E_6} \approx \frac{18}{E_4^2\,a}\,. \tag{4.46}
$$

Keeping only the leading terms in (4.41) and (4.43), and using (4.46), one finds

$$
\begin{aligned}
\mathcal{F}_2 &\approx \frac{21E_2}{32a^2} \\
\mathcal{F}_3 &\approx -\left(\frac{21E_2^2}{128} + \frac{1544857E_4}{330598817280}\right)\frac{1}{a^4}
\end{aligned}
\tag{4.47}
$$

The above formulae reproduce the result (3.12), with

$$
\beta = 1\,, \qquad \delta = 2\,, \qquad \gamma = \frac{9}{2}\,, \qquad \kappa = \frac{E_4}{E_2^2}\,, \qquad x = \frac{E_2\epsilon^2}{6a^2}\,, \tag{4.48}
$$

and

$$
c_0 = 1\,, \quad c_1 = -\frac{4634571}{10240}\,. \tag{4.49}
$$

Higher-genus prepotentials $\mathcal{F}_g$ can also be computed in exactly the same manner as in the previous examples.

4.6 $\mathbf{E_8}$ theory

The SW curve for the E_8 theory deformed by the twentyth-order mass Casimir is

$$
y^2 = 4x^3 - m^{20}x - u^5\,. \tag{4.50}
$$

In this case $g_2 = m^{20}$ and $g_3 = u^5$ leading to

$$
\begin{gathered}
u = \frac{E_6^{\frac{1}{5}}\,m^6}{3^{\frac{3}{10}}\,E_4^{\frac{3}{10}}}\,, \qquad \omega_1 = \left(\frac{\sqrt{2}E_4^{\frac{1}{4}}}{3^{\frac{1}{4}}m^5}\right)\,, \qquad \xi = \frac{5\,3^{\frac{11}{20}}\,E_4^{\frac{21}{20}}\,E_6^{\frac{4}{5}}}{\sqrt{2}m\,(E_6^2 - E_4^3)} \\
\mathcal{F}_1 = \frac{1}{12}\log\left(\frac{(E_4^3 - E_6^2)\,m^{90}}{E_4^{\frac{9}{2}}}\right) + \text{const}\,, \quad \Delta = 16(m^{60} - 27u^{10})
\end{gathered}
\tag{4.51}
$$

[15] Their values are clearly the same as in the $\mathcal{H}_1$ theory.

Here the holomorphic ambiguity takes again the form of the first expression in (3.15). Solving recursively (2.10) one finds the first few terms

$$\begin{aligned}
\mathcal{F}_2 &= \frac{\xi^2}{24\,12^2}\left[\frac{5E_2^3}{3}+\left(\frac{24E_4^2}{5E_6}-\frac{9E_6}{5E_4}\right)E_2^2+\left(\frac{39E_6^2}{5E_4^2}-\frac{314E_4}{5}\right)E_2+h_2(q)\right]\\
\mathcal{F}_3 &= \frac{\xi^4}{24\,12^4}\left[\frac{5E_2^6}{6}+\frac{10\left(2E_4^3+E_6^2\right)E_2^5}{3E_4E_6}+\left(\frac{1776E_4^4}{150E_6^2}+\frac{4088E_4}{150}+\frac{361E_6^2}{150E_4^2}\right)E_2^4\right.\\
&\quad+\frac{\left(2592E_4^9-85236E_6^2E_4^6-119429E_4^3E_6^4+573E_6^6\right)E_2^3}{1125E_6^3E_4^3}-\left(-\frac{307080E_4^5}{750E_6^2}+\frac{1603513E_4^2}{750}-\frac{468E_6^4}{750E_4^4}\right)E_2^2\\
&\quad\left.+\left(\frac{373864E_6^2}{750E_4}\right)E_2^2-\frac{\left(-13392E_4^9+705816E_6^2E_4^6+1812719E_6^4E_4^3+165107E_6^6\right)E_2}{1875E_4^2E_6^3}+h_3(q)\right] \qquad (4.52)\\
\mathcal{F}_4 &= \frac{\xi^6}{24\,12^6}\left[\frac{1105E_2^9}{1296}+\left(\frac{197E_4^2}{18E_6}+\frac{745E_6}{96E_4}\right)E_2^8+\left(\frac{80144E_4^4}{1800E_6^2}+\frac{205727E_4}{1800}+\frac{34279E_6^2}{1800E_4^2}\right)E_2^7\right.\\
&\quad+\frac{\left(1660608E_4^9+13257736E_6^2E_4^6+495277E_6^4E_4^3+11244E_6^6\right)E_2^6}{27000E_6^3E_4^3}\\
&\quad+\frac{\left(8688384E_4^{12}+117129744E_6^2E_4^9-201837711E_6^4E_4^6-104795596E_6^6E_4^3+2173929E_6^8\right)E_2^5}{450000E_4^4E_6^4}\\
&\quad-\frac{\left(120661632E_4^{12}+1894444360E_6^2E_4^9+2294021113E_6^4E_4^6+241621016E_6^6E_4^3-1371E_6^8\right)E_2^4}{2880E_4^4E_6^4}+\frac{39E_6^6E_2^3}{500E_4^6}\\
&\quad-\frac{\left(657891072E_4^{12}+43610234472E_4^9E_6^2+155402608452E_4^6E_6^4+70330783117E_4^3E_6^6+2529429287E_6^8\right)E_2^3}{675000E_6^4E_4^3}\\
&\quad-\frac{\left(211652806254E_4^9+3413887736674E_6^2E_4^6+5487101754954E_6^4E_4^3+1215593719179E_6^6\right)E_2^2}{42525000E_4E_6^3}\\
&\quad++\frac{2142744527E_6^5E_2^2}{6075000E_4^4}+\frac{\left(2590531017336E_4^{10}+248754470552736E_6^2E_4^7+1207090705967416E_6^4E_4^4\right)E_2}{425250000E_6^4}\\
&\quad\left.+\frac{35035331393449E_6^2E_4E_2}{8859375}+\frac{\left(70923207769701E_6^8E_4^3+31893238160E_6^{10}\right)E_2}{425250000E_4^5E_6^4}+h_4(q)\right] \qquad (4.53)
\end{aligned}$$

The ambiguous part is given by

$$\begin{aligned}
h_2 &= \frac{124E_4^3}{25E_6}-\frac{917E_6}{75}\\
h_3 &= \frac{21308148037E_4^3}{2835000}+\frac{11593299743E_6^2}{2835000}+\frac{56925211E_6^4}{1215000E_4^3}+\frac{7183179683E_4^6}{8505000E_6^2}\\
h_4 &= -\frac{392331535221859E_6^3}{273375000}-\frac{460255802444281E_6E_4^3}{1913625000}-\frac{8109292812051391E_4^6}{3827250000E_6}\\
&\quad-\frac{460255802444281E_4^9}{3827250000E_6^3}-\frac{12356384824399E_6^5}{273375000E_4^3}+\frac{4482151319E_6^7}{109350000E_4^6} \qquad (4.54)
\end{aligned}$$

which has been determined by imposing the gap conditions (2.29).

The conformal limit. The theory becomes conformal in the limit $m \to 0$ and fits into the class **A** according to (3.2). In this limit $\tau \to e^{\pi i/3}$. Therefore E_2, E_6 become constants and E_4 vanishes. More precisely, using (4.51), we find

$$\begin{aligned} u &\approx \frac{a^6}{2^9\, 3^3\, E_6} \\ E_4 &\approx \frac{3^9\, 2^{30}\, E_6^4\, m^{20}}{a^{20}} \\ \xi &\approx \frac{5\, 2^{31}\, 3^{10}\, E_6^3\, m^{20}}{a^{21}} \end{aligned} \tag{4.55}$$

From the above formulae we notice that while both E_4 and ξ go to zero in the limit $m \to 0$, their ratio stays finite and goes like

$$\frac{\xi}{E_4} \approx \frac{30}{E_6\, a}\,. \tag{4.56}$$

Keeping only the leading terms in (4.52), (4.53) and (4.54), and using (4.56), one finds

$$\begin{aligned} \mathcal{F}_2 &\approx \frac{65E_2}{32a^2} \\ \mathcal{F}_3 &\approx -\frac{65E_2^2}{64a^4} \\ \mathcal{F}_4 &\approx \left(\frac{1625E_2^3}{2048} + \frac{22410756595E_6}{53747712}\right)\frac{1}{a^6} \end{aligned} \tag{4.57}$$

The above formulae reproduce the result (3.12), with

$$\beta = 1\,, \qquad \delta = 3\,, \qquad \gamma = \frac{15}{2}\,, \qquad \kappa = \frac{E_6}{E_2^3}\,, \qquad x = \frac{E_2\epsilon^2}{6a^2}\,, \tag{4.58}$$

and

$$c_0 = 1\,, \quad c_1 = \frac{22410756595}{248832}\,. \tag{4.59}$$

Higher-genus prepotentials $\mathcal{F}_g$ can also be computed in exactly the same manner as in the previous examples.

5 NS limit: $\beta = 0$

In this section we consider the NS limit $\epsilon_1 \to 0$, i.e. $\beta \to 0$ [76]. In this limit the prepotential takes the form

$$\mathcal{F} = \sum_{g=0} \epsilon_2^{2g} \mathcal{F}_g \tag{5.1}$$

where

$$\mathcal{F}_g = \mathcal{F}_{0,g} = \lim_{\beta\to 0} \beta^{2g}\, \mathcal{F}_g(\beta) \tag{5.2}$$

The holomorphic anomaly (2.9) for $\widehat{\mathcal{F}} = \mathcal{F} - \mathcal{F}_0$ becomes

$$\partial_{E_2}\widehat{\mathcal{F}} = \frac{1}{24}\left(\partial_a \widehat{\mathcal{F}}\right)^2 \tag{5.3}$$

or equivalently, using (5.1),

$$\partial_{E_2}\widehat{\mathcal{F}}_g = \frac{1}{24}\sum_{g'=1}^{g-1}\partial_a\widehat{\mathcal{F}}_{g'}\partial_a\widehat{\mathcal{F}}_{g-g'} \tag{5.4}$$

This equation allows to compute all $\widehat{\mathcal{F}}_g$ terms recursively starting from $\mathcal{F}_1$, given in (2.18). Sending $\beta \to 0$ and using (2.17), we get

$$\widehat{\mathcal{F}}_1 = \frac{1}{24}\log\frac{1024(E_4^3-E_6^2)}{27\omega_1^{12}} \underset{m\to 0}{\approx} \tilde{\gamma}\log\left(\frac{a}{\epsilon_2}\right) + \text{const} \tag{5.5}$$

with

$$\tilde{\gamma} = \lim_{\beta\to 0}\beta^2\gamma = \frac{d-1}{2}\,. \tag{5.6}$$

Equation (5.4) has been extensively studied in the context of the WKB expansion for a certain class of quantum mechanical operators corresponding to AD theories with some deformations [74, 75, 83–85]. See also [32, 86–89] for other works relating WKB and non-lagrangian theories.

Here we are interested in the conformal limit where such deformationss are turned off. From the point of view of quantum mechanics this corresponds to having a potential with a single term of the form $V(x) = x^n$, $n \geq 3$. Parallel to (3.10) we make the following Ansatz to capture the conformal limit[16]

$$\epsilon_1^{-2}\mathcal{F} = \frac{\tilde{\gamma}}{2}\log(E_2) + \sum_{n\geq 0}\left(\frac{E_{2\delta}}{E_2^\delta}\right)^n f_n(x)\,, \qquad x = \frac{E_2\epsilon_2^2}{6a^2}\,. \tag{5.7}$$

Equation (5.3) can then be solved order by order in $E_{2\delta}$. At zero order we have[17]

$$\tilde{\gamma} - 2x^3 f_0'(x)^2 + 2x f_0'(x) = 0 \tag{5.8}$$

where the boundary conditions are chosen such that the solution does not have negative power-like behavior as $x \to 0$. This gives

$$f_0(x) = \frac{\sqrt{2\tilde{\gamma}x+1} - x\tilde{\gamma}\log\left(\frac{\tilde{\gamma}x+\sqrt{2\tilde{\gamma}x+1}+1}{\tilde{\gamma}x}\right) - 1}{2x}\,. \tag{5.9}$$

At the first order in $E_{2\delta}$, (5.3) gives

$$x\left(2x^2 f_0'(x) - 1\right) f_1'(x) + \delta f_1(x) = 0\,. \tag{5.10}$$

Using (5.9) we obtain

$$f_1(x) = c_1\left(\frac{\tilde{\gamma}x - \sqrt{2\tilde{\gamma}x+1}+1}{\tilde{\gamma}x}\right)^\delta\,, \tag{5.11}$$

[16] Recall that $\mathcal{F}$ in this paper is defined up to a multiplicative constant.

[17] We recall that in this conformal limit a and E_2 can be treated as independent variables.

where c_1 is the integration constant. Likewise, at second order we find

$$f_2(x) = \left(\frac{\gamma x - \sqrt{2\gamma x + 1} + 1}{\gamma x}\right)^{2\delta} \left(c_2 - c_1^2 \frac{\delta^2}{\gamma\sqrt{2\gamma x + 1}}\right) \tag{5.12}$$

where c_2 is the integration constant and c_1 is as in (5.11). In principle higher-order f_n terms can also be obtained in a similar manner. However, in contrast to the case of finite β, when $\beta = 0$ we do not have a general form for f_n. The value of the integration constants c_n is fixed by the holomorphic ambiguity.

The example of $\mathcal{H}_0$. Let us spell out some detail for the case of $\mathcal{H}_0$. The starting point of the recursion is given in (5.5), where m is the relevant coupling $c^{5/4}$. The special geometry relations are as in (4.2) and we have $\tilde{\gamma} = 1/10$, $\delta = 3$. The ambiguity is of the form (3.15) (first line) and the gap condition is given in (2.30). Using these initial conditions and running the equation (5.3), we obtain

$$\begin{aligned}
\mathcal{F}_2 &= \frac{1}{24\ 12^2}\xi^2\left[\frac{E_2E_6^2}{3456E_4^2} + h_2(q)\right] \\
\mathcal{F}_3 &= \frac{1}{24\ 12^4}\xi^4\left[-\frac{E_2^3E_6^3}{4458050224128E_4^3} - \frac{E_2^2\left(6E_4^3E_6^2+5E_6^4\right)}{1486016741376E_4^4} - \frac{E_2\left(474E_4^3E_6+1427E_6^3\right)}{7430083706880E_4^2} + h_3(q)\right] \\
\mathcal{F}_4 &= \frac{1}{24\ 12^4}\xi^4\left[\frac{E_2^5E_6^4}{739537035580145664E_4^4} + \frac{E_2^4\left(24E_4^3E_6^3+23E_6^5\right)}{739537035580145664E_4^5}\right. \\
&\quad + \frac{E_2^2\left(5688E_4^6E_6+55638E_4^3E_6^3+47141E_6^5\right)}{1848842588950364160E_4^4} + \frac{E_2^3\left(2502E_4^6E_6^2+8011E_4^3E_6^4+1250E_6^6\right)}{5546527766851092480E_4^6} \\
&\quad \left. + \frac{E_2\left(1572732E_4^9+95314012E_4^6E_6^2+199451203E_4^3E_6^4+24620960E_6^6\right)}{129418981226525491200E_4^5} + h_4(q)\right]
\end{aligned} \tag{5.13}$$

with

$$\xi = \frac{3^{\frac{7}{4}}\,E_4^{\frac{9}{4}}}{2^{\frac{1}{2}}c^{\frac{5}{4}}(E_6^2 - E_4^3)}\,, \tag{5.14}$$

and

$$\begin{aligned}
h_2 &= \frac{79E_6}{5} \\
h_3 &= -\frac{21983E_6^4}{22290251120640E_4^3} - \frac{5611E_4^3}{8668430991360} - \frac{47731E_6^2}{11145125560320} \\
h_4 &= \frac{8670019E_6^7}{19412847183978823680E_4^6} + \frac{382204771E_6^5}{18488425889503641600E_4^3} \\
&\quad + \frac{107731843E_4^3E_6}{8088686326657843200} + \frac{706159453E_6^3}{13866319417127731200}\,.
\end{aligned} \tag{5.15}$$

JHEP07(2023)195

The results perfectly agree with those obtained in [74]. The conformal limit $c \to 0$ can be computed using (4.5) and (4.6), and gives

$$
\begin{aligned}
\widehat{\mathcal{F}} \approx\ & \frac{\log(a)}{10} + \frac{E_2}{2400a^2} - \frac{E_2^2}{288000a^4} + \frac{4375E_2^3+8670019E_6}{90720000000a^6} \\
& - \frac{34680076E_2E_6+6125E_2^4}{7257600000000a^8} + \frac{26010057E_2^2E_6+2450E_2^5}{145152000000000a^{10}} \\
& - \frac{8523712429375E_2^3E_6+516140625E_2^6+261612601805031778E_6^2}{1401079680000000000000a^{12}} \\
& + \frac{155131566214625E_2^4E_6+1466105490089461119 1E_2E_6^2+6709828125E_2^7}{784604620800000000000000a^{14}} + O\left(\frac{1}{a^{16}}\right) .
\end{aligned}
\tag{5.16}
$$

This agrees with (5.7) for

$$
c_1 = \frac{8670019}{52500} \qquad , \qquad c_2 = -\frac{1458581050220478983}{2627625000} . \tag{5.17}
$$

6 Outlook

This paper exploits the holomorphic anomaly equation to compute the partition function of intrinsically strongly coupled SCFT's with eight supercharges living on a generic Ω-background. We studied one-parameter deformations of such theories allowing for an exact integration of the anomaly equation, order by order in the ϵ expansion of the Ω-deformed prepotential. Within this framework, we observed important simplifications at the conformal point. The Ω-deformed prepotential is given by the elegant formula (1.1) in terms of hypergeometric functions, with coefficients c_n determined by the gap conditions. It would be interesting to understand whether this non-trivial re-organization of the ϵ expansion of the partition function can improve the precision in the computation of extremal correlators made in [45, 62], and shed light on the analytic structure of the exact answer. We will report on this in [90].

In the NS phase of the Ω background ($\epsilon_1 = 0$), we show that (1.1) undergo a non-trivial re-organization in which the hypergeometric functions become simpler functions, see e.g (5.9) and (5.11). This results are relevant for the study of quantum periods of anharmonic oscillators. We will report more on this in [90].

Acknowledgments

We would like to thank Santiago Codesido for useful discussions. The authors thank the CERN theory division and INFN Tor Vergata for hospitality at various stages of this work. The work of FF and JFM is partially supported by the MIUR PRIN Grant 2020KR4KN2 "String Theory as a bridge between Gauge Theories and Quantum Gravity". The work of AG is partially supported by the Swiss National Science Foundation Grant No. 185723 and the NCCR SwissMAP.

A SW curves for SQCD

In this appendix we review the SQCD/AD dictionary for SW curves. The SW curves for a $SU(2)$ gauge theory with $0 < N_f < 4$ hypermultiplets transforming in the fundamental representation of the gauge group are given by

$$\hat{y}^2 + \hat{y}P(x) + q\prod_{i=1}^{N_f}(x - m_i) = 0 \tag{A.1}$$

with $q = \Lambda^{4-N_f}/4$ and

$$P(x) = \begin{cases} x^2 - u & N_f = 1\,, \\ x^2 - u + q & N_f = 2\,, \\ x^2 - u + q(x - \sum_i m_i) & N_f = 3\,. \end{cases} \tag{A.2}$$

is chosen in such a way that $u = \frac{1}{2}\mathrm{tr}\varphi^2$. The periods of the holomorphic one-form are

$$\frac{\partial a(u)}{\partial u} = \frac{1}{2\pi i}\oint_\alpha \frac{dx}{w(x)}\,, \qquad \frac{\partial a_D(u)}{\partial u} = \frac{1}{2\pi i}\oint_\beta \frac{dx}{w(x)} \tag{A.3}$$

with

$$w(x)^2 = d_0\prod_{i=1}^{4}(x - e_i) = \sum_{i=0}^{4} d_i x^{4-i} = P(x)^2 - 4q\prod_{i=1}^{N_f}(x - m_i) \tag{A.4}$$

The quantum correlator of the gauge theory can be obtained from the large x-expansion of the SW differential

$$-2\pi \mathrm{i}\lambda = x\frac{d\log\hat{y}(x)}{dx} \approx \sum_{n=0}^{\infty}\frac{\langle \mathrm{tr}\varphi^n\rangle}{x^n} \approx 2 + \frac{2u}{x^2} + \dots \tag{A.5}$$

leading to $u = \frac{1}{2}\mathrm{tr}\varphi^2$. To write the elliptic curve (A.4) into the Weierstrass form, we introduce the variables (y, z) related to (w, x) via

$$\frac{1}{x - e_4} = \frac{z}{\nu} + \delta\,, \qquad w = \frac{\mathrm{i}\,\nu\, y}{2(z + \nu\delta)^2} \tag{A.6}$$

with

$$\nu = d_0\prod_{i=1}^{3}(e_i - e_4)\,, \qquad \delta = \frac{1}{3}\sum\frac{1}{e_i - e_4} \tag{A.7}$$

In the new variables the SW curve takes the Weierstrass form

$$y^2 = 4z^3 - g_2 z - g_3 \tag{A.8}$$

with

$$\begin{aligned} g_2 &= \frac{4d_2^2}{3} - 4d_1d_3 + 16d_0d_4 \\ g_3 &= \frac{8d_2^3}{27} - \frac{4}{3}d_1d_3d_2 - \frac{32d_0d_2d_4}{3} + 4d_3^2d_0 + 4d_1^2d_4 \end{aligned} \tag{A.9}$$

Finally the discriminant of the Weierstrass is given by

$$\Delta = 16(g_2^3 - 27g_3^2) \tag{A.10}$$

A.1 $\mathcal{H}_0$ theory

The AD $\mathcal{H}_0$ theory can be obtained by tuning the parameters spanning the moduli space of $SU(2)$ with $N_f = 1$ fundamental flavors. For $N_f = 1$, the elliptic curve is given by

$$y^2 = 4z^3 - g_2 z - g_3 \tag{A.11}$$

with

$$\begin{aligned} g_2 &= \frac{64u^2}{3} + 16\Lambda^3 m \\ g_3 &= \frac{512u^3}{27} + \frac{64}{3}\Lambda^3 mu + 4\Lambda^6 \end{aligned} \tag{A.12}$$

The AD point is obtained by taking

$$\begin{aligned} u &= \frac{3\Lambda^2}{4} + u_{\text{AD}}\frac{\Lambda^{\frac{4}{5}}}{4} - c_{\text{AD}}\frac{\Lambda^{\frac{6}{5}}}{4}\,, & m &= -\frac{3\Lambda}{4} + c_{\text{AD}}\frac{\Lambda^{\frac{1}{5}}}{4} \\ z &= \tilde{z}\Lambda^{\frac{8}{5}}\,, & y &= \tilde{y}\Lambda^{\frac{12}{5}} \end{aligned} \tag{A.13}$$

and keeping the leading order in $u_{\text{AD}}, c_{\text{AD}} \to 0$. This leads to

$$\tilde{y}^2 = 4\tilde{z}^3 + 4c_{\text{AD}}\tilde{z} - 4u_{\text{AD}} \tag{A.14}$$

The same SW curve can be obtained from the quartic expression

$$w^2 = z^8\left(\frac{1}{z^7} + \frac{c_{\text{AD}}}{z^5} + \frac{u_{\text{AD}}}{z^4}\right)\,, \tag{A.15}$$

where the SW differential is given by [91] (see also [92] for a review)

$$\lambda = \frac{w}{z^4}\mathrm{d}z \tag{A.16}$$

A.2 $\mathcal{H}_1$ theory

The AD $\mathcal{H}_1$ theory can be obtained by tuning the parameters spanning the moduli space of $SU(2)$ with $N_f = 2$ flavors transforming in the fundamental representation of the gauge group. The elliptic curve is given by

$$y^2 = 4z^3 - g_2 z - g_3 \tag{A.17}$$

with

$$\begin{aligned} g_2 &= \frac{4}{3}\left(\Lambda^2\left(\Lambda^2 + 12\mu^2 - 12m^2\right) + 16u^2 - 4\Lambda^2 u\right) \\ g_3 &= \frac{8}{27}\left(\Lambda^4\left(\Lambda^2 + 18\mu^2 + 36m^2\right) - 6\Lambda^2 u\left(\Lambda^2 - 12\mu^2 + 12m^2\right) + 64u^3 - 24\Lambda^2 u^2\right) \end{aligned}$$

with $m = \frac{1}{2}(m_1 + m_2)$, $\mu = \frac{1}{2}(m_1 - m_2)$. The AD point is obtained by taking

$$\begin{aligned} u &= \frac{\Lambda^2}{2} + \Lambda^{\frac{2}{3}}u_{\text{AD}} + \Lambda\mu - \frac{\Lambda^{\frac{4}{3}}}{4}c\,, & m &= \frac{\Lambda}{2} - \frac{\Lambda^{\frac{1}{3}}}{4}c \\ z &= \tilde{z}\Lambda^{\frac{4}{3}}\,, & y &= \tilde{y}\Lambda^2 \end{aligned} \tag{A.18}$$

and taking $u_{\rm AD}$, c and μ small

$$\tilde{y}^2 = 4\tilde{z}^3 - 4\left(4u_{\rm AD} + \frac{c^2}{3}\right)\tilde{z} - 4\left(\frac{2}{27}c^3 - \frac{8}{3}c\,u_{\rm AD} + \mu^2\right) \tag{A.19}$$

The same curve is obtained starting from the standard $\mathcal{H}_1$ quartic form

$$w^2 = z^8\left(\frac{1}{z^8} + \frac{c}{z^6} + \frac{\mu}{z^5} + \frac{u_{\rm AD}}{z^4}\right) \tag{A.20}$$

A.3 $\mathcal{H}_2$ theory

The AD $\mathcal{H}_2$ theory can be obtained by tuning the parameters spanning the moduli space of $SU(2)$ with $N_f = 3$ flavors transforming in the fundamental representation of the gauge group. The elliptic curve is given by

$$y^2 = 4z^3 - g_2 z - g_3 \tag{A.21}$$

$$\begin{aligned}
g_2 &= \frac{64u^2}{3} + \frac{8\Lambda}{3}\left(2C_3 - 3C_2 m + 6m\left(m^2 + u\right)\right) + \Lambda^2\left(C_2 + 6m^2 - \frac{4u}{3}\right) - \frac{\Lambda^3 m}{2} + \frac{\Lambda^4}{192} \\
g_3 &= \frac{512u^3}{27} + \frac{32}{9}\Lambda u\left(2C_3 + 6m\left(m^2 + u\right) - 3C_2 m\right) - \frac{\Lambda^5 m}{96} + \frac{\Lambda^6}{13824} \\
&\quad + \Lambda^2\left(C_2^2 - 4C_2 m^2 - \frac{16C_3 m}{3} - \frac{8C_2 u}{3} + 20m^4 + \frac{20u^2}{9}\right) \\
&\quad + \frac{\Lambda^3}{18}\left(-21C_2 m + 2C_3 - 30m^3 + 30mu\right) + \frac{\Lambda^4}{144}\left(3C_2 + 54m^2 - 4u\right)
\end{aligned} \tag{A.22}$$

with

$$m = \frac{1}{3}\sum_i m_i\,, \qquad C_2 = \sum_i (m_i - m)^2 \qquad C_3 = \sum_i (m_i - m)^3 \tag{A.23}$$

The AD point is obtained by taking

$$\begin{aligned}
u &= \frac{5\Lambda^2}{64} - \Lambda^{\frac{1}{2}}\left(u_{\rm AD} + \frac{c^3}{24}\right) + \frac{3\Lambda^{\frac{3}{2}}}{16}c + \frac{\Lambda}{16}c^2\,, \qquad & m &= -\frac{\Lambda}{8} - \frac{\Lambda^{\frac{1}{2}}}{4}c \\
z &= \tilde{z}\Lambda\,, & y &= \tilde{y}\Lambda^{\frac{3}{2}}
\end{aligned} \tag{A.24}$$

and taking $u_{\rm AD}$, c and C_2, C_3 small

$$\tilde{y}^2 = 4\tilde{z}^3 + \tilde{z}\left(4cu_{\rm AD} + \frac{c^4}{12} - 2C_2\right) - 4u_{\rm AD}^2 + \frac{c^6}{432} - \frac{c^2 C_2}{6} - \frac{4C_3}{3} \tag{A.25}$$

The same curve is obtained starting from the standard $\mathcal{H}_2$ quartic form

$$w^2 = z^6\left(\frac{1}{z^6} + \frac{c}{z^5} + \frac{\mu}{z^4} + \frac{u_{\rm AD}}{z^3} + \frac{M^2}{z^2}\right) \tag{A.26}$$

after the identification

$$\begin{aligned}
C_2 &= \frac{1}{24}\left(c^4 + 16\mu^2 + 192M^2 - 8\mu c^2\right) \\
C_3 &= \frac{1}{288}\left(64\mu^3 - c^6 - 24c^2\mu^2 + 576c^2M^2 - 2304\mu M^2 + 12c^4\mu - 24c^2\mu^2\right)
\end{aligned} \tag{A.27}$$

JHEP07(2023)195

B Modular functions

In this section we collect some definitions and useful modular identities. The Eisenstein series are defined as

$$E_k(q) = 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \frac{n^{k-1} q^{2n}}{1-q^{2n}} \tag{B.1}$$

A basis of modular forms is given by E_4, E_6 and the quasi-modular form E_2. They are related to the theta functions via

$$\begin{aligned} E_4 &= \frac{1}{2}(\theta_2^8 + \theta_3^8 + \theta_4^8) \\ E_6^2 &= \frac{1}{8}\left[(\theta_2^8 + \theta_3^8 + \theta_4^8)^3 - 54\, 2^8\, \eta^{24}\right] \\ E_2 &= 12 q \partial_q \log \eta(q) \end{aligned} \tag{B.2}$$

We introduce the functions

$$K_2 = \theta_3^4 + \theta_4^4\,, \qquad L_2 = \theta_2^4 \tag{B.3}$$

Under S-duality they transform as

$$\begin{aligned} K_2(-1/\tau) &= -\tau^2 \frac{K_2(\tau) + 3L_2(\tau)}{2} \\ L_2(-1/\tau) &= -\tau^2 \frac{K_2(\tau) - L_2(\tau)}{2} \end{aligned} \tag{B.4}$$

whereas under T-duality they transform as

$$\begin{aligned} K_2(\tau+1) &= K_2(\tau) \\ L_2(\tau+1) &= -L_2(\tau) \end{aligned} \tag{B.5}$$

In terms of these variables the Eisenstein series read

$$E_4 = \frac{K_2^2 + 3L_2^2}{4}\,, \qquad E_6 = \frac{K_2(K_2^2 - 9L_2^2)}{8} \tag{B.6}$$

C c-deformation of $\mathcal{H}_1$

In this appendix we work at $\beta = 1$ and consider a deformation of $\mathcal{H}_1$ obtained by turning on the IR-relevant coupling c. The SW curve is now described by a cubic curve with

$$g_2 = u\,, \qquad g_3 = c\,u - 4c^3 \tag{C.1}$$

and discriminant

$$\Delta = 16(u - 3c^2)(u - 12c^2)^2 \tag{C.2}$$

We notice that we have now two monopole points $u = 3c^2$ and $u = 12c^2$. It is convenient in this case to introduce the modular functions K_2 and L_2 related to E_4 and E_6 via

$$E_4 = \frac{K_2^2 + 3L_2^2}{4}\,, \qquad E_6 = \frac{K_2(K_2^2 - 9L_2^2)}{8} \tag{C.3}$$

Plugging this into (2.16) and solving for ω_1 and u one finds three inequivalent solutions

$$\begin{aligned} u &= 3c^2\left(1+\frac{3L_2^2}{K_2^2}\right)\,, & \omega_1 &= \mathrm{i}\sqrt{\frac{K_2}{3c}} \\ u &= \frac{12c^2\left(K_2^2+3L_2^2\right)}{(K_2-3L_2)^2}\,, & \omega_1 &= \mathrm{i}\frac{\sqrt{3L_2-K_2}}{\sqrt{6}\sqrt{c}} \\ u &= \frac{12c^2\left(K_2^2+3L_2^2\right)}{(K_2+3L_2)^2}\,, & \omega_1 &= -\mathrm{i}\frac{\sqrt{-K_2-3L_2}}{\sqrt{6}\sqrt{c}}\,. \end{aligned} \tag{C.4}$$

These solutions correspond to three different duality frames related by S, T transformations (B.4), (B.5). In order to make contact with the results derived in section 4.2 for the mass-deformed $\mathcal{H}_1$ theory, we choose the duality frame given by the second line in (C.4). To make formulae simpler, it is convenient to define

$$\hat{K}_2 \equiv -\frac{K_2-3L_2}{2}\,, \qquad \hat{L}_2 \equiv -\frac{K_2+L_2}{2}\,, \tag{C.5}$$

which correspond to the ST transformations of K_2 and L_2 respectively. This leads to

$$\mathcal{F}_1 = \frac{1}{12}\log\left(c^9\frac{\hat{L}_2^2(\hat{L}_2^2-\hat{K}_2^2)^2}{\hat{K}_2^9}\right)\,, \qquad \xi = \frac{2}{\mathrm{i}}\frac{\hat{K}_2^{\frac{5}{2}}}{\sqrt{27c^3}\hat{L}_2^2(\hat{K}_2^2-\hat{L}_2^2)} \tag{C.6}$$

Likewise we define u_D, $\omega_{1D} = da_D/du$, $\mathcal{F}_g^D$ by the same formulae (C.4) replacing $K_2 \to \hat{K}_2$ and $L_2 \to \hat{L}_2$.

The holomorphic ambiguity for the theory has the following form (see [74])

$$h_g = \sum_{i=0}^{3g-4} \hat{L}_2^{2i}\hat{K}_2^{3(g-1)-2i} h_{g,i} \tag{C.7}$$

with coefficients $h_{g,i}$ determined by requiring that both $\mathcal{F}_g$ and $\mathcal{F}_g^D$ satisfy the gap conditions when $a \to 0$ and $a_D \to 0$ respectively, i.e. $q \to 0$ or $q_D \to 0$. Solving recursively, the holomorphic anomaly equation (2.10) one finds the first few terms

$$\begin{aligned} \mathcal{F}_2 &= \frac{\xi^2}{24^3}\left(\frac{5E_2^3}{3} + \frac{3E_2^2\left(\hat{K}_2^2+\hat{L}_2^2\right)}{2\hat{K}_2} + E_2\left(-\frac{81\hat{L}_2^4}{4\hat{K}_2^2} - \frac{55\hat{K}_2^2}{4} + 15\hat{L}_2^2\right) + h_2(q)\right) \\ \mathcal{F}_3 &= \frac{\xi^4}{24^5}\left(\frac{5E_2^6}{6} + E_2^5\left(5\hat{K}_2 - \frac{20\hat{L}_2^2}{3\hat{K}_2}\right) + E_2^4\left(\frac{59\hat{L}_2^4}{8\hat{K}_2^2} + \frac{83\hat{K}_2^2}{8} - \frac{263\hat{L}_2^2}{12}\right)\right. \\ &\quad + E_2^3\left(\frac{447\hat{L}_2^6}{8\hat{K}_2^3} - \frac{403\hat{K}_2^3}{18} - \frac{77\hat{L}_2^4}{2\hat{K}_2} + \frac{465\hat{K}_2\hat{L}_2^2}{8}\right) + h_3(q) \\ &\quad + \frac{E_2\left(199822\hat{K}_2^8 - 779495\hat{K}_2^6\hat{L}_2^2 + 1133751\hat{K}_2^4\hat{L}_2^4 - 801225\hat{K}_2^2\hat{L}_2^6 - 376245\hat{L}_2^8\right)}{480\hat{K}_2^3} \\ &\quad \left. - \frac{E_2^2\left(77573\hat{K}_2^8 - 254926\hat{K}_2^6\hat{L}_2^2 + 292815\hat{K}_2^4\hat{L}_2^4 - 129600\hat{K}_2^2\hat{L}_2^6 + 65610\hat{L}_2^8\right)}{480\hat{K}_2^4}\right) \end{aligned} \tag{C.8}$$

where

$$h_2 = \frac{1619\hat{K}_2^3}{120} - \frac{279\hat{L}_2^4}{8\hat{K}_2} - \frac{111\hat{K}_2\hat{L}_2^2}{4} \tag{C.9}$$

$$h_3 = -\frac{11660261\hat{K}_2^6}{40320} - \frac{3753\hat{L}_2^{10}}{8\hat{K}_2^4} + \frac{20885\hat{K}_2^4\hat{L}_2^2}{16} - \frac{303615\hat{L}_2^8}{128\hat{K}_2^2} - \frac{733469\hat{K}_2^2\hat{L}_2^4}{320} + \frac{31887\hat{L}_2^6}{16} \tag{C.10}$$

These are such that

$$\begin{aligned}\mathcal{F}_g^D &= (-1)^{g-1}2^{2g-1}\frac{B_{2g}}{2g(1-g)}\frac{1}{a_D^{2g-1}} + \mathcal{O}(a_D^0)\\ \mathcal{F}_g &= (-1)^{g-1}2^{2g-2}\frac{B_{2g}}{2g(1-g)}\frac{1}{a^{2g-1}} + \mathcal{O}(a^0)\,.\end{aligned} \tag{C.11}$$

The conformal limit. This is a theory of type **B**, hence $\tau^* = \mathrm{i}$ at the conformal point. In particular at this point both L_2 and K_2 are finite with

$$K_2\Big|_{\tau=\mathrm{i}} = 3L_2\Big|_{\tau=\mathrm{i}}, \qquad L_2\Big|_{\tau=\mathrm{i}} = 3^{1/2}\sqrt{E_4}\Big|_{\tau=\mathrm{i}} \tag{C.12}$$

and

$$a \approx \frac{32i\sqrt{2}c^{3/2}K_2^2}{27\left(L_2 - \frac{K_2}{3}\right)^{3/2}} + \frac{4i\sqrt{2}c^{3/2}(6E_2+4K_2)}{9\sqrt{L_2-\frac{K_2}{3}}} + O\left(\sqrt{L_2 - \frac{K_2}{3}}\right) \tag{C.13}$$

as well as

$$\begin{aligned}&\mathcal{F}_2 \approx -\frac{E_2}{96a^2}\,, \qquad \mathcal{F}_3 \approx \left(-\frac{E_2^2}{1152} - \frac{139E_4}{34992}\right)\frac{1}{a^4}\,, \qquad \mathcal{F}_4 \approx -\frac{E_2\left(7533E_2^2+106752E_4\right)}{40310784a^6}\,,\\ &\mathcal{F}_5 \approx \frac{-597051E_2^4 - 17173728E_2^2E_4 - 44454429E_4^2}{8707129344a^8}\,,\dots\end{aligned} \tag{C.14}$$

which agrees with the results of section 4.2. This provides an explicitly test that the conformal limit is independent of the deformation we perform.

D $\mathrm{SQCD}_{N_f=3}$ at the conformal point

The SW curve of SQCD with $N_f = 3$ flavors of equal mass $m = -\frac{\Lambda}{8}$ is described by a curve in the Weirstrass form with

$$\begin{aligned}g_2 &= \frac{1}{192}\left(64u - 5\Lambda^2\right)^2\,, \qquad g_3 = \frac{\left(64u-5\Lambda^2\right)^2\left(17\Lambda^2+128u\right)}{27648}\\ \Delta &= -\frac{\Lambda^2\left(64u-5\Lambda^2\right)^4\left(7\Lambda^2+256u\right)}{65536}\end{aligned} \tag{D.1}$$

For this choice, formula (2.16) can be explicitly solved and one finds

$$u = -\frac{\Lambda^2\left(17\,E_4^{3/2} + 10E_6\right)}{128\left(E_4^{3/2} - E_6\right)}\,, \qquad \xi = \frac{16\sqrt{\frac{2}{3}\left(E_4^{3/2}-E_6\right)}}{3\Lambda(E_4^{3/2}+E_6)} \tag{D.2}$$

and

$$\mathcal{F}_1 = \frac{1}{12}\log\left(\frac{\Lambda^{18}E_4^9\left(E_4^{3/2}+E_6\right)}{\left(E_4^{3/2}-E_6\right)^8}\right) \tag{D.3}$$

Plugging this into the anomaly equation, one finds for the first few gravitational corrections at $\beta = 1$

$$\begin{aligned}
\mathcal{F}_2 =& \frac{\xi^2}{24\,12^2}\left[\frac{5E_2^3}{3}+\frac{3E_2^2\left(3E_4^{3/2}+4E_6\right)}{E_4}-\frac{E_2\left(-36E_4^{3/2}E_6+7E_4^3+12E_6^2\right)}{E_4^2}+h_2(q)\right]\\
\mathcal{F}_3 =& \frac{\xi^4}{24\,12^4}\left[\frac{5E_2^6}{6}+\left(\frac{5E_6}{E_4}-5E_4^{\frac{1}{2}}\right)E_2^5-\left(\frac{8E_6^2}{E_4^2}+\frac{49E_6}{E_4^{\frac{1}{2}}}-\frac{E_4}{2}\right)E_2^4\right.\\
&+\left(\frac{68E_6^3}{3E_4^3}+\frac{48E_6^2}{E_4^{\frac{3}{2}}}-\frac{520E_6}{9}+96E_4^{\frac{3}{2}}\right)E_2^3-\left(\frac{48E_6^4}{E_4^4}+\frac{144E_6^3}{E_4^{\frac{5}{2}}}+\frac{482E_6^2}{E_4}-\frac{7108}{5}E_4^{\frac{1}{2}}E_6+\frac{4669e_4^2}{6}\right)E_2^2\\
&\left.+\left(\frac{12E_6^3}{E_4^2}-\frac{9676E_6^2}{5E_4^{\frac{1}{2}}}+\frac{101773E_4E_6}{15}-\frac{22947E_4^{\frac{5}{2}}}{5}\right)E_2+h_3(q)\right]\\
\mathcal{F}_4 =& \frac{\xi^6}{24\,12^6}\left[\frac{1105E_2^9}{1296}+E_2^8\left(\frac{5E_6}{E_4}-\frac{625\sqrt{E_4}}{48}\right)+\frac{E_2^7\left(-3543E_4^{3/2}E_6+3130E_4^3-270E_6^2\right)}{36E_4^2}\right.\\
&+\frac{1}{108}E_2^6\left(\frac{9912E_6^2}{E_4^{3/2}}-31530E_4^{3/2}+\frac{2448E_6^3}{E_4^3}+73105E_6\right)\\
&+\frac{E_2^5\left(128520E_4^{3/2}E_6^3+232890E_4^{9/2}E_6+172505E_4^6+235218E_4^3E_6^2+27480E_6^4\right)}{360E_4^4}\\
&+\frac{E_2^4\left(442080E_4^{3/2}E_6^4+2633742E_4^{9/2}E_6^2+5296335E_4^{15/2}-7947226E_4^6E_6+939180E_4^3E_6^3+94080E_6^5\right)}{360E_4^5}+\\
&\frac{E_2^2\left(2372448E_4^{3/2}E_6^4-696242070E_4^{9/2}E_6^2-897505518E_4^{15/2}+1568619971E_4^6E_6+52768708E_4^3E_6^3+733600E_6^5\right)}{1260E_4^4}\\
&-\frac{E_2^3}{540E_4^6}\left(1555200E_4^{3/2}E_6^5+2157384E_4^{9/2}E_6^3-54053697E_4^{15/2}E_6+27173594E_4^9+20978368E_4^6E_6^2\right.\\
&\qquad\left.+3306360E_4^3E_6^4+311040E_6^6\right)\\
&-\frac{E_2}{25200E_4^5}\left(512517120E_4^{3/2}E_6^5-3951672720E_4^{9/2}E_6^3-95049030780E_4^{15/2}E_6+53185189825E_4^9\right.\\
&\qquad\left.+49201422412E_4^6E_6^2+1174650320E_4^3E_6^4+92288000E_6^6\right)\\
&\left.+h_4(q)\right]
\end{aligned} \tag{D.4}$$

The ambiguous part is given by

$$\begin{aligned}
h_2 =& \frac{1106E_6}{15}-\frac{171E_4^{\frac{3}{2}}}{5}\\
h_3 =& -\frac{1648E_6^4}{9E_4^3}-\frac{24144E_6^3}{35E_4^{3/2}}-\frac{1234978E_6^2}{315}+\frac{292119}{35}E_4^{\frac{3}{2}}E_6-\frac{850279E_4^3}{126}\\
h_4 =& \frac{795392E_6^7}{27E_4^6}+\frac{2311168E_6^6}{15E_4^{9/2}}+\frac{13813852E_6^5}{45E_4^3}+\frac{79514072E_6^4}{315E_4^{3/2}}+\frac{9197665261E_6^3}{28350}\\
&-\frac{3057458963E_4^{3/2}E_6^2}{1260}+\frac{15201332353E_4^3E_6}{3780}-\frac{3669761651E_4^{9/2}}{1680}
\end{aligned} \tag{D.5}$$

JHEP07(2023)195

The conformal limit. This is a theory of type **A**, hence $\tau^* = e^{\frac{\pi i}{3}}$. By perturbing around this point we get

$$a \approx \frac{9\sqrt{3}E_4}{8\sqrt{E_6}} \tag{D.6}$$

and

$$\begin{aligned} &\mathcal{F}_2 \approx -\frac{E_2}{128a^2} && \mathcal{F}_3 \approx -\frac{E_2^2}{2048a^4} \\ &\mathcal{F}_4 \approx \frac{3107E_6}{663552a^6} - \frac{3E_2^3}{32768a^6} && \mathcal{F}_5 \approx \frac{34177E_2E_6}{5308416a^8} - \frac{97E_2^4}{3145728a^8} \\ &\vdots \end{aligned} \tag{D.7}$$

These can be resummed using the hypergeometric functions (3.12) and in agreement with (4.28). This provides an explicitly test that the conformal limit is independent of the deformation we perform.

E Holomorphic ambiguities for $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2$ at $\beta = 1$

The holomorphic anomaly equation determines $\widehat{\mathcal{Z}}(a, \beta)$ up to the E_2-independent part of (3.12), i.e.

$$\widehat{\mathcal{Z}}(a,\beta)|_{E_2 \to 0} = \sum_{n=0}^{\infty} c_n \left(-\frac{\epsilon_1\epsilon_2}{6a^2}\right)^{n\delta - \frac{\gamma}{2}} E_{2\delta}^n \,. \tag{E.1}$$

In this appendix we list the first few c_n coefficients at $\beta = 1$ for the three AD theories we analyzed in this paper.

$\mathcal{H}_0$.

$$\begin{aligned} c_2 &= -\frac{3411230845030961039}{2^{17}3^2 5^{14}}\,, \\ c_3 &= -\frac{1122841639515124786024331406 7849}{2^{25}3^4 5^{21}}\,, \\ c_4 &= -\frac{336921369293660561201677735133941404089137439}{2^{35}3^5 5^{28}}\,, \\ c_5 &= -\frac{54446679876958884558177953879909686803701101902116733352249}{2^{43}3^6 5^{36}}\,. \end{aligned} \tag{E.2}$$

These coefficients determine the behavior of $\mathfrak{f}_g$ in (3.9) up to $g = 18$.

$\mathcal{H}_1$.

$$\begin{aligned} c_2 &= -\frac{399471589}{60466176}\,, \\ c_3 &= -\frac{231844286893415}{176319369216} \end{aligned} \tag{E.3}$$

These coefficients determine the behavior of $\mathfrak{f}_g$ in (3.9) up to $g = 7$.

$\mathcal{H}_2$.

$$\begin{aligned} c_2 &= -\frac{23495274215}{2^{21}3^2}\,, \\ c_3 &= -\frac{412067029272808 6475}{2^{31}3^4}\,, \\ c_4 &= -\frac{6480114817503034769242602575}{2^{43}3^5}\,. \end{aligned} \tag{E.4}$$

These coefficients determine the behavior of $\mathfrak{f}_g$ in (3.9) up to $g = 15$.

References

[1] P.C. Argyres and M.R. Douglas, *New phenomena in $SU(3)$ supersymmetric gauge theory*, *Nucl. Phys. B* **448** (1995) 93 [hep-th/9505062] [INSPIRE].

[2] P.C. Argyres, M.R. Plesser, N. Seiberg and E. Witten, *New $N = 2$ superconformal field theories in four-dimensions*, *Nucl. Phys. B* **461** (1996) 71 [hep-th/9511154] [INSPIRE].

[3] T. Eguchi and K. Hori, *$N = 2$ superconformal field theories in four-dimensions and A-D-E classification*, in the proceedings of the *Conference on the Mathematical Beauty of Physics (In Memory of C. Itzykson)*, Saclay France, June 5–7 (1996), p. 67–82 [hep-th/9607125] [INSPIRE].

[4] T. Eguchi, K. Hori, K. Ito and S.-K. Yang, *Study of $N = 2$ superconformal field theories in four-dimensions*, *Nucl. Phys. B* **471** (1996) 430 [hep-th/9603002] [INSPIRE].

[5] P. Argyres, M. Lotito, Y. Lü and M. Martone, *Geometric constraints on the space of $\mathcal{N} = 2$ SCFTs. Part I: physical constraints on relevant deformations*, *JHEP* **02** (2018) 001 [arXiv:1505.04814] [INSPIRE].

[6] P.C. Argyres, M. Lotito, Y. Lü and M. Martone, *Geometric constraints on the space of $\mathcal{N} = 2$ SCFTs. Part II: construction of special Kähler geometries and RG flows*, *JHEP* **02** (2018) 002 [arXiv:1601.00011] [INSPIRE].

[7] P.C. Argyres, M. Lotito, Y. Lü and M. Martone, *Expanding the landscape of $\mathcal{N} = 2$ rank 1 SCFTs*, *JHEP* **05** (2016) 088 [arXiv:1602.02764] [INSPIRE].

[8] P. Argyres, M. Lotito, Y. Lü and M. Martone, *Geometric constraints on the space of $\mathcal{N} = 2$ SCFTs. Part III: enhanced Coulomb branches and central charges*, *JHEP* **02** (2018) 003 [arXiv:1609.04404] [INSPIRE].

[9] O. Chacaltana, J. Distler and A. Trimm, *A Family of 4D $\mathcal{N} = 2$ Interacting SCFTs from the Twisted A_{2N} Series*, arXiv:1412.8129 [INSPIRE].

[10] M. Martone, *Testing our understanding of SCFTs: a catalogue of rank-2 $\mathcal{N} = 2$ theories in four dimensions*, *JHEP* **07** (2022) 123 [arXiv:2102.02443] [INSPIRE].

[11] D. Xie, *General Argyres-Douglas Theory*, *JHEP* **01** (2013) 100 [arXiv:1204.2270] [INSPIRE].

[12] S. Cecotti and M. Del Zotto, *Infinitely many $N = 2$ SCFT with ADE flavor symmetry*, *JHEP* **01** (2013) 191 [arXiv:1210.2886] [INSPIRE].

[13] S. Cecotti, M. Del Zotto and S. Giacomelli, *More on the $N = 2$ superconformal systems of type $D_p(G)$*, *JHEP* **04** (2013) 153 [arXiv:1303.3149] [INSPIRE].

[14] Y. Wang and D. Xie, *Classification of Argyres-Douglas theories from M5 branes*, *Phys. Rev. D* **94** (2016) 065012 [arXiv:1509.00847] [INSPIRE].

[15] J.A. Minahan and D. Nemeschansky, *Superconformal fixed points with E_n global symmetry*, *Nucl. Phys. B* **489** (1997) 24 [hep-th/9610076] [INSPIRE].

[16] R. Rattazzi, V.S. Rychkov, E. Tonni and A. Vichi, *Bounding scalar operator dimensions in 4D CFT*, *JHEP* **12** (2008) 031 [arXiv:0807.0004] [INSPIRE].

JHEP07(2023)195

[17] C. Beem et al., *The $\mathcal{N} = 2$ superconformal bootstrap*, *JHEP* **03** (2016) 183 [arXiv:1412.7541] [INSPIRE].

[18] M. Cornagliotto, M. Lemos and P. Liendo, *Bootstrapping the (A_1, A_2) Argyres-Douglas theory*, *JHEP* **03** (2018) 033 [arXiv:1711.00016] [INSPIRE].

[19] A. Gimenez-Grau and P. Liendo, *Bootstrapping Coulomb and Higgs branch operators*, *JHEP* **01** (2021) 175 [arXiv:2006.01847] [INSPIRE].

[20] D. Simmons-Duffin, *The Conformal Bootstrap*, in the proceedings of the *Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings*, Boulder U.S.A., June 1–26 (2015), p. 1–74 [DOI:10.1142/9789813149441_0001] [arXiv:1602.07982] [INSPIRE].

[21] L.F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville Correlation Functions from Four-dimensional Gauge Theories*, *Lett. Math. Phys.* **91** (2010) 167 [arXiv:0906.3219] [INSPIRE].

[22] N. Wyllard, *A(N-1) conformal Toda field theory correlation functions from conformal $N = 2$ $SU(N)$ quiver gauge theories*, *JHEP* **11** (2009) 002 [arXiv:0907.2189] [INSPIRE].

[23] D. Gaiotto and J. Teschner, *Irregular singularities in Liouville theory and Argyres-Douglas type gauge theories, I*, *JHEP* **12** (2012) 050 [arXiv:1203.1052] [INSPIRE].

[24] G. Bonelli, K. Maruyoshi and A. Tanzini, *Wild Quiver Gauge Theories*, *JHEP* **02** (2012) 031 [arXiv:1112.1691] [INSPIRE].

[25] G. Bonelli et al., *On Painlevé/gauge theory correspondence*, *Lett. Matth. Phys.* **107** (2017) 2359 [arXiv:1612.06235] [INSPIRE].

[26] T. Kimura, T. Nishinaka, Y. Sugawara and T. Uetoko, *Argyres-Douglas theories, S-duality and AGT correspondence*, *JHEP* **04** (2021) 205 [arXiv:2012.14099] [INSPIRE].

[27] H. Nagoya, *Irregular conformal blocks, with an application to the fifth and fourth Painlevé equations*, *J. Math. Phys.* **56** (2015) 123505 [arXiv:1505.02398] [INSPIRE].

[28] H. Nagoya, *Remarks on irregular conformal blocks and Painlevé III and II tau functions*, arXiv:1804.04782 [INSPIRE].

[29] T. Nishinaka and C. Rim, *Matrix models for irregular conformal blocks and Argyres-Douglas theories*, *JHEP* **10** (2012) 138 [arXiv:1207.4480] [INSPIRE].

[30] C. Rim, *Irregular Conformal States and Spectral Curve: Irregular Matrix Model Approach*, *SIGMA* **13** (2017) 012 [arXiv:1612.00348] [INSPIRE].

[31] H. Itoyama, T. Oota and K. Yano, *Discrete Painlevé system for the partition function of $N_f = 2$ $SU(2)$ supersymmetric gauge theory and its double scaling limit*, *J. Phys. A* **52** (2019) 415401 [arXiv:1812.00811] [INSPIRE].

[32] A. Grassi and J. Gu, *Argyres-Douglas theories, Painlevé II and quantum mechanics*, *JHEP* **02** (2019) 060 [arXiv:1803.02320] [INSPIRE].

[33] T. Nishinaka and T. Uetoko, *Argyres-Douglas theories and Liouville Irregular States*, *JHEP* **09** (2019) 104 [arXiv:1905.03795] [INSPIRE].

[34] H. Itoyama, T. Oota and K. Yano, *Multicritical points of unitary matrix model with logarithmic potential identified with Argyres-Douglas points*, *Int. J. Mod. Phys. A* **35** (2020) 2050146 [arXiv:1909.10770] [INSPIRE].

JHEP07(2023)195

[35] T. Oota, *Perturbation of multi-critical unitary matrix models, double scaling limits, and Argyres-Douglas theories*, Nucl. Phys. B **976** (2022) 115718 [arXiv:2112.14441] [INSPIRE].

[36] H. Itoyama and K. Yano, *Theory space of one unitary matrix model and its critical behavior associated with Argyres-Douglas theory*, Int. J. Mod. Phys. A **36** (2021) 2150227 [arXiv:2103.11428] [INSPIRE].

[37] F. Fucito, J.F. Morales and R. Poghossian, *On irregular states and Argyres-Douglas theories*, arXiv:2306.05127 [INSPIRE].

[38] S. Hellerman, D. Orlando, S. Reffert and M. Watanabe, *On the CFT Operator Spectrum at Large Global Charge*, JHEP **12** (2015) 071 [arXiv:1505.01537] [INSPIRE].

[39] A. Bourget, D. Rodriguez-Gomez and J.G. Russo, *A limit for large R-charge correlators in* $\mathcal{N} = 2$ *theories*, JHEP **05** (2018) 074 [arXiv:1803.00580] [INSPIRE].

[40] A. Monin, D. Pirtskhalava, R. Rattazzi and F.K. Seibold, *Semiclassics, Goldstone Bosons and CFT data*, JHEP **06** (2017) 011 [arXiv:1611.02912] [INSPIRE].

[41] L. Alvarez-Gaume, O. Loukas, D. Orlando and S. Reffert, *Compensating strong coupling with large charge*, JHEP **04** (2017) 059 [arXiv:1610.04495] [INSPIRE].

[42] D. Jafferis, B. Mukhametzhanov and A. Zhiboedov, *Conformal Bootstrap At Large Charge*, JHEP **05** (2018) 043 [arXiv:1710.11161] [INSPIRE].

[43] S. Hellerman and S. Maeda, *On the Large R-charge Expansion in* $\mathcal{N} = 2$ *Superconformal Field Theories*, JHEP **12** (2017) 135 [arXiv:1710.07336] [INSPIRE].

[44] S. Hellerman et al., *Universal correlation functions in rank 1 SCFTs*, JHEP **12** (2019) 047 [arXiv:1804.01535] [INSPIRE].

[45] A. Grassi, Z. Komargodski and L. Tizzano, *Extremal correlators and random matrix theory*, JHEP **04** (2021) 214 [arXiv:1908.10306] [INSPIRE].

[46] M. Beccaria, F. Galvagno and A. Hasan, $\mathcal{N} = 2$ *conformal gauge theories at large R-charge: the* $SU(N)$ *case*, JHEP **03** (2020) 160 [arXiv:2001.06645] [INSPIRE].

[47] A. Losev, N. Nekrasov and S.L. Shatashvili, *Issues in topological gauge theory*, Nucl. Phys. B **534** (1998) 549 [hep-th/9711108] [INSPIRE].

[48] G.W. Moore, N. Nekrasov and S. Shatashvili, *Integrating over Higgs branches*, Commun. Math. Phys. **209** (2000) 97 [hep-th/9712241] [INSPIRE].

[49] A. Lossev, N. Nekrasov and S.L. Shatashvili, *Testing Seiberg-Witten solution*, *NATO Sci. Ser. C* **520** (1999) 359 [hep-th/9801061] [INSPIRE].

[50] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, Adv. Theor. Math. Phys. **7** (2003) 831 [hep-th/0206161] [INSPIRE].

[51] R. Flume and R. Poghossian, *An Algorithm for the microscopic evaluation of the coefficients of the Seiberg-Witten prepotential*, Int. J. Mod. Phys. A **18** (2003) 2541 [hep-th/0208176] [INSPIRE].

[52] U. Bruzzo, F. Fucito, J.F. Morales and A. Tanzini, *Multiinstanton calculus and equivariant cohomology*, JHEP **05** (2003) 054 [hep-th/0211108] [INSPIRE].

[53] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, Prog. Math. **244** (2006) 525 [hep-th/0306238] [INSPIRE].

JHEP07(2023)195

[54] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, Commun. Math. Phys. **313** (2012) 71 [arXiv:0712.2824] [INSPIRE].

[55] F. Fucito, J.F. Morales, R. Poghossian and D. Ricci Pacifici, *Exact results in $\mathcal{N}=2$ gauge theories*, JHEP **10** (2013) 178 [arXiv:1307.6612] [INSPIRE].

[56] F. Fucito, J.F. Morale and R. Poghossian, *Wilson Loops and Chiral Correlators on Squashed Spheres*, J. Geom. Phys. **118** (2017) 169 [arXiv:1603.02586] [INSPIRE].

[57] E. Gerchkovitz et al., *Correlation Functions of Coulomb Branch Operators*, JHEP **01** (2017) 103 [arXiv:1602.05971] [INSPIRE].

[58] D. Rodriguez-Gomez and J.G. Russo, *Operator mixing in large N superconformal field theories on S^4 and correlators with Wilson loops*, JHEP **12** (2016) 120 [arXiv:1607.07878] [INSPIRE].

[59] D. Rodriguez-Gomez and J.G. Russo, *Large N Correlation Functions in Superconformal Field Theories*, JHEP **06** (2016) 109 [arXiv:1604.07416] [INSPIRE].

[60] M. Billò et al., *Two-point correlators in $N=2$ gauge theories*, Nucl. Phys. B **926** (2018) 427 [arXiv:1705.02909] [INSPIRE].

[61] M. Billò et al., *Two-point correlators in non-conformal $\mathcal{N}=2$ gauge theories*, JHEP **05** (2019) 199 [arXiv:1901.09693] [INSPIRE].

[62] A. Bissi et al., *OPE coefficients in Argyres-Douglas theories*, JHEP **06** (2022) 085 [arXiv:2112.11899] [INSPIRE].

[63] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Holomorphic anomalies in topological field theories*, Nucl. Phys. B **405** (1993) 279 [hep-th/9302103] [INSPIRE].

[64] M.-X. Huang and A. Klemm, *Holomorphic Anomaly in Gauge Theories and Matrix Models*, JHEP **09** (2007) 054 [hep-th/0605195] [INSPIRE].

[65] M.-X. Huang and A. Klemm, *Holomorphicity and Modularity in Seiberg-Witten Theories with Matter*, JHEP **07** (2010) 083 [arXiv:0902.1325] [INSPIRE].

[66] M.-X. Huang, A.-K. Kashani-Poor and A. Klemm, *The Ω deformed B-model for rigid $\mathcal{N}=2$ theories*, Annales Henri Poincare **14** (2013) 425 [arXiv:1109.5728] [INSPIRE].

[67] M.-X. Huang, *Modular anomaly from holomorphic anomaly in mass deformed $\mathcal{N}=2$ superconformal field theories*, Phys. Rev. D **87** (2013) 105010 [arXiv:1302.6095] [INSPIRE].

[68] M. Billò et al., *Deformed $N=2$ theories, generalized recursion relations and S-duality*, JHEP **04** (2013) 039 [arXiv:1302.0686] [INSPIRE].

[69] M. Billò et al., *Modular anomaly equation, heat kernel and S-duality in $N=2$ theories*, JHEP **11** (2013) 123 [arXiv:1307.6648] [INSPIRE].

[70] S.K. Ashok et al., *Modular anomaly equations and S-duality in $\mathcal{N}=2$ conformal SQCD*, JHEP **10** (2015) 091 [arXiv:1507.07476] [INSPIRE].

[71] M. Billó et al., *S-duality and the prepotential in $\mathcal{N}=2^\star$ theories (I): the ADE algebras*, JHEP **11** (2015) 024 [arXiv:1507.07709] [INSPIRE].

[72] M. Billó et al., *S-duality and the prepotential of $\mathcal{N}=2^\star$ theories (II): the non-simply laced algebras*, JHEP **11** (2015) 026 [arXiv:1507.08027] [INSPIRE].

[73] D. Krefl and J. Walcher, *Extended Holomorphic Anomaly in Gauge Theory*, Lett. Math. Phys. **95** (2011) 67 [arXiv:1007.0263] [INSPIRE].

JHEP07(2023)195

[74] S. Codesido and M. Marino, *Holomorphic Anomaly and Quantum Mechanics*, *J. Phys. A* **51** (2018) 055402 [arXiv:1612.07687] [INSPIRE].

[75] F. Fischbach, A. Klemm and C. Nega, *WKB Method and Quantum Periods beyond Genus One*, *J. Phys. A* **52** (2019) 075402 [arXiv:1803.11222] [INSPIRE].

[76] N.A. Nekrasov and S.L. Shatashvili, *Quantization of Integrable Systems and Four Dimensional Gauge Theories*, in the proceedings of the *16th International Congress on Mathematical Physics*, Prague Czechia, August 3–8 (2009), p. 265–289 [DOI:10.1142/9789814304634_0015] [arXiv:0908.4052] [INSPIRE].

[77] E. Witten, *Quantum background independence in string theory*, in the proceedings of the *Conference on Highlights of Particle and Condensed Matter Physics (SALAMFEST)*, Trieste Italy, March 8–12 (1993) [hep-th/9306122] [INSPIRE].

[78] J. Aspman, E. Furrer and J. Manschot, *Cutting and gluing with running couplings in $N = 2$ QCD*, *Phys. Rev. D* **105** (2022) 025021 [arXiv:2107.04600] [INSPIRE].

[79] K. Hori et al., *Mirror symmetry*, AMS, Providence, U.S.A. (2003) [INSPIRE].

[80] T. Banks, M.R. Douglas and N. Seiberg, *Probing F theory with branes*, *Phys. Lett. B* **387** (1996) 278 [hep-th/9605199] [INSPIRE].

[81] G.W. Moore and I. Nidaiev, *The Partition Function Of Argyres-Douglas Theory On A Four-Manifold*, arXiv:1711.09257 [INSPIRE].

[82] T. Okuda and V. Pestun, *On the instantons and the hypermultiplet mass of $N = 2^*$ super Yang-Mills on S^4*, *JHEP* **03** (2012) 017 [arXiv:1004.1222] [INSPIRE].

[83] S. Codesido, M. Marino and R. Schiappa, *Non-Perturbative Quantum Mechanics from Non-Perturbative Strings*, *Annales Henri Poincare* **20** (2019) 543 [arXiv:1712.02603] [INSPIRE].

[84] S. Codesido Sanchez, *A geometric approach to non-perturbative quantum mechanics*, Ph.D. thesis, Faculté des Sciences, DPNC, Université de Genève, Geneva, Switzerland (2018) [INSPIRE].

[85] J. Gu and M. Marino, *On the resurgent structure of quantum periods*, arXiv:2211.03871 [INSPIRE].

[86] D. Gaiotto, *Opers and TBA*, arXiv:1403.6137 [INSPIRE].

[87] K. Ito and H. Shu, *ODE/IM correspondence and the Argyres-Douglas theory*, *JHEP* **08** (2017) 071 [arXiv:1707.03596] [INSPIRE].

[88] L. Hollands and A. Neitzke, *Exact WKB and abelianization for the T_3 equation*, *Commun. Math. Phys.* **380** (2020) 131 [arXiv:1906.04271] [INSPIRE].

[89] L. Hollands, P. Rüter and R.J. Szabo, *A geometric recipe for twisted superpotentials*, *JHEP* **12** (2021) 164 [arXiv:2109.14699] [INSPIRE].

[90] F. Fucito, A. Grassi, J.F. Morales and R. Savelli, in preparation.

[91] D. Gaiotto, *$N = 2$ dualities*, *JHEP* **08** (2012) 034 [arXiv:0904.2715] [INSPIRE].

[92] Y. Tachikawa, *$N = 2$ supersymmetric dynamics for pedestrians*, arXiv:1312.2684 [DOI:10.1007/978-3-319-08822-8] [INSPIRE].

JHEP07(2023)195