Published for SISSA by Springer

Received: April 26, 2022
Accepted: June 12, 2022
Published: July 15, 2022

TCFHs, hidden symmetries and type II theories

L. Grimanellis, G. Papadopoulos and J. Phillips

*Department of Mathematics, King's College London,
Strand, London WC2R 2LS, U.K.*

E-mail: loukas.grimanellis@kcl.ac.uk, george.papadopoulos@kcl.ac.uk, jake.phillips@kcl.ac.uk

Abstract: We present the twisted covariant form hierarchies (TCFH) of type IIA and IIB 10-dimensional supergravities and show that all form bilinears of supersymmetric backgrounds satisfy the conformal Killing-Yano equation with respect to a TCFH connection. We also compute the Killing-Stäckel, Killing-Yano and closed conformal Killing-Yano tensors of all spherically symmetric type II brane backgrounds and demonstrate that the geodesic flow on these solutions is completely integrable by giving all independent charges in involution. We then identify all form bilinears of common sector and D-brane backgrounds which generate hidden symmetries for particle and string probe actions. We also explore the question on whether charges constructed from form bilinears are sufficient to prove the integrability of probes on supersymmetric backgrounds.

Keywords: Conformal and W Symmetry, P-Branes, Supergravity Models, Superstring Vacua

ArXiv ePrint: 2111.15405

https://doi.org/10.1007/JHEP07(2022)097

Contents

1 Introduction

It has been known for sometime that some gravitational backgrounds admit Killing-Stäckel (KS) and Killing-Yano (KY) tensors, see [1]–[9], the reviews [10] and [11] and the references within. These are used to demonstrate the separability and integrability of classical equations, such as the geodesic, Hamilton-Jacobi and Dirac equations, on these backgrounds. A key property of KS tensors is that they generate hidden symmetries for relativistic particles while KY tensors generate hidden symmetries [12] for spinning particles [13] propagating on gravitational backgrounds.

It has been shown in [14] that the conditions imposed by the gravitino Killing spinor equation (KSE) on the (Killing spinor) form bilinears can be arranged as a twisted covariant form hierarchy (TCFH) [15]. This means that there is a connection, $\mathcal{D}^{\mathcal{F}}$, on the space of spacetime forms which depends on the fluxes, $\mathcal{F}$, of the theory such that the highest weight representation of $\mathcal{D}^{\mathcal{F}}\Omega$ vanishes, where Ω is a collection of forms of various degrees and $\mathcal{D}^{\mathcal{F}}$ may not be form degree preserving. Equivalently, this condition can be written as

$$\mathcal{D}^{\mathcal{F}}_X\Omega = i_X\mathcal{P} + X \wedge \mathcal{Q}\,, \tag{1.1}$$

for every spacetime vector field X, where $\mathcal{P}$ and $\mathcal{Q}$ are appropriate multi-forms and X also denotes the associated 1-form constructed from the vector field X after using the spacetime metric g, $X(Y) = g(X, Y)$. The proof of this result is rather general and includes supergravities on spacetimes of any signature as well as the effective theories of strings which include higher order curvature corrections. It also puts the conditions imposed by the KSEs on the form bilinears on a firm geometric basis.

One consequence of the TCFH is that the form bilinears satisfy a generalisation of the conformal Killing-Yano (CKY) equation with respect to the connection $\mathcal{D}^{\mathcal{F}}$. This can be easily seen after taking the skew-symmetric part and contraction with respect to the metric g of (1.1), and so one identifies $\mathcal{P}$ with an exterior derivative constructed from $\mathcal{D}^{\mathcal{F}}$ and $\mathcal{Q}$ with a formal adjoint of $\mathcal{D}^{\mathcal{F}}$ acting on Ω. This raises the question on whether the form bilinears generate hidden symmetries for worldvolume actions which describe the propagation of certain probes in supersymmetric backgrounds. This question was first investigated in the context of 5- and 4-dimensional supergravities in [16].

The purpose of this paper is twofold. One is to present the TCFHs of IIA and IIB supergravities and to discuss some of the properties of the TCFH connections $\mathcal{D}^{\mathcal{F}}$, like for example their holonomy, on generic as well as on some special supersymmetric backgrounds. As a consequence we demonstrate that the form bilinears of these theories satisfy a CKY equation with respect to $\mathcal{D}^{\mathcal{F}}$ in agreement with the general result of [14]. Another purpose of this paper is to give the KS tensors of type II branes[1] [19]–[27] and to use them to prove the complete integrability of the geodesic flow of those solutions that are spherically symmetric, i.e. those that depend on a harmonic function with one centre. In addition, the KY tensors that square to the KS tensors of these backgrounds will be given and the symmetries of spinning particles propagating on these backgrounds will be explored. Furthermore we shall investigate the conditions required for the TCFH to yield symmetries for particle and string probes propagating in common sector and D-brane backgrounds. Finally we shall compare the results we have obtained from the point of view of KS and KY tensors with those that arise from the TCFHs.

To investigate under which conditions the form bilinears generate symmetries for certain probe actions propagating in type II supersymmetric backgrounds, we shall match the conditions required for certain probe actions to be invariant under transformations generated by form bilinears with those imposed on them by the TCFHs. For the common sector of type II theories, it is shown that all form bilinears which are covariantly constant with respect to a connection with torsion given by the NS-NS 3-form field strength generate symmetries for string and spinning particle probes propagating on these backgrounds. Common sector backgrounds also admit form bilinears which are not covariantly constant and instead satisfy a general TCFH. These form bilinears may not generate symmetries for probes propagating in common sector backgrounds but nevertheless are part of their geometric structure. In particular the form bilinears of the fundamental string and NS5-brane solutions that are allowed to depend on multi-centre harmonic functions have been computed. It has been found that the type II fundamental string solution admits 2^7 covariantly constant

[1] Brane solutions have been instrumental in the understanding of string dualities [17, 18].

independent form bilinears while the type II NS5-brane solution admits 2^5 covariantly constant independent form bilinears. All these forms generate (hidden) symmetries for probe string and spinning particle actions propagating on these backgrounds.

A similar analysis is presented for all type II D-branes. In particular, the form bilinears of all D-branes are computed. It is found that the requirement for these to generate symmetries for spinning particle probes propagating on these backgrounds is rather restrictive. This is due to the difficulties of constructing probe actions which exhibit appropriate form couplings. Nevertheless all type II D-branes, which may depend on multi-centre harmonic functions, admit form bilinears which generate symmetries for spinning particle probe actions. It turns out that all such form bilinears have components only along the worldvolume directions of the D-branes. A comparison of the symmetries we have found generated by the KS and KY tensors and those generated by the form bilinears in type II brane backgrounds will be presented in the conclusions.

This paper is organised as follows. In sections 2 and 3, we give the TCFHs of IIA and IIB supergravities and discuss some of the properties of the TCFH connections. In section 4, after a summary of the properties of the KS and KY tensors, we present the KS and KY tensors of all type II branes. In addition, we prove the complete integrability of the geodesic flow in all type II branes that depend on a harmonic function with one centre by presenting all the independent conserved charges which are in involution. In section 5, we demonstrate that all covariantly constant form bilinears with respect to a connection with skew-symmetric torsion generate symmetries for certain probe string and particle actions propagating on common sector backgrounds. In addition, we explicitly give all the covariantly constant form bilinears for the type II fundamental string and NS5-brane solutions. In sections 6 and 7, we identify the form bilinears which generate symmetries for spinning particle actions propagating on type II D-brane backgrounds. In section 8, we give our conclusions. In appendices A and B, we give all the form bilinears of type II common sector branes and type II D-branes, respectively.

2 The TCFH of (massive) IIA supergravity

The KSEs of massive IIA supergravity [28] are given by the vanishing conditions of the supersymmetry variations of the gravitino and dilatino fields evaluated at the locus that all fermions are set to zero. The KSE associated with the gravitino field is a parallel transport equation for the supercovariant connection $\mathcal{D}$. In the string frame, this is given by

$$\begin{aligned}\mathcal{D}_M &:= \nabla_M + \frac{1}{8}H_{MPQ}\Gamma^{PQ}\Gamma_{11} + \frac{1}{8}e^{\Phi}S\Gamma_M \\ &\quad + \frac{1}{16}e^{\Phi}F_{PQ}\Gamma^{PQ}\Gamma_M\Gamma_{11} + \frac{1}{8\cdot 4!}e^{\Phi}G_{P_1\dots P_4}\Gamma^{P_1\dots P_4}\Gamma_M\,,\end{aligned} \tag{2.1}$$

see e.g. [29], where H is the NS-NS 3-form field strength, Φ is the dilaton, and F and G are the 2-form and 4-form R-R field strengths, respectively. In addition, ∇ is the Levi-Civita connection induced on the spinor bundle and $S = e^{\Phi}m$, where m is a constant which is non-zero in massive IIA and vanishes in the standard IIA supergravity. Furthermore Γ denotes the gamma matrices which satisfy the Clifford algebra relation $\Gamma_A\Gamma_B + \Gamma_B\Gamma_A = 2\eta_{AB}$ and in our conventions $\Gamma_{11} := \Gamma_{012\ldots9}$. In what follows, we shall not make a sharp distinction between spacetime and frame indices but we shall always assume that the indices of gamma matrices are frame indices. It turns out that $\mathcal{D}$ is a connection on the spin bundle over the spacetime associated with the Majorana (real) representation of $\mathfrak{spin}(9,1)$. The (reduced) holonomy of $\mathcal{D}$ for generic backgrounds is $SL(32,\mathbb{R})$ [30], see [31–33] for the computation of the holonomy of the supercovariant derivative of 11-dimensional supergravity.

The Killing spinors ϵ satisfy the gravitino KSE, $\mathcal{D}\epsilon = 0$, as well as the dilatino KSE which is an algebraic equation. Backgrounds that admit such Killing spinors are special and both the spacetime metric and fluxes are suitably restricted, see [34–36] where the IIA KSEs have been solved for one Killing spinor. The TCFHs are associated with the gravitino KSE which we shall focus on in what follows.

Given N Killing spinors ϵ^r, $r = 1, \ldots, N$, one can construct the form bilinears

$$\phi^{rs} = \frac{1}{k!}\langle \epsilon^r, \Gamma_{A_1\ldots A_k}\epsilon^s\rangle_D \, e^{A_1} \wedge \cdots \wedge e^{A_k} \,, \tag{2.2}$$

where $\langle \cdot, \cdot\rangle_D$ denotes that Dirac inner product and e^A is a suitable spacetime frame, $g_{MN} = \eta_{AB} e^A_M e^B_N$. As

$$\nabla_M \phi^{rs}_{A_1\ldots A_k} = \langle \nabla_M \epsilon^r, \Gamma_{A_1\ldots A_k}\epsilon^s\rangle_D + \langle \epsilon^r, \Gamma_{A_1\ldots A_k}\nabla_M\epsilon^s\rangle_D \,, \tag{2.3}$$

one can use the gravitino KSE, $\mathcal{D}\epsilon = 0$, and (2.1) to express the right-hand side of the above equation in terms of the fluxes and form bilinears of the theory. In [14] has been shown that these equations can be organised as TCFH.

Using the reality condition on ϵ, there are form bilinears which are either symmetric or skew-symmetric in the exchange of spinors ϵ^r and ϵ^s in (2.2). As a consequence the TCFH of the IIA supergravity factorises in two parts. A basis in form bilinears, up to a Hodge duality[2] operation, which are symmetric in the exchange of the two Killing spinors ϵ^r and ϵ^s is

$$\begin{aligned}
&\tilde{\sigma}^{rs} = \langle \epsilon^r, \Gamma_{11}\epsilon^s\rangle_D \,, && k^{rs} = \langle \epsilon^r, \Gamma_N \epsilon^s\rangle_D \; e^N, \quad \tilde{k} = \langle \epsilon^r, \Gamma_N\Gamma_{11}\epsilon^s\rangle_D \; e^N, \\
&\omega^{rs} = \frac{1}{2}\langle \epsilon^r, \Gamma_{NR}\epsilon^s\rangle_D \; e^N \wedge e^R, && \tilde{\zeta}^{rs} = \frac{1}{4!}\langle \epsilon^r, \Gamma_{N_1\ldots N_4}\Gamma_{11}\epsilon^s\rangle_D \; e^{N_1} \wedge \cdots \wedge e^{N_4}, \\
&\tau^{rs} = \frac{1}{5!}\langle \epsilon^r, \Gamma_{N_1\ldots N_5}\epsilon^s\rangle_D \; e^{N_1} \wedge \cdots \wedge e^{N_5} \,.
\end{aligned} \tag{2.4}$$

A direct computation reveals that the TCFH is

$$\mathcal{D}^{\mathcal{F}}_M \tilde{\sigma} := \nabla_M \tilde{\sigma} = -\frac{1}{4} H_{MPQ}\omega^{PQ} + \frac{1}{4} e^{\Phi} S \tilde{k}_M - \frac{1}{4} e^{\Phi} F_{MP} k^P + \frac{1}{4\cdot 5!} {}^{\star}G_{MP_1\ldots P_5}\tau^{P_1\ldots P_5} \,, \tag{2.5}$$

$$\begin{aligned}
\mathcal{D}^{\mathcal{F}}_M k_N := \nabla_M k_N = &-\frac{1}{2} H_{MNP}\tilde{k}^P + \frac{1}{4} e^{\Phi} S \omega_{MN} + \frac{1}{8} e^{\Phi} F_{PQ}\tilde{\zeta}^{PQ}{}_{MN} + \frac{1}{4} e^{\Phi} F_{MN}\tilde{\sigma} \\
&- \frac{1}{4\cdot 4!} e^{\Phi\star} G_{MNP_1\ldots P_4}\tilde{\zeta}^{P_1\ldots P_4} + \frac{1}{8} e^{\Phi} G_{MNPQ}\omega^{PQ} \,,
\end{aligned} \tag{2.6}$$

[2] Our convention for the Hodge duality operation is ${}^{\star}\omega_{N_1\ldots N_{n-p}} = \frac{1}{p!}\omega_{P_1\ldots P_p}\epsilon^{P_1\ldots P_p}{}_{N_1\ldots N_{n-p}}$ with $\epsilon_{012\ldots(n-1)} = -1$, where n is the spacetime dimension.

$$\begin{aligned}
\mathcal{D}^{\mathcal{F}}_M\tilde{k}_N &:= \nabla_M\tilde{k}_N - \frac{1}{2}e^{\Phi}F_{MP}\omega^P{}_N - \frac{1}{12}e^{\Phi}G_{MPQR}\tilde{\zeta}^{PQR}{}_N = -\frac{1}{2}H_{MNP}k^P \\
&\quad + \frac{1}{4}e^{\Phi}g_{MN}S\tilde{\sigma} + \frac{1}{8}e^{\Phi}g_{MN}F_{PQ}\omega^{PQ} - \frac{1}{2}e^{\Phi}F_{[M|P|}\omega^P{}_{N]} \\
&\quad + \frac{1}{4\cdot 4!}e^{\Phi}g_{MN}G_{P_1\dots P_4}\tilde{\zeta}^{P_1\dots P_4} - \frac{1}{12}e^{\Phi}G_{[M|PQR|}\tilde{\zeta}^{PQR}{}_{N]}\,, \qquad (2.7)
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}^{\mathcal{F}}_M\omega_{NR} &:= \nabla_M\omega_{NR} + \frac{1}{4}H_{MPQ}\tilde{\zeta}^{PQ}{}_{NR} + e^{\Phi}F_{M[N}\tilde{k}_{R]} - \frac{1}{12}e^{\Phi}G_{MP_1P_2P_3}\tau^{P_1P_2P_3}{}_{NR} \\
&= \frac{1}{2}H_{MNR}\tilde{\sigma} + \frac{1}{2}e^{\Phi}Sg_{M[N}k_{R]} + \frac{3}{4}e^{\Phi}F_{[MN}\tilde{k}_{R]} + \frac{1}{2}e^{\Phi}g_{M[N}F_{R]P}\tilde{k}^P \\
&\quad - \frac{1}{4\cdot 5!}e^{\Phi\star}F_{MNRP_1\dots P_5}\tau^{P_1\dots P_5} + \frac{1}{2\cdot 4!}e^{\Phi}g_{M[N}G_{|P_1\dots P_4|}\tau^{P_1\dots P_4}{}_{R]} \\
&\quad - \frac{1}{8}e^{\Phi}G_{[M|P_1P_2P_3|}\tau^{P_1P_2P_3}{}_{NR]} - \frac{1}{4}e^{\Phi}G_{MNRP}k^P\,, \qquad (2.8)
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}^{\mathcal{F}}_M\tilde{\zeta}_{N_1\dots N_4} &:= \nabla_M\tilde{\zeta}_{N_1\dots N_4} - \frac{1}{3}{}^\star H_{M[N_1N_2N_3|PQR|}\tilde{\zeta}^{PQR}{}_{N_4]} - 3H_{M[N_1N_2}\omega_{N_3N_4]} \\
&\quad + \frac{1}{2}e^{\Phi}F_{MP}\tau^P{}_{N_1\dots N_4} + \frac{1}{2}e^{\Phi\star}G_{M[N_1N_2|PQR|}\tau^{PQR}{}_{N_3N_4]} + 2e^{\Phi}G_{M[N_1N_2N_3}\tilde{k}_{N_4]} \\
&= \frac{1}{12}g_{M[N_1}{}^\star H_{N_2N_3N_4]P_1\dots P_4}\tilde{\zeta}^{P_1\dots P_4} - \frac{5}{12}{}^\star H_{[MN_1N_2N_3|PQR|}\tilde{\zeta}^{PQR}{}_{N_4]} \\
&\quad + \frac{1}{4\cdot 5!}e^{\Phi\star}S_{MN_1\dots N_4P_1\dots P_5}\tau^{P_1\dots P_5} - \frac{1}{2}e^{\Phi}g_{M[N_1}F_{|PQ|}\tau^{PQ}{}_{N_2N_3N_4]} \\
&\quad + \frac{5}{8}e^{\Phi}F_{[M|P|}\tau^P{}_{N_1\dots N_4]} + \frac{5}{12}e^{\Phi\star}G_{[MN_1N_2|PQR|}\tau^{PQR}{}_{N_3N_4]} + 3e^{\Phi}g_{M[N_1}F_{N_2N_3}k_{N_4]} \\
&\quad - \frac{1}{8}e^{\Phi}g_{M[N_1}{}^\star G_{N_2N_3|P_1\dots P_4|}\tau^{P_1\dots P_4}{}_{N_4]} + \frac{1}{4}e^{\Phi\star}G_{MN_1\dots N_4P}k^P \\
&\quad + \frac{5}{4}e^{\Phi}G_{[MN_1N_2N_3}\tilde{k}_{N_4]} + e^{\Phi}g_{M[N_1}G_{N_2N_3N_4]P}\tilde{k}^P\,, \qquad (2.9)
\end{aligned}$$

$$\begin{aligned}
&\mathcal{D}^{\mathcal{F}}_M\tau_{N_1\dots N_5} := \nabla_M\tau_{N_1\dots N_5} - \frac{5}{6}{}^\star H_{M[N_1N_2N_3|PQR|}\tau^{PQR}{}_{N_4N_5]} - \frac{5}{2}e^{\Phi}F_{M[N_1}\tilde{\zeta}_{N_2\dots N_5]} \\
&\quad - \frac{5}{2}e^{\Phi\star}G_{M[N_1N_2N_3|PQ|}\tilde{\zeta}^{PQ}{}_{N_4N_5]} + 5e^{\Phi}G_{M[N_1N_2N_3}\omega_{N_4N_5]} = -\frac{5}{4}{}^\star H_{[MN_1N_2N_3|PQR|}\tau^{PQR}{}_{N_4N_5]} \\
&\quad + \frac{5}{12}g_{M[N_1}{}^\star H_{N_2N_3N_4|P_1\dots P_4|}\tau^{P_1\dots P_4}{}_{N_5]} - \frac{1}{4\cdot 4!}e^{\Phi\star}S_{MN_1\dots N_5P_1\dots P_4}\tilde{\zeta}^{P_1\dots P_4} \\
&\quad + \frac{1}{8}e^{\Phi\star}F_{MN_1\dots N_5}{}^{PQ}\omega_{PQ} - 5e^{\Phi}g_{M[N_1}F_{N_2|P|}\tilde{\zeta}^P{}_{N_3N_4N_5]} - \frac{15}{4}e^{\Phi}F_{[MN_1}\tilde{\zeta}_{N_2\dots N_5]} \\
&\quad - \frac{15}{8}e^{\Phi\star}G_{[MN_1N_2N_3|PQ|}\tilde{\zeta}^{PQ}{}_{N_4N_5]} - \frac{1}{4}e^{\Phi\star}G_{MN_1\dots N_5}\tilde{\sigma} - \frac{5}{6}e^{\Phi}g_{M[N_1}{}^\star G_{N_2N_3N_4|PQR|}\tilde{\zeta}^{PQR}{}_{N_5]} \\
&\quad + \frac{15}{4}e^{\Phi}G_{[MN_1N_2N_3}\omega_{N_4N_5]} + 5e^{\Phi}g_{M[N_1}G_{N_2N_3N_4|P|}\omega^P{}_{N_5]}\,, \qquad (2.10)
\end{aligned}$$

where for simplicity we have suppressed the r, s indices on the form bilinears which count the different Killing spinors. The connection $\mathcal{D}^{\mathcal{F}}$ is the minimal connection of the TCFH, see [14] for the definition. As it has been explained in the introduction, the above TCFH implies that the form bilinears (2.4) satisfy a generalisation of the CKY with respect to the connection $\mathcal{D}^{\mathcal{F}}$. As expected k is Killing, $\nabla_{(M}k_{N)} = 0$.

A basis in the form bilinears, up to a Hodge duality operation, which are skew-symmetric in the exchange of the two Killing spinors is

$$\begin{aligned}
\sigma^{rs} &= \langle\epsilon^r, \epsilon^s\rangle_D\,, & \tilde{\omega}^{rs} &= \frac{1}{2}\langle\epsilon^r, \Gamma_{NR}\Gamma_{11}\epsilon^s\rangle_D\, e^N\wedge e^R\,, \\
\pi^{rs} &= \frac{1}{3!}\langle\epsilon^r, \Gamma_{NRS}\epsilon^s\rangle_D\, e^N\wedge e^R\wedge e^S\,, & \tilde{\pi}^{rs} &= \frac{1}{3!}\langle\epsilon^r, \Gamma_{NRS}\Gamma_{11}\epsilon^s\rangle_D\, e^N\wedge e^R\wedge e^S\,, \\
\zeta^{rs} &= \frac{1}{4!}\langle\epsilon^r, \Gamma_{N_1\dots N_4}\epsilon^s\rangle_D\, e^{N_1}\wedge\dots\wedge e^{N_4}\,. & & \qquad (2.11)
\end{aligned}$$

The associated TCFH with respect to the minimal connection is

$$
\mathcal{D}^{\mathcal{F}}_M\sigma := \nabla_M\sigma = -\frac{1}{4}H_{MPQ}\tilde{\omega}^{PQ} - \frac{1}{8}e^{\Phi}F_{PQ}\tilde{\pi}^{PQ}{}_M + \frac{1}{4!}e^{\Phi}G_{MPQR}\pi^{PQR}\,, \tag{2.12}
$$

$$
\begin{aligned}
\mathcal{D}^{\mathcal{F}}_M\tilde{\omega}_{NR} :=& \nabla_M\tilde{\omega}_{NR} + \frac{1}{4}H_{MPQ}\zeta^{PQ}{}_{NR} + \frac{1}{2}e^{\Phi}F_{MP}\pi^P{}_{NR} - \frac{1}{2}e^{\Phi}G_{M[N|PQ|}\tilde{\pi}^{PQ}{}_{R]}\\
=& \frac{1}{2}H_{MNR}\sigma + \frac{1}{4}e^{\Phi}S\tilde{\pi}_{MNR} - \frac{1}{4}e^{\Phi}g_{M[N}F_{|PQ|}\pi^{PQ}{}_{R]} + \frac{3}{4}e^{\Phi}F_{[M|P|}\pi^P{}_{NR]}\\
&- \frac{1}{4!}e^{\Phi\star}G_{MNRP_1P_2P_3}\pi^{P_1P_2P_3} - \frac{1}{12}e^{\Phi}g_{M[N}G_{R]P_1P_2P_3}\tilde{\pi}^{P_1P_2P_3}\\
&- \frac{3}{8}e^{\Phi}G_{[MN|PQ|}\tilde{\pi}^{PQ}{}_{R]}\,,
\end{aligned} \tag{2.13}
$$

$$
\begin{aligned}
\mathcal{D}^{\mathcal{F}}_M\pi_{NRS} :=& \nabla_M\pi_{NRS} + \frac{3}{2}H_{M[N|P|}\tilde{\pi}^P{}_{RS]} - \frac{3}{2}e^{\Phi}F_{M[N}\tilde{\omega}_{RS]} - \frac{3}{4}e^{\Phi}G_{M[N|PQ|}\zeta^{PQ}{}_{RS]}\\
=& \frac{1}{4}e^{\Phi}S\zeta_{MNRS} - \frac{1}{4\cdot 4!}e^{\Phi\star}F_{MNRSP_1\dots P_4}\zeta^{P_1\dots P_4} - \frac{3}{2}e^{\Phi}g_{M[N}F_{R|P|}\tilde{\omega}^P{}_{S]}\\
&- \frac{3}{2}e^{\Phi}F_{[MN}\tilde{\omega}_{RS]} - \frac{1}{4}e^{\Phi}G_{MNRS}\sigma + \frac{1}{8}e^{\Phi\star}G_{MNRSPQ}\tilde{\omega}^{PQ}\\
&- \frac{1}{4}e^{\Phi}g_{M[N}G_{R|P_1P_2P_3|}\zeta^{P_1P_2P_3}{}_{S]} - \frac{3}{4}e^{\Phi}G_{[MN|PQ|}\zeta^{PQ}{}_{RS]}\,,
\end{aligned} \tag{2.14}
$$

$$
\begin{aligned}
\mathcal{D}^{\mathcal{F}}_M\tilde{\pi}_{NRS} :=& \nabla_M\tilde{\pi}_{NRS} + \frac{3}{2}H_{M[N|P|}\pi^P{}_{RS]} - \frac{1}{2}e^{\Phi}F_{MP}\zeta^P{}_{NRS} + \frac{3}{2}e^{\Phi}G_{M[NR|P|}\tilde{\omega}^P{}_{S]}\\
&- \frac{1}{4}e^{\Phi\star}G_{M[NR|P_1P_2P_3|}\zeta^{P_1P_2P_3}{}_{S]} = +\frac{3}{4}e^{\Phi}Sg_{M[N}\tilde{\omega}_{RS]} + \frac{1}{2}e^{\Phi}F_{MP}\zeta^P{}_{NRS}\\
&+ \frac{3}{8}e^{\Phi}g_{M[N}F_{|PQ|}\zeta^{PQ}{}_{RS]} - e^{\Phi}F_{[M|P|}\zeta^P{}_{NRS]} - \frac{3}{4}e^{\Phi}g_{M[N}F_{RS]}\sigma\\
&- \frac{3}{8}e^{\Phi}g_{M[N}G_{RS]PQ}\tilde{\omega}^{PQ} + e^{\Phi}G_{[MNR|P|}\tilde{\omega}^P{}_{S]}\\
&+ \frac{1}{32}e^{\Phi}g_{M[N}{}^{\star}G_{RS]P_1\dots P_4}\zeta^{P_1\dots P_4} - \frac{1}{6}e^{\Phi\star}G_{[MNR|P_1P_2P_3|}\zeta^{P_1P_2P_3}{}_{S]}\,,
\end{aligned} \tag{2.15}
$$

$$
\begin{aligned}
\mathcal{D}^{\mathcal{F}}_M\zeta_{N_1\dots N_4} :=& \nabla_M\zeta_{N_1\dots N_4} - \frac{1}{3}{}^{\star}H_{M[N_1N_2N_3|PQR|}\zeta^{PQR}{}_{N_4]} - 3H_{M[N_1N_2}\tilde{\omega}_{N_3N_4]}\\
&+ 2e^{\Phi}F_{M[N_1}\tilde{\pi}_{N_2N_3N_4]} + 3e^{\Phi}G_{M[N_1N_2|P|}\pi^P{}_{N_3N_4]} + e^{\Phi\star}G_{M[N_1N_2N_3|PQ|}\tilde{\pi}^{PQ}{}_{N_4]}\\
=& \frac{1}{12}g_{M[N_1}{}^{\star}H_{N_2N_3N_4]P_1\dots P_4}\zeta^{P_1\dots P_4} - \frac{5}{12}{}^{\star}H_{[MN_1N_2N_3|PQR|}\zeta^{PQR}{}_{N_4]}\\
&+ e^{\Phi}Sg_{M[N_1}\pi_{N_2N_3N_4]} + \frac{1}{4!}e^{\Phi\star}F_{MN_1\dots N_4PQR}\pi^{PQR} + 3e^{\Phi}g_{M[N_1}F_{N_2|P|}\tilde{\pi}^P{}_{N_3N_4]}\\
&+ \frac{5}{2}e^{\Phi}F_{[MN_1}\tilde{\pi}_{N_2N_3N_4]} + \frac{1}{6}e^{\Phi}g_{M[N_1}{}^{\star}G_{N_2N_3N_4]PQR}\tilde{\pi}^{PQR} + \frac{5}{8}e^{\Phi\star}G_{[MN_1N_2N_3|PQ|}\tilde{\pi}^{PQ}{}_{N_4]}\\
&- \frac{3}{2}e^{\Phi}g_{M[N_1}G_{N_2N_3|PQ|}\pi^{PQ}{}_{N_4]} + \frac{5}{2}e^{\Phi}G_{[MN_1N_2|P|}\pi^P{}_{N_3N_4]}\,.
\end{aligned} \tag{2.16}
$$

As in the previous case, a consequence of the TCFH above is that the forms (2.11) satisfy a generalisation of the CKY equation with respect to the connection $\mathcal{D}^{\mathcal{F}}$. Later we shall demonstrate that in some cases the forms (2.4) and (2.11) generate symmetries in string and particle actions probing some IIA backgrounds.

The factorisation of the domain that the minimal TCFH connection $\mathcal{D}^{\mathcal{F}}$ acts as in (2.4) and (2.11) can be understood as follows. The product of two Majorana representations Δ_{32} in terms of forms is $\otimes^2\Delta_{32} = \Lambda^*(\mathbb{R}^{9,1})$. Therefore the form bilinears of all spinor span all spacetime forms. Therefore generically the TCFH connection acts on the space

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of all spacetime forms. However we have seen that the TCFH connection preserves the forms which are symmetric (skew-symmetric) in the exchange of the two Killing spinors, i.e. it preserves that symmetrised $S^2(\Delta_{32})$ and skew-symmetrised $\Lambda^2(\Delta_{32})$ subspaces of the product. As $\dim S^2(\Delta_{32}) = 528$ and $\dim \Lambda^2(\Delta_{32}) = 496$, the (reduced) holonomy of $\mathcal{D}^{\mathcal{F}}$ is included in $\mathrm{GL}(528) \times \mathrm{GL}(496)$. In fact the holonomy[3] of the minimal connection $\mathcal{D}^{\mathcal{F}}$ reduces further to $SO(9,1) \times \mathrm{GL}(517) \times \mathrm{GL}(495)$ as it acts trivially on the scalars σ and $\tilde{\sigma}$ and does not mix k with the other from bilinears. Of course the holonomy of $\mathcal{D}^{\mathcal{F}}$ reduces even further for special backgrounds.

3 The TCFH of IIB supergravity

The KSEs of IIB supergravity [37] are again associated with the vanishing conditions of the gravitino and dilatino supersymmetry variations. The gravitino KSE is a parallel transport equation for the supercovariant derivative $\mathcal{D}$ of the theory. In the string frame, this can be expressed [38] as

$$\begin{aligned}\mathcal{D}_M := \nabla_M &- \frac{1}{8}\,\Gamma^{N_1N_2}\,H_{MN_1N_2}\,\sigma_3 - \frac{1}{4}\,e^{\Phi}\,\Gamma^N{}_M\,G^{(1)}_N\,(i\sigma_2) - \frac{1}{4}\,e^{\Phi}\,G^{(1)}_M\,(i\sigma_2)\\ &- \frac{1}{24}\,e^{\Phi}\,\Gamma^{N_1N_2N_3}{}_M\,G^{(3)}_{N_1N_2N_3}\,\sigma_1 - \frac{1}{8}\,e^{\Phi}\,\Gamma^{N_1N_2}G^{(3)}_{MN_1N_2}\,\sigma_1\\ &- \frac{1}{96}\,e^{\Phi}\,\Gamma^{N_1\dots N_4}\,G^{(5)}_{MN_1\dots N_4}\,(i\sigma_2)\,,\end{aligned} \tag{3.1}$$

where H and $G^{(n)}$ are the 3-form and n-form, for $n = 1, 3, 5$, NS-NS and R-R field strengths of the theory, respectively, Φ is that dilaton and σ^i, $i = 1, 2, 3$ are the Pauli matrices. The field strength $G^{(5)}$ is anti-self-dual.[4] $\mathcal{D}$ is a connection of the spin bundle over the spacetime associated to two copies, $\oplus^2\Delta^+_{16}$, of the positive chirality Majorana-Weyl representation, Δ^+_{16}, of $\mathfrak{spin}(9,1)$. The (reduced) holonomy of $\mathcal{D}$ for generic IIB backgrounds is included in $SL(32,\mathbb{R})$ [30]. The KSEs of IIB supergravity have been solved for one Killing spinor in [39, 40].

As expected from the general result in [14], the conditions imposed on the form bilinears by the gravitino KSE, $\mathcal{D}\epsilon = 0$, can be organised as a TCFH. Given any two spinors ϵ^r and ϵ^s, the form bilinears are given by

$$\begin{aligned} k^{rs} &= \delta_{ab}\,\left\langle \epsilon^{ra}, \Gamma_P\,\epsilon^{sb}\right\rangle_D\, e^P\,, \qquad k^{(i)rs} = \delta_{ab}\,\left\langle \epsilon^{ra}, \Gamma_P\,(\sigma^i\,\epsilon^s)^b\right\rangle_D\, e^P\,,\\ \pi^{rs} &= \frac{1}{3!}\delta_{ab}\,\left\langle \epsilon^{ra}, \Gamma_{P_1P_2P_3}\,\epsilon^{sb}\right\rangle_D\, e^{P_1}\wedge e^{P_2}\wedge e^{P_3}\,,\\ \pi^{(i)rs} &= \frac{1}{3!}\,\delta_{ab}\,\left\langle \epsilon^{ra}, \Gamma_{P_1P_2P_3}\,(\sigma^i\,\epsilon^s)^b\right\rangle_D\, e^{P_1}\wedge e^{P_2}\wedge e^{P_3}\,,\\ \tau^{rs} &= \frac{1}{5!}\,\delta_{ab}\,\left\langle \epsilon^{ra}, \Gamma_{P_1P_2P_3P_4P_5}\,\epsilon^{sb}\right\rangle_D\, e^{P_1}\wedge\cdots\wedge e^{P_5}\,,\\ \tau^{(i)rs} &= \frac{1}{5!}\,\delta_{ab}\,\left\langle \epsilon^{ra}, \Gamma_{P_1P_2P_3P_4P_5}\,(\sigma^i\epsilon^s)^b\right\rangle_D\, e^{P_1}\wedge\cdots\wedge e^{P_5}\,,\end{aligned} \tag{3.2}$$

[3]Note though that the (reduced) holonomy of the maximal TCFH connection, see [14] for definition, is included in $\mathrm{GL}(528) \times \mathrm{GL}(496)$.

[4]Our Hodge duality conventions are as in the IIA theory.

where $\langle \sigma^i \alpha, \beta\rangle_D = \langle \alpha, \sigma^i \beta\rangle_D$ as the Pauli matrices are hermitian and $a, b = 1, 2$. Note that the forms k, $k^{(1)}$, $k^{(3)}$, $\pi^{(2)}$, τ, $\tau^{(1)}$ and $\tau^{(3)}$ are symmetric in the exchange of ϵ^r and ϵ^s while the rest are skew symmetric.

The forms $k^{(2)}$, $\pi^{(2)}$ and $\tau^{(2)}$ are purely imaginary while the rest are real. One could multiply them with the imaginary unit i so they become real but in such case the expression for the TCFH below would have been more involved. So we shall not do this here but later when we consider applications, we shall replace $k^{(2)}$, $\pi^{(2)}$ and $\tau^{(2)}$ with $ik^{(2)}$, $i\pi^{(2)}$ and $i\tau^{(2)}$.

Using the gravitino KSE, $\mathcal{D}\epsilon = 0$, one can show that the TCFH of IIB supergravity expressed in terms of the minimal connection $\mathcal{D}^{\mathcal{F}}$ is

$$\begin{aligned}\mathcal{D}^{\mathcal{F}}_M k_P &:= \nabla_M k_P = \frac{1}{2} H_{MP}{}^N k^{(3)}_N + \frac{i}{2} e^\Phi G^{(1)N} \pi^{(2)}_{NMP} + \frac{1}{12} e^\Phi G^{(3)N_1N_2N_3} \tau^{(1)}_{N_1N_2N_3MP} \\ &\quad + \frac{1}{2} e^\Phi G^{(3)}_{MP}{}^N k^{(1)}_N + \frac{i}{12} e^\Phi G^{(5)}_{MP}{}^{N_1N_2N_3} \pi^{(2)}_{N_1N_2N_3}\,,\end{aligned} \tag{3.3}$$

$$\begin{aligned}\mathcal{D}^{\mathcal{F}}_M k^{(i)}_P &:= \nabla_M k^{(i)}_P + \frac{i}{4} \varepsilon_{3ij} H_M{}^{N_1N_2} \pi^{(j)}_{PN_1N_2} - \varepsilon_{2ij} e^\Phi G^{(1)}_M k^{(j)}_P \\ &\quad + \frac{i}{2} \varepsilon_{1ij} e^\Phi G^{(3)N_1N_2}_M \pi^{(j)}_{PN_1N_2} - \frac{1}{48} \varepsilon_{2ij} e^\Phi G^{(5)N_1\ldots N_4}_M \tau^{(j)}_{PN_1\ldots N_4} \\ &= \frac{1}{2} \delta_{i3} H_{MP}{}^N k_N + \frac{i}{2} \delta_{i2} e^\Phi G^{(1)N} \pi_{MPN} - \varepsilon_{2ij} e^\Phi G^{(1)}_{[M} k^{(j)}_{P]} \\ &\quad - \frac{1}{2} \varepsilon_{2ij} e^\Phi g_{MP} G^{(1)N} k^{(j)}_N + \frac{1}{12} \delta_{i1} e^\Phi G^{(3)N_1N_2N_3} \tau_{MPN_1N_2N_3} \\ &\quad + \frac{i}{12} \varepsilon_{1ij} e^\Phi g_{MP} G^{(3)N_1N_2N_3} \pi^{(j)}_{N_1N_2N_3} + \frac{i}{2} \varepsilon_{1ij} e^\Phi G^{(3)N_1N_2}{}_{[M} \pi^{(j)}_{P]N_1N_2} \\ &\quad + \frac{1}{2} \delta_{i1} e^\Phi G^{(3)}_{MP}{}^N k_N + \frac{i}{12} \delta_{i2} e^\Phi G^{(5)}_{MP}{}^{N_1N_2N_3} \pi_{N_1N_2N_3}\,,\end{aligned} \tag{3.4}$$

$$\begin{aligned}\mathcal{D}^{\mathcal{F}}_M \pi_{P_1P_2P_3} &:= \nabla_M \pi_{P_1P_2P_3} - \frac{3}{2} H_{M[P_1}{}^N \pi^{(3)}_{P_2P_3]N} - 3 e^\Phi G^{(3)}_{M[P_1}{}^N \pi^{(1)}_{P_2P_3]N} \\ &\quad - \frac{i}{4} e^\Phi G^{(5)}_{M[P_1}{}^{N_1N_2N_3} \tau^{(2)}_{P_2P_3]N_1N_2N_3} \\ &= \frac{i}{2} e^\Phi G^{(1)N} \tau^{(2)}_{MP_1P_2P_3N} + 3i\, e^\Phi g_{M[P_1} G^{(1)}_{P_2} k^{(2)}_{P_3]} \\ &\quad - \frac{1}{12} e^\Phi\, {}^\star G^{(7)}_{MP_1P_2P_3}{}^{N_1N_2N_3} \pi^{(1)}_{N_1N_2N_3} + \frac{3}{2} e^\Phi g_{M[P_1} G^{(3)N_1N_2}_{P_2} \pi^{(1)}_{P_3]N_1N_2} \\ &\quad + 3 e^\Phi G^{(3)}_{[P_1P_2}{}^N \pi^{(1)}_{P_3M]N} - \frac{i}{2} e^\Phi G^{(5)}_{MP_1P_2P_3}{}^N k^{(2)}_N\,,\end{aligned} \tag{3.5}$$

$$\begin{aligned}\mathcal{D}^{\mathcal{F}}_M \pi^{(i)}_{P_1P_2P_3} &:= \nabla_M \pi^{(i)}_{P_1P_2P_3} - \frac{3}{2} \delta_{i3} H_{M[P_1}{}^N \pi_{P_2P_3]N} + \frac{i}{4} \varepsilon_{3ij} H_{MN_1N_2} \tau^{(j)}_{P_1P_2P_3}{}^{N_1N_2} \\ &\quad - \frac{3i}{2} \varepsilon_{3ij} H_{M[P_1P_2} k^{(j)}_{P_3]} - \varepsilon_{2ij} e^\Phi G^{(1)}_M \pi^{(j)}_{P_1P_2P_3} - 3\,\delta_{i1} e^\Phi G^{(3)}_{M[P_1}{}^N \pi_{P_2P_3]N} \\ &\quad + \frac{i}{2} \varepsilon_{1ij} e^\Phi G^{(3)N_1N_2}_M \tau^{(j)}_{P_1P_2P_3N_1N_2} - 3i\, \varepsilon_{1ij} e^\Phi G^{(3)}_{M[P_1P_2} k^{(j)}_{P_3]} \\ &\quad + \frac{3}{2} \varepsilon_{2ij} e^\Phi G^{(5)}_{M[P_1P_2}{}^{N_1N_2} \pi^{(j)}_{P_3]N_1N_2} - \frac{i}{4} \delta_{i2} e^\Phi G^{(5)}_{M[P_1}{}^{N_1N_2N_3} \tau_{P_2P_3]N_1N_2N_3} \\ &= \frac{i}{2} \delta_{i2} e^\Phi G^{(1)N} \tau_{MP_1P_2P_3N} + 3i\, \delta_{i2} e^\Phi g_{M[P_1} G^{(1)}_{P_2} k_{P_3]} + 2\,\varepsilon_{2ij} e^\Phi G^{(1)}_{[P_1} \pi^{(j)}_{P_2P_3M]} \\ &\quad - \frac{3}{2} \varepsilon_{2ij} e^\Phi G^{(1)N} g_{M[P_1} \pi^{(j)}_{P_2P_3]N} - \frac{1}{12} \delta_{i1} e^\Phi\, {}^\star G^{(7)}_{MP_1P_2P_3}{}^{N_1N_2N_3} \pi_{N_1N_2N_3} \\ &\quad + \frac{3}{2} \delta_{i1} e^\Phi g_{M[P_1} G^{(3)N_1N_2}_{P_2} \pi_{P_3]N_1N_2} + 3\,\delta_{i1} e^\Phi G^{(3)}_{[P_1P_2}{}^N \pi_{P_3M]N} \\ &\quad + \frac{i}{4} \varepsilon_{1ij} e^\Phi G^{(3)N_1N_2N_3} g_{M[P_1} \tau^{(j)}_{P_2P_3]N_1N_2N_3} - i\, \varepsilon_{1ij} e^\Phi G^{(3)}_{[P_1}{}^{N_1N_2} \tau^{(j)}_{P_2P_3M]N_1N_2}\end{aligned}$$

$$
\begin{aligned}
&- \frac{3i}{2}\, \varepsilon_{1ij}\, e^{\Phi}\, g_{M[P_1}\, G^{(3)}_{P_2P_3]}{}^{N}\, k^{(j)}_{N} + 2i\, \varepsilon_{1ij}\, e^{\Phi}\, G^{(3)}_{[P_1P_2P_3}\, k^{(j)}_{M]} \\
&- \frac{i}{2}\, \delta_{i2}\, e^{\Phi}\, G^{(5)}_{MP_1P_2P_3}{}^{N}\, k_N + \frac{1}{4}\, \varepsilon_{2ij}\, e^{\Phi}\, g_{M[P_1}\, G^{(5)}_{P_2P_3]}{}^{N_1N_2N_3}\, \pi^{(j)}_{N_1N_2N_3} \\
&- \varepsilon_{2ij}\, e^{\Phi}\, G^{(5)}_{[P_1P_2P_3}{}^{N_1N_2}\, \pi^{(j)}_{M]N_1N_2}\,, \qquad (3.6)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}^{\mathcal{F}}_M \tau_{P_1\dots P_5} :=& \nabla_M\, \tau_{P_1\dots P_5} + \frac{5}{2}\, H_M{}^{N}{}_{[P_1}\, \tau^{(3)}_{P_2\dots P_5]N} - 5\, e^{\Phi}\, G^{(3)}_{M[P_1}{}^{N}\, \tau^{(1)}_{P_2\dots P_5]N} \\
&+ 10i\, e^{\Phi}\, G^{(5)}_{M[P_1P_2P_3}{}^{N}\, \pi^{(2)}_{P_4P_5]N} \\
=& - \frac{i}{12}\, e^{\Phi}\, {}^{\star}G^{(9)}_{MP_1\dots P_5}{}^{N_1N_2N_3}\, \pi^{(2)}_{N_1N_2N_3} + 10i\, e^{\Phi}\, g_{M[P_1}\, G^{(1)}_{P_2}\, \pi^{(2)}_{P_3P_4P_5]} \\
&+ \frac{1}{2}\, e^{\Phi}\, {}^{\star}G^{(7)}_{MP_1\dots P_5}{}^{N}\, k^{(1)}_{N} + \frac{15}{2}\, e^{\Phi}\, G^{(3)}_{[P_1P_2}{}^{N}\, \tau^{(1)}_{P_3P_4P_5M]N} \\
&+ 5\, e^{\Phi}\, g_{M[P_1}\, G^{(3)}_{P_2}{}^{N_1N_2}\, \tau^{(1)}_{P_3P_4P_5]N_1N_2} - 10\, e^{\Phi}\, g_{M[P_1}\, G^{(3)}_{P_2P_3P_4}\, k^{(1)}_{P_5]} \\
&- 5i\, e^{\Phi}\, g_{M[P_1}\, G^{(5)}_{P_2P_3P_4}{}^{N_1N_2}\, \pi^{(2)}_{P_5]N_1N_2} - \frac{15i}{2}\, e^{\Phi}\, G^{(5)}_{[P_1\dots P_4}{}^{N}\, \pi^{(2)}_{P_5M]}\,, \qquad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}^{\mathcal{F}}_M \tau^{(i)}_{P_1\dots P_5} :=& \nabla_M\, \tau^{(i)}_{P_1\dots P_5} + \frac{5}{2}\, \delta_{i3}\, H_M{}^{N}{}_{[P_1}\, \tau_{P_2\dots P_5]N} + \frac{5i}{4}\, \varepsilon_{3ij}\, {}^{\star}H_{M[P_1\dots P_4}{}^{N_1N_2}\, \pi^{(j)}_{P_5]N_1N_2} \\
&- 5i\, \varepsilon_{3ij}\, H_{M[P_1P_2}\, \pi^{(j)}_{P_3P_4P_5]} - \varepsilon_{2ij}\, e^{\Phi}\, G^{(1)}_{M}\, \tau^{(j)}_{P_1\dots P_5} - 5\, \delta_{i1}\, e^{\Phi}\, G^{(3)}_{M[P_1}{}^{N}\, \tau_{P_2\dots P_5]N} \\
&+ \frac{5i}{2}\, \varepsilon_{1ij}\, e^{\Phi}\, {}^{\star}G^{(7)}_{M[P_1\dots P_4}{}^{N_1N_2}\, \pi^{(j)}_{P_5]N_1N_2} - 10i\, \varepsilon_{1ij}\, e^{\Phi}\, G^{(3)}_{M[P_1P_2}\, \pi^{(j)}_{P_3P_4P_5]} \\
&- 5\, \varepsilon_{2ij}\, e^{\Phi}\, G^{(5)}_{M[P_1\dots P_4}\, k^{(j)}_{P_5]} + \frac{5}{2}\, \varepsilon_{2ij}\, e^{\Phi}\, G^{(5)}_{M[P_1P_2}{}^{N_1N_2}\, \tau^{(j)}_{P_3P_4P_5]N_1N_2} \\
&+ 10i\, \delta_{i2}\, e^{\Phi}\, G^{(5)}_{M[P_1P_2P_3}{}^{N}\, \pi_{P_4P_5]N} \\
=& \frac{5i}{12} \varepsilon_{3ij}\, g_{M[P_1}\, {}^{\star}H_{P_2\dots P_5]}{}^{N_1N_2N_3}\, \pi^{(j)}_{N_1N_2N_3} - \frac{3i}{2}\, \varepsilon_{3ij}\, {}^{\star}H_{[P_1\dots P_5}{}^{N_1N_2}\, \pi^{(j)}_{M]N_1N_2} \\
&- \frac{i}{12}\, \delta_{i2}\, e^{\Phi}\, {}^{\star}G^{(9)}_{MP_1\dots P_5}{}^{N_1N_2N_3}\, \pi_{N_1N_2N_3} + 10i\, \delta_{i2}\, e^{\Phi}\, g_{M[P_1}\, G^{(1)}_{P_2}\, \pi_{P_3P_4P_5]} \\
&- \frac{5}{2}\, \varepsilon_{2ij}\, e^{\Phi}\, G^{(1)N}\, g_{M[P_1}\, \tau^{(j)}_{P_2\dots P_5]N} + 3\, \varepsilon_{2ij}\, e^{\Phi}\, G^{(1)}_{[P_1}\, \tau^{(j)}_{P_2\dots P_5M]} \\
&+ \frac{1}{2} \delta_{i1}\, e^{\Phi}\, {}^{\star}G^{(7)}_{MP_1\dots P_5}{}^{N}\, k_N + 5\, \delta_{i1}\, e^{\Phi}\, g_{M[P_1}\, G^{(3)}_{P_2}{}^{N_1N_2}\, \tau_{P_3P_4P_5]N_1N_2} \\
&+ \frac{15}{2} \delta_{i1}\, e^{\Phi}\, G^{(3)}_{[P_1P_2}{}^{N}\, \tau_{P_3P_4P_5M]N} - 10\delta_{i1}\, e^{\Phi}\, g_{M[P_1}\, G^{(3)}_{P_2P_3P_4}\, k_{P_5]} \\
&+ 10i\, \varepsilon_{1ij}\, e^{\Phi}\, G^{(3)}_{[P_1P_2P_3}\, \pi^{(j)}_{P_4P_5M]} - 15i\, \varepsilon_{1ij}\, e^{\Phi}\, g_{M[P_1}\, G^{(3)}_{P_2P_3}{}^{N}\, \pi^{(j)}_{P_4P_5]N} \\
&+ \frac{5i}{12}\, \varepsilon_{1ij} e^{\Phi}\, g_{M[P_1}\, {}^{\star}G^{(7)}_{P_2\dots P_5]}{}^{N_1N_2N_3}\, \pi^{(j)}_{N_1N_2N_3} - \frac{3i}{2}\, \varepsilon_{1ij}\, e^{\Phi}\, {}^{\star}G^{(7)}_{[P_1\dots P_5}{}^{N_1N_2}\, \pi^{(j)}_{M]N_1N_2} \\
&- \frac{5}{2}\, \varepsilon_{2ij}\, e^{\Phi}\, g_{M[P_1}\, G^{(5)}_{P_2\dots P_5]}{}^{N}\, k^{(j)}_{N} + 3\, \varepsilon_{2ij}\, e^{\Phi}\, G^{(5)}_{[P_1\dots P_5}\, k^{(j)}_{M]} \\
&- 5i\, \delta_{i2}\, e^{\Phi}\, g_{M[P_1} G^{(5)}_{P_2P_3P_4}{}^{N_1N_2}\, \pi_{P_5]N_1N_2} - \frac{15i}{2} \delta_{i2}\, e^{\Phi}\, G^{(5)}_{[P_1\dots P_4}{}^{N}\, \pi_{P_5M]N}\,, \qquad (3.8)
\end{aligned}
$$

where for simplicity we have suppressed the r, s indices on the form bilinears that label the Killing spinors. Although it is not manifest from the expression of TCFH above, the TCFH preserves the form bilinears that are either symmetric or skew-symmetric in the exchange of the two Killing spinors. Moreover all terms of the IIB TCFH can be arranged to be real. The imaginary unit that appears in some terms can be eliminated after replacing the purely imaginary forms $k^{(2)}$, $\pi^{(2)}$ and $\tau^{(2)}$ with the real forms $ik^{(2)}$, $i\pi^{(2)}$ and $i\tau^{(2)}$. A consequence of the TCFH is that all form bilinears satisfy a generalisation of the CKY equation with respect to the minimal connection $\mathcal{D}^{\mathcal{F}}$. In particular k is Killing, $\nabla_{(M}\, k^{rs}_{P)} = 0$, as expected.

To understand the factorisation of the domain that $\mathcal{D}^{\mathcal{F}}$ acts note that the product of two Majorana-Weyl representations Δ^+_{16} of $\mathfrak{spin}(9,1)$ decomposes as

$$\otimes^2\Delta^+_{16} = \Lambda^1(\mathbb{R}^{9,1}) \oplus \Lambda^3(\mathbb{R}^{9,1}) \oplus \Lambda^{5-}(\mathbb{R}^{9,1})\,, \tag{3.9}$$

where $\Lambda^{5-}(\mathbb{R}^{9,1})$ is the space of anti-self-dual 5-forms on $\mathbb{R}^{9,1}$. The Killing spinors lie in two copies of Δ^+_{16}, i.e. $\Delta^+_{32} = \oplus^2\Delta^+_{16}$. Therefore the space of all IIB form bilinears is identified with the product $\otimes^2\Delta^+_{32}$. This can be decomposed in terms of spacetime forms as indicated above. Indeed notice that $\dim\left(\otimes^2 \Delta^+_{32}\right) = 32\cdot 32 = 4[\dim\left(\Lambda^1(\mathbb{R}^{9,1})\right) + \dim\left(\Lambda^3(\mathbb{R}^{9,1})\right) + \dim\left(\Lambda^{5-}(\mathbb{R}^{9,1})\right)]$. The minimal connection $\mathcal{D}^{\mathcal{F}}$ of the TCFH preserves the symmetric $S^2(\Delta^+_{32})$ and skew-symmetric $\Lambda^2(\Delta^+_{32})$ subspaces of $\otimes^2\Delta^+_{32}$. As a consequence the (reduced) holonomy of $\mathcal{D}^{\mathcal{F}}$ for a generic background is included in $GL(528)\times GL(496)$. As in IIA case investigated in the previous section, the (reduced) holonomy[5] of the minimal TCFH connection reduces further to $SO(9,1)\times GL(518)\times GL(496)$ as it does not mix k with the other form bilinears.

4 Particles and integrability of type II branes

Before we proceed to investigate the symmetries of particle and string probes generated by the TCFHs of type II theories, we shall summarise some of the properties of KS, KY and CKY tensors and their applications to generating symmetries for particle actions, for more detailed studies, see reviews [10] and [11] and the references within. We shall also present some of the particle actions that are invariant under the symmetries generated by such tensors. Then we shall construct the KS and KY tensors of type II brane solutions which to our knowledge have not presented before. We shall use these to argue that the geodesic flow of some of these solutions is completely integrable and we shall give the associated independent conserved charges in involution.

4.1 Killing-Stäckel and Killing-Yano tensors

4.1.1 Definitions and outline of properties

A rank k conformal Killing-Stäckel (k-CKS) tensor is a symmetric $(0,k)$ tensor d on a n-dimensional spacetime M with metric g which satisfies the equation

$$\nabla_{(M} d_{N_1 N_2\cdots N_k)} = g_{(MN_1} q_{N_2\cdots N_k)}\,, \tag{4.1}$$

where q is a symmetric $(0,k-1)$ tensor and ∇ is the Levi-Civita connection of g. For $k=1$, the equation reduces to that of a conformal Killing vector field. If q vanishes, $q=0$, then d will be a Killing-Stäckel (KS) tensor.

Furthermore observe that if d and e are $k-$ and $\ell-$ CKS (KS) tensors on M, then

$$(d\otimes_s e)_{N_1\cdots N_{k+\ell}} := d_{(N_1\cdots N_k} e_{N_{k+1}\cdots N_{k+\ell})}\,, \tag{4.2}$$

is a $(k+\ell)-$CKS (KS) tensor on M.

[5]Notice that the (reduced) holonomy of the maximal TCFH connection, see [14], is included in $GL(528)\times GL(496)$.

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KS tensors are associated with conserved charges of test particle systems. Indeed consider the action

$$A = \frac{1}{2}\int d\tau\, g_{MN}\, \dot{x}^M\, \dot{x}^N\,, \tag{4.3}$$

which describes the geodesic flow[6] on a spacetime (manifold) M with metric g, where $\dot{x}$ denotes the derivative of the coordinate x with respect to the affine parameter τ. It is straightforward to show that if the spacetime M admits a KS tensor d, then

$$Q(d) = d_{N_1 N_2 \cdots N_k}\, \dot{x}^{N_1}\, \dot{x}^{N_2} \cdots \dot{x}^{N_k}\,, \tag{4.4}$$

is conserved along the geodesic flow, i.e. $\dot{Q}(d) = 0$ subject to the geodesic equations with affine parameter τ. This charge generates the infinitesimal transformation

$$\delta x^M = \epsilon\, d^M{}_{N_1 \cdots N_{k-1}} \dot{x}^{N_1} \cdots \dot{x}^{N_{k-1}}\,, \tag{4.5}$$

which is a symmetry of the action (4.3) with infinitesimal parameter ϵ.

A rank k conformal Killing-Yano (k-CKY) tensor is a k-form, α, which satisfies the condition

$$\nabla_M \alpha_{N_1 N_2 \cdots N_k} = \frac{1}{k+1} d\alpha_{M N_1 \dots N_k} - \frac{k}{n-k+1} g_{M[N_1} \delta\alpha_{N_2 \cdots N_k]}\,. \tag{4.6}$$

If α is co-closed, $\delta\alpha = 0$, then α is a Killing-Yano (KY) form while if α is closed, $d\alpha = 0$, α is a closed conformal Killing-Yano (CCKY) form. It turns out that if α is KY, then the Hodge dual, ${}^\star\alpha$, of α is CCKY form.

Furthermore, if α and β are k-CKY (k-KY) forms, then

$$\alpha_{(M}{}^{L_1 \cdots L_{k-1}} \beta_{N) L_1 \cdots L_{k-1}}\,, \tag{4.7}$$

is a 2-CKS (2-KS) tensor. In addition, if α and β are CCKY forms of rank k and ℓ, respectively, then $\alpha \wedge \beta$ is a $(k+\ell)$-CCKY form.

KY forms generate symmetries [12] for spinning particle actions [13]. These are supersymmetric extensions of (4.3). Such an action is

$$A = -\frac{i}{2}\int d\tau\, d\theta\, g_{MN}\, Dx^M\, \dot{x}^N\,, \tag{4.8}$$

where x are superfields $x = x(\tau, \theta)$, τ is the even and θ is the odd coordinate of the worldline superspace, and the superspace derivative D satisfies $D^2 = i\partial_\tau$. In particular, the KY form α generates the infinitesimal symmetry

$$\delta x^M = \epsilon\, \alpha^M{}_{N_1 \cdots N_{k-1}} Dx^{N_1} \cdots Dx^{N_{k-1}}\,, \tag{4.9}$$

for the action (4.8), where ϵ is an infinitesimal parameter. The associated conserved charge is

$$\begin{aligned} Q(\alpha) &= (k+1)\alpha_{N_1 N_2 \cdots N_k} \partial_\tau x^{N_1} Dx^{N_2} \cdots Dx^{N_k} \\ &\quad - \frac{i}{k+1} (d\alpha)_{N_1 N_2 \cdots N_{k+1}} Dx^{N_1} Dx^{N_2} \cdots Dx^{N_{k+1}}\,. \end{aligned} \tag{4.10}$$

Observe that $Q(\alpha)$ is conserved, $DQ(\alpha) = 0$, subject to the equations of motion of (4.8).

[6]When viewing the geodesic flow as a dynamical system, M is identified with its configuration space.

Note that if the KY form α is closed, $d\alpha = 0$, and so α is covariantly constant (or equivalently parallel) with respect to the Levi-Civita connection, then

$$\tilde{Q}(\alpha) = \alpha_{N_1 N_2 \cdots N_k} Dx^{N_1} Dx^{N_2} \cdots Dx^{N_k} \ , \tag{4.11}$$

is also conserved subject to the field equations of (4.8), $\partial_\tau \tilde{Q}(\alpha) = 0$. This gives the conservation of two charges $\tilde{Q}(\alpha)$ and $D\tilde{Q}(\alpha)$. The latter is proportional to that in (4.10) with $d\alpha = 0$.

There are several generalisations of CKY tensors [41–46]. One of the most common ones is to replace the Levi-Civita connection that appears in the definition (4.6) with another connection, for example a connection with skew-symmetric torsion. Some of the properties mentioned above extend to the generalised KY tensors. For an application of the KY forms to G-structures see [47, 48].

4.1.2 Integrability and separability

A dynamical system with a $2n$-dimensional phase space P is completely integrable according to Liouville provided it admits n independent constants of motion, Q^r, $r = 1, \ldots, n$, including the Hamiltonian H, in involution. Independence means that the map $Q : P \to \mathbb{R}^n$ is of rank n, where $Q = (Q^1, \ldots, Q^n)$, and in involution means that the Poisson bracket algebra of the constants of motion Q^r vanishes

$$\{Q^r, Q^s\}_{\text{PB}} = 0 \ . \tag{4.12}$$

Returning to the particle system described by the action (4.3), the conserved charges (4.4) can be written as functions on phase space, T^*M, as

$$Q(d) = d^{N_1 \cdots N_k} p_{N_1} \ldots p_{N_k} \ , \tag{4.13}$$

where p_M is the conjugate momentum of x^M. It turns out that if $Q(d)$ and $Q(e)$ are conserved charges associated with KS tensors d and e, then $\{Q(d), Q(e)\}_{\text{PB}}$ is associated with the KS tensor given in terms of the Nijenhuis-Schouten bracket

$$([d,e]_{\text{NS}})^{N_1 \cdots N_{k+\ell-1}} = k d^{M(N_1 \cdots N_{k-1}} \partial_M e^{N_k \cdots N_{k+\ell-1})} - \ell e^{M(N_1 \cdots N_{\ell-1}} \partial_M d^{N_k \cdots N_{k+\ell-1})} \ , \tag{4.14}$$

of d and e. Therefore, one has

$$\{Q(d), Q(e)\}_{\text{PB}} = Q([d,e]_{\text{NS}}) \ . \tag{4.15}$$

Observe that if d is a vector, then $[d,e]_{\text{NS}} = \mathcal{L}_d e$, i.e. the Nijenhuis-Schouten bracket is the Lie derivative of e with respect to the vector field d. So two charges are in involution provided that the Nijenhuis-Schouten bracket of the associated KS tensors vanishes.

Completely integrable systems are special. There are difficulties in both finding conserved charges in involution and in proving that they are independent. For example if $Q(d)$ and $Q(e)$ are conserved charges, $Q(d)Q(e)$ is not an independent conserved charge, as its inclusion in the map $Q : P \to \mathbb{R}^n$ does not alter its rank. However for the geodesic flow described by the action (4.3) that we shall investigate below, there is a simplifying feature.

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The spacetimes we shall be considering admit a non-abelian group of isometries. For every isometry generated by a Killing vector field K_r, there is an associated conserved charge

$$Q_r = K_r^M p_M \ . \tag{4.16}$$

Of course these charges may not be in involution. However note that the charges Q_r written in phase space do not depend on the spacetime metric. They only depend on the way that the isometry group acts on the spacetime. Typically there are many metrics for which Q_r are constants of motion for the action (4.3). Of course any polynomial of Q_r is also conserved and is independent from the metric of the particle system. We shall refer to these charges as *orbital* to emphasise their independence from the spacetime metric. In many occasions, it is possible to find polynomials of Q_r which are independent and are in involution. Suppose that one can find $n-1$ such independent (polynomial) orbital charges in involution and the Hamiltonian,

$$H = \frac{1}{2} g^{MN} p_M p_N \ , \tag{4.17}$$

is independent from the orbital charges. Then the geodesic flow is completely integrable because the orbital charges will Poisson commute with the Hamiltonian. Of course the Hamiltonian depends on the spacetime metric. To distinguish the conserved charges which depend on the spacetime metric from the orbital ones we shall refer to former as *Hamiltonian*. We shall demonstrate that this strategy of proving complete integrability of a geodesic flow based on non-abelian isometries is particularly effective whenever the non-abelian group of isometries has a principal orbit in a spacetime of codimension of at most one. The complete integrability of geodesic flows on homogeneous manifolds has been extensively investigated in the mathematics literature, see e.g. [49].

4.1.3 An example

Before we proceed to investigate the KS and KY tensors and the integrability of the geodesic flow on some type II backgrounds, let us present an example. The standard example is that of the Kerr black hole. However more suitable for the examples that follow is to consider $\mathbb{R}^{2n}$ with a conformally flat metric

$$g = h(|y|) \delta_{ij} dy^i dy^j \ , \tag{4.18}$$

where $|y|$ is the length of the coordinate y with respect to the Euclidean norm and $h > 0$.

A direct computation reveals that the following tensors

$$d_{i_1 \dots i_k} = h^k(|y|) \ y^{j_1} \dots y^{j_q} a_{j_1 \dots j_q, i_1 \dots i_k} \ , \tag{4.19}$$

are KS tensors provided that the coefficients a are constant and satisfy

$$a_{(j_1 \dots j_q, i_1) \dots i_k} = a_{j_1 \dots (j_q, i_1 \dots i_k)} = 0 \ . \tag{4.20}$$

For each of these KS tensors, there is an associated conserved charge $Q(d)$ given in (4.13) of the geodesic flow on $\mathbb{R}^{2n}$ with metric (4.18). These generate an infinite dimensional

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symmetry algebra for the action (4.3) with metric (4.18) which is isomorphic to the Poisson algebra of $Q(d)$'s up to terms proportional to the equations of motion, i.e. the algebra of symmetry transformations is isomorphic on-shell to the Poisson bracket algebra of the charges. The conserved charges $Q(d)$ may neither be independent nor in involution.

Next let us turn to find the KY and CCKY tensors on $\mathbb{R}^{2n}$ with metric (4.18). After some computation, one finds that

$$\alpha = h^{\frac{k}{2}} i_Y \varphi \,, \qquad \beta = h^{\frac{k+2}{2}} Y \wedge \varphi \,, \tag{4.21}$$

are KY and CCKY forms, respectively, for any constant k-form φ on $\mathbb{R}^{2n}$, where Y is either the vector field $Y = y^i \partial_i$ or the one-form $Y = y_i dy^i$; it is clear from the context what Y denotes in each case.

For each KY tensor above, one can construct the infinitesimal variation (4.9) which is a symmetry of the action (4.8). However the commutator of two such infinitesimal transformations does not close to an infinitesimal transformation of the same type. Typically, the right-hand side of the commutator will involve a term polynomial in Dx as well as a term which is linear in the velocity $\dot{x}$. A systematic exploration of such commutators in a related context can be found in [50, 51].

Next let us turn to investigate the integrability of the geodesic flow of the metric (4.18). The geodesic equations can be easily integrated in angular coordinates. However it is instructive to provide a symmetry argument for the complete integrability of the geodesic equations. The isometry group of the above backgrounds is SO($2n$). The Killing vector fields are

$$k_{ij} = y_i \partial_j - y_j \partial_i \,, \quad i < j \,, \tag{4.22}$$

where $y_i = y^i$. The associated conserved charges are

$$Q_{ij} = Q(k_{ij}) = y_i p_j - y_j p_i \,. \tag{4.23}$$

Notice that all these conserved charges are orbital as they do not depend on the metric (4.18). As $\mathcal{L}_{k_{ij}} g = 0$, one can show that Q_{ij} commute with the Hamiltonian $H = \frac{1}{2} h^{-1} \delta^{ij} p_i p_j$, i.e. $\{H, Q_{ij}\}_{\rm PB} = 0$.

The conserved charges Q_{ij} are not in involution as $\{Q(k_{ij}), Q(k_{pq})\}_{\rm PB} = Q([k_{ij}, k_{pq}])$. However using these, one can verify that the $2n-1$ orbital conserved charges

$$D_m = \frac{1}{4} \sum_{i,j \geq 2n+1-m} (Q_{ij})^2 \,, \quad m = 2, 3, \dots, 2n \,, \tag{4.24}$$

are in involution. These together with the Hamiltonian $H = \frac{1}{2} h^{-1} \delta^{ij} p_i p_j$ give $2n$ charges in involution. Therefore the geodesic flow of the metric (4.18) is completely (Liouville) integrable.

An alternative way to think about the complete integrability of the geodesic flow on $\mathbb{R}^{2n}$ with metric (4.18) is to consider it as a motion along the round sphere S^{2n-1} in $\mathbb{R}^{2n}$ and as a motion along the radial direction $r = |y|$. For this write the metric (4.18) as

$$g = h(r) (dr^2 + r^2 g(S^{2n-1})) \,, \tag{4.25}$$

where $g(S^{2n-1})$ is the metric on the round S^{2n-1} sphere. It is well known that the vector fields (4.22) are tangential to S^{2n-1} and leave the round metric on S^{2n-1} invariant. The associated conserved charges are as in (4.23) and they are functions of T^*S^{2n-1}, i.e. they do not depend on the radial direction p_r of the momentum p. One can proceed to define (4.24) and in turn show that the geodesic flow on S^{2n-1} is completely integrable. Notice that D_{2n} is the Hamiltonian of the geodesic flow on S^{2n-1}. All these charges including the Hamiltonian on S^{2n-1} are orbital as they do not depend on the metric (4.18). As there are $2n-1$ independent charges in involution associated with the geodesic flow on S^{2n-1}, the addition of the Hamiltonian $H = \frac{1}{2}h^{-1}\delta^{ij}p_i p_j$ of the geodesic flow on $\mathbb{R}^{2n}$ gives $2n$ independent conserved charges in involution proving the complete integrability of the geodesic flow of the metric (4.18).

This construction can be reversed engineered and generalised. In particular consider a metric on a n-dimensional manifold M^n

$$g(M^n) = dz^2 + g(N^{n-1})(z)\,, \tag{4.26}$$

where z is a coordinate and $g(N^{n-1})(z)$ is a metric on the submanifold N^{n-1} of M^n which may depend on z. Suppose now there is a group of isometries on M^n which has as a principal orbit N^{n-1}. Clearly the associated conserved charges $Q = K^M p_M$, for each Killing vector field K, will be functions on T^*N. If one is able to find orbital conserved charges D_m, $m = 1, \dots, n-1$ in involution, then the geodesic flow on M^n will be completely integrable after the inclusion of the Hamiltonian H of the geodesic flow on M^n as an additional conserved charge. This is because H is a function on T^*M^n and so it is independent from D_m which are functions on T^*N^{n-1}. Moreover $\{D_m, H\}_{\text{PB}} = 0$ as D_m are constructed as polynomials of the conserved charges associated with the isometries on M^n. This argument will be repeatedly used to prove complete integrability of geodesic flows of brane backgrounds and clearly can be adapted to all manifolds which have a principal orbit of codimension at most one with respect to a group action.

4.2 D-branes

4.2.1 The KS and CCKY tensors of D-branes

The metric of type II Dp-branes in the string frame [22–27] is

$$g = h^{-\frac{1}{2}} \sum_{a,b=0}^{p} \eta_{ab} d\sigma^a d\sigma^b + h^{\frac{1}{2}} \sum_{i,j=1}^{9-p} \delta_{ij} dy^i dy^j\,, \tag{4.27}$$

where $p = 0, \dots, 8$ with p even (odd) for IIA (IIB) D-branes, σ^a are the worldvolume coordinates, y^i are the transverse coordinates and $h = h(y)$ is a harmonic function $\delta^{ij}\partial_i\partial_j h = 0$. Apart from the metric, the solutions depend on a non-vanishing dilaton field and an appropriate form field strength which we suppress. For planar branes located at different points y_s in $\mathbb{R}^{9-p}$, one takes for $p \leq 6$

$$h = 1 + \sum_s \frac{q_s}{|y - y_s|^{7-p}}\,, \tag{4.28}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^{9-p}$ and q_s is a constant proportional to the charge density of the branes. The solution is invariant under the action of the Poincaré group, $SO(p,1) \ltimes \mathbb{R}^{p,1}$, acting on the worldvolume coordinates σ^a. If the harmonic function is chosen such that $h = h(|y|)$,[7] then the solution will be invariant under the action of $SO(9-p)$ group acting on the transverse coordinates y.

Considering the Dp-branes (4.27) with $h = h(|y|)$, the KS tensors which are invariant under the worldvolume symmetry of the solution are

$$d_{a_1\dots a_{2m} i_1\dots i_k} = h^{\frac{1}{4}(k-m)}(|y|)\; y^{j_1}\dots y^{j_q} a_{j_1\dots j_q, i_1\dots i_k} \eta_{(a_1 a_2}\dots \eta_{a_{2m-1} a_{2m})}\,, \tag{4.29}$$

provided that the constant coefficients a satisfy

$$a_{(j_1\dots j_q, i_1)\dots i_k} = a_{j_1\dots (j_q, i_1\dots i_k)} = 0\;. \tag{4.30}$$

Each of these KS tensors will generate a symmetry of the relativistic particle action (4.3). As a result each such action on a D-brane background admits an infinite number of symmetries. The algebra of the associated transformations is on-shell isomorphic to that of the Poisson bracket algebra of the associated charges.

To investigate the symmetries of the spinning particles (4.8) propagating on D-branes, it suffices to find the KY tensors of these backgrounds. For this, one begins with an ansatz which respects the worldvolume isometries of the solutions. As the KY tensors are dual to CCKY ones, let us focus on the latter. It turns out that

$$\beta(\varphi) = h^{\frac{k+1-p}{4}}(|y|)\; Y \wedge \varphi \wedge d\text{vol}(\mathbb{R}^{p,1})\,, \tag{4.31}$$

is a CCKY tensor for any constant k-form φ on $\mathbb{R}^{8-p}$, where $d\text{vol}(\mathbb{R}^{p,1})$ is the volume form of $\mathbb{R}^{p,1}$ with respect to the flat metric and $Y = \delta_{ij} y^i dy^j$. Therefore Dp-branes admit 2^{8-p} linearly independent KY forms each generating a symmetry of the action (4.8) of spinning particle probes in these backgrounds. The associated conserved charges are given in (4.10).

4.2.2 Complete integrability of geodesic flow

The geodesic flow on all Dp-brane backgrounds with $h = h(|y|)$ is completely integrable. Of course one can separate the geodesic equation in angular variables. Here we shall give all the charges which are in involution. As we have already mentioned, the isometry group of such a Dp-brane solution is $SO(p,1) \ltimes \mathbb{R}^{p,1} \times SO(9-p)$. Such a group has a codimension one principal orbit $\mathbb{R}^{p,1} \times S^{8-p}$ in the Dp-brane background. In particular, the Killing vectors generated by the translations along the worldvolume coordinates are $k_a = \partial_a$ and those generated by $SO(9-p)$ rotations on the transverse coordinates are

$$k_{ij} = y_i \partial_j - y_j \partial_i\,, \quad i<j\,, \tag{4.32}$$

where $y_i = y^i$. The associated conserved charges written in terms of the momenta are

$$Q_a = p_a\,, \quad Q_{ij} = Q(k_{ij}) = y_i p_j - y_j p_i\;. \tag{4.33}$$

[7] The harmonic function is $h = 1 + \frac{q}{|y|^{7-p}}$ for $p = 0,\dots,6$, $h = 1 + q\log|y|$ for $= 7$ and $h = 1 + q|y|$ for $p = 8$.

These charges are not in involution. However, one can verify that the 9 conserved charges

$$Q_a\,, \quad D_m = \frac{1}{4} \sum_{i,j \geq 10-p-m} (Q_{ij})^2\,, \quad m = 2, 3, \dots, 9-p\,, \tag{4.34}$$

are all orbital, independent and in involution. These together with the Hamiltonian of (4.3) yield 10 charges in involution and the geodesic flow on all such Dp-brane solutions is completely integrable.

4.3 Common sector branes

4.3.1 KS and KY tensors of common sector branes

The metric of the fundamental string solution [19] is

$$g = h^{-1} \eta_{ab} d\sigma^a d\sigma^b + \delta_{ij} dy^i dy^i\,, \tag{4.35}$$

where $a, b = 0, 1$ and $i, j = 1, \dots 8$ and h is a harmonic function on $\mathbb{R}^8$, $\delta^{ij}\partial_i\partial_j h = 0$. We have suppressed the other two fields of the solution the dilaton and 3-form field strength.

As for D-branes consider the fundamental string solution with $h = h(|y|) = 1 + \frac{q}{|y|^6}$. Such a solution admits the same isometry group as that of D1-brane. Then one can demonstrate that the KS tensors that preserve the worldvolume symmetry of the fundamental string are

$$d_{a_1 \dots a_{2m} i_1 \dots i_k} = h^{-m}(|y|) y^{j_1} \dots y^{j_q} a_{j_1 \dots j_q, i_1 \dots i_k} \eta_{(a_1 a_2} \dots \eta_{a_{2m-1} a_{2m})}\,, \tag{4.36}$$

provided that the constant coefficients satisfy $a_{(j_1 \dots j_q, i_1) \dots i_k} = a_{j_1 \dots (j_q, i_1 \dots i_k)} = 0$. As a result a relativistic particle whose dynamics is described by the action (4.3) on such a background admits an infinite number of symmetries generated by these KS tensors.

After some computation, one can verify that CCKY forms of the fundamental string solution are

$$\beta(\varphi) = h^{-1}(|y|)\, Y \wedge \varphi \wedge d\sigma^0 \wedge d\sigma^1\,, \tag{4.37}$$

for any constant k-form φ on $\mathbb{R}^8$, where $Y = \delta_{ij} y^i dy^j$. These give rise to 2^7 linearly independent dual KY forms which generate symmetries for a spinning particle with action (4.8) propagating on this background.

The metric of the NS5-brane solution [20, 21] is

$$g = \eta_{ab} d\sigma^a d\sigma^b + h \delta_{ij} dy^i dy^j\,, \tag{4.38}$$

where $a, b = 0, \dots, 5$, $i, j = 1, 2, 3, 4$ and h is a harmonic function on $\mathbb{R}^4$. We have again suppressed the dilaton and 3-form fields of the solution. For $h = h(|y|) = 1 + \frac{q}{|y|^2}$, the solution has the same isometry group as that of the D5-brane.

As for the fundamental string solution above, the KS tensors that preserve the worldvolume symmetry of the NS5-brane are

$$d_{a_1 \dots a_{2m} i_1 \dots i_k} = h^k(|y|)\; y^{j_1} \dots y^{j_q} a_{j_1 \dots j_q, i_1 \dots i_k} \eta_{(a_1 a_2} \dots \eta_{a_{2m-1} a_{2m})}\,, \tag{4.39}$$

provided that the constant tensors a satisfy $a_{(j_1\dots j_q,i_1)\dots i_k} = a_{j_1\dots(j_q,i_1\dots i_k)} = 0$. Therefore the action (4.3) of a relativistic particle action propagating in this background admits an infinite number of symmetries generated by these KS tensors.

The CCKY forms of the NS5-brane are

$$\beta(\varphi) = h^{\frac{k+2}{2}}(|y|)\; Y \wedge \varphi \wedge d\mathrm{vol}(\mathbb{R}^{5,1})\,, \tag{4.40}$$

for any constant k-form φ on $\mathbb{R}^4$, where $Y = \delta_{ij} y^i dy^j$ and $d\mathrm{vol}(\mathbb{R}^{5,1})$ is the volume form of the worldvolume of the NS5-branes with respect to the flat metric. These give rise to 2^3 linearly independent dual KY forms that generated the symmetries of a spinning particle with action (4.8) propagating on the background.

4.3.2 Complete integrability of geodesic flow

Consider a relativistic particle propagating on the fundamental string solution with $h = h(|y|)$. The worldsheet translations and transverse coordinate SO(8) rotations give rise to the conserved charges

$$Q_a = p_a\,, \quad a = 0, 1\;; \qquad Q_{ij} = y_i p_j - y_j p_i\,, \quad i, j = 1, \dots, 8\,, \tag{4.41}$$

respectively. From these one can construct the following nine independent, orbital, conserved charges

$$Q_a\,, \quad D_m = \frac{1}{4} \sum_{i,j \geq 9-m} (Q_{ij})^2\,, \quad m = 2, \dots, 8\,, \tag{4.42}$$

which are independent and in involution. These together with the Hamiltonian of the relativistic particle (4.3) lead to the integrability of the geodesic flow on the fundamental string background.

Similarly, the conserved charges of a relativistic particle propagating on a NS5-brane background associated with the worldvolume translations and transverse SO(4) rotations are

$$Q_a = p_a\,, \quad a = 0, \dots, 5\;; \quad Q_{ij} = y_i p_j - y_j p_i\,, \quad i, j = 1, 2, 3, 4\;. \tag{4.43}$$

These give rise to nine independent, orbital, conserved charges

$$Q_a\,, \quad D_m = \frac{1}{4} \sum_{i,j \geq 5-m} (Q_{ij})^2\,, \quad m = 2, \dots, 4\,, \tag{4.44}$$

which are independent and in involution. These together with the Hamiltonian of the relativistic particle imply the complete integrability of the geodesic flow of NS5-brane.

5 Common sector and TCFHs

The simplest sector to explore the TCFH of type II supergravities is the common sector. For this sector, all fields vanish apart from the metric, dilaton and the NS-NS 3-form field strength H, $dH = 0$. A direct inspection of the TCFH of type II supergravities reveals that some of the spinor bilinears are covariantly constant with respect to a connection with

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skew-symmetric torsion while some others satisfy a more general TCFH. The former are well known, especially in the context of string compactifications, and have been extensively investigated in the sigma model approach to string theory. They generate additional supersymmetries of the worldvolume actions as well as W-type of symmetries [50, 51]. Here we shall demonstrate that string probes on all common sector supersymmetric solutions admit W-type of symmetries generated by the form bilinears.

5.1 Probes

Before we proceed with the details of describing how the TCFHs generate symmetries for probes in supersymmetric backgrounds, we shall first describe the probe actions that we shall be considering. The main focus will be on string and particle probes. The dynamics of string probes propagating on a spacetime with metric g and a 2-form gauge potential b [52–55] is described by the action

$$A = \int d^2\rho\, d^2\theta\, (g + b)_{MN}\, D_+ x^M\, D_- x^N\,, \tag{5.1}$$

where $x = x(\rho, \theta)$ are real superfields that depend on the worldsheet superspace with commuting (ρ^0, ρ^1) and anti-commuting (θ^+, θ^-) real coordinates. The action above has been given as in [56, 57], where one defines the lightcone coordinates, $\rho^{\ddagger} = \rho^0 + \rho^1$, $\rho^{=} = -\rho^0 + \rho^1$, and the algebra of superspace derivatives is $D_-^2 = i\partial_=$, $D_+^2 = i\partial_{\ddagger}$ and $D_+D_- + D_-D_+ = 0$. Note that the sign labelling of the worldsheet superspace coordinates denotes $\mathfrak{spin}(1,1)$ chirality.

The infinitesimal symmetries of (5.1) that we shall be considering are given by

$$\delta x^M = \epsilon^{(+)} \beta^M{}_{P_1 \dots P_k}\, D_+ x^{P_1} \dots D_+ x^{P_k}\,, \tag{5.2}$$

where β is a spacetime $(k+1)$-form and $\epsilon^{(+)}$ is an infinitesimal parameter; the superscript $(+)$ indicates that the weight of the infinitesimal parameter ϵ is such that the right-hand side of (5.2) is a $\mathfrak{spin}(1,1)$ scalar. The action (5.1) is invariant under such transformations provided that

$$\nabla^{(+)}_M \beta_{P_1 \dots P_{k+1}} = 0\,, \tag{5.3}$$

where

$$\nabla^{(\pm)} = \nabla \pm \frac{1}{2} C\,, \tag{5.4}$$

with $C = db$, i.e. $\nabla^{(\pm)}_M Y^N = \nabla_M Y^N \pm \frac{1}{2} C^N{}_{MR} Y^R$. Therefore β generates a symmetry provided it is a $\nabla^{(+)}$-covariantly constant form.

One can also consider symmetries of (5.1) generated by the infinitesimal transformation

$$\delta x^M = \epsilon^{(-)} \beta^M{}_{P_1 \dots P_k} D_- x^{P_1} \dots D_- x^{P_k}\,, \tag{5.5}$$

where $\epsilon^{(-)}$ is an infinitesimal parameter. The condition for invariance of the action in such a case is

$$\nabla^{(-)}_M \beta_{P_1 \dots P_{k+1}} = 0\,, \tag{5.6}$$

i.e. β is a $\nabla^{(-)}$-covariantly constant form. In many examples that follow the spacetime will admit several $\nabla^{(\pm)}$-covariantly constant forms which generate symmetries of the string probe action (5.1). All $\nabla^{(+)}$-covariantly constant forms of the common sector backgrounds coincide with those of heterotic supersymmetric backgrounds. In turn these can be computed using the classification results of [58, 59] for all heterotic background Killing spinors. The $\nabla^{(-)}$-covariantly constant forms of common sector backgrounds can also be read from the classification results of [58, 59]. One can easily investigate the commutators of these symmetries (5.2) and (5.5). In general these symmetries are of W-type and have been previously explored in [50, 51] both in the context of string compactifications and special geometric structures.

Actions of spinning particle probes are also invariant under the symmetries generated by either $\nabla^{(+)}-$ or $\nabla^{(-)}-$ covariantly constant forms β. One such worldline probe action is

$$A = \int d\tau\, d^2\theta\, (g+b)_{MN} D_+ x^M D_- x^N\,, \tag{5.7}$$

which in addition to the metric exhibits a 2-form coupling b, where the superfields $x^M = x^M(\tau,\theta)$ depend on the worldline superspace with commuting τ and anti-commuting (θ^+,θ^-) real coordinates; see [60] for a systematic description of spinning particle actions with form and other couplings. The algebra of the worldline superspace derivatives is $D_+^2 = D_-^2 = i\partial_\tau$ and $D_+D_- + D_-D_+ = 0$. The signs on $\theta^\pm$ are just labels - there is no chirality in one dimension. The infinitesimal variation of the superfields is as in either (5.2) or (5.5), but now the fields are worldline superfields and the superspace derivatives are those of the worldline superspace. The conditions for invariance of the action above are given in either (5.3) or (5.6), respectively.

Another class of spinning particle probes we shall be considering are described by the action [60]

$$A = -\frac{1}{2}\int d\tau\, d\theta \left(i g_{MN} Dx^M \partial_\tau x^N + \frac{1}{6} C_{MNR} Dx^M Dx^N Dx^R \right), \tag{5.8}$$

where g is the spacetime metric and C is a 3-form on the spacetime - C is not a necessarily closed 3-form. Moreover x^M is a superfield that depends on the worldline superspace coordinates (τ,θ) and $D^2 = i\partial_\tau$. Given a $(k{+}1)$-form β one can construct the infinitesimal transformation

$$\delta x^M = \alpha\, \beta^M{}_{P_1\dots P_k} Dx^{P_1}\dots Dx^{P_k}\,, \tag{5.9}$$

where α is an infinitesimal parameter. The conditions required for this action to be invariant under the transformation (5.9) can be arranged in two different ways. One way is to require, as in previous cases, that β is $\nabla^{(+)}$-covariantly constant. An alternative way to arrange the conditions for invariance of (5.8) is

$$\begin{aligned} \nabla^{(+)}_M \beta_{P_1\dots P_{k+1}} &= \nabla^{(+)}_{[M} \beta_{P_1\dots P_{k+1}]}\,, \\ d i_\beta C + (-1)^k \frac{k+2}{2} i_\beta dC &= 0\,. \end{aligned} \tag{5.10}$$

These conditions and an explanation of the notation can be found in [45]. Therefore this set of conditions implies that β is a $\nabla^{(+)}$-KY form. For $C=0$, one obtains that β is a KY form as for the spinning particles described by the action (4.8).

5.2 IIA common sector

5.2.1 The TCFH

The TCFH of the common sector can be written as

$$\nabla_M \phi_{N_1\dots N_p} - \frac{p}{2} H^P{}_{M[N_1}\tilde{\phi}_{|P|\dots N_p]} = 0\,, \quad \nabla_M \tilde{\phi}_{N_1\dots N_p} - \frac{p}{2} H^P{}_{M[N_1}\phi_{|P|\dots N_p]} = 0\,, \tag{5.11}$$

for $\phi = k, \pi, \tau$ and

$$\nabla_M \tilde{\sigma} = -\frac{1}{4} H_{MPQ}\omega^{PQ}\,, \quad \nabla_M \omega_{NR} + \frac{1}{4} H_{MPQ}\tilde{\zeta}^{PQ}{}_{NR} = \frac{1}{2} H_{MNR}\tilde{\sigma}\,, \tag{5.12}$$

$$\begin{aligned} &\nabla_M \tilde{\zeta}_{N_1\dots N_4} - \frac{1}{3}{}^\star H_{M[N_1N_2N_3|PQR|}\tilde{\zeta}^{PQR}{}_{N_4]} - 3H_{M[N_1N_2}\,\omega_{N_3N_4]} = \\ &\quad \frac{1}{12} g_{M[N_1}{}^\star H_{N_2N_3N_4]P_1\dots P_4}\tilde{\zeta}^{P_1\dots P_4} - \frac{5}{12}{}^\star H_{[MN_1N_2N_3|PQR|}\tilde{\zeta}^{PQR}{}_{N_4]}\,, \end{aligned} \tag{5.13}$$

$$\nabla_M \sigma = -\frac{1}{4} H_{MPQ}\tilde{\omega}^{PQ}\,, \quad \nabla_M \tilde{\omega}_{NR} + \frac{1}{4} H_{MPQ}\zeta^{PQ}{}_{NR} = \frac{1}{2} H_{MNR}\sigma\,, \tag{5.14}$$

$$\begin{aligned} &\nabla_M \zeta_{N_1\dots N_4} - \frac{1}{3}{}^\star H_{M[N_1N_2N_3|PQR|}\zeta^{PQR}{}_{N_4]} - 3H_{M[N_1N_2}\tilde{\omega}_{N_3N_4]} = \\ &\frac{1}{12} g_{M[N_1}{}^\star H_{N_2N_3N_4]P_1\dots P_4}\zeta^{P_1\dots P_4} - \frac{5}{12}{}^\star H_{[MN_1N_2N_3|PQR|}\zeta^{PQR}{}_{N_4]}\,. \end{aligned} \tag{5.15}$$

These can be easily derived from the general IIA TCFH in section 2 upon setting all other fields apart from the metric, dilaton and NS-NS 3-form to zero.

It is clear from the TCFH that $k^{\pm rs} = k^{rs} \pm \tilde{k}^{rs}$, $\pi^{\pm rs} = \pi^{rs} \pm \tilde{\pi}^{rs}$ and $\tau^{\pm rs} = \tau^{rs} \pm \tilde{\tau}^{rs}$ are covariantly constant

$$\nabla^{(\pm)} k^{\pm rs} = \nabla^{(\pm)} \pi^{\pm rs} = \nabla^{(\pm)} \tau^{\pm rs} = 0\,, \tag{5.16}$$

with respect to the connections

$$\nabla^{(\pm)} = \nabla \pm \frac{1}{2} H\,. \tag{5.17}$$

These are the forms that have mostly been explored in the literature. Although the rest do not satisfy such a straightforward condition they are nevertheless part of the geometric structure of the common sector backgrounds. A consequence of the TCFH above is that the (reduced) holonomy of the connection[8] of a generic common sector background is included in $SO(9,1)\times SO(9,1)\times GL(255)\times GL(255)$. The subgroup $SO(9,1)\times SO(9,1)$ is the holonomy of the connections $\nabla^{(\pm)}$ as expected for the common sector. Here in addition we have demonstrated that the holonomy of the TCFH connection factorizes because of the way that it acts on the 2- and 4-form bilinears yielding the $GL(255)\times GL(255)$ subgroup.

[8]Note that the TCFH connection as stated above is not the minimal on k and $\tilde{k}$.

5.2.2 Probe hidden symmetries generated by the TCFH

After identifying the 3-form coupling $C = db$ of the probe actions (5.7) and (5.1) with the 3-form field strength H of common sector backgrounds, $C = H$, the conditions on the form bilinears $k^{\pm rs}$, $\pi^{\pm rs}$ and $\tau^{\pm rs}$ imposed by the TCFH (5.16) coincide with those in (5.3) and (5.6) as required for the invariance of these probe actions. Therefore the $\nabla^{(\pm)}$-covariantly constant form bilinears $k^{\pm rs}$, $\pi^{\pm rs}$ and $\tau^{\pm rs}$ generate symmetries for the particle (5.7) and string (5.1) probe actions. These are given by the infinitesimal transformations

$$\begin{aligned}\delta x^M &= \epsilon^{(\pm)}_{rs} (k^{\pm rs})^M \,, \quad \delta x^M = \epsilon^{(\pm)}_{rs} (\pi^{\pm rs})^M{}_{PQ} D_\pm x^P D_\pm x^Q \,, \\ \delta x^M &= \epsilon^{(\pm)}_{rs} (\tau^{\pm rs})^M{}_{N_1 \dots N_4} D_\pm x^{N_1} \dots D_\pm x^{N_4} \,,\end{aligned} \tag{5.18}$$

where $\epsilon^{(\pm)}_{rs}$ are the infinitesimal parameters.

Similarly after identifying C with H the spinning particle probes described by the action (5.8) are invariant under symmetries generated by the $\nabla^{(+)}$-covariantly constant forms k^{+rs}, π^{+rs} and τ^{+rs}. The infinitesimal variations are given as in (5.18) after replacing the worldsheet superfields with the worldline ones and the superspace derivative D_+ with D. The $\nabla^{(-)}$-covariantly constant forms k^{-rs}, π^{-rs} and τ^{-rs} also generate symmetries but for the spinning particle probe with action given in (5.8) but now with coupling C identified with $-H$, $C = -H$.

The interpretation of the rest of the form bilinears satisfying the TCFH conditions (5.12)–(5.15) as generators of symmetries of worldvolume probe actions is not apparent. For generic common sector backgrounds, these bilinears do not generate symmetries for the probe actions we have considered here. Nevertheless, they may generate symmetries for probes on some special backgrounds, as some terms in the TCFH may vanish and so the remaining TCFH conditions can be interpreted as invariance conditions of some worldvolume probe action.

5.2.3 Hidden symmetries of probes on common sector IIA branes

We have demonstrated that particle and string probes in common sector backgrounds exhibit a large number of symmetries generated by the $\nabla^{(\pm)}$-covariantly constant forms $k^{\pm rs}$, $\pi^{\pm rs}$ and $\tau^{\pm rs}$. To present some examples, we shall explore the symmetries generated by the form bilinears of the fundamental string and NS5-brane. For this, we have to compute the form bilinears of these two backgrounds.

To begin, let us assume that the worldsheet directions of the fundamental string are along 05. Then the Killing spinors of the solution can be written as $\epsilon = h^{-\frac{1}{4}} \epsilon_0$, where ϵ_0 is a constant spinor that satisfies the condition $\Gamma_0 \Gamma_5 \Gamma_{11} \epsilon_0 = \pm \epsilon_0$ with the gamma matrices in a frame basis.[9] The metric of the solution is given in (4.35) after changing the worldvolume directions from 01 to 05 and taking h to be any harmonic function on $\mathbb{R}^8$, e.g. h can be a multi-centred harmonic function as in (4.28) for $p = 1$. The choice of worldsheet directions we have made for the string above may be thought as unconventional. However,

[9] This will be the case for the conditions on the Killing spinors of all brane solutions that we shall investigate from now on.

it turns out that such a choice is aligned with the basis used in spinorial geometry [61] to construct realisations of Clifford algebras in terms of forms; for a review on spinorial geometry techniques see [62]. We shall use spinorial geometry to solve the condition on ϵ_0 and so this labelling of the coordinates is convenient.

Indeed choosing the plus sign in the condition on ϵ_0 and using the realisation of spinors in terms of forms[10] write $\epsilon_0 = \eta + e_5 \wedge \lambda$, where η and λ are constant Majorana $\mathfrak{spin}(8)$ spinors. Then the condition $\Gamma_0\Gamma_5\Gamma_{11}\epsilon_0 = \epsilon_0$ restricts η and λ to be positive chirality Majorana-Weyl spinors of $\mathfrak{spin}(8)$, i.e. $\eta, \lambda \in \Delta_8^+ \equiv \Lambda^{\text{ev}}(\mathbb{R}\langle e_1, e_2, e_3, e_4\rangle)$. Thus the most general solution of $\Gamma_0\Gamma_5\Gamma_{11}\epsilon_0 = \epsilon_0$ is

$$\epsilon_0 = \eta + e_5 \wedge \lambda\,, \tag{5.19}$$

where η and λ are positive chirality Majorana-Weyl spinors of $\mathfrak{spin}(8)$.

Using (5.19) one can easily express all the form bilinears of the fundamental string background in terms of the form bilinears of η and λ. The explicit expressions have been collected in appendix A. Using these one finds that

$$\begin{aligned}
&k^{+rs} = 2h^{-\frac{1}{2}}\langle \eta^r, \eta^s\rangle(e^0 - e^5)\,, && k^{-rs} = h^{-\frac{1}{2}}\langle \lambda^r, \lambda^s\rangle(e^0 + e^5)\,,\\
&\pi^{+rs} = h^{-\frac{1}{2}}\langle \eta^r, \Gamma_{ij}\eta^s\rangle(e^0 - e^5)\wedge e^i \wedge e^j\,, && \pi^{-rs} = h^{-\frac{1}{2}}\langle \lambda^r, \Gamma_{ij}\lambda^s\rangle(e^0 + e^5)\wedge e^i \wedge e^j\,,\\
&\tau^{+rs} = \frac{2}{4!}h^{-\frac{1}{2}}\langle \eta^r, \Gamma_{ijk\ell}\eta^s\rangle(e^0 - e^5)\wedge e^i \wedge e^j \wedge e^k \wedge e^\ell\,, &&\\
&\tau^{-rs} = \frac{2}{4!}h^{-\frac{1}{2}}\langle \lambda^r, \Gamma_{ijk\ell}\lambda^s\rangle(e^0 + e^5)\wedge e^i \wedge e^j \wedge e^k \wedge e^\ell\,, &&
\end{aligned} \tag{5.20}$$

where (e^0, e^5, e^i) is a pseudo-orthonormal frame for the metric (4.35), i.e. $g = -(e^0)^2 + (e^5)^2 + \sum_i (e^i)^2$, and $\langle\cdot,\cdot\rangle$ is the $\mathfrak{spin}(8)$-invariant (Hermitian) inner product on Δ_8^+. Both $k^{\pm rs}$ are along the worldvolume directions and Killing. This implies that both k and $\tilde{k}$ are Killing as well. This is expected for k but not for $\tilde{k}$. Nevertheless $\tilde{k}$ is Killing because the fundamental string is a special background. Observe that the $\nabla^{(+)}$- ($\nabla^{(-)}$-) parallel form bilinears are left- (right-) handed from the string worldvolume perspective as indicated by their dependence on the worldsheet lightcone directions.

It remains to compute the bilinears of $\mathfrak{spin}(8)$ Majorana-Weyl spinors η and λ. These can be obtained using the decomposition of the product of two positive chirality Majorana-Weyl representations Δ_8^+ in terms of forms on $\mathbb{R}^8$ as

$$\Delta_8^+ \otimes \Delta_8^+ = \Lambda^0(\mathbb{R}^8) \oplus \Lambda^2(\mathbb{R}^8) \oplus \Lambda^{4+}(\mathbb{R}^8)\,, \tag{5.21}$$

where $\Lambda^{4+}(\mathbb{R}^8)$ are the self-dual 4-forms on $\mathbb{R}^8$. As η and λ are in Δ_8^+ and otherwise unrestricted, their bilinears span all 0-, 2- and self-dual 4-forms in $\mathbb{R}^8$. As a consequence, the string probe (5.1) and particle probe (5.7) actions are invariant under 2^7 independent symmetries.

[10]In spinorial geometry the Dirac spinors of $\mathfrak{spin}(9,1)$ are identified with $\Lambda^*(\mathbb{C}^5)$. The Gamma matrices are realised on $\Lambda^*(\mathbb{C}^5)$ using the exterior multiplication and inner derivation operations with respect to a Hermitian basis $(e_1,\dots,e_5)$ in $\mathbb{C}^5$. The Majorana spinors satisfy the reality condition $\Gamma_{6789} * \epsilon = \epsilon$. For more details see e.g. appendix B of [62].

Next let us turn to the symmetries of probes on the NS5-brane background. Choosing the worldvolume of the NS5-brane along the 012567 directions, the Killing spinors $\epsilon = \epsilon_0$ of the background satisfy the condition $\Gamma_{3489}\Gamma_{11}\epsilon_0 = \pm\epsilon_0$, where ϵ_0 is a constant Majorana spinor. The metric of the solution is given in (4.38) after changing the worldvolume directions from 012345 to 012567 for similar reasons as those explained for the fundamental string above and after taking h to be a harmonic function on $\mathbb{R}^4$ as in (4.28) for $p = 5$. Choosing the plus sign, the condition $\Gamma_{3489}\Gamma_{11}\epsilon_0 = \epsilon_0$ can be solved using spinorial geometry. It is convenient to first solve this condition for Dirac spinors and then impose the reality condition on ϵ. The solution can be expressed as

$$\epsilon = \eta^1 + e_{34} \wedge \lambda^1 + e_3 \wedge \eta^2 + e_4 \wedge \lambda^2 \,, \tag{5.22}$$

where η and λ are positive chirality Weyl spinors of $\mathfrak{spin}(5,1)$, i.e. $\eta, \lambda \in \Delta^+_{(6)} \equiv \Lambda^{\rm ev}(\mathbb{C}\langle e_1, e_2, e_5\rangle)$. Imposing the reality condition on ϵ, $\Gamma_{6789} * \epsilon = \epsilon$, one finds that

$$\lambda^1 = -\Gamma_{67}(\eta^1)^* \,, \quad \lambda^2 = -\Gamma_{67}(\eta^2)^* \,. \tag{5.23}$$

So the Killing spinor ϵ is completely determined by the (complex) positive chirality $\mathfrak{spin}(5,1)$ spinors η^1 and η^2.

Using (5.22), one can easily compute all the form bilinears of the NS5-brane background and express them in terms of the form bilinears of η^1 and η^2. All these can be found in appendix A.

In particular the $\nabla^{(\pm)}$-covariantly constant spinor bilinears are

$$k^{+rs} = 4\mathrm{Re}\langle\eta^{1r}, \Gamma_a\eta^{1s}\rangle_D e^a \,, \quad k^{-rs} = 4\mathrm{Re}\langle\eta^{2r}, \Gamma_a\eta^{2s}\rangle_D\, e^a \,, \tag{5.24}$$

$$\begin{aligned}\pi^{+rs} = {} & \frac{2}{3}\mathrm{Re}\langle\eta^{1r}, \Gamma_{abc}\eta^{1s}\rangle_D\, e^a \wedge e^b \wedge e^c - 4\mathrm{Re}\langle\eta^{1r}, \Gamma_a\lambda^{1s}\rangle_D\, (e^3 \wedge e^4 - e^8 \wedge e^9) \wedge e^a \\ & - 4\mathrm{Im}\langle\eta^{1r}, \Gamma_a\eta^{1s}\rangle_D\, (e^3 \wedge e^8 + e^4 \wedge e^9) \wedge e^a \\ & - 4\mathrm{Im}\langle\eta^{1r}, \Gamma_a\lambda^{1s}\rangle_D\, (e^3 \wedge e^9 - e^4 \wedge e^8) \wedge e^a \,, \end{aligned} \tag{5.25}$$

$$\begin{aligned}\pi^{-rs} = {} & \frac{2}{3}\mathrm{Re}\langle\eta^{2r}, \Gamma_{abc}\eta^{2r}\rangle_D\, e^a \wedge e^b \wedge e^c + 4\mathrm{Re}\langle\eta^{2r}, \Gamma_a\lambda^{2s}\rangle_D\, (e^3 \wedge e^4 + e^8 \wedge e^9) \wedge e^a \\ & + 4\mathrm{Im}\langle\eta^{2r}, \Gamma_a\eta^{2s}\rangle_D\, (e^3 \wedge e^8 - e^4 \wedge e^9) \wedge e^a \\ & + 4\mathrm{Im}\langle\eta^{2r}, \Gamma_a\lambda^{2s}\rangle_D\, (e^3 \wedge e^9 + e^4 \wedge e^8) \wedge e^a \,, \end{aligned} \tag{5.26}$$

$$\begin{aligned}\tau^{+rs} = {} & k^{+rs} \wedge e^3 \wedge e^4 \wedge e^8 \wedge e^9 - \frac{2}{3}\mathrm{Re}\langle\eta^{1r}, \Gamma_{abc}\lambda^{1s}\rangle_D\, (e^3 \wedge e^4 - e^8 \wedge e^9) \wedge e^a \wedge e^b \wedge e^c \\ & - \frac{2}{3}\mathrm{Im}\langle\eta^{1r}, \Gamma_{abc}\eta^{1s}\rangle_D\, (e^3 \wedge e^8 + e^4 \wedge e^9) \wedge e^a \wedge e^b \wedge e^c \\ & - \frac{2}{3}\mathrm{Im}\langle\eta^{1r}, \Gamma_{abc}\lambda^{1s}\rangle_D\, (e^3 \wedge e^9 - e^4 \wedge e^8) \wedge e^a \wedge e^b \wedge e^c \\ & + \frac{4}{5!}\mathrm{Re}\langle\eta^{1r}, \Gamma_{a_1\dots a_5}\eta^{1s}\rangle_D\, e^{a_1} \wedge \dots \wedge e^{a_5} \,, \end{aligned} \tag{5.27}$$

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$$
\begin{aligned}
\tau^{-rs} = & - k^{-rs} \wedge e^3 \wedge e^4 \wedge e^8 \wedge e^9 + \frac{2}{3}\mathrm{Re}\langle \eta^{2r}, \Gamma_{abc}\lambda^{2s}\rangle_D (e^3 \wedge e^4 + e^8 \wedge e^9) \wedge e^a \wedge e^b \wedge e^c \\
& + \frac{2}{3}\mathrm{Im}\langle \eta^{2r}, \Gamma_{abc}\eta^{2s}\rangle_D (e^3 \wedge e^8 - e^4 \wedge e^9) \wedge e^a \wedge e^b \wedge e^c \\
& + \frac{2}{3}\mathrm{Im}\langle \eta^{2r}, \Gamma_{abc}\lambda^{2s}\rangle_D (e^3 \wedge e^9 + e^4 \wedge e^8) \wedge e^a \wedge e^b \wedge e^c \\
& + \frac{4}{5!}\mathrm{Re}\langle \eta^{2r}, \Gamma_{a_1\dots a_5}\eta^{2s}\rangle_D \; e^{a_1} \wedge \dots \wedge e^{a_5} \,, \qquad (5.28)
\end{aligned}
$$

where $a,b,c = 0,1,2,5,6,7$ are the worldvolume directions, $(e^a, e^3, e^4, e^8, e^9)$ is a pseudo-orthonormal frame for the metric (4.38), $\langle \cdot, \cdot \rangle_D$ is the $\mathfrak{spin}(5,1)$ invariant Dirac inner product and $\epsilon_{3489} = 1$. Both $k^{\pm rs}$ are along the worldvolume directions of the brane and are Killing. This in turn implies that both k and $\tilde{k}$ are Killing as well. Again $\tilde{k}$ is Killing because the NS5-brane is a special background. The 3- and 5-forms have mixed components along both worldvolume and transverse directions. Note that the anti-self-dual and self-dual 2-forms along the transverse directions contribute to $\nabla^{(+)}$ and $\nabla^{(-)}$ covariantly constant forms, respectively.

Therefore the NS5-brane form bilinears have been expressed in terms of those of two positive chirality Weyl $\mathfrak{spin}(5,1)$ spinors. The decomposition of two positive chirality Weyl $\mathfrak{spin}(5,1)$ representations, Δ_4^+, into forms on $\mathbb{C}^6$ is given by

$$
\otimes^2 \Delta_4^+ = \Lambda^1(\mathbb{C}^6) \oplus \Lambda^{3+}(\mathbb{C}^6) \qquad (5.29)
$$

Therefore the string probe with action (5.1) and particle probe with action (5.7) are invariant under 2^5 symmetries counted over the reals. To see this, observe that from the decomposition above all 1- and self-dual 3-forms along the NS5-brane worldvolume are spanned by these spinors. So there are $6 + 10 = 2^4$ independent symmetries generated by the $\nabla^{(+)}$-covariantly constant forms and similarly for the $\nabla^{(-)}$-covariantly constant forms yielding 2^5 in total. These generate a symmetry algebra of W-type [50, 51]. For the remaining form bilinears in appendix A, there is not a straightforward way to relate them to symmetries of particle or string probe actions.

5.3 IIB common sector

5.3.1 The TCFH and probe hidden symmetries

The TCFH of IIB common sector can be written as

$$
\nabla_M \phi^{rs}_{N_1\dots N_p} - \frac{p}{2} H_{M[N_1}{}^P \phi^{(3)rs}_{|P|\dots N_p]} = 0 \,, \quad \nabla_M \phi^{(3)rs}_{N_1\dots N_p} - \frac{p}{2} H_{M[N_1}{}^P \phi^{rs}_{|P|\dots N_p]} = 0 \,, \qquad (5.30)
$$

for $\phi = k, \pi$ and τ. The rest of the TCFH is

$$
\nabla_M k^{(\alpha)rs}_P + \frac{i}{4} \varepsilon_{\alpha\beta} H_M{}^{N_1N_2} \pi^{(\beta)rs}_{PN_1N_2} = 0 \,, \qquad (5.31)
$$

$$
\nabla_M \pi^{(\alpha)rs}_{P_1P_2P_3} + \frac{i}{4} \varepsilon_{\alpha\beta} H_{MN_1N_2} \tau^{(\beta)rs}_{P_1P_2P_3}{}^{N_1N_2} - \frac{3i}{2} \varepsilon_{\alpha\beta} H_{M[P_1P_2} k^{(\beta)rs}_{P_3]} = 0 \,, \qquad (5.32)
$$

$$\nabla_M \tau^{(\alpha)rs}_{P_1\dots P_5} + \frac{5i}{4}\,\varepsilon_{\alpha\beta}\,{}^\star H_{M[P_1\dots P_4}{}^{N_1N_2}\,\pi^{(\beta)rs}_{P_5]N_1N_2} - 5i\,\varepsilon_{\alpha\beta}\,H_{M[P_1P_2}\,\pi^{(\beta)rs}_{P_3P_4P_5]} =$$
$$+\frac{5i}{12}\varepsilon_{\alpha\beta}\,g_{M[P_1}\,{}^\star H_{P_2\dots P_5]}{}^{N_1N_2N_3}\,\pi^{(\beta)rs}_{N_1N_2N_3} - \frac{3i}{2}\,\varepsilon_{\alpha\beta}\,{}^\star H_{[P_1\dots P_5}{}^{N_1N_2}\,\pi^{(\beta)rs}_{M]N_1N_2}\ . \tag{5.33}$$

where $\alpha, \beta = 1, 2$ and $\varepsilon_{12} = 1$. As it has already been explained the TCFH is real after replacing the purely imaginary form bilinears $k^{(2)}, \pi^{(2)}$ and $\tau^{(2)}$ with $ik^{(2)}, i\pi^{(2)}$ and $i\tau^{(2)}$.

It is clear from the TCFH above, that the forms $k^\pm := k \pm k^{(3)}$, $\pi^\pm := \pi \pm \pi^{(3)}$ and $\tau^\pm := \tau \pm \tau^{(3)}$ are covariantly constant with respect to the $\nabla^{(\pm)}$ connection defined in (5.17). As a result these form bilinears generate symmetries in the worldvolume probe actions given in (5.1) and (5.7). These are the form bilinears that have mostly been explored in the literature. The remaining form bilinears in the TCFH do not have such an apparent interpretation. Nevertheless they are part of the geometric structure of the common sector backgrounds.

A consequence of the TCFH above is that the holonomy of the connection[11] $\mathcal{D}^{\mathcal{F}}$ of a generic common sector IIB background is included in $SO(9,1)\times SO(9,1)\times GL(256)\times GL(256)$. As in the IIA case, the subgroup $SO(9,1) \times SO(9,1)$ is the (reduced) holonomy of the $\nabla^{(\pm)}$ connections while the subgroup $GL(256) \times GL(256)$ arises from the way that the TCFH connection acts on the $\pi^{(\alpha)}$ and $\tau^{(\alpha)}$ form bilinears. Therefore the holonomy of the IIB TCFH connection factorises as that of the IIA theory. However note that the holonomy of the IIA common sector minimal connection is included in $SO(9,1) \times SO(9,1) \times GL(255) \times GL(255)$. The difference is that the action of $\mathcal{D}^{\mathcal{F}}$ on the IIA 0-form bilinears σ and $\tilde\sigma$ is via a partial derivative and so the holonomy is trivial. However if instead we had considered the maximal TCFH connections, see [14], of the IIA and IIB common sector both would have reduced holonomy contained in $SO(9,1) \times SO(9,1) \times GL(256) \times GL(256)$.

5.3.2 Hidden symmetries of probes on common sector IIB branes

As an example, we shall explicitly give the symmetries of string and particle probes on the IIB fundamental string and NS5-brane backgrounds. For this one has to calculate the form bilinears of these solutions. Starting with the fundamental string and choosing the worldsheet along the 05 directions as in the IIA case, the Killing spinors of the background are $\epsilon = h^{-\frac{1}{4}}\epsilon_0$, where the constant spinor ϵ_0, $\epsilon_0^t = (\epsilon_0^1, \epsilon_0^2)$, and the two components of ϵ_0 satisfy the conditions

$$\Gamma_{05}\epsilon_0^1 = \pm\epsilon_0^1\,, \quad \Gamma_{05}\epsilon_0^2 = \mp\epsilon_0^2\ . \tag{5.34}$$

Both ϵ_0^1 and ϵ_0^2 are Majorana-Weyl $\mathfrak{spin}(9,1)$ spinors. The metric of the solution is described in (4.35) after changing the worldsheet directions from 01 to 05 and h is taken to be a general harmonic function on $\mathbb{R}^8$, given in (4.28) for $p = 1$. To solve the above condition, we shall again use spinorial geometry [61]. In particular choosing the plus sign in (5.34) and writing $\epsilon_0 = \eta + e_5 \wedge \lambda$, where η (λ) is a doublet of chiral (anti-chiral) Majorana-Weyl $\mathfrak{spin}(8)$ spinors, one finds that

$$\epsilon_0^1 = \eta^1 = \eta\,, \quad \epsilon_0^2 = e_5 \wedge \lambda^2 = e_5 \wedge \lambda\,, \tag{5.35}$$

[11] Note that this is not the minimal connection on k and $k^{(3)}$.

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i.e. the condition on the Killing spinor implies $\lambda^1 = \eta^2 = 0$. One can use the solution (5.35) to express the bilinears of the Killing spinors in terms of those of independent $\mathfrak{spin}(8)$ spinors η and λ. The results can be found in appendix A.

In particular one finds that the $\nabla^{(\pm)}$-covariantly constant form bilinears can be expressed as

$$\begin{aligned}
k^{+rs} &= 2h^{-\frac{1}{2}}\langle\eta^r,\eta^s\rangle(e^0-e^5)\,, & k^{-rs} &= 2h^{-\frac{1}{2}}\langle\lambda^r,\lambda^s\rangle(e^0+e^5)\,,\\
\pi^{+rs} &= h^{-\frac{1}{2}}\langle\eta^r,\Gamma_{ij}\eta^s\rangle(e^0-e^5)\wedge e^i\wedge e^j\,, & \pi^{-rs} &= h^{-\frac{1}{2}}\langle\lambda^r,\Gamma_{ij}\lambda^s\rangle(e^0+e^5)\wedge e^i\wedge e^j\,,\\
\tau^{+rs} &= \frac{2}{4!}h^{-\frac{1}{2}}\langle\eta^r,\Gamma_{i_1\dots i_4}\eta^s\rangle(e^0-e^5)\wedge e^{i_1}\wedge\dots\wedge e^{i_4}\,,\\
\tau^{-rs} &= \frac{2}{4!}h^{-\frac{1}{2}}\langle\lambda^r,\Gamma_{i_1\dots i_4}\lambda^s\rangle(e^0+e^5)\wedge e^{i_1}\wedge\dots\wedge e^{i_4}\,,
\end{aligned} \tag{5.36}$$

where (e^0, e^5, e^i) is a pseudo-orthonormal frame for the metric (4.35) and $\langle\cdot,\cdot\rangle$ is the $\mathfrak{spin}(8)$ invariant inner product. As in the IIA case both $k^{\pm rs}$ are along the worldvolume directions and Killing which in turn implies that k and $k^{(3)}$ are Killing as well. The latter property is a special property of the IIB fundamental string solution. In addition, as in the IIA case, the $\nabla^{(+)}-$ ($\nabla^{(-)}-$) parallel form bilinears are left- (right-) handed from the string worldvolume perspective as indicated by their dependence on the worldsheet lightcone directions.

It remains to find the form bilinears of the $\mathfrak{spin}(8)$ spinors η and λ. These can be identified from the decomposition of the product of two chiral $\Delta^+_{(8)}$ and two anti-chiral Δ^-_8 Majorana-Weyl representations of $\mathfrak{spin}(8)$. It is well known that

$$\Delta^\pm_8\otimes\Delta^\pm_8 = \Lambda^0(\mathbb{R}^8)\oplus\Lambda^2(\mathbb{R}^8)\oplus\Lambda^{4\pm}(\mathbb{R}^8)\ . \tag{5.37}$$

Therefore these bilinears span all constant 0-, 2- and self-dual or anti-self-dual 4-forms on $\mathbb{R}^8$. As a result, the probe actions (5.1) and (5.7) admit 2^7 independent symmetries generated by these forms. Commutators of symmetries generated by $\nabla^{(\pm)}$-covariantly constant forms have been examined in [50, 51] and it was found that they are of W-type. After some investigation it has been found that the remaining form bilinears do not generate symmetries in particle and string probe actions like (5.1), (5.7) and (5.8).

Next let us turn to investigate the form bilinears of the IIB NS5-brane. Choosing the worldvolume along the directions 051627, the Killing spinors ϵ of the solution are constant, $\epsilon = \epsilon_0$, and satisfy the condition $\Gamma_{3489}\epsilon^1 = \pm\epsilon^1$ and $\Gamma_{3489}\epsilon^2 = \mp\epsilon^2$, where both ϵ^1 and ϵ^2 are Majorana-Weyl $\mathfrak{spin}(9,1)$ spinors. Choosing the first sign, one can solve the above conditions using spinorial geometry [61]. As in the IIA case, it is best to first solve the condition for ϵ complex and then impose the reality condition. The solution is

$$\epsilon^1 = \eta^1 + e_{34}\wedge\lambda^1\,,\quad \epsilon^2 = e_3\wedge\eta^2 + e_4\wedge\lambda^2\,, \tag{5.38}$$

where η^1,λ^1 are positive chirality $\mathfrak{spin}(5,1)$ spinors, i.e. $\eta^1,\lambda^1\in\Lambda^{\rm ev}(\mathbb{C}\langle e_1,e_2,e_5\rangle)$, and η^2,λ^2 are negative chirality $\mathfrak{spin}(5,1)$ spinors, i.e. $\eta^2,\lambda^2\in\Lambda^{\rm odd}(\mathbb{C}\langle e_1,e_2,e_5\rangle)$. Moreover the reality condition on the ϵ^1 and ϵ^2 spinors implies that

$$\lambda^1 = -\Gamma_{67}(\eta^1)^*\,,\quad \lambda^2 = -\Gamma_{67}(\eta^2)^*\ . \tag{5.39}$$

Using (5.38), one can easily compute the form bilinears in terms of those of η^1 and η^2. These can be found in appendix A.

Using the expressions of the form bilinears in appendix A, one finds that the $\nabla^{(\pm)}$-covariant constant bilinears are

$$k^{+rs} = 4\mathrm{Re}\langle\eta^{1r},\Gamma_a\eta^{1s}\rangle_D\, e^a\,, \quad k^{-rs} = 4\mathrm{Re}\langle\eta^{2r},\Gamma_a\eta^{2s}\rangle_D\, e^a\,, \tag{5.40}$$

$$\begin{aligned}\pi^{+rs} =& -4\mathrm{Re}\langle\eta^{1r},\Gamma_a\lambda^{1s}\rangle_D\, e^a\wedge(e^3\wedge e^4-e^8\wedge e^9) - 4\mathrm{Im}\langle\eta^{1r},\Gamma_a\eta^{1s}\rangle_D\, e^a\wedge(e^3\wedge e^8+e^4\wedge e^9)\\ &-4\mathrm{Im}\langle\eta^{1r},\Gamma_a\lambda^{1s}\rangle_D\, e^a\wedge(e^3\wedge e^9-e^4\wedge e^8)\\ &+\frac{2}{3}\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\eta^{1s}\rangle_D e^a\wedge e^b\wedge e^c\,,\end{aligned} \tag{5.41}$$

$$\begin{aligned}\pi^{-rs} =& 4\mathrm{Re}\langle\eta^{2r},\Gamma_a\lambda^{2s}\rangle_D\, e^a\wedge(e^3\wedge e^4+e^8\wedge e^9) + 4\mathrm{Im}\langle\eta^{2r},\Gamma_a\eta^{2s}\rangle_D\, e^a\wedge(e^3\wedge e^8-e^4\wedge e^9)\\ &+4\mathrm{Im}\langle\eta^{2r},\Gamma_a\lambda^{2s}\rangle_D\, e^a\wedge(e^3\wedge e^9+e^4\wedge e^8)\\ &+\frac{2}{3}\mathrm{Re}\langle\eta^{2r},\Gamma_{abc}\eta^{2s}\rangle_D e^a\wedge e^b\wedge e^c\,,\end{aligned} \tag{5.42}$$

$$\begin{aligned}\tau^{+rs} =& 4\mathrm{Re}\langle\eta^{1r},\Gamma_a\eta^{1s}\rangle_D\, e^a\wedge e^3\wedge e^4\wedge e^8\wedge e^9 + \frac{4}{5!}\mathrm{Re}\langle\eta^{1r},\Gamma_{a_1\dots a_5}\eta^{1s}\rangle_D e^{a_1}\wedge\cdots\wedge e^{a_5}\\ &-\frac{2}{3}\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\lambda^{1s}\rangle_D\, e^a\wedge e^b\wedge e^c\wedge(e^3\wedge e^4-e^8\wedge e^9)\\ &-\frac{2}{3}\mathrm{Im}\langle\eta^{1r},\Gamma_{abc}\eta^{1s}\rangle_D\, e^a\wedge e^b\wedge e^c\wedge(e^3\wedge e^8+e^4\wedge e^9)\\ &-\frac{2}{3}\mathrm{Im}\langle\eta^{1r},\Gamma_{abc}\lambda^{1s}\rangle_D\, e^a\wedge e^b\wedge e^c\wedge(e^3\wedge e^9-e^4\wedge e^8)\,,\end{aligned} \tag{5.43}$$

$$\begin{aligned}\tau^{-rs} =& -4\mathrm{Re}\langle\eta^{2r},\Gamma_a\eta^{2s}\rangle_D\, e^a\wedge e^3\wedge e^4\wedge e^8\wedge e^9 + \frac{4}{5!}\mathrm{Re}\langle\eta^{2r},\Gamma_{a_1\dots a_5}\eta^{2s}\rangle_D e^{a_1}\wedge\cdots\wedge e^{a_5}\\ &+\frac{2}{3}\mathrm{Re}\langle\eta^{2r},\Gamma_{abc}\lambda^{2s}\rangle_D\, e^a\wedge e^b\wedge e^c\wedge(e^3\wedge e^4+e^8\wedge e^9)\\ &+\frac{2}{3}\mathrm{Im}\langle\eta^{2r},\Gamma_{abc}\eta^{2s}\rangle_D\, e^a\wedge e^b\wedge e^c\wedge(e^3\wedge e^8-e^4\wedge e^9)\\ &+\frac{2}{3}\mathrm{Im}\langle\eta^{2r},\Gamma_{abc}\lambda^{2s}\rangle_D\, e^a\wedge e^b\wedge e^c\wedge(e^3\wedge e^9+e^4\wedge e^8)\,,\end{aligned} \tag{5.44}$$

where $(e^a, e^3, e^4, e^8, e^9)$ is a pseudo-orthonormal frame of the NS5-brane metric (4.38) and $\langle\cdot,\cdot\rangle_D$ is the $\mathfrak{spin}(5,1)$ invariant Dirac inner product. Clearly $k^\pm$ are Killing which implies that both k and $k^{(3)}$ are Killing as well. As in all previous common sector branes, the latter generates an additional symmetry for particle and string actions on NS5-brane backgrounds. The 3- and 5-form bilinears above have mixed components along both the worldvolume and transverse directions, and the anti-self-dual and self-dual 2-forms along the transverse directions contribute to $\nabla^{(+)}$- and $\nabla^{(-)}$- covariantly constant forms, respectively.

We have expressed the $\nabla^{(\pm)}$-covariantly constant bilinears in terms of the bilinears of the chiral and anti-chiral $\mathfrak{spin}(5,1)$ spinors η^1 and η^2, respectively. To determine those note that

$$\otimes^2\Delta_4^\pm = \Lambda^1(\mathbb{C}^6)\oplus\Lambda^{3\pm}(\mathbb{C}^6)\,. \tag{5.45}$$

Therefore these span all 1-forms and 3-forms on the worldvolume of the NS5-brane. In particular, they generate 2^5 independent symmetries, counting over the real numbers, for the spinning particle and string probe actions in (5.1) and (5.7). The algebra of these symmetries is of W-type [50, 51]. An investigation reveals that the remaining bilinears do not generate symmetries for the (5.1) and (5.7) probe actions.

6 IIA D-branes

There is no classification of IIA supersymmetric backgrounds. So to give more examples for which the TCFH can be interpreted as invariance condition for probe particle and string actions under symmetries generated by the form bilinears, we shall turn to some special solutions and in particular to the D-branes.[12] It is convenient to organise the investigation in electric-magnetic brane pairs as the non-vanishing fields that appear in TCFH are the same. The TCFH for each D-brane pair can be easily found from that of the IIA TCFH given in (2.5)–(2.10) and (2.13)–(2.16) upon setting all the form field strengths to zero apart from those associated to the D-brane under investigation.

6.1 D0- and D6-branes

6.1.1 D0-branes

The Killing spinors of the D0-brane are given by $\epsilon = h^{-\frac{1}{8}}\epsilon_0$, where ϵ_0 is a constant spinor restricted as $\Gamma_0\Gamma_{11}\epsilon_0 = \pm\epsilon_0$, the worldline is along the 0-th direction and h is a multi-centred harmonic function as in (4.28) for $p = 0$. Choosing the plus sign and using spinorial geometry [61], one can solve this condition by setting

$$\epsilon_0 = \eta - e_5 \wedge \Gamma_{11}\eta\,, \tag{6.1}$$

where $\eta \in \Lambda^*(\mathbb{R}\langle e_1, \dots, e_4\rangle)$ and the reality condition is imposed by $\Gamma_{6789} * \eta = \eta$. Using this, one can compute the form bilinears. These are given in appendix B.

As expected k is a Killing vector. As a result k generates a symmetry in all probe actions (5.1), (5.7) and (5.8) after setting the form couplings to zero. It also generates a symmetry in the probe action of [16] with the 2-form coupling; the D0-brane 2-form field strength $F = F_{0i}\, e^0 \wedge e^i$ is invariant under the action of k. An investigation of the TCFH for the rest of the form bilinears using that $F_{0i} \neq 0$ reveals that these do not generate symmetries for the probe actions we have been considering. Because of this we postpone a more detailed analysis of the TCFH for later and in particular for the D6- and D2-branes.

6.1.2 D6-brane

Choosing the transverse directions of the D6-brane along 549, the Killing spinor $\epsilon = h^{-\frac{1}{8}}\epsilon_0$ satisfies the condition

$$\Gamma_{549}\Gamma_{11}\epsilon_0 = \pm\epsilon_0\,, \tag{6.2}$$

where ϵ_0 is a constant spinor and h is a multi-centred harmonic function as in (4.28) with $p = 6$. To solve this condition with the plus sign using spinorial geometry, set

$$\epsilon_0 = \eta + e_4 \wedge \lambda\,, \tag{6.3}$$

[12] A consequence of this investigation is that we shall find all form bilinears of the type II D-brane solutions.

where $\eta, \lambda \in \Lambda^*(\mathbb{C}\langle e_1, e_2, e_3, e_5\rangle)$. Then the condition (6.2) gives

$$\Gamma_5\Gamma_{11}\eta = -i\eta\,, \quad \Gamma_5\Gamma_{11}\lambda = i\lambda\ . \tag{6.4}$$

One can proceed to expand η and λ as $\eta = \eta^1 + e_5 \wedge \eta^2$ and $\lambda = \lambda^1 + e_5 \wedge \lambda^2$ in which case the conditions (6.4) give $\eta^2 = i\Gamma_{11}\eta^1$ and $\lambda^2 = -i\Gamma_{11}\lambda^1$, where $\eta^1, \lambda^1 \in \Lambda^*(\mathbb{C}\langle e_1, e_2, e_3\rangle)$ are the independent spinors. However if one proceeds in this way the form bilinears will not be manifestly worldvolume Lorentz covariant, as the 0-th direction will be separated from the rest. Because of this, we shall not solve (6.4) and do the computation with η and λ. After the computation of the form bilinears, one can substitute in the formulae the solution of (6.4) in terms of η^1 and λ^1. However this is not necessary for the purpose of this paper. It remains to impose the reality condition on ϵ_0. This gives $\eta = -i\Gamma_{678} * \lambda$ or equivalently $\lambda = -i\Gamma_{678} * \eta$. The form bilinears are given in appendix B.

The TCFH for k on a background with a 2-form field strength is

$$\nabla_M k_N = \frac{1}{8} e^\Phi F_{PQ} \tilde{\zeta}^{PQ}{}_{MN} + \frac{1}{4} e^\Phi F_{MN} \tilde{\sigma}\ . \tag{6.5}$$

As expected k generates isometries and so symmetries in all the probe actions (5.7), (5.1) and (5.8) with vanishing form couplings. It also generates a symmetry for the probe action of [16] with the 2-form coupling, as the D6-brane 2-form field strength $F = \frac{1}{2} F_{ij}\, e^i \wedge e^j$ is invariant under the action of k. In what follows we shall be mostly concerned with the symmetries generated by the form bilinears for the probe action (5.8). The invariance of this action imposes the weakest conditions on the form bilinears amongst all probe actions that we have been investigating.

Next consider the $\tilde{k}$ and ω bilinears on a background with a 2-form field strength. The TCFH for these is

$$\nabla_M \tilde{k}_N - \frac{1}{2} e^\Phi F_{MP} \omega^P{}_N = \frac{1}{8} e^\Phi g_{MN} F_{PQ} \omega^{PQ} - \frac{1}{2} e^\Phi F_{[M|P|} \omega^P{}_{N]}\,, \tag{6.6}$$

$$\nabla_M \omega_{NR} + e^\Phi F_{M[N} \tilde{k}_{R]} = \frac{3}{4} e^\Phi F_{[MN} \tilde{k}_{R]} + \frac{1}{2} e^\Phi g_{M[N} F_{R]P} \tilde{k}^P - \frac{1}{4 \cdot 5!} e^\Phi {}^\star F_{MNRP_1\dots P_5} \tau^{P_1\dots P_5}\ . \tag{6.7}$$

For $\tilde{k}$ to generate symmetries in probe action (5.8) with $C = 0$, it must be a KY tensor. As for D6-branes $F_{ij} \neq 0$, the term proportional to the spacetime metric in the first of the equations above must vanish. This requires that $\omega_{ij} = 0$. Then from the expressions of the form bilinears of D6-brane in appendix B and (6.4), one concludes that $\tilde{k} = 0$. Therefore $\tilde{k}$ does not generate symmetries for the probe action (5.8).

Similarly for ω to generate a symmetry for probe action (5.8) with $C = 0$, one finds from the last TCFH above that $\tilde{k} = 0$. Then from the expressions for the D6-brane form bilinears in appendix B, this implies that $\omega_{ij} = 0$ or equivalently

$$\langle \eta^r, \Gamma_{11} \lambda^s \rangle_D = \mathrm{Im}\, \langle \eta^r, \eta^s \rangle_D = 0\ . \tag{6.8}$$

Then

$$\omega = \frac{1}{2} \omega_{ab}\, e^a \wedge e^b = h^{-\frac{1}{4}} \mathrm{Re} \langle \eta^r, \Gamma_{ab} \eta^s \rangle_D\, e^a \wedge e^b\,, \tag{6.9}$$

is a KY form and generates a (hidden) symmetry for the probe action (5.8) with $C = 0$. Note that there are Killing spinors for which $\omega \neq 0$ even though $\omega_{ij} = 0$. Indeed, take $\eta^r = \eta^s = 1 + e_1 + e_5 \wedge (i1 - ie_1)$.

The TCFH for the bilinears $\tilde{\omega}$ and π is

$$\nabla_M \tilde{\omega}_{NR} + \frac{1}{2} e^{\Phi} F_{MP} \pi^P{}_{NR} = -\frac{1}{4} e^{\Phi} g_{M[N} F_{|PQ|} \pi^{PQ}{}_{R]} + \frac{3}{4} e^{\Phi} F_{[M|P|} \pi^P{}_{NR]} \,, \tag{6.10}$$

$$\begin{aligned}\nabla_M \pi_{NRS} - \frac{3}{2} e^{\Phi} F_{M[N} \tilde{\omega}_{RS]} = &- \frac{1}{4 \cdot 4!} e^{\Phi \star} F_{MNRSP_1 \dots P_4} \zeta^{P_1 \dots P_4} \\ &- \frac{3}{2} e^{\Phi} g_{M[N} F_{R|P|} \tilde{\omega}^P{}_{S]} - \frac{3}{2} e^{\Phi} F_{[MN} \tilde{\omega}_{RS]} \,. \end{aligned} \tag{6.11}$$

For $\tilde{\omega}$ to be a KY form and so generate a symmetry in the probe action (5.8) with $C = 0$, $\pi_{aij} = 0$. As it can be seen from the D6-brane bilinears in appendix B after using (6.4), this implies that $\tilde{\omega} = 0$ and so $\tilde{\omega}$ does not generate any symmetries. Turning to π, one finds that this is a KY tensor provided that $\tilde{\omega} = 0$ which implies that $\pi_{aij} = 0$ or equivalently

$$\langle \eta^r, \Gamma_a \Gamma_5 \lambda^s \rangle_D = \mathrm{Im} \langle \eta^r, \Gamma_a \eta^s \rangle_D = 0 \,. \tag{6.12}$$

The remaining components of π,

$$\pi = \frac{1}{3!} \pi_{abc} e^a \wedge e^b \wedge e^c = \frac{1}{3} h^{-\frac{1}{4}} \mathrm{Re} \langle \eta^r, \Gamma_{abc} \eta^s \rangle_D e^a \wedge e^b \wedge e^c \,, \tag{6.13}$$

generate a (hidden) symmetry for the probe action (5.8) with $C = 0$. There are Killing spinors such that they satisfy (6.12) and $\pi \neq 0$, e.g. $\eta^r = 1 + e_1 + ie_5 \wedge (1 - e_1)$ and $\eta^s = i(1 - e_1) - e_5 \wedge (1 + e_1)$.

From now on to simplify the analysis that follows on the symmetries generated by TCFHs for all IIA D-branes, we shall only mention the components of the form bilinears that are required to vanish in order for some others become KY forms. In particular, we shall not give the explicit expressions for the vanishing components of the form bilinears and those of the KY forms in terms of the Killing spinors as we have done in e.g. (6.12) and (6.13), respectively. These can be easily read from the expressions of the form bilinears of D-branes given in appendix B.

The TCFH for the bilinears ζ and $\tilde{\pi}$ is

$$\begin{aligned}\nabla_M \tilde{\pi}_{NRS} - \frac{1}{2} e^{\Phi} F_{MP} \zeta^P{}_{NRS} = &\frac{3}{8} e^{\Phi} g_{M[N} F_{|PQ|} \zeta^{PQ}{}_{RS]} \\ &- e^{\Phi} F_{[M|P|} \zeta^P{}_{NRS]} - \frac{3}{4} e^{\Phi} g_{M[N} F_{RS]} \sigma \,, \end{aligned} \tag{6.14}$$

$$\begin{aligned}\nabla_M \zeta_{N_1 \dots N_4} + 2 e^{\Phi} F_{M[N_1} \tilde{\pi}_{N_2 N_3 N_4]} = &\frac{1}{4!} e^{\Phi \star} F_{MN_1 \dots N_4 PQR} \pi^{PQR} \\ &+ 3 e^{\Phi} g_{M[N_1} F_{N_2|P|} \tilde{\pi}^P{}_{N_3 N_4]} + \frac{5}{2} e^{\Phi} F_{[MN_1} \tilde{\pi}_{N_2 N_3 N_4]} \,. \end{aligned} \tag{6.15}$$

A similar analysis to the one presented above reveals that $\tilde{\pi}$ does not generate symmetries in the probe actions we have been considering. While for ζ to be a KY form, and so generate a (hidden) symmetry for the probe action (5.8) with $C = 0$, one requires that $\tilde{\pi} = 0$. This in turn implies that $\zeta_{abij} = 0$. So there is the possibility that $\zeta = \frac{1}{24}\zeta_{a_1\dots a_4} e^{a_1} \wedge \dots \wedge e^{a_4}$ is a KY form. But one can verify after some computation[13] that there are not Killing spinors such that $\zeta_{abij} = 0$ with $\zeta \neq 0$.

The TCFH for $\tilde{\zeta}$ and τ is

$$
\begin{aligned}
\nabla_M \tilde{\zeta}_{N_1\dots N_4} + \frac{1}{2}e^\Phi F_{MP}\tau^P{}_{N_1\dots N_4} =& -\frac{1}{2}e^\Phi g_{M[N_1}F_{|PQ|}\tau^{PQ}{}_{N_2N_3N_4]} + \frac{5}{8}e^\Phi F_{[M|P|}\tau^P{}_{N_1\dots N_4]} \\
&+ 3e^\Phi g_{M[N_1}F_{N_2N_3}k_{N_4]}\,, \qquad (6.16) \\
\nabla_M \tau_{N_1\dots N_5} - \frac{5}{2}e^\Phi F_{M[N_1}\tilde{\zeta}_{N_2\dots N_5]} =& \frac{1}{8}e^{\Phi\star}F_{MN_1\dots N_5}{}^{PQ}\omega_{PQ} \\
&- 5e^\Phi g_{M[N_1}F_{N_2|P|}\tilde{\zeta}^P{}_{N_3N_4N_5]} - \frac{15}{4}e^\Phi F_{[MN_1}\tilde{\zeta}_{N_2\dots N_5]}\,. \qquad (6.17)
\end{aligned}
$$

For $\tilde{\zeta}$ to generate a symmetry, the above TCFH requires $k_a = 0$ and $\tau_{abcij} = 0$. These imply that $\tilde{\zeta} = 0$ and so this bilinear does not generate a symmetry. It turns out that τ is a KY form provided that $\tilde{\zeta}_{abci} = 0$. As a result $\tau_{abcij} = 0$. The remaining non-vanishing components of τ, $\tau = \frac{1}{5!}\tau_{a_1\dots a_5} e^{a_1} \wedge \dots \wedge e^{a_5}$ potentially generates a (hidden) symmetry of the probe action (5.8) with $C = 0$. But after some computation one can verify that there are no Killing spinors such that $\tau_{abcij} = 0$ and $\tau \neq 0$.

It is clear from the TCFH in (6.5)–(6.7), (6.10), (6.11), (6.14), (6.15), (6.16) and (6.17) that the holonomy of the minimal connection reduces for backgrounds with only a 2-form field strength. In particular, the (reduced) holonomy of the minimal connection reduces to a subgroup of $SO(9,1) \times GL(55) \times GL(165) \times GL(330) \times GL(462)$. For completeness we state the TCFH on the scalar bilinears

$$
\nabla_M \tilde{\sigma} = -\frac{1}{4}e^\Phi F_{MP}k^P\,, \quad \nabla_M \sigma = -\frac{1}{8}e^\Phi F_{PQ}\tilde{\pi}^{PQ}{}_M\,. \qquad (6.18)
$$

These give a trivial contribution to the holonomy of the minimal connection.

To summarise the results of this section, we have concluded as a consequence of the TCFH that there are Killing spinors such that k, ω and π, which have non-vanishing components only along the worldvolume directions of the D6-brane, are KY forms. Therefore they generate symmetries for the probe described by the action (5.8) with $C = 0$ in a D6-brane background. This is the case for any multi-centred harmonic function h that the D6-brane solution depends on.

6.2 D2 and D4-branes

6.2.1 D2 brane

Choosing the worldvolume directions of the D2-brane along 051, the Killing spinors $\epsilon = h^{-\frac{1}{8}}\epsilon_0$ of the solution satisfy the condition

$$
\Gamma_{051}\epsilon_0 = \pm\epsilon_0\,, \qquad (6.19)
$$

[13] To prove this one uses spinorial geometry techniques and the freedom to choose a pseudo-orthonormal frame to find a representative for η^{1r}. Then one solves for all conditions arising from $\zeta_{abij} = 0$. This restricts η^{1s} and leads to $\zeta = 0$. It is a lengthy computation that will not be presented here. See also section 7.2 for a similar computation.

where ϵ_0 is a constant spinor and h is given in (4.28) for $p = 2$. To solve this condition with the plus sign using spinorial geometry, set

$$\epsilon_0 = \eta + e_5 \wedge \lambda\,, \tag{6.20}$$

to find that the remaining restrictions on η and λ are

$$\Gamma_1 \eta = \eta\,, \quad \Gamma_1 \lambda = \lambda\,, \tag{6.21}$$

where $\eta, \lambda \in \Lambda^*(\mathbb{R}\langle e_1, e_2, e_3, e_4\rangle)$; the reality condition is imposed with $\Gamma_{6789} * \eta = \eta$ and $\Gamma_{6789} * \lambda = \lambda$. As in the D6-brane case, the remaining condition on η and λ can be solved by setting $\eta = \eta^1 + e_1 \wedge \eta^1$ and $\lambda = \lambda^1 + e_1 \wedge \lambda^1$, where $\eta^1, \lambda^1 \in \Lambda^*(\mathbb{R}\langle e_2, e_3, e_4\rangle)$ label the independent solutions of (6.19). However, we shall perform the computation of the form bilinears using (6.20) as otherwise their expression will not be manifestly covariant along the transverse directions of the D2-brane, e.g. the 6-th direction will have to be treated separately from the rest. The form bilinears of the D2-brane can be found in appendix B.

D2-branes exhibit a non-vanishing 4-form field strength $G_{015i} \neq 0$. As the probe actions we have been considering do not exhibit such a coupling, the only remaining coupling is that of the spacetime metric. Therefore for the form bilinears to generate a symmetry, they must be KY forms. To investigate which of the form bilinears are KY, we shall organise the TCFH according to the domain that the minimal connection acts on. As expected the TCFH

$$\nabla_M k_N = -\frac{1}{4\cdot 4!} e^{\Phi} {}^\star G_{MNP_1\dots P_4} \tilde\zeta^{P_1\dots P_4} + \frac{1}{8} e^{\Phi} G_{MNPQ} \omega^{PQ}\,, \tag{6.22}$$

implies that k is a Killing 1-form. As a result it generates symmetries in all probe action (5.7), (5.1) and (5.8) after setting $b = C = 0$.

Next observe that

$$\nabla_M \tilde k_N - \frac{1}{12} e^{\Phi} G_{MPQR} \tilde\zeta^{PQR}{}_N = \frac{1}{4\cdot 4!} e^{\Phi} g_{MN} G_{P_1\dots P_4} \tilde\zeta^{P_1\dots P_4} - \frac{1}{12} e^{\Phi} G_{[M|PQR|} \tilde\zeta^{PQR}{}_{N]}\,, \tag{6.23}$$

$$\begin{aligned}
&\nabla_M \tilde\zeta_{N_1\dots N_4} + \frac{1}{2} e^{\Phi} {}^\star G_{M[N_1N_2|PQR|} \tau^{PQR}{}_{N_3N_4]} + 2e^{\Phi} G_{M[N_1N_2N_3} \tilde k_{N_4]} \\
&= -\frac{1}{8} e^{\Phi} g_{M[N_1} {}^\star G_{N_2N_3|P_1\dots P_4|} \tau^{P_1\dots P_4}{}_{N_4]} \\
&\quad + \frac{5}{12} e^{\Phi} {}^\star G_{[MN_1N_2|PQR|} \tau^{PQR}{}_{N_3N_4]} + \frac{1}{4} e^{\Phi} {}^\star G_{MN_1\dots N_4P} k^P \\
&\quad + \frac{5}{4} e^{\Phi} G_{[MN_1N_2N_3} \tilde k_{N_4]} + e^{\Phi} g_{M[N_1} G_{N_2N_3N_4]P} \tilde k^P\,.
\end{aligned} \tag{6.24}$$

$$\begin{aligned}
&\nabla_M \tau_{N_1\dots N_5} - \frac{5}{2} e^{\Phi} {}^\star G_{M[N_1N_2N_3|PQ|} \tilde\zeta^{PQ}{}_{N_4N_5]} + 5e^{\Phi} G_{M[N_1N_2N_3} \omega_{N_4N_5]} \\
&= -\frac{15}{8} e^{\Phi} {}^\star G_{[MN_1N_2N_3|PQ|} \tilde\zeta^{PQ}{}_{N_4N_5]} \\
&\quad - \frac{1}{4} e^{\Phi} {}^\star G_{MN_1\dots N_5} \tilde\sigma - \frac{5}{6} e^{\Phi} g_{M[N_1} {}^\star G_{N_2N_3N_4|PQR|} \tilde\zeta^{PQR}{}_{N_5]} + \frac{15}{4} e^{\Phi} G_{[MN_1N_2N_3} \omega_{N_4N_5]} \\
&\quad + 5e^{\Phi} g_{M[N_1} G_{N_2N_3N_4|P|} \omega^P{}_{N_5]}\,,
\end{aligned} \tag{6.25}$$

$$\nabla_M \omega_{NR} - \frac{1}{12} e^{\Phi} G_{MP_1P_2P_3} \tau^{P_1P_2P_3}{}_{NR} = \frac{1}{2\cdot 4!} e^{\Phi} g_{M[N} G_{|P_1\dots P_4|} \tau^{P_1\dots P_4}{}_{R]}$$
$$-\frac{1}{8} e^{\Phi} G_{[M|P_1P_2P_3|} \tau^{P_1P_2P_3}{}_{NR]} - \frac{1}{4} e^{\Phi} G_{MNRP} k^P \,, \tag{6.26}$$

and so the minimal connection acts on the domain of $\tilde{k}$, $\tilde{\zeta}$, τ and ω form bilinears. Using that for D2-branes $G_{015i} \neq 0$ and the explicit expression for the form bilinears in appendix B, one finds that the TCFH implies that the form bilinears $\tilde{k}$, $\tilde{\zeta}$ and τ cannot be KY tensors. So these do not generate a symmetry in probe actions. On the other hand for ω to be a KY tensor, the TCFH implies that $\tau_{abcij} = 0$. This in turn implies that $\omega_{ij} = 0$. As a result $\omega = \frac{1}{2}\omega_{ab} e^a \wedge e^b$ is a KY form and generates a (hidden) symmetry in the probe action (5.8). The condition $\tau_{abcij} = 0$ on the Killing spinors and the expression for ω_{ab} in terms of Killing spinors can be easily read from the expressions of these form bilinears in appendix B. There are Killing spinors such that $\tau_{abcij} = 0$ and $\omega \neq 0$. For example set $\eta^r = \lambda^r$ and $\eta^s = \lambda^s$ with $\langle \eta^r, \eta^s \rangle \neq 0$.

The TCFH on the remaining form bilinears is

$$\nabla_M \pi_{NRS} - \frac{3}{4} e^{\Phi} G_{M[N|PQ|} \zeta^{PQ}{}_{RS]} = -\frac{1}{4} e^{\Phi} G_{MNRS} \sigma + \frac{1}{8} e^{\Phi} {}^{\star}G_{MNRSPQ} \tilde{\omega}^{PQ}$$
$$- \frac{1}{4} e^{\Phi} g_{M[N} G_{R|P_1P_2P_3|} \zeta^{P_1P_2P_3}{}_{S]} - \frac{3}{4} e^{\Phi} G_{[MN|PQ|} \zeta^{PQ}{}_{RS]} \,, \tag{6.27}$$

$$\nabla_M \zeta_{N_1\dots N_4} + 3 e^{\Phi} G_{M[N_1N_2|P|} \pi^{P}{}_{N_3N_4]} + e^{\Phi} {}^{\star}G_{M[N_1N_2N_3|PQ|} \tilde{\pi}^{PQ}{}_{N_4]}$$
$$= \frac{1}{6} e^{\Phi} g_{M[N_1} {}^{\star}G_{N_2N_3N_4]PQR} \tilde{\pi}^{PQR} + \frac{5}{8} e^{\Phi} {}^{\star}G_{[MN_1N_2N_3|PQ|} \tilde{\pi}^{PQ}{}_{N_4]}$$
$$- \frac{3}{2} e^{\Phi} g_{M[N_1} G_{N_2N_3|PQ|} \pi^{PQ}{}_{N_4]} + \frac{5}{2} e^{\Phi} G_{[MN_1N_2|P|} \pi^{P}{}_{N_3N_4]} \,, \tag{6.28}$$

$$\nabla_M \tilde{\pi}_{NRS} + \frac{3}{2} e^{\Phi} G_{M[NR|P|} \tilde{\omega}^{P}{}_{S]} - \frac{1}{4} e^{\Phi} {}^{\star}G_{M[NR|P_1P_2P_3|} \zeta^{P_1P_2P_3}{}_{S]}$$
$$= -\frac{3}{8} e^{\Phi} g_{M[N} G_{RS]PQ} \tilde{\omega}^{PQ} + e^{\Phi} G_{[MNR|P|} \tilde{\omega}^{P}{}_{S]} + \frac{1}{32} e^{\Phi} g_{M[N} {}^{\star}G_{RS]P_1\dots P_4} \zeta^{P_1\dots P_4}$$
$$- \frac{1}{6} e^{\Phi} {}^{\star}G_{[MNR|P_1P_2P_3|} \zeta^{P_1P_2P_3}{}_{S]} \,, \tag{6.29}$$

$$\nabla_M \tilde{\omega}_{NR} - \frac{1}{2} e^{\Phi} G_{M[N|PQ|} \tilde{\pi}^{PQ}{}_{R]} = -\frac{1}{4!} e^{\Phi} {}^{\star}G_{MNRP_1P_2P_3} \pi^{P_1P_2P_3}$$
$$- \frac{1}{12} e^{\Phi} g_{M[N} G_{R]P_1P_2P_3} \tilde{\pi}^{P_1P_2P_3} - \frac{3}{8} e^{\Phi} G_{[MN|PQ|} \tilde{\pi}^{PQ}{}_{R]} \,. \tag{6.30}$$

Requiring that these form bilinears must be KY tensors, the above TCFH together with the explicit expressions for the D2-brane form bilinears in B reveal that $\zeta = \tilde{\pi} = \tilde{\omega} = 0$. For π to be a KY form, the TCFH implies that $\zeta_{ijab} = 0$ which in turn gives $\pi_{ija} = 0$. The remaining non-vanishing component of π, $\pi = \frac{1}{3!}\pi_{abc} e^a \wedge e^b \wedge e^c$, is a KY tensor and generates a (hidden) symmetry in the probe action (5.8) with $C = 0$. Again the expression of the conditions $\zeta_{ijab} = 0$ and that of π in terms of the Killing spinors can be found in appendix B. There are Killing spinors such that $\zeta_{ijab} = 0$ and $\pi \neq 0$. Indeed set $\lambda^r = -\eta^r$, $\lambda^s = \eta^s$ and $\eta^r = \eta^s = 1 + e_{234} + e_1 \wedge (1 + e_{234})$.

It is clear that the holonomy of the minimal connection of the TCFH with only the 4-form field strength reduces. In particular, the reduced holonomy is included in $SO(9,1) \times GL(517) \times GL(495)$. For completeness we give the TCFH on the scalars as

$$\nabla_M \tilde{\sigma} = \frac{1}{4\cdot 5!} {}^\star G_{MP_1\dots P_5} \tau^{P_1\dots P_5}\,, \quad \nabla_M \sigma = \frac{1}{4!} e^{\Phi} G_{MPQR} \pi^{PQR}\,, \tag{6.31}$$

which give a trivial contribution in the holonomy of the minimal connection.

To summarise the results of this section, we have shown that there are choices of Killing spinors such that ω and π, with non-vanishing components only along the worldvolume directions of the D2-brane, are KY tensors. Therefore these bilinears generate (hidden) symmetries for a probe described by the action (5.8) with $C = 0$ on all D2-brane backgrounds, including those that depend on a multi-centred harmonic function h.

6.2.2 D4 brane

Choosing the transverse directions of the D4-brane as 23849, the Killing spinors $\epsilon = h^{-\frac{1}{8}} \epsilon_0$ of the solution satisfy the condition

$$\Gamma_{23849} \epsilon_0 = \pm \epsilon_0\,, \tag{6.32}$$

where ϵ_0 is a constant spinor and h is a harmonic function as in (4.28) for $p = 4$. To solve this condition with the plus sign using spinorial geometry write

$$\epsilon_0 = \eta^1 + e_{34} \wedge \eta^2 + e_3 \wedge \lambda^1 + e_4 \wedge \lambda^2\,, \tag{6.33}$$

where $\eta, \lambda \in \Lambda^*(\mathbb{C}\langle e_5, e_1, e_2\rangle)$. Substituting this into (6.32), one finds that

$$\Gamma_2 \eta^1 = -\eta^1\,, \quad \Gamma_2 \lambda^1 = -\lambda^1\,, \tag{6.34}$$

and similarly for η^2 and λ^2. The reality condition on ϵ implies that $\eta^1 = \Gamma_{67} * \eta^2$ and $\lambda^1 = \Gamma_{67} * \lambda^2$. The remaining conditions (6.34) can be solved by setting $\eta^1 = \rho - e_2 \wedge \rho$, where $\rho \in \Lambda^*\langle e_5, e_1\rangle$, and similarly for the rest of the spinors. However as for the D2-brane, we shall not do this as otherwise the expression for the form bilinears will not be manifestly covariant in the worldvolume directions because the 6-th direction will have to be treated separately from the rest. The form bilinears of the D4-brane can be expressed in terms of those of η and λ spinors. Their expressions can be found in appendix B.

As in the D2-brane case, the form bilinears generate symmetries in the probe actions we have been considering provided that they are KY forms. This condition requires that certain terms in the TCFH must vanish. Using that for the D4-brane solution $G_{ijkl} \neq 0$ and the explicit expression of the form bilinears in appendix B, one finds after a detailed analysis of the TCFH that only k, $\tilde{\zeta}$, τ, $\tilde{\omega}$ and π can be KY tensors while the rest of the bilinears vanish. In particular, as expected, k is Killing and so generates a symmetry for the probe actions we have been considering.

For $\tilde{\zeta}$ to be a KY tensor, the TCFH requires that $\tilde{k} = 0$, $\tau_{ija_1a_2a_3} = 0$ and $\tau_{a_1\dots a_5} = 0$. These imply that $\tilde{\zeta}_{ija_1a_2} = 0$. The non-vanishing component of $\tilde{\zeta}$, $\tilde{\zeta} = \frac{1}{4!} \tilde{\zeta}_{a_1\dots a_4} e^{a_1} \wedge \cdots \wedge e^{a_4}$, generates a (hidden) symmetry for the probe action (5.8) with $C = 0$. Similarly for

τ to be a KY form, the TCFH requires that $\omega = 0$ and $\tilde{\zeta}_{ijab} = 0$. These imply that $\tau = \frac{1}{5!}\tau_{a_1\dots\tau_5} e^{a_1} \wedge \cdots \wedge e^{a_5}$ is a KY form and generates a (hidden) symmetry for the probe action (5.8) with $C = 0$.

For $\tilde{\omega}$ to be a KY form the TCFH requires that $\tilde{\pi}_{ijk} = 0$, which in turn implies that $\tilde{\omega}_{ij} = 0$. The remaining component of $\tilde{\omega} = \frac{1}{2}\tilde{\omega}_{ab} e^a \wedge e^b$ is a KY tensor and generates a symmetry for probe action (5.8) with $C = 0$. There are Killing spinors such that $\tilde{\pi}_{ijk} = 0$ while $\tilde{\omega} \neq 0$. Indeed take $\eta^{1r} = x1 + ye_1 - e_2 \wedge (x1 + ye_1)$ and $\eta^{1s} = -ix1 + iye_1 - e_2 \wedge (-ix1 + iye_1)$, $x, y \in \mathbb{C} - \{0\}$ and $\lambda^{1r} = \lambda^{1s} = 0$.

Similarly for π to be a KY form, the TCFH requires that $\zeta_{aijk} = 0$, which in turn gives $\pi_{aij} = 0$. Then $\pi = \frac{1}{3!}\pi_{abc} e^a \wedge e^b \wedge e^c$ is a KY form and generates a (hidden) symmetry for the probe action (5.8) with $C = 0$. In all the above cases, the explicit expressions for the vanishing conditions on some of the components of the form bilinears, as well as the expressions of KY forms in terms of the Killing spinors, can be easily read from the results of appendix B and so they will not be repeated here. For $\tilde{\zeta}$, τ and π we have not verified whether there exist Killing spinors such that these are non-vanishing KY forms. A preliminary investigation has revealed that they do not exist.

To summarise the results of this section, there are Killing spinors such that k, and $\tilde{\omega}$ with non-vanishing components only along the worldvolume directions of D4-brane, are KY tensors. Therefore, they generate (hidden) symmetries for the probe described by the action (5.8) with $C = 0$ on any D4-brane background depending of a harmonic function h as in (4.28) for $p = 4$.

6.3 D8-brane

To derive the TCFH on D8-brane type of backgrounds set all the IIA form fields strengths to zero apart from S. Then the IIA TCFH in section 2 reduces to

$$\nabla_M \tilde{\sigma} = \frac{1}{4} e^\Phi S \tilde{k}_M \,, \qquad \nabla_M k_N = \frac{1}{4} e^\Phi S \omega_{MN} \,, \quad \nabla_M \tilde{k}_N = \frac{1}{4} e^\Phi g_{MN} S \tilde{\sigma} \,, \tag{6.35}$$

$$\nabla_M \omega_{NR} = \frac{1}{2} e^\Phi S g_{M[N} k_{R]} \,, \quad \nabla_M \tilde{\zeta}_{N_1\dots N_4} = \frac{1}{4\cdot 5!} e^\Phi {}^\star S_{MN_1\dots N_4 P_1\dots P_5} \tau^{P_1\dots P_5} \,, \tag{6.36}$$

$$\nabla_M \tau_{N_1\dots N_5} = -\frac{1}{4\cdot 4!} e^\Phi {}^\star S_{MN_1\dots N_5 P_1\dots P_4} \tilde{\zeta}^{P_1\dots P_4} \,, \qquad \nabla_M \sigma = 0 \,,$$

$$\nabla_M \tilde{\omega}_{NR} = \frac{1}{4} e^\Phi S \tilde{\pi}_{MNR} \,, \tag{6.37}$$

$$\nabla_M \pi_{NRS} = \frac{1}{4} e^\Phi S \zeta_{MNRS} \,, \qquad \nabla_M \tilde{\pi}_{NRS} = \frac{3}{4} e^\Phi S g_{M[N} \tilde{\omega}_{RS]} \,,$$

$$\nabla_M \zeta_{N_1\dots N_4} = e^\Phi S g_{M[N_1} \pi_{N_2 N_3 N_4]} \,. \tag{6.38}$$

It is clear from this that k, $\tilde{\zeta}$, τ, $\tilde{\omega}$ and π are KY tensors and generate a (hidden) symmetry of the probe action (5.8) with $C = 0$. Note that all these form bilinears k, $\tilde{\zeta}$, τ, $\tilde{\omega}$ and π have components only along the worldvolume directions of the D8-brane. Notice also that the (reduced) holonomy of the minimal TCFH connection is included in SO(9, 1).

To find an explicit expression of the form bilinears of D8-brane solution choose the worldvolume directions along 012346789. The Killing spinors $\epsilon = h^{-\frac{1}{8}}\epsilon_0$ of the solution satisfy the condition $\Gamma_5\epsilon_0 = \pm\epsilon_0$, where ϵ_0 is a constant spinor and $h = 1 + \sum_\ell q_\ell |y - y_\ell|$. Taking the plus sign, this condition can be solved using spinorial geometry by setting

$$\epsilon_0 = \eta + e_5 \wedge \eta\,, \tag{6.39}$$

where $\eta \in \Lambda^*(\mathbb{R}\langle e_1, e_2, e_3, e_4\rangle)$ after imposing the reality condition $\Gamma_{6789} * \eta = \eta$. Using the solution for ϵ_0 above, one can easily compute the form bilinears of D8-brane in terms of those of η. Their expressions can be found in appendix B. Imposing the condition that the remaining form bilinears $\tilde{k}$, ζ, ω and $\tilde{\pi}$ must be KY forms, the TCFH together with their explicit expressions in B imply that they should vanish. Therefore they do not generate symmetries for probe actions. However as a consequence of the TCFH above $\tilde{k}$, ζ, ω and $\tilde{\pi}$ are CCKY forms and so their spacetimes duals are KY forms.

7 TCFH and probe symmetries on IIB D-branes

As in the IIA, there is no classification of IIB supersymmetric backgrounds. So we shall turn to IIB D-branes to give more examples of backgrounds for which the TCFH can be interpreted as the condition for invariance of particle and string probe actions under symmetries generated by the form bilinears. The computation will again be organised in D-brane electric-magnetic pairs. The TCFH for each pair can be easily found from that of the IIB TCFH given in (3.3)–(3.8) upon setting all the form field strengths to zero apart from those associated to the D-brane under investigation.

7.1 D1- and D5-branes

7.1.1 The TCFH of D1- and D5-branes

To illustrate the construction of symmetries for probes propagating on D1- and D5-brane backgrounds using the IIB TCFH, we shall present the D1- and D5-brane TCFH. This is easily derived from (3.3)–(3.8) upon setting $G^{(1)} = G^{(5)} = 0$. After a re-arrangement of terms so that $e^\Phi G^{(3)}$ can be interpreted as torsion of a TCFH connection, one finds

$$\nabla_M k^{rs}_P - \frac{1}{2} e^\Phi G^{(3)}_{MP}{}^N k^{(1)rs}_N = \frac{1}{12} e^\Phi G^{(3)N_1N_2N_3} \tau^{(1)rs}_{N_1N_2N_3MP}$$

$$\begin{aligned}
&\nabla_M k^{(i)rs}_P - \frac{1}{2} \delta_{i1} e^\Phi G^{(3)}_{MP}{}^N k^{rs}_N + \frac{i}{2} \varepsilon_{1ij} e^\Phi G^{(3)}_M{}^{N_1N_2} \pi^{(j)rs}_{PN_1N_2}\\
&= \frac{1}{12} \delta_{i1} e^\Phi G^{(3)N_1N_2N_3} \tau^{rs}_{MPN_1N_2N_3}\\
&\quad + \frac{i}{12} \varepsilon_{1ij} e^\Phi g_{MP} G^{(3)N_1N_2N_3} \pi^{(j)rs}_{N_1N_2N_3} + \frac{i}{2} \varepsilon_{1ij} e^\Phi G^{(3)N_1N_2}{}_{[M} \pi^{(j)rs}_{P]N_1N_2}
\end{aligned}$$

$$\begin{aligned}
&\nabla_M \pi^{rs}_{P_1P_2P_3} - 3 e^\Phi G^{(3)}_{M[P_1}{}^N \pi^{(1)rs}_{P_2P_3]N} = -\frac{1}{12} e^\Phi \star G^{(7)}_{MP_1P_2P_3}{}^{N_1N_2N_3} \pi^{(1)rs}_{N_1N_2N_3}\\
&\quad + \frac{3}{2} e^\Phi g_{M[P_1} G^{(3)N_1N_2}_{P_2} \pi^{(1)rs}_{P_3]N_1N_2} + 3 e^\Phi G^{(3)}_{[P_1P_2}{}^N \pi^{(1)rs}_{P_3M]N}
\end{aligned}$$

$$\begin{aligned}
&\nabla_M \pi^{(i)rs}_{P_1P_2P_3} - 3 \delta_{i1} e^\Phi G^{(3)}_{M[P_1}{}^N \pi^{rs}_{P_2P_3]N} + \frac{i}{2} \varepsilon_{1ij} e^\Phi G^{(3)N_1N_2}_M \tau^{(j)rs}_{P_1P_2P_3N_1N_2}\\
&\quad - 3i\, \varepsilon_{1ij} e^\Phi G^{(3)}_{M[P_1P_2} k^{(j)rs}_{P_3]} = -\frac{1}{12} \delta_{i1} e^\Phi \star G^{(7)}_{MP_1P_2P_3}{}^{N_1N_2N_3} \pi^{rs}_{N_1N_2N_3}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{3}{2}\,\delta_{i1}\,e^{\Phi}\,g_{M[P_1}\,G^{(3)N_1N_2}_{P_2}\,\pi^{rs}_{P_3]N_1N_2} + 3\,\delta_{i1}\,e^{\Phi}\,G^{(3)}_{[P_1P_2}{}^{N}\,\pi^{rs}_{P_3M]N} \\
&+ \frac{i}{4}\,\varepsilon_{1ij}\,e^{\Phi}\,G^{(3)N_1N_2N_3}\,g_{M[P_1}\,\tau^{(j)rs}_{P_2P_3]N_1N_2N_3} - i\,\varepsilon_{1ij}\,G^{(3)}_{[P_1}{}^{N_1N_2}\,\tau^{(j)rs}_{P_2P_3M]N_1N_2} \\
&- \frac{3i}{2}\,\varepsilon_{1ij}\,e^{\Phi}\,g_{M[P_1}\,G^{(3)}_{P_2P_3]}{}^{N}\,k^{(j)rs}_{N} + 2i\,\varepsilon_{1ij}\,e^{\Phi}\,G^{(3)}_{[P_1P_2P_3}\,k^{(j)rs}_{M]}\,,
\end{aligned}
$$

$$
\begin{aligned}
&\nabla_M\,\tau^{rs}_{P_1\dots P_5} - 5\,e^{\Phi}\,G^{(3)}_{M[P_1}{}^{N}\,\tau^{(1)rs}_{P_2\dots P_5]N} = \frac{1}{2}\,e^{\Phi}\,{}^{\star}G^{(7)}_{MP_1\dots P_5}{}^{N}\,k^{(1)rs}_{N} \\
&+ \frac{15}{2}\,e^{\Phi}\,G^{(3)}_{[P_1P_2}{}^{N}\,\tau^{(1)rs}_{P_3P_4P_5M]N} + 5\,e^{\Phi}\,g_{M[P_1}\,G^{(3)N_1N_2}_{P_2}\,\tau^{(1)rs}_{P_3P_4P_5]N_1N_2} - 10\,e^{\Phi}\,g_{M[P_1}\,G^{(3)}_{P_2P_3P_4}\,k^{(1)rs}_{P_5]}
\end{aligned}
$$

$$
\begin{aligned}
&\nabla_M\,\tau^{(i)rs}_{P_1\dots P_5} - 5\,\delta_{i1}\,e^{\Phi}\,G^{(3)}_{M[P_1}{}^{N}\,\tau^{rs}_{P_2\dots P_5]N} + \frac{5i}{2}\,\varepsilon_{1ij}\,e^{\Phi}\,{}^{\star}G^{(7)}_{M[P_1\dots P_4}{}^{N_1N_2}\,\pi^{(j)rs}_{P_5]N_1N_2} \\
&- 10i\,\varepsilon_{1ij}\,e^{\Phi}\,G^{(3)}_{M[P_1P_2}\,\pi^{(j)rs}_{P_3P_4P_5]} = \frac{1}{2}\delta_{i1}\,e^{\Phi}\,{}^{\star}G^{(7)}_{MP_1\dots P_5}{}^{N}\,k^{rs}_{N} + 5\,\delta_{i1}\,e^{\Phi}\,g_{M[P_1}\,G^{(3)N_1N_2}_{P_2}\,\tau^{rs}_{P_3P_4P_5]N_1N_2} \\
&+ \frac{15}{2}\delta_{i1}\,e^{\Phi}\,G^{(3)}_{[P_1P_2}{}^{N}\,\tau^{rs}_{P_3P_4P_5M]N} - 10\delta_{i1}\,e^{\Phi}\,g_{M[P_1}\,G^{(3)}_{P_2P_3P_4}\,k^{rs}_{P_5]} \\
&+ 10i\,\varepsilon_{1ij}\,e^{\Phi}\,G^{(3)}_{[P_1P_2P_3}\,\pi^{(j)rs}_{P_4P_5M]} - 15i\,\varepsilon_{1ij}\,e^{\Phi}\,g_{M[P_1}\,G^{(3)}_{P_2P_3}{}^{N}\,\pi^{(j)rs}_{P_4P_5]N} \\
&+ \frac{5i}{12}\,\varepsilon_{1ij}\,g_{M[P_1}\,{}^{\star}G^{(7)}_{P_2\dots P_5]}{}^{N_1N_2N_3}\,\pi^{(j)rs}_{N_1N_2N_3} - \frac{3i}{2}\,\varepsilon_{1ij}\,e^{\Phi}\,{}^{\star}G^{(7)}_{[P_1\dots P_5}{}^{N_1N_2}\,\pi^{(j)rs}_{M]N_1N_2}\,. \qquad (7.1)
\end{aligned}
$$

Clearly the (reduced) holonomy of the TCFH connection for generic backgrounds with only $G^{(3)}$ non-vanishing is included in $\times^2\mathrm{SO}(9,1)\times^2\mathrm{GL}(256)$. The TCFH connection acting on π and $\pi^{(1)}$ is the same as that acting on τ and $\tau^{(1)}$ but it is different from that acting on k and $k^{(1)}$.

The difficulties that one encounters when interpreting the TCFH above as invariance conditions for a particle probe described by an action,[14] like (5.8), for symmetries generated by the form bilinears are twofold. One is that the TCFH connection contains terms that involve double and higher contractions of indices between the $G^{(3)}$ field strength and the form bilinears. The other is that the right-hand side of the TCFH involves terms that contain the spacetime metric. Terms such as these do not occur as invariance conditions for actions like (5.8) under symmetries generated by spacetime forms, see (5.10). The only option is to set both such terms to zero. As $G^{(3)}$ is given for each solution, this puts restrictions on the form bilinears and, in turn, on the choice of Killing spinors used to construct these bilinears.

7.1.2 D1-brane

To find the form bilinears of the D1-brane, choose the worldsheet along the directions 05. The Killing spinors of the solution are $\epsilon = h^{-\frac{1}{8}}\epsilon_0$, where the constant spinor $\epsilon_0 = (\epsilon_0^1, \epsilon_0^2)^t$ is a doublet of Majorana-Weyl $\mathfrak{spin}(9,1)$ spinors satisfying the additional condition

$$\Gamma_{05}\sigma_1\epsilon_0 = \pm\epsilon_0\,, \qquad (7.2)$$

and h is a harmonic function on $\mathbb{R}^8$ as in (4.28) for $p=1$. The metric of the D1-brane is given in (4.27) for $p=1$. Choosing the plus sign in the condition above the components

[14] A probe with action (5.8) is chosen because it gives the weakest invariance conditions on the couplings and on the forms that generate the symmetries.

of the doublet ϵ_0 are restricted as $\Gamma_{05}\epsilon_0^1 = \epsilon_0^2$ and $\Gamma_{05}\epsilon_0^2 = \epsilon_0^1$. As in previous cases, these conditions are solved using spinorial geometry [61]. After a short computation, one finds that

$$\epsilon_0^1 = \eta + e_5 \wedge \lambda\,, \quad \epsilon_0^2 = \eta - e_5 \wedge \lambda\,, \tag{7.3}$$

where $\eta \in \Delta^+_{(8)} = \Lambda^{\rm ev}(\mathbb{R}\langle e_1, e_2, e_3, e_4\rangle)$ and $\lambda \in \Delta^+_{(8)} = \Lambda^{\rm odd}(\mathbb{R}\langle e_1, e_2, e_3, e_4\rangle)$ are chiral and anti-chiral Majorana-Weyl $\mathfrak{spin}(8)$ spinors, respectively. The form bilinears of ϵ can be computed in terms of those of η and λ. The result can be found in appendix B.

For the D-string, the non-vanishing components of $G^{(3)}$ are proportional to $G^{(3)}_{05i}$. Using these, and the expression for the form bilinears in appendix B, one concludes from the TCFH that

$$\nabla_M\, k^{rs}_P - \frac{1}{2}\, e^\Phi\, G^{(3)}_{MP}{}^N\, k^{(1)rs}_N = 0\,, \quad \nabla_M\, k^{(1)rs}_P - \frac{1}{2}\, e^\Phi\, G^{(3)}_{MP}{}^N\, k^{rs}_N = 0\,. \tag{7.4}$$

Therefore both $\tilde{k}^\pm = k \pm k^{(1)}$ are covariantly constant with respect to the connection $\nabla^{(\pm)}$, as in (5.17), but now with torsion $e^\Phi\, G^{(3)}$. As $d(e^\Phi\, G^{(3)}) = 0$, $\tilde{k}^\pm$ generate symmetries for the probe actions (5.1) and (5.7), where the coupling b is given by $e^\Phi\, G^{(3)} = db$. Furthermore $\tilde{k}^+$ ($\tilde{k}^-$) generates a symmetry for the probe action (5.8), where the coupling C is $C = e^\Phi\, G^{(3)}$ ($C = -e^\Phi\, G^{(3)}$). Note that both $\tilde{k}^\pm$ have components along the worldsheet directions of the D-string.

It can be shown that the remaining form bilinears do not generate symmetries for the probe actions (5.1), (5.7) and (5.8). The details of this analysis is similar to those explained for IIA D-branes and they will not be presented here.

7.1.3 D5-brane

Choosing the transverse directions of the D5-brane as 3489, the condition on the Killing spinors $\epsilon = h^{-\frac{1}{8}}\epsilon_0$ for the D5-branes is

$$\Gamma_{3489}\sigma_1\epsilon_0 = \pm\epsilon_0\,, \tag{7.5}$$

where $\epsilon_0 = (\epsilon_0^1, \epsilon_0^2)^t$ is a doublet of constant Majorana-Weyl $\mathfrak{spin}(9,1)$ spinors and h is a harmonic function as in (4.28) for $p = 5$. This condition with the plus sign can be solved using spinorial geometry to yield

$$\epsilon_0^1 = \eta^1 + e_{34} \wedge \lambda^1 + e_3 \wedge \eta^2 + e_4 \wedge \lambda^2\,, \quad \epsilon_0^2 = \eta^1 + e_{34} \wedge \lambda^1 - e_3 \wedge \eta^2 - e_4 \wedge \lambda^2\,, \tag{7.6}$$

where η^1, λ^1 (η^2, λ^2) are positive (negative) chirality spinors of $\mathfrak{spin}(5,1)$. The reality condition on ϵ_0 implies that $\lambda^1 = -\Gamma_{67} * \eta^1$ and $\lambda^2 = -\Gamma_{67} * \eta^2$. Using this, one can calculate the form bilinears of the D5-brane solution. These have been presented in appendix B.

As for D1-branes, let us define $\tilde{k}^\pm = k \pm k^{(1)}$. The TCFH together with the expression of the form bilinears for this background in appendix B give

$$\nabla^{(\pm)}_M \tilde{k}^\pm_N = \nabla^{(\pm)}_{[M} \tilde{k}^\pm_{N]}\,. \tag{7.7}$$

Therefore $\tilde{k}^\pm$ satisfy the KY equation with respect to the connection $\nabla^{(\pm)}$ as in (5.17) with torsion $\pm e^\Phi G^{(3)}$. A consequence of this is that $\tilde{k}^\pm$ generate symmetries in the particle probe action (5.8) with 3-form coupling $\pm e^{\Phi}G^{(3)}$. Note that the second condition in (5.10) required for this is also satisfied as $i_{\tilde{k}^{\pm}}d(e^{\Phi}G^{(3)}) = 0$ and $i_{\tilde{k}^{\pm}}G^{(3)} = 0$ because $\tilde{k}^{\pm}$ have components only along the worldvolume directions of the D5-brane. A similar investigation reveals that $k^{(2)}$ and $k^{(3)}$ do not generate symmetries for the probe actions we are considering.

Next define $\tilde{\pi}^{\pm} = \pi \pm \pi^{(1)}$. The TCFH can be re-organised as a KY equation with respect to a connection with skew-symmetric torsion provided that the term proportional to the spacetime metric g vanishes. For this the $\tilde{\pi}^{\pm}_{Mij}$ components of the 3-form bilinears should vanish. In particular, $\tilde{\pi}^{+}$ is a KY form with respect to $\nabla^{(+)}$ connection provided that

$$\langle \eta^{1r}, \Gamma_a \lambda^{1s}\rangle_D = \mathrm{Im}\langle \eta^{1r}, \Gamma_a \eta^{1s}\rangle_D = 0\,, \tag{7.8}$$

and similarly $\tilde{\pi}^{-}$ is a KY form with respect to $\nabla^{(-)}$ connection provided that

$$\langle \eta^{2r}, \Gamma_a \lambda^{2s}\rangle_D = \mathrm{Im}\langle \eta^{2r}, \Gamma_a \eta^{2s}\rangle_D = 0\ . \tag{7.9}$$

The remaining non-vanishing components of $\tilde{\pi}^{\pm}$ are

$$\begin{aligned}
\tilde{\pi}^{+rs} &= \frac{4}{3}\, h^{-1/4}\,\mathrm{Re}\left\langle \eta^{1r}, \Gamma_{abc}\eta^{1s}\right\rangle_D e^a\wedge e^b\wedge e^c\,,\\
\tilde{\pi}^{-rs} &= \frac{4}{3}\, h^{-1/4}\,\mathrm{Re}\left\langle \eta^{2r}, \Gamma_{abc}\eta^{2s}\right\rangle_D e^a\wedge e^b\wedge e^c\ .
\end{aligned} \tag{7.10}$$

The conditions (7.8) and (7.9) may impose additional restrictions on $\tilde{\pi}^{\pm}$ above. Focusing in $\tilde{\pi}^{+}$, let us solve (7.8). For this note that $\mathfrak{spin}(5,1) = \mathfrak{sl}(2,\mathbb{H})$ and that the positive chirality representation of $\mathfrak{spin}(5,1)$ can be identified with $\mathbb{H}^2$. Therefore up to a $\mathfrak{spin}(5,1)$ rotation, we can choose without loss of generality $\eta^{1r} = 1$. Setting $\eta^{1s} = w1 + ye_{12} + fe_{15} + ze_{25}$, $w, y, f, z \in \mathbb{C}$, the first condition in (7.8) implies that $y = f = z = 0$. Thus $\eta^{1s} = w1$. Then it follows that the second condition in (7.8) implies that $\tilde{\pi}^{+} = 0$. A similar argument also implies that (7.9) gives $\tilde{\pi}^{-} = 0$. Therefore the form bilinears $\tilde{\pi}^{\pm}$ do not generate symmetries for the spinning particle probe action (5.8). In addition an investigation reveals that $\pi^{(2)}$ and $\pi^{(3)}$ do not generate symmetries for the probe actions we have been considering. The same applies for all four 5-form bilinears.

To summarise the results of this section, we have demonstrated that there are Killing spinors such that the form bilinears $\tilde{k}^{\pm}$ are KY forms with respect to connections with skew-symmetric torsion proportional to $\pm e^{\Phi}G^{(3)}$. It turns out that these forms $\tilde{k}^{\pm}$ generate symmetries for the probes described by action (5.8) with form coupling C equal to $\pm e^{\Phi}G^{(3)}$.

7.2 D3-brane

Choosing the worldvolume directions of the D3-brane as 0549, the Killing spinors, $\epsilon = h^{-\frac{1}{8}}\epsilon_0$, of this solution satisfy the condition

$$\Gamma_{0549}\epsilon_0^1 = \pm\epsilon_0^2\,, \tag{7.11}$$

where $\epsilon_0 = (\epsilon_0^1, \epsilon_0^2)^t$ is a doublet of constant Majorana-Weyl spinors of $\mathfrak{spin}(9,1)$ and h a harmonic function as in (4.28) with $p = 3$. This condition with the plus sign can be solved using spinorial geometry as

$$\epsilon_0^1 = \eta^1 + e_{45}\wedge\lambda^1 + e_4\wedge\eta^2 + e_5\wedge\lambda^2\,,\quad \epsilon_0^2 = i\eta^1 + ie_{45}\wedge\lambda^1 - ie_4\wedge\eta^2 - ie_5\wedge\lambda^2\,, \tag{7.12}$$

where $\eta^1, \lambda^1 \in \Lambda^{\text{ev}}(\mathbb{C}\langle e_1, e_2, e_3\rangle)$ ($\eta^2, \lambda^2 \in \Lambda^{\text{odd}}(\mathbb{C}\langle e_1, e_2, e_3\rangle)$) are positive (negative) chirality Weyl spinors of $\mathfrak{spin}(6)$. Furthermore the reality condition on ϵ implies that

$$\eta^2 = -i\Gamma_{678} * \eta^1 \,, \quad \lambda^2 = i\Gamma_{678} * \lambda^1 \,. \tag{7.13}$$

Using these, one can easily express the form bilinears of the D3-brane solution in terms of those of the η and λ $\mathfrak{spin}(6)$ spinors. The form bilinears can be found in appendix B.

As the probe actions (5.1), (5.7) and (5.8) do not exhibit a 5-form coupling, the only coupling one should consider is that of the spacetime metric. For the form bilinears to generate a symmetry for the probe described by the action (5.8), they must be KY tensors. To see whether this is the case, let us begin with the 1-form bilinears k and $k^{(2)}$. The TCFH[15] gives

$$\begin{aligned} \nabla_M k^{rs}_P &= \frac{1}{12} e^{\Phi} G^{(5)}_{MP}{}^{N_1N_2N_3} \pi^{(2)rs}_{N_1N_2N_3} \,, \\ \nabla_M k^{(2)rs}_P &= -\frac{1}{12} e^{\Phi} G^{(5)}_{MP}{}^{N_1N_2N_3} \pi^{rs}_{N_1N_2N_3} \,. \end{aligned} \tag{7.14}$$

Clearly both are KY tensors and so generate symmetries for the probe action (5.8) with $C = 0$. Using that the components $G^{(5)}_{a_1\dots a_4 i}$ and $G^{(5)}_{i_1\dots i_5}$ of the 5-form field strength of the D3-brane solution do not vanish and the expressions for the bilinears in appendix B, one can show that the remaining two 1-form bilinears do not generate a symmetry for the probe actions we have been considering.

Next let us turn to the 3-form bilinears π and $\pi^{(2)}$. The TCFH on a D3-background reads

$$\nabla_M \pi^{rs}_{P_1P_2P_3} - \frac{1}{4} e^{\Phi} G^{(5)}_{M[P_1}{}^{N_1N_2N_3} \tau^{(2)rs}_{P_2P_3]N_1N_2N_3} = -\frac{1}{2} e^{\Phi} G^{(5)}_{MP_1P_2P_3}{}^{N} k^{(2)rs}_N \,, \tag{7.15}$$

$$\nabla_M \pi^{(2)rs}_{P_1P_2P_3} + \frac{1}{4} e^{\Phi} G^{(5)}_{M[P_1}{}^{N_1N_2N_3} \tau^{rs}_{P_2P_3]N_1N_2N_3} = \frac{1}{2} e^{\Phi} G^{(5)}_{MP_1P_2P_3}{}^{N} k^{rs}_N \,. \tag{7.16}$$

For either π or $\pi^{(2)}$ be KY forms, the connection term involving $G^{(5)}$ in the TCFH must vanish. For π, this requires that $\tau^{(2)}_{ijabc} = 0$ which in turn implies that

$$\begin{aligned} \text{Re}\left\langle \eta^{1r}, \Gamma_{ij}\eta^{1s}\right\rangle = \text{Re}\left\langle \lambda^{1r}, \Gamma_{ij}\lambda^{1s}\right\rangle &= 0 \,, \\ \text{Re}\left\langle \eta^{1r}, \Gamma_{ij}\lambda^{1s}\right\rangle + \text{Re}\left\langle \lambda^{1r}, \Gamma_{ij}\eta^{1s}\right\rangle &= 0 \,, \\ \text{Im}\left\langle \eta^{1r}, \Gamma_{ij}\lambda^{1s}\right\rangle - \text{Im}\left\langle \lambda^{1r}, \Gamma_{ij}\eta^{1s}\right\rangle &= 0 \,. \end{aligned} \tag{7.17}$$

Imposing the above conditions, the non-vanishing components of π are

$$\begin{aligned} \pi^{rs} =& -4h^{-\frac{1}{4}}\, \text{Im}\left\langle \eta^{1r}, \eta^{1s}\right\rangle (e^0 - e^5) \wedge e^4 \wedge e^9 \\ &+ 4h^{-\frac{1}{4}}\, \text{Im}\left\langle \lambda^{1r}, \lambda^{1s}\right\rangle (e^0 + e^5) \wedge e^4 \wedge e^9 \\ &+ 4h^{-\frac{1}{4}} \left(\text{Re}\left\langle \eta^{1r}, \lambda^{1s}\right\rangle - \text{Re}\left\langle \lambda^{1r}, \eta^{1s}\right\rangle\right) e^0 \wedge e^5 \wedge e^4 \\ &+ 4h^{-\frac{1}{4}} \left(\text{Im}\left\langle \eta^{1r}, \lambda^{1s}\right\rangle + \text{Im}\left\langle \lambda^{1r}, \eta^{1s}\right\rangle\right) e^0 \wedge e^5 \wedge e^9 \,. \end{aligned} \tag{7.18}$$

[15] We have replaced $k^{(2)}, \pi^{(2)}$ and $\tau^{(2)}$ with $ik^{(2)}, i\pi^{(2)}$ and $i\tau^{(2)}$ so that the TCFH for the D3-brane and later for the D7-brane to be manifestly real.

The conditions (7.17) may impose additional restrictions on the components of π above. Indeed as $\mathfrak{spin}(6) = \mathfrak{su}(4)$ and the positivity chirality representation of $\mathfrak{spin}(6)$ is identified with the fundamental representation of $\mathfrak{su}(4)$, one can choose without loss of generality $\eta^{1r} = a1$, $a \in \mathbb{C}$, up to a $\mathfrak{spin}(6)$ rotation. As the isotropy algebra of $\eta^{1r} = a1$ is $\mathfrak{su}(3)$, one can again choose without loss of generality $\eta^{1s} = b\,1 + c\,e_{12}$, $b, c \in \mathbb{C}$. Then the first condition in (7.17) for $i = 1, j = 2$ and $i = 1, j = 7$ implies that $a\bar{c} = 0$. Taking $a \neq 0$, i.e. $\eta^{1r} \neq 0$, we find that $c = 0$. Therefore $\eta^{1s} = b1$. Then, one can demonstrate using the first condition in (7.17) for $i = 1$ and $j = 6$ that the component of π^{rs} which depends on $\langle \eta, \eta \rangle$ vanishes. A similar argument implies that the component of π which depends on $\langle \lambda, \lambda \rangle$ vanishes as well. To prove that the remaining components of π vanish, we utilise the result we have proven above that we can always choose $\eta^r = a1$ and $\eta^s = b1$. Then the last two equation in (7.17) for $i = 1$ and $j = 6$ imply that the remaining components of π vanish. The same conclusion holds in the case that $\eta^{1r} = 0$. Therefore the conditions in (7.17) imply that $\pi = 0$ and so it does not generate a symmetry in the spinning particle action (5.8) with $C = 0$.

Similarly for $\pi^{(2)}$ to be a KY form, $\tau_{ijabc} = 0$. The conditions on the spinors are given as in (7.17) after replacing Re with Im and vice versa. After imposing these conditions, the non-vanishing components of $\pi^{(2)}$ are

$$\begin{aligned}\pi^{(2)rs} =& - 4h^{-\frac{1}{4}}\, \mathrm{Re}\left\langle \eta^{1r}, \eta^{1s} \right\rangle (e^0 - e^5) \wedge e^4 \wedge e^9 \\ &+ 4h^{-\frac{1}{4}}\, \mathrm{Re}\left\langle \lambda^{1r}, \lambda^{1s} \right\rangle (e^0 + e^5) \wedge e^4 \wedge e^9 \\ &- 4h^{-\frac{1}{4}} \left(\mathrm{Im}\left\langle \eta^{1r}, \lambda^{1s} \right\rangle - \mathrm{Im}\left\langle \lambda^{1r}, \eta^{1s} \right\rangle \right) e^0 \wedge e^5 \wedge e^4 \\ &+ 4h^{-\frac{1}{4}} \left(\mathrm{Re}\left\langle \eta^{1r}, \lambda^{1s} \right\rangle + \mathrm{Re}\left\langle \lambda^{1r}, \eta^{1s} \right\rangle \right) e^0 \wedge e^5 \wedge e^9 \,. \end{aligned} \tag{7.19}$$

A similar argument as that used to determine the component of π implies that $\pi^{(2)} = 0$. Therefore $\pi^{(2)}$ does not generate symmetries for the spinning particle action (5.8) with $C = 0$.

Next let us focus on the two remaining 3-form bilinears $\pi^{(1)}$ and $\pi^{(3)}$. It turns out that they do not generate symmetries for the probe action (5.8) that we are considering. In particular for $\pi^{(1)}$ to be a KY form, the TCFH requires that $\pi^{(3)} = 0$. This in turn implies that $\pi^{(1)} = 0$. To establish the latter the Hodge duality properties of the transverse components of $\pi^{(3)}$ have to be used.

To find the conditions for τ and $\tau^{(2)}$ be KY forms, the TCFH for these bilinears on a D3-brane background is

$$\begin{aligned}\nabla_M\, \tau^{rs}_{P_1\dots P_5} + 10\, e^{\Phi}\, G^{(5)}_{M[P_1P_2P_3}{}^{N}\, \pi^{(2)rs}_{P_4P_5]N} &= -5\, e^{\Phi}\, g_{M[P_1}\, G^{(5)}_{P_2P_3P_4}{}^{N_1N_2}\, \pi^{(2)rs}_{P_5]N_1N_2} \\ &\quad - \frac{15}{2}\, e^{\Phi}\, G^{(5)}_{[P_1\dots P_4}{}^{N}\, \pi^{(2)rs}_{P_5M]N}\,, \end{aligned} \tag{7.20}$$

$$\begin{aligned}\nabla_M\, \tau^{(2)rs}_{P_1\dots P_5} - 10\, e^{\Phi}\, G^{(5)}_{M[P_1P_2P_3}{}^{N}\, \pi^{rs}_{P_4P_5]N} &= 5\, e^{\Phi}\, g_{M[P_1}\, G^{(5)}_{P_2P_3P_4}{}^{N_1N_2}\, \pi^{rs}_{P_5]N_1N_2} \\ &\quad + \frac{15}{2}\, e^{\Phi}\, G^{(5)}_{[P_1\dots P_4}{}^{N}\, \pi^{rs}_{P_5M]N}\,. \end{aligned} \tag{7.21}$$

It turns out that for τ to be a KY tensor, $\pi^{(2)} = 0$. Using the chirality of η^1 and η^2 as $\mathfrak{spin}(6)$ spinors, one concludes that $\tau = 0$, and so there are no symmetries generated by this

5-form bilinear. Similarly, $\tau^{(2)}$ does not generate any symmetries for the probe action (5.8) we have been considering.

Finally, let us turn to investigate the TCFH of $\tau^{(1)}$ and $\tau^{(3)}$ on a D3-brane background. One finds that

$$\begin{aligned}&\nabla_M \tau^{(1)rs}_{P_1\dots P_5} + 5\, e^{\Phi}\, G^{(5)}_{M[P_1\dots P_4}\, k^{(3)rs}_{P_5]} - \frac{5}{2}\, e^{\Phi}\, G^{(5)}_{M[P_1 P_2}{}^{N_1 N_2}\, \tau^{(3)rs}_{P_3 P_4 P_5] N_1 N_2} \\ &\quad = \frac{5}{2}\, e^{\Phi}\, g_{M[P_1}\, G^{(5)}_{P_2\dots P_5]}{}^{N}\, k^{(3)rs}_{N} - 3\, e^{\Phi}\, G^{(5)}_{[P_1\dots P_5}\, k^{(3)rs}_{M]}\,, \qquad (7.22)\\ &\nabla_M \tau^{(3)rs}_{P_1\dots P_5} - 5\, e^{\Phi}\, G^{(5)}_{M[P_1\dots P_4}\, k^{(1)rs}_{P_5]} + \frac{5}{2}\, e^{\Phi}\, G^{(5)}_{M[P_1 P_2}{}^{N_1 N_2}\, \tau^{(1)rs}_{P_3 P_4 P_5] N_1 N_2} \\ &\quad = -\frac{5}{2}\, e^{\Phi}\, g_{M[P_1}\, G^{(5)}_{P_2\dots P_5]}{}^{N}\, k^{(1)rs}_{N} + 3\, e^{\Phi}\, G^{(5)}_{[P_1\dots P_5}\, k^{(1)rs}_{M]}\,. \qquad (7.23)\end{aligned}$$

Focusing on the former condition, $\tau^{(1)}$ is a KY tensor provided that $k^{(3)} = 0$ and $\tau^{(3)}_{ijkab} = 0$. Using the chirality of η^2 and λ^2 as $\mathfrak{spin}(6)$ spinors, one finds that $\tau^{(1)} = 0$. A similar calculation for $\tau^{(3)}$ reveals that $\tau^{(3)} = 0$. These two forms do not generate symmetries for the probe action (5.8).

To summarise the results of this section, we have demonstrated that there are Killing spinors such that the form bilinears k and $k^{(2)}$ of the D3-brane background are KY forms and so generate (hidden) symmetries for the probe described by the action (5.8) with $C = 0$. These forms have components only along the worldvolume directions of the D3-brane.

7.3 D7-brane

Choosing the transverse directions of the D7-brane as 49, the Killing spinor $\epsilon = h^{-\frac{1}{8}}\epsilon_0$ of the solution satisfies the condition

$$\Gamma_{49}\epsilon^1_0 = \pm\epsilon^2_0\,, \qquad (7.24)$$

where $\epsilon_0 = (\epsilon^1_0, \epsilon^2_0)^t$ is a constant doublet of Majorana-Weyl $\mathfrak{spin}(9,1)$ spinors and $h = 1 + \sum_\ell q_\ell \log|y - y_\ell|$. This condition with the plus sign can be solved using spinorial geometry as

$$\epsilon^1_0 = \eta + e_4 \wedge \lambda\,, \quad \epsilon^2_0 = i\eta - ie_4 \wedge \lambda\,, \qquad (7.25)$$

where η (λ) is a positive, $\eta \in \Lambda^{\mathrm{ev}}(\mathbb{C}\langle e_1, e_2, e_3, e_5\rangle)$, (negative, $\lambda \in \Lambda^{\mathrm{odd}}(\mathbb{C}\langle e_1, e_2, e_3, e_5\rangle)$,) chirality $\mathfrak{spin}(7,1)$ Weyl spinors. The reality condition on ϵ implies that

$$\lambda = -i\Gamma_{678} * \eta\,. \qquad (7.26)$$

Using the above expression for the Killing spinors, the form bilinears can be easily computed and can be found in appendix B.

The TCFH for the form bilinears k and $k^{(2)}$ gives

$$\nabla_M\, k^{rs}_P = \frac{1}{2}\, e^{\Phi}\, G^{(1)N}\, \pi^{(2)rs}_{NMP}\,, \quad \nabla_M k^{(2)rs}_P = -\frac{1}{2}\, e^{\Phi}\, G^{(1)N}\, \pi^{rs}_{NMP}\,. \qquad (7.27)$$

As a result, they are both KY forms. Therefore both generate symmetries for the probe action (5.8) with $C = 0$. It can be shown that the remaining two 1-form bilinears $k^{(1)}$ and $k^{(3)}$ do not generate symmetries for the probe actions we are considering.

Similarly the TCFH of π and $\pi^{(2)}$ on D7-brane background reads

$$\begin{aligned}\nabla_M \pi^{rs}_{P_1P_2P_3} &= \frac{1}{2}\, e^{\Phi}\, G^{(1)N}\, \tau^{(2)rs}_{MP_1P_2P_3N} + 3\, e^{\Phi}\, g_{M[P_1}\, G^{(1)}_{P_2}\, k^{(2)rs}_{P_3]}\,,\\ \nabla_M \pi^{(2)rs}_{P_1P_2P_3} &= -\frac{1}{2}\, e^{\Phi}\, G^{(1)N}\, \tau^{rs}_{MP_1P_2P_3N} - 3\, e^{\Phi}\, g_{M[P_1}\, G^{(1)}_{P_2}\, k^{rs}_{P_3]}\,. \end{aligned} \tag{7.28}$$

For these to be KY forms, it is required that the terms of the TCFH that explicitly contain the spacetime metric must vanish. As the form field strength for the D7-brane $G^{(1)} \neq 0$, for π this leads to the condition $k^{(2)} = 0$, or equivalently,

$$\mathrm{Im}\,\langle \eta^r, \Gamma_a \eta^s \rangle_D = 0\,. \tag{7.29}$$

Therefore

$$\pi^{rs} = \frac{2}{3} h^{-\frac{1}{4}}\, \mathrm{Re}\,\langle \eta^r, \Gamma_{abc}\eta^s\rangle_D\, e^a \wedge e^b \wedge e^c\,, \tag{7.30}$$

is a KY form and generates a (hidden) symmetry for the particle probe described by the action (5.8) with $C = 0$. There are Killing spinors such that (7.29) is satisfied and $\pi \neq 0$, e.g. $\eta^r = 1$ and $\eta^s = e_{12}$.

Similarly the condition for $\pi^{(2)}$ to be a KY form is

$$\mathrm{Re}\,\langle \eta^r, \Gamma_a \eta^s \rangle_D = 0\,. \tag{7.31}$$

As a result

$$\pi^{(2)rs} = -\frac{2}{3} h^{-\frac{1}{4}}\, \mathrm{Im}\,\langle \eta^r, \Gamma_{abc}\eta^s\rangle_D\, e^a \wedge e^b \wedge e^c\,, \tag{7.32}$$

is a KY form and generates a (hidden) symmetry for the particle probe described by the action (5.8) with $C = 0$. Again there are Killing spinors such that $\pi^{(2)}$ above is non-vanishing, e.g. $\eta^r = i1$ and $\eta^s = e_{12}$. The remaining two 3-form bilinears $\pi^{(1)}$ and $\pi^{(3)}$ do not generate symmetries for the probe action we have been considering.

It remains to investigate whether any of the 5-form bilinears generate symmetries for probe action (5.8). To begin consider τ and $\tau^{(2)}$. The TCFH for these in a D7-background is

$$\begin{aligned}\nabla_M \tau^{rs}_{P_1\dots P_5} &= -\frac{1}{12}\, e^{\Phi\star} G^{(9)}_{MP_1\dots P_5}{}^{N_1N_2N_3}\, \pi^{(2)rs}_{N_1N_2N_3} + 10\, e^{\Phi}\, g_{M[P_1}\, G^{(1)}_{P_2}\, \pi^{(2)rs}_{P_3P_4P_5]}\,,\\ \nabla_M \tau^{(2)rs}_{P_1\dots P_5} &= \frac{1}{12}\, e^{\Phi\star} G^{(9)}_{MP_1\dots P_5}{}^{N_1N_2N_3}\, \pi^{rs}_{N_1N_2N_3} - 10\, e^{\Phi}\, g_{M[P_1}\, G^{(1)}_{P_2}\, \pi^{rs}_{P_3P_4P_5]}\,. \end{aligned} \tag{7.33}$$

For τ, the vanishing of the last term in the first TCFH that contains the metric leads to the condition $\pi^{(2)}_{abc} = 0$. This condition in turn implies that $\tau = 0$. In particular observe that τ_{ijabc} vanishes as a consequence of $\pi^{(2)}_{abc} = 0$ and $\tau_{a_1\dots a_5}$ vanishes because it is worldvolume dual to $\pi^{(2)}_{abc}$. To show the latter, one has to use the $\mathfrak{spin}(7,1)$ chirality of the spinors η.

Similarly for $\tau^{(2)}$ to be a KY form, one finds that $\pi_{abc} = 0$. An argument similar to the one presented above implies that $\tau^{(2)} = 0$

To summarise the results of this section, we have demonstrated that there are Killing spinors such that the form bilinears k, $k^{(2)}$, π and $\pi^{(2)}$ of the D7-brane background are KY forms and so generate (hidden) symmetries for the probe described by the action (5.8) with $C = 0$. All these forms have components only along the worldvolume directions of the D7-brane.

8 Concluding remarks

We have presented the TCFH of both IIA and IIB supergravities and demonstrated that the form bilinears satisfy a generalisation of the CKY equation with respect to the minimal TCFH connection in agreement with the general theorem in [14]. Then prompted by the well-known result that KY forms generate (hidden) symmetries in spinning particle actions, we explored the question on whether the form bilinears of some known supergravity backgrounds, which include all type II branes, generate symmetries for various particle and string probes propagating on these backgrounds.

We have also explored the complete integrability of geodesic flow on all type II brane backgrounds. We have demonstrated that if the harmonic function that the solutions depend on has at most one centre, i.e. they are spherically symmetric, then the geodesic flow is completely integrable. We have explicitly given all independent conserved charges in involution. We have also presented the KS, KY and CCKY tensors of these brane backgrounds associated with their integrability structure.

Returning to the symmetries generated by the TCFH, supersymmetric type II common sector backgrounds admit form bilinears which are covariantly constant with respect to a connection with skew-symmetric torsion given by the NS-NS 3-form field strength. All these bilinears generate (hidden) symmetries for string and particle probe actions with 3-form couplings. The type II fundamental string and NS5-brane background form bilinears have explicitly been given. Common sector backgrounds admit additional form bilinears which satisfy a TCFH but they are not covariantly constant with respect to a connection with skew-symmetric torsion. Although these forms are part of the geometric structure of common sector backgrounds, their geometric interpretation is less straightforward.

Moreover we found that there are Killing spinors in all Dp-brane backgrounds, for $p \neq 1, 5$, such that the associated bilinears are KY forms and so generate (hidden) symmetries for spinning particle probes. All these form bilinears have components only along the worldvolume directions of the Dp-branes. A similar conclusion holds for the D1- and D5-brane solutions, only that in this case the form bilinears are KY forms with respect to a connection with skew-symmetric torsion that is determined by the 3-form field strength of the backgrounds. These form bilinears have non-vanishing components only along the worldvolume directions of the D-branes and generate (hidden) symmetries for particle probes described by the action (5.8) with a non-vanishing 3-form coupling.

It is fruitful to compare the KY forms we have obtained from the TCFH with those that are needed to investigate the integrability of the geodesic flow in type II brane backgrounds. TCFH KY forms exist for any choice of the harmonic function that the brane solutions depend on. Moreover, as we have mentioned, these KY forms have non-vanishing components only along the worldvolume directions of D-branes. It is clear from this that although they generate symmetries for particle probes propagating on D-brane backgrounds these symmetries are not necessarily connected to the integrability properties of such dynamical systems. This is because it is not expected, for example, that the geodesic flow of brane solutions which depend on a multi-centred harmonic function to be completely integrable. Indeed the KS and KY tensors we have found that are responsible for the integrability of the geodesic flow on spherically symmetric branes also have components along the transverse directions of these solutions. As the brane metrics have a non-trivial dependence on the transverse coordinates, this is essential for proving the integrability of the geodesic flow. Therefore one concludes that although the form bilinears of supersymmetric backgrounds can generate symmetries in string and particle probes propagating in these backgrounds, they are not sufficient to prove the complete integrability of probe dynamics. Nevertheless the TCFH KY tensors, when they exist, are associated with symmetries of probes propagating on brane backgrounds which are not necessarily spherically symmetric.

To find TCFH KY tensors, we have imposed a rather stringent set of conditions on the form bilinears. In particular in several D-brane backgrounds, we set all terms of the minimal TCFH connection that depend on a form field strength to zero. It is likely that such a restriction can be lifted and the only condition necessary for invariance of a probe action will be that the terms in the TCFH which contain explicitly the metric should vanish. For this a new set a probe actions should be found that have couplings which depend on the form field strengths of the supergravity theories and generalise (5.8) which exhibits only a 3-form coupling. We hope to report on such a development in the future.

Acknowledgments

JP is supported by the EPSRC grant EP/R513064/1.

A Common sector brane form bilinears

A.1 Form bilinears of IIA branes

A.1.1 Fundamental String

A direct computation using (5.19) reveals that the form bilinears of IIA fundamental string are

$$
\begin{aligned}
\sigma^{rs} &= h^{-\frac{1}{2}}\big(-\langle\eta^r,\lambda^s\rangle + \langle\lambda^r,\eta^s\rangle\big)\ ,\\
k^{rs} &= h^{-\frac{1}{2}}\langle\eta^r,\eta^s\rangle(e^0-e^5) + h^{-\frac{1}{2}}\langle\lambda^r,\lambda^s\rangle(e^0+e^5)\,,\\
\omega^{rs} &= h^{-\frac{1}{2}}\big(\langle\eta^r,\lambda^s\rangle + \langle\lambda^r,\eta^s\rangle\big)e^0\wedge e^5 + \frac{1}{2}h^{-\frac{1}{2}}\big(-\langle\eta^r,\Gamma_{ij}\lambda^s\rangle + \langle\lambda^r,\Gamma_{ij}\eta^s\rangle\big)e^i\wedge e^j\,,\\
\pi^{rs} &= \frac{1}{2}h^{-\frac{1}{2}}\langle\eta^r,\Gamma_{ij}\eta^s\rangle\rangle(e^0-e^5)\wedge e^i\wedge e^j + \frac{1}{2}h^{-\frac{1}{2}}\langle\lambda^r,\Gamma_{ij}\lambda^s\rangle(e^0+e^5)\wedge e^i\wedge e^j\,,\\
\zeta^{rs} &= \frac{1}{2}h^{-\frac{1}{2}}\big(\langle\eta^r,\Gamma_{ij}\lambda^s\rangle + \langle\lambda^r,\Gamma_{ij}\eta^s\rangle\big)e^0\wedge e^5\wedge e^i\wedge e^j\\
&\quad + \frac{1}{4!}h^{-\frac{1}{2}}\big(-\langle\eta^r,\Gamma_{ijk\ell}\lambda^s\rangle + \langle\lambda^r,\Gamma_{ijk\ell}\eta^s\rangle\big)e^i\wedge e^j\wedge e^k\wedge e^\ell\,,\\
\tau^{rs} &= \frac{1}{4!}h^{-\frac{1}{2}}\langle\eta^r,\Gamma_{ijk\ell}\eta^s\rangle(e^0-e^5)\wedge e^i\wedge e^j\wedge e^k\wedge e^\ell\\
&\quad + \frac{1}{4!}h^{-\frac{1}{2}}\langle\lambda^r,\Gamma_{ijk\ell}\lambda^s\rangle(e^0+e^5)\wedge e^i\wedge e^j\wedge e^k\wedge e^\ell\,,
\end{aligned}
\tag{A.1}
$$

where $i,j,k,\ell = 1,2,3,4,6,7,8,9$ are the transverse directions of the string and (e^0,e^5,e^i) is a pseudo-orthonormal frame of the fundamental string metric (4.35), i.e $g = -(e^0)^2 +$

$(e^5)^2 + \sum_i (e^i)^2$. The remaining form bilinears $\tilde{\sigma}$, $\tilde{k}$, $\tilde{\omega}$, $\tilde{\pi}$, $\tilde{\zeta}$ and $\tilde{\tau}$ can be obtained from the expressions above upon setting λ^s to $-\lambda^s$.

A.1.2 NS5-brane

A direct computation using (5.22) reveals that the form bilinears of NS5-brane are

$$
\begin{aligned}
k^{rs} &= 2\big(\mathrm{Re}\langle\eta^{1r},\Gamma_a\eta^{1s}\rangle_D + \mathrm{Re}\langle\eta^{2r},\Gamma_a\eta^{2s}\rangle_D\big)\ e^a\,, && \text{(A.2)}\\
\omega^{rs} &= 2\big(\mathrm{Re}\langle\eta^{1r},\Gamma_a\eta^{2s}\rangle_D + \mathrm{Re}\langle\eta^{2r},\Gamma_a\eta^{1s}\rangle_D\big)\ e^a\wedge e^3\\
&\quad + 2\big(\mathrm{Re}\langle\eta^{1r},\Gamma_a\lambda^{2s}\rangle_D - \mathrm{Re}\langle\eta^{2r},\Gamma_a\lambda^{1s}\rangle_D\big)\ e^a\wedge e^4\\
&\quad + 2\big(\mathrm{Im}\langle\eta^{1r},\Gamma_a\eta^{2s}\rangle_D - \mathrm{Im}\langle\eta^{2r},\Gamma_a\eta^{1s}\rangle_D\big)\ e^a\wedge e^8\\
&\quad + 2\big(\mathrm{Im}\langle\eta^{1r},\Gamma_a\lambda^{2s}\rangle_D - \mathrm{Im}\langle\eta^{2r},\Gamma_a\lambda^{1s}\rangle_D\big)\ e^a\wedge e^9\,, && \text{(A.3)}\\
\pi^{rs} &= \frac{1}{3}\big(\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\eta^{1s}\rangle_D + \mathrm{Re}\langle\eta^{2r},\Gamma_{abc}\eta^{2r}\rangle_D\big)e^a\wedge e^b\wedge e^c\\
&\quad - 2\mathrm{Re}\langle\eta^{1r},\Gamma_a\lambda^{1s}\rangle_D(e^3\wedge e^4 - e^8\wedge e^9)\wedge e^a\\
&\quad + 2\mathrm{Re}\langle\eta^{2r},\Gamma_a\lambda^{2s}\rangle_D(e^3\wedge e^4 + e^8\wedge e^9)\wedge e^a\\
&\quad - 2\mathrm{Im}\langle\eta^{1r},\Gamma_a\eta^{1s}\rangle_D(e^3\wedge e^8 + e^4\wedge e^9)\wedge e^a\\
&\quad + 2\mathrm{Im}\langle\eta^{2r},\Gamma_a\eta^{2s}\rangle_D(e^3\wedge e^8 - e^4\wedge e^9)\wedge e^a\\
&\quad - 2\mathrm{Im}\langle\eta^{1r},\Gamma_a\lambda^{1s}\rangle_D(e^3\wedge e^9 - e^4\wedge e^8)\wedge e^a\\
&\quad + 2\mathrm{Im}\langle\eta^{2r},\Gamma_a\lambda^{2s}\rangle_D(e^3\wedge e^9 + e^4\wedge e^8)\wedge e^a\,, && \text{(A.4)}\\
\zeta^{rs} &= \frac{1}{6}\tilde{\omega}^{rs}_{a\ell}\epsilon^{\ell}{}_{ijk}e^a\wedge e^i\wedge e^j\wedge e^k\\
&\quad + \frac{1}{3}\big(\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\eta^{2s}\rangle_D + \mathrm{Re}\langle\eta^{2r},\Gamma_a\eta^{1s}\rangle_D\big)\ e^a\wedge e^b\wedge e^c\wedge e^3\\
&\quad + \frac{1}{3}\big(\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\lambda^{2s}\rangle_D - \mathrm{Re}\langle\eta^{2r},\Gamma_{abc}\lambda^{1s}\rangle_D\big)\ e^a\wedge e^b\wedge e^c\wedge e^4\\
&\quad + \frac{1}{3}\big(\mathrm{Im}\langle\eta^{1r},\Gamma_{abc}\eta^{2s}\rangle_D - \mathrm{Im}\langle\eta^{2r},\Gamma_{abc}\eta^{1s}\rangle_D\big)\ e^a\wedge e^b\wedge e^c\wedge e^8\\
&\quad + \frac{1}{3}\big(\mathrm{Im}\langle\eta^{1r},\Gamma_{abc}\lambda^{2s}\rangle_D - \mathrm{Im}\langle\eta^{2r},\Gamma_{abc}\lambda^{1s}\rangle_D\big)\ e^a\wedge e^b\wedge e^c\wedge e^9\,, && \text{(A.5)}\\
\tau^{rs} &= \tilde{k}^{rs}\wedge e^3\wedge e^4\wedge e^8\wedge e^9\\
&\quad - \frac{1}{3}\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\lambda^{1s}\rangle_D(e^3\wedge e^4 - e^8\wedge e^9)\wedge e^a\wedge e^b\wedge e^c\\
&\quad + \frac{1}{3}\mathrm{Re}\langle\eta^{2r},\Gamma_{abc}\lambda^{2s}\rangle_D(e^3\wedge e^4 + e^8\wedge e^9)\wedge e^a\wedge e^b\wedge e^c\\
&\quad - \frac{1}{3}\mathrm{Im}\langle\eta^{1r},\Gamma_{abc}\eta^{1s}\rangle_D(e^3\wedge e^8 + e^4\wedge e^9)\wedge e^a\wedge e^b\wedge e^c\\
&\quad + \frac{1}{3}\mathrm{Im}\langle\eta^{2r},\Gamma_{abc}\eta^{2s}\rangle_D(e^3\wedge e^8 - e^4\wedge e^9)\wedge e^a\wedge e^b\wedge e^c\\
&\quad - \frac{1}{3}\mathrm{Im}\langle\eta^{1r},\Gamma_{abc}\lambda^{1s}\rangle_D(e^3\wedge e^9 - e^4\wedge e^8)\wedge e^a\wedge e^b\wedge e^c\\
&\quad + \frac{1}{3}\mathrm{Im}\langle\eta^{2r},\Gamma_{abc}\lambda^{2s}\rangle_D(e^3\wedge e^9 + e^4\wedge e^8)\wedge e^a\wedge e^b\wedge e^c\\
&\quad + \frac{2}{5!}\big(\mathrm{Re}\langle\eta^{1r},\Gamma_{a_1\dots a_5}\eta^{1s}\rangle_D + \mathrm{Re}\langle\eta^{2r},\Gamma_{a_1\dots a_5}\eta^{2s}\rangle_D\big)\ e^{a_1}\wedge\dots\wedge e^{a_5}\,, && \text{(A.6)}
\end{aligned}
$$

where $a, b, c = 0, 1, 2, 5, 6, 7$ are the worldvolume directions, $\epsilon_{3489} = 1$ and $(e^a, e^3, e^4, e^8, e^9)$ is a pseudo-orthonormal frame for the NS5-brane metric (4.38). The remaining form bilinears $\tilde{\sigma}$, $\tilde{k}$, $\tilde{\omega}$, $\tilde{\pi}$, $\tilde{\zeta}$ and $\tilde{\tau}$ bilinears can be constructed from those above upon replacing both η^{2s} and λ^{2s} with $-\eta^{2s}$ and $-\lambda^{2s}$, respectively.

A.2 Form bilinears IIB branes

All the bilinears below are manifestly real. In particular we have replaced $k^{(2)}, \pi^{(2)}$ and $\tau^{(2)}$ with $ik^{(2)}, i\pi^{(2)}$ and $i\tau^{(2)}$, respectively.

A.2.1 Fundamental string

Choosing the worldvolume and transverse directions of the IIB fundamental string as in the IIA case, a direct computation using (5.35) reveals that the form bilinears of IIB fundamental string are

$$\begin{aligned} k^{rs} &= h^{-\frac{1}{2}}\langle\eta^r,\eta^s\rangle(e^0-e^5)+h^{-\frac{1}{2}}\langle\lambda^r,\lambda^s\rangle(e^0+e^5)\,,\\ \pi^{rs} &= \frac{1}{2}h^{-\frac{1}{2}}\langle\eta^r,\Gamma_{ij}\eta^s\rangle(e^0-e^5)\wedge e^i\wedge e^j+\frac{1}{2}h^{-\frac{1}{2}}\langle\lambda^r,\Gamma_{ij}\lambda^s\rangle(e^0+e^5)\wedge e^i\wedge e^j\,,\\ \tau^{rs} &= \frac{1}{4!}h^{-\frac{1}{2}}\langle\eta^r,\Gamma_{i_1\dots i_4}\eta^s\rangle(e^0-e^5)\wedge e^{i_1}\wedge\dots\wedge e^{i_4}\\ &\quad+\frac{1}{4!}h^{-\frac{1}{2}}\langle\lambda^r,\Gamma_{i_1\dots i_4}\lambda^s\rangle(e^0+e^5)\wedge e^{i_1}\wedge\dots\wedge e^{i_4}\,, \end{aligned} \tag{A.7}$$

where again (e^0, e^5, e^i) is a pseudo-orthonormal frame of (4.35). The $k^{(3)}, \pi^{(3)}$ and $\tau^{(3)}$ bilinears can be obtained from those above upon replacing λ^s with $-\lambda^s$.

For the remaining form bilinears a direct computation yields

$$k^{(1)rs} = h^{-\frac{1}{2}}\langle\eta^r,\Gamma_i\lambda^s\rangle\,e^i+h^{-\frac{1}{2}}\langle\lambda^r,\Gamma_i\eta^s\rangle\,e^i\,, \tag{A.8}$$

$$\begin{aligned} \pi^{(1)rs} &= -h^{-\frac{1}{2}}\langle\eta^r,\Gamma_i\lambda^s\rangle e^0\wedge e^5\wedge e^i+\frac{1}{3!}h^{-\frac{1}{2}}\langle\eta^r,\Gamma_{ijk}\lambda^s\rangle e^i\wedge e^j\wedge e^k\\ &\quad+h^{-\frac{1}{2}}\langle\lambda^r,\Gamma_i\eta^s\rangle e^0\wedge e^5\wedge e^i+\frac{1}{3!}h^{-\frac{1}{2}}\langle\lambda^r,\Gamma_{ijk}\eta^s\rangle e^i\wedge e^j\wedge e^k\,, \end{aligned} \tag{A.9}$$

$$\begin{aligned} \tau^{(1)rs} &= -\frac{1}{3!}h^{-\frac{1}{2}}\langle\eta^r,\Gamma_{ijk}\lambda^s\rangle e^0\wedge e^5\wedge e^i\wedge e^j\wedge e^k\\ &\quad+\frac{1}{5!}h^{-\frac{1}{2}}\langle\eta^r,\Gamma_{i_1\dots i_5}\lambda^s\rangle e^{i_1}\wedge\dots\wedge e^{i_5}\\ &\quad+\frac{1}{3!}h^{-\frac{1}{2}}\langle\lambda^r,\Gamma_{ijk}\eta^s\rangle e^0\wedge e^5\wedge e^i\wedge e^j\wedge e^k\\ &\quad+\frac{1}{5!}h^{-\frac{1}{2}}\langle\lambda^r,\Gamma_{i_1\dots i_5}\eta^s\rangle e^{i_1}\wedge\dots\wedge e^{i_5}\;. \end{aligned} \tag{A.10}$$

The $k^{(2)}, \pi^{(2)}$ and $\tau^{(2)}$ bilinears can be obtained from those above upon replacing η^s with $-\eta^s$.

A.2.2 NS5-brane

Choosing the worldvolume and transverse directions as in the IIA case above, a direct computation using (5.38) reveals that the form bilinears of IIB NS5-brane are

$$k^{rs} = 2\mathrm{Re}\langle\eta^{1r},\Gamma_a\eta^{1s}\rangle_D\, e^a + 2\mathrm{Re}\langle\eta^{2r},\Gamma_a\eta^{2s}\rangle_D\, e^a\,, \tag{A.11}$$

$$\begin{aligned}
\pi^{rs} =& -2\mathrm{Re}\langle\eta^{1r},\Gamma_a\lambda^{1s}\rangle_D\, e^a\wedge(e^3\wedge e^4 - e^8\wedge e^9)\\
&+2\mathrm{Re}\langle\eta^{2r},\Gamma_a\lambda^{2s}\rangle_D\, e^a\wedge(e^3\wedge e^4 + e^8\wedge e^9)\\
&-2\mathrm{Im}\langle\eta^{1r},\Gamma_a\eta^{1s}\rangle_D\, e^a\wedge(e^3\wedge e^8 + e^4\wedge e^9)\\
&+2\mathrm{Im}\langle\eta^{2r},\Gamma_a\eta^{2s}\rangle_D\, e^a\wedge(e^3\wedge e^8 - e^4\wedge e^9)\\
&-2\mathrm{Im}\langle\eta^{1r},\Gamma_a\lambda^{1s}\rangle_D\, e^a\wedge(e^3\wedge e^9 - e^4\wedge e^8)\\
&+2\mathrm{Im}\langle\eta^{2r},\Gamma_a\lambda^{2s}\rangle_D\, e^a\wedge(e^3\wedge e^9 + e^4\wedge e^8)\\
&+\frac{1}{3}(\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\eta^{1s}\rangle_D + \mathrm{Re}\langle\eta^{2r},\Gamma_{abc}\eta^{2s}\rangle_D)e^a\wedge e^b\wedge e^c\,,
\end{aligned} \tag{A.12}$$

$$\begin{aligned}
\tau^{rs} =& \,2\big(\mathrm{Re}\langle\eta^{1r},\Gamma_a\eta^{1s}\rangle_D - \mathrm{Re}\langle\eta^{2r},\Gamma_a\eta^{2s}\rangle_D\big)\, e^a\wedge e^3\wedge e^4\wedge e^8\wedge e^9\\
&-\frac{1}{3}\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\lambda^{1s}\rangle_D\, e^a\wedge e^b\wedge e^c\wedge(e^3\wedge e^4 - e^8\wedge e^9)\\
&+\frac{1}{3}\mathrm{Re}\langle\eta^{2r},\Gamma_{abc}\lambda^{2s}\rangle_D\, e^a\wedge e^b\wedge e^c\wedge(e^3\wedge e^4 + e^8\wedge e^9)\\
&-\frac{1}{3}\mathrm{Im}\langle\eta^{1r},\Gamma_{abc}\eta^{1s}\rangle_D\, e^a\wedge e^b\wedge e^c\wedge(e^3\wedge e^8 + e^4\wedge e^9)\\
&+\frac{1}{3}\mathrm{Im}\langle\eta^{2r},\Gamma_{abc}\eta^{2s}\rangle_D\, e^a\wedge e^b\wedge e^c\wedge(e^3\wedge e^8 - e^4\wedge e^9)\\
&-\frac{1}{3}\mathrm{Im}\langle\eta^{1r},\Gamma_{abc}\lambda^{1s}\rangle_D\, e^a\wedge e^b\wedge e^c\wedge(e^3\wedge e^9 - e^4\wedge e^8)\\
&+\frac{1}{3}\mathrm{Im}\langle\eta^{2r},\Gamma_{abc}\lambda^{2s}\rangle_D\, e^a\wedge e^b\wedge e^c\wedge(e^3\wedge e^9 + e^4\wedge e^8)\\
&+\frac{2}{5!}(\mathrm{Re}\langle\eta^{1r},\Gamma_{a_1\dots a_5}\eta^{1s}\rangle_D + \mathrm{Re}\langle\eta^{2r},\Gamma_{a_1\dots a_5}\eta^{2s}\rangle_D)e^{a_1}\wedge\dots\wedge e^{a_5}\,,
\end{aligned} \tag{A.13}$$

where $(e^a, e^3, e^4, e^8, e^9)$ is a pseudo-orthonormal frame of the metric (4.38). The form bilinears $k^{(3)}$, $\pi^{(3)}$ and $\tau^{(3)}$ can be constructed from those above after changing the sign in front of the terms containing the inner products $\langle\eta^2, Q\eta^2\rangle_D$ and $\langle\eta^2, Q\lambda^2\rangle_D$ for all Clifford elements Q.

The remaining bilinears can be obtained in a similar way to find

$$\begin{aligned}
k^{(1)rs} =& \,2\big(\mathrm{Re}\langle\eta^{1r},\eta^{2s}\rangle_D - \mathrm{Re}\langle\eta^{2r},\eta^{1s}\rangle_D\big)\, e^3 + 2\big(\mathrm{Re}\langle\eta^{1r},\lambda^{2s}\rangle_D + \mathrm{Re}\langle\eta^{2r},\lambda^{1s}\rangle_D\big)\, e^4\\
&+2\big(\mathrm{Im}\langle\eta^{1r},\eta^{2s}\rangle_D + \mathrm{Im}\langle\eta^{2r},\eta^{1s}\rangle_D\big)\, e^8\\
&+2\big(\mathrm{Im}\langle\eta^{1r},\lambda^{2s}\rangle_D + \mathrm{Im}\langle\eta^{2r},\lambda^{1s}\rangle_D\big)\, e^9\,,
\end{aligned} \tag{A.14}$$

$$\begin{aligned}
\pi^{(1)rs} =& \Big[\big(\mathrm{Re}\langle\eta^{1r},\Gamma_{ab}\eta^{2s}\rangle_D - \mathrm{Re}\langle\eta^{2r},\Gamma_{ab}\eta^{1s}\rangle_D\big)\, e^3\\
&+\big(\mathrm{Re}\langle\eta^{1r},\Gamma_{ab}\lambda^{2s}\rangle_D + \mathrm{Re}\langle\eta^{2r},\Gamma_{ab}\lambda^{1s}\rangle_D\big)\, e^4\\
&+\big(\mathrm{Im}\langle\eta^{1r},\Gamma_{ab}\eta^{2s}\rangle_D + \mathrm{Im}\langle\eta^{2r},\Gamma_{ab}\eta^{1s}\rangle_D\big)\, e^8\\
&+\big(\mathrm{Im}\langle\eta^{1r},\Gamma_{ab}\lambda^{2s}\rangle_D + \mathrm{Im}\langle\eta^{2r},\Gamma_{ab}\lambda^{1s}\rangle_D\big)\, e^9\Big]\wedge e^a\wedge e^b\\
&+2\big(-\mathrm{Re}\langle\eta^{1r},\eta^{2s}\rangle_D - \mathrm{Re}\langle\eta^{2r},\eta^{1s}\rangle_D\big)\, e^4\wedge e^8\wedge e^9
\end{aligned}$$

$$
\begin{aligned}
&-2\big(-\mathrm{Re}\langle\eta^{1r},\lambda^{2s}\rangle_D+\mathrm{Re}\langle\eta^{2r},\lambda^{1s}\rangle_D\big)\,e^3\wedge e^8\wedge e^9\\
&+2\big(-\mathrm{Im}\langle\eta^{1r},\eta^{2s}\rangle_D+\mathrm{Im}\langle\eta^{2r},\eta^{1s}\rangle_D\big)\,e^3\wedge e^4\wedge e^9\\
&-2\big(-\mathrm{Im}\langle\eta^{1r},\lambda^{2s}\rangle_D+\mathrm{Im}\langle\eta^{2r},\lambda^{1s}\rangle_D\big)\,e^3\wedge e^4\wedge e^8\,,
\end{aligned}
\tag{A.15}
$$

$$
\begin{aligned}
\tau^{(1)rs}=\;&\frac{1}{12}\Big[\big(\mathrm{Re}\langle\eta^{1r},\Gamma_{a_1\dots a_4}\eta^{2s}\rangle_D-\mathrm{Re}\langle\eta^{2r},\Gamma_{a_1\dots a_4}\eta^{1s}\rangle_D\big)\,e^3\\
&+\big(\mathrm{Re}\langle\eta^{1r},\Gamma_{a_1\dots a_4}\lambda^{2s}\rangle_D+\mathrm{Re}\langle\eta^{2r},\Gamma_{a_1\dots a_4}\lambda^{1s}\rangle_D\big)\,e^4\\
&+\big(\mathrm{Im}\langle\eta^{1r},\Gamma_{a_1\dots a_4}\eta^{2s}\rangle_D+\mathrm{Im}\langle\eta^{2r},\Gamma_{a_1\dots a_4}\eta^{1s}\rangle_D\big)\,e^8\\
&+\big(\mathrm{Im}\langle\eta^{1r},\Gamma_{a_1\dots a_4}\lambda^{2s}\rangle_D+\mathrm{Im}\langle\eta^{2r},\Gamma_{a_1\dots a_4}\lambda^{1s}\rangle_D\big)\,e^9\Big]\wedge e^{a_1}\wedge\cdots\wedge e^{a_4}\\
&+\big(-\mathrm{Re}\langle\eta^{1r},\Gamma_{ab}\eta^{2s}\rangle_D-\mathrm{Re}\langle\eta^{2r},\Gamma_{ab}\eta^{1s}\rangle_D\big)\;e^a\wedge e^b\wedge e^4\wedge e^8\wedge e^9\\
&-\big[\big(-\mathrm{Re}\langle\eta^{1r},\Gamma_{ab}\lambda^{2s}\rangle_D+\mathrm{Re}\langle\eta^{2r},\Gamma_{ab}\lambda^{1s}\rangle_D\big)\;e^a\wedge e^b\wedge e^3\wedge e^8\wedge e^9\\
&+\big(-\mathrm{Im}\langle\eta^{1r},\Gamma_{ab}\eta^{2s}\rangle_D+\mathrm{Im}\langle\eta^{2r},\Gamma_{ab}\eta^{1s}\rangle_D\big)\,e^a\wedge e^b\wedge\,e^3\wedge e^4\wedge e^9\\
&-\big(-\mathrm{Im}\langle\eta^{1r},\Gamma_{ab}\lambda^{2s}\rangle_D+\mathrm{Im}\langle\eta^{2r},\Gamma_{ab}\lambda^{1s}\rangle_D\big)\big]\,e^a\wedge e^b\wedge e^3\wedge e^4\wedge e^8\,,
\end{aligned}
\tag{A.16}
$$

where the form bilinears $k^{(2)}$, $\pi^{(2)}$ and $\tau^{(2)}$ can be obtained from those above upon changing the sign in front of the components containing the bilinears $\langle\eta^2,Q\eta^1\rangle$, $\langle\eta^2,Q\lambda^1\rangle$ and $\langle\lambda^2,Q\eta^1\rangle$, where Q is a Clifford element.

B Form bilinears of D-branes

B.1 Form bilinears of IIA D-branes

B.1.1 D0-brane

Using the expression for the Killing spinors of the D0-brane (6.1), one finds that the non-vanishing from bilinears of the solution are

$$\tilde\sigma^{rs}=-2h^{-\frac14}\,\langle\eta^r,\eta^s\rangle\,,\qquad k^{rs}=2h^{-\frac14}\,\langle\eta^r,\eta^s\rangle\,e^0\,,\tag{B.1}$$

$$\tilde k^{rs}=-2h^{-\frac14}\langle\eta^r,\Gamma_{11}\eta^s\rangle\,e^5+2h^{-\frac14}\langle\eta^r,\Gamma_i\eta^s\rangle\,e^i\,,\tag{B.2}$$

$$\omega^{rs}=-2h^{-\frac14}\,\langle\eta^r,\Gamma_{11}\eta^s\rangle\,e^0\wedge e^5+2h^{-\frac14}\langle\eta^r,\Gamma_i\eta^s\rangle e^0\wedge e^i\,,\tag{B.3}$$

$$\tilde\omega^{rs}=-2h^{-\frac14}\,\langle\eta^r,\Gamma_i\Gamma_{11}\eta^s\rangle\,e^5\wedge e^i-h^{-\frac14}\langle\eta^r,\Gamma_{ij}\eta^s\rangle e^i\wedge e^j\,,\tag{B.4}$$

$$\pi^{rs}=2h^{-\frac14}\,\langle\eta^r,\Gamma_i\Gamma_{11}\eta^s\rangle\,e^0\wedge e^5\wedge e^i+h^{-\frac14}\langle\eta^r,\Gamma_{ij}\eta^s\rangle e^0\wedge e^i\wedge e^j\,,\tag{B.5}$$

$$\tilde\pi^{rs}=-h^{-\frac14}\langle\eta^r,\Gamma_{ij}\Gamma_{11}\eta^s\rangle\,e^5\wedge e^i\wedge e^j+\frac13 h^{-\frac14}\langle\eta^r,\Gamma_{ijk}\eta^s\rangle e^i\wedge e^j\wedge e^k\,,\tag{B.6}$$

$$\zeta^{rs}=-h^{-\frac14}\,\langle\eta^r,\Gamma_{ij}\Gamma_{11}\eta^s\rangle\,e^0\wedge e^5\wedge e^i\wedge e^j+\frac13 h^{-\frac14}\langle\eta^r,\Gamma_{ijk}\eta^s\rangle\,e^0\wedge e^i\wedge e^j\wedge e^k\,,\tag{B.7}$$

$$\tilde\zeta^{rs}=-\frac13 h^{-\frac14}\,\langle\eta^r,\Gamma_{ijk}\Gamma_{11}\eta^s\rangle\,e^5\wedge e^i\wedge e^j\wedge e^k-\frac{2}{4!}h^{-\frac14}\,\langle\eta^r,\Gamma_{i_1\dots i_4}\eta^s\rangle e^{i_1}\wedge\cdots\wedge e^{i_4}\,,\tag{B.8}$$

$$
\begin{aligned}
\tau^{rs}=\;&\frac13 h^{-\frac14}\,\langle\eta^r,\Gamma_{ijk}\Gamma_{11}\eta^s\rangle\,e^0\wedge e^5\wedge e^i\wedge e^j\wedge e^k\\
&+\frac{2}{4!}h^{-\frac14}\langle\eta^r,\Gamma_{i_1\dots i_4}\eta^s\rangle\,e^0\wedge e^{i_1}\wedge\cdots\wedge e^{i_4}\,,
\end{aligned}
\tag{B.9}
$$

where $i,j,k=1,2,3,4,6,7,8,9$ and (e^0,e^5,e^i) is a pseudo-orthonormal frame of the D0-brane metric (4.27) for $p=0$.

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B.1.2 D6-brane

Using the expression for the Killing spinors in (6.3), one can easily compute the non-vanishing form bilinears of D6-brane as follows

$$\sigma^{rs} = 2h^{-\frac{1}{4}}\mathrm{Re}\langle\eta^r,\eta^s\rangle_D\,, \quad k^{rs} = 2h^{-\frac{1}{4}}\mathrm{Re}\langle\eta^r,\Gamma_a\eta^s\rangle_D\, e^a\,, \tag{B.10}$$

$$\tilde{k}^{rs} = -2h^{-\frac{1}{4}}\mathrm{Re}\langle\eta^r,\Gamma_{11}\lambda^s\rangle_D\, e^4 + 2h^{-\frac{1}{4}}\mathrm{Im}\,\langle\eta^r,\eta^s\rangle_D\; e^5 - 2h^{-\frac{1}{4}}\mathrm{Im}\langle\eta^r,\Gamma_{11}\lambda^s\rangle_D\, e^9\,, \tag{B.11}$$

$$\begin{aligned}\omega^{rs} &= -2h^{-\frac{1}{4}}\mathrm{Re}\langle\eta^r,\Gamma_5\lambda^s\rangle_D e^4\wedge e^5 - 2h^{-\frac{1}{4}}\mathrm{Im}\langle\eta^r,\eta^s\rangle_D e^4\wedge e^9\\ &\quad + 2h^{-\frac{1}{4}}\mathrm{Im}\langle\eta^r,\Gamma_5\lambda^s\rangle_D e^5\wedge e^9 + h^{-\frac{1}{4}}\mathrm{Re}\langle\eta^r,\Gamma_{ab}\eta^s\rangle_D e^a\wedge e^b\,,\end{aligned} \tag{B.12}$$

$$\begin{aligned}\tilde{\omega}^{rs} &= -2h^{-\frac{1}{4}}\mathrm{Re}\langle\eta^r,\Gamma_a\Gamma_{11}\lambda^s\rangle_D e^a\wedge e^4 + 2h^{-\frac{1}{4}}\mathrm{Im}\langle\eta^r,\Gamma_a\eta^s\rangle_D e^a\wedge e^5\\ &\quad - 2h^{-\frac{1}{4}}\mathrm{Im}\langle\eta^r,\Gamma_a\Gamma_{11}\lambda^s\rangle_D e^a\wedge e^9\,,\end{aligned} \tag{B.13}$$

$$\begin{aligned}\pi^{rs} &= -2h^{-\frac{1}{4}}\mathrm{Re}\langle\eta^r,\Gamma_a\Gamma_5\lambda^s\rangle_D e^a\wedge e^4\wedge e^5 - 2h^{-\frac{1}{4}}\mathrm{Im}\langle\eta^r,\Gamma_a\eta^s\rangle_D e^a\wedge e^4\wedge e^9\\ &\quad + 2h^{-\frac{1}{4}}\mathrm{Im}\langle\eta^r,\Gamma_a\Gamma_5\lambda^s\rangle_D e^a\wedge e^5\wedge e^9 + \frac{1}{3}h^{-\frac{1}{4}}\mathrm{Re}\langle\eta^r,\Gamma_{abc}\eta^s\rangle_D e^a\wedge e^b\wedge e^c\,,\end{aligned} \tag{B.14}$$

$$\begin{aligned}\tilde{\pi}^{rs} &= -2h^{-\frac{1}{4}}\mathrm{Re}\langle\eta^r,\eta^s\rangle_D e^4\wedge e^5\wedge e^9 - h^{-\frac{1}{4}}\mathrm{Re}\langle\eta^r,\Gamma_{ab}\Gamma_{11}\lambda^s\rangle_D e^a\wedge e^b\wedge e^4\\ &\quad + h^{-\frac{1}{4}}\mathrm{Im}\langle\eta^r,\Gamma_{ab}\eta^s\rangle_D e^a\wedge e^b\wedge e^5 - h^{-\frac{1}{4}}\mathrm{Im}\langle\eta^r,\Gamma_{ab}\Gamma_{11}\lambda^s\rangle_D e^a\wedge e^b\wedge e^9\,,\end{aligned} \tag{B.15}$$

$$\begin{aligned}\zeta^{rs} &= -h^{-\frac{1}{4}}\mathrm{Re}\langle\eta^r,\Gamma_{ab}\Gamma_5\lambda^s\rangle_D e^a\wedge e^b\wedge e^4\wedge e^5 - h^{-\frac{1}{4}}\mathrm{Im}\langle\eta^r,\Gamma_{ab}\eta^s\rangle_D e^a\wedge e^b\wedge e^4\wedge e^9\\ &\quad + h^{-\frac{1}{4}}\mathrm{Im}\langle\eta^r,\Gamma_{ab}\Gamma_5\lambda^s\rangle_D e^a\wedge e^b\wedge e^5\wedge e^9\\ &\quad + \frac{1}{12}h^{-\frac{1}{4}}\mathrm{Re}\langle\eta^r,\Gamma_{abcd}\eta^s\rangle_D e^a\wedge e^b\wedge e^c\wedge e^d\,,\end{aligned} \tag{B.16}$$

$$\begin{aligned}\tilde{\zeta}^{rs} &= -2h^{-\frac{1}{4}}\mathrm{Re}\langle\eta^r,\Gamma_a\eta^s\rangle_D e^a\wedge e^4\wedge e^5\wedge e^9 - \frac{1}{3}h^{-\frac{1}{4}}\mathrm{Re}\langle\eta^r,\Gamma_{abc}\Gamma_{11}\lambda^s\rangle_D e^a\wedge e^b\wedge e^c\wedge e^4\\ &\quad + \frac{1}{3}h^{-\frac{1}{4}}\mathrm{Im}\langle\eta^r,\Gamma_{abc}\eta^s\rangle_D e^a\wedge e^b\wedge e^c\wedge e^5\\ &\quad - \frac{1}{3}h^{-\frac{1}{4}}\mathrm{Im}\langle\eta^r,\Gamma_{abc}\Gamma_{11}\lambda^s\rangle_D e^a\wedge e^b\wedge e^c\wedge e^9\,,\end{aligned} \tag{B.17}$$

$$\begin{aligned}\tau^{rs} &= -\frac{1}{3}h^{-\frac{1}{4}}\mathrm{Re}\langle\eta^r,\Gamma_{abc}\Gamma_5\lambda^s\rangle_D e^a\wedge e^b\wedge e^c\wedge e^4\wedge e^5\\ &\quad - \frac{1}{3}h^{-\frac{1}{4}}\mathrm{Im}\langle\eta^r,\Gamma_{abc}\eta^s\rangle_D e^a\wedge e^b\wedge e^c\wedge e^4\wedge e^9\\ &\quad + \frac{1}{3}h^{-\frac{1}{4}}\mathrm{Im}\langle\eta^r,\Gamma_{abc}\Gamma_5\lambda^s\rangle_D e^a\wedge e^b\wedge e^c\wedge e^5\wedge e^9\\ &\quad + \frac{2}{5!}h^{-\frac{1}{4}}\mathrm{Re}\langle\eta^r,\Gamma_{a_1\dots a_5}\eta^s\rangle_D e^{a_1}\wedge\cdots\wedge e^{a_5}\,,\end{aligned} \tag{B.18}$$

where $a,b,c = 0,1,2,3,6,7,8$ and (e^a,e^5,e^4,e^9) is a pseudo-orthonormal frame of the metric (4.27) with $p=6$.

B.1.3 D2-brane

Using the Killing spinors (6.20), one finds that the non-vanishing form bilinears of the D2-brane solution are as follows

$$\sigma^{rs} = h^{-\frac{1}{4}}(-\langle\eta^r,\lambda^s\rangle + \langle\lambda^r,\eta^s\rangle)\,, \quad \tilde{k}^{rs} = h^{-\frac{1}{4}}(-\langle\eta^r,\Gamma_i\Gamma_{11}\lambda^s\rangle + \langle\lambda^r,\Gamma_i\Gamma_{11}\eta^s\rangle)\,e^i\,, \tag{B.19}$$

$$\begin{aligned} k^{rs} &= h^{-\frac{1}{4}}(\langle\eta^r,\eta^s\rangle + \langle\lambda^r,\lambda^s\rangle)\,e^0 + h^{-\frac{1}{4}}(-\langle\eta^r,\eta^s\rangle + \langle\lambda^r,\lambda^s\rangle)\,e^5 \\ &\quad + h^{-\frac{1}{4}}(\langle\eta^r,\lambda^s\rangle + \langle\lambda^r,\eta^s\rangle)\,e^1\,, \end{aligned} \tag{B.20}$$

$$\begin{aligned} \omega^{rs} &= \frac{h^{-\frac{1}{4}}}{2}(-\langle\eta^r,\Gamma_{ij}\lambda^s\rangle + \langle\lambda^r,\Gamma_{ij}\eta^s\rangle)\,e^i\wedge e^j + h^{-\frac{1}{4}}(\langle\eta^r,\lambda^s\rangle + \langle\lambda^r,\eta^s\rangle)\,e^0\wedge e^5 \\ &\quad + h^{-\frac{1}{4}}(\langle\eta^r,\eta^s\rangle - \langle\lambda^r,\lambda^s\rangle)\,e^0\wedge e^1 + h^{-\frac{1}{4}}(\langle\eta^r,\eta^s\rangle + \langle\lambda^r,\lambda^s\rangle)\,e^1\wedge e^5\,, \end{aligned} \tag{B.21}$$

$$\begin{aligned} \tilde{\omega}^{rs} &= -h^{-\frac{1}{4}}(\langle\eta^r,\Gamma_i\Gamma_{11}\eta^s\rangle + \langle\lambda^r,\Gamma_i\Gamma_{11}\lambda^s\rangle)\,e^i\wedge e^0 + h^{-\frac{1}{4}}(\langle\eta^r,\Gamma_i\Gamma_{11}\eta^s\rangle \\ &\quad - \langle\lambda^r,\Gamma_i\Gamma_{11}\lambda^s\rangle)\,e^i\wedge e^5 - h^{-\frac{1}{4}}(\langle\eta^r,\Gamma_i\Gamma_{11}\lambda^s\rangle + \langle\lambda^r,\Gamma_i\Gamma_{11}\eta^s\rangle)\,e^i\wedge e^1\,, \end{aligned} \tag{B.22}$$

$$\begin{aligned} \pi^{rs} &= h^{-\frac{1}{4}}(-\langle\eta^r,\lambda^s\rangle + \langle\lambda^r,\eta^s\rangle)\,e^0\wedge e^5\wedge e^1 \\ &\quad + \frac{h^{-\frac{1}{4}}}{2}\Big((\langle\eta^r,\Gamma_{ij}\eta^s\rangle + \langle\lambda^r,\Gamma_{ij}\lambda^s\rangle)\,e^0 + (-\langle\eta^r,\Gamma_{ij}\eta^s\rangle + \langle\lambda^r,\Gamma_{ij}\lambda^s\rangle)\,e^5 \\ &\quad + (\langle\eta^r,\Gamma_{ij}\lambda^s\rangle + \langle\lambda^r,\Gamma_{ij}\eta^s\rangle)\,e^1\Big)\wedge e^i\wedge e^j\,, \end{aligned} \tag{B.23}$$

$$\begin{aligned} \tilde{\pi}^{rs} &= h^{-\frac{1}{4}}(\langle\lambda^r,\Gamma_i\Gamma_{11}\eta^s\rangle + \langle\eta^r,\Gamma_i\Gamma_{11}\lambda^s\rangle)\,e^0\wedge e^5\wedge e^i \\ &\quad + h^{-\frac{1}{4}}(\langle\eta^r,\Gamma_i\Gamma_{11}\eta^s\rangle - \langle\lambda^r,\Gamma_i\Gamma_{11}\lambda^s\rangle)\,e^0\wedge e^1\wedge e^i \\ &\quad - h^{-\frac{1}{4}}(\langle\eta^r,\Gamma_i\Gamma_{11}\eta^s\rangle + \langle\lambda^r,\Gamma_i\Gamma_{11}\lambda^s\rangle)\,e^5\wedge e^1\wedge e^i \\ &\quad + \frac{h^{-\frac{1}{4}}}{3!}(-\langle\eta^r,\Gamma_{ijk}\Gamma_{11}\lambda^s\rangle + \langle\lambda^r,\Gamma_{ijk}\Gamma_{11}\eta^s\rangle)\,e^i\wedge e^j\wedge e^k\,, \end{aligned} \tag{B.24}$$

$$\begin{aligned} \zeta^{rs} &= \frac{h^{-\frac{1}{4}}}{4!}(-\langle\eta^r,\Gamma_{i_1\dots i_4}\lambda^s\rangle + \langle\lambda^r,\Gamma_{i_1\dots i_4}\eta^s\rangle)\,e^{i_1}\wedge\cdots\wedge e^{i_4} \\ &\quad + \frac{h^{-\frac{1}{4}}}{2}\Big((\langle\eta^r,\Gamma_{ij}\lambda^s\rangle + \langle\lambda^r,\Gamma_{ij}\eta^s\rangle)\,e^0\wedge e^5 + (\langle\eta^r,\Gamma_{ij}\eta^s\rangle - \langle\lambda^r,\Gamma_{ij}\lambda^s\rangle)\,e^0\wedge e^1 \\ &\quad + (\langle\eta^r,\Gamma_{ij}\eta^s\rangle + \langle\lambda^r,\Gamma_{ij}\lambda^s\rangle)\,e^1\wedge e^5\Big)\wedge e^i\wedge e^j\,, \end{aligned} \tag{B.25}$$

$$\begin{aligned} \tilde{\zeta}^{rs} &= \frac{h^{-\frac{1}{4}}}{3!}\Big(-(\langle\eta^r,\Gamma_{ijk}\Gamma_{11}\eta^s\rangle + \langle\lambda^r,\Gamma_{ijk}\Gamma_{11}\lambda^s\rangle)\,e^i\wedge e^j\wedge e^k\wedge e^0 \\ &\quad + (\langle\eta^r,\Gamma_{ijk}\Gamma_{11}\eta^s\rangle - \langle\lambda^r,\Gamma_{ijk}\Gamma_{11}\lambda^s\rangle)\,e^i\wedge e^j\wedge e^k\wedge e^5 \\ &\quad - (\langle\eta^r,\Gamma_{ijk}\Gamma_{11}\lambda^s\rangle + \langle\lambda^r,\Gamma_{ijk}\Gamma_{11}\eta^s\rangle)\Big)\,e^i\wedge e^j\wedge e^k\wedge e^1 \\ &\quad + h^{-\frac{1}{4}}(-\langle\eta^r,\Gamma_i\Gamma_{11}\lambda^s\rangle + \langle\lambda^r,\Gamma_i\Gamma_{11}\eta^s\rangle)\,e^0\wedge e^5\wedge e^1\wedge e^i\,, \end{aligned} \tag{B.26}$$

$$\begin{aligned} \tau^{rs} &= \frac{h^{-\frac{1}{4}}}{2}(-\langle\eta^r,\Gamma_{ij}\lambda^s\rangle + \langle\lambda^r,\Gamma_{ij}\eta^s\rangle)\,e^0\wedge e^5\wedge e^1\wedge e^i\wedge e^j \\ &\quad + \frac{h^{-\frac{1}{4}}}{4!}\Big((\langle\eta^r,\Gamma_{i_1\dots i_4}\eta^s\rangle + \langle\lambda^r,\Gamma_{i_1\dots i_4}\lambda^s\rangle)\,e^0 \\ &\quad + (-\langle\eta^r,\Gamma_{i_1\dots i_4}\eta^s\rangle + \langle\lambda^r,\Gamma_{i_1\dots i_4}\lambda^s\rangle)\,e^5 \\ &\quad + (\langle\eta^r,\Gamma_{i_1\dots i_4}\lambda^s\rangle + \langle\lambda^r,\Gamma_{i_1\dots i_4}\eta^s\rangle)\,e^1\Big)\wedge e^{i_1}\wedge\cdots\wedge e^{i_4}\,, \end{aligned} \tag{B.27}$$

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where $i,j,k = 2,3,4,6,7,8,9$ and (e^0, e^5, e^1, e^i) is a pseudo-orthonormal frame of the metric (4.27) with $p=2$.

B.1.4 D4-brane

Using the Killing spinors (6.33), one finds that the non-vanishing form bilinears of the D4-brane solution are as follows

$$\tilde{\sigma}^{rs} = 2h^{-\frac{1}{4}}\mathrm{Re}\langle\eta^{1r},\Gamma_{11}\eta^{1s}\rangle_D + 2h^{-\frac{1}{4}}\mathrm{Re}\langle\lambda^{1r},\Gamma_{11}\lambda^{1s}\rangle_D\,, \tag{B.28}$$

$$k^{rs} = 2h^{-\frac{1}{4}}(\mathrm{Re}\langle\eta^{1r},\Gamma_a\eta^{1s}\rangle_D + \mathrm{Re}\langle\lambda^{1r},\Gamma_a\lambda^{1s}\rangle_D)\,e^a\,, \tag{B.29}$$

$$\begin{aligned}\tilde{k}^{rs} = {} & 2h^{-\frac{1}{4}}(\mathrm{Re}\langle\eta^{1r},\Gamma_{11}\eta^{1s}\rangle_D - \mathrm{Re}\langle\lambda^{1r},\Gamma_{11}\lambda^{1s}\rangle_D)\,e^2\\
&+2h^{-\frac{1}{4}}(-\mathrm{Re}\langle\eta^{1r},\Gamma_{11}\lambda^{2s}\rangle_D + \mathrm{Re}\langle\lambda^{1r},\Gamma_{11}\eta^{2s}\rangle_D)\,e^4\\
&-2h^{-\frac{1}{4}}(\mathrm{Re}\langle\eta^{1r},\Gamma_{11}\lambda^{1s}\rangle_D + \mathrm{Re}\langle\lambda^{1r},\Gamma_{11}\eta^{1s}\rangle_D)\,e^3\\
&+2h^{-\frac{1}{4}}(-\mathrm{Im}\langle\eta^{1r},\Gamma_{11}\lambda^{1s}\rangle_D + \mathrm{Im}\langle\lambda^{1r},\Gamma_{11}\eta^{1s}\rangle_D)\,e^8\\
&+2h^{-\frac{1}{4}}(-\mathrm{Im}\langle\eta^{1r},\Gamma_{11}\lambda^{2s}\rangle_D + \mathrm{Im}\langle\lambda^{1r},\Gamma_{11}\eta^{2s}\rangle_D)\,e^9\,,\end{aligned} \tag{B.30}$$

$$\begin{aligned}\omega^{rs} = {} & 2h^{-\frac{1}{4}}(-\mathrm{Re}\langle\eta^{1r},\Gamma_a\eta^{1s}\rangle_D + \mathrm{Re}\langle\lambda^{1r},\Gamma_a\lambda^{1s}\rangle_D)\,e^a\wedge e^2\\
&+2h^{-\frac{1}{4}}(\mathrm{Re}\langle\eta^{1r},\Gamma_a\lambda^{1s}\rangle_D + \mathrm{Re}\langle\lambda^{1r},\Gamma_a\eta^{1s}\rangle_D)\,e^a\wedge e^3\\
&+2h^{-\frac{1}{4}}(\mathrm{Re}\langle\eta^{1r},\Gamma_a\lambda^{2s}\rangle_D - \mathrm{Re}\langle\lambda^{1r},\Gamma_a\eta^{2s}\rangle_D)\,e^a\wedge e^4\\
&+2h^{-\frac{1}{4}}(\mathrm{Im}\langle\eta^{1r},\Gamma_a\lambda^{1s}\rangle_D - \mathrm{Im}\langle\lambda^{1r},\Gamma_a\eta^{1s}\rangle_D)\,e^a\wedge e^8\\
&+2h^{-\frac{1}{4}}(\mathrm{Im}\langle\eta^{1r},\Gamma_a\lambda^{2s}\rangle_D - \mathrm{Im}\langle\lambda^{1r},\Gamma_a\eta^{2s}\rangle_D)\,e^a\wedge e^9\,,\end{aligned} \tag{B.31}$$

$$\begin{aligned}\tilde{\omega}^{rs} = {} & h^{-\frac{1}{4}}(\mathrm{Re}\langle\eta^{1r},\Gamma_{ab}\Gamma_{11}\eta^{1s}\rangle_D + \mathrm{Re}\langle\lambda^{1r},\Gamma_{ab}\Gamma_{11}\lambda^{1s}\rangle_D)\,e^a\wedge e^b\\
&-2h^{-\frac{1}{4}}(\mathrm{Re}\langle\eta^{1r},\Gamma_{11}\lambda^{1s}\rangle_D - \mathrm{Re}\langle\lambda^{1r},\Gamma_{11}\eta^{1s}\rangle_D)\,e^2\wedge e^3\\
&+2h^{-\frac{1}{4}}(-\mathrm{Re}\langle\eta^{1r},\Gamma_{11}\lambda^{2s}\rangle_D - \mathrm{Re}\langle\lambda^{1r},\Gamma_{11}\eta^{2s}\rangle_D)\,e^2\wedge e^4\\
&-2h^{-\frac{1}{4}}(\mathrm{Im}\langle\eta^{1r},\Gamma_{11}\lambda^{1s}\rangle_D + \mathrm{Im}\langle\lambda^{1r},\Gamma_{11}\eta^{1s}\rangle_D)\,e^2\wedge e^8\\
&-2h^{-\frac{1}{4}}(\mathrm{Im}\langle\eta^{1r},\Gamma_{11}\lambda^{2s}\rangle_D + \mathrm{Im}\langle\lambda^{1r},\Gamma_{11}\eta^{2s}\rangle_D)\,e^2\wedge e^9\\
&+2h^{-\frac{1}{4}}(-\mathrm{Re}\langle\eta^{1r},\Gamma_{11}\eta^{2s}\rangle_D + \mathrm{Re}\langle\lambda^{1r},\Gamma_{11}\lambda^{2s}\rangle_D)\,e^3\wedge e^4\\
&+2h^{-\frac{1}{4}}(-\mathrm{Im}\langle\eta^{1r},\Gamma_{11}\eta^{1s}\rangle_D + \mathrm{Im}\langle\lambda^{1r},\Gamma_{11}\lambda^{1s}\rangle_D)\,e^3\wedge e^8\\
&+2h^{-\frac{1}{4}}(-\mathrm{Im}\langle\eta^{1r},\Gamma_{11}\eta^{2s}\rangle_D + \mathrm{Im}\langle\lambda^{1r},\Gamma_{11}\lambda^{2s}\rangle_D)\,e^3\wedge e^9\\
&+2h^{-\frac{1}{4}}(\mathrm{Im}\langle\eta^{1r},\Gamma_{11}\eta^{2s}\rangle_D + \mathrm{Im}\langle\lambda^{1r},\Gamma_{11}\lambda^{2s}\rangle_D)\,e^4\wedge e^8\\
&-2h^{-\frac{1}{4}}(\mathrm{Im}\langle\eta^{1r},\Gamma_{11}\eta^{1s}\rangle_D + \mathrm{Im}\langle\lambda^{1r},\Gamma_{11}\lambda^{1s}\rangle_D)\,e^4\wedge e^9\\
&+2h^{-\frac{1}{4}}(\mathrm{Re}\langle\eta^{1r},\Gamma_{11}\eta^{2s}\rangle_D + \mathrm{Re}\langle\lambda^{1r},\Gamma_{11}\lambda^{2s}\rangle_D)\,e^8\wedge e^9\,,\end{aligned} \tag{B.32}$$

$$\begin{aligned}\pi^{rs} = {} & \frac{h^{-\frac{1}{4}}}{3}(\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\eta^{1s}\rangle_D + \mathrm{Re}\langle\lambda^{1r},\Gamma_{abc}\lambda^{1s}\rangle_D)\,e^a\wedge e^b\wedge e^c\\
&-2h^{-\frac{1}{4}}(\mathrm{Re}\langle\eta^{1r},\Gamma_a\lambda^{1s}\rangle_D - \mathrm{Re}\langle\lambda^{1r},\Gamma_a\eta^{1s}\rangle_D)\,e^2\wedge e^3\wedge e^a\\
&+2h^{-\frac{1}{4}}(-\mathrm{Re}\langle\eta^{1r},\Gamma_a\lambda^{2s}\rangle_D - \mathrm{Re}\langle\lambda^{1r},\Gamma_a\eta^{2s}\rangle_D)\,e^2\wedge e^4\wedge e^a\\
&-2h^{-\frac{1}{4}}(\mathrm{Im}\langle\eta^{1r},\Gamma_a\lambda^{1s}\rangle_D + \mathrm{Im}\langle\lambda^{1r},\Gamma_a\eta^{1s}\rangle_D)\,e^2\wedge e^8\wedge e^a\\
&-2h^{-\frac{1}{4}}(\mathrm{Im}\langle\eta^{1r},\Gamma_a\lambda^{2s}\rangle_D + \mathrm{Im}\langle\lambda^{1r},\Gamma_a\eta^{2s}\rangle_D)\,e^2\wedge e^9\wedge e^a\\
&+2h^{-\frac{1}{4}}(-\mathrm{Re}\langle\eta^{1r},\Gamma_a\eta^{2s}\rangle_D + \mathrm{Re}\langle\lambda^{1r},\Gamma_a\lambda^{2s}\rangle_D)\,e^3\wedge e^4\wedge e^a\\
&+2h^{-\frac{1}{4}}(-\mathrm{Im}\langle\eta^{1r},\Gamma_a\eta^{1s}\rangle_D + \mathrm{Im}\langle\lambda^{1r},\Gamma_a\lambda^{1s}\rangle_D)\,e^3\wedge e^8\wedge e^a\end{aligned}$$

$$
\begin{aligned}
&+2h^{-\frac{1}{4}}(-\mathrm{Im}\langle\eta^{1r},\Gamma_a\eta^{2s}\rangle_D+\mathrm{Im}\langle\lambda^{1r},\Gamma_a\lambda^{2s}\rangle_D)\,e^3\wedge e^9\wedge e^a\\
&+2h^{-\frac{1}{4}}(\mathrm{Im}\langle\eta^{1r},\Gamma_a\eta^{2s}\rangle_D+\mathrm{Im}\langle\lambda^{1r},\Gamma_a\lambda^{2s}\rangle_D)\,e^4\wedge e^8\wedge e^a\\
&-2h^{-\frac{1}{4}}(\mathrm{Im}\langle\eta^{1r},\Gamma_a\eta^{1s}\rangle_D+\mathrm{Im}\langle\lambda^{1r},\Gamma_a\lambda^{1s}\rangle_D)\,e^4\wedge e^9\wedge e^a\\
&+2h^{-\frac{1}{4}}(\mathrm{Re}\langle\eta^{1r},\Gamma_a\eta^{2s}\rangle_D+\mathrm{Re}\langle\lambda^{1r},\Gamma_a\lambda^{2s}\rangle_D)\,e^8\wedge e^9\wedge e^a \qquad \text{(B.33)}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\pi}^{rs} = {}& h^{-\frac{1}{4}}\Big((\mathrm{Re}\langle\eta^{1r},\Gamma_{ab}\Gamma_{11}\eta^{1s}\rangle_D-\mathrm{Re}\langle\lambda^{1r},\Gamma_{ab}\Gamma_{11}\lambda^{1s}\rangle_D)\,e^2\\
&+(-\mathrm{Re}\langle\eta^{1r},\Gamma_{ab}\Gamma_{11}\lambda^{2s}\rangle_D+\mathrm{Re}\langle\lambda^{1r},\Gamma_{ab}\Gamma_{11}\eta^{2s}\rangle_D)\,e^4\\
&-(\mathrm{Re}\langle\eta^{1r},\Gamma_{ab}\Gamma_{11}\lambda^{1s}\rangle_D+\mathrm{Re}\langle\lambda^{1r},\Gamma_{ab}\Gamma_{11}\eta^{1s}\rangle_D)\,e^3\\
&+(-\mathrm{Im}\langle\eta^{1r},\Gamma_{ab}\Gamma_{11}\lambda^{1s}\rangle_D+\mathrm{Im}\langle\lambda^{1r},\Gamma_{ab}\Gamma_{11}\eta^{1s}\rangle_D)\,e^8\\
&+(-\mathrm{Im}\langle\eta^{1r},\Gamma_{ab}\Gamma_{11}\lambda^{2s}\rangle_D+\mathrm{Im}\langle\lambda^{1r},\Gamma_{ab}\Gamma_{11}\eta^{2s}\rangle_D)\,e^9\Big)\wedge e^a\wedge e^b\\
&+\frac{1}{2\cdot 3!}\epsilon_{ijk}{}^{mn}\tilde{\omega}^{rs}_{mn}\,e^i\wedge e^i\wedge e^k\,, \qquad \text{(B.34)}
\end{aligned}
$$

$$
\begin{aligned}
\zeta^{rs} = {}& \frac{h^{-\frac{1}{4}}}{3}(-\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\eta^{1s}\rangle_D+\mathrm{Re}\langle\lambda^{1r},\Gamma_{abc}\lambda^{1s}\rangle_D)\,e^a\wedge e^b\wedge e^c\wedge e^2\\
&+\frac{h^{-\frac{1}{4}}}{3}(\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\lambda^{1s}\rangle_D+\mathrm{Re}\langle\lambda^{1r},\Gamma_{abc}\eta^{1s}\rangle_D)\,e^a\wedge e^b\wedge e^c\wedge e^3\\
&+\frac{h^{-\frac{1}{4}}}{3}(\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\lambda^{2s}\rangle_D-\mathrm{Re}\langle\lambda^{1r},\Gamma_{abc}\eta^{2s}\rangle_D)\,e^a\wedge e^b\wedge e^c\wedge e^4\\
&+\frac{h^{-\frac{1}{4}}}{3}(\mathrm{Im}\langle\eta^{1r},\Gamma_{abc}\lambda^{1s}\rangle_D-\mathrm{Im}\langle\lambda^{1r},\Gamma_{abc}\eta^{1s}\rangle_D)\,e^a\wedge e^b\wedge e^c\wedge e^8\\
&+\frac{h^{-\frac{1}{4}}}{3}(\mathrm{Im}\langle\eta^{1r},\Gamma_{abc}\lambda^{2s}\rangle_D-\mathrm{Im}\langle\lambda^{1r},\Gamma_{abc}\eta^{2s}\rangle_D)\,e^a\wedge e^b\wedge e^c\wedge e^9\\
&-\frac{1}{12}\pi^{rs}_{amn}\epsilon_{ijk}{}^{mn}e^a\wedge e^i\wedge e^j\wedge e^k\,, \qquad \text{(B.35)}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\zeta}^{rs} = {}& \frac{2h^{-\frac{1}{4}}}{4!}(\mathrm{Re}\langle\eta^{1r},\Gamma_{a_1\dots a_4}\Gamma_{11}\eta^{1s}\rangle_D+\mathrm{Re}\langle\lambda^{1r},\Gamma_{a_1\dots a_4}\Gamma_{11}\lambda^{1s}\rangle_D)\,e^{a_1}\wedge\dots\wedge e^{a_4}\\
&-h^{-\frac{1}{4}}(\mathrm{Re}\langle\eta^{1r},\Gamma_{ab}\Gamma_{11}\lambda^{1s}\rangle_D-\mathrm{Re}\langle\lambda^{1r},\Gamma_{ab}\Gamma_{11}\eta^{1s}\rangle_D)\,e^a\wedge e^b\wedge e^2\wedge e^3\\
&+h^{-\frac{1}{4}}(-\mathrm{Re}\langle\eta^{1r},\Gamma_{ab}\Gamma_{11}\lambda^{2s}\rangle_D-\mathrm{Re}\langle\lambda^{1r},\Gamma_{ab}\Gamma_{11}\eta^{2s}\rangle_D)\,e^a\wedge e^b\wedge e^2\wedge e^4\\
&-h^{-\frac{1}{4}}(\mathrm{Im}\langle\eta^{1r},\Gamma_{ab}\Gamma_{11}\lambda^{1s}\rangle_D+\mathrm{Im}\langle\lambda^{1r},\Gamma_{ab}\Gamma_{11}\eta^{1s}\rangle_D)\,e^a\wedge e^b\wedge e^2\wedge e^8\\
&-h^{-\frac{1}{4}}(\mathrm{Im}\langle\eta^{1r},\Gamma_{ab}\Gamma_{11}\lambda^{2s}\rangle_D+\mathrm{Im}\langle\lambda^{1r},\Gamma_{ab}\Gamma_{11}\eta^{2s}\rangle_D)\,e^a\wedge e^b\wedge e^2\wedge e^9\\
&+h^{-\frac{1}{4}}(-\mathrm{Re}\langle\eta^{1r},\Gamma_{ab}\Gamma_{11}\eta^{2s}\rangle_D+\mathrm{Re}\langle\lambda^{1r},\Gamma_{ab}\Gamma_{11}\lambda^{2s}\rangle_D)\,e^a\wedge e^b\wedge e^3\wedge e^4\\
&+h^{-\frac{1}{4}}(-\mathrm{Im}\langle\eta^{1r},\Gamma_{ab}\Gamma_{11}\eta^{1s}\rangle_D+\mathrm{Im}\langle\lambda^{1r},\Gamma_{ab}\Gamma_{11}\lambda^{1s}\rangle_D)\,e^a\wedge e^b\wedge e^3\wedge e^8\\
&+h^{-\frac{1}{4}}(-\mathrm{Im}\langle\eta^{1r},\Gamma_{ab}\Gamma_{11}\eta^{2s}\rangle_D+\mathrm{Im}\langle\lambda^{1r},\Gamma_{ab}\Gamma_{11}\lambda^{2s}\rangle_D)\,e^a\wedge e^b\wedge e^3\wedge e^9\\
&+h^{-\frac{1}{4}}(\mathrm{Im}\langle\eta^{1r},\Gamma_{ab}\Gamma_{11}\eta^{2s}\rangle_D+\mathrm{Im}\langle\lambda^{1r},\Gamma_{ab}\Gamma_{11}\lambda^{2s}\rangle_D)\,e^a\wedge e^b\wedge e^4\wedge e^8\\
&-h^{-\frac{1}{4}}(\mathrm{Im}\langle\eta^{1r},\Gamma_{ab}\Gamma_{11}\eta^{1s}\rangle_D+\mathrm{Im}\langle\lambda^{1r},\Gamma_{ab}\Gamma_{11}\lambda^{1s}\rangle_D)\,e^a\wedge e^b\wedge e^4\wedge e^9\\
&+h^{-\frac{1}{4}}(\mathrm{Re}\langle\eta^{1r},\Gamma_{ab}\Gamma_{11}\eta^{2s}\rangle_D+\mathrm{Re}\langle\lambda^{1r},\Gamma_{ab}\Gamma_{11}\lambda^{2s}\rangle_D)\,e^a\wedge e^b\wedge e^8\wedge e^9\\
&-\frac{1}{4!}\epsilon_{i_1\dots i_4}{}^j\tilde{k}^{rs}_j\,e^{i_1}\wedge\dots\wedge e^{i_4}\,, \qquad \text{(B.36)}
\end{aligned}
$$

$$
\begin{aligned}
\tau^{rs} =& \frac{2h^{-\frac{1}{4}}}{5!}(\mathrm{Re}\langle\eta^{1r},\Gamma_{a_1\dots a_5}\eta^{1s}\rangle_D+\mathrm{Re}\langle\lambda^{1r},\Gamma_{a_1\dots a_5}\lambda^{1s}\rangle_D)\, e^{a_1}\wedge\cdots\wedge e^{a_5}\\
&-\frac{h^{-\frac{1}{4}}}{3}(\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\lambda^{1s}\rangle_D-\mathrm{Re}\langle\lambda^{1r},\Gamma_{abc}\eta^{1s}\rangle_D)\, e^2\wedge e^3\wedge e^a\wedge e^b\wedge e^c\\
&+\frac{h^{-\frac{1}{4}}}{3}(-\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\lambda^{2s}\rangle_D-\mathrm{Re}\langle\lambda^{1r},\Gamma_{abc}\eta^{2s}\rangle_D)\, e^2\wedge e^4\wedge e^a\wedge e^b\wedge e^c\\
&-\frac{h^{-\frac{1}{4}}}{3}(\mathrm{Im}\langle\eta^{1r},\Gamma_{abc}\lambda^{1s}\rangle_D+\mathrm{Im}\langle\lambda^{1r},\Gamma_{abc}\eta^{1s}\rangle_D)\, e^2\wedge e^8\wedge e^a\wedge e^b\wedge e^c\\
&-\frac{h^{-\frac{1}{4}}}{3}(\mathrm{Im}\langle\eta^{1r},\Gamma_{abc}\lambda^{2s}\rangle_D+\mathrm{Im}\langle\lambda^{1r},\Gamma_{abc}\eta^{2s}\rangle_D)\, e^2\wedge e^9\wedge e^a\wedge e^b\wedge e^c\\
&+\frac{h^{-\frac{1}{4}}}{3}(-\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\eta^{2s}\rangle_D+\mathrm{Re}\langle\lambda^{1r},\Gamma_{abc}\lambda^{2s}\rangle_D)\, e^3\wedge e^4\wedge e^a\wedge e^b\wedge e^c\\
&+\frac{h^{-\frac{1}{4}}}{3}(-\mathrm{Im}\langle\eta^{1r},\Gamma_{abc}\eta^{1s}\rangle_D+\mathrm{Im}\langle\lambda^{1r},\Gamma_{abc}\lambda^{1s}\rangle_D)\, e^3\wedge e^8\wedge e^a\wedge e^b\wedge e^c\\
&+\frac{h^{-\frac{1}{4}}}{3}(-\mathrm{Im}\langle\eta^{1r},\Gamma_{abc}\eta^{2s}\rangle_D+\mathrm{Im}\langle\lambda^{1r},\Gamma_{abc}\lambda^{2s}\rangle_D)\, e^3\wedge e^9\wedge e^a\wedge e^b\wedge e^c\\
&+\frac{h^{-\frac{1}{4}}}{3}(\mathrm{Im}\langle\eta^{1r},\Gamma_{abc}\eta^{2s}\rangle_D+\mathrm{Im}\langle\lambda^{1r},\Gamma_{abc}\lambda^{2s}\rangle_D)\, e^4\wedge e^8\wedge e^a\wedge e^b\wedge e^c\\
&-\frac{h^{-\frac{1}{4}}}{3}(\mathrm{Im}\langle\eta^{1r},\Gamma_{abc}\eta^{1s}\rangle_D+\mathrm{Im}\langle\lambda^{1r},\Gamma_{abc}\lambda^{1s}\rangle_D)\, e^4\wedge e^9\wedge e^a\wedge e^b\wedge e^c\\
&+\frac{h^{-\frac{1}{4}}}{3}(\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\eta^{2s}\rangle_D+\mathrm{Re}\langle\lambda^{1r},\Gamma_{abc}\lambda^{2s}\rangle_D)\, e^8\wedge e^9\wedge e^a\wedge e^b\wedge e^c\\
&+\frac{1}{4!}\epsilon_{i_1\dots i_4}{}^{j}\omega_{aj}e^a\wedge e^{i_1}\wedge\cdots\wedge e^{i_4}\,,
\end{aligned}
\tag{B.37}
$$

where $\epsilon_{23849}=1$, $a,b,c=0,5,1,6,7$, $i,j,k=2,3,4,8,9$ and (e^a,e^i) is a pseudo-orthonormal frame of the metric (4.27) for $p=4$.

B.1.5 D8-brane

Using the Killing spinors (6.39), the non-vanishing form bilinears of D8-brane are as follows

$$
\begin{aligned}
\tilde{\sigma}^{rs} &= 2h^{-\frac{1}{4}}\langle\eta^r,\Gamma_{11}\eta^s\rangle\,, \qquad k^{rs}=2h^{-\frac{1}{4}}\langle\eta^r,\eta^s\rangle\, e^0+2h^{-\frac{1}{4}}\langle\eta^r,\Gamma_{a'}\eta^s\rangle\, e^{a'}\,,\\
\tilde{k}^{rs} &= -2h^{-\frac{1}{4}}\langle\eta^r,\Gamma_{11}\eta^s\rangle\, e^5\,, \quad \omega^{rs}=2h^{-\frac{1}{4}}\langle\eta^r,\eta^s\rangle\ e^0\wedge e^5+2h^{-\frac{1}{4}}\langle\eta^r,\Gamma_{a'}\eta^s\rangle\ e^{a'}\wedge e^5\,,\\
\tilde{\omega}^{rs} &= 2h^{-\frac{1}{4}}\langle\eta^r,\Gamma_{a'}\Gamma_{11}\eta^s\rangle\ e^0\wedge e^{a'}+h^{-\frac{1}{4}}\langle\eta^r,\Gamma_{a'b'}\Gamma_{11}\eta^s\rangle\ e^{a'}\wedge e^{b'}\,,\\
\pi^{rs} &= h^{-\frac{1}{4}}\langle\eta^r,\Gamma_{b'c'}\eta^s\rangle\ e^0\wedge e^{b'}\wedge e^{c'}+\frac{h^{-\frac{1}{4}}}{3}\langle\eta^r,\Gamma_{a'b'c'}\eta^s\rangle\ e^{a'}\wedge e^{b'}\wedge e^{c'}\,,\\
\tilde{\pi}^{rs} &= -2h^{-\frac{1}{4}}\langle\eta^r,\Gamma_{a'}\Gamma_{11}\eta^s\rangle\ e^0\wedge e^{a'}\wedge e^5-h^{-\frac{1}{4}}\langle\eta^r,\Gamma_{ab}\Gamma_{11}\eta^s\rangle\ e^a\wedge e^b\wedge e^5\,,\\
\zeta^{rs} &= h^{-\frac{1}{4}}\langle\eta^r,\Gamma_{b'c'}\eta^s\rangle\ e^0\wedge e^{b'}\wedge e^{c'}\wedge e^5+\frac{h^{-\frac{1}{4}}}{3}\langle\eta^r,\Gamma_{a'b'c'}\eta^s\rangle\ e^{a'}\wedge e^{b'}\wedge e^{c'}\wedge e^5\,,\\
\tilde{\zeta}^{rs} &= \frac{h^{-\frac{1}{4}}}{3}\langle\eta^r,\Gamma_{a'b'c'}\Gamma_{11}\eta^s\rangle\ e^0\wedge e^{a'}\wedge e^{b'}\wedge e^{c'}+\frac{h^{-\frac{1}{4}}}{12}\langle\eta^r,\Gamma_{a'_1\dots a'_4}\Gamma_{11}\eta^s\rangle\ e^{a'_1}\wedge\cdots\wedge e^{a'_4}\,,\\
\tau^{rs} &= \frac{h^{-\frac{1}{4}}}{12}\langle\eta^r,\Gamma_{a'_1\dots a'_4}\eta^s\rangle\ e^0\wedge e^{a'_1}\wedge\cdots\wedge e^{a'_4}+\frac{2h^{-\frac{1}{4}}}{5!}\langle\eta^r,\Gamma_{a'_1\dots a'_5}\eta^s\rangle\ e^{a'_1}\wedge\cdots\wedge e^{a'_5}\,,
\end{aligned}
\tag{B.38}
$$

where $a', b', c' = 1, 6, 2, 7, 3, 8, 4, 9$ and (e^a, e^5) is a pseudo-orthonormal frame of the D8-brane metric (4.27) for $p = 8$.

B.2 Form bilinears of IIB D-branes

As for the common sector IIB branes, all the bilinears below are manifestly real. In particular, $k^{(2)}, \pi^{(2)}$ and $\tau^{(2)}$ have been replaced with $ik^{(2)}, i\pi^{(2)}$ and $i\tau^{(2)}$, respectively.

B.2.1 D-string

Using the Killing spinors (7.3), one can easily compute the form bilinears of the D-string background to find

$$k^{rs} = 2h^{-\frac{1}{4}}(\langle\eta^r,\eta^s\rangle + \langle\lambda^r,\lambda^s\rangle)e^0 + 2h^{-\frac{1}{4}}(-\langle\eta^r,\eta^s\rangle + \langle\lambda^r,\lambda^s\rangle)e^5\,, \tag{B.39}$$

$$k^{(3)rs} = 2h^{-\frac{1}{4}}(\langle\eta^r,\Gamma_i\lambda^s\rangle + \langle\lambda^r,\Gamma_i\eta^s\rangle)e^i\,, \tag{B.40}$$

$$k^{(1)rs} = 2h^{-\frac{1}{4}}(\langle\eta^r,\eta^s\rangle - \langle\lambda^r,\lambda^s\rangle)e^0 - 2h^{-\frac{1}{4}}(\langle\eta^r,\eta^s\rangle + \langle\lambda^r,\lambda^s\rangle)e^5\,, \tag{B.41}$$

$$k^{(2)rs} = 2h^{-\frac{1}{4}}(-\langle\eta^r,\Gamma_i\lambda^s\rangle + \langle\lambda^r,\Gamma_i\eta^s\rangle)e^i\,, \tag{B.42}$$

$$\begin{aligned}\pi^{rs} &= h^{-\frac{1}{4}}(\langle\eta^r,\Gamma_{ij}\eta^s\rangle + \langle\lambda^r,\Gamma_{ij}\lambda^s\rangle)e^0\wedge e^i\wedge e^j\\ &\quad + h^{-\frac{1}{4}}(-\langle\eta^r,\Gamma_{ij}\eta^s\rangle + \langle\lambda^r,\Gamma_{ij}\lambda^s\rangle)e^5\wedge e^i\wedge e^j\,,\end{aligned} \tag{B.43}$$

$$\begin{aligned}\pi^{(3)rs} &= 2h^{-\frac{1}{4}}(-\langle\eta^r,\Gamma_i\lambda^s\rangle + \langle\lambda^r,\Gamma_i\eta^s\rangle)e^0\wedge e^5\wedge e^i\\ &\quad + \frac{1}{3}h^{-\frac{1}{4}}(\langle\eta^r,\Gamma_{ijk}\lambda^s\rangle + \langle\lambda^r,\Gamma_{ijk}\eta^s\rangle)e^i\wedge e^j\wedge e^k\,,\end{aligned} \tag{B.44}$$

$$\begin{aligned}\pi^{(1)rs} &= h^{-\frac{1}{4}}(\langle\eta^r,\Gamma_{ij}\eta^s\rangle - \langle\lambda^r,\Gamma_{ij}\lambda^s\rangle)e^0\wedge e^i\wedge e^j\\ &\quad - h^{-\frac{1}{4}}(\langle\eta^r,\Gamma_{ij}\eta^s\rangle + \langle\lambda^r,\Gamma_{ij}\lambda^s\rangle)e^5\wedge e^i\wedge e^j\,,\end{aligned} \tag{B.45}$$

$$\begin{aligned}\pi^{(2)rs} &= 2h^{-\frac{1}{4}}(\langle\eta^r,\Gamma_i\lambda^s\rangle + \langle\lambda^r,\Gamma_i\eta^s\rangle)e^0\wedge e^5\wedge e^i\\ &\quad + \frac{1}{3}h^{-\frac{1}{4}}(-\langle\eta^r,\Gamma_{ijk}\lambda^s\rangle + \langle\lambda^r,\Gamma_{ijk}\eta^s\rangle)e^i\wedge e^j\wedge e^k\,,\end{aligned} \tag{B.46}$$

$$\begin{aligned}\tau^{rs} &= \frac{2}{4!}h^{-\frac{1}{4}}(\langle\eta^r,\Gamma_{i_1\dots i_4}\eta^s\rangle + \langle\lambda^r,\Gamma_{i_1\dots i_4}\lambda^s\rangle)e^0\wedge e^{i_1}\wedge\cdots\wedge e^{i_4}\\ &\quad + \frac{2}{4!}h^{-\frac{1}{4}}(-\langle\eta^r,\Gamma_{i_1\dots i_4}\eta^s\rangle + \langle\lambda^r,\Gamma_{i_1\dots i_4}\lambda^s\rangle)e^5\wedge e^{i_1}\wedge\cdots\wedge e^{i_4}\,,\end{aligned} \tag{B.47}$$

$$\begin{aligned}\tau^{(3)rs} &= \frac{1}{3}h^{-\frac{1}{4}}(-\langle\eta^r,\Gamma_{ijk}\lambda^s\rangle + \langle\lambda^r,\Gamma_{ijk}\eta^s\rangle)e^0\wedge e^5\wedge e^i\wedge e^j\wedge e^k\\ &\quad + \frac{2}{5!}h^{-\frac{1}{4}}(\langle\eta^r,\Gamma_{i_1\dots i_5}\lambda^s\rangle + \langle\lambda^r,\Gamma_{i_1\dots i_5}\eta^s\rangle)e^{i_1}\wedge\cdots\wedge e^{i_5}\,,\end{aligned} \tag{B.48}$$

$$\begin{aligned}\tau^{(1)rs} &= \frac{2}{4!}h^{-\frac{1}{4}}(\langle\eta^r,\Gamma_{i_1\dots i_4}\eta^s\rangle - \langle\lambda^r,\Gamma_{i_1\dots i_4}\lambda^s\rangle)e^0\wedge e^{i_1}\wedge\cdots\wedge e^{i_4}\\ &\quad - \frac{2}{4!}h^{-\frac{1}{4}}(\langle\eta^r,\Gamma_{i_1\dots i_4}\eta^s\rangle + \langle\lambda^r,\Gamma_{i_1\dots i_4}\lambda^s\rangle)e^5\wedge e^{i_1}\wedge\cdots\wedge e^{i_4}\,,\end{aligned} \tag{B.49}$$

$$\begin{aligned}\tau^{(2)rs} &= \frac{1}{3}h^{-\frac{1}{4}}(\langle\eta^r,\Gamma_{ijk}\lambda^s\rangle + \langle\lambda^r,\Gamma_{ijk}\eta^s\rangle)e^0\wedge e^5\wedge e^i\wedge e^j\wedge e^k\\ &\quad + \frac{2}{5!}h^{-\frac{1}{4}}(-\langle\eta^r,\Gamma_{i_1\dots i_5}\lambda^s\rangle + \langle\lambda^r,\Gamma_{i_1\dots i_5}\eta^s\rangle)e^{i_1}\wedge\cdots\wedge e^{i_5}\,,\end{aligned} \tag{B.50}$$

where $i, j, k = 1, 6, 2, 7, 3, 8, 4, 9$ and (e^0, e^5, e^i) is a pseudo-orthonormal frame of the D-string metric (4.27) for $p = 1$.

B.2.2 D5-brane

Using (7.6), one can find that the form bilinears of the D5-brane background are

$$
\begin{aligned}
k^{rs} &= 4h^{-1/4}\left(\mathrm{Re}\left\langle\eta^{1r},\Gamma_a\eta^{1s}\right\rangle_D+\mathrm{Re}\left\langle\eta^{2r},\Gamma_a\eta^{2s}\right\rangle_D\right)e^a\,, && \text{(B.51)}\\
\pi^{rs} &= 4h^{-1/4}\Big(-\mathrm{Re}\left\langle\eta^{1r},\Gamma_a\lambda^{1s}\right\rangle_D e^a\wedge(e^3\wedge e^4-e^8\wedge e^9)\\
&\quad+\mathrm{Re}\left\langle\eta^{2r},\Gamma_a\lambda^{2s}\right\rangle_D e^a\wedge(e^3\wedge e^4+e^8\wedge e^9)\\
&\quad-\mathrm{Im}\left\langle\eta^{1r},\Gamma_a\eta^{1s}\right\rangle_D e^a\wedge(e^3\wedge e^8+e^4\wedge e^9)\\
&\quad+\mathrm{Im}\left\langle\eta^{2r},\Gamma_a\eta^{2s}\right\rangle_D e^a\wedge(e^3\wedge e^8-e^4\wedge e^9)\\
&\quad-\mathrm{Im}\left\langle\eta^{1r},\Gamma_a\lambda^{1s}\right\rangle_D e^a\wedge(e^3\wedge e^9-e^4\wedge e^8)\\
&\quad+\mathrm{Im}\left\langle\eta^{2r},\Gamma_a\lambda^{2s}\right\rangle_D e^a\wedge(e^3\wedge e^9+e^4\wedge e^8)\Big)\\
&\quad+\frac{2}{3}h^{-1/4}\left(\mathrm{Re}\left\langle\eta^{1r},\Gamma_{abc}\eta^{1s}\right\rangle_D+\mathrm{Re}\left\langle\eta^{2r},\Gamma_{abc}\eta^{2s}\right\rangle_D\right)e^a\wedge e^b\wedge e^c\,, && \text{(B.52)}\\
\tau^{rs} &= -4h^{-1/4}\left(-\mathrm{Re}\left\langle\eta^{1r},\Gamma_a\eta^{1s}\right\rangle_D+\mathrm{Re}\left\langle\eta^{2r},\Gamma_a\eta^{2s}\right\rangle_D\right)e^a\wedge e^3\wedge e^4\wedge e^8\wedge e^9\\
&\quad+\frac{2}{3}h^{-1/4}\Big(-\mathrm{Re}\left\langle\eta^{1r},\Gamma_{abc}\lambda^{1s}\right\rangle_D(e^3\wedge e^4-e^8\wedge e^9)\\
&\quad+\mathrm{Re}\left\langle\eta^{2r},\Gamma_{abc}\lambda^{2s}\right\rangle_D(e^3\wedge e^4+e^8\wedge e^9)-\mathrm{Im}\left\langle\eta^{1r},\Gamma_{abc}\eta^{1s}\right\rangle_D(e^3\wedge e^8+e^4\wedge e^9)\\
&\quad+\mathrm{Im}\left\langle\eta^{2r},\Gamma_{abc}\eta^{2s}\right\rangle_D(e^3\wedge e^8-e^4\wedge e^9)-\mathrm{Im}\left\langle\eta^{1r},\Gamma_{abc}\lambda^{1s}\right\rangle_D(e^3\wedge e^9-e^4\wedge e^8)\\
&\quad+\mathrm{Im}\left\langle\eta^{2r},\Gamma_{abc}\lambda^{2s}\right\rangle_D(e^3\wedge e^9+e^4\wedge e^8)\Big)\wedge e^a\wedge e^b\wedge e^c\\
&\quad+\frac{4}{5!}h^{-1/4}\left(\mathrm{Re}\left\langle\eta^{1r},\Gamma_{a_1\dots a_5}\eta^{1s}\right\rangle_D+\mathrm{Re}\left\langle\eta^{2r},\Gamma_{a_1\dots a_5}\eta^{2s}\right\rangle_D\right)e^{a_1}\wedge\cdots\wedge e^{a_5}\,, && \text{(B.53)}
\end{aligned}
$$

where $a,b,c=0,5,1,6,2,7$ and (e^a,e^3,e^4,e^8,e^9) is a pseudo-orthonormal frame of the D5-brane metric (4.27) for $p=5$. The formula for the form bilinears $k^{(1)},\pi^{(1)}$ and $\tau^{(1)}$ can be obtained from that of k, π and τ by changing the sign in front of the $\langle\eta^2,Q\eta^2\rangle_D$ and $\langle\eta^2,Q\lambda^2\rangle_D$ terms.

The rest of the form bilinears are

$$
\begin{aligned}
k^{(3)rs} &= 4h^{-1/4}\Big(\left(\mathrm{Re}\left\langle\eta^{1r},\eta^{2s}\right\rangle_D-\mathrm{Re}\left\langle\eta^{2r},\eta^{1s}\right\rangle_D\right)e^3\\
&\quad+\left(\mathrm{Re}\left\langle\eta^{1r},\lambda^{2s}\right\rangle_D+\mathrm{Re}\left\langle\eta^{2r},\lambda^{1s}\right\rangle_D\right)e^4\\
&\quad+\left(\mathrm{Im}\left\langle\eta^{1r},\eta^{2s}\right\rangle_D+\mathrm{Im}\left\langle\eta^{2r},\eta^{1s}\right\rangle_D\right)e^8\\
&\quad+\left(\mathrm{Im}\left\langle\eta^{1r},\lambda^{2s}\right\rangle_D+\mathrm{Im}\left\langle\eta^{2r},\lambda^{1s}\right\rangle_D\right)\Big)e^9\,, && \text{(B.54)}\\
\pi^{(3)rs} &= 2h^{-1/4}\Big(\left(\mathrm{Re}\left\langle\eta^{1r},\Gamma_{ab}\eta^{2s}\right\rangle_D-\mathrm{Re}\left\langle\eta^{2r},\Gamma_{ab}\eta^{1s}\right\rangle_D\right)e^3\wedge e^a\wedge e^b\\
&\quad+\left(\mathrm{Re}\left\langle\eta^{1r},\Gamma_{ab}\lambda^{2s}\right\rangle_D+\mathrm{Re}\left\langle\eta^{2r},\Gamma_{ab}\lambda^{1s}\right\rangle_D\right)e^4\wedge e^a\wedge e^b\\
&\quad+\left(\mathrm{Im}\left\langle\eta^{1r},\Gamma_{ab}\eta^{2s}\right\rangle_D+\mathrm{Im}\left\langle\eta^{2r},\Gamma_{ab}\eta^{1s}\right\rangle_D\right)e^8\wedge e^a\wedge e^b\\
&\quad+\left(\mathrm{Im}\left\langle\eta^{1r},\Gamma_{ab}\lambda^{2s}\right\rangle_D+\mathrm{Im}\left\langle\eta^{2r},\Gamma_{ab}\lambda^{1s}\right\rangle_D\right)e^9\wedge e^a\wedge e^b
\end{aligned}
$$

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$$+ 2\left(-\operatorname{Re}\left\langle\eta^{1r},\eta^{2s}\right\rangle_D - \operatorname{Re}\left\langle\eta^{2r},\eta^{1s}\right\rangle_D\right) e^4\wedge e^8\wedge e^9$$
$$- 2\left(-\operatorname{Re}\left\langle\eta^{1r},\lambda^{2s}\right\rangle_D + \operatorname{Re}\left\langle\eta^{2r},\lambda^{1s}\right\rangle_D\right) e^3\wedge e^8\wedge e^9$$
$$- 2\left(\operatorname{Im}\left\langle\eta^{1r},\eta^{2s}\right\rangle_D - \operatorname{Im}\left\langle\eta^{2r},\eta^{1s}\right\rangle_D\right) e^3\wedge e^4\wedge e^9$$
$$+ 2\left(\operatorname{Im}\left\langle\eta^{1r},\lambda^{2s}\right\rangle_D - \operatorname{Im}\left\langle\eta^{2r},\lambda^{1s}\right\rangle_D\right))e^3\wedge e^4\wedge e^8\,, \tag{B.55}$$

$$\tau^{(3)rs} = -\frac{1}{6}h^{-1/4}\Big(\left(\operatorname{Re}\left\langle\eta^{1r},\Gamma_{a_1\dots a_4}\eta^{2s}\right\rangle_D - \operatorname{Re}\left\langle\eta^{2r},\Gamma_{a_1\dots a_4}\eta^{1s}\right\rangle_D\right) e^3$$
$$+ \left(\operatorname{Re}\left\langle\eta^{1r},\Gamma_{a_1\dots a_4}\lambda^{2s}\right\rangle_D + \operatorname{Re}\left\langle\eta^{2r},\Gamma_{a_1\dots a_4}\lambda^{1s}\right\rangle_D\right) e^4$$
$$+ \left(\operatorname{Im}\left\langle\eta^{1r},\Gamma_{a_1\dots a_4}\eta^{2s}\right\rangle_D + \operatorname{Im}\left\langle\eta^{2r},\Gamma_{a_1\dots a_4}\eta^{1s}\right\rangle_D\right) e^8$$
$$+ \left(\operatorname{Im}\left\langle\eta^{1r},\Gamma_{a_1\dots a_4}\lambda^{2s}\right\rangle_D + \operatorname{Im}\left\langle\eta^{2r},\Gamma_{a_1\dots a_4}\lambda^{1s}\right\rangle_D\right) e^9\Big)\wedge e^{a_1}\wedge\cdots\wedge e^{a_4}$$
$$+ 2h^{-1/4}\Big(\left(-\operatorname{Re}\left\langle\eta^{1r},\Gamma_{ab}\eta^{2s}\right\rangle_D - \operatorname{Re}\left\langle\eta^{2r},\Gamma_{ab}\eta^{1s}\right\rangle_D\right) e^4\wedge e^8\wedge e^9$$
$$- \left(-\operatorname{Re}\left\langle\eta^{1r},\Gamma_{ab}\lambda^{2s}\right\rangle_D + \operatorname{Re}\left\langle\eta^{2r},\Gamma_{ab}\lambda^{1s}\right\rangle_D\right) e^3\wedge e^8\wedge e^9$$
$$- \left(\operatorname{Im}\left\langle\eta^{1r},\Gamma_{ab}\eta^{2s}\right\rangle_D - \operatorname{Im}\left\langle\eta^{2r},\Gamma_{ab}\eta^{1s}\right\rangle_D\right) e^3\wedge e^4\wedge e^9$$
$$+ \left(\operatorname{Im}\left\langle\eta^{1r},\Gamma_{ab}\lambda^{2s}\right\rangle_D - \operatorname{Im}\left\langle\eta^{2r},\Gamma_{ab}\lambda^{1s}\right\rangle_D\right) e^3\wedge e^4\wedge e^8\Big)\wedge e^a\wedge e^b\,. \tag{B.56}$$

The formula for the form bilinears $k^{(2)}, \pi^{(2)}$ and $\tau^{(2)}$ can be obtained from that of $k^{(3)}, \pi^{(3)}$ and $\tau^{(3)}$ by changing the sign in front of the $\langle\eta^1, Q\eta^2\rangle_D$ and $\langle\eta^1, Q\lambda^2\rangle_D$ terms.

B.2.3 D3-brane

Using for the Killing spinors (7.12), one finds that the form bilinears of the D3-brane solution are as follows

$$k^{rs} = 4h^{-\frac{1}{4}}\Big(\operatorname{Re}\left\langle\eta^{1r},\eta^{1s}\right\rangle(e^0-e^5) + \operatorname{Re}\left\langle\lambda^{1r},\lambda^{1s}\right\rangle(e^0+e^5)$$
$$- \left(\operatorname{Re}\left\langle\eta^{1r},\lambda^{1s}\right\rangle + \operatorname{Re}\left\langle\lambda^{1r},\eta^{1s}\right\rangle\right) e^4$$
$$- \left(\operatorname{Im}\left\langle\eta^{1r},\lambda^{1s}\right\rangle - \operatorname{Im}\left\langle\lambda^{1r},\eta^{1s}\right\rangle\right) e^9\Big)\,, \tag{B.57}$$

$$k^{(1)rs} = 4h^{-\frac{1}{4}}\left(\operatorname{Im}\left\langle\eta^{1r},\Gamma_i\lambda^{2s}\right\rangle + \operatorname{Im}\left\langle\lambda^{1r},\Gamma_i\eta^{2s}\right\rangle\right) e^i\,, \tag{B.58}$$

$$\pi^{rs} = h^{-\frac{1}{4}}\Big(2\operatorname{Re}\left\langle\eta^{1r},\Gamma_{ij}\eta^{1s}\right\rangle(e^0-e^5)\wedge e^i\wedge e^j$$
$$+ 2\operatorname{Re}\left\langle\lambda^{1r},\Gamma_{ij}\lambda^{1s}\right\rangle(e^0+e^5)\wedge e^i\wedge e^j$$
$$- 2\left(\operatorname{Re}\left\langle\eta^{1r},\Gamma_{ij}\lambda^{1s}\right\rangle + \operatorname{Re}\left\langle\lambda^{1r},\Gamma_{ij}\eta^{1s}\right\rangle\right) e^4\wedge e^i\wedge e^j$$
$$- 2\left(\operatorname{Im}\left\langle\eta^{1r},\Gamma_{ij}\lambda^{1s}\right\rangle - \operatorname{Im}\left\langle\lambda^{1r},\Gamma_{ij}\eta^{1s}\right\rangle\right) e^9\wedge e^i\wedge e^j$$
$$- 4\operatorname{Im}\left\langle\eta^{1r},\eta^{1s}\right\rangle(e^0-e^5)\wedge e^4\wedge e^9$$
$$+ 4\operatorname{Im}\left\langle\lambda^{1r},\lambda^{1s}\right\rangle(e^0+e^5)\wedge e^4\wedge e^9$$
$$+ 4\left(\operatorname{Re}\left\langle\eta^{1r},\lambda^{1s}\right\rangle - \operatorname{Re}\left\langle\lambda^{1r},\eta^{1s}\right\rangle\right) e^0\wedge e^5\wedge e^4$$
$$+ 4\left(\operatorname{Im}\left\langle\eta^{1r},\lambda^{1s}\right\rangle + \operatorname{Im}\left\langle\lambda^{1r},\eta^{1s}\right\rangle\right) e^0\wedge e^5\wedge e^9\Big)\,, \tag{B.59}$$

$$\begin{aligned}
\pi^{(1)rs} = h^{-\frac{1}{4}}\Big(&-4\,\mathrm{Im}\left\langle \eta^{1r},\Gamma_i\eta^{2s}\right\rangle (e^0-e^5)\wedge e^4\wedge e^i\\
&-4\,\mathrm{Im}\left\langle \lambda^{1r},\Gamma_i\lambda^{2s}\right\rangle (e^0+e^5)\wedge e^4\wedge e^i\\
&+4\,\mathrm{Re}\left\langle \eta^{1r},\Gamma_i\eta^{2s}\right\rangle (e^0-e^5)\wedge e^9\wedge e^i\\
&-4\,\mathrm{Re}\left\langle \lambda^{1r},\Gamma_i\lambda^{2s}\right\rangle (e^0+e^5)\wedge e^9\wedge e^i\\
&-4\left(\mathrm{Im}\left\langle \eta^{1r},\Gamma_i\lambda^{2s}\right\rangle-\mathrm{Im}\left\langle \lambda^{1r},\Gamma_i\eta^{2s}\right\rangle\right) e^0\wedge e^5\wedge e^i\\
&+4\left(\mathrm{Re}\left\langle \eta^{1r},\Gamma_i\lambda^{2s}\right\rangle-\mathrm{Re}\left\langle \lambda^{1r},\Gamma_i\eta^{2s}\right\rangle\right) e^4\wedge e^9\wedge e^i\\
&+\frac{2}{3}\left(\mathrm{Im}\left\langle \eta^{1r},\Gamma_{ijk}\lambda^{2s}\right\rangle+\mathrm{Im}\left\langle \lambda^{1r},\Gamma_{ijk}\eta^{2s}\right\rangle\right) e^i\wedge e^j\wedge e^k\Big)\,, \qquad \text{(B.60)}\\
\tau^{rs} = \frac{h^{-\frac{1}{4}}}{6}\Big(&\mathrm{Re}\left\langle \eta^{1r},\Gamma_{i_1\ldots i_4}\eta^{1s}\right\rangle (e^0-e^5)+\mathrm{Re}\left\langle \lambda^{1r},\Gamma_{i_1\ldots i_4}\lambda^{1s}\right\rangle (e^0+e^5)\\
&-\left(\mathrm{Re}\left\langle \eta^{1r},\Gamma_{i_1\ldots i_4}\lambda^{1s}\right\rangle+\mathrm{Re}\left\langle \lambda^{1r},\Gamma_{i_1\ldots i_4}\eta^{1s}\right\rangle\right) e^4\\
&-\left(\mathrm{Im}\left\langle \eta^{1r},\Gamma_{i_1\ldots i_4}\lambda^{1s}\right\rangle-\mathrm{Im}\left\langle \lambda^{1r},\Gamma_{i_1\ldots i_4}\eta^{1s}\right\rangle\right) e^9\Big)\wedge e^{i_1}\wedge\cdots\wedge e^{i_4}\\
&-2h^{-\frac{1}{4}}\,\mathrm{Im}\left\langle \eta^{1r},\Gamma_{ij}\eta^{1s}\right\rangle (e^0-e^5)\wedge e^4\wedge e^9\wedge e^i\wedge e^j\\
&+2h^{-\frac{1}{4}}\,\mathrm{Im}\left\langle \lambda^{1r},\Gamma_{ij}\lambda^{1s}\right\rangle (e^0+e^5)\wedge e^4\wedge e^9\wedge e^i\wedge e^j\\
&+2h^{-\frac{1}{4}}\left(\mathrm{Re}\left\langle \eta^{1r},\Gamma_{ij}\lambda^{1s}\right\rangle-\mathrm{Re}\left\langle \lambda^{1r},\Gamma_{ij}\eta^{1s}\right\rangle\right) e^0\wedge e^5\wedge e^4\wedge e^i\wedge e^j\\
&+2h^{-\frac{1}{4}}\left(\mathrm{Im}\left\langle \eta^{1r},\Gamma_{ij}\lambda^{1s}\right\rangle+\mathrm{Im}\left\langle \lambda^{1r},\Gamma_{ij}\eta^{1s}\right\rangle\right) e^0\wedge e^5\wedge e^9\wedge e^i\wedge e^j\,, \qquad \text{(B.61)}\\
\tau^{(1)rs} = &-4h^{-\frac{1}{4}}\left(\mathrm{Re}\left\langle \eta^{1r},\Gamma_i\lambda^{2s}\right\rangle+\mathrm{Re}\left\langle \lambda^{1r},\Gamma_i\eta^{2s}\right\rangle\right) e^0\wedge e^5\wedge e^4\wedge e^9\wedge e^i\\
&\frac{2h^{-\frac{1}{4}}}{3}\Big(-\mathrm{Im}\left\langle \eta^{1r},\Gamma_{ijk}\eta^{2s}\right\rangle (e^0-e^5)\wedge e^4-\mathrm{Im}\left\langle \lambda^{1r},\Gamma_{ijk}\lambda^{2s}\right\rangle (e^0+e^5)\wedge e^4\\
&+\mathrm{Re}\left\langle \eta^{1r},\Gamma_{ijk}\eta^{2s}\right\rangle (e^0-e^5)\wedge e^9-\mathrm{Re}\left\langle \lambda^{1r},\Gamma_{ijk}\lambda^{2s}\right\rangle (e^0+e^5)\wedge e^9\\
&-\left(\mathrm{Im}\left\langle \eta^{1r},\Gamma_{ijk}\lambda^{2s}\right\rangle-\mathrm{Im}\left\langle \lambda^{1r},\Gamma_{ijk}\eta^{2s}\right\rangle\right) e^0\wedge e^5\\
&+\left(\mathrm{Re}\left\langle \eta^{1r},\Gamma_{ijk}\lambda^{2s}\right\rangle-\mathrm{Re}\left\langle \lambda^{1r},\Gamma_{ijk}\eta^{2s}\right\rangle\right) e^4\wedge e^9\Big)\wedge e^i\wedge e^j\wedge e^k\\
&+\frac{4h^{-\frac{1}{4}}}{5!}\left(\mathrm{Im}\left\langle \eta^{1r},\Gamma_{i_1\ldots i_5}\lambda^{2s}\right\rangle+\mathrm{Im}\left\langle \lambda^{1r},\Gamma_{i_1\ldots i_5}\eta^{2s}\right\rangle\right) e^{i_1}\wedge\cdots\wedge e^{i_5}\,, \qquad \text{(B.62)}
\end{aligned}$$

where $i,j,k = 1,6,2,7,3,8$ and (e^0,e^5,e^4,e^9) is a pseudo-orthonormal frame of the D3-brane metric (4.27) for $p=3$. The $k^{(2)},\pi^{(2)}$, and $\tau^{(2)}$ form bilinears can be obtained from k,π, and τ, and the $k^{(3)},\pi^{(3)}$, and $\tau^{(3)}$ form bilinears can be obtained from $k^{(1)},\pi^{(1)}$, and $\tau^{(1)}$ after replacing Re and Im with Im and $-$Re, respectively.

B.2.4 D7-brane

Using the Killing spinors (7.25), one can compute the form bilinears of the D7-brane to find

$$k^{rs} = 4h^{-\frac{1}{4}}\,\mathrm{Re}\,\langle \eta^r,\Gamma_a\eta^s\rangle_D\, e^a\,, \qquad \text{(B.63)}$$

$$\pi^{rs} = -4h^{-\frac{1}{4}}\,\mathrm{Im}\,\langle \eta^r,\Gamma_a\eta^s\rangle_D\, e^a\wedge e^4\wedge e^9+\frac{2}{3}h^{-\frac{1}{4}}\,\mathrm{Re}\,\langle \eta^r,\Gamma_{abc}\eta^s\rangle_D\, e^a\wedge e^b\wedge e^c\,, \qquad \text{(B.64)}$$

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$$\tau^{rs} = -\frac{2}{3}h^{-\frac{1}{4}}\,\mathrm{Im}\,\langle\eta^r, \Gamma_{abc}\eta^s\rangle_D\, e^a\wedge e^b\wedge e^c\wedge e^4\wedge e^9$$
$$+\frac{4}{5!}h^{-\frac{1}{4}}\,\mathrm{Re}\,\langle\eta^r, \Gamma_{a_1\ldots a_5}\eta^s\rangle_D\, e^{a_1}\wedge\cdots\wedge e^{a_5}\,, \tag{B.65}$$

where $a, b, c = 0, 5, 1, 6, 2, 7, 3, 8$ and (e^a, e^4, e^9) is a pseudo-orthonormal frame of the metric (4.27) for $p = 7$. The form bilinears $k^{(2)}, \pi^{(2)}$ and $\tau^{(2)}$ can be obtained from k, π and τ after replacing Re and Im with Im and $-$Re, respectively, in all the above expressions.

The rest bilinears are given by

$$k^{(3)rs} = 4h^{-\frac{1}{4}}\,\mathrm{Re}\,\langle\eta^r, \lambda^s\rangle_D\, e^4 + 4h^{-\frac{1}{4}}\,\mathrm{Im}\,\langle\eta^r, \lambda^s\rangle_D\, e^9\,, \tag{B.66}$$
$$\pi^{(3)rs} = 2h^{-\frac{1}{4}}\,\mathrm{Re}\,\langle\eta^r, \Gamma_{ab}\lambda^s\rangle_D\, e^a\wedge e^b\wedge e^4 + 2h^{-\frac{1}{4}}\,\mathrm{Im}\,\langle\eta^r, \Gamma_{ab}\lambda^s\rangle_D\, e^a\wedge e^b\wedge e^9\,, \tag{B.67}$$
$$\tau^{(3)rs} = \frac{1}{6}h^{-\frac{1}{4}}\,\mathrm{Re}\,\langle\eta^r, \Gamma_{a_1\ldots a_4}\lambda^s\rangle_D\, e^{a_1}\wedge\cdots\wedge e^{a_4}\wedge e^4$$
$$+\frac{1}{6}h^{-\frac{1}{4}}\,\mathrm{Im}\,\langle\eta^r, \Gamma_{a_1\ldots a_4}\lambda^s\rangle_D\, e^{a_1}\wedge\cdots\wedge e^{a_4}\wedge e^9\,. \tag{B.68}$$

Again the bilinears $k^{(1)}, \pi^{(1)}$ and $\tau^{(1)}$ can be derived from $k^{(3)}, \pi^{(3)}$ and $\tau^{(3)}$ after replacing Re and Im with Im and $-$Re, respectively, in all three expressions above.

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