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Reduction of one-loop integrals with higher poles by unitarity cut method

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ABSTRACT: Unitarity cut method has been proved to be very useful in the computation of one-loop integrals. In this paper, we generalize the method to the situation where the powers of propagators in the denominator are larger than one in general. We show how to use the trick of differentiation over masses to translate the problem to the integrals where all powers are just one. Then by using the unitarity cut method, we can find the wanted reduction coefficients of all basis except the tadpole. Using this method, we calculate the reduction of scalar bubble, scalar triangle, scalar box and scalar pentagon with general power of propagators.

KEYWORDS: Scattering Amplitudes, Renormalization Regularization and Renormalons

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1 Motivation

Unitarity cut method [1–3] has been proved to be very efficient method to calculate one-loop amplitudes. The power of this method is that using the holomorphic anomaly [4], one can analytically carry out the reduced phase space integration with double cuts, thus to extract the wanted reduction coefficients of master basis. The phase space integration method has been systematically developed for pure $4D$ -dimension in [5, 6] and for general $(4 - 2\epsilon)$ -dimension in [7–9]. With these results, pure analytic algebraic expressions for reduction coefficients have been given in [9–11] and their properties have been studied in [12, 13].

All above works have been assumed that the power of propagators in the denominator of one-loop integrals is just one. This assumption is harmless for general situations, but higher power of propagators does appear in some situation,¹ for example, in higher loops or in the middle steps if one use the IBP method to do the reduction. Thus it is naturally to ask if the unitarity cut method can be applied to these more general situations.

¹Some works for the higher power of propagators can be found in [14–16].

In this paper, we consider the reduction of one-loop integrals with general pole structures, i.e., propagators could have general power. The general integration will be given by

$$\mathcal{M}[\ell] \equiv \int \frac{d^D \ell}{(2\pi)^{D/2}} \frac{\mathcal{N}[\ell]}{\prod_{j=1}^n ((\ell - K_j)^2 - m_j^2 + i\epsilon)^{a_j}}, \quad a_i \geq 1 \quad (1.1)$$

where $\mathcal{N}[\ell]$ is an arbitrary polynomial function of ℓ . According to the PV-reduction method [17], we can decompose

$$\mathcal{M}[\ell] = \sum_t c_t \mathcal{I}_t[\ell] \quad (1.2)$$

where c_t 's are the rational functions and $\mathcal{I}_t$ are scalar basis of tadpole, bubble, triangle, box and pentagon. Let us consider the c_i 's first. The data entering the integral (1.1) are the external momenta, the masses and the polarization vectors (the coupling constants are overall factor, so can be dropped in the discussion). The c_i should be the rational functions of these data with Lorentz invariant contractions.

The idea of unitarity cut method is to compare the imaginary part of both sides of (1.2). Since c_t 's are rational functions without imaginary contributions, we have

$$\text{Im}(\mathcal{M}[\ell]) = \sum_t c_t \text{Im}(\mathcal{I}_t[\ell]) \quad (1.3)$$

Different master integrals have different analytic structures for the imaginary part (we will call them as the “signature”), thus if we can analytically computer the left hand side, we can do the splitting according to the analytic signature of each master integral and find expansion coefficients c_t at the right hand side. With this thought, the key of reduction by unitarity cut method is to compute the imaginary part of $\mathcal{M}[\ell]$. When all $a_j = 1$, the computation of left hand side (1.3) is transformed to the reduced phase space integration with double cuts, which we know how to do it as reviewed in the first paragraph of this section. For general a_i , we use following trick to solve the problem. Noticing that²

$$\begin{aligned} & \int \frac{d^D \ell}{(2\pi)^{D/2}} \frac{\mathcal{N}[\ell]}{\prod_{j=1}^n ((\ell - K_j)^2 - m_j^2 + i\epsilon)^{a_j}} \\ &= \left\{ \prod_{j=1}^n \frac{1}{(a_j - 1)!} \frac{d^{a_j-1}}{d\eta_j^{a_j-1}} \int \frac{d^D \ell}{(2\pi)^{D/2}} \frac{\mathcal{N}[\ell]}{\prod_{j=1}^n ((\ell - K_j)^2 - m_j^2 - \eta_j + i\epsilon)} \right\} \Big|_{\eta_j \rightarrow 0} \end{aligned} \quad (1.4)$$

the computation of the imaginary part of an one-loop integral is transformed to the reduced phase space integration with double cut for the case with all $a_i = 1$. More explicitly, let us separate both sides of (1.4) to the real and imaginary part, we have

$$\text{Re}[L] + i\text{Im}[L] = \left\{ \prod_{j=1}^n \frac{1}{(a_j - 1)!} \frac{d^{a_j-1}}{d\eta_j^{a_j-1}} (\text{Re}[R] + i\text{Im}[R]) \right\} \Big|_{\eta_j \rightarrow 0} \quad (1.5)$$

²The same trick has also been used to get the homogenous solution of differential equations by maximum cut in [18].

Since the η_i 's are defined to take real values, we have

$$Re[L] + iIm[L] = \left\{ \prod_{j=1}^n \frac{1}{(a_j - 1)!} \frac{d^{a_j-1}}{d\eta_j^{a_j-1}} Re[R] \right\} \Big|_{\eta_j \rightarrow 0} + i \left\{ \prod_{j=1}^n \frac{1}{(a_j - 1)!} \frac{d^{a_j-1}}{d\eta_j^{a_j-1}} Im[R] \right\} \Big|_{\eta_j \rightarrow 0} \quad (1.6)$$

thus we get

$$Im[L] = \left\{ \prod_{j=1}^n \frac{1}{(a_j - 1)!} \frac{d^{a_j-1}}{d\eta_j^{a_j-1}} Im[R] \right\} \Big|_{\eta_j \rightarrow 0} \quad (1.7)$$

For general $\mathcal{N}[\ell]$ in (1.4), we know the expansion

$$Im[R] = \sum_t c_t Im(\mathcal{I}_t[\ell]) \quad (1.8)$$

and the action of $\frac{d}{d\eta}$ will act on both c_t and $Im(\mathcal{I}_t[\ell])$. Since the function c_t 's have been given in [9–11], the unknown piece is the action of $\frac{d}{d\eta}$ on $Im(\mathcal{I}_t[\ell])$ and its expansion. In another words, we just need to consider the reduction of general power with $\mathcal{N}[\ell] = 1$ in (1.1) for $n \leq 5$.

The plan of the paper is following. In the section two, we consider the reduction of bubbles with higher poles. We establish the general recurrence relation and check our results with some examples. Same method has been applied to triangles, boxes and pentagons in the section three, four and five. A brief conclusion is given in the section six.

2 Bubble

For bubble topology, let us define

$$I_2(a, b)[K; M_1, M_2] \equiv \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{(p^2 - M_1^2)^a ((p - K)^2 - M_2^2)^b} \quad (2.1)$$

to be the general scalar bubble integral with higher power of propagators. The master integral of bubble is the case $n = m = 1$ and for this special case, sometimes we just write it as $\mathcal{I}_2[K; M_1, M_2]$ or just $\mathcal{I}_2$ for simplicity. The reduction of $I_2(a, b)[K; M_1, M_2]$ will be the following expansion

$$I_2(a, b)[K; M_1, M_2] = c_{2 \rightarrow 2}(a, b) \mathcal{I}_2[K; M_1, M_2] + \sum_{i=1}^2 c_{2 \rightarrow 1; i}(a, b) \mathcal{I}_{1; i}[M_i] \quad (2.2)$$

where $c_{2 \rightarrow 1; i}$ means that when reducing the bubble to the tadpole, the i -th propagator has been kept. The tadpole $\mathcal{I}_{1; i}[M_i]$ should be written in the standard form with the proper momentum shifting, i.e.,

$$\mathcal{I}_{1; i}[M_i] \equiv \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{(p^2 - M_i^2)} \quad (2.3)$$

Expansion coefficients in (2.2) can be found by various methods, for example, the IBP method. However, in this paper, we will try to use the unitarity cut method to find

expansion coefficients. When we use the unitarity cut method, the tadpole can not be detected, thus tadpole coefficients can not be found by this way. Although we will not consider the tadpole coefficients in this paper, we want to point out that some efforts have been done to fill the gap by using the single cut [19–21].

The unitarity cut of $I_2(1, 1)$ is given by (see references [7–9])³

$$\mathcal{C}[\mathcal{I}_2] = (K^2)^{-1+\epsilon} \Delta^{\frac{1}{2}-\epsilon} \int_0^1 du u^{-1-\epsilon} \sqrt{1-u} \quad (2.4)$$

where

$$\begin{aligned} \Delta[K; M_1, M_2] &= (K^2)^2 + (M_1^2)^2 + (M_2^2)^2 - 2M_1^2 M_2^2 - 2K^2 M_1^2 - 2K^2 M_2^2 \\ &= -4M_1^2 M_2^2 \left| \begin{array}{cc} 1 & -\frac{K^2 - M_1^2 - M_2^2}{2M_1 M_2} \\ -\frac{K^2 - M_1^2 - M_2^2}{2M_1 M_2} & 1 \end{array} \right|, \end{aligned} \quad (2.5)$$

which is the Landau surface of bubble of the first type of singularities. For the later convenience, let us define

$$Bub^{(n)} = \frac{1}{2(n-\epsilon)} \int_0^1 du \frac{u^{n-\epsilon}}{\sqrt{1-u}} = \int_0^1 du u^{-1-\epsilon} u^n \sqrt{1-u}. \quad (2.6)$$

Thus by comparing (2.6) with (2.4) we see that $\mathcal{C}[\mathcal{I}_2] = (K^2)^{-1+\epsilon} \Delta^{\frac{1}{2}-\epsilon} Bub^{(0)}$. The $Bub^{(n)}$ is well defined for $n \geq 0$ and it is easy to derive a recursion relation by integration-by-part

$$Bub^{(n)} = \frac{(n-1-\epsilon)}{(n+\frac{1}{2}-\epsilon)} Bub^{(n-1)}. \quad (2.7)$$

Solving it we get

$$Bub^{(n)} = F^{(n)} Bub^{(0)}, \quad F^{(n)} = \frac{\Gamma(3/2-\epsilon)\Gamma(n-\epsilon)}{\Gamma(-\epsilon)\Gamma(n+3/2-\epsilon)}. \quad (2.8)$$

We find that when doing the reduction for triangles and boxes, we will meet the form $Bub^{(n)}$ with the negative integer n . For this case, we can use (2.7) to analytically continue from positive n to negative n . For example, using $n = 0$ in (2.7), we get

$$Bub^{(-1)} = \frac{(1-2\epsilon)}{2(-1-\epsilon)} Bub^{(0)} \quad (2.9)$$

Having above preparation, now we consider the reduction of $I_2(a, b)$. According to our general idea in (1.7), we should calculate⁴ $\left(\frac{\partial}{\partial M_1^2}\right)^{a-1} \left(\frac{\partial}{\partial M_2^2}\right)^{b-1} \Delta^{\frac{1}{2}-\epsilon}$. To get a better idea, let us start with $\left(\frac{\partial}{\partial M_1^2}\right)^n \Delta^{\frac{1}{2}-\epsilon}$. Before doing this, we rewrite Δ as

$$\Delta = (M_1^2 - M_2^2 - K^2 - \sqrt{2K^2 M_2^2})(M_1^2 - M_2^2 - K^2 + \sqrt{2K^2 M_2^2}) \quad (2.10)$$

³In the reference [9] when we do the general one-loop unitarity cut phase space integration, an overall factor $\left(\frac{\Delta[K, M_1, M_2]}{K^2}\right)^{-\epsilon}$ has been neglected in later computation (see the equation after eq. (9)). This is fine for the work in [9], but since it depends on the masses, it is crucial for current computation and we must include it back.

⁴From the discussion of (1.4) one can see that the role of η is identical with m^2 . For scalar basis, m does not appear in other places, so we can take the derivative of m^2 instead of η without making any mistake.

where each factor is linear in M_1^2 . It is easy to get the n -th derivative of M_1^2 is given by

$$\begin{aligned} \left(\frac{\partial}{\partial M_1^2}\right)^n \Delta^{\frac{1}{2}-\epsilon} &= \sum_{\lambda=0}^n C_n^\lambda \left(\frac{\partial}{\partial M_1^2}\right)^{n-\lambda} \left\{ \left(M_1^2 - M_2^2 - K^2 - \sqrt{2K^2 M_2^2}\right)^{\frac{1}{2}-\epsilon} \right\} \\ &\quad \times \left(\frac{\partial}{\partial M_1^2}\right)^\lambda \left\{ \left(M_1^2 - M_2^2 - K^2 + \sqrt{2K^2 M_2^2}\right)^{\frac{1}{2}-\epsilon} \right\} \end{aligned} \quad (2.11)$$

Using $\Gamma(x+1) = x\Gamma(x)$, we have

$$\frac{d^n(x-a)^b}{dx^n} = b(b-1)\dots(b-n+1)(x-a)^{b-n} = \frac{\Gamma(b+1)}{\Gamma(b+1-n)}(x-a)^{b-n} \quad (2.12)$$

thus (2.11) is

$$\left(\frac{\partial}{\partial M_1^2}\right)^n \Delta^{\frac{1}{2}-\epsilon} = \Delta^{\frac{1}{2}-\epsilon} \left\{ \sum_{\lambda=0}^n C_n^\lambda \frac{\Gamma(\frac{3}{2}-\epsilon)}{\Gamma(\frac{3}{2}-\epsilon-(n-\lambda))} \Delta_{M_1,-}^{-(n-\lambda)} \frac{\Gamma(\frac{3}{2}-\epsilon)}{\Gamma(\frac{3}{2}-\epsilon-\lambda)} \Delta_{M_1,+}^{-\lambda} \right\} \quad (2.13)$$

where we have defined

$$\Delta_{M_1,\pm} = M_1^2 - M_2^2 - K^2 \pm \sqrt{2K^2 M_2^2} \quad (2.14)$$

Using above result, we can find the reduction coefficient of $I_2(n+1, 1)$ by (1.7) since

$$\mathcal{C}[I_2(n+1, 1)] = \frac{1}{n!} \left(\frac{\partial}{\partial M_1^2}\right)^n \mathcal{C}[\mathcal{I}_2] = c_{2 \rightarrow 2}(n+1, 1) \mathcal{C}[\mathcal{I}_2] \quad (2.15)$$

where

$$c_{2 \rightarrow 2}(n+1, 1)[K; M_1, M_2] = \frac{1}{n!} \left\{ \sum_{\lambda=0}^n C_n^\lambda \frac{\Gamma(\frac{3}{2}-\epsilon)}{\Gamma(\frac{3}{2}-\epsilon-(n-\lambda))} \Delta_{M_1,-}^{-(n-\lambda)} \frac{\Gamma(\frac{3}{2}-\epsilon)}{\Gamma(\frac{3}{2}-\epsilon-\lambda)} \Delta_{M_1,+}^{-\lambda} \right\} \quad (2.16)$$

by the result (2.13).

For the general bubble coefficients of the reduction of $I_2(n+1, m+1)$, using the relation

$$\mathcal{C}[I_2(n+1, m+1)] = \frac{1}{m!n!} \left(\frac{\partial}{\partial M_2^2}\right)^m \left(\frac{\partial}{\partial M_1^2}\right)^n \mathcal{C}[I_2(1, 1)] \quad (2.17)$$

we get reduction coefficient as

$$c_{2 \rightarrow 2}(n+1, m+1)[K; M_1, M_2] = \frac{1}{m!n!\Delta^{\frac{1}{2}-\epsilon}} \left(\frac{\partial}{\partial M_2^2}\right)^m \left(\frac{\partial}{\partial M_1^2}\right)^n \Delta^{\frac{1}{2}-\epsilon} \quad (2.18)$$

The general analytic expression of $c_{2 \rightarrow 2}(n+1, m+1)$ will be complicated to write down.

Now we check the result (2.18). The first check is by the symmetry. Noticing that when shifting $p \rightarrow p+K$, the (2.1) becomes

$$\int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{1}{((p+K)^2 - M_1^2)^a (p^2 - M_2^2)^b} \quad (2.19)$$

thus we have $I_2(a, b)[K; M_1, M_2] = I_2(b, a)[-K; M_2, M_1]$. When doing reduction at both sides, we must have the bubble coefficient to be same, i.e.,

$$c_{2 \rightarrow 2}(n+1, m+1)[K; M_1, M_2] = c_{2 \rightarrow 2}(m+1, n+1)[-K; M_2, M_1] \quad (2.20)$$

From the first line of (2.18), the Δ is invariant under the replacement $[K; M_1, M_2] \rightarrow [-K; M_2, M_1]$, so it does satisfy the relation (2.20).

The second check is following. For one loop massless integral (example 5.2 of [22])

$$F_{5.2}(a_1, a_2) = \int \frac{d^d k}{(k^2)^{a_1} [(q-k)^2]^{a_2}} \quad (2.21)$$

by the IBP method, one can derive the relation

$$F(a_1, a_2 > 1) = -\frac{1}{(a_2 - 1)q^2} [(d - 2a_1 - a_2 + 1)F(a_1, a_2 - 1) - (a_2 - 1)F(a_1 - 1, a_2)] \quad (2.22)$$

and when $a_2 = 1$ we have

$$F(a_1, 1) = -\frac{d - a_1 - 1}{(a_1 - 1)q^2} F(a_1 - 1, 1). \quad (2.23)$$

To compare with our calculation, we should take $M_1, M_2 \rightarrow 0$, $K \rightarrow q$ and $d \rightarrow 4 - 2\epsilon$ after all derivatives related to M_1^2, M_2^2 having been done. We have used the Mathematica to check the massless case of our general result (2.18). The third check is that we have used the LiteRed [23] to explicitly calculate some examples with nonzero M_1, M_2 and we do find the match.

2.1 Recurrence relation

Like the IBP relation, the idea of (1.7) can be used to establish the recurrence relation for reduction coefficients. Let us start from the reduction of $I_2(a, 1)$. Noticing that

$$\begin{aligned} I_2(a, 1) &= \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{(p^2 - M_1^2)^a ((p-K)^2 - M_2^2)} \\ &= \frac{1}{(a-1)} \frac{d}{d(M_1^2)} \frac{1}{(a-2)!} \frac{d^{a-2}}{d(M_1^2)^{a-2}} \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{(p^2 - M_1^2)((p-K)^2 - M_2^2)} \\ &= \frac{1}{(a-1)} \frac{d}{d(M_1^2)} I_2(a-1, 1), \end{aligned} \quad (2.24)$$

if we know

$$I_2(a-1, 1) = c_{2 \rightarrow 2}(a-1, 1)[K; M_1, M_2] I_2(1, 1) + \dots \quad (2.25)$$

where $\dots$ is for the tadpole part, we can write

$$I_2(a, 1) = \left(\frac{1}{(a-1)} \frac{dc_{2 \rightarrow 2}(a-1, 1)}{d(M_1^2)} \right) I_2(1, 1) + \frac{c_{2 \rightarrow 2}(a-1, 1)}{(a-1)} I_2(2, 1) + \dots \quad (2.26)$$

where for simplicity we have dropped the dependence of K, M . From (2.26), we can read out a recurrence relation for the coefficient⁵

$$c_{2 \rightarrow 2}(a, 1)[K; M_1, M_2] = \left(\frac{1}{(a-1)} \frac{dc_{2 \rightarrow 2}(a-1, 1)}{d(M_1^2)} \right) + \frac{c_{2 \rightarrow 2}(a-1, 1)}{(a-1)} c_{2 \rightarrow 2}(2, 1) \quad (2.27)$$

⁵The relation (2.27) holds for $a \geq 2$, but when $a = 2$, the $c_{2 \rightarrow 2}(1, 1) = 1$ and the identity is trivial.

Relation (2.27) tell us that staring from $c_{2 \rightarrow 2}(2, 1)$, we can write down all $c_{2 \rightarrow 2}(a, 1)$.

The same idea can be used to write down the recurrence relation of general $c_{2 \rightarrow 2}(a, b)$. Noticing that

$$\begin{aligned} I_2(a, b) &= \frac{1}{(b-1)!} \frac{d^{b-1}}{d(M_2^2)^{b-1}} I_2(a, 1) = \frac{1}{(b-1)!} \frac{d^{b-1}}{d(M_2^2)^{b-1}} \{c_{2 \rightarrow 2}(a, 1) I_2(1, 1) + \dots\} \\ &= \frac{1}{(b-1)!} \sum_{t=0}^{b-1} C_{b-1}^t \frac{d^t c_{2 \rightarrow 2}(a, 1)}{d(M_2^2)^t} (b-1-t)! I_2(1, b-t) + \dots \end{aligned} \quad (2.28)$$

we get immediately the recurrence relation

$$c_{2 \rightarrow 2}(a, b) = \frac{1}{(b-1)!} \sum_{t=0}^{b-1} C_{b-1}^t (b-1-t)! c_{2 \rightarrow 2}(1, b-t) \frac{d^t c_{2 \rightarrow 2}(a, 1)}{d(M_2^2)^t} \quad (2.29)$$

where $c_{2 \rightarrow 2}(1, m)$ can be obtained from $c_{2 \rightarrow 2}(m, 1)$ by proper replacement as given in (2.20). Result (2.29) shows that knowing the coefficient $c_{2 \rightarrow 2}(2, 1)$ is enough for general reduction coefficients. The similar idea can be used to the reduction for triangle, box and pentagon as will be shown later.

3 Triangle

For triangle topology, let us define⁶

$$\begin{aligned} I_3(n_1, n_2, n_3)[K_1, K_2, K_3; M_1, M_2, m_1] \\ = \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{(p^2 - M_1^2)^{n_1} ((p - K_1)^2 - M_2^2)^{n_2} ((p + K_3)^2 - m_1^2)^{n_3}} \end{aligned} \quad (3.1)$$

For the special case, i.e., $n_1 = n_2 = n_3 = 1$, we get the familiar scalar triangle basis, which for simplicity, we will denote as $\mathcal{I}_3[K_1, K_2, K_3; M_1, M_2, m_1]$. The unitarity cut of $\mathcal{I}_3$ with the cut momentum K_1 has been given by [9]

$$\mathcal{C}(I_3) = - \left(\frac{4K_1^2}{\Delta[K_1, M_1, M_2]} \right)^\epsilon \frac{1}{\sqrt{\Delta_{3;m=0}}} \text{Tri}^{(0)}(Z) \quad (3.2)$$

with

$$\text{Tri}^{(n)}(Z) = \int_0^1 du u^{-1-\epsilon} u^n \ln \left(\frac{Z + \sqrt{1-u}}{Z - \sqrt{1-u}} \right) = \frac{-Z}{n-\epsilon} \int_0^1 du \frac{u^{n-\epsilon}}{\sqrt{1-u} (1-u-Z^2)}, \quad n \geq 0 \quad (3.3)$$

where Z is given by

$$Z = - \frac{(2K_1 \cdot Q) K_1^2}{\sqrt{\Delta_{3;m=0} \Delta[K_1, M_1, M_2]}} \quad (3.4)$$

⁶Although by momentum conservation, we will have $K_1 + K_2 + K_3 = 0$. Keeping them all in the parameter will make the symmetry more transparent.

with⁷

$$\Delta_{3;m=0} = 4((K_1 \cdot K_3)^2 - K_1^2 K_3^2) = -4 \det \begin{pmatrix} K_1^2 & K_1 \cdot K_3 \\ K_1 \cdot K_3 & K_3^2 \end{pmatrix} \quad (3.5)$$

$$\Delta[K_1, M_1, M_2] = (K_1^2)^2 + (M_1^2)^2 + (M_2^2)^2 - 2K_1^2 M_1^2 - 2K_1^2 M_2^2 - 2M_1^2 M_2^2 \quad (3.6)$$

$$K_1 \cdot Q = \frac{K_1^2 + M_1^2 - M_2^2}{K_1^2} (K_1 \cdot K_3) + (K_3^2 + M_1^2 - m_1^2) \quad (3.7)$$

The $Tri^{(n)}(Z)$ has the recurrence relation [8, 9]

$$Tri^{(n)}(Z) = -\frac{(Z^2 - 1)(n - 1 - \epsilon)}{n - \epsilon} Tri^{(n-1)}(Z) + \frac{2Z(n - 1 - \epsilon)}{(n - \epsilon)} Bub^{(n-1)} \quad (3.8)$$

which will be useful later.

For general n_i , the $I_3(n_1, n_2, n_3)$ will be reduced to the sum of one triangle I_3 , three bubbles $I_{2;\bar{i}}, i = 1, 2, 3$ (where $\bar{i}$ means to set $n_i = 0$ in (3.1) and other n_j 's one) and three tadpoles $I_{1;i}, i = 1, 2, 3$ (where i means to set $n_i = 1$ in (3.1) and other n_j 's zero). In other words, we will have the expansion

$$I_3(n_1, n_2, n_3) = c_{3 \rightarrow 3}(n_1, n_2, n_3) \mathcal{I}_3 + \sum_{i=1}^3 c_{3 \rightarrow 2;\bar{i}}(n_1, n_2, n_3) \mathcal{I}_{2;\bar{i}} + \sum_{i=1}^3 c_{3 \rightarrow 1;i}(n_1, n_2, n_3) \mathcal{I}_{1;i} \quad (3.9)$$

where for simplicity we have dropped the dependence over K, M of coefficients c 's. We want to emphasize that when writing down $\mathcal{I}_{2;\bar{i}}$, we need to do the proper momentum shifting to reach the standard form (2.1). More explicitly, we will have

$$\mathcal{I}_{2;\bar{3}} = \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{(p^2 - M_1^2)((p - K_1)^2 - M_2^2)} \quad (3.10)$$

and

$$\mathcal{I}_{2;\bar{1}} = \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{(p^2 - M_2^2)((p - K_2)^2 - m_1^2)} \quad (3.11)$$

by using the shifting $p \rightarrow p + K_1$, and finally

$$\mathcal{I}_{2;\bar{2}} = \int \frac{d^{4-2\epsilon} p^4}{(2\pi)^{4-2\epsilon}} \frac{1}{((p - K_3)^2 - M_1^2)(p^2 - m_1^2)} \quad (3.12)$$

by using the shifting $p \rightarrow p + K_1 + K_2$. For later convenience, let us give a more general bubbles coming from the triangle reduction (3.1)

$$\begin{aligned} I_{2;\bar{3}}(n_1, n_2)[K_1; M_1, M_2] &= \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{(p^2 - M_1^2)^{n_1} ((p - K_1)^2 - M_2^2)^{n_2}} \\ I_{2;\bar{1}}(n_1, n_2)[K_2; M_2, m_1] &= \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{(p^2 - M_2^2)^{n_1} ((p - K_2)^2 - m_1^2)^{n_2}} \\ I_{2;\bar{2}}(n_2, n_1)[K_3; m_1, M_1] &= \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{((p - K_3)^2 - M_1^2)^{n_1} (p^2 - m_1^2)^{n_2}} \end{aligned} \quad (3.13)$$

⁷The $\Delta[K_1, M_1, M_2]$ reflects the singularity of bubble as given in (2.5), while $\Delta_{3;m=0}$ corresponds the pure second-type Landau singularity of triangle.

When we use the unitarity cut method to find the reduction coefficients, a given cut can only detect these coefficients of basis which have the corresponding cut. For each cut, it can detect the triangle coefficient and one bubble coefficient, while all tadpole coefficients can not be found by this way. For triangle, there are three different cuts, i.e. K_1, K_2, K_3 . the coefficient of bubble $\mathcal{I}_{2;\bar{3}}$ can only be detected by the cut with momentum K_1 and similarly bubble coefficients of $\mathcal{I}_{2;\bar{1}}$ and $\mathcal{I}_{2;\bar{2}}$ by the cut K_2 and K_3 respectively. Thus using them all, we can get reduction coefficients of triangle and bubbles. Furthermore, there are overlaps of detected reduction coefficients between different cuts. For the triangle case, all three cuts can detect the same triangle coefficient, so we can use the overlap as the cross check.

3.1 Recurrence relation

Before going to explicit calculation, let us establish some recurrence relations like the one (2.27) and (2.29) in the previous section. To prepare for this task, we discuss some symmetric properties of the integral (3.1) first. By shifting the momentum p , we can rewrite (3.1) to different forms, for example, by shifting $p \rightarrow p + K_1$ we get

$$\begin{aligned} I_3(n_1, n_2, n_3)[K_1, K_2, K_3; M_1, M_2, m_1] \\ = \int \frac{d^{4-2\epsilon} p^4}{(2\pi)^{4-2\epsilon}} \frac{1}{((p+K_1)^2 - M_1^2)^{n_1} (p^2 - M_2^2)^{n_2} ((p-K_2)^2 - m_1^2)^{n_3}} \\ = I_3(n_2, n_3, n_1)[K_2, K_3, K_1; M_2, m_1, M_1] \end{aligned} \quad (3.14)$$

and similarly for the shifting $p \rightarrow p + K_1 + K_2$. Above relations (3.14) by shifting can be summarized as the cyclic symmetry Z_3 of three ordered lists

$$g_3 : \{(n_1, n_2, n_3); (K_1, K_2, K_3); (M_1, M_2, m_1)\} \rightarrow \{(n_2, n_3, n_1); (K_2, K_3, K_1); (M_2, m_1, M_1)\} \quad (3.15)$$

We can also consider the variable changing $p \rightarrow -p$ in (3.1) to get⁸

$$\begin{aligned} I_3(n_1, n_2, n_3)[K_1, K_2, K_3; M_1, M_2, m_1] \\ = \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{(p^2 - M_1^2)^{n_1} ((p+K_1)^2 - M_2^2)^{n_2} ((p-K_3)^2 - m_1^2)^{n_3}} \\ = I_3(n_1, n_3, n_2)[K_3, K_2, K_1; M_1, m_1, M_2] \end{aligned} \quad (3.16)$$

which can be summarized as the reflection symmetry Z_2 of three ordered lists

$$g_2 : \{(n_1, n_2, n_3); (K_1, K_2, K_3); (M_1, M_2, m_1)\} \rightarrow \{(n_1, n_3, n_2); (K_3, K_2, K_1); (M_1, m_1, M_2)\} \quad (3.17)$$

When combining g_2, g_3 together, we generate the permutation group S_3 . Using the symmetric property, we can connect the reduction of one integral to another integral. For example,

⁸When changing $p \rightarrow \tilde{p} = -p$, the $\int_{-\infty}^{+\infty} dp \rightarrow \int_{+\infty}^{-\infty} -d\tilde{p} = \int_{-\infty}^{+\infty} d\tilde{p}$, i.e., the measure is invariant under the reflection.

we know the expansion of $I_3(1, 1, n)[K_1, K_2, K_3; M_1, M_2, m_1]$ and we want to calculate the expansion of $I_3(1, n, 1)[K_1, K_2, K_3; M_1, M_2, m_1]$. Using (3.14), we have

$$\begin{aligned} I_3(1, n, 1)[K_1, K_2, K_3; M_1, M_2, m_1] &= g_3 \{I_3(1, 1, n)[K_3, K_1, K_2; m_1, M_1, M_2]\} \\ &= g_3 \left\{ c_{3 \rightarrow 3}(n_1, n_2, n_3) \mathcal{I}_3 + \sum_{i=1}^3 c_{3 \rightarrow 2; \bar{i}}(n_1, n_2, n_3) \mathcal{I}_{2; \bar{i}} + \sum_{i=1}^3 c_{3 \rightarrow 1; i}(n_1, n_2, n_3) \mathcal{I}_{1; i} \right\}. \end{aligned} \quad (3.18)$$

Noticing that $\mathcal{I}_3[K_1, K_2, K_3; M_1, M_2, m_1] = g_3 \{ \mathcal{I}_3[K_3, K_1, K_2; m_1, M_1, M_2] \}$, we have

$$c_{3 \rightarrow 3}(1, n, 1)[K_1, K_2, K_3; M_1, M_2, m_1] = g_3 \{ c_{3 \rightarrow 3}(1, 1, n)[K_3, K_1, K_2; m_1, M_1, M_2] \}. \quad (3.19)$$

Similarly, for bubble part we have

$$\mathcal{I}_{2; \bar{i}}[K_1, K_2, K_3; M_1, M_2, m_1] = g_3 \left\{ \mathcal{I}_{2; \bar{i}}[K_3, K_1, K_2; m_1, M_1, M_2] \right\},$$

thus

$$c_{3 \rightarrow 2; \bar{i}}(1, n, 1)[K_1, K_2, K_3; M_1, M_2, m_1] = g_3 \left\{ c_{3 \rightarrow 2; \bar{i}}(1, 1, n)[K_3, K_1, K_2; m_1, M_1, M_2] \right\}. \quad (3.20)$$

One simple consequence of above symmetry property is that if we know the expansion of $I_3(1, 1, 2)$, we know also $I_3(1, 2, 1)$ and $I_3(2, 1, 1)$.

Now we show how to use $I_3(1, 1, 2)$, $I_3(1, 2, 1)$ and $I_3(2, 1, 1)$ to get the expansion of general $I_3(n_1, n_2, n_3)$ using only differentiation. Let us start from $I_3(1, 1, n_3)$ first. It is easy to see that

$$\begin{aligned} I_3(1, 1, n_3) &= \frac{1}{(n_3 - 1)} \frac{d}{d(m_1^2)} I_3(1, 1, n_3 - 1) \\ &= \frac{1}{(n_3 - 1)} \frac{d}{d(m_1^2)} \left\{ c_{3 \rightarrow 3}(1, 1, n_3 - 1) \mathcal{I}_3 + \sum_{i=1}^3 c_{3 \rightarrow 2; \bar{i}}(1, 1, n_3 - 1) \mathcal{I}_{2; \bar{i}} + \dots \right\} \\ &= \frac{1}{(n_3 - 1)} \frac{dc_{3 \rightarrow 3}(1, 1, n_3 - 1)}{d(m_1^2)} \mathcal{I}_3 + \frac{c_{3 \rightarrow 3}(1, 1, n_3 - 1)}{(n_3 - 1)} I_3(1, 1, 2) \\ &\quad + \sum_{i=1}^3 \frac{dc_{3 \rightarrow 2; \bar{i}}(1, 1, n_3 - 1)}{(n_3 - 1) d(m_1^2)} \mathcal{I}_{2; \bar{i}} + \frac{c_{3 \rightarrow 2; \bar{1}}(1, 1, n_3 - 1)}{(n_3 - 1)} I_{2; \bar{1}}(1, 2) \\ &\quad + \frac{c_{3 \rightarrow 2; \bar{2}}(1, 1, n_3 - 1)}{(n_3 - 1)} I_{2; \bar{2}}(1, 2) + \dots \end{aligned} \quad (3.21)$$

where in the last line, since the $\mathcal{I}_{2; \bar{3}}$ does not depend on m_1 , the action of $\frac{d}{d(m_1^2)}$ is zero (see (3.10), (3.11) and (3.12)). Now, using the expansion of $I_3(1, 1, 2)$, $I_2(1, 2)$ and $I_2(2, 1)$, we get immediately the recurrence relation

$$\begin{aligned} c_3(1, 1, n_3) &= \frac{1}{(n_3 - 1)} \frac{dc_{3 \rightarrow 3}(1, 1, n_3 - 1)}{d(m_1^2)} + \frac{c_{3 \rightarrow 3}(1, 1, n_3 - 1)}{(n_3 - 1)} c_{3 \rightarrow 3}(1, 1, 2) \\ c_{3 \rightarrow 2; \bar{1}}(1, 1, n_3) &= \frac{c_{3 \rightarrow 3}(1, 1, n_3 - 1)}{(n_3 - 1)} c_{3 \rightarrow 2; \bar{1}}(1, 1, 2) + \frac{1}{(n_3 - 1)} \frac{dc_{3 \rightarrow 2; \bar{1}}(1, 1, n_3 - 1)}{d(m_1^2)} \\ &\quad + \frac{c_{3 \rightarrow 2; \bar{1}}(1, 1, n_3 - 1)}{(n_3 - 1)} c_{2 \rightarrow 2; \bar{1}}(1, 2) \end{aligned}$$

$$\begin{aligned}
 c_{3 \rightarrow 2; \bar{2}}(1, 1, n_3) &= \frac{c_{3 \rightarrow 3}(1, 1, n_3 - 1)}{(n_3 - 1)} c_{3 \rightarrow 2; \bar{2}}(1, 1, 2) + \frac{1}{(n_3 - 1)} \frac{dc_{3 \rightarrow 2; \bar{2}}(1, 1, n_3 - 1)}{d(m_1^2)} \\
 &\quad + \frac{c_{3 \rightarrow 2; \bar{2}}(1, 1, n_3 - 1)}{(n - 1)} c_{2 \rightarrow 2; \bar{2}}(2, 1) \\
 c_{3 \rightarrow 2; \bar{3}}(1, 1, n_3) &= \frac{c_{3 \rightarrow 3}(1, 1, n_3 - 1)}{(n_3 - 1)} c_{3 \rightarrow 2; \bar{3}}(1, 1, 2) + \frac{1}{(n_3 - 1)} \frac{dc_{3 \rightarrow 2; \bar{3}}(1, 1, n_3 - 1)}{d(m_1^2)} \quad (3.22)
 \end{aligned}$$

Knowing (3.22), it is easy to use (3.15) and (3.17) to get reduction coefficients for $I_3(1, n, 1)$ and $I_3(n, 1, 1)$ as shown in (3.18). One point we want to emphasize is that since g_2, g_3 acts on K, M also, the kinematic dependence of K, M in (3.22) should be carefully identified although for simplicity we have not written them down explicitly.

Knowing the reduction of $I_3(1, 1, n_3)$, now we consider the case $I_3(1, n_2, n_3)$. Similarly to (3.21), we have

$$I_3(1, n_2, n_3) = \frac{1}{(n_2 - 1)} \frac{\partial}{\partial M_2^2} \left\{ c_{3 \rightarrow 3}(1, n_2 - 1, n_3) I_3 + \sum_{i=1}^3 c_{3 \rightarrow 2; i}(1, n_2 - 1, n_3) I_{2; i} + \dots \right\} \quad (3.23)$$

thus we can get the recurrence relation for reduction coefficients

$$\begin{aligned}
 c_{3 \rightarrow 3}(1, n_2, n_3) &= \frac{1}{n_2 - 1} \left(\frac{\partial c_{3 \rightarrow 3}(1, n_2 - 1, n_3)}{\partial M_2^2} + c_{3 \rightarrow 3}(1, n_2 - 1, n_3) c_{3 \rightarrow 3}(1, 2, 1) \right) \\
 c_{3 \rightarrow 2; \bar{1}}(1, n_2, n_3) &= \frac{1}{n_2 - 1} \left(c_{3 \rightarrow 3}(1, n_2 - 1, n_3) c_{3 \rightarrow 2; \bar{1}}(1, 2, 1) + \frac{\partial c_{3 \rightarrow 2; \bar{1}}(1, n_2 - 1, n_3)}{\partial M_2^2} \right. \\
 &\quad \left. + c_{3 \rightarrow 2; \bar{1}}(1, n_2 - 1, n_3) c_{2 \rightarrow 2; \bar{1}}(2, 1) \right) \\
 c_{3 \rightarrow 2; \bar{2}}(1, n_2, n_3) &= \frac{1}{n_2 - 1} \left(c_{3 \rightarrow 3}(1, n_2 - 1, n_3) c_{3 \rightarrow 2; \bar{2}}(1, 2, 1) + \frac{\partial c_{3 \rightarrow 2; \bar{2}}(1, n_2 - 1, n_3)}{\partial M_2^2} \right) \\
 c_{3 \rightarrow 2; \bar{3}}(1, n_2, n_3) &= \frac{1}{n_2 - 1} \left(c_{3 \rightarrow 3}(1, n_2 - 1, n_3) c_{3 \rightarrow 2; \bar{3}}(1, 2, 1) + \frac{\partial c_{3 \rightarrow 2; \bar{3}}(1, n_2 - 1, n_3)}{\partial M_2^2} \right. \\
 &\quad \left. + c_{3 \rightarrow 2; \bar{3}}(1, n_2 - 1, n_3) c_{2 \rightarrow 2}(1, 2) \right) \quad (3.24)
 \end{aligned}$$

So if we have known the expansion coefficients of $I_3(1, n_2 - 1, n_3)$ to the basis I_3 and I_2 , we can derive the expansion of $I_3(1, n_2, n_3)$.

Finally, for $I_3(n_1, n_2, n_3)$ with $n_1 > 1$, similar action lead to the recurrence relation

$$\begin{aligned}
 c_{3 \rightarrow 3}(n_1, n_2, n_3) &= \frac{1}{n_1 - 1} \left(\frac{\partial c_{3 \rightarrow 3}(n_1 - 1, n_2, n_3)}{\partial M_1^2} + c_{3 \rightarrow 3}(n_1 - 1, n_2, n_3) c_{3 \rightarrow 3}(2, 1, 1) \right) \\
 c_{3 \rightarrow 2; \bar{1}} &= \frac{1}{n_1 - 1} \left(c_{3 \rightarrow 3}(n_1 - 1, n_2, n_3) c_{3 \rightarrow 2; \bar{1}}(2, 1, 1) + \frac{\partial c_{3 \rightarrow 2; \bar{1}}(n_1 - 1, n_2, n_3)}{\partial M_1^2} \right) \\
 c_{3 \rightarrow 2; \bar{2}} &= \frac{1}{n_1 - 1} \left(c_{3 \rightarrow 3}(n_1 - 1, n_1, n_3) c_{3 \rightarrow 2; \bar{2}}(2, 1, 1) + \frac{\partial c_{3 \rightarrow 2; \bar{2}}(n_1 - 1, n_2, n_3)}{\partial M_1^2} \right. \\
 &\quad \left. + c_{3 \rightarrow 2; \bar{2}}(n_1 - 1, n_2, n_3) c_{2 \rightarrow 2; \bar{2}}(2, 1) \right)
 \end{aligned}$$

$$c_{2,\bar{3}} = \frac{1}{n_1 - 1} \left(c_3(n_1 - 1, n_2, n_3) c_{2,\bar{3}}(2, 1, 1) + \frac{\partial c_{2,\bar{3}}(n_1 - 1, n_2, n_3)}{\partial M_1^2} \right. \\ \left. + c_{2,\bar{3}}(n_1 - 1, n_2, n_3) c_{2,1}[K_1, M_1, M_2] \right) \quad (3.25)$$

3.2 Triangle(1,1,2)

From the recurrence relation in previous subsection, we see that all computations have been reduced to the reduction of $I_3(1, 1, 2)$. Although there are other methods to reduce $I_3(1, 1, 2)$ such as the IBP method. In this part, we show how to use unitarity cut method to fulfill the task. Using our idea, we can write

$$I_3(1, 1, 2) = \frac{\partial}{\partial m_1^2} I_3(1, 1, 1). \quad (3.26)$$

To get all reduction coefficients, we need to calculate all three cuts. However, different cuts can be related using the symmetry discussed in previous subsection, thus we will focus on only one cut.

3.2.1 Cut K_1

For this cut, the r.h.s. is given by

$$\mathcal{C}_{K_1}(I_3(1, 1, 2)) = \frac{\partial}{\partial m_1^2} \mathcal{C}(I_3) = - \left(\frac{4K_1^2}{\Delta[K_1, M_1, M_2]} \right)^\epsilon \frac{1}{\sqrt{\Delta_{3;m=0}}} \frac{\partial}{\partial m_1^2} Tri^{(0)}(Z) \quad (3.27)$$

where we have used the fact that only Z contains the m_1 . Carrying out the derivative, we get

$$\frac{\partial}{\partial m_1^2} Tri^{(0)}(Z) = \frac{\partial Z}{\partial m_1^2} \frac{\partial}{\partial Z} Tri^{(0)}(Z), \quad \frac{\partial Z}{\partial m_1^2} = \frac{2K_1^2}{\sqrt{\Delta_{3;m=0}} \Delta[K_1, M_1, M_2]} \\ \frac{\partial}{\partial Z} Tri^{(0)}(Z) = \int_0^1 du u^{-1-\epsilon} \left[\frac{1}{Z + \sqrt{1-u}} - \frac{1}{Z - \sqrt{1-u}} \right] \\ = 2 \int_0^1 du u^{-1-\epsilon} \frac{1-u}{\sqrt{1-u}(1-u-Z^2)} \quad (3.28)$$

Among two terms of $\frac{\partial}{\partial Z} Tri^{(0)}(Z)$, by comparing with (3.3), the second term is

$$2 \int_0^1 du u^{-1-\epsilon} \frac{-u}{\sqrt{1-u}(1-u-Z^2)} = -\frac{2\epsilon}{Z} Tri^{(0)}(Z) \quad (3.29)$$

For the first term, noting that

$$\int_0^1 du u^{-1-\epsilon} \frac{1}{\sqrt{1-u}} = (1-Z^2) \int_0^1 du u^{-1-\epsilon} \frac{1}{\sqrt{1-u}(1-u-Z^2)} - \int_0^1 du u^{-\epsilon} \frac{1}{\sqrt{1-u}(1-u-Z^2)} \quad (3.30)$$

we have

$$\frac{1}{1-Z^2} \frac{\epsilon}{Z} Tri^{(0)}(Z) + \frac{1-2\epsilon}{1-Z^2} Bub^{(0)} \quad (3.31)$$

Putting all together, we get

$$\frac{\partial}{\partial Z} Tri^{(0)}(Z) = \frac{2(1-2\epsilon)}{1-Z^2} Bub^{(0)} + \frac{2Z\epsilon}{1-Z^2} Tri^{(0)}(Z) \quad (3.32)$$

By identifying

$$\frac{\partial}{\partial m_1^2} Cut I_3(1, 1, 1) = c_{3 \rightarrow 3; K_1}(1, 1, 2) Cut I_3(1, 1, 1) + c_{3 \rightarrow 2; \bar{3}; K_1}(1, 1, 2) Cut I_2(1, 1) \quad (3.33)$$

we get the reduction coefficients as

$$c_{3 \rightarrow 3; K_1}(1, 1, 2) = \frac{4K_1^2}{\sqrt{\Delta_{3; m=0} \Delta[K_1, M_1, M_2]}} \frac{Z\epsilon}{1-Z^2} \quad (3.34)$$

$$c_{3 \rightarrow 2; \bar{3}; K_1}(1, 1, 2) = -\frac{4K_1^2}{\Delta[K_1, M_1, M_2] \Delta_{3; m=0}} \frac{1-2\epsilon}{1-Z^2} \quad (3.35)$$

where for simplicity we have not expanded Z further (see (3.4)). We want to remark that the coefficients $c_{3 \rightarrow 3; K_1}$ and $c_{3 \rightarrow 2; \bar{3}; K_1}$ are nothing, but the $c_{3 \rightarrow 3}$ and $c_{3 \rightarrow 2; \bar{3}}$ is the reduction

$$I_3(1, 1, 2) = c_{3 \rightarrow 3}(1, 1, 2) I_3 + c_{3 \rightarrow 2; \bar{3}}(1, 1, 2) I_{3; \bar{3}} + \dots \quad (3.36)$$

The reason that we have added subscript K_1 in (3.34) and (3.35) is to emphasize that these two expressions are calculated using the cut K_1 .

3.2.2 Cut K_2

No we consider the cut K_2 . The use the cut result (3.2), we need to rewrite the form (3.1) into the standard form for the cut K_2 by shifting the integral momentum $p \rightarrow p + K_1$, thus it becomes

$$\begin{aligned} I_3(n_1, n_2, n_3)[K_1, K_2, K_3; M_1, M_2, m_1] &= \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{((p+K_1)^2 - M_1^2)^{n_1} (p^2 - M_2^2)^{n_2} ((p-K_2)^2 - m_1^2)^{n_3}} \\ &= I_3(n_2, n_3, n_1)[K_2, K_3, K_1; M_2, m_1, M_1] \\ &= g_3 \{ I_3(n_1, n_2, n_3)[K_1, K_2, K_3; M_1, M_2, m_1] \} \end{aligned} \quad (3.37)$$

For the case $n_3 = 2, n_1 = n_2 = 1$

$$\begin{aligned} I_3(1, 1, 2)[K_1, K_2, K_3; M_1, M_2, m_1] &= \frac{\partial}{\partial m_1^2} I_3(1, 1, 1)[K_1, K_2, K_3; M_1, M_2, m_1] \\ &= \frac{\partial}{\partial m_1^2} I_3(1, 1, 1)[K_2, K_3, K_1; M_2, m_1, M_1] \\ &= g_3 \left\{ \frac{\partial}{\partial M_2^2} I_3(1, 1, 1)[K_1, K_2, K_3; M_1, M_2, m_1] \right\} \end{aligned} \quad (3.38)$$

Thus we have reduced the problem to the cut K_1 , but with different action $\frac{\partial}{\partial M_2^2}$.

For the action of $\frac{\partial}{\partial M_2^2}$, there are two terms containing the parameter M_2 in the $Cut_{K_1}(I_3(1, 1, 2))$.

$$\begin{aligned} \frac{\partial}{\partial M_2^2} Cut_{K_1}(I_3(1, 1, 2)) = & -\frac{1}{\sqrt{\Delta_{3;m=0}}} \left(\frac{\partial}{\partial M_2^2} \left(\frac{\Delta}{4K_1^2} \right)^{-\epsilon} \right) Tri^{(0)}(Z) \\ & + \left(-\frac{1}{\sqrt{\Delta_{3;m=0}}} \right) \left(\frac{\Delta}{4K_1^2} \right)^{-\epsilon} \frac{\partial}{\partial M_2^2} Tri^{(0)}(Z) \end{aligned} \quad (3.39)$$

where the Δ means $\Delta[K_1, M_1, M_2]$. The first term in (3.39) is trivial, while the derivative part in the second term is just $\frac{\partial Z}{\partial M_2^2} \frac{\partial}{\partial Z} Tri^{(0)}(Z)$, where $\frac{\partial}{\partial Z} Tri^{(0)}(Z)$ has been given in (3.32). Put them altogether, we get

$$\frac{\partial}{\partial M_2^2} C_{K_1}(I_3) = \tilde{c}_3 C(I_3) + \tilde{c}_2 C(I_2) \quad (3.40)$$

with

$$\tilde{c}_3 = \Delta^\epsilon \frac{\partial}{\partial M_2^2} \Delta^{-\epsilon} + \frac{\partial Z}{\partial M_2^2} \frac{2Z\epsilon}{1-Z^2}, \quad \tilde{c}_2 = -\frac{1}{\sqrt{\Delta_{3;m=0}}} \frac{K_1^2}{\sqrt{\Delta}} \frac{\partial Z}{\partial M_2^2} \frac{2(1-2\epsilon)}{1-Z^2} \quad (3.41)$$

Of course, the $\tilde{c}_2$ and $\tilde{c}_3$ in (3.41) are not the final result, since we have to do the permutation g_3 to get $c_{3 \rightarrow 3}(1, 1, 2) = g_3(\tilde{c}_2)$ and $c_{3 \rightarrow 2; \bar{1}}(1, 1, 2) = g_3(\tilde{c}_3)$. One can find that the $g_3(\tilde{c}_2)$ is equal to $c_{3 \rightarrow 3}(1, 1, 2)$ given in (3.34), thus we have passed the first consistent check.

3.2.3 Cut K_3

Now we consider the cut K_3 . Again we need to rewrite the form (3.1) into the standard form for the cut K_3 by shifting the integral momentum $p \rightarrow p + K_1 + K_2$. With similar argument we will have

$$\begin{aligned} I_3(1, 1, 2)[K_1, K_2, K_3; M_1, M_2, m_1] &= \frac{\partial}{\partial m_1^2} I_3(1, 1, 1)[K_3, K_1, K_2; m_1, M_1, M_2] \\ &= g_3^{-1} \left\{ \frac{\partial}{\partial M_1^2} I_3(1, 1, 2)[K_1, K_2, K_3; M_1, M_2, m_1] \right\} \end{aligned} \quad (3.42)$$

thus the cut K_3 has been reduced to the cut K_1 with the action $\frac{\partial}{\partial M_1^2}$. The calculation is similar to the previous subsection. After simplification, we have

$$\frac{\partial}{\partial M_1^2} C_{K_1}(I_3) = \hat{c}_3 C_{K_1}(I_3) + \hat{c}_2 C_{K_1}(I_2) \quad (3.43)$$

with the coefficients

$$\begin{aligned} \hat{c}_3 &= -\epsilon \Delta^{-1} \frac{\partial \Delta}{\partial M_1^2} + \frac{\partial Z}{\partial M_1^2} \frac{2Z\epsilon}{1-Z^2} \\ \hat{c}_2 &= \left(-\frac{1}{\sqrt{\Delta_{3;m=0}}} \right) \frac{K_1^2}{\sqrt{\Delta}} \frac{\partial Z}{\partial M_1^2} \frac{2(1-2\epsilon)}{1-Z^2} \end{aligned} \quad (3.44)$$

After doing the permutation g_3^{-1} on the coefficients above, we get the $c_{3 \rightarrow 3}$ and $c_{3 \rightarrow 2; \bar{2}}$ in this cut. Again, one can check the $c_{3 \rightarrow 3}$ is the same with $c_{3 \rightarrow 3; K_1}$ in (3.34).

For the case $I_3(1, 1, 2)$ there is another consistent check we can do. Noticing that when we do following changing variable $p \rightarrow -p + K_1$, the (3.1) becomes to

$$I_3(n_1, n_2, n_3)[1] = \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{1}{((p - K_1)^2 - M_1^2)^{n_1} (p^2 - M_2^2)^{n_2} ((p + K_2) - m_1^2)^{n_3}} \quad (3.45)$$

Thus for $(n_1, n_2, n_3) = (1, 1, 2)$, there is the symmetry among the reduction coefficients, i.e., under the permutation $K_2 \longleftrightarrow K_3, M_1 \longleftrightarrow M_2$, we should have $c_{3 \rightarrow 3}$ invariant and $c_{3 \rightarrow 2; \bar{2}} \leftrightarrow c_{3 \rightarrow 2; \bar{1}}$. This can be easily checked by MATHEMATICA for our results.

3.2.4 A short summary

In the subsection, we have reduced the $I_3(1, 1, 2)$ to triangle I_3 and bubble I_2 . After taking three different cuts, we get the all needed triangle and bubble coefficients of the reduction of $I_3(1, 1, 2)$

$$I_3(1, 1, 2) = c_{3 \rightarrow 3}(1, 1, 2)I_3 + \sum_{i=1}^3 c_{3 \rightarrow 2; \bar{i}}(1, 1, 2)I_{2; \bar{i}} + \dots \quad (3.46)$$

where the $\dots$ represents the tadpoles neglected in the whole paper. The coefficients in (3.46) are

$$\begin{aligned} c_{3 \rightarrow 3}(1, 1, 2) &= \frac{4K_1^2}{\sqrt{\Delta_{3; m=0}} \Delta[K_1, M_1, M_2]} \frac{Z\epsilon}{1 - Z^2} \\ c_{3 \rightarrow 2; \bar{3}}(1, 1, 2) &= -\frac{4K_1^4}{\Delta[K_1, M_1, M_2] \Delta_{3; m=0}} \frac{1 - 2\epsilon}{1 - Z^2} \\ c_{3 \rightarrow 2; \bar{1}}(1, 1, 2) &= \hat{g}_3 \left(-\frac{1}{\sqrt{\Delta_{3; m=0}}} \frac{K_1^2}{\sqrt{\Delta[K_1, M_1, M_2]}} \frac{\partial Z}{\partial M_2^2} \frac{2(1 - 2\epsilon)}{1 - Z^2} \right) \\ c_{3 \rightarrow 2; \bar{2}}(1, 1, 2) &= \hat{g}_3^{-1} \left(-\frac{1}{\sqrt{\Delta_{3; m=0}}} \frac{K_1^2}{\sqrt{\Delta[K_1, M_1, M_2]}} \frac{\partial Z}{\partial M_1^2} \frac{2(1 - 2\epsilon)}{1 - Z^2} \right) \end{aligned} \quad (3.47)$$

The result is confirmed with IBP method using LiteRed [23]. We have also carried out the reduction of $I_3(1, 2, 1)$, $I_3(2, 1, 1)$ and $I_3(1, 2, 2)$ using the method laid out in the paper and found perfect match with the IBP method.

4 Box

For the box topology, we define

$$\begin{aligned} I_4(n_1, n_2, n_3, n_4) \\ = \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{1}{(p^2 - M_1^2)^{n_1} ((p - K_1)^2 - M_2^2)^{n_2} ((p - K_1 - K_2)^2 - m_1^2)^{n_3} ((p + K_4)^2 - m_2^2)^{n_4}} \end{aligned} \quad (4.1)$$

or

$$\begin{aligned} I_4(n_1, n_2, n_3, n_4) \\ = \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{1}{(p^2 - M_1^2)^{n_1} ((p - K)^2 - M_2^2)^{n_2} ((p - P_1)^2 - m_1^2)^{n_3} ((p - P_2)^2 - m_2^2)^{n_4}} \end{aligned} \quad (4.2)$$

for later convenience, where we have labeled the cut momentum as K and the masses of two cut propagators as M_1 and M_2 respectively. For the other two momenta we denote

them as P_1 and P_2 respectively. For example, if we choose K_1 as the cut momentum, we will have $P_1 = K_1 + K_2$ and $P_2 = -K_4$, but if the cut momentum is $K_1 + K_2$, we will have $P_1 = K_1, P_2 = K_1 + K_2 + K_3$.

The special case with $n_i = 1, i = 1, 2, 3, 4$ gives the scalar basis, which we will denote as I_4 . The unitarity cut of I_4 with the cut momentum K has been given by [9]

$$C(I_4) = \left(\frac{\Delta}{4K^2}\right)^{-\epsilon} \frac{b}{2K^2} \int_0^1 du u^{-1-\epsilon} \frac{1}{\sqrt{B-Au}} \ln \left(\frac{D-Cu + \sqrt{1-u}\sqrt{B-Au}}{D-Cu - \sqrt{1-u}\sqrt{B-Au}} \right) \quad (4.3)$$

with the parameters Δ in (2.5) and⁹

$$\begin{aligned} A &= -\frac{b^4}{K^2} \det \begin{pmatrix} P_1^2 & P_1 \cdot P_2 & P_1 \cdot K \\ P_1 \cdot P_2 & P_2^2 & P_2 \cdot K \\ P_1 \cdot K & P_2 \cdot K & K^2 \end{pmatrix}, & B &= -\det \begin{pmatrix} R_1^2 & R_1 \cdot R_2 \\ R_1 \cdot R_2 & R_2^2 \end{pmatrix} \\ C &= \frac{b^2}{K^2} \det \begin{pmatrix} P_1 \cdot P_2 & P_1 \cdot K \\ P_2 \cdot K & K^2 \end{pmatrix}, & D &= R_1 \cdot R_2 \end{aligned} \quad (4.4)$$

where

$$b = \frac{\sqrt{\Delta[K, M_1, M_2]}}{K^2}, \quad a_i = \frac{P_i^2 + M_1^2 - m_1^2}{K^2}, \quad R_i = -bP_i + \left(a_i - \frac{P_i \cdot K}{K^2} (a-b)\right) K \quad (4.5)$$

With the definition

$$Box^{(n)} = \int_0^1 du u^{-1-\epsilon} \frac{u^n}{\sqrt{B-Au}} \ln \left(\frac{D-Cu + \sqrt{1-u}\sqrt{B-Au}}{D-Cu - \sqrt{1-u}\sqrt{B-Au}} \right) \quad (4.6)$$

we can write $C(I_4)$ as

$$C(I_4) = \left(\frac{\Delta}{4K^2}\right)^{-\epsilon} \frac{b}{2K^2} Box^{(0)} \quad (4.7)$$

and there is also a recurrence relation of $Box^{(n)}$, which is given by [8, 9]

$$Box^{(n)} = \frac{n-1-\epsilon}{n-\frac{1}{2}-\epsilon} \frac{B}{A} Box^{(n-1)} - \frac{n-1-\epsilon}{n-\frac{1}{2}-\epsilon} \frac{C_{Z_1}}{AZ_1} Tri^{(n-1)}(Z_1) - \frac{n-1-\epsilon}{n-\frac{1}{2}-\epsilon} \frac{C_{Z_2}}{AZ_2} Tri^{(n-1)}(Z_2) \quad (4.8)$$

with the parameter

$$C_{Z_i} = D + (Z_i^2 - 1)C \quad (4.9)$$

To be calar, we list six possible cuts of a box, with K_1, K_2, K_3, K_4 in clockwise ordering. And there will be two cut triangles related to each given cut momentum K of box:

Box Cut K	P_1	P_2	Triangle One's (K_1, K_3)	Triangle Two's (K_1, K_3)
K_1	K_{12}	$-K_4$	(K_1, K_{34})	(K_1, K_4)
K_2	K_{23}	$-K_1$	(K_2, K_{41})	(K_2, K_1)
K_3	K_{34}	$-K_2$	(K_3, K_{12})	(K_3, K_2)
K_4	K_{41}	$-K_3$	(K_4, K_{23})	(K_4, K_3)
K_{12}	K_1	$-K_4$	(K_{34}, K_2)	(K_{12}, K_4)
K_{23}	K_2	$-K_1$	(K_{41}, K_3)	(K_{23}, K_1)

⁹The A corresponds to the second-type singularity of box.

Symmetry analysis. Similar to the case of triangle, there are some symmetries of box by momentum shifting and reflection. Let us define two generators

$$\begin{aligned}\hat{g}_4 : & \{((n_1, n_2, n_3, n_4); (K_1, K_2, K_3, K_4); (M_1, M_2, m_1, m_2))\} \\ & \rightarrow \{(n_2, n_3, n_4, n_1); (K_2, K_3, K_4, K_1); (M_2, m_1, m_2, M_1)\}\end{aligned}\quad (4.10)$$

and

$$\begin{aligned}\hat{g}_2 : & \{((n_1, n_2, n_3, n_4); (K_1, K_2, K_3, K_4); (M_1, M_2, m_1, m_2))\} \\ & \rightarrow \{(n_1, n_4, n_3, n_2); (K_4, K_3, K_2, K_1); (M_1, m_2, m_1, M_2)\}\end{aligned}\quad (4.11)$$

Using them we get the dihedral group D_4 and following relations

$$\begin{aligned}I_4(n_2, n_3, n_4, n_1)[K_2, K_3, K_4, K_1; M_2, m_1, m_2, M_1] \\ &= \hat{g}_4 I_4(n_1, n_2, n_3, n_4)[K_1, K_2, K_3, K_4; M_1, M_2, m_1, m_2] \\ I_4(n_3, n_4, n_1, n_2)[K_3, K_4, K_1, K_2; m_1, m_2, M_1, M_2] \\ &= \hat{g}_4^2 I_4(n_1, n_2, n_3, n_4)[K_1, K_2, K_3, K_4; M_1, M_2, m_1, m_2] \\ I_4(n_4, n_1, n_2, n_3)[K_4, K_1, K_2, K_3; m_2, M_1, M_2, m_1] \\ &= \hat{g}_4^3 I_4(n_1, n_2, n_3, n_4)[K_1, K_2, K_3, K_4; M_1, M_2, m_1, m_2]\end{aligned}\quad (4.12)$$

and

$$\begin{aligned}I_4(n_1, n_4, n_3, n_2)[K_4, K_3, K_2, K_1; M_1, m_2, m_1, M_2] \\ &= \hat{g}_2 I_4(n_1, n_2, n_3, n_4)[K_1, K_2, K_3, K_4; M_1, M_2, m_1, m_2] \\ I_4(n_4, n_3, n_2, n_1)[K_3, K_2, K_1, K_4; m_2, m_1, M_2, M_1] \\ &= \hat{g}_4 \hat{g}_2 I_4(n_1, n_2, n_3, n_4)[K_1, K_2, K_3, K_4; M_1, M_2, m_1, m_2] \\ I_4(n_3, n_2, n_1, n_4)[K_2, K_1, K_4, K_3; m_1, M_2, M_1, m_2] \\ &= \hat{g}_4^2 \hat{g}_2 I_4(n_1, n_2, n_3, n_4)[K_1, K_2, K_3, K_4; M_1, M_2, m_1, m_2] \\ I_4(n_2, n_1, n_4, n_3)[K_1, K_4, K_3, K_2; M_2, M_1, m_2, m_1] \\ &= \hat{g}_4^3 \hat{g}_2 I_4(n_1, n_2, n_3, n_4)[K_1, K_2, K_3, K_4; M_1, M_2, m_1, m_2]\end{aligned}\quad (4.13)$$

Furthermore, using the same idea in the subsection 3.1 we can write down similar recurrence relation of reduction coefficients for general $I_4(n_1, n_2, n_3, n_4)$ using the expansion of $I_4(2, 1, 1, 1)$, $I_4(1, 2, 1, 1)$, $I_4(1, 1, 2, 1)$ and $I_4(1, 1, 1, 2)$. However, by relation (4.12) and (4.13), all three cases $I_4(2, 1, 1, 1)$, $I_4(1, 2, 1, 1)$ and $I_4(1, 1, 2, 1)$ can be reduced to the reduction of $I_4(1, 1, 1, 2)$, thus we need only to deal with the reduction of $I_4(1, 1, 1, 2)$.

4.1 Box $I_4(1, 1, 1, 2)$

Now, we use our method to calculate the reduction coefficients of $I_4(1, 1, 1, 2)$. According our idea, we can write

$$I_4(1, 1, 1, 2) = \frac{\partial}{\partial m_2^2} I_4(1, 1, 1, 1) \quad (4.14)$$

Comparing with the calculation of triangle, we need to consider six different cuts: four cuts with nearby propagators and two with opposite propagators. By the symmetry, we could just calculate one for each type of cuts, and get the others by proper permutation. To demonstrate our method, let us show the calculation for the cut K_1 only. The reduction coefficients calculated for $I_4(1, 1, 1, 2)$ has been checked using the LiteRed [23].

For the cut K_1 , we have $K = K_1$, $P_1 = K_1 + K_2$ and $P_2 = -K_4$ in (4.2). To calculate $\frac{\partial}{\partial m_2^2} C(I_4)$, since the parameter Δ and b do not contain the m_2 , we could just write it as

$$\frac{\partial}{\partial m_2^2} C(I_4) = \left(\frac{\Delta}{4K^2} \right)^{-\epsilon} \frac{b}{2K^2} \frac{\partial}{\partial m_2^2} Box^{(0)} \quad (4.15)$$

To simplify $\frac{\partial}{\partial m_2^2} Box^{(0)}$, first we rewrite $Box^{(0)}$ as by partial integration and algebraic separation as

$$\begin{aligned} Box^{(0)} = & \frac{2}{-A} \left[(1+\epsilon) \int_0^1 du u^{-2-\epsilon} \sqrt{B-Au} \ln \left(\frac{D-Cu + \sqrt{1-u}\sqrt{B-Au}}{D-Cu - \sqrt{1-u}\sqrt{B-Au}} \right) \right. \\ & + \frac{\epsilon C_{Z_1}}{Z_1(Z_1^2-1)} Tri^{(0)}(Z_1) + \frac{\epsilon C_{Z_2}}{Z_2(Z_2^2-1)} Tri^{(0)}(Z_2) \\ & \left. + \left(\frac{C_{Z_1}(1-2\epsilon)}{Z_1^2-1} + \frac{C_{Z_2}(1-2\epsilon)}{Z_2^2-1} \right) Bub^{(0)} \right] \end{aligned} \quad (4.16)$$

where the bubble and two triangles are specified by our cut K . Now we consider the action of $\frac{\partial}{\partial m_2^2}$. Since only B and D contains the parameter m_2 , we can write directly

$$\begin{aligned} \frac{\partial}{\partial m_2^2} Box^{(0)} = & \frac{2}{-A} \left[(1+\epsilon) \int_0^1 du u^{-2-\epsilon} \frac{B'_{m_2}}{2\sqrt{B-Au}} \ln \left(\frac{D-Cu + \sqrt{1-u}\sqrt{B-Au}}{D-Cu - \sqrt{1-u}\sqrt{B-Au}} \right) \right. \\ & + (1+\epsilon) \int_0^1 du u^{-2-\epsilon} \frac{(1-u) \left(u \left(2AD'_{(m_2^2)} - B'_{(m_2^2)} C \right) + \left(B'_{(m_2^2)} D - 2BD'_{(m_2^2)} \right) \right)}{\sqrt{1-u} \left((D-Cu)^2 - (1-u)(B-Au) \right)} \\ & + \frac{\partial}{\partial m_2^2} \left(\frac{\epsilon C_{Z_1}}{Z_1(Z_1^2-1)} \right) Tri^{(0)}(Z_1) + \frac{\partial}{\partial m_2^2} \left(\frac{\epsilon C_{Z_2}}{Z_2(Z_2^2-1)} \right) Tri^{(0)}(Z_2) \\ & + \frac{\epsilon C_{Z_1}}{Z_1(Z_1^2-1)} \frac{\partial}{\partial m_2^2} Tri^{(0)}(Z_1) + \frac{\epsilon C_{Z_2}}{Z_2(Z_2^2-1)} \frac{\partial}{\partial m_2^2} Tri^{(0)}(Z_2) \\ & \left. + \frac{\partial}{\partial m_2^2} \left(\frac{C_{Z_1}(1-2\epsilon)}{Z_1^2-1} + \frac{C_{Z_2}(1-2\epsilon)}{Z_2^2-1} \right) Bub^{(0)} \right] \end{aligned} \quad (4.17)$$

where the $B'_{(m_2^2)}$ and $D'_{(m_2^2)}$ means $\frac{\partial}{\partial m_2^2} B$ and $\frac{\partial}{\partial m_2^2} D$. There are five terms in the bracket. Since the third line and fifth line are just some simple differentiation over coefficients, we calculate the first, second and four lines only:

- (1) For the first term, it is $\frac{(1+\epsilon)B'_{(m_2^2)}}{2} Box^{(-1)}$, while using (4.8), we can expand

$Box^{(-1)}$ as

$$Box^{(-1)} = \frac{\frac{1}{2} + \epsilon}{1 + \epsilon} \frac{A}{B} Box^{(0)} + \frac{\epsilon}{1 + \epsilon} \frac{C_{Z_1}}{B Z_1 (1 - Z_1^2)} Tri^{(0)}(Z_1) \quad (4.18)$$

$$+ \frac{\epsilon}{1 + \epsilon} \frac{C_{Z_2}}{B Z_2 (1 - Z_2^2)} Tri^{(0)}(Z_2) + \frac{1 - 2\epsilon}{1 + \epsilon} \left(\frac{C_{Z_1}}{(1 - Z_1^2)B} + \frac{C_{Z_2}}{(1 - Z_2^2)B} \right) Bub^{(0)}$$

- (2) For the second term, using the trick of splitting terms and the recurrence relation, we get

$$C_{3;Z_1} Tri^{(0)}(Z_1) + C_{3;Z_2} Tri^{(0)}(Z_2) + C_2 Bub^{(0)} \quad (4.19)$$

with

$$C_{3;Z_1;K_1} = \frac{\epsilon Z_1 \left(\frac{\partial B}{\partial m_2^2} D - 2B \frac{\partial D}{\partial m_2^2} + (Z_1^2 - 1) \left(\frac{\partial B}{\partial m_2^2} C - 2A \frac{\partial D}{\partial m_2^2} \right) \right)}{\beta_1 \beta_2 (Z_1^2 - 1)^2 (Z_1^2 - Z_2^2)}$$

$$C_{3;Z_2;K_2} = \frac{\epsilon Z_2 \left(\left(\frac{\partial B}{\partial m_2^2} D - 2B \frac{\partial D}{\partial m_2^2} \right) + (Z_2^2 - 1) \left(\frac{\partial B}{\partial m_2^2} C - 2A \frac{\partial D}{\partial m_2^2} \right) \right)}{\beta_1 \beta_2 (Z_2^2 - 1)^2 (Z_2^2 - Z_1^2)}$$

$$C_{2;K_1} = \frac{n_{2;K_1}}{2\beta_1 \beta_2 (Z_1^2 - 1)^2 (Z_2^2 - 1)^2 (\epsilon + 1)} \quad (4.20)$$

where

$$n_{2;K_1} = (2\epsilon - 1) \left(2 \frac{\partial B}{\partial m_2^2} C (Z_1^2 - 1) (Z_2^2 - 1) (1 + \epsilon) - 4A \frac{\partial D}{\partial m_2^2} (Z_1^2 - 1) (Z_2^2 - 1) (1 + \epsilon) \right. \\ \left. + \frac{\partial B}{\partial m_2^2} D (-3 - 4\epsilon + Z_2^2 (1 + 2\epsilon) + Z_1^2 (1 + Z_2^2 + 2\epsilon)) \right. \\ \left. - 2B \frac{\partial D}{\partial m_2^2} (-3 - 4\epsilon + Z_2^2 (1 + 2\epsilon) + Z_1^2 (1 + Z_2^2 + 2\epsilon)) \right)$$

- (3) For the fourth line with $\frac{\partial}{\partial m_2^2} Tri^{(0)}$, we could use result (3.32) and (3.28) to rewrite it as

$$- \frac{\partial Z_1}{\partial m_2^2} \frac{2\epsilon^2 C_{Z_1}}{(Z_1^2 - 1)^2} Tri^{(0)}(Z_1) - \frac{\partial Z_2}{\partial m_2^2} \frac{2\epsilon^2 C_{Z_2}}{(1 - Z_2^2)^2} Tri^{(0)}(Z_2) \\ - \left(\frac{\partial Z_1}{\partial m_2^2} \frac{2\epsilon(1 - 2\epsilon)C_{Z_1}}{Z_1(Z_1^2 - 1)^2} + \frac{\partial Z_2}{\partial m_2^2} \frac{2\epsilon(1 - 2\epsilon)C_{Z_2}}{Z_2(Z_2^2 - 1)^2} \right) Bub^{(0)}$$

Now collecting all results together for (4.17) we get our final result in the cut K_1 :

$$\frac{\partial}{\partial m_2^2} I_4 = c_{4 \rightarrow 4; K_1} I_4 + c_{4 \rightarrow 3; \bar{4}; K_1} I_{3; \bar{4}} + c_{4 \rightarrow 3; \bar{3}; K_1} I_{3; \bar{3}} + c_{4 \rightarrow 2; 12; K_1} I_{2; 12; K_1} + \dots \quad (4.21)$$

where

$$I_{3; \bar{4}} = I_4(1, 1, 1, 0), \quad I_{3; \bar{3}} = I_4(1, 1, 0, 1), \quad I_{2; 12; K_1} = I_4(1, 1, 0, 0) \quad (4.22)$$

with the coefficients

$$c_{4 \rightarrow 4; K_1} = - \frac{\frac{\partial B}{\partial m_2^2} (\frac{1}{2} + \epsilon)}{B}$$

$$c_{4 \rightarrow 3; \bar{4}; K_1} = \frac{-b\sqrt{\Delta_{3; m=0}[Z_1]}}{2K_1^2} \left(\frac{-\epsilon C_{Z_1} \frac{\partial B}{\partial m_2^2}}{AB Z_1 (1 - Z_1^2)} - \frac{2(1 + \epsilon)}{A} C_{3; Z_1} - \frac{2}{A} \frac{\partial}{\partial m_2^2} \left(\frac{\epsilon C_{Z_1}}{Z_1 (Z_1^2 - 1)} \right) + \frac{2}{A} \frac{\partial Z_1}{\partial m_2^2} \frac{2\epsilon^2 C_{Z_1}}{(Z_1^2 - 1)^2} \right)$$

$$\begin{aligned}
 c_{4 \rightarrow 3; \bar{3}; K_1} &= \frac{-b\sqrt{\Delta_{3;m=0}[Z_2]}}{2K_1^2} \left(\frac{-\epsilon C_{Z_2} \frac{\partial B}{\partial m_2^2}}{ABZ_2(1-Z_2^2)} - \frac{2(1+\epsilon)}{A} C_{3;Z_2} - \frac{2}{A} \frac{\partial}{\partial m_2^2} \left(\frac{\epsilon C_{Z_2}}{Z_2(Z_2^2-1)} \right) + \frac{2}{A} \frac{\partial Z_2}{\partial m_2^2} \frac{2\epsilon^2 C_{Z_2}}{(1-Z_2^2)^2} \right) \\
 c_{4 \rightarrow 2; 12; K_1} &= \frac{1}{2K_1^2} \left[-(1-2\epsilon) \left(\frac{\frac{\partial B}{\partial m_2^2} C_{Z_1}}{AB(1-Z_1^2)} + \frac{\frac{\partial B}{\partial m_2^2} C_{Z_2}}{AB(1-Z_2^2)} \right) - \frac{2(1+\epsilon)}{A} C_2 \right. \\
 &\quad \left. + \frac{2}{A} \left(\frac{\partial Z}{\partial m_2^2} \frac{2\epsilon(1-2\epsilon) C_{Z_1}}{Z_1(Z_1^2-1)^2} + \frac{\partial Z}{\partial m_2^2} \frac{2\epsilon(1-2\epsilon) C_{Z_2}}{Z_2(Z_2^2-1)^2} \right) - \frac{2}{A} \frac{\partial}{\partial m_2^2} \left(\frac{C_{Z_1}(1-2\epsilon)}{Z_1^2-1} + \frac{C_{Z_2}(1-2\epsilon)}{Z_2^2-1} \right) \right]
 \end{aligned} \tag{4.23}$$

where $C_{3;Z_1}$, $C_{3;Z_2}$ and C_2 given in eq. (4.20), and

$$\Delta_{3;m=0}[Z_1] = 4((K_1 \cdot K_{34})^2 - K_1^2 K_{34}^2), \quad \Delta_{3;m=0}[Z_2] = 4((K_1 \cdot K_4)^2 - K_1^2 K_4^2) \tag{4.24}$$

Having given details for the cut K_1 , the computation of other cuts will be similar, as shown in the section of triangle. For simplicity we will not list them one by one. All coefficients have been checked using LiteRed [23].

5 Pentagon

For the pentagon, let us define

$$\begin{aligned}
 I_5(n_1, n_2, n_3, n_4, n_5)[K_1, K_2, K_3, K_4, K_5; M_1, M_2, m_1, m_2, m_3] \\
 = \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{(p^2 - M_1^2)^{n_1} ((p-K_1)^2 - M_2^2)^{n_2} ((p-K_{12})^2 - m_1^2)^{n_3} ((p-K_{123})^2 - m_2^2)^{n_4} ((p+K_5)^2 - m_3^2)^{n_5}} \\
 = \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{(p^2 - M_1^2)^{n_1} ((p-K)^2 - M_2^2)^{n_2} ((p-P_1)^2 - m_1^2)^{n_3} ((p-P_2)^2 - m_2^2)^{n_4} ((p-P_3)^2 - m_3^2)^{n_5}} \tag{5.1}
 \end{aligned}$$

where the second form is suitable for the discussion of unitarity cut with cut momentum K in various cuts. The master basis of pentagon is given by $I_5 \equiv I_5(1, 1, 1, 1, 1)$ and the cut part with K is given by [9]

$$\begin{aligned}
 C(I_5) &= \left(\frac{\Delta}{4K^2} \right)^{-\epsilon} (-b) \int_0^1 du u^{-1-\epsilon} \frac{\sqrt{1-u}}{(K^2)^2} \left(\frac{S[Q_3, Q_2, Q_1, K]}{4\sqrt{(Q_3 \cdot Q_2)^2 - Q_3^2 Q_2^2}} \ln \left(\frac{Q_3 \cdot Q_2 - \sqrt{(Q_3 \cdot Q_2)^2 - Q_3^2 Q_2^2}}{Q_3 \cdot Q_2 + \sqrt{(Q_3 \cdot Q_2)^2 - Q_3^2 Q_2^2}} \right) \right. \\
 &\quad + \frac{S[Q_3, Q_1, Q_2, K]}{4\sqrt{(Q_3 \cdot Q_1)^2 - Q_3^2 Q_1^2}} \ln \left(\frac{Q_3 \cdot Q_1 - \sqrt{(Q_3 \cdot Q_1)^2 - Q_3^2 Q_1^2}}{Q_3 \cdot Q_1 + \sqrt{(Q_3 \cdot Q_1)^2 - Q_3^2 Q_1^2}} \right) \\
 &\quad \left. + \frac{S[Q_2, Q_1, Q_3, K]}{4\sqrt{(Q_2 \cdot Q_1)^2 - Q_2^2 Q_1^2}} \ln \left(\frac{Q_2 \cdot Q_1 - \sqrt{(Q_2 \cdot Q_1)^2 - Q_2^2 Q_1^2}}{Q_2 \cdot Q_1 + \sqrt{(Q_2 \cdot Q_1)^2 - Q_2^2 Q_1^2}} \right) \right) \tag{5.2}
 \end{aligned}$$

where $Q_i = -(b\sqrt{1-u}) P_i + \frac{P_i^2 + M_1^2 - m_i^2 - 2z(K \cdot P_i)}{K^2} K$ (see also (4.5)) and $S[Q_3, Q_2, Q_1, K]$ is a rational function defined as follows

$$\begin{aligned}
 S[Q_j, Q_i, Q_k, K] &= \frac{T_{jik}}{T_2}, \quad T_{jik} = -8 \det \begin{pmatrix} Q_k \cdot K & Q_j \cdot K & Q_i \cdot K \\ Q_j \cdot Q_k & Q_j^2 & Q_j \cdot Q_i \\ Q_i \cdot Q_k & Q_j \cdot Q_i & Q_i^2 \end{pmatrix}, \\
 T_2 &= -4 \det \begin{pmatrix} Q_3^2 & Q_2 \cdot Q_3 & Q_1 \cdot Q_3 \\ Q_2 \cdot Q_3 & Q_2^2 & Q_2 \cdot Q_1 \\ Q_1 \cdot Q_3 & Q_2 \cdot Q_1 & Q_1^2 \end{pmatrix}
 \end{aligned}$$

The form of $C(I_5)$ is like an addition of three different $C(I_4)$, with the common factor $\sqrt{1-u}$. To simplify further, noticing that there is a common factor $(1-u)^2$ between T_{jik} and T_2 , we can define (please notice that $T_{jik} = T_{ijk}$)

$$T_{2,re} = \frac{T_2}{(1-u)^2} = H_0 + H_1 u, \quad T_{12} = \frac{T_{213}}{(1-u)^2}, \quad T_{13} = \frac{T_{312}}{(1-u)^2}, \quad T_{23} = \frac{T_{321}}{(1-u)^2} \quad (5.3)$$

where T_{ij} is independent of u and $T_{2,re}$ is a liner function of u . Furthermore, with parameters

$$a = \frac{K^2 + M_1^2 - M_2^2}{K^2}, \quad b = \frac{\sqrt{\Delta[K, M_1, M_2]}}{K^2}, \quad a_i \equiv \frac{P_i^2 + M_1^2 - m_i^2}{K^2} \quad (5.4)$$

we define

$$\alpha_i \equiv a_i K^2 - a P_i \cdot K, \quad \beta_i \equiv b^2 \left(P_i^2 - \frac{(P_i \cdot K)^2}{K^2} \right), \quad \gamma_{ij} \equiv b^2 \left((P_i \cdot P_j)^2 - \frac{(P_i \cdot K)(P_j \cdot K)}{K^2} \right) \quad (5.5)$$

thus A, B, C, D defined in (4.4) for box cut and (5.3) can be simplified as

$$\begin{aligned} A_{12} &= \gamma_{12}^2 - \beta_1 \beta_2, & B_{12} &= \frac{-\alpha_2^2 \beta_1 - \alpha_1^2 \beta_2 + 2\alpha_1 \alpha_2 \gamma_{12}}{K^2} + \gamma_{12}^2 - \beta_1 \beta_2, \\ C_{12} &= \gamma_{12}, & D_{12} &= \gamma_{12} + \frac{\alpha_1 \alpha_2}{K^2} \end{aligned} \quad (5.6)$$

and

$$\begin{aligned} H_1 &= 4\beta_1 \beta_2 \beta_3 + 8\gamma_{12} \gamma_{13} \gamma_{23} - 4\beta_3 \gamma_{12}^2 - 4\beta_2 \gamma_{13}^2 - 4\beta_1 \gamma_{23}^2 \\ H_0 &= -H_1 + \frac{4}{K^2} \left(\alpha_1^2 A_{23} + \alpha_2^2 A_{13} + \alpha_3^2 A_{12} \right) \\ &\quad + 8 \left((D_{23} - C_{23})(\beta_1 C_{23} - C_{12} C_{13}) + (D_{13} - C_{13})(\beta_2 C_{13} - C_{12} C_{23}) \right. \\ &\quad \left. + (D_{12} - C_{12})(\beta_3 C_{12} - C_{13} C_{23}) \right) \\ T_{12} &= 8(\alpha_3 A_{12} + \alpha_1 \beta_2 \gamma_{13} - \alpha_2 \gamma_{12} \gamma_{13} + \alpha_2 \beta_1 \gamma_{23} - \alpha_1 \gamma_{12} \gamma_{23}) \\ T_{23} &= 8(\alpha_1 A_{23} + \alpha_2 \beta_3 \gamma_{12} + \alpha_3 \beta_2 \gamma_{13} - \alpha_3 \gamma_{12} \gamma_{23} - \alpha_2 \gamma_{13} \gamma_{23}) \\ T_{13} &= 8(\alpha_2 A_{13} + \alpha_1 \beta_3 \gamma_{12} + \alpha_3 \beta_1 \gamma_{23} - \alpha_1 \gamma_{13} \gamma_{23} - \alpha_3 \gamma_{12} \gamma_{13}) \end{aligned} \quad (5.7)$$

With above new notations, we can rewrite the expression of $C(I_5)$ as

$$\begin{aligned} C(I_5) &= \left(\frac{\Delta}{4K^2} \right)^{-\epsilon} \frac{b}{4(K^2)^2} \int_0^1 du u^{-1-\epsilon} \\ &\quad \times \frac{1}{H_0 + H_1 u} \left\{ \frac{T_{23}}{\sqrt{B_{23} - A_{23}u}} \ln \left(\frac{D_{23} - C_{23}u + \sqrt{1-u}\sqrt{B_{23} - A_{23}u}}{D_{23} - C_{23}u - \sqrt{1-u}\sqrt{B_{23} - A_{23}u}} \right) \right. \\ &\quad + \frac{T_{13}}{\sqrt{B_{13} - A_{13}u}} \ln \left(\frac{D_{13} - C_{13}u + \sqrt{1-u}\sqrt{B_{13} - A_{13}u}}{D_{13} - C_{13}u - \sqrt{1-u}\sqrt{B_{13} - A_{13}u}} \right) \\ &\quad \left. + \frac{T_{12}}{\sqrt{B_{12} - A_{12}u}} \ln \left(\frac{D_{12} - C_{12}u + \sqrt{1-u}\sqrt{B_{12} - A_{12}u}}{D_{12} - C_{12}u - \sqrt{1-u}\sqrt{B_{12} - A_{12}u}} \right) \right\} \end{aligned} \quad (5.8)$$

We can see that in (5.8) only parameters α_i contains m_i^2 , so

$$\frac{\partial}{\partial m_i^2} = \frac{\partial}{\partial \alpha_i} \frac{\partial \alpha_i}{\partial a_i} \frac{\partial a_i}{\partial m_i^2} = -\frac{\partial}{\partial \alpha_i} \quad (5.9)$$

For later convenience, we define three functions

$$\begin{aligned} pen_{ij}^{(n)} &= \int_0^1 du u^{n-1-\epsilon} \frac{1}{H_0 + H_1 u} \frac{1}{\sqrt{B_{ij} - A_{ij}u}} \ln \left(\frac{D_{ij} - C_{ij}u + \sqrt{1-u}\sqrt{B_{ij} - A_{ij}u}}{D_{ij} - C_{ij}u - \sqrt{1-u}\sqrt{B_{ij} - A_{ij}u}} \right) \\ &= \int_0^1 du u^{n-1-\epsilon} \frac{1}{H_0 + H_1 u} \frac{1}{\sqrt{B_{ij} - A_{ij}u}} \ln(*_{ij}), \quad *_{ij} \\ &\equiv \frac{D_{ij} - C_{ij}u + \sqrt{1-u}\sqrt{B_{ij} - A_{ij}u}}{D_{ij} - C_{ij}u - \sqrt{1-u}\sqrt{B_{ij} - A_{ij}u}} \end{aligned} \quad (5.10)$$

thus the $C(I_5)$ could be written as

$$C(I_5) = \left(\frac{\Delta}{4K^2} \right)^{-\epsilon} \frac{b}{4(K^2)^2} \left(T_{23} pen_{23}^{(0)} + T_{13} pen_{13}^{(0)} + T_{12} pen_{12}^{(0)} \right) \quad (5.11)$$

From the definition of (5.10), one can easily establish the recursion relation by rewriting $u = \frac{(H_0 + H_1 u) - H_0}{H_1}$. The cancelation of denominator $H_0 + H_1 u$ in (5.10) is nothing, but the cut of corresponding box (see (4.6)).

Similar to other sections, with momentum shifting and reflection, we can define the action

$$\begin{aligned} \hat{g}_5 : \quad & \{(n_1, n_2, n_3, n_4, n_5); (K_1, K_2, K_3, K_4, K_5); (M_1, M_2, m_1, m_2, m_3)\} \\ & \rightarrow \{(n_2, n_3, n_4, n_5, n_1); (K_2, K_3, K_4, K_5, K_1); (M_2, m_1, m_2, m_3, M_1)\} \\ \hat{g}_2 : \quad & \{(n_1, n_2, n_3, n_4, n_5); (K_1, K_2, K_3, K_4, K_5); (M_1, M_2, m_1, m_2, m_3)\} \\ & \rightarrow \{(n_2, n_1, n_5, n_4, n_3); (K_1, K_5, K_4, K_3, K_2); (M_2, M_1, m_3, m_2, m_1)\} \end{aligned} \quad (5.12)$$

thus if we write $\hat{I}_5 \equiv I_5(n_1, n_2, n_3, n_4, n_5)[K_1, K_2, K_3, K_4, K_5; M_1, M_2, m_1, m_2, m_3]$ we will have

$$\begin{aligned} I_5(n_2, n_3, n_4, n_5, n_1)[K_2, K_3, K_4, K_5, K_1; M_2, m_1, m_2, m_3, M_1] &= \hat{g}_5 \hat{I}_5 \\ I_5(n_3, n_4, n_5, n_1, n_2)[K_3, K_4, K_5, K_1, K_2; m_1, m_2, m_3, M_1, M_2] &= \hat{g}_5^2 \hat{I}_5 \\ I_5(n_4, n_5, n_1, n_2, n_3)[K_4, K_5, K_1, K_2, K_3; m_2, m_3, M_1, M_2, m_1] &= \hat{g}_5^3 \hat{I}_5 \\ I_5(n_5, n_1, n_2, n_3, n_4)[K_5, K_1, K_2, K_3, K_4; m_3, M_1, M_2, m_1, m_2] &= \hat{g}_5^4 \hat{I}_5 \end{aligned} \quad (5.13)$$

and

$$\begin{aligned} I_5(n_2, n_1, n_5, n_4, n_3)[K_1, K_5, K_4, K_3, K_2; M_2, M_1, m_3, m_2, m_1] &= \hat{g}_2 \hat{I}_5 \\ I_5(n_3, n_2, n_1, n_5, n_4)[K_2, K_1, K_5, K_4, K_3; m_1, M_2, M_1, m_3, m_2] &= \hat{g}_5^{-1} \hat{g}_2 \hat{I}_5 \\ I_5(n_4, n_3, n_2, n_1, n_5)[K_3, K_2, K_1, K_5, K_4; m_2, m_1, M_2, M_1, m_3] &= \hat{g}_5^{-2} \hat{g}_2 \hat{I}_5 \\ I_5(n_5, n_4, n_3, n_2, n_1)[K_4, K_3, K_2, K_1, K_5; m_3, m_2, m_1, M_2, M_1] &= \hat{g}_5^{-3} \hat{g}_2 \hat{I}_5 \\ I_5(n_1, n_5, n_4, n_3, n_2)[K_5, K_4, K_3, K_2, K_1; M_1, m_3, m_2, m_1, M_2] &= \hat{g}_5^{-4} \hat{g}_2 \hat{I}_5 \end{aligned} \quad (5.14)$$

Furthermore, using the same idea in the subsection 3.1 we can write down similar recurrence relation for general $I_5(n_1, n_2, n_3, n_4, n_5)$ using the expansion of $I_5(2, 1, 1, 1, 1)$, $I_5(1, 2, 1, 1, 1)$, $I_5(1, 1, 2, 1, 1)$, $I_5(1, 1, 1, 2, 1)$ and $I_5(1, 1, 1, 1, 2)$. However, by relation (5.13) and (5.14), all other four cases can be reduced to the reduction of $I_5(1, 1, 1, 1, 2)$.

To reduce $I_5(1, 1, 1, 1, 2)$, according to our ideas, we should calculate $\frac{\partial}{\partial m_3^2} I_5(1, 1, 1, 1, 1)$ in $\binom{5}{2} = 10$ different cuts. Again, in the main part, we present only the computation of the cut K_1 . For this case, since the analytic checking using LiteRed [23] is too hard, we have checked only numerically.

5.1 Cut K_1 of $I_2(1, 1, 1, 1, 2)$

For this cut, we will choose the parameters of the second form in (5.1) as $K = K_1$, $P_1 = K_{12}$, $P_2 = K_{123}$ and $P_3 = -K_5$. Since the m_3 is not contained in $(\frac{\Delta}{4K^2})^{-\epsilon}$, we could just drop the factor $(\frac{\Delta}{4K^2})^{-\epsilon}$. Furthermore the parameter m_3 is only contained in B , D , H_0 and T_{12} , T_{13} and T_{23} , thus we have

$$\begin{aligned} \frac{\partial}{\partial m_3^2} C(I_5) = \frac{b}{4(K^2)^2} & \left(\frac{\partial T_{23}}{\partial m_3^2} pen_{23}^{(0)} + \frac{\partial T_{13}}{\partial m_3^2} pen_{13}^{(0)} + \frac{\partial T_{12}}{\partial m_3^2} pen_{12}^{(0)} \right. \\ & \left. + T_{23} \frac{\partial}{\partial m_3^2} pen_{23}^{(0)} + T_{13} \frac{\partial}{\partial m_3^2} pen_{13}^{(0)} + T_{12} \frac{\partial}{\partial m_3^2} pen_{12}^{(0)} \right) \end{aligned} \quad (5.15)$$

The result of $\frac{\partial T_{ij}}{\partial m_3^2}$ is very simple and given by

$$\frac{\partial T_{23}}{\partial m_3^2} = 8(\gamma_{12}\gamma_{23} - \beta_2\gamma_{13}), \quad \frac{\partial T_{13}}{\partial m_3^2} = 8(\gamma_{12}\gamma_{13} - \beta_1\gamma_{23}), \quad \frac{\partial T_{12}}{\partial m_3^2} = 8(\beta_1\beta_2 - \gamma_{12}^2) \quad (5.16)$$

To calculate $\frac{\partial}{\partial m_3^2} pen_{ij}^{(0)}$, noticing that

$$\begin{aligned} \frac{\partial}{\partial m_3^2} pen^{(0)} = \int_0^1 du u^{-1-\epsilon} & \frac{1}{(H_0 + H_1 u)^2} \frac{-\frac{\partial H_0}{\partial m_3^2}}{\sqrt{B - Au}} \ln(*) \\ & + \int_0^1 du u^{-1-\epsilon} \frac{1}{H_0 + H_1 u} \frac{-\frac{1}{2} \frac{\partial B}{\partial m_3^2}}{\sqrt{B - Au}^3} \ln(*) \\ & + \int_0^1 du u^{-1-\epsilon} \frac{1}{H_0 + H_1 u} \frac{1}{\sqrt{B - Au}} \frac{\partial}{\partial m_3^2} \ln(*) \equiv L_1 + L_2 + L_3 \end{aligned} \quad (5.17)$$

we need to calculate these three terms respectively.

To prepare the reduction of three integral L_1 , L_2 , and L_3 , we rewrite $pen^{(n)}$, which is defined in (5.10), as following To prepare the reduction of three integral L_1 , L_2 , and L_3 , we want to rewrite $pen^{(n)}$ defined in (5.10) in the following by doing the partial integration $du^{n-\epsilon} = (n - \epsilon)u^{n-1-\epsilon}du$

$$\begin{aligned} pen^{(n)} = \frac{1}{n - \epsilon} & \left\{ H_1 \int_0^1 du u^{n-\epsilon} \frac{1}{(H_0 + H_1 u)^2} \frac{1}{\sqrt{B - Au}} \ln(*) \right. \\ & - \frac{A}{2} \int_0^1 du u^{n-\epsilon} \frac{1}{H_0 + H_1 u} \frac{1}{\sqrt{B - Au}^3} \ln(*) \\ & \left. + \int_0^1 du u^{n-\epsilon} \frac{AD + BD - 2BC + u(AC + BC - 2AD)}{(H_0 + H_1 u)(B - Au)} \frac{1}{\sqrt{1 - u}} \right\} \end{aligned} \quad (5.18)$$

where

$$\$_{ij} = (D_{ij} - C_{ij}u)^2 - (1-u)(B_{ij} - A_{ij}u) = \beta_i\beta_j(1-u-Z_i^2)(1-u-Z_j^2) \quad (5.19)$$

The first line in (5.18) has the form of L_1 in (5.17) since H_0 does not depend on u as defined in (5.3).

Now we consider the third line in (5.18). Using algebraic separation it becomes

$$\begin{aligned} & \int_0^1 duu^{n-\epsilon} \frac{AD+BD-2BC+u(AC+BC-2AD)}{(H_0+H_1u)(B-Au)} \frac{1}{\sqrt{1-u}} \\ &= \frac{(AD+BD-2BC)H_1-H_0(AC+BC-2AD)}{AH_0+BH_1} \int_0^1 duu^{n-\epsilon} \frac{1}{(H_0+H_1u)\sqrt{1-u}} \\ &+ \frac{(AD+BD-2BC)A+B(AC+BC-2AD)}{AH_0+BH_1} \int_0^1 duu^{n-\epsilon} \frac{1}{(B-Au)\sqrt{1-u}} \end{aligned} \quad (5.20)$$

Among these two terms in (5.20), the second term will be canceled by the second line in (5.18). For the first term, using the factorization form of $\$_{ij}$ in (5.19) we get

$$\begin{aligned} \frac{1}{(H_0+H_1u)\sqrt{1-u}\$_{ij}} &= \frac{1}{\sqrt{1-u}\beta_i\beta_j(Z_i^2-Z_j^2)} \times \frac{1}{H_0+H_1u} \left(\frac{1}{1-u-Z_i^2} - \frac{1}{1-u-Z_j^2} \right) \\ &= W_{0;ij} \frac{1}{\sqrt{1-u}(H_0+H_1u)} + W_{i;ij} \frac{1}{\sqrt{1-u}(1-u-Z_i^2)} + W_{j;ij} \frac{1}{\sqrt{1-u}(1-u-Z_j^2)} \end{aligned} \quad (5.21)$$

with the coefficients

$$\begin{aligned} W_{0;ij} &= \frac{1}{\beta_i\beta_j(Z_i^2-Z_j^2)} \left(\frac{H_1}{H_0+(1-Z_i^2)H_1} - \frac{H_1}{H_0+(1-Z_j^2)H_1} \right) \\ W_{i;ij} &= \frac{1}{\beta_i\beta_j(Z_i^2-Z_j^2)} \frac{1}{H_0+(1-Z_i^2)H_1}, \quad W_{j;ij} = \frac{1}{\beta_i\beta_j(Z_i^2-Z_j^2)} \frac{-1}{H_0+(1-Z_j^2)H_1} \end{aligned} \quad (5.22)$$

Among three terms in (5.21), the first term is spurious and will be canceled when summing contributions from three pen_{ij} terms. The second and third terms are essentially the triangle part. Putting all coefficients back, we find the third line in (5.18) is given by

$$\begin{aligned} & \int_0^1 duu^{n-\epsilon} \frac{AD+BD-2BC+u(AC+BC-2AD)}{(H_0+H_1u)(B-Au)} \frac{1}{\sqrt{1-u}} \\ &= \frac{(A-B)(AD-BC)}{AH_0+BH_1} \int_0^1 duu^{n-\epsilon} \frac{1}{(B-Au)\sqrt{1-u}} \\ &+ G_{ij}W_{0;ij} \int_0^1 duu^{n-\epsilon} \frac{1}{\sqrt{1-u}(H_0+H_1u)} - \frac{G_{ij}W_{i;ij}(n-\epsilon)}{Z_i} Tr i^{(n)}(Z_i) \\ &- \frac{G_{ij}W_{j;ij}(n-\epsilon)}{Z_j} Tr i^{(n)}(Z_j) \end{aligned} \quad (5.23)$$

where

$$G_{ij} \equiv \frac{(A_{ij}D_{ij} + B_{ij}D_{ij} - 2B_{ij}C_{ij})H_1 - H_0(A_{ij}C_{ij} + B_{ij}C_{ij} - 2A_{ij}D_{ij})}{(A_{ij}H_0 + B_{ij}H_1)} \quad (5.24)$$

For the second line in (5.18), we will use the rewriting of box

$$Box^{(n)} = \frac{1}{n-\epsilon} \times \left\{ -\frac{A}{2} \int_0^1 du u^{n-\epsilon} \frac{1}{\sqrt{B-Au}^3} \ln(*) \right. \\ \left. + \int_0^1 du u^{n-\epsilon} \frac{AD+BD-2BC+u(AC+BC-2AD)}{(B-Au)\sqrt{1-u}} \right\} \quad (5.25)$$

which is obtained by the similar method as in (5.18). Using the similar splitting technique done in previous paragraphes to the second term in (5.25), we will arrive

$$Box_{ij}^{(n)} = \frac{1}{n-\epsilon} \left\{ -\frac{A}{2} \int_0^1 du u^{n-\epsilon} \frac{1}{\sqrt{B-Au}^3} \ln(*) \right. \\ + \frac{(A-B)(AD-BC)}{A} \int_0^1 du u^{n-\epsilon} \frac{1}{(B-Au)\sqrt{1-u}} \\ - \frac{2AD-AC-BC}{A} \frac{1}{\beta_i \beta_j (Z_i^2 - Z_j^2)} \frac{n-\epsilon}{Z_i} Tri^{(n)}(Z_i) \\ \left. + \frac{2AD-AC-BC}{A} \frac{1}{\beta_i \beta_j (Z_i^2 - Z_j^2)} \frac{n-\epsilon}{Z_j} Tri^{(n)}(Z_j) \right\} \quad (5.26)$$

For the first term in (5.26), we do following manipulation

$$\int_0^1 du u^{n-\epsilon} \frac{1}{\sqrt{B-Au}^3} \ln(*) = \int_0^1 du u^{n-\epsilon} \frac{H_0+H_1 u}{(H_0+H_1 u)\sqrt{B-Au}^3} \ln(*) \\ = \int_0^1 du u^{n-\epsilon} \frac{H_0+H_1 \frac{Au-B+B}{A}}{(H_0+H_1 u)\sqrt{B-Au}^3} \ln(*) \\ = (H_0 + \frac{BH_1}{A}) \int_0^1 du u^{n-\epsilon} \frac{1}{(H_0+H_1 u)\sqrt{B-Au}^3} \ln(*) - \frac{H_1}{A} pen^{(n+1)} \quad (5.27)$$

Thus the second line in (5.18) becomes

$$-\frac{A}{2} \int_0^1 du u^{n-\epsilon} \frac{1}{(H_0+H_1 u)\sqrt{B-Au}^3} \ln(*) = \frac{A(n-\epsilon)}{AH_0+BH_1} Box^{(n)} - \frac{AH_1}{2(AH_0+BH_1)} pen^{(n+1)} \\ - \frac{(A-B)(AD-BC)}{(AH_0+BH_1)} \int_0^1 du u^{n-\epsilon} \frac{1}{(B-Au)\sqrt{1-u}} \\ - \frac{A\lambda_{i;j}(n-\epsilon)}{2} Tri^{(n)}(Z_i) - \frac{A\lambda_{j;i}(n-\epsilon)}{2} Tri^{(n)}(Z_j) \quad (5.28)$$

with the coefficients

$$\lambda_{i;j} = \frac{2(AC+BC-2AD)}{A(AH_0+BH_1)} \frac{1}{\beta_i \beta_j (Z_i^2 - Z_j^2)} \frac{1}{Z_i}, \\ \lambda_{j;i} = \frac{2(AC+BC-2AD)}{A(AH_0+BH_1)} \frac{1}{\beta_i \beta_j (Z_j^2 - Z_i^2)} \frac{1}{Z_j} \quad (5.29)$$

Now we can explicitly see that the third term in (5.28) cancel the first term in (5.23).

Putting all together, we have

$$\begin{aligned}
 pen^{(n)} = & \frac{H_1}{n-\epsilon} \int_0^1 duu^{n-\epsilon} \frac{1}{(H_0 + H_1 u)^2} \frac{1}{\sqrt{B-Au}} \ln(*) + \frac{A}{AH_0 + BH_1} Box^{(n)} \\
 & - \frac{AH_1}{2(AH_0 + BH_1)} \frac{1}{n-\epsilon} pen^{(n+1)} - \left(\frac{A\lambda_{i;ij}}{2} + \frac{G_{ij}W_{i;ij}}{Z_i} \right) Tri^{(n)}(Z_i) \\
 & - \left(\frac{A\lambda_{j;ij}}{2} + \frac{G_{ij}W_{j;ij}}{Z_j} \right) Tri^{(n)}(Z_j) + \frac{G_{ij}W_{0;ij}}{n-\epsilon} \int_0^1 duu^{n-\epsilon} \frac{1}{\sqrt{1-u}(H_0 + H_1 u)}
 \end{aligned} \quad (5.30)$$

with the coefficients given by (5.22), (5.24) and (5.29).

Now we can consider the L_1, L_2, L_3 terms. Let us start from the L_1 term. By comparing L_1 with the first term in (5.17), we see that

$$\begin{aligned}
 L_1 = & -\frac{\partial H_0}{\partial m_3^2} \int_0^1 duu^{-1-\epsilon} \frac{1}{(H_0 + H_1 u)^2} \frac{1}{\sqrt{B-Au}} \ln(*) \\
 = & \frac{\partial H_0}{\partial m_3^2} \frac{(1+\epsilon)}{H_1} pen^{(-1)} - \frac{\partial H_0}{\partial m_3^2} \frac{A(1+\epsilon)}{H_1(AH_0 + BH_1)} Box^{(-1)} - \frac{\partial H_0}{\partial m_3^2} \frac{A}{2(AH_0 + BH_1)} pen^{(0)} \\
 & + \frac{\partial H_0}{\partial m_3^2} \left(\frac{A\lambda_{i;ij}}{2} + \frac{G_{ij}W_{i;ij}}{Z_i} \right) \frac{(1+\epsilon)}{H_1} Tri^{(-1)}(Z_i) \\
 & + \frac{\partial H_0}{\partial m_3^2} \left(\frac{A\lambda_{j;ij}}{2} + \frac{G_{ij}W_{j;ij}}{Z_j} \right) \frac{(1+\epsilon)}{H_1} Tri^{(-1)}(Z_j) \\
 & + \frac{\partial H_0}{\partial m_3^2} \frac{G_{ij}W_{0;ij}}{H_1} \int_0^1 duu^{-1-\epsilon} \frac{1}{\sqrt{1-u}(H_0 + H_1 u)}
 \end{aligned} \quad (5.31)$$

For the L_2 , which is written as

$$L_2 = -\frac{1}{2} \frac{\partial B}{\partial m_3^2} \int_0^1 duu^{-1-\epsilon} \frac{1}{(H_0 + H_1 u)\sqrt{B-Au}^3} \ln(*) \quad (5.32)$$

using the (5.27) with $n = -1$ we have

$$\begin{aligned}
 L_2 = & -\frac{\partial B}{\partial m_3^2} \frac{(1+\epsilon)}{AH_0 + BH_1} Box^{(-1)} - \frac{1}{2} \frac{\partial B}{\partial m_3^2} \frac{H_1}{AH_0 + BH_1} pen^{(0)} \\
 & - \frac{\partial B}{\partial m_3^2} \frac{(A-B)(AD-BC)}{A(AH_0 + BH_1)} \int_0^1 duu^{-1-\epsilon} \frac{1}{(B-Au)\sqrt{1-u}} \\
 & + \frac{1}{2} \frac{\partial B}{\partial m_3^2} \lambda_{i;ij}(1+\epsilon) Tri^{(-1)}(Z_i) + \frac{1}{2} \frac{\partial B}{\partial m_3^2} \lambda_{j;ij}(1+\epsilon) Tri^{(-1)}(Z_j)
 \end{aligned} \quad (5.33)$$

For the integral L_3

$$L_3 = \int_0^1 duu^{-1-\epsilon} \frac{1}{(H_0 + H_1 u)} \frac{(1-u)[B'D - 2BD' + u(2AD' - B'C)]}{(B-Au)\sqrt{1-u}} \quad (5.34)$$

where $B' \equiv \frac{\partial B}{\partial m_3^2}$, $D' \equiv \frac{\partial D}{\partial m_3^2}$, using the splitting of § in (5.19) it is given by

$$\begin{aligned}
 L_3 = & r_1 \int_0^1 duu^{-1-\epsilon} \frac{1}{\sqrt{1-u}} + r_2 \int_0^1 duu^{-1-\epsilon} \frac{1}{(B-Au)\sqrt{1-u}} \\
 & + r_3 \int_0^1 duu^{-1-\epsilon} \frac{1}{(H_0 + H_1 u)\sqrt{1-u}}
 \end{aligned} \quad (5.35)$$

with the coefficients

$$\begin{aligned} r_1 &= \frac{2AD' - B'C}{AH_1}, \quad r_2 = \frac{B'(A - B)(AD - BC)}{A(AH_0 + BH_1)} \\ r_3 &= \frac{(H_0 + H_1) \left[(B'D - 2BD')H_1 - (2AD' - B'C)H_0 \right]}{H_1(AH_0 + BH_1)} \end{aligned} \quad (5.36)$$

There are three terms. The first term could be split into two triangles, as we have done before. The second term is a spurious term, and is canceled with the same term in the integral L_2 . And the last term could also be split into three pieces by using (5.23) with $n = -1$. Putting all together we have

$$\begin{aligned} L_3 &= r_2 \int_0^1 du u^{-1-\epsilon} \frac{1}{(B - Au) \sqrt{1-u}} + r_3 W_{0;ij} \int_0^1 du u^{-1-\epsilon} \frac{1}{\sqrt{1-u} (H_0 + H_1 u)} \\ &+ \left(\frac{r_1}{\beta_i \beta_j (Z_i^2 - Z_j^2)} + r_3 W_{i;ij} \right) \frac{1+\epsilon}{Z_i} Tri^{(-1)}(Z_i) \\ &+ \left(\frac{r_1}{\beta_i \beta_j (Z_j^2 - Z_i^2)} + r_3 W_{j;ij} \right) \frac{1+\epsilon}{Z_j} Tri^{(-1)}(Z_j) \end{aligned} \quad (5.37)$$

Collecting above results for L_1, L_2, L_3 we have

$$\begin{aligned} \frac{\partial}{\partial m_3^2} pen_{ij}^{(0)} &= \frac{H'_0(1+\epsilon)}{H_1} pen^{(-1)} - \frac{(AH'_0 + H_1 B')}{2(AH_0 + BH_1)} pen^{(0)} + \frac{(1+\epsilon)}{AH_0 + BH_1} \left(-\frac{AH'_0}{H_1} - B' \right) Box^{(-1)} \\ &+ \frac{(1+\epsilon)}{2} \left(\frac{2r_3 W_{i;ij}}{Z_i} + \frac{2r_1}{\beta_i \beta_j Z_i (Z_i^2 - Z_j^2)} + B' \lambda_{i;ij} + \frac{H'_0 (2G_{ij} W_{i;ij} + A Z_i \lambda_{i;ij})}{H_1 Z_i} \right) Tri^{(-1)}(Z_i) \\ &+ \frac{(1+\epsilon)}{2} \left(\frac{2r_3 W_{j;ij}}{Z_j} + \frac{2r_1}{\beta_i \beta_j Z_j (Z_j^2 - Z_i^2)} + B' \lambda_{j;ij} + \frac{H'_0 (2G_{ij} W_{j;ij} + A Z_j \lambda_{j;ij})}{H_1 Z_j} \right) Tri^{(-1)}(Z_j) \\ &+ \left(\frac{H'_0 G_{ij} W_{0;ij}}{H_1} + r_3 W_{0;ij} \right) \int_0^1 du u^{-1-\epsilon} \frac{1}{\sqrt{1-u} (H_0 + H_1 u)} \end{aligned} \quad (5.38)$$

In (5.38), the last term will be canceled when summing over three $\frac{\partial}{\partial m_3^2} pen_{ij}^{(0)}$ in (5.15). To continue, we need to reduce $Tri^{(-1)}$, $Box^{(-1)}$ and $pen^{(-1)}$ to our scalar basis. Using

$$H_0 pen^{(n)} + H_1 pen^{(n+1)} = Box^{(n)} \quad (5.39)$$

with $n = -1$,

$$Tri^{(-1)}(Z) = \frac{\epsilon}{(\epsilon + 1)(1 - Z^2)} Tri^{(0)}(Z) + \frac{(1 - 2\epsilon)Z}{(1 - Z^2)(\epsilon + 1)} Bub^{(0)} \quad (5.40)$$

and

$$\begin{aligned} Box^{(-1)} &= \frac{\frac{1}{2} + \epsilon}{1 + \epsilon} \frac{A}{B} Box^{(0)} + \frac{\epsilon}{1 + \epsilon} \frac{C_{Z_1}}{B Z_1 (1 - Z_1^2)} Tri^{(0)}(Z_1) + \frac{\epsilon}{1 + \epsilon} \frac{C_{Z_2}}{B Z_2 (1 - Z_2^2)} Tri^{(0)}(Z_2) \\ &+ \frac{1 - 2\epsilon}{1 + \epsilon} \left(\frac{C_{Z_1}}{(1 - Z_1^2)B} + \frac{C_{Z_2}}{(1 - Z_2^2)B} \right) Bub^{(0)} \end{aligned} \quad (5.41)$$

we have

$$\frac{\partial}{\partial m_3^2} pen_{ij}^{(0)} = q_{ij;5} pen_{ij}^{(0)} + q_{ij;4} Box_{ij}^{(0)} + q_{ij;Z_i} Tri^{(0)}(Z_i) + q_{ij;Z_j} Tri^{(0)}(Z_j) + q_{ij;2} Bub^{(0)} \quad (5.42)$$

with the coefficients

$$\begin{aligned} q_{ij;5} &= l_{ij;5} = -\frac{H'_0(1+\epsilon)}{H_0} - \frac{(A_{ij}H'_0 + H_1B'_{ij})}{2(A_{ij}H_0 + B_{ij}H_1)} \\ q_{ij;4} &= \frac{\frac{1}{2} + \epsilon}{1 + \epsilon} \frac{A}{B} l_{ij;4} = \frac{A_{ij}(B_{ij}H'_0 - B'_{ij}H_0)(\frac{1}{2} + \epsilon)}{B_{ij}H_0(A_{ij}H_0 + B_{ij}H_1)} \\ q_{ij;Z_i} &= l_{ij;4} \times \frac{\epsilon}{1 + \epsilon} \frac{C_{Z_i;ij}}{B_{ij}Z_i(1 - Z_i^2)} + l_{ij;Z_i} \times \frac{\epsilon}{(\epsilon + 1)(1 - Z_i^2)} \\ &= \frac{q_{i;ij;n}}{B_{ij}H_0(A_{ij}H_0 + B_{ij}H_1)Z_i(Z_i^2 - 1)(H_0 + H_1(1 - Z_i^2))(Z_i^2 - Z_j^2)\beta_i\beta_j} \\ q_{ij;Z_j} &= l_{ij;4} \times \frac{\epsilon}{1 + \epsilon} \frac{C_{Z_j;ij}}{B_{ij}Z_j(1 - Z_j^2)} + l_{ij;Z_j} \times \frac{\epsilon}{(\epsilon + 1)(1 - Z_j^2)} \\ &= \frac{q_{j;ij;n}}{B_{ij}H_0(A_{ij}H_0 + B_{ij}H_1)Z_j(Z_j^2 - 1)(H_0 + H_1(1 - Z_j^2))(Z_j^2 - Z_i^2)\beta_i\beta_j} \\ q_{ij;2} &= l_{ij;4} \times \frac{1 - 2\epsilon}{1 + \epsilon} \left(\frac{C_{Z_i;ij}}{(1 - Z_i^2)B_{ij}} + \frac{C_{Z_j;ij}}{(1 - Z_j^2)B_{ij}} \right) \\ &\quad + l_{ij;Z_i} \times \frac{(1 - 2\epsilon)Z_i}{(1 - Z_i^2)(\epsilon + 1)} + l_{ij;Z_j} \times \frac{(1 - 2\epsilon)Z_j}{(1 - Z_j^2)(\epsilon + 1)} \end{aligned} \quad (5.43)$$

where

$$\begin{aligned} q_{i;ij;n} &= \epsilon \left[B^2 H_0 (-DH'_0 + 2D'H_1 Z_i^2 + CH'_0(1 + Z_i^2)) \right. \\ &\quad + B' C_{Z_i} H_0 (H_0 + H_1(1 - Z_i^2))(Z_i^2 - Z_j^2) \beta_i \beta_j \\ &\quad + B \left(AH_0 (DH'_0 + (2D'H_0 - 2DH'_0)Z_i^2 + CH_0(Z_i^2 - 1)) \right. \\ &\quad + B'H_0 (D(H_0 + H_1(1 - 2Z_i^2)) - C(H_0 + H_1 + H_1 Z_i^2 - H_1 Z_i^2)) \\ &\quad \left. \left. - C_{Z_i} H'_0 (H_0 + H_1 - H_1 Z_i^2)(Z_i^2 - Z_j^2) \beta_i \beta_j \right) \right] \\ q_{j;ij;n} &= q_{i;ij;n}|_{i \leftrightarrow j}, \quad B' \equiv \frac{\partial B}{\partial m_3^2}, \quad D' \equiv \frac{\partial D}{\partial m_3^2}, \quad H'_0 \equiv \frac{\partial H_0}{\partial m_3^2}, \quad C_{Z_k;ij} = D_{ij} + (Z_k^2 - 1)C_{ij} \end{aligned} \quad (5.44)$$

Now we put the result (5.42) to (5.15) to get

$$\begin{aligned} \frac{\partial}{\partial m_3^2} C(I_5) &= \frac{b}{4(K^2)^2} \left\{ \left(\frac{\partial T_{12}}{\partial m_3^2} + q_{12;5} T_{12} \right) pen_{12}^{(0)} + \left(\frac{\partial T_{13}}{\partial m_3^2} + q_{13;5} T_{13} \right) pen_{13}^{(0)} \right. \\ &\quad + \left(\frac{\partial T_{23}}{\partial m_3^2} + q_{23;5} T_{23} \right) pen_{23}^{(0)} + q_{12;4} T_{12} Box_{12}^{(0)} + q_{13;4} T_{13} Box_{13}^{(0)} \\ &\quad + q_{23;4} T_{23} Box_{23}^{(0)} + (q_{12;Z_1} T_{12} + q_{13;Z_1} T_{13}) Tri^{(0)}(Z_1) \\ &\quad + (q_{12;Z_2} T_{12} + q_{23;Z_2} T_{23}) Tri^{(0)}(Z_2) + (q_{13;Z_3} T_{13} + q_{23;Z_3} T_{23}) Tri^{(0)}(Z_3) \\ &\quad \left. + (q_{12;2} T_{12} + q_{13;2} T_{13} + q_{23;2} T_{23}) Bub^{(0)} \right\} \end{aligned} \quad (5.45)$$

where $K = K_1$ and the parameters are given in (5.43) and (5.7). One important point is that the sum of the first three terms in (5.45) gives exactly the cut of pentagon in (5.11). Thus we have the result

$$I_5(1, 1, 1, 1, 2) = c_{5 \rightarrow 5; K_1} I_5 + c_{5 \rightarrow 4; \bar{5}; K_1} I_{4; \bar{5}} + c_{5 \rightarrow 3; \bar{4}; K_1} I_{4; \bar{4}} + c_{5 \rightarrow 4; \bar{3}; K_1} I_{4; \bar{3}} \\ + c_{5 \rightarrow 3; \bar{4}\bar{5}; K_1} I_{3; \bar{4}\bar{5}} + c_{5 \rightarrow 3; \bar{3}\bar{5}; K_1} I_{3; \bar{3}\bar{5}} + c_{5 \rightarrow 3; \bar{3}\bar{4}; K_1} I_{3; \bar{3}\bar{4}} + c_{5 \rightarrow 2; \bar{3}\bar{4}\bar{5}; K_1} I_{2; \bar{3}\bar{4}\bar{5}} + \dots \quad (5.46)$$

where the elips represents the tadpoles and the coefficients are given by

$$c_{5 \rightarrow 5; K_1} = \frac{1}{T_{12}} \frac{\partial T_{12}}{\partial m_3^2} + q_{12} = \frac{1}{T_{13}} \frac{\partial T_{13}}{\partial m_3^2} + q_{13} = \frac{1}{T_{23}} \frac{\partial T_{23}}{\partial m_3^2} + q_{23} \\ c_{5 \rightarrow 4; \bar{3}; K_1} = \frac{1}{2K^2} q_{23; 4} T_{23}, \quad c_{5 \rightarrow 4; \bar{4}} = \frac{1}{2K^2} q_{13; 4} T_{13}, \quad c_{5 \rightarrow 4\bar{5}} = \frac{1}{2K^2} q_{12; 4} T_{12} \\ c_{5 \rightarrow 3; \bar{4}\bar{5}; K_1} = \frac{-b\sqrt{\Delta_{3; m=0}[Z_1]}}{4K^4} (q_{12; Z_1} T_{12} + q_{13; Z_1} T_{13}), \quad \Delta_{3; m=0}[Z_1] = 4[(K \cdot P_1)^2 - K^2 P_1^2] \\ c_{5 \rightarrow 3; \bar{3}\bar{5}; K_1} = \frac{-b\sqrt{\Delta_{3; m=0}[Z_2]}}{4K^4} (q_{12; Z_2} T_{12} + q_{23; Z_2} T_{23}), \quad \Delta_{3; m=0}[Z_2] = 4[(K \cdot P_2)^2 - K^2 P_2^2] \\ c_{5 \rightarrow 3; \bar{3}\bar{4}; K_1} = \frac{-b\sqrt{\Delta_{3; m=0}[Z_3]}}{4K^4} (q_{13; Z_3} T_{13} + q_{23; Z_3} T_{23}), \quad \Delta_{3; m=0}[Z_3] = 4[(K \cdot P_3)^2 - K^2 P_3^2] \\ c_{5 \rightarrow 2; \bar{3}\bar{4}\bar{5}; K_1} = \frac{1}{4K^4} (q_{12; 2} T_{12} + q_{13; 2} T_{13} + q_{23; 2} T_{23}), \quad P_1 = K_{12}, \quad P_2 = K_{123}, \quad P_3 = K_{1234} \quad (5.47)$$

with the parameters given in (5.43). The corresponding basis are

$$I_{4; \bar{5}} = I_5(1, 1, 1, 1, 0), \quad I_{4; \bar{4}} = I_5(1, 1, 1, 0, 1), \quad I_{4; \bar{3}} = I_5(1, 1, 0, 1, 1) \quad (5.48) \\ I_{3; \bar{4}\bar{5}} = I_5(1, 1, 1, 0, 0), \quad I_{3; \bar{3}\bar{5}} = I_5(1, 1, 0, 1, 0), \quad I_{3; \bar{3}\bar{4}} = I_5(1, 1, 0, 0, 1), \quad I_{2; \bar{3}\bar{4}\bar{5}} = I_5(1, 1, 0, 0, 0)$$

Again, having presented details for the cut K_1 , the computation of other cuts will be similar, as shown in the section of triangle. For simplicity we will not list them one by one. All coefficients have been checked using LiteRed [23] numerically.

6 Conclusion

In this paper, we have considered the reduction of one-loop integrals with higher poles using the unitarity cut method. By the trick of differentiation over auxiliary masses, we have translate the problem to the decomposition of differentiation of imaginary part of scalar basis. Furthermore, from the angle of differentiation, recurrence relation can be established and the whole reduction can be reduced to the reduction of basic integral, i.e., one and only one propagator with power two. We demonstrate our method by carrying out the reduction of basic example for the scalar bubbles, triangles, boxes and pentagons and give analytic expression for reduction coefficients.

One of unsolved problems of our algorithm is the analytic tadpole coefficients of the reduction. Although the recurrence relation can be established for the tadpole coefficients, the tadpole coefficients for the basic integrals, for examples $I_2(1, 2)$, $I_3(1, 1, 2)$, $I_4(1, 1, 1, 2)$

and $I_5(1, 1, 1, 1, 2)$ can not be found by unitarity cut method. One can use the technique developed in [19–21] or the familiar IBP method to calculate them to complete the whole reduction program for higher poles.

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