Correlation functions of spinor current multiplets in $\mathcal{N} = 1$ superconformal theory

Evgeny I. Buchbinder, Jessica Hutomo and Sergei M. Kuzenko

*Department of Physics M013, The University of Western Australia,
35 Stirling Highway, Crawley W.A. 6009, Australia*

E-mail: evgeny.buchbinder@uwa.edu.au, jessica.hutomo@uwa.edu.au,
sergei.kuzenko@uwa.edu.au

ABSTRACT: We consider $\mathcal{N} = 1$ superconformal field theories in four dimensions possessing an additional conserved spinor current multiplet S_α and study three-point functions involving such an operator. A conserved spinor current multiplet naturally exists in superconformal theories with $\mathcal{N} = 2$ supersymmetry and contains the current of the second supersymmetry. However, we do not assume $\mathcal{N} = 2$ supersymmetry. We show that the three-point function of two spinor current multiplets and the $\mathcal{N} = 1$ supercurrent depends on three independent tensor structures and, in general, is not contained in the three-point function of the $\mathcal{N} = 2$ supercurrent. It then follows, based on symmetry considerations only, that the existence of one more Grassmann odd current multiplet in $\mathcal{N} = 1$ superconformal field theory does not necessarily imply $\mathcal{N} = 2$ superconformal symmetry.

KEYWORDS: Extended Supersymmetry, Superspaces

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1 Introduction

The study of the general structure of correlation functions of primary fields in conformal field theory has a long history. Early theoretical ideas and results can be traced back to the 1970s [1–9] (see also a review [10] and references therein). A systematic approach to determining two- and three-point functions of conserved currents, such as the energy-momentum tensor and vector current, was undertaken by Osborn and Petkou in [11] and

by Erdmenger and Osborn in [12]. Such correlation functions are severely constrained by conformal symmetry and the conservation properties and are fixed up to finitely many coefficients. In certain cases these coefficients are related to Weyl or chiral anomalies of a conformal field theory coupled to an external curved background or an external gauge field.

Since the late 1990s, the approach of [11, 12] was generalised to (super)conformal field theories in diverse dimensions [13–24]. In supersymmetric case the energy-momentum tensor and vector currents must be supplemented with new fermionic conserved currents. The energy-momentum tensor is then replaced with the supercurrent and vector currents are replaced with flavour current multiplets. Thus, the supercurrent and flavour current multiplets contain fundamental information about the symmetries of a given supersymmetric field theory. The concept of supercurrent was introduced in [25], while flavour current multiplets were discussed for the first time in [26]. For discussions of the most general four-dimensional $\mathcal{N} = 1$ non-conformal supercurrents, see [27–30]. Recently, general properties of superconformal theories in diverse dimensions were studied in [31–35].

Most analysis of correlation functions in (super)conformal field theories has been performed in coordinate space. A formalism in momentum space was recently developed in [36–42].

The aim of this paper is to study four-dimensional $\mathcal{N} = 1$ superconformal field theory possessing a conserved *spinor current multiplet*. Such a current naturally exists in $\mathcal{N} = 2$ superconformal field theory and is a part of the $\mathcal{N} = 2$ supercurrent $\mathcal{J}$ [43, 44]. In general, $\mathcal{J}$ contains several conserved $\mathcal{N} = 1$ currents. To describe them let us split the $\mathcal{N} = 2$ covariant derivatives $(D_\alpha^i, \bar{D}_i^{\dot{\alpha}})$ into $(D_\alpha^1 = D_\alpha, \bar{D}_1^{\dot{\alpha}} = \bar{D}^{\dot{\alpha}})$ and $(D_\alpha^2, \bar{D}_2^{\dot{\alpha}})$ and define the $\mathcal{N} = 1$ bar-projection $U| = U(x, \theta_i^\alpha, \bar{\theta}_{\dot{\alpha}}^j)|_{\theta_2 = \bar{\theta}^2 = 0}$. Then $\mathcal{J}$ is composed out of the three independent $\mathcal{N} = 1$ currents [18]:

$$J = \mathcal{J}|, \quad J_\alpha = D_\alpha^2 \mathcal{J}|, \quad J_{\alpha\dot{\alpha}} = \frac{1}{2}[D_\alpha^2, \bar{D}_{\dot{\alpha}2}] \mathcal{J}| - \frac{1}{6}[D_\alpha^1, \bar{D}_{\dot{\alpha}1}] \mathcal{J}|. \quad (1.1)$$

Here $J_{\alpha\dot{\alpha}}$ is the $\mathcal{N} = 1$ supercurrent containing the energy-momentum tensor, the current of the first supersymmetry and the $\mathcal{N} = 1$ R -symmetry current [25]. The spinor current multiplet J_α contains the current of the second supersymmetry and two of the three $SU(2)$ R -symmetry currents. Finally, J contains the current corresponding to the combined $\mathcal{N} = 2$ $U(1)$ R -transformations and $SU(2)$ z -rotations which leave θ_1 and $\bar{\theta}^1$ invariant. In this paper, however, we do not assume $\mathcal{N} = 2$ supersymmetry. We study just $\mathcal{N} = 1$ superconformal field theory which also has a Grassmann odd spinor current multiplet S_α and ask the question whether the existence of such a current implies $\mathcal{N} = 2$ superconformal symmetry. For this purpose we study three-point functions involving S_α and find their most general form consistent with $\mathcal{N} = 1$ superconformal symmetry. If existence of S_α implies $\mathcal{N} = 2$ superconformal symmetry, such three-point functions should be all obtained from the three-point function of the $\mathcal{N} = 2$ supercurrent $\mathcal{J}$ by superspace reduction.

A surprising result is that the answer to the above question is negative. More precisely, we find that the three point function

$$\langle \bar{S}_{\dot{\alpha}}(z_1) S_\beta(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle \quad (1.2)$$

is, in general, inconsistent with $\mathcal{N} = 2$ superconformal symmetry because it has more independent tensor structures. That is, despite the fact that S_α in components contains a conserved spin- $\frac{3}{2}$ current, its existence does not imply $\mathcal{N} = 2$ superconformal symmetry. We would like to stress that our results are based on the symmetry considerations only and we do not know how to realise them in local field theory. We were not able to find explicit field theoretic examples of $\mathcal{N} = 1$ superconformal theories possessing a spinor current multiplet, yet without being $\mathcal{N} = 2$ supersymmetric. On the other hand, we demonstrate that the three-point function

$$\langle \bar{S}_\alpha(z_1) S_\beta(z_2) L(z_3) \rangle, \quad (1.3)$$

where L is a flavour current superfield in an $\mathcal{N} = 1$ superconformal theory, is fixed up to two real parameters. This result is in agreement with $\mathcal{N} = 2$ superconformal symmetry, in the sense that the three-point function $\langle \bar{J}_\alpha(z_1) J_\beta(z_2) J(z_3) \rangle$ also possesses two linearly independent structures.

One natural question which arises in this context is whether our results are intrinsic for $\mathcal{N} = 1$ superconformal symmetry or also hold in non-supersymmetric case. That is, let us consider a non-supersymmetric conformal theory which also possesses a conserved Grassmann odd spin- $\frac{3}{2}$ current $Q_{m\alpha}$. Based on symmetries only, does the existence of $Q_{m\alpha}$ imply $\mathcal{N} = 1$ supersymmetry? If the answer is positive the three-point function

$$\langle \bar{Q}_{m\dot{\alpha}}(x_1) Q_{n\beta}(x_2) T_{pq}(x_3) \rangle, \quad (1.4)$$

where T_{pq} is the energy-momentum tensor, must be contained in the three-point function of the $\mathcal{N} = 1$ supercurrent. This question is left for future research.¹ One more natural question is whether our results are intrinsic to four-dimensional (super)conformal symmetry or hold in say three and six dimensional (super)conformal theories. This question is also left for future research.

The paper is organised as follows. In section 2 we review the general construction of three-point functions in superconformal theories. In our review we follow [13–15, 18] where additional details can be found. In section 3 we discuss three-point functions of spinor current multiplets and the supercurrent in $\mathcal{N} = 1$ superconformal theories. Our main result is that the three point function (1.2) is fixed up to three independent parameters. In section 4 we consider three-point functions of spinor current and the flavour current multiplets. In section 5 we discuss remaining three-point correlators involving just the spinor current multiplet. In section 6 we study whether our results in sections 3 and 4 are consistent with $\mathcal{N} = 2$ superconformal symmetry. For this we perform superspace reduction from $\mathcal{N} = 2$ to $\mathcal{N} = 1$. We show that our result for the three point function (1.2), in general, is not consistent with $\mathcal{N} = 2$ superconformal symmetry. Finally, in section 7 we discuss the reduction of the superfield correlator (1.2) to correlators of component currents. More precisely, we concentrate on the three-point functions of two U(1) vector currents contained in S_α with the axial current and the energy-momentum tensor contained in $J_{\gamma\dot{\gamma}}$. We show that our results are in complete agreement with the general structure of these three-point functions derived in [11, 12].

¹To best of our knowledge this correlation function has not been previously studied.

2 Superconformal building blocks

In this section we provide a brief review of two and three-point superconformal structures in 4D $\mathcal{N}$ -extended superspace, which were introduced in [13, 14] in the $\mathcal{N} = 1$ case, and later generalised to arbitrary $\mathcal{N}$ in [15] (see also [18] for a review). These superconformal structures are important for constructing correlation functions of conserved current multiplets in later sections. This section closely follows [18].

2.1 Infinitesimal superconformal transformations

Consider $\mathcal{N}$ -extended Minkowski superspace $\mathbb{M}^{4|4\mathcal{N}}$ parametrised by $z^A = (x^a, \theta_i^\alpha, \bar{\theta}_{\dot{\alpha}}^i)$, where $a = 0, 1, \dots, 3$, $\alpha, \dot{\alpha} = 1, 2$ and $i = \underline{1}, \underline{2}, \dots, \underline{\mathcal{N}}$. The Grassmann variables θ_i^α and $\bar{\theta}_{\dot{\alpha}}^i$ are related to each other by complex conjugation: $\overline{\theta_i^\alpha} = \bar{\theta}^{\dot{\alpha}i}$. Infinitesimal superconformal transformations

$$\delta z^A = \xi z^A, \quad (2.1)$$

are generated by conformal Killing supervector fields [18, 45]

$$\xi = \bar{\xi} = \xi^a(z) \partial_a + \xi_i^\alpha(z) D_\alpha^i + \bar{\xi}_{\dot{\alpha}}^i(z) \bar{D}_{\dot{\alpha}}^i \quad (2.2)$$

which satisfy the equation

$$[\xi, D_\alpha^i] \propto D_\beta^j. \quad (2.3)$$

Eq. (2.3) implies that the spinor parameters can be expressed in terms of the vector ones

$$\xi_i^\alpha = -\frac{i}{8} \bar{D}_{\dot{\beta}i} \xi^{\dot{\beta}\alpha}, \quad \bar{D}_{\dot{\beta}j} \xi_i^\alpha = 0. \quad (2.4)$$

The vector parameters satisfy

$$D_{(\alpha}^i \xi_{\beta)}^{\dot{\beta}} = \bar{D}_{i(\dot{\alpha}} \xi_{\beta\dot{\beta})} = 0, \quad (2.5)$$

which result in the conformal Killing equation

$$\partial_a \xi_b + \partial_b \xi_a = \frac{1}{2} \eta_{ab} \partial_c \xi^c. \quad (2.6)$$

The general solution to eq. (2.5) was given in [45] for $\mathcal{N} = 1$ and in [15] for $\mathcal{N} > 1$. The superalgebra of $\mathcal{N}$ -extended conformal Killing supervector fields is isomorphic to $\mathfrak{su}(2, 2|\mathcal{N})$. We have the relation

$$[\xi, D_\alpha^i] = -(D_\alpha^i \xi_j^\beta) D_\beta^j = \hat{\omega}_\alpha{}^\beta D_\beta^i - \frac{1}{\mathcal{N}} \left((\mathcal{N} - 2) \sigma + 2\bar{\sigma} \right) D_\alpha^i - i \hat{\Lambda}_j{}^i D_\alpha^j. \quad (2.7)$$

Here the superfield parameters are expressed in terms of $\xi^A = (\xi^a, \xi_i^\alpha, \bar{\xi}_{\dot{\alpha}}^i)$ as

$$\begin{aligned} \hat{\omega}_{\alpha\beta}(z) &= -\frac{1}{\mathcal{N}} D_{(\alpha}^i \xi_{\beta)}^{\dot{\beta}} i, & \sigma(z) &= \frac{1}{\mathcal{N}(\mathcal{N} - 4)} \left(\frac{1}{2} (\mathcal{N} - 2) D_\alpha^i \xi_i^\alpha - \bar{D}_{\dot{\alpha}}^i \bar{\xi}_{\dot{\alpha}}^i \right), \\ \hat{\Lambda}_j{}^i(z) &= -\frac{1}{32} \left([D_\alpha^i, \bar{D}_{\dot{\alpha}j}] - \frac{1}{\mathcal{N}} \delta_j^i [D_\alpha^k, \bar{D}_{\dot{\alpha}k}] \right) \xi^{\dot{\alpha}\alpha}. \end{aligned} \quad (2.8)$$

The z -dependent parameters $\hat{\omega}_{\alpha\beta}(z), \sigma(z), \hat{\Lambda}_j{}^i(z)$ correspond to the ‘local’ Lorentz, scale and $SU(\mathcal{N})$ R -symmetry transformations, respectively. One may also check the following properties:

$$\begin{aligned}\bar{D}_{\dot{\alpha}i}\hat{\omega}_{\alpha\beta} &= 0, \quad \bar{D}_{\dot{\alpha}i}\sigma = 0, \\ \hat{\Lambda}^\dagger &= \hat{\Lambda}, \quad \text{tr } \hat{\Lambda} = 0, \quad D_\alpha^k \hat{\Lambda}_j{}^i = 2i \left(\delta_j^k D_\alpha^i - \frac{1}{\mathcal{N}} \delta_j^i D_\alpha^k \right) \sigma.\end{aligned}\tag{2.9}$$

For $\mathcal{N} = 2$, the latter property leads to the analyticity condition

$$D_\alpha^{(i} \hat{\Lambda}^{jk)} = \bar{D}_{\dot{\alpha}}^{(i} \hat{\Lambda}^{jk)} = 0.\tag{2.10}$$

The above formalism cannot be directly applied to the case $\mathcal{N} = 4$. In this paper, we are interested in the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ cases.

2.2 Two-point structures

Let z_1 and z_2 be two different points in superspace. In constructing correlation functions of primary superfields, all building blocks are composed of the following two-point structures

$$x_{12}^a = -x_{2\bar{1}}^a = x_{1-}^a - x_{2+}^a + 2i\theta_{2i}\sigma^a\bar{\theta}_1^i,\tag{2.11a}$$

$$\theta_{12} = \theta_1 - \theta_2, \quad \bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2,\tag{2.11b}$$

where $x_\pm^a = x^a \pm i\theta_i\sigma^a\bar{\theta}^i$. The former can also be expressed in spinor notation as

$$(x_{12})^{\dot{\alpha}\alpha} = (\tilde{\sigma}_a)^{\dot{\alpha}\alpha} x_{12}^a,\tag{2.12a}$$

$$(x_{2\bar{1}})_{\alpha\dot{\alpha}} = (\sigma_a)_{\alpha\dot{\alpha}} x_{2\bar{1}}^a = -(\sigma_a)_{\alpha\dot{\alpha}} x_{12}^a = -\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}(x_{12})^{\dot{\beta}\beta},\tag{2.12b}$$

$$(x_{12}^\dagger)^{\dot{\alpha}\alpha} = -(x_{2\bar{1}})^{\dot{\alpha}\alpha}.\tag{2.12c}$$

The notation ‘ x_{12} ’ means that x_{12} is antichiral with respect to z_1 and chiral with respect to z_2 . That is,

$$D_{(1)\alpha}^i x_{12} = 0, \quad \bar{D}_{(2)\dot{\alpha}i} x_{12} = 0,\tag{2.13}$$

where $D_{(1)\alpha}^i$ and $\bar{D}_{(1)\dot{\alpha}i}$ are the superspace covariant spinor derivatives acting on the point z_1 and similarly, $D_{(2)\alpha}^i$ and $\bar{D}_{(2)\dot{\alpha}i}$ act on the point z_2 ,

$$D_\alpha^i = \frac{\partial}{\partial \theta_i^\alpha} + i(\sigma^a)_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}i} \frac{\partial}{\partial x^a}, \quad \bar{D}_{\dot{\alpha}i} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}i}} - i\theta_i^\alpha (\sigma^a)_{\alpha\dot{\alpha}} \frac{\partial}{\partial x^a}.\tag{2.14}$$

We will also introduce the inverse of $\tilde{x}_{12}$:

$$(\tilde{x}_{12}^{-1})_{\alpha\dot{\alpha}} = \frac{1}{x_{12}^2} (x_{2\bar{1}})_{\alpha\dot{\alpha}}.\tag{2.15}$$

Note that $(x_{12})^{\dot{\alpha}\alpha} (x_{2\bar{1}})_{\alpha\dot{\beta}} = x_{12}^2 \delta^{\dot{\alpha}}_{\dot{\beta}}$. We will also often make use of matrix-like conventions of [14, 18] when the spinor indices are not explicitly written. We denote

$$\psi = (\psi^\alpha), \quad \tilde{\psi} = (\psi_\alpha), \quad \bar{\psi} = (\bar{\psi}^{\dot{\alpha}}), \quad \tilde{\bar{\psi}} = (\bar{\psi}_{\dot{\alpha}}),\tag{2.16a}$$

$$x = (x_{\alpha\dot{\alpha}}), \quad \tilde{x} = (x^{\dot{\alpha}\alpha}).\tag{2.16b}$$

Since $x^2 \equiv x^a x_a = -\frac{1}{2} \text{tr}(\tilde{x}x)$, it follows that $\tilde{x}^{-1} = -x/x^2$.

The two-point structures transform under the superconformal group as follows

$$\delta x_{12}^{\dot{\alpha}\alpha} = -\left(\hat{\omega}^{\dot{\alpha}}_{\dot{\beta}}(z_1) - \delta^{\dot{\alpha}}_{\dot{\beta}} \bar{\sigma}(z_1)\right) x_{12}^{\dot{\beta}\alpha} - x_{12}^{\dot{\alpha}\beta} \left(\hat{\omega}_{\beta}^{\alpha}(z_2) - \delta_{\beta}^{\alpha} \sigma(z_2)\right) \quad (2.17a)$$

$$\begin{aligned} \delta \theta_{12i}^{\alpha} &= i \hat{\Lambda}_i^j(z_1) + \left(\frac{2}{\mathcal{N}} (\bar{\sigma}(z_1) - \sigma(z_1)) \delta_i^j\right) \theta_{12j}^{\alpha} - i \hat{\eta}_{\dot{\beta}i}^{\dot{\alpha}}(z_1) x_{12}^{\dot{\beta}\alpha} \\ &\quad - \theta_{12i}^{\beta} (\hat{\omega}_{\beta}^{\alpha}(z_2) - \delta_{\beta}^{\alpha} \sigma(z_2)) , \end{aligned} \quad (2.17b)$$

with $\hat{\eta}_{\alpha}^i(z) := \frac{1}{2} D_{\alpha}^i \sigma(z)$. It is useful to define

$$(\hat{x}_{2\bar{1}})_{\alpha\dot{\alpha}} = \frac{(x_{2\bar{1}})_{\alpha\dot{\alpha}}}{(x_{12}^2)^{\frac{1}{2}}} \in \text{SL}(2, \mathbb{C}) , \quad (\hat{x}_{1\bar{2}})_{\alpha\dot{\alpha}} = -(\hat{x}_{2\bar{1}}^{\dagger})_{\alpha\dot{\alpha}} . \quad (2.18)$$

From (2.17a), it follows that x_{12}^2 and $(\hat{x}_{2\bar{1}})_{\alpha\dot{\alpha}}$ transform covariantly under superconformal transformations:

$$\delta x_{12}^2 = 2(\bar{\sigma}(z_1) + \sigma(z_2)) x_{12}^2 , \quad (2.19a)$$

$$\delta (\hat{x}_{2\bar{1}})_{\alpha\dot{\alpha}} = (\hat{x}_{2\bar{1}})_{\alpha\dot{\gamma}} \hat{\omega}^{\dot{\gamma}}_{\dot{\alpha}}(z_1) + \hat{\omega}_{\alpha}^{\gamma}(z_2) (\hat{x}_{2\bar{1}})_{\gamma\dot{\alpha}} . \quad (2.19b)$$

Hence, they appear as building blocks in correlation functions of primary operators.

Another important structure is the conformally covariant $\mathcal{N} \times \mathcal{N}$ matrix

$$u_i^j(z_{12}) = \delta_i^j + 4i \frac{\theta_{12i} x_{2\bar{1}} \bar{\theta}_{12}^j}{x_{12}^2} = \delta_i^j + 4i \theta_{12i} \tilde{x}_{12}^{-1} \bar{\theta}_{12}^j , \quad (2.20)$$

which satisfies

$$u^{\dagger}(z_{12}) u(z_{12}) = 1 , \quad u^{-1}(z_{12}) = u(z_{21}) , \quad \det u(z_{12}) = \frac{x_{12}^2}{x_{21}^2} . \quad (2.21)$$

The unimodular unitary matrix

$$\hat{u}_i^j(z_{12}) = \left(\frac{x_{21}^2}{x_{12}^2}\right)^{1/\mathcal{N}} u_i^j(z_{12}) \quad (2.22)$$

has the transformation rule

$$\delta \hat{u}_i^j(z_{12}) = i \hat{\Lambda}_i^k(z_1) \hat{u}_k^j(z_{12}) - i \hat{u}_i^k(z_{12}) \hat{\Lambda}_k^j(z_2) . \quad (2.23)$$

As follows from (2.19b) and (2.23), the two-point structures $(\hat{x}_{2\bar{1}})_{\alpha\dot{\alpha}}$ and $\hat{u}_i^j(z_{12})$ transform covariantly with respect to the superconformal group, i.e., as tensors with Lorentz and $\text{SU}(\mathcal{N})$ indices at both superspace points.

In the $\mathcal{N} = 2$ case, using the $\text{SU}(2)$ -invariant tensors $\varepsilon_{ij} = -\varepsilon_{ji}$ and $\varepsilon^{ij} = -\varepsilon^{ji}$, with $\varepsilon^{12} = \varepsilon_{21} = 1$, we can raise and lower isindices

$$C^i = \varepsilon^{ij} C_j , \quad C_i = \varepsilon_{ij} C^j . \quad (2.24)$$

We can then write the condition of unimodularity of the matrix $\hat{u}$ as

$$(\hat{u}^{-1}(z_{12}))_i^j = \hat{u}_i^j(z_{21}) = \varepsilon^{jk} \hat{u}_k^l(z_{12}) \varepsilon_{li} , \quad (2.25)$$

or, equivalently

$$\hat{u}_{ji}(z_{21}) = -\hat{u}_{ij}(z_{12}). \quad (2.26)$$

To finish this subsection let us also note several useful differential identities:

$$D_{(1)\alpha}^i (x_{\bar{2}1})^{\dot{\beta}\beta} = 4i\delta_\alpha^\beta \bar{\theta}_{12}^{\dot{\beta}i}, \quad \bar{D}_{(1)\dot{\alpha}i} (x_{\bar{1}2})^{\dot{\beta}\beta} = 4i\delta_{\dot{\alpha}}^{\dot{\beta}} \theta_{12i}^\beta, \quad (2.27a)$$

$$D_{(1)\alpha}^i \left(\frac{1}{x_{\bar{2}1}^2} \right) = -\frac{4i}{x_{\bar{2}1}^2} (\tilde{x}_{\bar{2}1}^{-1})_{\alpha\dot{\beta}} \bar{\theta}_{12}^{\dot{\beta}i}, \quad \bar{D}_{(1)\dot{\alpha}i} \left(\frac{1}{x_{\bar{1}2}^2} \right) = -\frac{4i}{x_{\bar{1}2}^2} (\tilde{x}_{\bar{1}2}^{-1})_{\alpha\dot{\alpha}} \theta_{12i}^\alpha, \quad (2.27b)$$

Here and throughout the paper we assume that the space points are not coincident, $x_1 \neq x_2$.

2.3 Three-point structures

Associated with three superspace points z_1, z_2 and z_3 are the following superconformally covariant bosonic and fermionic variables $\mathbf{Z}_1, \mathbf{Z}_2$ and $\mathbf{Z}_3$, where $\mathbf{Z}_1 = (\mathbf{X}_1^a, \Theta_1^{\alpha i}, \bar{\Theta}_{1i}^{\dot{\alpha}})$ (see [13, 14, 18] for details):

$$\mathbf{X}_1 = \tilde{x}_{\bar{2}1}^{-1} \tilde{x}_{\bar{2}3} \tilde{x}_{\bar{1}3}^{-1}, \quad \tilde{\Theta}_1^i = i(\tilde{x}_{\bar{2}1}^{-1} \bar{\theta}_{12}^i - \tilde{x}_{\bar{3}1}^{-1} \bar{\theta}_{13}^i), \quad \tilde{\bar{\Theta}}_{1i} = i(\theta_{12i} \tilde{x}_{\bar{1}2}^{-1} - \theta_{13i} \tilde{x}_{\bar{1}3}^{-1}), \quad (2.28a)$$

$$\mathbf{X}_2 = \tilde{x}_{\bar{3}2}^{-1} \tilde{x}_{\bar{3}1} \tilde{x}_{\bar{2}1}^{-1}, \quad \tilde{\Theta}_2^i = i(\tilde{x}_{\bar{3}2}^{-1} \bar{\theta}_{23}^i - \tilde{x}_{\bar{1}2}^{-1} \bar{\theta}_{21}^i), \quad \tilde{\bar{\Theta}}_{2i} = i(\theta_{23i} \tilde{x}_{\bar{2}3}^{-1} - \theta_{21i} \tilde{x}_{\bar{2}1}^{-1}), \quad (2.28b)$$

$$\mathbf{X}_3 = \tilde{x}_{\bar{1}3}^{-1} \tilde{x}_{\bar{1}2} \tilde{x}_{\bar{3}2}^{-1}, \quad \tilde{\Theta}_3^i = i(\tilde{x}_{\bar{1}3}^{-1} \bar{\theta}_{31}^i - \tilde{x}_{\bar{2}3}^{-1} \bar{\theta}_{32}^i), \quad \tilde{\bar{\Theta}}_{3i} = i(\theta_{31i} \tilde{x}_{\bar{3}1}^{-1} - \theta_{32i} \tilde{x}_{\bar{3}2}^{-1}). \quad (2.28c)$$

Clearly, the structures in (2.28b) and (2.28c) are obtained through cyclic permutations of superspace points. Thus, it suffices to study the properties of (2.28a). We also define

$$\bar{\mathbf{X}}_1 = \mathbf{X}_1^\dagger = -\tilde{x}_{\bar{3}1}^{-1} \tilde{x}_{\bar{3}2} \tilde{x}_{\bar{1}2}^{-1}, \quad (2.29)$$

and similar relations hold for $\bar{\mathbf{X}}_2, \bar{\mathbf{X}}_3$. The structures (2.28a) transform as tensors at z_1

$$\delta \mathbf{X}_{1\alpha\dot{\alpha}} = \left(\hat{\omega}_\alpha^{\dot{\beta}}(z_1) - \delta_\alpha^{\dot{\beta}} \sigma(z_1) \right) \mathbf{X}_{1\beta\dot{\alpha}} + \mathbf{X}_{1\alpha\dot{\beta}} \left(\hat{\omega}^{\dot{\beta}}_{\dot{\alpha}}(z_1) - \delta^{\dot{\beta}}_{\dot{\alpha}} \bar{\sigma}(z_1) \right), \quad (2.30a)$$

$$\delta \Theta_{1\alpha}^i = \hat{\omega}_\alpha^{\dot{\beta}}(z_1) \Theta_{1\dot{\beta}}^i - i \Theta_{1\alpha}^j \hat{\Lambda}_j^i(z_1) - \frac{1}{\mathcal{N}} \left((\mathcal{N} - 2) \sigma(z_1) + 2 \bar{\sigma}(z_1) \right) \Theta_{1\alpha}^i. \quad (2.30b)$$

By cyclic permutation of labels, we may see that $\mathbf{Z}_i$ transform as tensors at the point z_i , $i = 1, 2, 3$.

Using the matrices $u(z_{rs})$, $r, s = 1, 2, 3$, it is possible to construct unitary matrices

$$\begin{aligned} \mathbf{u}(\mathbf{Z}_3) &= u(z_{31})u(z_{12})u(z_{23}), & \mathbf{u}_i^j(\mathbf{Z}_3) &= \delta_i^j - 4i\tilde{\Theta}_{3i} \mathbf{X}_3^{-1} \tilde{\Theta}_3^j \\ \mathbf{u}^\dagger(\mathbf{Z}_3) &= u(z_{32})u(z_{21})u(z_{13}), & \mathbf{u}_i^{\dagger j}(\mathbf{Z}_3) &= \delta_i^j + 4i\tilde{\Theta}_{3i} \bar{\mathbf{X}}_3^{-1} \tilde{\Theta}_3^j \end{aligned} \quad (2.31)$$

transforming covariantly at z_3 only. Their properties are

$$\mathbf{u}^\dagger(\mathbf{Z}_3) = \mathbf{u}^{-1}(\mathbf{Z}_3), \quad \det \mathbf{u}(\mathbf{Z}_3) = \frac{\mathbf{X}_3^2}{\bar{\mathbf{X}}_3^2}. \quad (2.32)$$

In the $\mathcal{N} = 1$ case, there are several properties of $\mathbf{Z}$'s which might be useful later.

$$\mathbf{X}_1^2 = \frac{x_{\bar{2}3}^2}{x_{\bar{2}1}^2 x_{\bar{1}3}^2}, \quad \bar{\mathbf{X}}_1^2 = \frac{x_{\bar{3}2}^2}{x_{\bar{3}1}^2 x_{\bar{1}2}^2}, \quad (2.33a)$$

$$\bar{\mathbf{X}}_{1\alpha\dot{\alpha}} = \mathbf{X}_{1\alpha\dot{\alpha}} + i\mathbf{P}_{1\alpha\dot{\alpha}}, \quad \mathbf{P}_{1\alpha\dot{\alpha}} = -4\Theta_{1\alpha} \bar{\Theta}_{1\dot{\alpha}}, \quad \mathbf{P}_1^2 = -8\Theta_1^2 \bar{\Theta}_1^2, \quad (2.33b)$$

$$\frac{1}{\bar{\mathbf{X}}_1^2} = \frac{1}{\mathbf{X}_1^2} - 2i \frac{(\mathbf{P}_1 \cdot \mathbf{X}_1)}{(\mathbf{X}_1^2)^2}, \quad (\mathbf{P}_1 \cdot \mathbf{X}_1) = -\frac{1}{2} \mathbf{P}_1^{\dot{\alpha}\alpha} \mathbf{X}_{1\alpha\dot{\alpha}}. \quad (2.33c)$$

Note that $\bar{\mathbf{X}}$ is not an independent variable since it can be expressed in terms of $\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}$. The variables $\mathbf{Z}$ with different labels are related to each other via the identities

$$\tilde{x}_{\bar{1}3} \mathbf{X}_3 \tilde{x}_{\bar{3}1} = -\bar{\mathbf{X}}_1^{-1} = \frac{\tilde{\bar{\mathbf{X}}}_1}{\bar{\mathbf{X}}_1^2}, \quad \tilde{x}_{\bar{1}3} \bar{\mathbf{X}}_3 \tilde{x}_{\bar{3}1} = -\mathbf{X}_1^{-1} = \frac{\tilde{\mathbf{X}}_1}{\mathbf{X}_1^2}, \quad (2.34a)$$

$$\frac{x_{\bar{3}1}^2}{x_{\bar{1}3}^2} \tilde{x}_{\bar{1}3} \tilde{\boldsymbol{\Theta}}_3 = -\mathbf{X}_1^{-1} \tilde{\boldsymbol{\Theta}}_1, \quad \frac{x_{\bar{1}3}^2}{x_{\bar{3}1}^2} \tilde{\boldsymbol{\Theta}}_3 \tilde{x}_{\bar{3}1} = \tilde{\boldsymbol{\Theta}}_1 \bar{\mathbf{X}}_1^{-1}. \quad (2.34b)$$

Making use of (2.30) and (2.34), it follows that

$$\frac{\mathbf{X}_1^2}{\bar{\mathbf{X}}_1^2} = \frac{\mathbf{X}_2^2}{\bar{\mathbf{X}}_2^2} = \frac{\mathbf{X}_3^2}{\bar{\mathbf{X}}_3^2}, \quad (2.35)$$

and this combination is a superconformal invariant.

2.4 Correlation functions of primary superfields

Consider a tensor superfield $\mathcal{O}_{\mathcal{I}}^{\mathcal{A}}(z)$ transforming in a representation T of the Lorentz group with respect to the index $\mathcal{A}$, and in the representation D of the R -symmetry group $\text{SU}(\mathcal{N})$ with respect to the index $\mathcal{I}$.² Such a superfield is called primary if its infinitesimal superconformal transformation law reads

$$\begin{aligned} \delta \mathcal{O}_{\mathcal{I}}^{\mathcal{A}}(z) = & -\xi \mathcal{O}_{\mathcal{I}}^{\mathcal{A}}(z) + (\hat{\omega}^{\alpha\beta}(z) M_{\alpha\beta} + \hat{\bar{\omega}}^{\dot{\alpha}\dot{\beta}}(z) \bar{M}_{\dot{\alpha}\dot{\beta}})^{\mathcal{A}}{}_{\mathcal{B}} \mathcal{O}_{\mathcal{I}}^{\mathcal{B}}(z) \\ & + i \hat{\Lambda}^k{}_l(z) (R^l{}_k)_{\mathcal{I}}{}^{\mathcal{J}} \mathcal{O}_{\mathcal{J}}^{\mathcal{A}}(z) - 2(q\sigma(z) + \bar{q}\bar{\sigma}(z)) \mathcal{O}_{\mathcal{I}}^{\mathcal{A}}(z). \end{aligned} \quad (2.36)$$

In the above, ξ is the superconformal Killing vector, while $\hat{\omega}^{\alpha\beta}(z)$, $\sigma(z)$, $\hat{\Lambda}^k{}_l(z)$ are the z -dependent parameters associated with ξ , see eq. (2.8). The superscript ‘ $\mathcal{A}$ ’ collectively denotes the undotted and dotted spinor indices on which the Lorentz generators $M_{\alpha\beta}$ and $\bar{M}_{\dot{\alpha}\dot{\beta}}$ act. The matrices $R^i{}_j$ are the $\text{SU}(\mathcal{N})$ generators. The weights q and $\bar{q}$ determine the dimension $(q + \bar{q})$ and $\text{U}(1)$ R -symmetry charge $(q - \bar{q})$ of the superfield.

Various primary superfields, including conserved current multiplets, obey certain differential constraints imposed by their conservation equations. These constraints need to be taken into account when computing correlation functions. It proves beneficial to make use of these conformally covariant operators [14, 18]: $\mathcal{D}_{\bar{A}} = (\partial/\partial \mathbf{X}_3^a, \mathcal{D}_{\alpha i}, \bar{\mathcal{D}}^{\dot{\alpha} i})$ and $\mathcal{Q}_{\bar{A}} = (\partial/\partial \mathbf{X}_3^a, \mathcal{Q}_{\alpha i}, \bar{\mathcal{Q}}^{\dot{\alpha} i})$ defined by

$$\begin{aligned} \mathcal{D}_{\alpha i} &= \frac{\partial}{\partial \boldsymbol{\Theta}_3^{\alpha i}} - 2i(\sigma^a)_{\alpha\dot{\alpha}} \bar{\boldsymbol{\Theta}}_3^{\dot{\alpha} i} \frac{\partial}{\partial \mathbf{X}_3^a}, & \bar{\mathcal{D}}^{\dot{\alpha} i} &= \frac{\partial}{\partial \bar{\boldsymbol{\Theta}}_3^{\dot{\alpha} i}}, \\ \mathcal{Q}_{\alpha i} &= \frac{\partial}{\partial \boldsymbol{\Theta}_3^{\alpha i}}, & \bar{\mathcal{Q}}^{\dot{\alpha} i} &= \frac{\partial}{\partial \bar{\boldsymbol{\Theta}}_3^{\dot{\alpha} i}} + 2i \boldsymbol{\Theta}_3^i{}_{\alpha} (\bar{\sigma}^a)^{\dot{\alpha}\alpha} \frac{\partial}{\partial \mathbf{X}_3^a}, \\ [\mathcal{D}_{\bar{A}}, \mathcal{Q}_{\bar{B}}] &= 0. \end{aligned} \quad (2.37)$$

They satisfy anti-commutation relations

$$\{\mathcal{D}_i^{\alpha}, \bar{\mathcal{D}}^{\dot{\alpha} j}\} = 2i \delta_i^j (\bar{\sigma}^a)^{\dot{\alpha}\alpha} \frac{\partial}{\partial \mathbf{X}^a}, \quad \{\mathcal{Q}_i^{\alpha}, \bar{\mathcal{Q}}^{\dot{\alpha} j}\} = -2i \delta_i^j (\bar{\sigma}^a)^{\dot{\alpha}\alpha} \frac{\partial}{\partial \mathbf{X}^a}. \quad (2.38)$$

²We assume the representations T and D are irreducible.

Given an arbitrary function $t(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3)$, the following differential identities hold

$$D_{(1)\alpha}^i t(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3) = -\frac{i}{x_{\bar{3}1}^2} (x_{1\bar{3}})_{\alpha\dot{\alpha}} u_j^i(z_{31}) \bar{D}^{\dot{\alpha}j} t(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3), \quad (2.39a)$$

$$\bar{D}_{(1)\dot{\alpha}i} t(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3) = -\frac{i}{x_{1\bar{3}}^2} (x_{3\bar{1}})_{\alpha\dot{\alpha}} u_i^j(z_{13}) \mathcal{D}_j^\alpha t(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3), \quad (2.39b)$$

$$D_{(2)\alpha}^i t(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3) = -\frac{i}{x_{\bar{3}2}^2} (x_{2\bar{3}})_{\alpha\dot{\alpha}} u_j^i(z_{32}) \bar{\mathcal{Q}}^{\dot{\alpha}j} t(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3), \quad (2.39c)$$

$$\bar{D}_{(2)\dot{\alpha}i} t(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3) = -\frac{i}{x_{\bar{2}3}^2} (x_{3\bar{2}})_{\alpha\dot{\alpha}} u_i^j(z_{23}) \mathcal{Q}_j^\alpha t(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3). \quad (2.39d)$$

The above identities can be derived using these relations:

$$\bar{D}_{(1)\dot{\alpha}i} \mathbf{X}_{3\gamma\dot{\gamma}} = -4(\tilde{x}_{1\bar{3}}^{-1})_{\gamma\dot{\alpha}} u_i^j(z_{13}) \bar{\boldsymbol{\Theta}}_{3\dot{\gamma}j}, \quad \bar{D}_{(1)\dot{\alpha}i} \boldsymbol{\Theta}_{3\gamma}^j = i(\tilde{x}_{1\bar{3}}^{-1})_{\gamma\dot{\alpha}} u_i^j(z_{13}), \quad (2.40a)$$

$$\bar{D}_{(1)\dot{\alpha}i} \bar{\mathbf{X}}_{3\gamma\dot{\gamma}} = 0, \quad \bar{D}_{(1)\dot{\alpha}i} \bar{\boldsymbol{\Theta}}_{3\dot{\gamma}j} = 0, \quad (2.40b)$$

$$D_{(2)\beta}^i \mathbf{X}_{3\gamma\dot{\gamma}} = 4(\tilde{x}_{3\bar{2}}^{-1})_{\beta\dot{\gamma}} \boldsymbol{\Theta}_{3\gamma}^j u_j^i(z_{32}), \quad D_{(2)\beta}^i \bar{\boldsymbol{\Theta}}_{3\dot{\gamma}j} = i(\tilde{x}_{3\bar{2}}^{-1})_{\beta\dot{\gamma}} u_j^i(z_{32}), \quad (2.40c)$$

$$D_{(2)\beta}^i \bar{\mathbf{X}}_{3\gamma\dot{\gamma}} = 0, \quad D_{(2)\beta}^i \boldsymbol{\Theta}_{3\gamma}^j = 0. \quad (2.40d)$$

In the $\mathcal{N} = 1$ case, we arrive at the following

$$D_{(1)\alpha} t(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3) = -\frac{i}{x_{\bar{1}3}^2} (x_{1\bar{3}})_{\alpha\dot{\alpha}} \bar{D}^{\dot{\alpha}} t(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3), \quad (2.41a)$$

$$\bar{D}_{(1)\dot{\alpha}} t(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3) = -\frac{i}{x_{3\bar{1}}^2} (x_{3\bar{1}})_{\alpha\dot{\alpha}} \mathcal{D}^\alpha t(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3), \quad (2.41b)$$

$$D_{(2)\alpha} t(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3) = \frac{i}{x_{\bar{2}3}^2} (x_{2\bar{3}})_{\alpha\dot{\alpha}} \bar{\mathcal{Q}}^{\dot{\alpha}} t(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3), \quad (2.41c)$$

$$\bar{D}_{(2)\dot{\alpha}} t(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3) = \frac{i}{x_{3\bar{2}}^2} (x_{3\bar{2}})_{\alpha\dot{\alpha}} \mathcal{Q}^\alpha t(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3). \quad (2.41d)$$

In accordance with the general prescription of [13–15, 18], the two-point correlation function of a primary superfield $\mathcal{O}_{\mathcal{I}}^{\mathcal{A}}$ with its conjugate $\bar{\mathcal{O}}_{\mathcal{B}}^{\mathcal{J}}$ takes the form

$$\langle \mathcal{O}_{\mathcal{I}}^{\mathcal{A}}(z_1) \bar{\mathcal{O}}_{\mathcal{B}}^{\mathcal{J}}(z_2) \rangle = C_{\mathcal{O}} \frac{T^{\mathcal{A}}_{\mathcal{B}}(\hat{x}_{1\bar{2}}, \hat{x}_{2\bar{1}}) D_{\mathcal{I}}^{\mathcal{J}}(\hat{u}(z_{12}))}{(x_{1\bar{2}}^2)^{\bar{q}} (x_{2\bar{1}}^2)^q}, \quad (2.42)$$

with $C_{\mathcal{O}}$ being a normalisation constant. Furthermore, the three-point correlation function of primary superfields $\Phi_{\mathcal{I}_1}^{\mathcal{A}_1}, \Psi_{\mathcal{I}_2}^{\mathcal{A}_2}$ and $\Pi_{\mathcal{I}_3}^{\mathcal{A}_3}$ has the general expression [13–15, 18]:

$$\begin{aligned} & \langle \Phi_{\mathcal{I}_1}^{\mathcal{A}_1}(z_1) \Psi_{\mathcal{I}_2}^{\mathcal{A}_2}(z_2) \Pi_{\mathcal{I}_3}^{\mathcal{A}_3}(z_3) \rangle \\ &= \frac{T^{(1)\mathcal{A}_1}_{\mathcal{B}_1}(\hat{x}_{1\bar{3}}, \hat{x}_{3\bar{1}}) T^{(2)\mathcal{A}_2}_{\mathcal{B}_2}(\hat{x}_{2\bar{3}}, \hat{x}_{3\bar{2}}) D^{(1)}_{\mathcal{I}_1}{}^{\mathcal{J}_1}(\hat{u}(z_{13})) D^{(2)}_{\mathcal{I}_2}{}^{\mathcal{J}_2}(\hat{u}(z_{23}))}{(x_{1\bar{3}}^2)^{\bar{q}_1} (x_{3\bar{1}}^2)^{q_1} (x_{2\bar{3}}^2)^{\bar{q}_2} (x_{3\bar{2}}^2)^{q_2}} \\ & \times H^{\mathcal{B}_1 \mathcal{B}_2 \mathcal{A}_3}_{\mathcal{J}_1 \mathcal{J}_2 \mathcal{I}_3}(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3). \end{aligned} \quad (2.43)$$

The functional form of the tensor $H^{\mathcal{B}_1 \mathcal{B}_2 \mathcal{A}_3}_{\mathcal{J}_1 \mathcal{J}_2 \mathcal{I}_3}$ is highly constrained by the superconformal symmetry as follows:

- (i) It possesses the homogeneity property

$$H_{\mathcal{J}_1 \mathcal{J}_2 \mathcal{I}_3}^{\mathcal{B}_1 \mathcal{B}_2 \mathcal{A}_3}(\lambda \bar{\lambda} \mathbf{X}, \lambda \boldsymbol{\Theta}, \bar{\lambda} \bar{\boldsymbol{\Theta}}) = \lambda^{2a} \bar{\lambda}^{2\bar{a}} H_{\mathcal{J}_1 \mathcal{J}_2 \mathcal{I}_3}^{\mathcal{B}_1 \mathcal{B}_2 \mathcal{A}_3}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}) \quad (2.44)$$

$$a - 2\bar{a} = \bar{q}_1 + \bar{q}_2 - q_3, \quad \bar{a} - 2a = q_1 + q_2 - \bar{q}_3,$$

which guarantees that the correlation function has the correct transformation law under the superconformal group. Note that eq. (2.43) by construction has the correct transformation properties at the points z_1 and z_2 . The above homogeneity property implies that it transforms correctly also at the point z_3 .

- (ii) If any of the superfields Φ , Ψ and Π satisfy differential equations (e.g. conservation laws of conserved current multiplets), then the tensor $H_{\mathcal{B}_1 \mathcal{B}_2 \mathcal{A}_3}^{\mathcal{J}_1 \mathcal{J}_2 \mathcal{I}_3}$ is constrained by certain differential equations as well. The latter may be derived using (2.39).
- (iii) If any (or all) of the superfields Φ , Ψ and Π coincide, the tensor $H_{\mathcal{B}_1 \mathcal{B}_2 \mathcal{A}_3}^{\mathcal{J}_1 \mathcal{J}_2 \mathcal{I}_3}$ obeys certain constraints, as a consequence of the symmetry under permutations of superspace points, e.g.

$$\langle \Phi_{\mathcal{I}}^{\mathcal{A}}(z_1) \Phi_{\mathcal{J}}^{\mathcal{B}}(z_2) \Pi_{\mathcal{K}}^{\mathcal{C}}(z_3) \rangle = (-1)^{\epsilon(\Phi)} \langle \Phi_{\mathcal{J}}^{\mathcal{B}}(z_2) \Phi_{\mathcal{I}}^{\mathcal{A}}(z_1) \Pi_{\mathcal{K}}^{\mathcal{C}}(z_3) \rangle, \quad (2.45)$$

where $\epsilon(\Phi)$ is the Grassmann parity of $\Phi_{\mathcal{I}}^{\mathcal{A}}$. We further note that under permutations of any two superspace points, the three-point building blocks transform as

$$\mathbf{X}_{3\alpha\dot{\alpha}} \xrightarrow{1\leftrightarrow 2} -\bar{\mathbf{X}}_{3\alpha\dot{\alpha}}, \quad \boldsymbol{\Theta}_{3\alpha}^i \xrightarrow{1\leftrightarrow 2} -\boldsymbol{\Theta}_{3\alpha}^i, \quad (2.46a)$$

$$\mathbf{X}_{3\alpha\dot{\alpha}} \xrightarrow{2\leftrightarrow 3} -\bar{\mathbf{X}}_{2\alpha\dot{\alpha}}, \quad \boldsymbol{\Theta}_{3\alpha}^i \xrightarrow{2\leftrightarrow 3} -\boldsymbol{\Theta}_{2\alpha}^i, \quad (2.46b)$$

$$\mathbf{X}_{3\alpha\dot{\alpha}} \xrightarrow{1\leftrightarrow 3} -\bar{\mathbf{X}}_{1\alpha\dot{\alpha}}, \quad \boldsymbol{\Theta}_{3\alpha}^i \xrightarrow{1\leftrightarrow 3} -\boldsymbol{\Theta}_{1\alpha}^i. \quad (2.46c)$$

The above conditions fix the functional form of $H_{\mathcal{B}_1 \mathcal{B}_2 \mathcal{A}_3}^{\mathcal{J}_1 \mathcal{J}_2 \mathcal{I}_3}$ (and therefore the three-point correlation function) up to a few arbitrary constants.

The main objective of this paper is to determine the constraints on the general structure of three-point correlation functions involving a conserved spinor current multiplet $S_{\alpha}(z)$ (and its conjugate $\bar{S}_{\dot{\alpha}}(z)$), imposed by $\mathcal{N} = 1$ superconformal symmetry. The spinor current superfield S_{α} obeys the following constraints

$$D^{\alpha} S_{\alpha} = 0 \quad \implies \quad D^2 S_{\alpha} = 0, \quad (2.47a)$$

$$\bar{D}^2 S_{\alpha} = 0. \quad (2.47b)$$

These conditions fix its superconformal transformation law to be

$$\delta S_{\alpha} = -\xi S_{\alpha} + \hat{\omega}_{\alpha}^{\beta} S_{\beta} - (3\sigma + 2\bar{\sigma}) S_{\alpha}, \quad (2.48)$$

and hence, S_{α} is a primary superfield with weights $(q, \bar{q}) = (\frac{3}{2}, 1)$ and dimension $\frac{5}{2}$.

It was first demonstrated in [18] (and also discussed in the introduction) that the spinor current multiplet S_α naturally arises from the reduction of the $\mathcal{N} = 2$ conformal supercurrent [43] to $\mathcal{N} = 1$ superspace.³ However, in this paper we will not assume $\mathcal{N} = 2$ superconformal symmetry. We will consider $\mathcal{N} = 1$ superconformal theory which also has a conserved spinor current multiplet S_α . Our aim is to find how three-point correlation functions involving S_α are determined by $\mathcal{N} = 1$ superconformal symmetry. A surprising result is that these correlation functions, in general, do not imply $\mathcal{N} = 2$ superconformal symmetry.

3 Correlation functions of spinor current multiplets with the supercurrent

Let us recall that the $\mathcal{N} = 1$ supercurrent is described by a primary real superfield $J_{\alpha\dot{\alpha}} = \bar{J}_{\dot{\alpha}\alpha}$, subject to the conservation law

$$D^\alpha J_{\alpha\dot{\alpha}} = \bar{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} = 0. \quad (3.1)$$

Its superconformal transformation is

$$\delta J_{\alpha\dot{\alpha}} = -\xi J_{\alpha\dot{\alpha}} + (\hat{\omega}_\alpha{}^\beta \delta_{\dot{\alpha}}^{\dot{\beta}} + \bar{\hat{\omega}}_{\dot{\alpha}}{}^{\dot{\beta}} \delta_\alpha^\beta) J_{\beta\dot{\beta}} - 3(\sigma + \bar{\sigma}) J_{\alpha\dot{\alpha}}. \quad (3.2)$$

The reality constraint and conservation law (3.1) imply that $J_{\alpha\dot{\alpha}}$ has weights $(q, \bar{q}) = (\frac{3}{2}, \frac{3}{2})$ and dimension 3.

3.1 Correlator $\langle \bar{S}_{\dot{\alpha}}(z_1) S_\beta(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle$

According to the general prescription (2.43), we shall look for the three-point function of the form

$$\langle \bar{S}_{\dot{\alpha}}(z_1) S_\beta(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle = \frac{(x_{3\bar{1}})_{\alpha\dot{\alpha}} (x_{2\bar{3}})_{\beta\dot{\beta}}}{(x_{1\bar{3}}^2)^2 (x_{3\bar{2}}^2)^2 x_{3\bar{1}}^2 x_{2\bar{3}}^2} H^{\dot{\beta}\alpha}{}_{\gamma\dot{\gamma}}(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3). \quad (3.3)$$

The tensor $H_{\alpha\dot{\beta}, \gamma\dot{\gamma}}$ has the following homogeneity property

$$H_{\alpha\dot{\beta}, \gamma\dot{\gamma}}(\lambda\bar{\lambda}\mathbf{X}, \lambda\boldsymbol{\Theta}, \bar{\lambda}\bar{\boldsymbol{\Theta}}) = \lambda^{-2}\bar{\lambda}^{-2} H_{\alpha\dot{\beta}, \gamma\dot{\gamma}}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}). \quad (3.4)$$

The correlator must obey the conservation equations (2.47) and (3.1). Specifically, we impose:

$$\bar{D}_{(1)}^{\dot{\alpha}} \langle \bar{S}_{\dot{\alpha}}(z_1) S_\beta(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle = 0, \quad D_{(1)}^2 \langle \bar{S}_{\dot{\alpha}}(z_1) S_\beta(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle = 0, \quad (3.5a)$$

$$D_{(2)}^\beta \langle \bar{S}_{\dot{\alpha}}(z_1) S_\beta(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle = 0, \quad \bar{D}_{(2)}^2 \langle \bar{S}_{\dot{\alpha}}(z_1) S_\beta(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle = 0, \quad (3.5b)$$

$$D_{(3)}^\gamma \langle \bar{S}_{\dot{\alpha}}(z_1) S_\beta(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle = 0, \quad \bar{D}_{(3)}^{\dot{\gamma}} \langle \bar{S}_{\dot{\alpha}}(z_1) S_\beta(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle = 0. \quad (3.5c)$$

Upon the use of identities (2.41), along with

$$D_{(1)}^\alpha \left[\frac{(x_{1\bar{3}})_{\alpha\dot{\alpha}}}{(x_{3\bar{1}}^2)^2} \right] = 0 \quad \Longleftrightarrow \quad \bar{D}_{(1)}^{\dot{\alpha}} \left[\frac{(x_{3\bar{1}})_{\alpha\dot{\alpha}}}{(x_{1\bar{3}}^2)^2} \right] = 0, \quad (3.6)$$

³In a more general setting, ref. [46] describes the structure of conformal current supermultiplets of arbitrary ranks in a supergravity background.

the constraints on the superspace points z_1 and z_2 are translated to

$$\mathcal{D}^\alpha H_{\alpha\dot{\beta},\gamma\dot{\gamma}} = 0, \quad (3.7a)$$

$$\bar{\mathcal{D}}^2 H_{\alpha\dot{\beta},\gamma\dot{\gamma}} = 0, \quad (3.7b)$$

$$\bar{\mathcal{Q}}^{\dot{\beta}} H_{\alpha\dot{\beta},\gamma\dot{\gamma}} = 0, \quad (3.7c)$$

$$\mathcal{Q}^2 H_{\alpha\dot{\beta},\gamma\dot{\gamma}} = 0. \quad (3.7d)$$

Imposing differential constraints (3.5c) is more complicated. We will take care of (3.5c) at the end.

Our aim is to solve for $H_{\alpha\dot{\beta},\gamma\dot{\gamma}}$ subject to the constraints (3.4) and (3.7). First, noting that $H_{\alpha\dot{\beta},\gamma\dot{\gamma}}$ is Grassmann even, the most general ansatz takes the form

$$\begin{aligned} H_{\alpha\dot{\beta},\gamma\dot{\gamma}}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}) &= F_{\alpha\dot{\beta},\gamma\dot{\gamma}}(\mathbf{X}) + A_{\alpha\dot{\beta},\gamma\dot{\gamma}}(\mathbf{X})\boldsymbol{\Theta}^2 + B_{\alpha\dot{\beta},\gamma\dot{\gamma}}(\mathbf{X})\bar{\boldsymbol{\Theta}}^2 + C_{\alpha\dot{\beta},\gamma\dot{\gamma}}\boldsymbol{\Theta}^2\bar{\boldsymbol{\Theta}}^2 \\ &\quad + G_{\alpha\dot{\beta},\gamma\dot{\gamma},\delta\dot{\delta}}(\mathbf{X})\boldsymbol{\Theta}^\delta\bar{\boldsymbol{\Theta}}^{\dot{\delta}}. \end{aligned} \quad (3.8)$$

It is easy to see that the conditions (3.7b) and (3.7d) lead to

$$A_{\alpha\dot{\beta},\gamma\dot{\gamma}}(\mathbf{X}) = 0, \quad B_{\alpha\dot{\beta},\gamma\dot{\gamma}}(\mathbf{X}) = 0, \quad C_{\alpha\dot{\beta},\gamma\dot{\gamma}}(\mathbf{X}) = 0, \quad (3.9)$$

and hence, $H_{\alpha\dot{\beta},\gamma\dot{\gamma}}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}})$ gets simplified to

$$H_{\alpha\dot{\beta},\gamma\dot{\gamma}}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}) = F_{\alpha\dot{\beta},\gamma\dot{\gamma}}(\mathbf{X}) + G_{\alpha\dot{\beta},\gamma\dot{\gamma},\delta\dot{\delta}}(\mathbf{X})\boldsymbol{\Theta}^\delta\bar{\boldsymbol{\Theta}}^{\dot{\delta}}. \quad (3.10)$$

In order to work out the structures of $F_{\alpha\dot{\beta},\gamma\dot{\gamma}}(\mathbf{X})$ and $G_{\alpha\dot{\beta},\gamma\dot{\gamma},\delta\dot{\delta}}(\mathbf{X})$, it is instructive to make use of vector notation, where we write

$$F_{\alpha\dot{\beta},\gamma\dot{\gamma}} = (\sigma^m)_{\alpha\dot{\beta}}(\sigma^n)_{\gamma\dot{\gamma}}F_{mn}, \quad (3.11a)$$

$$G_{\alpha\dot{\beta},\gamma\dot{\gamma},\delta\dot{\delta}} = (\sigma^m)_{\alpha\dot{\beta}}(\sigma^n)_{\gamma\dot{\gamma}}(\sigma^p)_{\delta\dot{\delta}}G_{mnp}. \quad (3.11b)$$

It is not difficult to construct the most general expressions for these tensors, which are consistent with eq. (3.4). We get

$$F_{mn} = c_1 \frac{\eta_{mn}}{\mathbf{X}^2} + c_2 \frac{\mathbf{X}_m \mathbf{X}_n}{(\mathbf{X}^2)^2}, \quad (3.12)$$

$$G_{mnp} = \text{id}_1 \frac{\mathbf{X}_m \mathbf{X}_n \mathbf{X}_p}{(\mathbf{X}^2)^3} + \text{id}_2 \frac{\eta_{mn} \mathbf{X}_p}{(\mathbf{X}^2)^2} + \text{id}_3 \frac{\eta_{mp} \mathbf{X}_n}{(\mathbf{X}^2)^2} + \text{id}_4 \frac{\eta_{np} \mathbf{X}_m}{(\mathbf{X}^2)^2} + d_5 \epsilon_{mnpq} \frac{\mathbf{X}^q}{(\mathbf{X}^2)^2}, \quad (3.13)$$

where c_n and d_n are some complex coefficients. Our next task is to rewrite (3.7a) and (3.7c) in vector notation to obtain constraints on the coefficients c_n and d_n . Making use of the expression (3.10), one can show that eq. (3.7a) is fulfilled under the conditions

$$\partial^{\delta\dot{\alpha}} G_{\alpha\dot{\beta},\gamma\dot{\gamma},\delta\dot{\delta}} = 0, \quad \varepsilon^{\alpha\delta} G_{\alpha\dot{\beta},\gamma\dot{\gamma},\delta\dot{\delta}} = 2i \varepsilon_{\delta\dot{\alpha}} \partial^{\dot{\alpha}\alpha} F_{\alpha\dot{\beta},\gamma\dot{\gamma}}. \quad (3.14)$$

These differential constraints are equivalent to

$$\eta^{mp} \partial_k G_{mnp} - \partial^m (G_{mnk} + G_{knm}) + i \epsilon^{pqm}{}_k \partial_q G_{mnp} = 0, \quad (3.15a)$$

$$G_{mnp} - G_{pnm} = 2i (\partial_p F_{mn} - \partial_m F_{pn}), \quad \eta^{mp} G_{mnp} = 2i \partial^m F_{mn}. \quad (3.15b)$$

On the other hand, eq. (3.7c) implies that

$$\partial^{\dot{\beta}\dot{\delta}} G_{\alpha\dot{\beta},\gamma\dot{\gamma},\delta\dot{\delta}} = 0, \quad \varepsilon^{\dot{\beta}\dot{\delta}} G_{\alpha\dot{\beta},\gamma\dot{\gamma},\delta\dot{\delta}} = 2i \varepsilon_{\delta\beta} \partial^{\beta\dot{\beta}} F_{\alpha\dot{\beta},\gamma\dot{\gamma}}, \quad (3.16a)$$

which read

$$\eta^{mp} \partial_k G_{mnp} - \partial^m (G_{mnk} + G_{knm}) - i \epsilon^{pqm}{}_k \partial_q G_{mnp} = 0, \quad (3.17a)$$

$$G_{mnp} - G_{pnm} = 2i (\partial_p F_{mn} - \partial_m F_{pn}), \quad \eta^{mp} G_{mnp} = 2i \partial^m F_{mn}. \quad (3.17b)$$

Thus, eqs. (3.15b) and (3.17b) are identical. It follows from eqs. (3.15a) and (3.17a) that

$$\epsilon^{pqm}{}_k \partial_q G_{mnp} = 0, \quad \eta^{mp} \partial_k G_{mnp} - \partial^m (G_{mnk} + G_{knm}) = 0, \quad (3.18)$$

which lead to

$$d_5 = 0, \quad d_1 = -2d_3. \quad (3.19)$$

Upon imposing (3.15b), we find that

$$c_1 = -\frac{1}{4}(d_2 + d_3), \quad c_2 = \frac{1}{2}(d_3 + d_4). \quad (3.20)$$

At this stage, we end up with three free coefficients, d_2, d_3, d_4 for our function H_{mn} :

$$\begin{aligned} H_{mn}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}) &= -\frac{1}{4}(d_2 + d_3) \frac{\eta_{mn}}{\mathbf{X}^2} + \frac{1}{2}(d_3 + d_4) \frac{\mathbf{X}_m \mathbf{X}_n}{(\mathbf{X}^2)^2} \\ &\quad + \frac{1}{2} d_4 \frac{i}{(\mathbf{X}^2)^2} \mathbf{X}_m \mathbf{P}_n + \frac{1}{2} d_3 \frac{i}{(\mathbf{X}^2)^2} \mathbf{X}_n \mathbf{P}_m \\ &\quad + \frac{1}{2} d_2 \frac{i}{(\mathbf{X}^2)^2} \eta_{mn} (\mathbf{P} \cdot \mathbf{X}) - d_3 \frac{i}{(\mathbf{X}^2)^3} (\mathbf{P} \cdot \mathbf{X}) \mathbf{X}_m \mathbf{X}_n, \end{aligned} \quad (3.21)$$

where we recalled that $\mathbf{P}_{\alpha\dot{\alpha}} = -4\boldsymbol{\Theta}_\alpha \bar{\boldsymbol{\Theta}}_{\dot{\alpha}}$. We can also express our result in spinor notation:

$$\begin{aligned} H^{\dot{\beta}\alpha}{}_{\gamma\dot{\gamma}}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}) &= \frac{1}{2}(d_2 + d_3) \frac{1}{\mathbf{X}^2} \delta_\gamma^\alpha \delta_{\dot{\gamma}}^{\dot{\beta}} + \frac{1}{2}(d_3 + d_4) \frac{1}{(\mathbf{X}^2)^2} \mathbf{X}^{\dot{\beta}\alpha} \mathbf{X}_{\gamma\dot{\gamma}} \\ &\quad + \frac{1}{2} d_4 \frac{i}{(\mathbf{X}^2)^2} \mathbf{X}^{\dot{\beta}\alpha} \mathbf{P}_{\gamma\dot{\gamma}} + \frac{1}{2} d_3 \frac{i}{(\mathbf{X}^2)^2} \mathbf{X}_{\gamma\dot{\gamma}} \mathbf{P}^{\dot{\beta}\alpha} \\ &\quad - d_2 \frac{i}{(\mathbf{X}^2)^2} (\mathbf{P} \cdot \mathbf{X}) \delta_\gamma^\alpha \delta_{\dot{\gamma}}^{\dot{\beta}} - d_3 \frac{i}{(\mathbf{X}^2)^3} (\mathbf{P} \cdot \mathbf{X}) \mathbf{X}_{\gamma\dot{\gamma}} \mathbf{X}^{\dot{\beta}\alpha}. \end{aligned} \quad (3.22)$$

It remains to impose conservation equations on the supercurrent $J_{\gamma\dot{\gamma}}$, eq. (3.5c). Checking conservation laws on z_3 is non-trivial since there are no identities that would allow differential operators acting on the z_3 dependence to pass through the prefactor of (3.3). To simplify our computation, let us first express our correlator (3.3) as

$$\langle \bar{S}_{\dot{\alpha}}(z_1) S_{\beta}(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle = \frac{1}{k_1} I_{\alpha\dot{\alpha}}(x_{3\bar{1}}) I_{\beta\dot{\beta}}(x_{2\bar{3}}) H^{\dot{\beta}\alpha}{}_{\gamma\dot{\gamma}}(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3), \quad (3.23)$$

where we have defined⁴

$$\begin{aligned} I_{\alpha\dot{\alpha}}(x_{3\bar{1}}) &= \frac{(x_{3\bar{1}})_{\alpha\dot{\alpha}}}{(x_{1\bar{3}}^2)^{1/2}}, & I_{\beta\dot{\beta}}(x_{2\bar{3}}) &= \frac{(x_{2\bar{3}})_{\beta\dot{\beta}}}{(x_{3\bar{2}}^2)^{1/2}}, \\ k_1 &:= (x_{1\bar{3}}^2)^{3/2} (x_{3\bar{2}}^2)^{3/2} x_{3\bar{1}}^2 x_{2\bar{3}}^2. \end{aligned} \quad (3.24)$$

⁴The operator $I_{\alpha\dot{\alpha}}(x_{3\bar{1}})$ was previously denoted by $(\hat{x}_{3\bar{1}})_{\alpha\dot{\alpha}}$ in subsection 2.2.

On the other hand, the same correlator can be written as follows

$$\begin{aligned}
 \langle \bar{S}_{\dot{\alpha}}(z_1) S_{\beta}(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle &= -\langle S_{\beta}(z_2) J_{\gamma\dot{\gamma}}(z_3) \bar{S}_{\dot{\alpha}}(z_1) \rangle \\
 &= -\frac{1}{k_2} I_{\beta\dot{\rho}}(x_{2\bar{1}}) I_{\gamma\dot{\delta}}(x_{3\bar{1}}) I_{\delta\dot{\gamma}}(x_{1\bar{3}}) \tilde{H}_{\dot{\alpha}}^{\dot{\rho}, \delta\dot{\delta}}(\mathbf{X}_1, \boldsymbol{\Theta}_1, \bar{\boldsymbol{\Theta}}_1), \quad (3.25) \\
 k_2 &:= (x_{1\bar{2}}^2)^{3/2} (x_{1\bar{3}}^2)^{3/2} (x_{3\bar{1}}^2)^{3/2} x_{2\bar{1}}^2,
 \end{aligned}$$

for some function $\tilde{H}_{\dot{\alpha}}^{\dot{\rho}, \delta\dot{\delta}}(\mathbf{X}_1, \boldsymbol{\Theta}_1, \bar{\boldsymbol{\Theta}}_1)$. The second line in eq. (3.25) has the same structure as the general expression (2.43) except now we are treating z_1 as the “third point”. Since eqs. (3.23) and (3.25) represent the same correlator the functions H and $\tilde{H}$ are related. Knowing H we can find $\tilde{H}$ by comparing eqs. (3.23) and (3.25). For this let us introduce the operators inverse to (3.24):

$$\bar{I}^{\dot{\alpha}\alpha}(\tilde{x}_{12}) = \frac{(x_{12})^{\dot{\alpha}\alpha}}{(x_{12}^2)^{1/2}}, \quad \bar{I}^{\dot{\alpha}\beta}(\tilde{x}_{12}) I_{\beta\dot{\beta}}(x_{2\bar{1}}) = \delta_{\dot{\beta}}^{\dot{\alpha}}. \quad (3.26)$$

It follows from (3.23), (3.24), (3.25) and (3.26) that the transformed function $\tilde{H}_{\dot{\alpha}}^{\dot{\rho}, \delta\dot{\delta}}$ can be directly computed from $H^{\dot{\beta}\alpha}_{\gamma\dot{\gamma}}(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3)$ using the relation

$$\begin{aligned}
 &\tilde{H}_{\dot{\alpha}}^{\dot{\lambda}, \dot{\sigma}\sigma}(\mathbf{X}_1, \boldsymbol{\Theta}_1, \bar{\boldsymbol{\Theta}}_1) \\
 &= -\frac{k_2}{k_1} [\bar{I}^{\dot{\lambda}\beta}(\tilde{x}_{12}) I_{\beta\dot{\beta}}(x_{2\bar{3}})] [I_{\alpha\dot{\alpha}}(x_{3\bar{1}}) \bar{I}^{\dot{\sigma}\gamma}(\tilde{x}_{13}) \bar{I}^{\dot{\gamma}\sigma}(\tilde{x}_{31})] H^{\dot{\beta}\alpha}_{\gamma\dot{\gamma}}(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3). \quad (3.27)
 \end{aligned}$$

Using the identity

$$\bar{I}^{\dot{\lambda}\beta}(\tilde{x}_{12}) I_{\beta\dot{\mu}}(x_{2\bar{3}}) \bar{I}^{\dot{\mu}\mu}(\tilde{x}_{31}) = \bar{I}^{\dot{\lambda}\mu}(\tilde{\mathbf{X}}_1), \quad \bar{I}^{\dot{\lambda}\mu}(\tilde{\mathbf{X}}_1) = \frac{\bar{\mathbf{X}}_1^{\dot{\lambda}\mu}}{(\bar{\mathbf{X}}_1^2)^{1/2}} \quad (3.28)$$

we can also present eq. (3.27) in the form

$$\begin{aligned}
 &\tilde{H}_{\dot{\alpha}}^{\dot{\lambda}, \dot{\sigma}\sigma}(\mathbf{X}_1, \boldsymbol{\Theta}_1, \bar{\boldsymbol{\Theta}}_1) \\
 &= -\frac{k_2}{k_1} \bar{I}^{\dot{\lambda}\mu}(\tilde{\mathbf{X}}_1) [I_{\alpha\dot{\alpha}}(x_{3\bar{1}}) I_{\mu\dot{\beta}}(x_{1\bar{3}}) \bar{I}^{\dot{\sigma}\gamma}(\tilde{x}_{13}) \bar{I}^{\dot{\gamma}\sigma}(\tilde{x}_{31})] H^{\dot{\beta}\alpha}_{\gamma\dot{\gamma}}(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3). \quad (3.29)
 \end{aligned}$$

To compute $\tilde{H}_{\dot{\alpha}}^{\dot{\lambda}, \dot{\sigma}\sigma}$, we will use the following identities [14]:

$$I_{\alpha\dot{\beta}}(x_{3\bar{1}}) I_{\beta\dot{\alpha}}(x_{1\bar{3}}) \mathbf{X}_3^{\dot{\alpha}\alpha} = -\frac{\mathbf{X}_1^I{}_{\beta\dot{\beta}}}{(\mathbf{X}_1^2 \bar{\mathbf{X}}_1^2 x_{1\bar{3}}^2 x_{3\bar{1}}^2)^{\frac{1}{2}}}, \quad (3.30a)$$

$$\bar{I}^{\dot{\alpha}\beta}(x_{3\bar{1}}) \bar{I}^{\dot{\beta}\alpha}(x_{1\bar{3}}) \mathbf{X}_3^{\alpha\dot{\alpha}} = -\frac{\mathbf{X}_1^{I\dot{\beta}\beta}}{(\mathbf{X}_1^2 \bar{\mathbf{X}}_1^2 x_{1\bar{3}}^2 x_{3\bar{1}}^2)^{\frac{1}{2}}}, \quad (3.30b)$$

$$I_{\alpha\dot{\alpha}}(x_{1\bar{3}}) \bar{\boldsymbol{\Theta}}_3^{\dot{\alpha}} = \left(\frac{x_{3\bar{1}}^2}{\bar{\mathbf{X}}_1^2} \right)^{\frac{1}{2}} \frac{\bar{\boldsymbol{\Theta}}_{1\alpha}^I}{x_{1\bar{3}}^2}, \quad \boldsymbol{\Theta}_3^{\alpha} I_{\alpha\dot{\alpha}}(x_{3\bar{1}}) = \left(\frac{x_{1\bar{3}}^2}{\mathbf{X}_1^2} \right)^{\frac{1}{2}} \frac{\boldsymbol{\Theta}_{1\dot{\alpha}}^I}{x_{3\bar{1}}^2}. \quad (3.30c)$$

Here we have defined

$$\mathbf{X}_1^I{}_{\alpha\dot{\alpha}} := -\left(\frac{\mathbf{X}_1^2}{\bar{\mathbf{X}}_1^2}\right)^{\frac{1}{2}} \bar{\mathbf{X}}_1{}_{\alpha\dot{\alpha}}, \quad (3.30d)$$

$$\Theta_{1\dot{\alpha}}^I := \Theta_1^\alpha I_{\alpha\dot{\alpha}}(-\mathbf{X}_1) = -\frac{\Theta_1^\alpha \mathbf{X}_{1\alpha\dot{\alpha}}}{(\mathbf{X}_1^2)^{\frac{1}{2}}}, \quad (3.30e)$$

$$\bar{\Theta}_{1\alpha}^I := I_{\alpha\dot{\alpha}}(\bar{\mathbf{X}}_1) \bar{\Theta}_1^{\dot{\alpha}} = \frac{\bar{\mathbf{X}}_{1\alpha\dot{\alpha}}}{(\bar{\mathbf{X}}_1^2)^{\frac{1}{2}}} \bar{\Theta}_1^{\dot{\alpha}}. \quad (3.30f)$$

After some calculations using the above identities we find that $\tilde{H}_\alpha^{\dot{\rho},\delta\delta}(\mathbf{X}_1, \Theta_1, \bar{\Theta}_1)$ takes the form

$$\begin{aligned} \tilde{H}_\alpha^{\dot{\rho},\delta\delta}(\mathbf{X}_1, \Theta_1, \bar{\Theta}_1) = & -\frac{1}{2}(d_2 + d_3) \frac{1}{\mathbf{X}_1^2 \bar{\mathbf{X}}_1^2} \bar{\mathbf{X}}_1^{\dot{\rho}\delta} \delta_\alpha^\delta + \frac{1}{2}(d_3 + d_4) \frac{1}{\mathbf{X}_1^2 \bar{\mathbf{X}}_1^2} \bar{\mathbf{X}}_1^{\delta\delta} \delta_\alpha^{\dot{\rho}} \\ & - \frac{i}{2} d_4 \frac{1}{(\mathbf{X}_1^2)^2 \bar{\mathbf{X}}_1^2} \bar{\mathbf{X}}_1^{\dot{\beta}\delta} \mathbf{P}_{1\beta\dot{\beta}} \mathbf{X}_1^{\delta\beta} \delta_\alpha^{\dot{\rho}} - \frac{i}{2} d_3 \frac{1}{(\mathbf{X}_1^2)^2 \bar{\mathbf{X}}_1^2} \bar{\mathbf{X}}_1^{\delta\delta} \mathbf{X}_{1\beta\dot{\alpha}} \mathbf{P}_1^{\dot{\rho}\beta} \\ & + i d_2 \frac{1}{(\mathbf{X}_1^2)^2 \bar{\mathbf{X}}_1^2} \bar{\mathbf{X}}_1^{\dot{\rho}\delta} (\mathbf{P}_1 \cdot \mathbf{X}_1) \delta_\alpha^\delta - i d_3 \frac{1}{(\mathbf{X}_1^2)^2 \bar{\mathbf{X}}_1^2} \bar{\mathbf{X}}_1^{\delta\delta} (\mathbf{P}_1 \cdot \mathbf{X}_1) \delta_\alpha^{\dot{\rho}}. \end{aligned} \quad (3.31)$$

Next, relabelling the superspace points $z_2 \rightarrow z_1$, $z_3 \rightarrow z_2$, $z_1 \rightarrow z_3$, along with eqs. (2.34), allows us to rewrite (3.25) in the equivalent form

$$\begin{aligned} \langle S_\beta(z_1) J_{\gamma\dot{\gamma}}(z_2) \bar{S}_\alpha(z_3) \rangle &= \frac{1}{k_3} I_{\beta\dot{\beta}}(x_{1\bar{3}}) I_{\gamma\dot{\gamma}}(x_{2\bar{3}}) I_{\delta\dot{\delta}}(x_{3\bar{2}}) \tilde{H}_\alpha^{\dot{\rho},\delta\delta}(\mathbf{X}_3, \Theta_3, \bar{\Theta}_3), \quad (3.32) \\ k_3 &:= (x_{\bar{3}1}^2)^{3/2} (x_{\bar{3}2}^2)^{3/2} (x_{\bar{2}3}^2)^{3/2} x_{1\bar{3}}^2. \end{aligned}$$

Now we can apply the conservation condition of $J_{\gamma\dot{\gamma}}$ using eqs. (2.41). We get

$$\mathcal{Q}_\delta \tilde{H}_\alpha^{\dot{\rho},\delta\delta}(\mathbf{X}_3, \Theta_3, \bar{\Theta}_3) = 0, \quad \bar{\mathcal{Q}}_{\dot{\delta}} \tilde{H}_\alpha^{\dot{\rho},\delta\delta}(\mathbf{X}_3, \Theta_3, \bar{\Theta}_3) = 0. \quad (3.33)$$

After some calculations, one may verify that (3.31) satisfies (3.33) for an arbitrary choice of d_2, d_3, d_4 .

Finally, we note that our three-point function $\langle \bar{S}_\alpha(z_1) S_\beta(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle$ satisfies the reality condition

$$\langle \bar{S}_\alpha(z_1) S_\beta(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle = \langle \bar{S}_\beta(z_2) S_\alpha(z_1) J_{\gamma\dot{\gamma}}(z_3) \rangle^*, \quad (3.34)$$

where $*$ means complex conjugation. It implies the following condition on the function $\bar{H}^{\dot{\alpha}\beta}{}_{\gamma\dot{\gamma}}$:

$$\bar{H}^{\dot{\alpha}\beta}{}_{\gamma\dot{\gamma}}(\mathbf{X}_3, \Theta_3, \bar{\Theta}_3) = H^{\dot{\alpha}\beta}{}_{\gamma\dot{\gamma}}(-\bar{\mathbf{X}}_3, -\Theta_3, -\bar{\Theta}_3), \quad (3.35)$$

where $\bar{H}^{\dot{\alpha}\beta}{}_{\gamma\dot{\gamma}}(\mathbf{X}_3, \Theta_3, \bar{\Theta}_3)$ is the complex conjugate of the expression (3.22). This condition gives the reality constraint on the coefficients:

$$\bar{d}_2 = d_2, \quad \bar{d}_3 = d_3, \quad \bar{d}_4 = d_4. \quad (3.36)$$

Therefore, our final expression for the three-point function $\langle \bar{S}_\alpha(z_1) S_\beta(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle$ is determined up to three independent, real coefficients:

$$\langle \bar{S}_\alpha(z_1) S_\beta(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle = \frac{I_{\alpha\dot{\alpha}}(x_{3\bar{1}}) I_{\beta\dot{\beta}}(x_{2\bar{3}})}{(x_{1\bar{3}}^2)^{3/2} (x_{\bar{3}2}^2)^{3/2} x_{\bar{3}1}^2 x_{\bar{2}3}^2} H^{\dot{\rho}\alpha}{}_{\gamma\dot{\gamma}}(\mathbf{X}_3, \Theta_3, \bar{\Theta}_3), \quad (3.37a)$$

with

$$\begin{aligned}
 H^{\dot{\beta}\alpha}{}_{\gamma\dot{\gamma}}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}) &= \frac{1}{2}(d_2 + d_3) \frac{1}{\mathbf{X}^2} \delta_{\gamma}^{\alpha} \delta_{\dot{\gamma}}^{\dot{\beta}} + \frac{1}{2}(d_3 + d_4) \frac{1}{(\mathbf{X}^2)^2} \mathbf{X}^{\dot{\beta}\alpha} \mathbf{X}_{\gamma\dot{\gamma}} \\
 &+ \frac{1}{2} d_4 \frac{i}{(\mathbf{X}^2)^2} \mathbf{X}^{\dot{\beta}\alpha} \mathbf{P}_{\gamma\dot{\gamma}} + \frac{1}{2} d_3 \frac{i}{(\mathbf{X}^2)^2} \mathbf{P}^{\dot{\beta}\alpha} \mathbf{X}_{\gamma\dot{\gamma}} \\
 &- d_2 \frac{i}{(\mathbf{X}^2)^2} (\mathbf{P} \cdot \mathbf{X}) \delta_{\gamma}^{\alpha} \delta_{\dot{\gamma}}^{\dot{\beta}} - d_3 \frac{i}{(\mathbf{X}^2)^3} (\mathbf{P} \cdot \mathbf{X}) \mathbf{X}^{\dot{\beta}\alpha} \mathbf{X}_{\gamma\dot{\gamma}}. \quad (3.37b)
 \end{aligned}$$

In vector notation, this is equivalent to

$$\begin{aligned}
 H_{mn}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}) &= \frac{1}{4} (\sigma_m)_{\alpha\dot{\beta}} (\tilde{\sigma}_n)^{\dot{\gamma}\gamma} H^{\dot{\beta}\alpha}{}_{\gamma\dot{\gamma}}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}) \\
 &= -\frac{1}{4} (d_2 + d_3) \frac{\eta_{mn}}{\mathbf{X}^2} + \frac{1}{2} (d_3 + d_4) \frac{\mathbf{X}_m \mathbf{X}_n}{(\mathbf{X}^2)^2} \\
 &+ \frac{1}{2} d_4 \frac{i}{(\mathbf{X}^2)^2} \mathbf{X}_m \mathbf{P}_n + \frac{1}{2} d_3 \frac{i}{(\mathbf{X}^2)^2} \mathbf{X}_n \mathbf{P}_m \\
 &+ \frac{1}{2} d_2 \frac{i}{(\mathbf{X}^2)^2} \eta_{mn} (\mathbf{P} \cdot \mathbf{X}) - d_3 \frac{i}{(\mathbf{X}^2)^3} (\mathbf{P} \cdot \mathbf{X}) \mathbf{X}_m \mathbf{X}_n, \quad (3.37c)
 \end{aligned}$$

Later we will see that eq. (3.37) cannot be obtained from three-point function of the $\mathcal{N} = 2$ supercurrent unless the coefficients d_2, d_3, d_4 satisfy an additional constraint $d_2 + d_3 + d_4 = 0$. However, $\mathcal{N} = 1$ superconformal symmetry alone does not impose this constraint.

3.2 Correlator $\langle S_{\alpha}(z_1) J_{\beta\dot{\beta}}(z_2) S_{\gamma}(z_3) \rangle$

In the remaining part of the section we will consider two other correlators involving the supercurrent and spinor currents. They both have non-compensating R -symmetry charge and are expected to vanish. We will check that it is indeed the case.

According to (2.43), we write the general ansatz

$$\langle S_{\alpha}(z_1) J_{\beta\dot{\beta}}(z_2) S_{\gamma}(z_3) \rangle = \frac{(x_{1\bar{3}})_{\alpha\dot{\alpha}} (x_{2\bar{3}})_{\beta\dot{\beta}} (x_{3\bar{2}})_{\delta\dot{\delta}}}{x_{1\bar{3}}^2 (x_{3\bar{1}}^2)^2 (x_{2\bar{3}}^2)^2 (x_{3\bar{2}}^2)^2} H^{\dot{\alpha}, \dot{\delta}\delta}{}_{\gamma}(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3). \quad (3.38)$$

As a result of imposing the conservation equations (2.47) and (3.1), the tensor $H^{\dot{\alpha}, \dot{\delta}\delta}{}_{\gamma}$ must satisfy the following constraints

$$\begin{aligned}
 \bar{\mathcal{Q}}^{\dot{\delta}} H_{\dot{\alpha}, \dot{\delta}\delta, \gamma} &= 0, \quad \mathcal{Q}^{\delta} H_{\dot{\alpha}, \dot{\delta}\delta, \gamma} = 0, \\
 \bar{\mathcal{D}}^{\dot{\alpha}} H_{\dot{\alpha}, \dot{\delta}\delta, \gamma} &= 0, \quad \mathcal{D}^2 H_{\dot{\alpha}, \dot{\delta}\delta, \gamma} = 0.
 \end{aligned} \quad (3.39)$$

It obeys the homogeneity property

$$H^{\dot{\alpha}, \dot{\delta}\delta}{}_{\gamma}(\lambda \bar{\lambda} \mathbf{X}, \lambda \boldsymbol{\Theta}, \bar{\lambda} \bar{\boldsymbol{\Theta}}) = \lambda^{-3} \bar{\lambda}^{-3} H^{\dot{\alpha}, \dot{\delta}\delta}{}_{\gamma}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}). \quad (3.40)$$

We also need to make sure that the correlator has the right symmetry property under the replacement of superspace points $z_1 \leftrightarrow z_3$,

$$\langle S_{\alpha}(z_1) J_{\beta\dot{\beta}}(z_2) S_{\gamma}(z_3) \rangle = -\langle S_{\gamma}(z_3) J_{\beta\dot{\beta}}(z_2) S_{\alpha}(z_1) \rangle, \quad (3.41)$$

since the superfield S_{α} is Grassmann odd.

As in the previous subsection, it is useful to switch to vector notation,

$$H_{\dot{\alpha}, \delta \dot{\delta}, \gamma} = (\sigma^m)_{\delta \dot{\delta}} (\sigma^n)_{\gamma \dot{\alpha}} H_{mn}. \quad (3.42)$$

Since $H_{mn}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}})$ is Grassmann even, the most general form we can write is

$$\begin{aligned} H_{mn}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}) = & A_{mn}(\mathbf{X}) + \boldsymbol{\Theta}^2 B_{mn}(\mathbf{X}) + \bar{\boldsymbol{\Theta}}^2 C_{mn}(\mathbf{X}) + \boldsymbol{\Theta}^2 \bar{\boldsymbol{\Theta}}^2 D_{mn}(\mathbf{X}) \\ & + \boldsymbol{\Theta}^\alpha \bar{\boldsymbol{\Theta}}^{\dot{\beta}} (\sigma^p)_{\alpha \dot{\beta}} F_{mnp}(\mathbf{X}). \end{aligned} \quad (3.43)$$

The requirements $\mathcal{Q}^\delta H_{\dot{\alpha}, \delta \dot{\delta}, \gamma} = 0$ and $\bar{\mathcal{D}}^{\dot{\alpha}} H_{\dot{\alpha}, \delta \dot{\delta}, \gamma} = 0$ lead to the algebraic constraints

$$B_{mn} = 0, \quad C_{mn} = 0, \quad D_{mn} = 0, \quad (3.44a)$$

$$\eta^{mp} F_{mnp} = \eta^{np} F_{mnp} = 0, \quad F_{mnp} = F_{mpn} = F_{pnm}, \quad (3.44b)$$

and hence, (3.44b) implies that $F_{mnp}(\mathbf{X})$ is completely symmetric and traceless. On the other hand, imposing $\bar{\mathcal{Q}}^{\dot{\delta}} H_{\dot{\alpha}, \delta \dot{\delta}, \gamma} = 0$ and $\mathcal{D}^2 H_{\dot{\alpha}, \delta \dot{\delta}, \gamma} = 0$ gives us several differential constraints:

$$\partial^m F_{mnp} = 0, \quad (3.45a)$$

$$\partial^m A_{mn} = \partial_{[p} A_{m]n} = 0, \quad \square A_{mn} = 0, \quad (3.45b)$$

where $\square := \partial^m \partial_m$.

Let us solve for $F_{mnp}(\mathbf{X})$. Being completely symmetric in m, n, p , the general solution for $F_{mnp}(\mathbf{X})$ compatible with the homogeneity property (3.40) reads

$$F_{mnp} = \frac{f_1}{(\mathbf{X}^2)^{7/2}} \mathbf{X}_m \mathbf{X}_n \mathbf{X}_p + \frac{f_2}{(\mathbf{X}^2)^{5/2}} (\eta_{mn} \mathbf{X}_p + \eta_{np} \mathbf{X}_m + \eta_{mp} \mathbf{X}_n) \quad (3.46)$$

Imposing (3.44b) and (3.45a) leads to

$$f_1 = f_2 = 0. \quad (3.47)$$

Next, we turn to $A_{mn}(\mathbf{X})$, for which we have the general expression

$$A_{mn} = a_1 \frac{\eta_{mn}}{(\mathbf{X}^2)^{3/2}} + a_2 \frac{\mathbf{X}_m \mathbf{X}_n}{(\mathbf{X}^2)^{5/2}}. \quad (3.48)$$

It is easy to check that the conditions $\partial^m A_{mn} = 0$ and $\partial_{[p} A_{m]n} = 0$ already give

$$a_1 = a_2 = 0. \quad (3.49)$$

Therefore, we conclude that

$$\langle S_\alpha(z_1) J_{\beta \dot{\beta}}(z_2) S_\gamma(z_3) \rangle = 0. \quad (3.50)$$

3.3 Correlator $\langle S_\alpha(z_1) J_{\beta\dot{\beta}}(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle$

According to eq. (2.43), the general expression of this three-point function is as follows

$$\langle S_\alpha(z_1) J_{\beta\dot{\beta}}(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle = \frac{(x_{1\bar{3}})_{\alpha\dot{\alpha}}(x_{2\bar{3}})_{\beta\dot{\beta}}(x_{3\bar{2}})_{\delta\dot{\delta}}}{x_{1\bar{3}}^2(x_{3\bar{1}}^2)^2(x_{2\bar{3}}^2)^2(x_{3\bar{2}}^2)^2} H^{\dot{\alpha},\dot{\delta}}{}_{\gamma\dot{\gamma}}(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3). \quad (3.51)$$

As a result of imposing the conservation equations (2.47) and (3.1), the tensor $H^{\dot{\alpha},\dot{\delta}}{}_{\gamma\dot{\gamma}}$ must satisfy the following constraints

$$\begin{aligned} \bar{\mathcal{Q}}^{\dot{\delta}} H_{\dot{\alpha},\dot{\delta},\gamma\dot{\gamma}} &= 0, \quad \mathcal{Q}^{\dot{\delta}} H_{\dot{\alpha},\dot{\delta},\gamma\dot{\gamma}} = 0, \\ \bar{\mathcal{D}}^{\dot{\alpha}} H_{\dot{\alpha},\dot{\delta},\gamma\dot{\gamma}} &= 0, \quad \mathcal{D}^2 H_{\dot{\alpha},\dot{\delta},\gamma\dot{\gamma}} = 0. \end{aligned} \quad (3.52)$$

It also obeys the homogeneity property

$$H^{\dot{\alpha},\dot{\delta}}{}_{\gamma\dot{\gamma}}(\lambda\bar{\lambda}\mathbf{X}, \lambda\boldsymbol{\Theta}, \bar{\lambda}\bar{\boldsymbol{\Theta}}) = \lambda^{-8/3}\bar{\lambda}^{-7/3} H^{\dot{\alpha},\dot{\delta}}{}_{\gamma\dot{\gamma}}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}). \quad (3.53)$$

The correlator also satisfies the symmetry under $z_2 \leftrightarrow z_3$:

$$\langle S_\alpha(z_1) J_{\beta\dot{\beta}}(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle = \langle S_\alpha(z_1) J_{\gamma\dot{\gamma}}(z_3) J_{\beta\dot{\beta}}(z_2) \rangle. \quad (3.54)$$

Let us now write

$$H_{\dot{\alpha},\dot{\delta},\gamma\dot{\gamma}} = (\sigma^m)_{\dot{\delta}\dot{\delta}}(\sigma^n)_{\gamma\dot{\gamma}} H_{mn,\dot{\alpha}}. \quad (3.55)$$

Since $H_{mn,\dot{\alpha}}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}})$ is Grassmann odd, the most general form we can have is

$$\begin{aligned} H_{mn,\dot{\alpha}}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}) &= \boldsymbol{\Theta}^\rho(\sigma^k)_{\rho\dot{\alpha}} A_{kmn}(\mathbf{X}) + \bar{\boldsymbol{\Theta}}^2 \boldsymbol{\Theta}^\rho(\sigma^k)_{\rho\dot{\alpha}} B_{kmn}(\mathbf{X}) \\ &+ \bar{\boldsymbol{\Theta}}^{\dot{\rho}} C_{mn,\dot{\rho},\dot{\alpha}}(\mathbf{X}) + \boldsymbol{\Theta}^2 \bar{\boldsymbol{\Theta}}^{\dot{\rho}} D_{mn,\dot{\rho},\dot{\alpha}}(\mathbf{X}). \end{aligned} \quad (3.56)$$

Imposing $\mathcal{Q}^{\dot{\delta}} H_{\dot{\alpha},\dot{\delta},\gamma\dot{\gamma}} = 0$ and $\bar{\mathcal{D}}^{\dot{\alpha}} H_{\dot{\alpha},\dot{\delta},\gamma\dot{\gamma}} = 0$ leads to the requirements

$$B_{kmn} = 0, \quad D_{mn,\dot{\rho},\dot{\alpha}} = 0, \quad (3.57a)$$

$$C_{mn,\dot{\rho},\dot{\alpha}} = \frac{1}{2} \varepsilon_{\dot{\rho}\dot{\lambda}} (\tilde{\sigma}^k \sigma^p - \tilde{\sigma}^p \sigma^k)^{\dot{\lambda}}{}_{\dot{\alpha}} \tilde{C}_{mn[kp]}, \quad (3.57b)$$

$$\eta^{km} A_{kmn} = 0, \quad A_{[km]n} = 0. \quad (3.57c)$$

The condition $\bar{\mathcal{Q}}^{\dot{\delta}} H_{\dot{\alpha},\dot{\delta},\gamma\dot{\gamma}} = 0$ is equivalent to

$$(\sigma^m \tilde{\sigma}^k \sigma^p) \tilde{C}_{mn[kp]} = 0, \quad \partial^m \tilde{C}_{mn[kp]} = 0, \quad \partial_{[r} \tilde{C}_{m]n[kp]} = 0, \quad (3.58a)$$

$$\partial^k A_{kmn} = 0, \quad \partial_{[r} A_{k]mn} = 0. \quad (3.58b)$$

It can be shown that $\mathcal{D}^2 H_{\dot{\alpha},\dot{\delta},\gamma\dot{\gamma}} = 0$ does not give further constraints.

Let us try to construct the explicit solution for $A_{kmn}(\mathbf{X})$. Since it has dimension -3 , we can have a general expression

$$\begin{aligned} A_{kmn} &= \frac{a_1}{(\mathbf{X}^2)^3} \mathbf{X}_k \mathbf{X}_m \mathbf{X}_n + \frac{1}{(\mathbf{X}^2)^2} \left(a_2 \eta_{km} \mathbf{X}_n + a_3 \eta_{kn} \mathbf{X}_m + a_4 \eta_{mn} \mathbf{X}_k \right. \\ &\quad \left. + a_5 \epsilon_{kmnp} \mathbf{X}^p \right) \end{aligned} \quad (3.59)$$

Eq. (3.57c) relates the coefficients in the following way

$$a_5 = 0, \quad a_3 = a_4 = -\frac{1}{2}(a_1 + 4a_2). \quad (3.60)$$

Demanding that $\partial^k A_{kmn} = 0$ leads to $a_2 = -\frac{1}{2}a_1$, while $\partial_{[r} A_{k]mn} = 0$ gives $a_1 = 0$. As a result,

$$A_{kmn}(\mathbf{X}) = 0. \quad (3.61)$$

Next, we solve for $\tilde{C}_{mn[kp]}(\mathbf{X})$, for which we have the general expression

$$\begin{aligned} \tilde{C}_{mn[kp]} &= \frac{c_1}{(\mathbf{X}^2)^{3/2}} \epsilon_{mnkp} + \frac{c_2}{(\mathbf{X}^2)^{3/2}} (\eta_{mk}\eta_{np} - \eta_{mp}\eta_{nk}) \\ &\quad + \frac{c_3}{(\mathbf{X}^2)^{5/2}} (\eta_{nk}\mathbf{X}_p\mathbf{X}_m - \eta_{np}\mathbf{X}_k\mathbf{X}_m) + \frac{c_4}{(\mathbf{X}^2)^{5/2}} (\eta_{mk}\mathbf{X}_p\mathbf{X}_n - \eta_{mp}\mathbf{X}_k\mathbf{X}_n). \end{aligned} \quad (3.62)$$

It can be shown that (3.62) solves $(\sigma^m \tilde{\sigma}^k \sigma^p) \tilde{C}_{mn[kp]} = 0$ for

$$c_1 = i \left(c_2 - \frac{c_3}{3} \right), \quad c_4 = -\frac{2}{3}c_3. \quad (3.63)$$

In deriving the above, we have made use of the identity

$$\sigma^m \tilde{\sigma}^k \sigma^p = \eta^{mp} \sigma^k - \eta^{kp} \sigma^m - \eta^{mk} \sigma^p + i\epsilon^{mkpq} \sigma_q. \quad (3.64)$$

On the other hand, the constraint $\partial^m \tilde{C}_{mn[kp]} = 0$ requires

$$c_1 = 0, \quad c_4 = -3c_2. \quad (3.65)$$

Taking into account (3.63), we immediately obtain

$$\tilde{C}_{mn[kp]}(\mathbf{X}) = 0. \quad (3.66)$$

Therefore, we conclude that

$$\langle S_\alpha(z_1) J_{\beta\dot{\beta}}(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle = 0. \quad (3.67)$$

4 Correlation functions of spinor current with the flavour current multiplets

The $\mathcal{N} = 1$ flavour current multiplet is a primary real superfield $L^{\bar{a}} = \bar{L}^{\bar{a}}$, where $\bar{a}$ is the index of a flavour symmetry group, subject to the conservation equation

$$D^2 L^{\bar{a}} = \bar{D}^2 L^{\bar{a}} = 0. \quad (4.1)$$

Its superconformal transformation law is

$$\delta L = -\xi L - 2(\sigma + \bar{\sigma})L. \quad (4.2)$$

It then follows that it has weights $(q, \bar{q}) = (1, 1)$ and dimension 2.

In this section, we consider three-point functions of spinor current multiplets $S_\alpha, \bar{S}_{\dot{\alpha}}$ with a U(1) flavour current L . Generalisations to the non-Abelian case are also straightforward.

4.1 Correlator $\langle \bar{S}_{\dot{\alpha}}(z_1) S_{\beta}(z_2) L(z_3) \rangle$

As in the previous section, we start with the general form of the three-point function

$$\langle \bar{S}_{\dot{\alpha}}(z_1) S_{\beta}(z_2) L(z_3) \rangle = \frac{(x_{3\bar{1}})_{\alpha\dot{\alpha}} (x_{2\bar{3}})_{\beta\dot{\beta}}}{(x_{1\bar{3}}^2)^2 x_{31}^2 x_{23}^2 (x_{32}^2)^2} H^{\dot{\beta}\alpha}(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3), \quad (4.3)$$

where $H^{\dot{\beta}\alpha}$ has the following homogeneity property

$$H^{\dot{\beta}\alpha}(\lambda \bar{\lambda} \mathbf{X}, \lambda \boldsymbol{\Theta}, \bar{\lambda} \bar{\boldsymbol{\Theta}}) = \lambda^{-3} \bar{\lambda}^{-3} H^{\dot{\beta}\alpha}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}). \quad (4.4)$$

By taking the complex conjugate of (4.3), we have the reality condition

$$\bar{H}^{\dot{\alpha}\beta}(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3) = H^{\dot{\alpha}\beta}(-\bar{\mathbf{X}}_3, -\boldsymbol{\Theta}_3, -\bar{\boldsymbol{\Theta}}_3). \quad (4.5)$$

As a consequence of the spinor current conservation law (2.47) (and its conjugate), the tensor $H_{\alpha\dot{\beta}}$ must satisfy the differential constraints

$$\begin{aligned} \mathcal{D}^{\alpha} H_{\alpha\dot{\beta}} &= 0, \quad \bar{\mathcal{D}}^2 H_{\alpha\dot{\beta}} = 0, \\ \bar{\mathcal{Q}}^{\dot{\beta}} H_{\alpha\dot{\beta}} &= 0, \quad \mathcal{Q}^2 H_{\alpha\dot{\beta}} = 0. \end{aligned} \quad (4.6)$$

The conservation equations for the flavour current,

$$D_{(3)}^2 \langle \bar{S}_{\dot{\alpha}}(z_1) S_{\beta}(z_2) L(z_3) \rangle = 0, \quad \bar{D}_{(3)}^2 \langle \bar{S}_{\dot{\alpha}}(z_1) S_{\beta}(z_2) L(z_3) \rangle = 0 \quad (4.7)$$

will be imposed at the end, by making use of a similar trick as in subsection 3.1.

Since $H_{\alpha\dot{\beta}}$ is Grassmann even we have the following expansion

$$\begin{aligned} H_{\alpha\dot{\beta}}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}) &= F_{\alpha\dot{\beta}}(\mathbf{X}) + A_{\alpha\dot{\beta}}(\mathbf{X}) \boldsymbol{\Theta}^2 + B_{\alpha\dot{\beta}}(\mathbf{X}) \bar{\boldsymbol{\Theta}}^2 + C_{\alpha\dot{\beta}}(\mathbf{X}) \boldsymbol{\Theta}^2 \bar{\boldsymbol{\Theta}}^2 \\ &\quad + G_{\alpha\dot{\beta}, \delta\dot{\delta}}(\mathbf{X}) \boldsymbol{\Theta}^{\delta} \bar{\boldsymbol{\Theta}}^{\dot{\delta}}. \end{aligned} \quad (4.8)$$

From the conditions $\bar{\mathcal{D}}^2 H_{\alpha\dot{\beta}} = 0$, $\mathcal{Q}^2 H_{\alpha\dot{\beta}} = 0$ we quickly obtain that

$$A_{\alpha\dot{\beta}}(\mathbf{X}) = B_{\alpha\dot{\beta}}(\mathbf{X}) = C_{\alpha\dot{\beta}}(\mathbf{X}) = 0 \quad (4.9)$$

which leaves us with

$$H_{\alpha\dot{\beta}}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}) = F_{\alpha\dot{\beta}}(\mathbf{X}) + G_{\alpha\dot{\beta}, \delta\dot{\delta}}(\mathbf{X}) \boldsymbol{\Theta}^{\delta} \bar{\boldsymbol{\Theta}}^{\dot{\delta}}. \quad (4.10)$$

Converting to vector notation

$$F_{\alpha\dot{\beta}} = (\sigma^m)_{\alpha\dot{\beta}} F_m, \quad G_{\alpha\dot{\beta}, \delta\dot{\delta}} = (\sigma^m)_{\alpha\dot{\beta}} (\sigma^n)_{\delta\dot{\delta}} G_{mn}, \quad (4.11)$$

we see that there are only few possible options for F_m and G_{mn} :

$$F_m = c_1 \frac{\mathbf{X}_m}{(\mathbf{X}^2)^2}, \quad G_{mn} = d_1 \frac{\eta_{mn}}{(\mathbf{X}^2)^2} + d_2 \frac{\mathbf{X}_m \mathbf{X}_n}{(\mathbf{X}^2)^3}, \quad (4.12)$$

where c_1, d_1, d_2 are arbitrary free coefficients. Substituting eqs. (4.10), (4.11), (4.12) into eq. (4.6) we find that $d_2 = -4d_1$. Using the reality property (4.5) we find that c_1 is imaginary and d_1 is real. Hence, we have the following solution

$$H_{\alpha\dot{\beta}} = ic_1 \frac{\mathbf{X}_{\alpha\dot{\beta}}}{(\mathbf{X}^2)^2} + \frac{d_1}{2(\mathbf{X}^2)^3} \left(\mathbf{X}^2 \mathbf{P}_{\alpha\dot{\beta}} - 4(\mathbf{P} \cdot \mathbf{X}) \mathbf{X}_{\alpha\dot{\beta}} \right), \quad (4.13)$$

where we redefined $c_1 \rightarrow ic_1$ so that c_1 and d_1 are now real, free coefficients.

We still must check the flavour current conservation equations (4.7). First, we will express our correlator (4.3) as

$$\begin{aligned} \langle \bar{S}_{\dot{\alpha}}(z_1) S_{\beta}(z_2) L(z_3) \rangle &= \frac{1}{k_1} I_{\alpha\dot{\alpha}}(x_{3\bar{1}}) I_{\beta\dot{\beta}}(x_{2\bar{3}}) H^{\dot{\beta}\alpha}(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3), \\ k_1 &:= (x_{\bar{1}3}^2)^{3/2} x_{\bar{3}1}^2 x_{2\bar{3}}^2 (x_{\bar{3}2}^2)^{3/2}. \end{aligned} \quad (4.14)$$

By rearranging the operators in the three-point function, we may write

$$\begin{aligned} \langle \bar{S}_{\dot{\alpha}}(z_1) S_{\beta}(z_2) L(z_3) \rangle &= -\langle S_{\beta}(z_2) L(z_3) \bar{S}_{\dot{\alpha}}(z_1) \rangle = -\frac{1}{k_2} I_{\beta\dot{\gamma}}(x_{2\bar{1}}) \tilde{H}^{\dot{\gamma}\alpha}(\mathbf{X}_1, \boldsymbol{\Theta}_1, \bar{\boldsymbol{\Theta}}_1), \\ k_2 &:= (x_{\bar{1}2}^2)^{3/2} x_{2\bar{1}}^2 x_{\bar{1}3}^2 x_{3\bar{1}}^2. \end{aligned} \quad (4.15)$$

Comparing eqs. (4.14) and (4.15) we find the relation between H and $\tilde{H}$:

$$\tilde{H}^{\dot{\gamma}\alpha}(\mathbf{X}_1, \boldsymbol{\Theta}_1, \bar{\boldsymbol{\Theta}}_1) = -\frac{k_2}{k_1} \tilde{I}^{\dot{\gamma}\delta}(\tilde{\mathbf{X}}_1) \left[I_{\delta\dot{\beta}}(x_{1\bar{3}}) I_{\alpha\dot{\alpha}}(x_{3\bar{1}}) H^{\dot{\beta}\alpha}(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3) \right], \quad (4.16)$$

where the identity (3.28) has also been used. Performing a similar calculation as in the previous section we find

$$\tilde{H}^{\dot{\gamma}\alpha}(\mathbf{X}_1, \boldsymbol{\Theta}_1, \bar{\boldsymbol{\Theta}}_1) = \frac{ic_1}{\mathbf{X}_1^2} \delta^{\dot{\gamma}\alpha} - \frac{d_1}{2(\mathbf{X}_1^2)^2} \left(\mathbf{P}_1^{\dot{\gamma}\gamma} \mathbf{X}_{1\gamma\dot{\alpha}} + 4\delta^{\dot{\gamma}\alpha} (\mathbf{P}_1 \cdot \mathbf{X}_1) \right). \quad (4.17)$$

Upon relabelling superspace points $z_2 \rightarrow z_1, z_3 \rightarrow z_2, z_1 \rightarrow z_3$, eq. (4.15) turns into

$$\begin{aligned} \langle S_{\beta}(z_1) L(z_2) \bar{S}_{\dot{\alpha}}(z_3) \rangle &= \frac{1}{k_3} I_{\beta\dot{\gamma}}(x_{1\bar{3}}) \tilde{H}^{\dot{\gamma}\alpha}(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3), \\ k_3 &:= (x_{\bar{3}1}^2)^{3/2} x_{3\bar{2}}^2 x_{2\bar{3}}^2 x_{\bar{1}3}^2. \end{aligned} \quad (4.18)$$

Now the conservation conditions (4.7) become

$$\mathcal{Q}^2 \tilde{H}^{\dot{\gamma}\alpha}(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3) = 0, \quad \bar{\mathcal{Q}}^2 \tilde{H}^{\dot{\gamma}\alpha}(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3) = 0. \quad (4.19)$$

By explicit calculations one can show that $\tilde{H}$ in eq. (4.17) satisfies (4.19) for arbitrary c_1 and d_1 .

Thus, in $\mathcal{N} = 1$ superconformal theory, the three-point function involving spinor current multiplets and a flavour current superfield has two linearly independent functional structures with real coefficients c_1 and d_1 :

$$\langle \bar{S}_{\dot{\alpha}}(z_1) S_{\beta}(z_2) L(z_3) \rangle = \frac{(x_{3\bar{1}})_{\alpha\dot{\alpha}} (x_{2\bar{3}})_{\beta\dot{\beta}}}{(x_{\bar{1}3}^2)^2 x_{\bar{3}1}^2 x_{2\bar{3}}^2 (x_{\bar{3}2}^2)^2} H^{\dot{\beta}\alpha}(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3), \quad (4.20)$$

where

$$H_{\alpha\dot{\beta}} = ic_1 \frac{\mathbf{X}_{\alpha\dot{\beta}}}{(\mathbf{X}^2)^2} + \frac{d_1}{2(\mathbf{X}^2)^3} \left(\mathbf{X}^2 \mathbf{P}_{\alpha\dot{\beta}} - 4(\mathbf{P} \cdot \mathbf{X}) \mathbf{X}_{\alpha\dot{\beta}} \right). \quad (4.21)$$

In section 6, we will study how this correlator is related to $\mathcal{N} = 2$ superconformal symmetry.

4.2 Correlator $\langle L(z_1)L(z_2)S_\alpha(z_3) \rangle$

This correlator carries a non-vanishing R -symmetry charge and we will show that it vanishes. Here we demonstrate that by imposing the conservation laws for $L(z_1)$ and $L(z_2)$, it is sufficient to see that $\langle L(z_1)L(z_2)S_\alpha(z_3) \rangle = 0$. As usual, we start with the general form

$$\langle L(z_1)L(z_2)S_\alpha(z_3) \rangle = \frac{1}{x_{13}^2 x_{31}^2 x_{23}^2 x_{32}^2} H_\alpha(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3), \quad (4.22)$$

where the tensor $H_\alpha(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3)$ obeys

$$H_\alpha(\lambda \bar{\lambda} \mathbf{X}, \lambda \boldsymbol{\Theta}, \bar{\lambda} \bar{\boldsymbol{\Theta}}) = \lambda^{-5/3} \bar{\lambda}^{-4/3} H_\alpha(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}), \quad (4.23)$$

and so its dimension is $-3/2$. The conservation equations for $J(z_1)$ and $J(z_2)$ result in

$$\begin{aligned} \bar{\mathcal{D}}^2 H_\alpha &= 0, & \mathcal{D}^2 H_\alpha &= 0, \\ \bar{\mathcal{Q}}^2 H_\alpha &= 0, & \mathcal{Q}^2 H_\alpha &= 0. \end{aligned} \quad (4.24)$$

Remembering that H_α is Grassmann odd, its general expression is given by

$$H_\alpha = \boldsymbol{\Theta}_\alpha A(\mathbf{X}) + \bar{\boldsymbol{\Theta}}^2 \boldsymbol{\Theta}_\alpha B(\mathbf{X}) + \bar{\boldsymbol{\Theta}}^{\dot{\alpha}} (\sigma^m)_{\alpha\dot{\alpha}} C_m(\mathbf{X}) + \boldsymbol{\Theta}^2 \bar{\boldsymbol{\Theta}}^{\dot{\alpha}} (\sigma^m)_{\alpha\dot{\alpha}} D_m(\mathbf{X}). \quad (4.25)$$

The second order constraints $\bar{\mathcal{D}}^2 H_\alpha = \mathcal{Q}^2 H_\alpha = 0$ set

$$B = 0, \quad D_m = 0. \quad (4.26)$$

Using the property (4.23), we can then write

$$H_\alpha = \frac{a_1}{\mathbf{X}^2} \boldsymbol{\Theta}_\alpha + \frac{c_1}{(\mathbf{X}^2)^{3/2}} \bar{\boldsymbol{\Theta}}^{\dot{\alpha}} \mathbf{X}_{\alpha\dot{\alpha}}. \quad (4.27)$$

From here, it is not hard to verify that the requirements $\mathcal{D}^2 H_\alpha = 0$ and $\bar{\mathcal{Q}}^2 H_\alpha = 0$ imply $a_1 = 0$ and $c_1 = 0$, respectively. Thus, we conclude that

$$\langle L(z_1)L(z_2)S_\alpha(z_3) \rangle = 0. \quad (4.28)$$

5 Correlators of the spinor current multiplets

For completeness, we will also consider three-point functions of three spinor current multiplets. Such correlators carry an R -symmetry charge and are expected to vanish. In this section, we will check that it is indeed true.

5.1 Correlator $\langle S_\alpha(z_1)S_\beta(z_2)S_\gamma(z_3) \rangle$

We begin with the general ansatz

$$\langle S_\alpha(z_1)S_\beta(z_2)S_\gamma(z_3) \rangle = \frac{(x_{13})_{\alpha\dot{\alpha}}(x_{23})_{\beta\dot{\beta}}}{x_{13}^2(x_{31}^2)^2 x_{23}^2(x_{32}^2)^2} H^{\dot{\alpha}, \dot{\beta}}{}_\gamma(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3). \quad (5.1)$$

As a result of imposing (2.47), the tensor $H_{\dot{\alpha},\gamma\dot{\beta}}$ must satisfy the following constraints

$$\begin{aligned}\bar{\mathcal{Q}}^{\dot{\beta}}H_{\dot{\alpha},\gamma\dot{\beta}} &= 0, & \mathcal{Q}^2H_{\dot{\alpha},\gamma\dot{\beta}} &= 0, \\ \bar{\mathcal{D}}^{\dot{\alpha}}H_{\dot{\alpha},\gamma\dot{\beta}} &= 0, & \mathcal{D}^2H_{\dot{\alpha},\gamma\dot{\beta}} &= 0.\end{aligned}\tag{5.2}$$

It also obeys the homogeneity property

$$H_{\dot{\alpha},\gamma\dot{\beta}}(\lambda\bar{\lambda}\mathbf{X},\lambda\boldsymbol{\Theta},\bar{\lambda}\bar{\boldsymbol{\Theta}}) = \lambda^{-3}\bar{\lambda}^{-2}H_{\dot{\alpha},\gamma\dot{\beta}}(\mathbf{X},\boldsymbol{\Theta},\bar{\boldsymbol{\Theta}}).\tag{5.3}$$

Note that we must impose one more condition: under permutation of superspace points z_1 and z_3 , the correlation function must satisfy

$$\langle S_{\alpha}(z_1)S_{\beta}(z_2)S_{\gamma}(z_3) \rangle = -\langle S_{\gamma}(z_3)S_{\beta}(z_2)S_{\alpha}(z_1) \rangle.\tag{5.4}$$

However, we will see that the constraints (5.2) and (5.3) are sufficient to show that the correlator vanishes.

Let us trade a pair of spinor indices with the vector one, i.e.

$$H_{\dot{\alpha},\gamma\dot{\beta}} = (\sigma^m)_{\gamma\dot{\beta}}H_{m,\dot{\alpha}},\tag{5.5}$$

where $H_{m,\dot{\alpha}}(\mathbf{X},\boldsymbol{\Theta},\bar{\boldsymbol{\Theta}})$ being Grassmann odd has the following expansion

$$\begin{aligned}H_{m,\dot{\alpha}}(\mathbf{X},\boldsymbol{\Theta},\bar{\boldsymbol{\Theta}}) &= \boldsymbol{\Theta}^{\delta}(\sigma^n)_{\delta\dot{\alpha}}A_{mn}(\mathbf{X}) + \bar{\boldsymbol{\Theta}}^2\boldsymbol{\Theta}^{\delta}(\sigma^n)_{\delta\dot{\alpha}}B_{mn}(\mathbf{X}) \\ &+ \bar{\boldsymbol{\Theta}}^{\dot{\delta}}C_{m,\dot{\delta},\dot{\alpha}}(\mathbf{X}) + \boldsymbol{\Theta}^2\bar{\boldsymbol{\Theta}}^{\dot{\delta}}D_{m,\dot{\delta},\dot{\alpha}}(\mathbf{X}).\end{aligned}\tag{5.6}$$

One can readily verify that the conditions $\mathcal{Q}^2H_{\dot{\alpha},\gamma\dot{\beta}} = 0$ and $\bar{\mathcal{D}}^{\dot{\alpha}}H_{\dot{\alpha},\gamma\dot{\beta}} = 0$ give

$$B_{mn} = 0, \qquad D_{m,\dot{\delta},\dot{\alpha}} = 0,\tag{5.7a}$$

$$\varepsilon^{\dot{\alpha}\dot{\delta}}C_{m,\dot{\delta},\dot{\alpha}} = 0.\tag{5.7b}$$

Thus, it suffices for us to determine the general ansatz for $A_{m,n}$ and $C_{m,(\dot{\delta}\dot{\alpha})}$. In order to be compatible with (5.3), we should write

$$A_{mn} = a_1\frac{\eta_{mn}}{(\mathbf{X}^2)^{3/2}} + a_2\frac{\mathbf{X}_m\mathbf{X}_n}{(\mathbf{X}^2)^{5/2}},\tag{5.8a}$$

$$C_{m,(\dot{\delta}\dot{\alpha})} = (\tilde{\sigma}^{nk})_{\dot{\delta}\dot{\alpha}}\left[\frac{c_1}{(\mathbf{X}^2)^2}(\eta_{mn}\mathbf{X}_k - \eta_{mk}\mathbf{X}_n) + \frac{c_2}{(\mathbf{X}^2)^2}\epsilon_{mnkp}\mathbf{X}^p\right].\tag{5.8b}$$

However, when the remaining constraints $\bar{\mathcal{Q}}^{\dot{\beta}}H_{\dot{\alpha},\gamma\dot{\beta}} = 0$ and $\mathcal{D}^2H_{\dot{\alpha},\gamma\dot{\beta}} = 0$ are taken into account, we find that $a_1 = a_2 = c_1 = c_2 = 0$ and, thus,

$$\langle S_{\alpha}(z_1)S_{\beta}(z_2)S_{\gamma}(z_3) \rangle = 0.\tag{5.9}$$

5.2 Correlator $\langle S_\alpha(z_1) \bar{S}_{\dot{\beta}}(z_2) S_\gamma(z_3) \rangle$

We again begin with the general ansatz

$$\langle S_\alpha(z_1) \bar{S}_{\dot{\beta}}(z_2) S_\gamma(z_3) \rangle = \frac{(x_{1\bar{3}})_{\alpha\dot{\alpha}} (x_{3\bar{2}})_{\beta\dot{\beta}}}{x_{1\bar{3}}^2 (x_{3\bar{1}}^2)^2 (x_{2\bar{3}}^2)^2 x_{3\bar{2}}^2} H^{\dot{\alpha},\beta}{}_\gamma(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3). \quad (5.10)$$

As a result of imposing (2.47), the tensor $H_{\beta,\gamma\dot{\alpha}}$ must satisfy the following constraints

$$\begin{aligned} \mathcal{Q}^\beta H_{\beta,\gamma\dot{\alpha}} &= 0, & \bar{\mathcal{Q}}^2 H_{\beta,\gamma\dot{\alpha}} &= 0, \\ \bar{\mathcal{D}}^{\dot{\alpha}} H_{\beta,\gamma\dot{\alpha}} &= 0, & \mathcal{D}^2 H_{\beta,\gamma\dot{\alpha}} &= 0. \end{aligned} \quad (5.11)$$

It also obeys the homogeneity property

$$H_{\beta,\gamma\dot{\alpha}}(\lambda\bar{\lambda}\mathbf{X}, \lambda\boldsymbol{\Theta}, \bar{\lambda}\bar{\boldsymbol{\Theta}}) = \lambda^{-8/3} \bar{\lambda}^{-7/3} H_{\beta,\gamma\dot{\alpha}}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}). \quad (5.12)$$

Let us trade a pair of spinor indices with the vector one, i.e.

$$H_{\beta,\gamma\dot{\alpha}} = (\sigma^m)_{\gamma\dot{\alpha}} H_{m,\beta}, \quad (5.13)$$

where $H_{m,\beta}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}})$ being Grassmann odd has the following expansion

$$\begin{aligned} H_{m,\beta}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}) &= \boldsymbol{\Theta}^\delta A_{m,\delta,\beta}(\mathbf{X}) + \bar{\boldsymbol{\Theta}}^2 \boldsymbol{\Theta}^\delta B_{m,\delta,\beta}(\mathbf{X}) \\ &+ \bar{\boldsymbol{\Theta}}^{\dot{\delta}} (\sigma^n)_{\beta\dot{\delta}} C_{mn}(\mathbf{X}) + \boldsymbol{\Theta}^2 \bar{\boldsymbol{\Theta}}^{\dot{\delta}} (\sigma^n)_{\beta\dot{\delta}} D_{mn}(\mathbf{X}). \end{aligned} \quad (5.14)$$

One can readily verify that the conditions $\bar{\mathcal{D}}^{\dot{\alpha}} H_{\beta,\gamma\dot{\alpha}} = 0$ and $\mathcal{Q}^\beta H_{\beta,\gamma\dot{\alpha}} = 0$ give

$$B_{m,\delta,\beta} = 0, \quad D_{mn} = 0, \quad \varepsilon^{\beta\delta} A_{m,\delta,\beta} = 0, \quad (5.15a)$$

$$\eta^{mn} C_{mn} = 0, \quad C_{[mn]} = 0. \quad (5.15b)$$

Thus, it suffices for us to determine the general ansatz for $A_{m,(\delta\beta)}$ and $C_{(mn)}$ only. In order to be compatible with (5.12) and (5.15b), we should write

$$A_{m,(\delta\beta)} = (\tilde{\sigma}^{nk})_{\delta\beta} \left[\frac{a_1}{(\mathbf{X}^2)^2} (\eta_{mn} \mathbf{X}_k - \eta_{mk} \mathbf{X}_n) + \frac{a_2}{(\mathbf{X}^2)^2} \epsilon_{mnkp} \mathbf{X}^p \right], \quad (5.16a)$$

$$C_{(mn)} = c_1 \left[\frac{\eta_{mn}}{(\mathbf{X}^2)^{3/2}} - 4 \frac{\mathbf{X}_m \mathbf{X}_n}{(\mathbf{X}^2)^{5/2}} \right]. \quad (5.16b)$$

However, it turns out that the second order constraints $\mathcal{D}^2 H_{\beta,\gamma\dot{\alpha}} = 0$ and $\bar{\mathcal{Q}}^2 H_{\beta,\gamma\dot{\alpha}} = 0$ are enough to show that $a_1 = a_2 = c_1 = 0$. Thus,

$$\langle S_\alpha(z_1) \bar{S}_{\dot{\beta}}(z_2) S_\gamma(z_3) \rangle = 0. \quad (5.17)$$

6 $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ superspace reduction

In this section we will discuss whether the three-point functions studied in the previous sections are consistent with $\mathcal{N} = 2$ superconformal symmetry. For this we will study if they are contained in the three-point function of the $\mathcal{N} = 2$ supercurrent.

6.1 The three-point function of the $\mathcal{N} = 2$ supercurrent

Let us recall that in $\mathcal{N} = 2$ superconformal field theory, the supercurrent [43, 44] is described by a primary real scalar superfield $\mathcal{J}(z)$ of dimension 2, subject to the conservation condition

$$D^{ij} \mathcal{J} = 0 \quad \Longleftrightarrow \quad \bar{D}^{ij} \mathcal{J} = 0. \quad (6.1)$$

Here $D_A = (\partial_a, D_\alpha^i, \bar{D}_{\dot{\alpha}}^i)$ are the $\mathcal{N} = 2$ supersymmetric covariant derivatives, $i = \underline{1}, \underline{2}$, and $D^{ij} = D^{\alpha(i} D_\alpha^{j)}$, $\bar{D}^{ij} = \bar{D}_{\dot{\alpha}}^{(i} \bar{D}^{j)\dot{\alpha}}$. The superconformal transformation law of $\mathcal{J}$, which is uniquely determined from the reality condition $\bar{\mathcal{J}} = \mathcal{J}$ and the conservation equation, reads

$$\delta \mathcal{J}(z) = -\xi \mathcal{J}(z) - 2(\sigma(z) + \bar{\sigma}(z)) \mathcal{J}(z). \quad (6.2)$$

As was discussed in the introduction, the $\mathcal{N} = 2$ supercurrent $\mathcal{J}$ is composed of three independent $\mathcal{N} = 1$ multiplets [18]. For convenience, we will write them again

$$J \equiv \mathcal{J}| = \bar{J}, \quad J_\alpha \equiv D_\alpha^2 \mathcal{J}|, \quad J_{\alpha\dot{\alpha}} \equiv \frac{1}{2} [D_\alpha^2, \bar{D}_{\dot{\alpha}2}] \mathcal{J}| - \frac{1}{6} [D_\alpha^1, \bar{D}_{\dot{\alpha}1}] \mathcal{J}| = \bar{J}_{\alpha\dot{\alpha}}. \quad (6.3)$$

It follows from eqs. (6.1) and (6.3) that the $\mathcal{N} = 1$ components of $\mathcal{J}$ satisfy the following conservation equations

$$\bar{D}^2 J = 0, \quad D^2 J = 0 \quad (6.4a)$$

$$D^\alpha J_\alpha = 0, \quad \bar{D}^2 J_\alpha = 0, \quad (6.4b)$$

$$\bar{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} = 0, \quad D^\alpha J_{\alpha\dot{\alpha}} = 0. \quad (6.4c)$$

From eqs. (6.2) and (6.3) one can also deduce the transformation properties

$$\delta J = -\xi J - 2(\sigma + \bar{\sigma}) J \quad (6.5a)$$

$$\delta J_\alpha = -\xi J_\alpha + \hat{\omega}_\alpha{}^\beta J_\beta - (3\sigma + 2\bar{\sigma}) J_\alpha \quad (6.5b)$$

$$\delta J_{\alpha\dot{\alpha}} = -\xi J_{\alpha\dot{\alpha}} + (\hat{\omega}_\alpha{}^\beta \delta_{\dot{\alpha}}{}^{\dot{\beta}} + \bar{\hat{\omega}}_{\dot{\alpha}}{}^{\dot{\beta}} \delta_\alpha{}^\beta) J_{\beta\dot{\beta}} - 3(\sigma + \bar{\sigma}) J_{\alpha\dot{\alpha}}. \quad (6.5c)$$

The above equations (6.4) and (6.5) imply that $J_{\alpha\dot{\alpha}}$ is the $\mathcal{N} = 1$ supercurrent, J_α is a spinor current and J is a flavour-like current corresponding to one of the $U(1)$ generators of the R -symmetry.

If an $\mathcal{N} = 1$ superconformal theory also has $\mathcal{N} = 2$ superconformal symmetry, all three-point functions involving any combination of $J, J_\alpha, J_{\alpha\dot{\alpha}}$ must then be contained in the three-point function of the $\mathcal{N} = 2$ supercurrent. More precisely, they can be found from $\langle \mathcal{J}(z_1) \mathcal{J}(z_2) \mathcal{J}(z_3) \rangle$ by making use of the $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ superspace reduction formalism of [18]. Hence, our aim will be to consider the relevant superspace reductions of $\langle \mathcal{J}(z_1) \mathcal{J}(z_2) \mathcal{J}(z_3) \rangle$ to see if we can reproduce the results for $\langle \bar{S}_{\dot{\alpha}}(z_1) S_\beta(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle$ and $\langle \bar{S}_{\dot{\alpha}}(z_1) S_\beta(z_2) L(z_3) \rangle$ obtained in the previous sections.

The three-point function of $\mathcal{N} = 2$ supercurrent was found in [18]. It is given by

$$\begin{aligned} \langle \mathcal{J}(z_1) \mathcal{J}(z_2) \mathcal{J}(z_3) \rangle &= \frac{1}{x_{13}^2 x_{31}^2 x_{23}^2 x_{32}^2} H(\mathbf{Z}_3), \\ H(\lambda \bar{\lambda} \mathbf{X}, \lambda \boldsymbol{\Theta}, \bar{\lambda} \bar{\boldsymbol{\Theta}}) &= (\lambda \bar{\lambda})^{-2} H(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}), \end{aligned} \quad (6.6)$$

where the real function $H(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}})$ is

$$\begin{aligned} H(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3) &= A \left(\frac{1}{\mathbf{X}_3^2} + \frac{1}{\bar{\mathbf{X}}_3^2} \right) + B \frac{\boldsymbol{\Theta}_3^{\alpha\beta} \mathbf{X}_{3\alpha\dot{\alpha}} \mathbf{X}_{3\beta\dot{\beta}} \bar{\boldsymbol{\Theta}}_3^{\dot{\alpha}\dot{\beta}}}{(\mathbf{X}_3^2)^3} \\ &= \left(A + \frac{B}{8} \right) \left(\frac{1}{\mathbf{X}_3^2} + \frac{1}{\bar{\mathbf{X}}_3^2} \right) - \frac{B}{8} (\bar{\mathbf{X}}_3 \cdot \mathbf{X}_3) \left(\frac{1}{(\mathbf{X}_3^2)^2} + \frac{1}{(\bar{\mathbf{X}}_3^2)^2} \right). \end{aligned} \quad (6.7)$$

In the above, A and B are arbitrary real parameters and

$$\boldsymbol{\Theta}_3^{\alpha\beta} = \boldsymbol{\Theta}_3^{(\alpha\beta)} = \boldsymbol{\Theta}_3^{\alpha i} \boldsymbol{\Theta}_{3i}^\beta, \quad \bar{\boldsymbol{\Theta}}_3^{\dot{\alpha}\dot{\beta}} = \bar{\boldsymbol{\Theta}}_3^{(\dot{\alpha}\dot{\beta})} = \bar{\boldsymbol{\Theta}}_{3i}^{\dot{\alpha}} \bar{\boldsymbol{\Theta}}_3^{\dot{\beta}i}. \quad (6.8)$$

The second line of (6.7) is obtained with the aid of the following identities [18]:

$$\begin{aligned} \frac{1}{\bar{\mathbf{X}}_3^2} &= \frac{1}{\mathbf{X}_3^2} - 4i \frac{\boldsymbol{\Theta}_3^i \mathbf{X}_3 \bar{\boldsymbol{\Theta}}_{3i}}{(\mathbf{X}_3^2)^2} - 8 \frac{\boldsymbol{\Theta}_3^{\alpha\beta} \mathbf{X}_{3\alpha\dot{\alpha}} \mathbf{X}_{3\beta\dot{\beta}} \bar{\boldsymbol{\Theta}}_3^{\dot{\alpha}\dot{\beta}}}{(\mathbf{X}_3^2)^3}, \\ \bar{\mathbf{X}}_3^a &= \mathbf{X}_3^a + 2i \boldsymbol{\Theta}_3^i \sigma^a \bar{\boldsymbol{\Theta}}_{3i}. \end{aligned} \quad (6.9)$$

In the next subsections we will extract $\langle \bar{J}_{\dot{\alpha}}(z_1) J_{\beta}(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle$ and $\langle \bar{J}_{\dot{\alpha}}(z_1) J_{\beta}(z_2) J(z_3) \rangle$ by superspace reduction and compare the results with eq. (3.37) and (4.21) respectively.

6.2 Correlator $\langle \bar{J}_{\dot{\alpha}}(z_1) J_{\beta}(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle$

Extracting three-point functions involving the $\mathcal{N} = 1$ supercurrent $J_{\alpha\dot{\alpha}}$ is technically quite difficult. Following similar considerations in [18], we define the operator

$$\Delta_{\alpha\dot{\alpha}} = \frac{1}{2} [D_{\alpha}^2, \bar{D}_{\dot{\alpha}2}] - \frac{1}{6} [D_{\alpha}^1, \bar{D}_{\dot{\alpha}1}], \quad (6.10)$$

and hence $J_{\alpha\dot{\alpha}}$ is obtained from $\mathcal{J}$ by applying the operator $\Delta_{\alpha\dot{\alpha}}$ and then switching off the Grassmann variables $\theta_2, \bar{\theta}^2$. As a first step, let us consider $\langle \Delta_{\alpha\dot{\alpha}} \mathcal{J}(z_1) \mathcal{J}(z_2) \mathcal{J}(z_3) \rangle$. In [18] it was found that

$$\begin{aligned} \langle \Delta_{\alpha\dot{\alpha}} \mathcal{J}(z_1) \mathcal{J}(z_2) \mathcal{J}(z_3) \rangle &= \frac{(\tilde{x}_{31}^{-1})_{\alpha\dot{\sigma}} (\tilde{x}_{13}^{-1})_{\sigma\dot{\alpha}}}{x_{13}^2 x_{31}^2 x_{23}^2 x_{32}^2} \\ &\times \left(\frac{64}{3} \theta_{132}^{\sigma} \bar{\theta}_{13}^{\dot{\sigma}2} H(\mathbf{Z}_3) + \frac{16}{3} \theta_{132}^{\sigma} u_k^2(z_{31}) \bar{D}^{\dot{\sigma}k} H(\mathbf{Z}_3) - \frac{16}{3} \bar{\theta}_{13}^{\dot{\sigma}2} u_{\underline{2}}^k(z_{13}) \mathcal{D}^{\sigma k} H(\mathbf{Z}_3) \right. \\ &\left. + \frac{1}{2} \left\{ u_{\underline{2}}^k(z_{13}) u_l^2(z_{31}) - \frac{1}{3} u_{\underline{1}}^k(z_{13}) u_l^1(z_{31}) \right\} [\mathcal{D}_k^{\sigma}, \bar{D}^{\dot{\sigma}l}] H(\mathbf{Z}_3) \right). \end{aligned} \quad (6.11)$$

Next, we may express $H(\mathbf{Z}_3)$, $\mathcal{D}_k^{\sigma} H(\mathbf{Z}_3)$ and $[\mathcal{D}_k^{\sigma}, \bar{D}^{\dot{\sigma}l}] H(\mathbf{Z}_3)$ as functions of $\mathbf{Z}_1$, by virtue of (2.34). Upon relabelling $z_1 \leftrightarrow z_3$ (and noting the relation (2.46c)), we then obtain

$$\langle \mathcal{J}(z_1) \mathcal{J}(z_2) \Delta_{\gamma\dot{\gamma}} \mathcal{J}(z_3) \rangle, \quad (6.12)$$

which is a function of $\mathbf{Z}_3$. Thus, the three-point function (3.3) can be derived by evaluating the bar-projection

$$\langle \bar{J}_{\dot{\alpha}}(z_1) J_{\beta}(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle = \bar{D}_{(1)\dot{\alpha}2} D_{(2)\beta}^2 \langle \mathcal{J}(z_1) \mathcal{J}(z_2) \Delta_{\gamma\dot{\gamma}} \mathcal{J}(z_3) \rangle, \quad (6.13)$$

and making use of the identities (2.39).

After quite a long calculation, the above procedure yields

$$\langle \bar{J}_{\dot{\alpha}}(z_1) J_{\beta}(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle = \frac{(x_{3\bar{1}})_{\alpha\dot{\alpha}} (x_{2\bar{3}})_{\beta\dot{\beta}}}{(x_{1\bar{3}}^2)^2 (x_{3\bar{2}}^2)^2 x_{\bar{3}1}^2 x_{\bar{2}3}^2} \mathcal{H}^{\dot{\beta}\alpha}_{\gamma\dot{\gamma}}(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3), \quad (6.14)$$

where

$$\begin{aligned} \mathcal{H}^{\dot{\beta}\alpha}_{\gamma\dot{\gamma}}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}) &= \frac{16}{3} \left(A - \frac{3B}{8} \right) \frac{1}{\mathbf{X}^2} \delta_{\gamma}^{\alpha} \delta_{\dot{\gamma}}^{\dot{\beta}} - \frac{16}{3} \left(A + \frac{B}{8} \right) \frac{1}{(\mathbf{X}^2)^2} \mathbf{X}^{\dot{\beta}\alpha} \mathbf{X}_{\gamma\dot{\gamma}} \\ &\quad - \frac{16}{3} \left(A - \frac{3B}{8} \right) \frac{i}{(\mathbf{X}^2)^2} \mathbf{X}^{\dot{\beta}\alpha} \mathbf{P}_{\gamma\dot{\gamma}} - \frac{8}{3} B \frac{i}{(\mathbf{X}^2)^2} \mathbf{P}^{\dot{\beta}\alpha} \mathbf{X}_{\gamma\dot{\gamma}} \\ &\quad - \frac{32}{3} \left(A + \frac{B}{8} \right) \frac{i}{(\mathbf{X}^2)^2} (\mathbf{P} \cdot \mathbf{X}) \delta_{\gamma}^{\alpha} \delta_{\dot{\gamma}}^{\dot{\beta}} + \frac{16}{3} B \frac{i}{(\mathbf{X}^2)^3} (\mathbf{P} \cdot \mathbf{X}) \mathbf{X}^{\dot{\beta}\alpha} \mathbf{X}_{\gamma\dot{\gamma}}. \end{aligned} \quad (6.15)$$

Thus, any $\mathcal{N} = 1$ superconformal field theory possessing a conserved spinor current multiplet S_{α} also has $\mathcal{N} = 2$ superconformal symmetry, if and only if the correlator $\langle \bar{S}_{\dot{\alpha}}(z_1) S_{\beta}(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle$ is of the form (6.14), (6.16). Comparing the above with (3.37), we see that they agree provided

$$d_2 = \frac{32}{3} \left(A + \frac{B}{8} \right), \quad d_3 = -\frac{16}{3} B, \quad d_4 = -\frac{32}{3} \left(A - \frac{3B}{8} \right). \quad (6.16)$$

This implies

$$d_2 + d_3 + d_4 = 0. \quad (6.17)$$

Therefore, we obtain the following surprising result: existence of a conserved spinor current in $\mathcal{N} = 1$ superconformal field theory, in general, does not imply $\mathcal{N} = 2$ superconformal symmetry. Only if the coefficients d_2, d_3, d_4 in (3.37) satisfy the additional constraint (6.17) the three-point function (3.37) is contained in the three-point function of the $\mathcal{N} = 2$ supercurrent and can be obtained by the superspace reduction. Otherwise, eq. (3.37) contains an extra independent tensor structure incompatible with $\mathcal{N} = 2$ superconformal symmetry. More precisely, the tensor H_{mn} in (3.37) may be expressed in the following way

$$H_{mn}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}) = \mathcal{H}_{mn}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}) + \frac{1}{2} (d_2 + d_3 + d_4) \frac{1}{(\mathbf{X}^2)^2} \{ \mathbf{X}_m \mathbf{X}_n + i \mathbf{X}_m \mathbf{P}_n \}, \quad (6.18)$$

where $\mathcal{H}_{mn}$ is eq. (6.16) written in vector notation, that is

$$\begin{aligned} \mathcal{H}_{mn}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}) &= -\frac{1}{4} (d_2 + d_3) \frac{\eta_{mn}}{\mathbf{X}^2} - \frac{1}{2} d_2 \frac{\mathbf{X}_m \mathbf{X}_n}{(\mathbf{X}^2)^2} \\ &\quad - \frac{1}{2} (d_2 + d_3) \frac{i}{(\mathbf{X}^2)^2} \mathbf{X}_m \mathbf{P}_n + \frac{1}{2} d_3 \frac{i}{(\mathbf{X}^2)^2} \mathbf{X}_n \mathbf{P}_m \\ &\quad + \frac{1}{2} d_2 \frac{i}{(\mathbf{X}^2)^2} \eta_{mn} (\mathbf{P} \cdot \mathbf{X}) - d_3 \frac{i}{(\mathbf{X}^2)^3} (\mathbf{P} \cdot \mathbf{X}) \mathbf{X}_m \mathbf{X}_n. \end{aligned} \quad (6.19)$$

In section 7, we will present two consistency checks of our result (3.37).

6.3 Correlator $\langle \bar{J}_{\dot{\alpha}}(z_1) J_{\beta}(z_2) J(z_3) \rangle$

In this subsection, we are going to show that the three-point function $\langle \bar{S}_{\dot{\alpha}}(z_1) S_{\beta}(z_2) J(z_3) \rangle$, eq. (4.20), is consistent with $\mathcal{N} = 2$ superconformal symmetry and must come from the superspace reduction of $\langle \mathcal{J}(z_1) \mathcal{J}(z_2) \mathcal{J}(z_3) \rangle$.

It follows from eqs. (6.3) that in $\mathcal{N} = 2$ superconformal theory, the three-point function $\langle \bar{J}_{\dot{\alpha}}(z_1) J_{\beta}(z_2) J(z_3) \rangle$ can be extracted using the following superspace reduction:

$$\langle \bar{J}_{\dot{\alpha}}(z_1) J_{\beta}(z_2) J(z_3) \rangle = \bar{D}_{(1)\dot{\alpha}2} D^2_{(2)\beta} \langle \mathcal{J}(z_1) \mathcal{J}(z_2) \mathcal{J}(z_3) \rangle \Big| . \quad (6.20)$$

Keeping in mind that

$$D^i_{(2)\beta} \bar{\mathbf{X}}_{3\gamma\dot{\gamma}} = 0, \quad \bar{D}_{(1)\dot{\alpha}i} \bar{\mathbf{X}}_{3\gamma\dot{\gamma}} = 0, \quad (6.21)$$

along with the identities (2.39), one obtains

$$\langle \bar{J}_{\dot{\alpha}}(z_1) J_{\beta}(z_2) J(z_3) \rangle = \frac{(x_{3\bar{1}})_{\alpha\dot{\alpha}} (x_{2\bar{3}})_{\beta\dot{\beta}}}{(x_{1\bar{3}}^2)^2 x_{3\bar{1}}^2 x_{2\bar{3}}^2 (x_{3\bar{2}}^2)^2} \mathcal{H}^{\dot{\beta}\alpha}(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3), \quad (6.22)$$

with

$$\mathcal{H}^{\dot{\beta}\alpha}(\mathbf{X}, \boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}) = -4i A \frac{\mathbf{X}^{\dot{\beta}\alpha}}{(\mathbf{X}^2)^2} + \frac{B}{2(\mathbf{X}^2)^3} \left(\mathbf{X}^2 \mathbf{P}^{\dot{\beta}\alpha} - 4(\mathbf{P} \cdot \mathbf{X}) \mathbf{X}^{\dot{\beta}\alpha} \right). \quad (6.23)$$

Comparing eqs. (6.22), (6.23) with the $\mathcal{N} = 1$ results (4.20), (4.21) we see that they are in perfect agreement, provided the coefficients are related via

$$c_1 = -4A, \quad d_1 = B. \quad (6.24)$$

That is, the $\mathcal{N} = 1$ correlator $\langle \bar{S}_{\dot{\alpha}}(z_1) S_{\beta}(z_2) J(z_3) \rangle$ is always consistent with $\mathcal{N} = 2$ superconformal symmetry.

6.4 Correlator $\langle \bar{J}_{\dot{\alpha}}(z_1) J_{\beta}(z_2) L(z_3) \rangle$

The $\mathcal{N} = 2$ flavour current superfield [18] is subject to the reality condition $\overline{\mathcal{L}_{ij}} = \mathcal{L}^{ij}$, and the conservation (analyticity) equation

$$D_{\alpha}^{(i} \mathcal{L}^{jk)} = \bar{D}_{\dot{\alpha}}^{(i} \mathcal{L}^{jk)} = 0. \quad (6.25)$$

These constraints fix its transformation law to be

$$\delta \mathcal{L}_{ij}(z) = -\xi \mathcal{L}_{ij}(z) + 2i \hat{\Lambda}_{(i}{}^k(z) \mathcal{L}_{j)k}(z) - 2(\sigma(z) + \bar{\sigma}(z)) \mathcal{L}_{ij}(z). \quad (6.26)$$

Its leading $\mathcal{N} = 1$ component

$$L \equiv i \mathcal{L}^{12}| = \bar{L} \quad (6.27)$$

obeys the conservation equation

$$D^2 L = \bar{D}^2 L = 0. \quad (6.28)$$

Hence, its superconformal transformation rule is similar to J (see eq. (6.5a))

$$\delta L = -\xi L - 2(\sigma + \bar{\sigma}) L. \quad (6.29)$$

It was demonstrated in [18] that the three-point function involving two $\mathcal{N} = 2$ supercurrent insertions and a flavour $\mathcal{N} = 2$ current superfield vanishes,

$$\langle \mathcal{J}(z_1) \mathcal{J}(z_2) \mathcal{L}_{ij}(z_3) \rangle = 0. \quad (6.30)$$

As a result,

$$\langle \bar{J}_{\dot{\alpha}}(z_1) J_{\beta}(z_2) L(z_3) \rangle = -i \bar{D}_{(1)\dot{\alpha}2} D^2_{(2)\beta} \langle \mathcal{J}(z_1) \mathcal{J}(z_2) \mathcal{L}_{21}(z_3) \rangle = 0. \quad (6.31)$$

7 Reduction to component currents

In $\mathcal{N} = 1$ superconformal field theory, the spinor current S_{α} and the supercurrent $J_{\alpha\dot{\alpha}}$ multiplets contain, among others, the following conserved currents:

$$\mathbb{V}_{\alpha\dot{\alpha}} := \bar{D}_{\dot{\alpha}} S_{\alpha}|, \quad (7.1)$$

$$R_{\alpha\dot{\alpha}} := J_{\alpha\dot{\alpha}}|, \quad T_{\alpha\beta\dot{\alpha}\dot{\beta}} := [D_{(\beta}, \bar{D}_{\dot{\beta})} J_{\alpha\dot{\alpha}}]|. \quad (7.2)$$

Here the bar-projection corresponds to setting the Grassmann coordinates to zero. The vector current $\mathbb{V}_{\alpha\dot{\alpha}}$ is a complex field,

$$\mathbb{V}_{\alpha\dot{\alpha}} = V_{\alpha\dot{\alpha}} + i\tilde{V}_{\alpha\dot{\alpha}}. \quad (7.3)$$

The components $R_{\alpha\dot{\alpha}}$ and $T_{\alpha\beta\dot{\alpha}\dot{\beta}}$ are the U(1) R -symmetry current and the energy-momentum tensor, respectively. This means that the correlator

$$\langle \bar{S}_{\dot{\alpha}}(z_1) S_{\beta}(z_2) J_{\gamma\dot{\gamma}}(z_3) \rangle \quad (7.4)$$

in particular contains the following component correlators

$$\langle V_{\alpha\dot{\alpha}}(x_1) V_{\beta\dot{\beta}}(x_2) R_{\gamma\dot{\gamma}}(x_3) \rangle, \quad \langle V_{\beta\dot{\beta}}(x_1) T_{\gamma\delta\dot{\gamma}\dot{\delta}}(x_2) V_{\alpha\dot{\alpha}}(x_3) \rangle. \quad (7.5)$$

The three-point functions of this kind were studied in [11, 12].⁵ It was shown that the three-point function of two U(1) vector currents and an axial current has only one independent structure, and the three-point function of two vector currents and the energy-momentum tensor is fixed up to two independent structures. On the other hand, the correlator (7.4) was shown in subsection 3.1 to depend on three linearly independent structures. Hence, it is not trivial to see if our results for the correlator (7.4) are consistent with the general structure of the correlators (7.5) derived in [11, 12]. In this section we will perform a component reduction of the superfield correlator (7.4) to extract the component correlators (7.5), thus providing consistency checks of our results.

⁵Three-point functions involving R -symmetry currents and various primary operators were recently studied in [47].

7.1 Correlator $\langle V_{\alpha\dot{\alpha}}(x_1)V_{\beta\dot{\beta}}(x_2)R_{\gamma\dot{\gamma}}(x_3) \rangle$

Our first task is to extract the correlator $\langle V_{\alpha\dot{\alpha}}(z_1)V_{\beta\dot{\beta}}(z_2)R_{\gamma\dot{\gamma}}(z_3) \rangle$ from (3.37), where $V_{\alpha\dot{\alpha}}$ denotes the real part of $\mathbb{V}_{\alpha\dot{\alpha}}$,

$$V_{\alpha\dot{\alpha}} = \text{Re}(\mathbb{V}_{\alpha\dot{\alpha}}) = \frac{1}{2}(\bar{D}_{\dot{\alpha}}S_{\alpha} - D_{\alpha}\bar{S}_{\dot{\alpha}})|. \quad (7.6)$$

The correlation function can then be computed using

$$\langle V_{\alpha\dot{\alpha}}(x_1)V_{\beta\dot{\beta}}(x_2)R_{\gamma\dot{\gamma}}(x_3) \rangle = \frac{1}{4} \left(D_{(1)\alpha}\bar{D}_{(2)\dot{\beta}} \langle \bar{S}_{\dot{\alpha}}(z_1)S_{\beta}(z_2)J_{\gamma\dot{\gamma}}(z_3) \rangle + \text{c.c.} \right). \quad (7.7)$$

In the above, we have used the fact that the correlators $\langle \bar{S}\bar{S}J_{\gamma\dot{\gamma}} \rangle = \langle SSJ_{\gamma\dot{\gamma}} \rangle$ are vanishing, see subsection 3.2. With the help of identities (2.41), it is easy to see that

$$D_{(1)\alpha}\bar{D}_{(2)\dot{\beta}} \langle \bar{S}_{\dot{\alpha}}(z_1)S_{\beta}(z_2)J_{\gamma\dot{\gamma}}(z_3) \rangle = \frac{(x_{1\bar{3}})_{\alpha\dot{\mu}}(x_{3\bar{1}})_{\delta\dot{\alpha}}(x_{3\bar{2}})_{\mu\dot{\beta}}(x_{2\bar{3}})_{\beta\dot{\delta}}}{(x_{1\bar{3}}^2)^3(x_{3\bar{2}}^2)^3x_{3\bar{1}}^2x_{2\bar{3}}^2} \bar{D}^{\dot{\mu}}\mathcal{Q}^{\mu}H^{\delta\delta}_{\gamma\dot{\gamma}}| \quad (7.8)$$

$$= -\frac{(x_{13})_{\alpha\dot{\mu}}(x_{13})_{\delta\dot{\alpha}}(x_{23})_{\mu\dot{\beta}}(x_{23})_{\beta\dot{\delta}}}{(x_{13}^2x_{23}^2)^4} \frac{\partial}{\partial\bar{\Theta}_{\dot{\mu}}}\frac{\partial}{\partial\Theta_{\mu}}H^{\delta\delta}_{\gamma\dot{\gamma}}|, \quad (7.9)$$

and hence, after taking complex conjugation into account,

$$\begin{aligned} & \langle V_{\alpha\dot{\alpha}}(x_1)V_{\beta\dot{\beta}}(x_2)R_{\gamma\dot{\gamma}}(x_3) \rangle \\ &= -\frac{1}{4} \frac{(x_{13})_{\alpha\dot{\mu}}(x_{13})_{\delta\dot{\alpha}}(x_{23})_{\mu\dot{\beta}}(x_{23})_{\beta\dot{\delta}}}{(x_{13}^2x_{23}^2)^4} \left\{ \frac{\partial}{\partial\bar{\Theta}_{\dot{\mu}}}\frac{\partial}{\partial\Theta_{\mu}}H^{\delta\delta}_{\gamma\dot{\gamma}}| - \frac{\partial}{\partial\bar{\Theta}_{\dot{\delta}}}\frac{\partial}{\partial\Theta_{\delta}}H^{\dot{\mu}\mu}_{\gamma\dot{\gamma}}| \right\}. \end{aligned} \quad (7.10)$$

Here we have defined $x_{1\bar{3}}| = x_{13} = x_1 - x_3 = -x_{31}$. We find that

$$\begin{aligned} \frac{\partial}{\partial\bar{\Theta}_{\dot{\mu}}}\frac{\partial}{\partial\Theta_{\mu}}H^{\delta\delta}_{\gamma\dot{\gamma}}| &= -2i \left\{ d_4 \delta^{\mu}_{\gamma} \delta^{\dot{\mu}}_{\dot{\gamma}} \frac{X^{\delta\delta}}{X^4} + d_2 \delta^{\delta}_{\gamma} \delta^{\dot{\delta}}_{\dot{\gamma}} \frac{X^{\dot{\mu}\mu}}{X^4} \right. \\ &\quad \left. + d_3 \left(\varepsilon^{\mu\delta} \varepsilon^{\dot{\mu}\dot{\delta}} \frac{X_{\gamma\dot{\gamma}}}{X^4} + \frac{X^{\delta\delta} X^{\dot{\mu}\mu} X_{\gamma\dot{\gamma}}}{X^6} \right) \right\}. \end{aligned} \quad (7.11)$$

It follows from (7.10) that there is just one linearly independent form for the correlator,

$$\langle V_{\alpha\dot{\alpha}}(x_1)V_{\beta\dot{\beta}}(x_2)R_{\gamma\dot{\gamma}}(x_3) \rangle = \frac{i}{2}(d_4 - d_2) \frac{(x_{13})_{\alpha\dot{\mu}}(x_{13})_{\delta\dot{\alpha}}(x_{23})_{\mu\dot{\beta}}(x_{23})_{\beta\dot{\delta}}}{(x_{13}^2x_{23}^2)^4} t^{\dot{\mu}\mu, \delta\delta}_{\gamma\dot{\gamma}}(X_3), \quad (7.12a)$$

where the function $t^{\dot{\mu}\mu, \delta\delta}_{\gamma\dot{\gamma}}(X)$ takes the form

$$t^{\dot{\mu}\mu, \delta\delta}_{\gamma\dot{\gamma}}(X) = \delta^{\mu}_{\gamma} \delta^{\dot{\mu}}_{\dot{\gamma}} \frac{X^{\delta\delta}}{X^4} - \delta^{\delta}_{\gamma} \delta^{\dot{\delta}}_{\dot{\gamma}} \frac{X^{\dot{\mu}\mu}}{X^4}, \quad (7.12b)$$

with $X_{\alpha\dot{\alpha}} = \mathbf{X}_{\alpha\dot{\alpha}}| = \bar{\mathbf{X}}_{\alpha\dot{\alpha}}|$. Hence, the above correlator indeed contains a single structure in agreement with ref. [12].

7.2 Correlator $\langle V_{\beta\dot{\beta}}(x_1) T_{\gamma\delta\dot{\gamma}\dot{\delta}}(x_2) V_{\alpha\dot{\alpha}}(x_3) \rangle$

To derive the three-point function involving two vector currents and the energy-momentum tensor from (3.37) it is convenient to start with the following equivalent expression

$$\langle S_{\beta}(z_1) J_{\gamma\dot{\gamma}}(z_2) \bar{S}_{\dot{\alpha}}(z_3) \rangle = \frac{(x_{1\bar{3}})_{\beta\dot{\mu}}(x_{2\bar{3}})_{\gamma\dot{\lambda}}(x_{3\bar{2}})_{\lambda\dot{\gamma}}}{x_{1\bar{3}}^2(x_{3\bar{1}}^2)^2(x_{2\bar{3}}^2)^2(x_{3\bar{2}}^2)^2} \tilde{H}_{\dot{\alpha}}^{\dot{\mu}, \dot{\lambda}\lambda}(\mathbf{X}_3, \boldsymbol{\Theta}_3, \bar{\boldsymbol{\Theta}}_3), \quad (7.13)$$

where the explicit form of $\tilde{H}_{\dot{\alpha}}^{\dot{\mu}, \dot{\lambda}\lambda}$ is given by (3.31). In order to obtain the correlation function $\langle V_{\beta\dot{\beta}}(x_1) T_{\gamma\delta\dot{\gamma}\dot{\delta}}(x_2) V_{\alpha\dot{\alpha}}(x_3) \rangle$, we must act with the following derivatives on (7.13):

$$\begin{aligned} & \langle V_{\beta\dot{\beta}}(x_1) T_{\gamma\delta\dot{\gamma}\dot{\delta}}(x_2) V_{\alpha\dot{\alpha}}(x_3) \rangle \\ &= -\frac{1}{4} D_{(3)\alpha} \bar{D}_{(1)\dot{\beta}} [D_{(2)(\delta}, \bar{D}_{(2)(\dot{\delta}}] \langle S_{\beta}(z_1) J_{\gamma\dot{\gamma}}(z_2) \bar{S}_{\dot{\alpha}}(z_3) \rangle + \text{c.c.} \end{aligned} \quad (7.14)$$

Let us first compute

$$\begin{aligned} [D_{(2)(\delta}, \bar{D}_{(2)(\dot{\delta}}] \langle S_{\beta}(z_1) J_{\gamma\dot{\gamma}}(z_2) \bar{S}_{\dot{\alpha}}(z_3) \rangle &= \frac{(x_{1\bar{3}})_{\beta\dot{\mu}}(x_{2\bar{3}})_{(\gamma\dot{\lambda}}(x_{2\bar{3}})_{\delta)\dot{\rho}}(x_{3\bar{2}})_{\lambda(\dot{\gamma}}(x_{3\bar{2}})_{\rho)\dot{\delta}}}{x_{1\bar{3}}^2(x_{3\bar{1}}^2)^2(x_{2\bar{3}}^2)^3(x_{3\bar{2}}^2)^3} \\ &\times \left(160 \theta_{23}^{\rho} \bar{\theta}_{23}^{\dot{\rho}} \tilde{H}_{\dot{\alpha}}^{\dot{\mu}, \dot{\lambda}\lambda} + 20 \frac{x_{23}^2}{x_{32}^2} \bar{\theta}_{23}^{\dot{\rho}} \mathcal{Q}^{\rho} \tilde{H}_{\dot{\alpha}}^{\dot{\mu}, \dot{\lambda}\lambda} \right. \\ &\quad \left. - 20 \frac{x_{32}^2}{x_{23}^2} \theta_{23}^{\rho} \bar{\mathcal{Q}}^{\dot{\rho}} \tilde{H}_{\dot{\alpha}}^{\dot{\mu}, \dot{\lambda}\lambda} + [\mathcal{Q}^{\rho}, \bar{\mathcal{Q}}^{\dot{\rho}}] \tilde{H}_{\dot{\alpha}}^{\dot{\mu}, \dot{\lambda}\lambda} \right). \end{aligned} \quad (7.15)$$

From (7.15), it is now clear that only the last two terms survive after acting with $D_{(3)\alpha} \bar{D}_{(1)\dot{\beta}}$ and then setting all the Grassmann variables to zero since neither $D_{(3)\alpha}$ nor $\bar{D}_{(1)\dot{\beta}}$ can act on $\bar{\theta}_{23}^{\dot{\rho}}$. Then we get

$$\begin{aligned} \langle V_{\beta\dot{\beta}}(x_1) T_{\gamma\delta\dot{\gamma}\dot{\delta}}(x_2) V_{\alpha\dot{\alpha}}(x_3) \rangle &= -\frac{1}{4} \frac{(x_{13})_{\beta\dot{\mu}}(x_{13})_{\mu\dot{\beta}}(x_{23})_{(\gamma\dot{\lambda}}(x_{23})_{\delta)\dot{\rho}}(x_{23})_{\lambda(\dot{\gamma}}(x_{23})_{\rho)\dot{\delta}}}{(x_{13}^2)^4(x_{23}^2)^6} \\ &\times \left\{ 20i (D_{(3)\alpha} \theta_{23}^{\rho}) \mathcal{D}^{\mu} \bar{\mathcal{Q}}^{\dot{\rho}} \tilde{H}_{\dot{\alpha}}^{\dot{\mu}, \dot{\lambda}\lambda} + 4i (D_{(3)\alpha} \theta_{13}^{\mu}) [\mathcal{Q}^{\rho}, \bar{\mathcal{Q}}^{\dot{\rho}}] \tilde{H}_{\dot{\alpha}}^{\dot{\mu}, \dot{\lambda}\lambda} \right. \\ &\quad \left. + i D_{(3)\alpha} \mathcal{D}^{\mu} [\mathcal{Q}^{\rho}, \bar{\mathcal{Q}}^{\dot{\rho}}] \tilde{H}_{\dot{\alpha}}^{\dot{\mu}, \dot{\lambda}\lambda} \right\} + \text{c.c.} \end{aligned} \quad (7.16)$$

In the first two terms we will use $D_{(3)\alpha} \theta_{23}^{\rho} = -\delta^{\rho}_{\alpha}$. In the last term only the structures linear in $\bar{\boldsymbol{\Theta}}_3$ will contribute. We then use the identity

$$D_{(3)\alpha} \bar{\boldsymbol{\Theta}}_{3\dot{\alpha}} = -X_{\alpha\dot{\alpha}}. \quad (7.17)$$

After a lengthy calculation, we obtain

$$\begin{aligned} \langle V_{\beta\dot{\beta}}(x_1) T_{\gamma\delta\dot{\gamma}\dot{\delta}}(x_2) V_{\alpha\dot{\alpha}}(x_3) \rangle &= \frac{(x_{13})_{\beta\dot{\mu}}(x_{13})_{\mu\dot{\beta}}(x_{23})_{(\gamma\dot{\lambda}}(x_{23})_{\delta)\dot{\rho}}(x_{23})_{\lambda(\dot{\gamma}}(x_{23})_{\rho)\dot{\delta}}}{(x_{13}^2)^4(x_{23}^2)^6} \\ &\times h_{\alpha\dot{\alpha}}^{\dot{\rho}\rho, \dot{\mu}\mu, \dot{\lambda}\lambda}(X_3), \end{aligned} \quad (7.18a)$$

where the tensor $h_{\alpha\dot{\alpha}}^{\dot{\rho}\rho, \dot{\mu}\mu, \dot{\lambda}\lambda}(X)$ is given by

$$\begin{aligned} h_{\alpha\dot{\alpha}}^{\dot{\rho}\rho, \dot{\mu}\mu, \dot{\lambda}\lambda}(X) = & 4d_3 \left\{ \frac{2}{X^6} \delta^\lambda_\alpha \delta^{\dot{\lambda}}_{\dot{\alpha}} X^{\dot{\mu}\rho} X^{\dot{\rho}\mu} + \frac{3}{X^6} \delta^\lambda_\alpha \delta^{\dot{\lambda}}_{\dot{\alpha}} X^{\dot{\mu}\mu} X^{\dot{\rho}\rho} \right. \\ & \left. + \frac{2}{X^6} \delta^\mu_\alpha \delta^{\dot{\mu}}_{\dot{\alpha}} X^{\dot{\lambda}\lambda} X^{\dot{\rho}\rho} - \frac{3}{X^8} X^{\dot{\mu}\rho} X^{\dot{\rho}\mu} X^{\dot{\lambda}\lambda} X_{\alpha\dot{\alpha}} \right\} \\ & - 4(d_2 + 3d_4 + 3d_3) \left\{ \frac{1}{X^6} \delta^\lambda_\alpha \delta^{\dot{\mu}}_{\dot{\alpha}} X^{\dot{\lambda}\rho} X^{\dot{\rho}\mu} + \frac{1}{X^6} \delta^\mu_\alpha \delta^{\dot{\lambda}}_{\dot{\alpha}} X^{\dot{\rho}\lambda} X^{\dot{\mu}\rho} \right\}. \end{aligned} \quad (7.18b)$$

Note that due to the symmetries of the prefactor only the symmetric part in (λ, ρ) and $(\dot{\lambda}, \dot{\rho})$ of the function $h_{\alpha\dot{\alpha}}^{\dot{\rho}\rho, \dot{\mu}\mu, \dot{\lambda}\lambda}(X)$ contributes. Also note that the correlator (7.18) is fixed up to two independent coefficients.

Let us now rewrite the tensor $h_{\alpha\dot{\alpha}}^{\dot{\rho}\rho, \dot{\mu}\mu, \dot{\lambda}\lambda}(X)$ in vector notation, where we define

$$h^{abcd} = \frac{1}{16} (\sigma^a)_{(\rho(\dot{\rho}} (\sigma^b)_{\lambda)\dot{\lambda}}) (\sigma^c)_{\mu\dot{\mu}} (\tilde{\sigma}^d)^{\dot{\alpha}\alpha} h_{\alpha\dot{\alpha}}^{\dot{\rho}\rho, \dot{\mu}\mu, \dot{\lambda}\lambda}(X). \quad (7.19)$$

The tensor h^{abcd} thus introduced is symmetric and traceless in (a, b) . We obtain

$$\begin{aligned} h^{abcd}(X) = & \frac{d_3}{X^4} (\eta^{ac} \eta^{bd} + \eta^{bc} \eta^{ad}) + \frac{1}{X^6} (d_2 + 3d_4 - 2d_3) (\eta^{ad} X^b X^c + \eta^{bd} X^a X^c) \\ & - \frac{1}{X^6} (d_2 + 3d_4) (\eta^{ac} X^b X^d + \eta^{bc} X^a X^d) + \frac{2}{X^6} (d_2 + 3d_4 + d_3) \eta^{cd} X^a X^b \\ & - 12d_3 \frac{X^a X^b X^c X^d}{X^8} - \frac{1}{2X^4} (d_2 + 3d_4 + 2d_3) \eta^{ab} \eta^{cd} + \frac{4d_3}{X^6} \eta^{ab} X^c X^d. \end{aligned} \quad (7.20)$$

Our final result in the form (7.20) agrees perfectly with the general structure of the three-point function involving two vector currents and the energy-momentum tensor constructed by Osborn and Petkou [11]. More precisely, we find the following relations:

$$d_3 = e, \quad (d_2 + 3d_4) = -c, \quad (7.21)$$

where c and e are the two linearly independent, real parameters appearing in the function $\tilde{t}^{abcd}(X)$ in eq. (3.13) in [11].

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