Critical behavior of the 2d scalar theory: resumming the N⁸LO perturbative mass gap

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ABSTRACT: We apply the optimized perturbation theory (OPT) to resum the perturbative series describing the mass gap of the bidimensional ϕ^4 theory in the $\mathbb{Z}_2$ symmetric phase. Already at NLO (one loop) the method is capable of generating a quite reasonable non-perturbative result for the critical coupling. At order- g^7 we obtain $g_c = 2.779(25)$ which compares very well with the state of the art N⁸LO result, $g_c = 2.807(34)$. As a novelty we investigate the supercritical region showing that it contains some useful complimentary information that can be used in extrapolations to arbitrarily high orders.

KEYWORDS: Discrete Symmetries, Nonperturbative Effects

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1 Introduction

The bidimensional scalar ϕ^4 model describes a simple non integrable super-renormalizable theory which displays rich phase transition patterns. When the original mass parameter, m^2 , is positive the model has a mass gap and remains invariant under the $\mathbb{Z}_2$ transformation $\phi \rightarrow -\phi$ as far as one remains within the weak coupling regime. As the coupling (g) increases the mass gap decreases until the symmetry gets ultimately broken through a second order phase transition when a critical value, g_c , is attained [1, 2]. The case $m^2 < 0$ displays an even richer phase transition structure in which the $\mathbb{Z}_2$ symmetry that is broken for $0 < g < \tilde{g}_c$ gets restored at $g = \tilde{g}_c$ through a second order transition [3]. As the coupling further increases the model returns to the broken phase at $g = \tilde{g}'_c$. An interesting duality between the $\mathbb{Z}_2$ broken and unbroken phases, which allows to relate the three different critical couplings, was discovered by Chang [1]. Since its β function vanishes at all orders in perturbation theory the model represents a conformal field theory at the critical coupling where it becomes gapless. It then lies within the same universality class as the bidimensional Ising model. These physically appealing characteristics combined with its simplicity suggest that the strongly coupled bidimensional ϕ^4 model provides an excellent framework to test how accurately different non-perturbative techniques describe critical parameters associated with the phase transitions. Indeed, a survey of the literature reveals that methods such as lattice simulations [4–7], Hamiltonian truncations (HT) [8–17] as well as other resummation schemes [18] have been recently used to determine the numerical value of g_c for the symmetric ($m^2 > 0$) case. In some of the most recent investigations [19, 20] the self energy contributions to the physical mass have been perturbatively evaluated up to the N⁸LO before being Borel resummed to yield $g_c = 2.807(34)$. The availability of such a perturbative series provides us with an excellent opportunity to test alternative resummation techniques as those prescribed by variational methods such as the optimized perturbation theory (OPT) [21, 22] to be considered here. Let us point out that similar

approximations appear under acronyms such as LDE (linear δ expansion [23]), VPT (variational perturbation theory [24]), and SPT (screened perturbation theory [25]).

The aim of most of these variational approximations is to resum an originally perturbative series by combining the easiness of perturbative evaluations with some optimization criterion in order to produce non-perturbative results. In this way the formal evaluations at each order generally involve only a handful of contributions which is certainly advantageous. A welcome feature is that the renormalization program can then be implemented by following the perturbative approach discussed in most standard textbooks *before* non-perturbative results be produced through optimization. One way to implement this kind of approximation is to deform the original theory by adding and subtracting a Gaussian term written in terms of an arbitrary variational (mass) parameter, η , which represents a Lagrange multiplier. For example, in the case of the ϕ^4 scalar theory to be considered here one deforms the original theory by shifting the harmonic term $m^2\phi^2 \rightarrow m^2\phi^2 + (1-\delta)\eta^2\phi^2$ while multiplying the original couplings by a dummy bookkeeping parameter, δ . Then, as one can easily check, the deformed Lagrangian density interpolates between a free theory ($\delta = 0$) and the original interacting case ($\delta = 1$). Next, a physical quantity, Φ , is evaluated *perturbatively* in powers of δ up to a given order k producing $\Phi^{(k)}(\delta, \eta)$. One then sets $\delta = 1$ (the original value) and fixes the optimum $\bar{\eta}$ by requiring that it satisfies the so called principle of minimal sensitivity (PMS), $(\partial\Phi)/(\partial\eta)|_{\bar{\eta}} = 0$ [26, 27]. In most situations this variational criterion produces non-perturbative results since the optimal $\bar{\eta}$ turns out to depend on the couplings in a non trivial way (often, via self consistent relations). In general, as a result of implementing such a procedure, one can manage to turn a highly divergent series into a convergent expansion. The method is also known for exactly reproducing large- N results already at the first non trivial order in many relevant situations [28]. Finite N non-perturbative corrections are easily taken into account by considering few higher loop contributions. Another advantage concerns the case of an originally massless theory for in this case the variational mass also acts as an infra red regulator. A possible concern regards the OPT convergence properties since arbitrary manipulations of divergent series can lead to wrong results. This important question has been addressed in the context of the anharmonic oscillator [29–31] where a rigorous convergence proof was established before being generalized to the case of renormalizable quantum field theories [32].

The OPT has been successfully employed in many different physical situations involving symmetry breaking and phase transitions. In particular, applications related to condensed matter physics have shown that by including finite N corrections in a non-perturbative fashion the method was able to produce very accurate results regarding the critical dopant concentration in polyacetylene [33], the critical temperature for homogeneous Bose gases [34], and the phase diagram of magnetized planar fermionic systems [35]. Applications to high energy physics include the evaluation of quark susceptibilities [36] and the phase diagram of effective QCD models [37] among others. Regarding gauge theories at finite temperatures and non vanishing baryonic densities a variant of the method, with renormalization group properties, has been recently applied to determine the QCD equation of state for dense hadronic matter at $T = 0$ [38] and $T \neq 0$ [39, 40]. This variant, which has been originally dubbed renormalization group optimized perturbation theory

(RGOPT) [41, 42], has produced results which are in excellent agreement with the state of the art lattice QCD predictions [39, 40].

In the present work the OPT capabilities at strong couplings will be tested primarily to evaluate g_c for the case $m^2 > 0$. With this aim we shall resum the N⁸LO mass gap perturbative series which became recently available [20]. Our other goal is to investigate if some extra useful information can be acquired by exploring the supercritical region. The work is organized as follows. In the next section we review the N⁸LO mass gap perturbative series presented in ref. [20]. In section 3 the OPT method is illustrated with an application to the mass gap at the two first non trivial orders. Numerical results for g_c at N⁸LO are obtained in section 4. Then, in section 5, we explore the region where $g > g_c$. Finally, our conclusions are presented in section 6.

2 Reviewing the perturbative mass gap series

The well known ϕ^4 model is described by the following $\mathbb{Z}_2$ invariant Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{m^2}{2} \phi^2 + \lambda \phi^4. \quad (2.1)$$

In 2d, where the theory is super-renormalizable, λ has canonical dimensions $[\lambda] = 2$. In this particular case the coupling is finite so that $\beta_\lambda = 0$ at all perturbative orders. Regarding the two point function the only primitive divergence stems from tadpole (direct) contributions which do not depend on the external momenta. Hence, no wave function renormalization is needed.

Let us start by reviewing the perturbative evaluation of the physical mass squared. At the lowest perturbative order, an explicit evaluation using dimensional regularization in the $\overline{\text{MS}}$ scheme gives

$$M_{PT}^2 = m^2 + \lambda \frac{3}{\pi} \left(\frac{1}{\epsilon} + L_m \right) + m_{ct}^2, \quad (2.2)$$

where we have defined

$$L_m \equiv \ln \frac{\mu^2}{m^2}, \quad (2.3)$$

with μ representing an arbitrary energy scale. Within the $\overline{\text{MS}}$ renormalization scheme only the pole is eliminated by the counterterm, m_{ct}^2 , so that the renormalized physical mass squared reads

$$M_{PT}^2 = m^2 + \lambda \frac{3}{\pi} L_m. \quad (2.4)$$

Requiring this quantity to satisfy the Callan-Symanzik equation

$$(\mu \partial_\mu + \beta_{m^2} \partial_{m^2}) M^2 = 0, \quad (2.5)$$

one can fix the mass anomalous dimension to [3]

$$\beta_{m^2} = -\lambda \frac{6}{\pi}. \quad (2.6)$$

Therefore,

$$m^2(\mu) = m^2(\mu_0) - \lambda \frac{3}{\pi} \ln \frac{\mu^2}{\mu_0^2}, \quad (2.7)$$

where μ_0 represents a reference scale. As already emphasized the only primitive divergence associated with the two point function is the one which appears in eq. (2.2). This means that by consistently considering the counterterm $m_{ct}^2 = -\lambda 3/(\pi\epsilon)$ the physical mass squared will remain finite at any perturbative order. Note also that β_{m^2} , as given by eq. (2.6), remains valid as higher orders are considered. In the case where tadpoles are present the finite perturbative physical mass squared at $\mathcal{O}(\lambda^8)$ reads [20]

$$\begin{aligned} M_{PT}^2 = & m^2 + \frac{3\lambda}{\pi} L_m - \frac{9\lambda^2}{\pi^2 m^2} L_m - \frac{3\lambda^2}{2m^2} \\ & + \frac{\lambda^3}{(m^2)^2} \left\{ \frac{9}{\pi} + \frac{63}{2\pi^3} \zeta(3) + \frac{27}{\pi^3} L_m + \frac{9}{2\pi} L_m + \frac{27}{2\pi^3} L_m^2 \right\} \\ & - \frac{\lambda^4}{(m^2)^3} \left\{ 14.655869(22) + \left(6 + 5\pi^2 + 14\zeta(3) \right) L_m + \frac{27}{2\pi^4} \left(9 + \pi^2 \right) L_m^2 + \frac{27}{\pi^4} L_m^3 \right\} \\ & + \frac{\lambda^5}{(m^2)^4} \left\{ 65.97308(43) + 51.538171(63) L_m + \frac{81}{4\pi^5} \left(36 + 17\pi^2 + 42\zeta(3) \right) L_m^2 \right. \\ & \left. + \frac{81}{2\pi^5} \left(11 + \pi^2 \right) L_m^3 + \frac{243}{4\pi^5} L_m^4 \right\} \\ & - \frac{\lambda^6}{(m^2)^5} \left\{ 347.8881(28) + 301.2139(16) L_m + 114.49791(12) L_m^2 \right. \\ & \left. + \frac{81}{2\pi^6} \left(105 + 37\pi^2 + 84\zeta(3) \right) L_m^3 + \frac{243}{4\pi^6} \left(25 + 2\pi^2 \right) L_m^4 + \frac{729}{5\pi^6} L_m^5 \right\} \\ & + \frac{\lambda^7}{(m^2)^6} \left\{ 2077.703(36) + 1948.682(14) L_m + 828.4327(39) L_m^2 + 205.20516(19) L_m^3 \right. \\ & \left. + \frac{243}{8\pi^7} \left(675 + 197\pi^2 + 420\zeta(3) \right) L_m^4 + \frac{729}{5\pi^5} \left(137 + 10\pi^2 \right) L_m^5 + \frac{729}{5\pi^7} L_m^6 \right\} \\ & - \frac{\lambda^8}{(m^2)^7} \left\{ 13771.04(54) + 13765.22(21) L_m + 6373.657(40) L_m^2 + 1778.1465(75) L_m^3 \right. \\ & \left. + 323.93839(27) L_m^4 + \frac{2187}{20\pi^8} \left(812 + 207\pi^2 + 420\zeta(3) \right) L_m^5 \right. \\ & \left. + \frac{2187}{20\pi^8} \left(147 + 10\pi^2 \right) L_m^6 + \frac{6561}{7\pi^8} L_m^7 \right\}. \quad (2.8) \end{aligned}$$

Note that by setting $\mu = m$ all L_m dependent (tadpole) terms appearing in eq. (2.8) vanish and one retrieves the series considered in ref. [19]. However, as we shall explicitly see, these terms should not be discarded prior to implementing the OPT mass shift. The reason is that after expanding in powers of δ the L_m terms become $\ln[\mu^2/(m^2 + \eta^2)]$ eventually giving non vanishing contributions even if one later chooses $\mu = m$.

3 OPT resummation

Now, to implement the OPT procedure one considers eq. (2.8) with the following replacements [21]

$$\begin{aligned} m^2(\mu) &= m^2(\mu_0) - \frac{3\lambda}{\pi} \ln \frac{\mu^2}{\mu_0^2} \longrightarrow m^2(\mu) + \eta^2(1 - \delta) = m^2(\mu_0) + \eta^2(1 - \delta) - \delta \frac{3\lambda}{\pi} \ln \frac{\mu^2}{\mu_0^2} \\ \lambda &\longrightarrow \delta\lambda, \end{aligned} \quad (3.1)$$

where μ_0 is a reference scale. Next, by reexpanding to a given order $\delta^{(k)}$ one obtains the OPT physical mass squared, M^2 . To understand how the method works it is convenient to extract the maximum of information in an analytical fashion. This can be achieved by considering the first two non trivial lowest order contributions given by

$$\begin{aligned} M^2(\mu) &= m^2(\mu_0) - \delta \frac{3\lambda}{\pi} \ln \frac{\mu^2}{\mu_0^2} + \eta^2(1 - \delta) + \delta \frac{3\lambda}{\pi} \ln \frac{\mu^2}{[m(\mu_0)^2 + \eta^2]} + \delta^2 \frac{3\lambda\eta^2}{\pi[m(\mu_0)^2 + \eta^2]} \\ &\quad - \delta^2 \frac{9\lambda^2}{\pi^2[m(\mu_0)^2 + \eta^2]} \ln \frac{\mu^2}{[m(\mu_0)^2 + \eta^2]} + \delta^2 \frac{9\lambda^2}{\pi^2[m(\mu_0)^2 + \eta^2]} \ln \frac{\mu^2}{\mu_0^2} \\ &\quad - \delta^2 \frac{3\lambda^2}{2[m(\mu_0)^2 + \eta^2]} + \mathcal{O}(\delta^3). \end{aligned} \quad (3.2)$$

It is now a trivial matter to rearrange the logarithms to see that at any arbitrary scale, μ' , the OPT mass can be written as¹ $M^2(\mu') = M^2(\mu_0)$ where

$$\begin{aligned} M^2(\mu_0) &= m^2(\mu_0) + \eta^2(1 - \delta) + \delta \frac{3\lambda}{\pi} L_\eta + \delta^2 \frac{3\lambda\eta^2}{\pi[m(\mu_0)^2 + \eta^2]} - \delta^2 \frac{9\lambda^2}{\pi^2[m(\mu_0)^2 + \eta^2]} L_\eta \\ &\quad - \delta^2 \frac{3\lambda^2}{2[m(\mu_0)^2 + \eta^2]} + \mathcal{O}(\delta^3). \end{aligned} \quad (3.3)$$

The following definition has been used in the previous relation

$$L_\eta \equiv \ln \frac{\mu_0^2}{[m(\mu_0)^2 + \eta^2]}. \quad (3.4)$$

Having understood that the complete standard perturbative renormalization procedure is not spoiled by the OPT simple replacements we can turn to the optimization procedure. With this aim let us set $\mu_0 = m(\mu_0)$ in eq. (3.3) so that our results can be directly compared to those of ref. [19] (or, equivalently, to those presented in ref. [20] for the particular case $\kappa = L_m \equiv 0$). Next, let us define the dimensionless coupling

$$g \equiv \frac{\lambda}{m^2}. \quad (3.5)$$

Then, in units of m^2 the physical mass squared becomes

$$\frac{M^2}{m^2} = 1 + \frac{\eta^2}{m^2}(1 - \delta) + \delta \frac{3g}{\pi} L_\eta + \delta^2 \frac{1}{(1 + \eta^2/m^2)} \left[g \frac{3\eta^2}{\pi m^2} - g^2 \frac{9}{\pi^2} L_\eta - g^2 \frac{3}{2} \right] + \mathcal{O}(\delta^3). \quad (3.6)$$

¹As in the purely perturbative case this feature remains valid as higher order contributions are considered.

Figure 1. The physical mass squared, in units of m^2 , as a function of g obtained with standard PT (dashed line) and OPT (continuous line) at the two loop level. The critical couplings occur at $g_c = \sqrt{2/3} = 0.82$ (PT) and $g_c = 1.511$ (OPT).

At order- δ the optimal solution is just $\bar{\eta} = 0$ implying that the trivial perturbative solution $M^2/m^2 = 1$ is recovered. A non trivial result is obtained at order- δ^2 where, after setting $\delta = 1$ and applying the variational criterion [26, 27]

$$\left. \frac{\partial M^2}{\partial \eta} \right|_{\bar{\eta}} = 0, \quad (3.7)$$

to eq. (3.6), one obtains as solutions the trivial $\bar{\eta} = 0$ as well as the highly non-perturbative relation

$$\bar{\eta}^2 = m^2 \left[\frac{3g}{\pi} W \left(\frac{\pi \exp[1 + \pi/(3g) + \pi^2/6]}{3g} \right) - 1 \right], \quad (3.8)$$

where W represents the Lambert- W function. Remark that exactly at $g = 0$ the non trivial solution, eq. (3.8), would give $\bar{\eta}^2 = -m^2$ leading to divergences but obviously in this case one has a free theory and the optimal mass is just $\bar{\eta} = 0$. Figure 1 compares the OPT result for $M^2(g)$ with the standard PT prediction at order- g^2 (two loop level) showing that a second order quantum transition takes place when $M^2(g_c) = 0$.

Before pushing the evaluation of M^2 to higher orders it is important to remark that the authors of ref. [20] noticed that by resumming the series for M , instead of the one for M^2 , much better results could be obtained due to the fact that M approaches the critical point smoothly. Therefore, for comparison purposes, we shall also resum this quantity to determine g_c from the condition $M(g_c) = 0$. After taking the square root of eq. (2.8), reexpanding to order λ^8 , and carrying out the OPT replacements represented by eqs. (3.1) one easily obtains the OPT series up to order- δ^8 . As we did in the M^2 case let us start by considering M expanded to order- δ^2 . The explicit result reads

$$\begin{aligned} \frac{M}{m} = & \sqrt{1 + \eta^2/m^2} - \delta \frac{1}{2\sqrt{1 + \eta^2/m^2}} \left(\frac{\eta^2}{m^2} - g \frac{3}{\pi} L_\eta \right) \\ & - \delta^2 \frac{\eta^2}{m^2(1 + \eta^2/m^2)^{3/2}} \left[\frac{\eta^2}{8m^2} - g \frac{1}{2\pi} \left(3 + \frac{1}{2} L_\eta \right) \right] \\ & - \delta^2 g^2 \frac{3}{(1 + \eta^2/m^2)^{3/2}} \left[\frac{1}{4} + \frac{3}{2\pi^2} L_\eta \left(1 + \frac{1}{4} L_\eta \right) \right] + \mathcal{O}(\delta^3). \end{aligned} \quad (3.9)$$

Figure 2. The optimized variational mass, in units of m , as a function of g obtained by resumming M (odd orders) and M^2 (even orders).

After setting $\delta = 1$ and applying the PMS condition one immediately notices that, contrary to what happens when resumming M^2 , optimizing M at order- δ yields $\bar{\eta} = 0$ together with the non trivial result

$$\bar{\eta}^2 = m^2 \left[\frac{3g}{\pi} W \left(\frac{\pi \exp[2 + \pi/(3g)]}{3g} \right) - 1 \right], \quad (3.10)$$

which is rather similar to the optimal mass obtained by resumming the squared mass at order- δ^2 . However, in opposition to what happens in the M^2 case, the optimization of M does not furnish a non trivial real result at order- δ^2 . Then, injecting $\bar{\eta}$ as given by eq. (3.10) back into eq. (3.9) at $\mathcal{O}(\delta)$ yields $g_c = 3.760$. Although numerically not so accurate (e.g., when compared to $g_c = 2.807(34)$ of ref. [19]), this is still a quite remarkable result given that it has been generated by resumming only a simple one loop contribution. This result also emphasizes the crucial role played by the OPT tadpole terms L_η which, in contrast to purely perturbative L_m , survive the scale choice $\mu = m$.

4 Numerical results at N⁸LO

Now, by following the steps which led to the order- δ^2 mass gap result it is a simple matter to consider higher order corrections up to $\mathcal{O}(\delta^8)$ starting from the perturbative result, eq. (2.8). The whole procedure can be easily carried out in a numerical fashion. Exactly as it happened at NLO it turns out that real solutions for $\bar{\eta}$ can only be obtained at even(odd) orders when $M^2(M)$ is optimized. The complete set of results for the optimal $\bar{\eta}$ is displayed in figure 2. Using the results for $\bar{\eta}$ at even orders one can then investigate the physical mass squared to determine the critical coupling from $M^2(g_c) = 0$ as figure 3 illustrates.

At the same time, at odd orders the critical coupling value should be determined from the condition $M(g_c) = 0$. Figure 4 shows $M(g)$ indicating that the value of g_c decreases as more orders are considered in opposition to what happens when M^2 is resummed.

Table 1 compares the OPT predicted values for $M(g)$, at some representative coupling values, with those furnished by Hamiltonian truncations [13, 14] and Borel resummation [19]. The comparison indicates a particularly good agreement between our results and those from ref. [19]. As far as the critical coupling value is concerned figure 5 compares the OPT predictions, at different δ orders, with the state of the art result obtained in ref. [19].

Figure 3. Left panel: the physical mass squared, in units of m^2 , as a function of g at even orders. Right panel: same as the left panel but zoomed.

Figure 4. Left panel: the physical mass, in units of m , as a function of g at odd orders. Right panel: same as the left panel but zoomed.

g	$M(g)$	Ref.
0.02	0.979733(5)	[13, 14]
	0.9797313(4)	[19]
	0.9797315(1)	this work
1	0.7494(2)	[13, 14]
	0.7507(5)	[19]
	0.750520(2)	this work
2	0.345(2)	[13, 14]
	0.357(5)	[19]
	0.352838(14)	this work

Table 1. Comparing some OPT values for $M(g)$, in units of m , with those predicted by Hamiltonian truncations [13, 14] and Borel resummation [19].

The figure clearly indicates that also within the OPT resumming M produces a much faster convergence than resumming M^2 . The actual numerical values are presented in table 2. Not surprisingly, in comparison with $g_c = 2.807(34)$, our most accurate prediction happens at N⁷LO which represents the highest available odd order for which M can be optimized in the present work.

Figure 5. Order by order predictions for the critical coupling. The dots correspond to values obtained by resumming M at odd orders while the crosses represent the values obtained by resumming M^2 at even orders. The straight line represents the prediction $g_c = 2.807(34)$ obtained in ref. [19] by resumming M to N⁸LO.

Resummed M		Resummed M^2	
order	g_c	order	g_c
1	3.76015	2	1.51147
3	2.89809	4	1.98859
5	2.80400	6	2.21970
7	2.77947	8	2.37301

Table 2. Critical couplings obtained from the OPT resummation of $M(M^2)$ at odd(even) orders.

5 Exploring the supercritical region

Having established the reliability of the OPT as an efficient resummation tool let us now investigate what happens at coupling values greater than $g_c \equiv g_c^{(w)}$ which represents a (supercritical) region that has been seldom explored. Our motivation stems from a very recent investigation [43] in the three dimensional case where the authors observed that a second transition, leading back to the symmetric phase, could in principle occur at some coupling value, $g_c^{(s)}$, greater than $g_c^{(w)}$. This intriguing possibility (which in that work occurs within a particular renormalization scheme) has not been fully explored in ref. [43] because $g_c^{(s)}$ is reached after the theory has passed a phase transition. In this case, resummations of perturbative series are not guaranteed to work since $g_c^{(w)}$ represents a non analytical point. Despite of that we shall see that by exploring this region one gets extra information that can be used to further improve convergence properties. To do that let us consider once again the physical mass squared, M^2 , since this quantity is often taken to represent the order parameter in studies related to phase transitions in scalar models [44] while M does not have a clear physical interpretation beyond g_c . Our NNLO result suggests that after being broken at $g_c^{(w)}$ the $\mathbb{Z}_2$ symmetry could get restored at a higher value, $g_c^{(s)}$. Since at this first non trivial order $g_c^{(s)} \simeq 14g_c^{(w)}$ one could dismiss the second transition on the grounds that it takes place at far too high couplings where the resummation is possibly less effective as already emphasized. Within this point of view it would be reasonable to expect that if

Order	$g_c^{(w)}$	$g_c^{(s)}$
2	1.51	21.4
4	1.98	5.89
6	2.21	4.16
8	2.37	3.53

Table 3. The first and second critical points, $g_c^{(w)}$ and $g_c^{(s)}$, from resumming M^2 .

Figure 6. Left panel: the physical mass squared, in units of m^2 , as a function of g at even orders. For clarity the order- g^2 result is not shown in this panel. Right panel: location of the minima from $\mathcal{O}(2)$ to $\mathcal{O}(8)$ (light dots). The dark dot locates the extrapolated minimum corresponding to an arbitrarily high order.

indeed the transition associated with $g_c^{(s)}$ emerges as an artifact of our approximation the difference between $g_c^{(s)}$ and $g_c^{(w)}$ should further increase until $g_c^{(s)}$ eventually disappears, as higher orders are considered.

However, table 3 reveals that at increasing orders the difference between the two critical coupling values decreases reaching $g_c^{(s)} \simeq 1.5g_c^{(w)}$ at N⁸LO. As a digression we note that, physically, the supercritical scenario observed in figure 6 reminds us of a reentrant phase which is a characteristic of many condensed matter systems. One of the best known examples is the symmetry pattern observed in potassium sodium tartrate tetrahydrate most commonly known as the Rochelle salt which, as the temperature increases, goes from a more symmetric orthorhombic crystalline structure to a less symmetric monoclinic structure at $T \simeq 255\text{ K}$ [45]. It then returns to be orthorhombic phase at $T \simeq 297\text{ K}$, till it melts at $T \simeq 348\text{ K}$ therefore exhibiting intermediary inverse symmetry breaking through a reentrant phase (see ref. [46] for a detailed list of different materials exhibiting this phenomenon). Naively, figure 6 could then lead us to think that a similar situation takes place in the present bidimensional case where, instead of the temperature, the coupling drives the (quantum) transition. However, although physically interesting, this exotic phase transition pattern is unlikely to emerge within the bidimensional ϕ^4 model since our present investigation of M , at odd orders, indicates only one value at which $M(g_c) = 0$. Moreover, the existence of a unique critical coupling is also supported by other robust investigations [13, 14, 19, 20]. Then, in the light of these results, we shall not assume that a reentrant transition really takes place within the simple theory being studied here

Method	Year, ref.	g_c
Lattice Monte Carlo	2009, [48]	$2.70^{+0.025}_{-0.013}$
Uniform matrix product states	2013, [5]	2.766(5)
Lattice Monte Carlo	2015, [4]	2.788(15)(8)
Resummed perturbation theory	2015, [49]	2.75(1)
LO renormalized HT	2015, [8]	2.97(14)
raw HT	2016, [10]	2.78(6)
NLO-HT	2017, [14]	2.76(3)
Borel resummation	2018, [19]	2.807(34)
OPT optimal M to N ⁷ LO	2021, This work	2.779(25)
OPT optimal (extrapolated) M^2	2021, This work	2.785

Table 4. Summary of recent predictions.

(in opposition to what happens in some $O(N) \times O(N)$ scalar models [47]). Nevertheless, even when this pragmatic view is adopted the supercritical region may still offer interesting possibilities regarding high order extrapolations. To see that let us first recall that, as seen in the previous section, resumming M seems to be more effective than resumming M^2 . Next, let us refer again to figure 6 to point out that each curve presents a minimum lying within the $g_c^{(w)} - g_c^{(s)}$ interval. As one considers more perturbative contributions both, the interval and $|M^2|$, decrease so it is plausible that at arbitrarily high orders the exact (and unique) critical value is the one which satisfies $g_c^{(w)} = g_c^{(s)} \equiv g_c$.

Assuming that this is true one can use the low order results to estimate the critical coupling value at arbitrarily high orders through a simple extrapolation (see right panel of figure 6). By interpolating over the four available (even) orders we predict $g_c = 2.785$ as the extrapolated value. This simple exercise shows that although M^2 displays a much slower convergence than M one can nevertheless use the information contained within the supercritical region to substantially improve the resummation of the physical mass squared. Table 4 compares our best results with the most recent predictions for the critical coupling.

Before closing this section it is important to emphasize that one should not interpret our extrapolated value, $g_c = 2.785$, as representing the exact one given that it has been obtained through a very simple numerical extrapolation procedure. Rather, one should take this estimate as an indication that the physical mass squared can also be used to predict accurate critical coupling values provided that one uses the information stored within the supercritical region to accelerate convergence.

6 Conclusions

We have applied the OPT method to the bidimensional ϕ^4 model in order to resum the N⁸LO perturbative series representing the mass gap within the $\overline{\text{MS}}$ renormalization scheme. Our main goal was to evaluate the numerical value of the critical coupling, g_c , at which a second order phase transition signals the breaking of the $\mathbb{Z}_2$ symmetry. When the physical

mass squared is considered the OPT optimization criterion has real solutions only at even orders. In this case, the value $g_c = 1.511$ found at NNLO increases as higher orders are considered reaching the value $g_c = 2.373$ at N⁸LO. Following ref. [19] we have also resummed the series representing the mass, M , observing that for this quantity the OPT optimization criterion has real solutions only at odd orders. Our investigation confirms that in this case a better convergence is achieved: as higher orders are considered, the value $g_c = 3.760$ found at NLO decreases to $g_c = 2.779$ at N⁷LO. It turns out that the latter value is in excellent agreement with the state of the art N⁸LO result, $g_c = 2.807(34)$, obtained in ref. [19] where M was (Borel) resummed.

Our prediction also agrees very well with results obtained with methods such as Hamiltonian truncation [8, 10, 14], lattice Monte Carlo [4, 48] and matrix product states [5] which predict g_c to lie within the range 2.75–2.788. As emphasized in the text, the logarithmic terms of the form $\ln(\mu^2/m^2)$ which are associated with tadpole contributions must be included in the perturbative evaluation (prior to the OPT resummation) even when one later decides to set $\mu = m$. The reason is that within the OPT these terms become $\ln[\mu^2/(m^2+\eta^2)]$ so that even when $\mu = m$ the logarithms give important contributions as we have explicitly observed when resumming M at NLO. In this case, an order- g logarithmic term was crucial to generate a highly non-perturbative variational mass allowing us to show that a decent prediction for g_c can readily be obtained even when only a simple tadpole contribution is considered.

Motivated by some recent observations [43] we have investigated the physical mass squared behavior in the supercritical domain. As a novelty, the present work shows that this (often neglected) region contains useful information which allows for high order extrapolations to be performed. Indeed, applying our extrapolation method to M^2 has substantially improved the N⁸LO prediction generating values which are compatible with the OPT resummation of M , at N⁷LO, as well as with other recent estimates. The present application demonstrates how a purely perturbative series can be efficiently resummed by the OPT procedure.

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References

- [1] S.-J. Chang, *Existence of a second-order phase transition in a two-dimensional φ^4 field theory*, *Phys. Rev. D* **13** (1976) 2778 [Erratum *ibid.* **16** (1977) 1979] [[INSPIRE](#)].
- [2] B. Simon and R.B. Griffiths, *The $(\phi^4)_2$ field theory as a classical Ising model*, *Commun. Math. Phys.* **33** (1973) 145 [[INSPIRE](#)].
- [3] M. Serone, G. Spada and G. Villadoro, *$\lambda\phi^4_2$ theory — Part II. the broken phase beyond NNNN(NNNN)LO*, *JHEP* **05** (2019) 047 [[arXiv:1901.05023](#)] [[INSPIRE](#)].
- [4] A. Milsted, J. Haegeman and T.J. Osborne, *Matrix product states and variational methods applied to critical quantum field theory*, *Phys. Rev. D* **88** (2013) 085030 [[arXiv:1302.5582](#)] [[INSPIRE](#)].
- [5] P. Bosetti, B. De Palma and M. Guagnelli, *Monte Carlo determination of the critical coupling in ϕ^4_2 theory*, *Phys. Rev. D* **92** (2015) 034509 [[arXiv:1506.08587](#)] [[INSPIRE](#)].
- [6] S. Bronzin, B. De Palma and M. Guagnelli, *New Monte Carlo determination of the critical coupling in ϕ^4_2 theory*, *Phys. Rev. D* **99** (2019) 034508 [[arXiv:1807.03381](#)] [[INSPIRE](#)].
- [7] D. Kadoh, Y. Kuramashi, Y. Nakamura, R. Sakai, S. Takeda and Y. Yoshimura, *Tensor network analysis of critical coupling in two dimensional ϕ^4 theory*, *JHEP* **05** (2019) 184 [[arXiv:1811.12376](#)] [[INSPIRE](#)].
- [8] S. Rychkov and L.G. Vitale, *Hamiltonian truncation study of the ϕ^4 theory in two dimensions*, *Phys. Rev. D* **91** (2015) 085011 [[arXiv:1412.3460](#)] [[INSPIRE](#)].
- [9] S. Rychkov and L.G. Vitale, *Hamiltonian truncation study of the ϕ^4 theory in two dimensions. II. The $\mathbb{Z}_2$ -broken phase and the Chang duality*, *Phys. Rev. D* **93** (2016) 065014 [[arXiv:1512.00493](#)] [[INSPIRE](#)].
- [10] Z. Bajnok and M. Lajer, *Truncated Hilbert space approach to the 2d ϕ^4 theory*, *JHEP* **10** (2016) 050 [[arXiv:1512.06901](#)] [[INSPIRE](#)].
- [11] M. Burkardt, S.S. Chabysheva and J.R. Hiller, *Two-dimensional light-front ϕ^4 theory in a symmetric polynomial basis*, *Phys. Rev. D* **94** (2016) 065006 [[arXiv:1607.00026](#)] [[INSPIRE](#)].
- [12] N. Anand, V.X. Genest, E. Katz, Z.U. Khandker and M.T. Walters, *RG flow from ϕ^4 theory to the 2D Ising model*, *JHEP* **08** (2017) 056 [[arXiv:1704.04500](#)] [[INSPIRE](#)].
- [13] J. Elias-Miro, S. Rychkov and L.G. Vitale, *High-Precision Calculations in Strongly Coupled Quantum Field Theory with Next-to-Leading-Order Renormalized Hamiltonian Truncation*, *JHEP* **10** (2017) 213 [[arXiv:1706.06121](#)] [[INSPIRE](#)].
- [14] J. Elias-Miro, S. Rychkov and L.G. Vitale, *NLO Renormalization in the Hamiltonian Truncation*, *Phys. Rev. D* **96** (2017) 065024 [[arXiv:1706.09929](#)] [[INSPIRE](#)].
- [15] A.L. Fitzpatrick, J. Kaplan, E. Katz, L.G. Vitale and M.T. Walters, *Lightcone effective Hamiltonians and RG flows*, *JHEP* **08** (2018) 120 [[arXiv:1803.10793](#)] [[INSPIRE](#)].
- [16] S.S. Chabysheva and J.R. Hiller, *Transitioning from equal-time to light-front quantization in ϕ^4_2* , *Phys. Rev. D* **102** (2020) 116010 [[arXiv:1811.01685](#)] [[INSPIRE](#)].
- [17] A.L. Fitzpatrick, E. Katz and M.T. Walters, *Nonperturbative Matching Between Equal-Time and Lightcone Quantization*, *JHEP* **10** (2020) 092 [[arXiv:1812.08177](#)] [[INSPIRE](#)].
- [18] P. Romatschke, *Simple non-perturbative resummation schemes beyond mean-field: case study for scalar ϕ^4 theory in 1+1 dimensions*, *JHEP* **03** (2019) 149 [[arXiv:1901.05483](#)] [[INSPIRE](#)].

- [19] M. Serone, G. Spada and G. Villadoro, $\lambda\phi^4$ Theory I: The Symmetric Phase Beyond NNNNNNNLO, *JHEP* **08** (2018) 148 [[arXiv:1805.05882](#)] [[INSPIRE](#)].
- [20] G. Sberveglieri, M. Serone and G. Spada, Renormalization scheme dependence, RG flow, and Borel summability in ϕ^4 Theories in $d < 4$, *Phys. Rev. D* **100** (2019) 045008 [[arXiv:1905.02122](#)] [[INSPIRE](#)].
- [21] A. Okopińska, Nonstandard expansion techniques for the effective potential in $\lambda\phi^4$ quantum field theory, *Phys. Rev. D* **35** (1987) 1835 [[INSPIRE](#)].
- [22] H. Yamada, Spontaneous symmetry breaking in QCD, *Z. Phys. C* **59** (1993) 67 [[INSPIRE](#)].
- [23] A. Duncan and H.F. Jones, Interpolating Lagrangians and U(1) Gauge Theory on the Lattice, *Nucl. Phys. B* **320** (1989) 189 [[INSPIRE](#)].
- [24] R.P. Feynman and H. Kleinert, Effective Classical Partition Functions, *Phys. Rev. A* **34** (1986) 5080 [[INSPIRE](#)].
- [25] F. Karsch, A. Patkos and P. Petreczky, Screened perturbation theory, *Phys. Lett. B* **401** (1997) 69 [[hep-ph/9702376](#)] [[INSPIRE](#)].
- [26] P.M. Stevenson, Optimized Perturbation Theory, *Phys. Rev. D* **23** (1981) 2916 [[INSPIRE](#)].
- [27] P. Stevenson, Sense and Nonsense in the Renormalization Scheme Dependence Problem, *Nucl. Phys. B* **203** (1982) 472 [[INSPIRE](#)].
- [28] S.K. Gandhi, H.F. Jones and M.B. Pinto, The δ -expansion in the large- N limit, *Nucl. Phys. B* **359** (1991) 429 [[INSPIRE](#)].
- [29] R. Guida, K. Konishi and H. Suzuki, Improved convergence proof of the delta expansion and order dependent mappings, *Annals Phys.* **249** (1996) 109 [[hep-th/9505084](#)] [[INSPIRE](#)].
- [30] B. Bellet, P. Garcia and A. Neveu, Convergent sequences of perturbative approximations for the anharmonic oscillator. 1. Harmonic approach, *Int. J. Mod. Phys. A* **11** (1996) 5587 [[hep-th/9507155](#)] [[INSPIRE](#)].
- [31] B. Bellet, P. Garcia and A. Neveu, Convergent sequences of perturbative approximations for the anharmonic oscillator. 2. Compact time approach, *Int. J. Mod. Phys. A* **11** (1996) 5607 [[hep-th/9507156](#)] [[INSPIRE](#)].
- [32] J.L. Kneur and D. Reynaud, (Borel) convergence of the variationally improved mass expansion and the $O(N)$ Gross-Neveu model mass gap, *Phys. Rev. D* **66** (2002) 085020 [[hep-th/0205133](#)] [[INSPIRE](#)].
- [33] H. Caldas, J.L. Kneur, M.B. Pinto and R.O. Ramos, Critical dopant concentration in polyacetylene and phase diagram from a continuous four-Fermi model, *Phys. Rev. B* **77** (2008) 205109 [[arXiv:0804.2675](#)] [[INSPIRE](#)].
- [34] J.-L. Kneur, A. Neveu and M.B. Pinto, Improved optimization of perturbation theory: Applications to the oscillator energy levels and Bose-Einstein condensate critical temperature, *Phys. Rev. A* **69** (2004) 053624 [[cond-mat/0401324](#)] [[INSPIRE](#)].
- [35] J.-L. Kneur, M.B. Pinto and R.O. Ramos, Phase diagram of the magnetized planar Gross-Neveu model beyond the large- N approximation, *Phys. Rev. D* **88** (2013) 045005 [[arXiv:1306.2933](#)] [[INSPIRE](#)].
- [36] T.E. Restrepo, J.C. Macias, M.B. Pinto and G.N. Ferrari, Dynamical Generation of a Repulsive Vector Contribution to the Quark Pressure, *Phys. Rev. D* **91** (2015) 065017 [[arXiv:1412.3074](#)] [[INSPIRE](#)].

- [37] L. Ferroni, V. Koch and M.B. Pinto, *Multiple Critical Points in Effective Quark Models*, *Phys. Rev. C* **82** (2010) 055205 [[arXiv:1007.4721](#)] [[INSPIRE](#)].
- [38] J.-L. Kneur, M.B. Pinto and T.E. Restrepo, *Renormalization group improved pressure for cold and dense QCD*, *Phys. Rev. D* **100** (2019) 114006 [[arXiv:1908.08363](#)] [[INSPIRE](#)].
- [39] J.-L. Kneur, M.B. Pinto and T.E. Restrepo, *QCD pressure: renormalization group optimized perturbation theory confronts lattice*, [arXiv:2101.02124](#) [[INSPIRE](#)].
- [40] J.-L. Kneur, M.B. Pinto and T.E. Restrepo, *Renormalization group improved pressure for hot and dense quark matter*, [arXiv:2101.08240](#) [[INSPIRE](#)].
- [41] J.L. Kneur and A. Neveu, *Renormalization Group Improved Optimized Perturbation Theory: Revisiting the Mass Gap of the $O(2N)$ Gross-Neveu Model*, *Phys. Rev. D* **81** (2010) 125012 [[arXiv:1004.4834](#)] [[INSPIRE](#)].
- [42] J.-L. Kneur and A. Neveu, α_S from F_π and Renormalization Group Optimized Perturbation Theory, *Phys. Rev. D* **88** (2013) 074025 [[arXiv:1305.6910](#)] [[INSPIRE](#)].
- [43] G. Sberveglieri, M. Serone and G. Spada, *Self-Dualities and Renormalization Dependence of the Phase Diagram in 3d $O(N)$ Vector Models*, *JHEP* **02** (2021) 098 [[arXiv:2010.09737](#)] [[INSPIRE](#)].
- [44] M.B. Pinto and R.O. Ramos, *High temperature resummation in the linear δ expansion*, *Phys. Rev. D* **60** (1999) 105005 [[hep-ph/9903353](#)] [[INSPIRE](#)].
- [45] F. Jona and G. Shirane, *Ferroelectric Crystals*, Monographs in Solid State Physics, Pergamon Press, Oxford U.K. (1962).
- [46] N. Schupper and N.M. Shnerb, *Inverse melting and inverse freezing: A spin model*, *Phys. Rev. E* **72** (2005) 046107 [[cond-mat/0502033](#)].
- [47] M.B. Pinto, R.O. Ramos and J.E. Parreira, *Phase transition patterns in relativistic and nonrelativistic multi-scalar-field models*, *Phys. Rev. D* **71** (2005) 123519 [[hep-th/0506131](#)] [[INSPIRE](#)].
- [48] D. Schaich and W. Loinaz, *An Improved lattice measurement of the critical coupling in ϕ^4_2 theory*, *Phys. Rev. D* **79** (2009) 056008 [[arXiv:0902.0045](#)] [[INSPIRE](#)].
- [49] A. Pelissetto and E. Vicari, *Critical mass renormalization in renormalized ϕ^4 theories in two and three dimensions*, *Phys. Lett. B* **751** (2015) 532 [[arXiv:1508.00989](#)] [[INSPIRE](#)].