$N = 4$ near-horizon geometries in $D = 11$ supergravity

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ABSTRACT: Extreme near-horizon geometries in $D = 11$ supergravity preserving four supersymmetries are classified. It is shown that the Killing spinors fall into three possible orbits, corresponding to pairs of spinors defined on the spatial cross-sections of the horizon which have isotropy groups $SU(3)$, G_2 , or $SU(4)$. In each case, the conditions on the geometry and the 4-form flux are determined. The integrability conditions obtained from the Killing spinor equations are also investigated.

KEYWORDS: Black Holes in String Theory, Supergravity Models

ARXIV EPRINT: [2104.05478](https://arxiv.org/abs/2104.05478)

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1 Introduction

Black hole uniqueness is of particular interest in the context of string theory. The strongest black hole uniqueness theorems are formulated for asymptotically flat solutions in four dimensions [1–6]. However, it is notable that the inclusion of a (negative) cosmological constant weakens these types of uniqueness theorems, even in four dimensions [7, 8]. The situation is furthermore complicated in relation to black holes in string theory, which frequently arise in higher dimensional supergravities. In this case the notion of black hole uniqueness is also weaker. The first signal of this was the discovery of asymptotically flat black rings (both with, and without, supersymmetry) [9–11], for which the event horizon spatial topology is $S^2 \times S^1$. Examples of black rings can be constructed in such a way as to have the same asymptotic charges as black holes with S^3 event horizon topology. Additional evidence of non-uniqueness has also been more recently discovered in [12, 13],

where a very large class of extreme, supersymmetric, black hole solutions has been found. These are both asymptotically flat, and have near-horizon geometries which include the geometry of the near-horizon BMPV solution [14] as a possibility, but which in addition have non-trivial topology located outside of the horizon. The charges associated with such solutions include the mass, angular momenta and electric charge which are evaluated at infinity; but in addition to these, there are also magnetic charges supported on certain, but arbitrarily many, 2-cycles. The thermodynamic properties of such solutions have also been considered in [15]. So, there is a large family of black hole geometries in five dimensions, which includes the BMPV black hole as a special case, but for which there is, a priori, no upper bound on the number of independent parameters which can be used to construct the geometries in this family. This is in marked contrast to the classical uniqueness theorems in four dimensions.

Nevertheless, in spite of this increased complexity, there are uniqueness results established when there are sufficiently many commuting rotational isometries [16–19]. Uniqueness theorems for static and asymptotically flat higher dimensional black hole solutions have also been established [20]. It is however notable that, even given such a rich structure of higher dimensional black hole solutions, there are currently no known analytic examples of asymptotically de-Sitter, or anti-de-Sitter, black rings in higher dimensions with regular horizons, and there are also non-existence theorems for such objects in certain theories [21–24]. It would appear that additional matter fields are required to be included for such solutions to exist.

Motivated by this, it is natural to investigate the issue of black hole uniqueness for extreme and supersymmetric black holes in eleven-dimensional supergravity. We remark that, in contrast to a large number of four and five-dimensional supergravity theories, black hole solutions in D=11 supergravity need not be extreme even if supersymmetry is imposed. This is because the algebraic identities satisfied by the gauge-invariant spinor bilinears differ between D=11 supergravity and four and five-dimensional theories. Hence, in this work, extremality will be imposed as an extra assumption in addition to supersymmetry, which enables the near-horizon limit of the black hole to be considered.

Although higher dimensional black holes are not determined uniquely by their near-horizon geometry, investigating the geometric structures associated with such near-horizon solutions is a necessary first step in establishing a systematic classification of these types of black holes. As we have mentioned above, even in the presence of supersymmetry, black hole uniqueness breaks down in higher dimensions. However, the conditions on the geometry following from supersymmetry can be used to put constraints on the possible solutions, and in some cases to eliminate certain geometries entirely [21]. More generally near-horizon geometries typically admit additional symmetries when compared to the bulk black hole geometry, and this makes the analysis more tractable.

In terms of D=11 supergravity, there has been some significant progress in understanding the properties of supersymmetric near-horizon geometries. In particular, in [25], it was proven that all supersymmetric near-horizon geometries in D=11 supergravity must admit an even number of supersymmetries, and furthermore all such solutions admit a $SL(2, \mathbb{R})$ symmetry algebra. The geometric structure of near-horizon solutions admitting $N = 2$

supersymmetry was classified in [25, 26]. Following on from this, analysis of the superalgebras associated with supersymmetric near-horizon geometries [27, 28] was used to further elucidate the nature of the symmetry algebras of solutions with $N > 2$ supersymmetry. In conjunction with the homogeneity theorem in [29], together with a classification of all compact homogeneous 9-dimensional Riemannian manifolds in [30], it was shown that there are no supersymmetric near horizon geometries with $16 < N < 32$ supersymmetries. Any maximally supersymmetric $N = 32$ near-horizon geometry must furthermore be one of the maximally supersymmetric geometries determined in [31].

It therefore remains to consider supersymmetric near horizon geometries with an even number $4 \leq N \leq 16$ of supersymmetries, and in this work we shall consider the case of $N = 4$. We remark that the cases of $6 \leq N \leq 16$ supersymmetries are not readily amenable to analysis via the techniques used in this paper. This is because, in order to determine the conditions obtained from the Killing spinor equations, we first explicitly integrate these equations along two of the light-cone directions which are generic to all near-horizon solutions. The remaining content of the Killing spinor equations reduces to finding spinors whose components depend only on the 9 remaining co-ordinates associated with the 9-dimensional spatial cross-section of the event horizon, which are parallel with respect to certain 9-dimensional supercovariant derivatives including contributions from the 4-form flux. To extract conditions on the geometry and fluxes from requiring the existence of such parallel spinors, Spin(9) gauge transformations are used to simplify the form of these spinors. Such simplifications can be implemented in the cases of $N = 2$ and $N = 4$ supersymmetry. However, the majority of this gauge freedom is used up in simplifying the spinors for the $N = 4$ case, and little further useful simplification can be made to the extra spinors for $N = 6$ solutions. Nevertheless, it is clear that for $6 \leq N \leq 16$, the additional supersymmetry will impose further conditions on the geometry, and it would be interesting to consider these cases in more detail in future work.

In this paper we determine the conditions on the geometry of the 9-dimensional spatial horizon section, and also the 4-form flux, of extreme supersymmetric black hole near-horizon geometries preserving $N = 4$ supersymmetry in $D = 11$ supergravity. To this end, we shall solve the horizon Killing spinor equations (KSE) using spinorial geometry techniques [32, 33]. This method has been used extensively to construct classifications of supergravity solutions (e.g. [34]), as well as to investigate properties of highly supersymmetric solutions [35] and black hole solutions [26]; see also [36] and references therein for a comprehensive list of applications. The key features of the spinorial geometry approach are firstly that appropriate gauge transformations are utilized to write explicitly the spinors in the simplest possible canonical form. Using this, the conditions obtained from the KSE can then be evaluated directly. This approach to analysing the KSE is particularly useful when investigating the properties of solutions with more than the minimal possible amount of supersymmetry, as it is not necessary to consider the rather detailed analysis of Fierz identities which would otherwise be required.

We apply these techniques to the $N = 4$ near-horizon geometries by determining explicitly the simple canonical forms for the spinors. In particular, the conditions required for $N = 4$ supersymmetry are equivalent to requiring the existence of two Majorana spinors

whose components depend only on the horizon section co-ordinates. These two spinors must be positive chirality, with respect to a certain light-cone chirality projection, and can be chosen to be orthonormal as a consequence of the global analysis of [25]. Using appropriate gauge transformations, we find that there are three different stabilizers, namely $SU(3)$, G_2 or $SU(4)$, for the two positive chirality spinors. The three cases are treated separately. In each case, using the spinorial geometry techniques, we find the constraints on the geometry and we express the fluxes in terms of the geometry in a covariant fashion using the gauge-invariant bilinears which are constructed from the spinors.

The paper is organized as follows. In section 2 we present the near-horizon fields of $D = 11$ supergravity and write the field equations, the Bianchi identities and the Killing spinor equations in terms of these near-horizon fields. In particular, the conditions required for supersymmetry reduce to finding spinors which depend only on the co-ordinates of the spatial horizon section, and are parallel with respect to certain supercovariant derivatives defined on the horizon sections. We also review some global and Killing superalgebra analysis. In section 3 we present the solution of the Killing spinor equations of near-horizon geometries preserving $N = 2$ supersymmetries, fully expressing it in terms of the $Spin(7)$ gauge-invariant bilinears constructed out of the spinors. In section 4 we solve the Killing spinor equations of near-horizon geometries preserving $N = 4$ supersymmetries, splitting the analysis in three distinct cases, $SU(3)$, G_2 or $SU(4)$, corresponding to the isotropy groups of the spinors associated with the $N = 4$ solutions. In section 5 we analyse the integrability conditions of near-horizon geometries in $D = 11$ supergravity. Using a purely local argument, we shall show that all the components of the Einstein equations are implied by the 11-dimensional KSE, the 4-form field equations and the Bianchi identities. The paper concludes with four appendices, where we have gathered some of the lengthier formulae, namely the $N = 4$ linear systems associated to the KSE, the expression of the 4-form in terms of the $SU(3)$ gauge-invariant bilinears, and various G_2 and $Spin(7)$ identities.

2 Supersymmetric near-horizon geometries

In this section, we collate and summarize the key results obtained from [25–28, 37], which we shall utilize in considering the analysis of near-horizon geometries.

Using Gaussian null coordinates [38, 39], the near horizon metric and the 4-form flux¹ of $D = 11$ supergravity can be written [25, 26, 37] in the following form

$$\begin{aligned} ds^2 &= 2\mathbf{e}^+ \mathbf{e}^- + \delta_{ij} \mathbf{e}^i \mathbf{e}^j = 2du \left(dr + rh - \frac{1}{2} r^2 \Delta du \right) + ds^2(\mathcal{S}), \\ F &= \mathbf{e}^+ \wedge \mathbf{e}^- \wedge Y + r \mathbf{e}^+ \wedge d_h Y + X, \end{aligned} \quad (2.1)$$

where the vielbein $\{\mathbf{e}^+, \mathbf{e}^-, \mathbf{e}^i\}$, with $i = 1, 2, 3, 4, 6, 7, 8, 9, \sharp$ is given by

$$\mathbf{e}^+ = du, \quad \mathbf{e}^- = dr + rh - \frac{1}{2} r^2 \Delta du, \quad \mathbf{e}^i = e^i_J dy^J, \quad g_{IJ} = \delta_{ij} e^i_I e^j_J, \quad (2.2)$$

and

$$ds^2(\mathcal{S}) = \delta_{ij} \mathbf{e}^i \mathbf{e}^j, \quad (2.3)$$

¹Let ω be a k -form, then $d_h \omega := d\omega - h \wedge \omega$.

is the metric of the horizon spatial cross-section $\mathcal{S}$ given by $r = u = 0$. In these expressions, $h = h_i \mathbf{e}^i$, Δ , $\mathbf{e}^i$, Y and X depend only on the co-ordinates y of $\mathcal{S}$. Furthermore, we assume that $\mathcal{S}$ is a compact 9-dimensional manifold without boundary, and that Δ , h , Y and X are globally defined and smooth 0-, 1-, 2- and 4-forms on $\mathcal{S}$, respectively.

2.1 Bianchi identities and field equations

The Bianchi identities and field equations of $D = 11$ supergravity [40] can be decomposed along the lightcone directions and those of the horizon section $\mathcal{S}$. For the Bianchi identities $dF = 0$, such a decomposition yields

$$dX = 0. \quad (2.4)$$

The field equation of the 4-form flux F reads

$$d \star_{11} F - \frac{1}{2} F \wedge F = 0, \quad (2.5)$$

where $\star_{11}$ is the Hodge star on 11-dimensional spacetime. Equation (2.5) can be written as

$$- \star_9 d_h Y - h \wedge \star_9 X + d \star_9 X = Y \wedge X, \quad (2.6)$$

and

$$- d \star_9 Y = \frac{1}{2} X \wedge X, \quad (2.7)$$

where $\star_9$ denotes the Hodge dual on $\mathcal{S}$. The volume form is chosen as $\epsilon_{11} = \mathbf{e}^+ \wedge \mathbf{e}^- \wedge \epsilon_{\mathcal{S}}$, where $\epsilon_{\mathcal{S}}$ is the volume form of $\mathcal{S}$.

The Einstein equation is

$$R_{MN} = \frac{1}{12} F_{ML_1 L_2 L_3} F_N{}^{L_1 L_2 L_3} - \frac{1}{144} g_{MN} F_{L_1 L_2 L_3 L_4} F^{L_1 L_2 L_3 L_4}, \quad (2.8)$$

for which the independent components are the ij components

$$\begin{aligned} \tilde{R}_{ij} + \tilde{\nabla}_{(i} h_{j)} - \frac{1}{2} h_i h_j &= -\frac{1}{2} Y_{i\ell} Y_j{}^\ell + \frac{1}{12} X_{i\ell_1 \ell_2 \ell_3} X_j{}^{\ell_1 \ell_2 \ell_3} \\ &\quad + \delta_{ij} \left(\frac{1}{12} Y_{\ell_1 \ell_2} Y^{\ell_1 \ell_2} - \frac{1}{144} X_{\ell_1 \ell_2 \ell_3 \ell_4} X^{\ell_1 \ell_2 \ell_3 \ell_4} \right), \end{aligned} \quad (2.9)$$

and the $+-$ component

$$\tilde{\nabla}^i h_i = 2\Delta + h^2 - \frac{1}{3} Y_{\ell_1 \ell_2} Y^{\ell_1 \ell_2} - \frac{1}{72} X_{\ell_1 \ell_2 \ell_3 \ell_4} X^{\ell_1 \ell_2 \ell_3 \ell_4}. \quad (2.10)$$

Here $\tilde{\nabla}$ is the Levi-Civita connection of the metric $ds^2(\mathcal{S})$ and $\tilde{R}_{ij}$ is the Ricci tensor of $\mathcal{S}$. The $++$ and $+i$ components of the Einstein equation hold as a consequence of (2.4), the 3-form field equations (2.6) and (2.7) and the components of the Einstein equation in (2.9) and (2.10). The $--$ and the $-i$ components of the Einstein equations are also satisfied automatically for all near-horizon solutions (2.1). Thus, the conditions on $ds^2(\mathcal{S})$, Δ , h , Y and X simplify to (2.4), (2.6), (2.7), (2.9) and (2.10).

2.2 Independent Killing spinor equations

The Killing spinor equations (KSE) of 11-dimensional supergravity [40] are

$$\nabla_M \epsilon + \left(-\frac{1}{288} \Gamma_M^{L_1 L_2 L_3 L_4} F_{L_1 L_2 L_3 L_4} + \frac{1}{36} F_{M L_1 L_2 L_3} \Gamma^{L_1 L_2 L_3} \right) \epsilon = 0, \quad (2.11)$$

where ∇ is the Levi-Civita connection. The KSE can be integrated along the two lightcone directions, as all the bosonic fields are independent of the u co-ordinate, and the dependence on the r co-ordinate is known explicitly. The remaining content of the KSE can then be further decomposed along the directions corresponding to the spatial horizon section $\mathcal{S}$. To begin, we write the Killing spinor ϵ as [26]

$$\epsilon = \epsilon_+ + \epsilon_-, \quad \Gamma_{\pm} \epsilon_{\pm} = 0. \quad (2.12)$$

Then after some computation, see [25, 26, 37], we find that

$$\epsilon_+ = \eta_+, \quad \epsilon_- = \eta_- + r \Gamma_- \Theta_+ \eta_+, \quad (2.13)$$

and

$$\eta_+ = \phi_+ + u \Gamma_+ \Theta_- \phi_-, \quad \eta_- = \phi_-, \quad (2.14)$$

where $\phi_{\pm} = \phi_{\pm}(y)$ do not depend on r or u and

$$\Theta_{\pm} = \left(\frac{1}{4} h_i \Gamma^i + \frac{1}{288} X_{\ell_1 \ell_2 \ell_3 \ell_4} \Gamma^{\ell_1 \ell_2 \ell_3 \ell_4} \pm \frac{1}{12} Y_{\ell_1 \ell_2} \Gamma^{\ell_1 \ell_2} \right). \quad (2.15)$$

Using the field equations and Bianchi identities, the independent KSE are

$$\nabla_i^{(\pm)} \phi_{\pm} \equiv \tilde{\nabla}_i \phi_{\pm} + \Psi_i^{(\pm)} \phi_{\pm} = 0, \quad (2.16)$$

where

$$\begin{aligned} \Psi_i^{(\pm)} = & \mp \frac{1}{4} h_i - \frac{1}{288} \Gamma_i^{\ell_1 \ell_2 \ell_3 \ell_4} X_{\ell_1 \ell_2 \ell_3 \ell_4} + \frac{1}{36} X_{i \ell_1 \ell_2 \ell_3} \Gamma^{\ell_1 \ell_2 \ell_3} \\ & \pm \frac{1}{24} \Gamma_i^{\ell_1 \ell_2} Y_{\ell_1 \ell_2} \mp \frac{1}{6} Y_{ij} \Gamma^j. \end{aligned} \quad (2.17)$$

2.3 Global and superalgebra analysis of supersymmetry

For each of the “horizon gravitino KSEs” on $\mathcal{S}$ given in (2.16), one can associate a “horizon Dirac equation” as

$$\mathcal{D}^{(\pm)} \phi_{\pm} \equiv \Gamma^i \nabla_i^{(\pm)} \phi_{\pm} = \Gamma^i \tilde{\nabla}_i \phi_{\pm} + \Psi^{(\pm)} \phi_{\pm} = 0, \quad (2.18)$$

where

$$\Psi^{(\pm)} = \Gamma^i \Psi_i^{(\pm)} = \mp \frac{1}{4} h_{\ell} \Gamma^{\ell} + \frac{1}{96} X_{\ell_1 \ell_2 \ell_3 \ell_4} \Gamma^{\ell_1 \ell_2 \ell_3 \ell_4} \pm \frac{1}{8} Y_{\ell_1 \ell_2} \Gamma^{\ell_1 \ell_2}. \quad (2.19)$$

These Dirac equations, in addition to the Levi-Civita connection, also depend on the fluxes of the supergravity theory restricted on the horizon section $\mathcal{S}$, and the properties of the horizon Dirac equations play an important role in counting the number of supersymmetries

preserved by the solutions. In particular, generalized Lichnerowicz Theorems established in [25] (see also analogous results for type II supergravity theories in [36, 41–43]) imply that

$$\nabla_i^{(\pm)} \phi_{\pm} = 0 \iff \mathcal{D}^{(\pm)} \phi_{\pm} = 0, \quad (2.20)$$

and furthermore, via an index theory argument, we have

$$\dim \ker \mathcal{D}^{(+)} = \dim \ker \mathcal{D}^{(-)}. \quad (2.21)$$

The global analysis which is used to establish the generalized Lichnerowicz theorems, also implies, by a maximum principle argument, that $\|\phi_+\|^2$ is constant, and henceforth we shall take all such positive chirality spinors to satisfy $\|\phi_+\|^2 = 1$. The number of positive chirality spinors which are parallel with respect to $\nabla_i^{(+)}$ is equal to the number of negative chirality spinors which are parallel with respect to $\nabla_i^{(-)}$, and for a supersymmetric near-horizon geometry preserving $2k$ supersymmetries, we take $\phi_{\pm}^r$ spinors for $r = 1, \dots, k$ with, without loss of generality, $\langle \phi_+^r, \phi_+^s \rangle = \delta^{rs}$, where $\langle, \rangle$ is a Spin(9)-invariant Dirac inner product.

There is furthermore an algebraic relation which can be used to construct ϕ_+^r spinors from ϕ_-^r spinors which are parallel with respect to $\nabla_i^{(\pm)}$ via

$$\phi_+^r = \Gamma_+ \Theta_- \phi_-^r \quad (2.22)$$

and in particular, it has been shown that if $\phi_-^r \neq 0$ then ϕ_+^r defined as above is non-vanishing. A further algebraic relation, based on a superalgebra computation [27, 28], has also been constructed which relates ϕ_-^r to ϕ_+^r via

$$2\phi_-^t - 2\|\phi_-^t\|^2 \Gamma_- \Theta_+ \phi_+^t + \check{V} \Gamma_- \phi_+^t = 0, \quad (2.23)$$

where $\check{V}$ is obtained from the horizon section isometries

$$V_i^{rs} \equiv \langle \Gamma_+ \phi_-^r, \Gamma_i \phi_+^s \rangle \quad (2.24)$$

via

$$V^{rs} = \delta^{rs} \check{V} + Z^{rs}, \quad (2.25)$$

where Z^{rs} is antisymmetric in the indices r, s . Moreover, the negative chirality spinors ϕ_-^r satisfy

$$\langle \phi_-^r, \phi_-^s \rangle = \delta^{rs} \|\phi_-^r\|^2, \quad (2.26)$$

and the positive chirality spinors ϕ_+^r satisfy

$$4\langle \Theta_+ \phi_+^r, \Theta_+ \phi_+^s \rangle = \delta^{rs} \Delta, \quad \langle \phi_+^r, \Gamma_i \Theta_+ \phi_+^s \rangle = 0. \quad (2.27)$$

The isometries V_i^{rs} on $\mathcal{S}$ are obtained from $D = 11$ spacetime isometries as set out in [27, 28], and both the $D = 11$ metric and 4-form F are invariant with respect to these isometries [44]. Consequently, the isometries $\check{V}$ and Z^{rs} leave invariant all of the near-horizon data:

$$\tilde{\nabla}_{(i} \check{V}_{j)} = 0, \quad \tilde{\mathcal{L}}_{\check{V}} h = 0, \quad \tilde{\mathcal{L}}_{\check{V}} \Delta = 0, \quad \tilde{\mathcal{L}}_{\check{V}} Y = 0, \quad \tilde{\mathcal{L}}_{\check{V}} X = 0, \quad (2.28)$$

and similarly

$$\tilde{\nabla}_{(i} Z_{j)}^{rs} = 0, \quad \tilde{\mathcal{L}}_{Z^{rs}} h = 0, \quad \tilde{\mathcal{L}}_{Z^{rs}} \Delta = 0, \quad \tilde{\mathcal{L}}_{Z^{rs}} Y = 0, \quad \tilde{\mathcal{L}}_{Z^{rs}} X = 0. \quad (2.29)$$

Furthermore, $\check{V}$ and Z^{rs} commute

$$[\check{V}, Z^{rs}] = 0. \quad (2.30)$$

The condition (2.23) does not however determine the ϕ_-^r spinors directly from the ϕ_+^r spinors because $\check{V}$ depends implicitly on ϕ_-^r via (2.25). The vector field $\check{V}$ may vanish, in such a case the near-horizon geometry is static, and is a warped product $AdS_2 \times_w \mathcal{S}$ [25]. Generically, a non-static $N = 4$ solution has non-vanishing commuting isometries $\check{V}$ and $Z = Z^{12}$, and therefore admits a $U(1) \times U(1)$ symmetry on the horizon spatial section $\mathcal{S}$. However, there are also special cases when this symmetry is reduced. Additional geometric conditions can be obtained from a detailed analysis of the Killing spinor equations, which we shall undertake in the following sections. We remark that it will be most straightforward to consider the conditions obtained by solving

$$\nabla_i^{(+)} \phi_+^r = 0, \quad (2.31)$$

where $r = 1$ for $N = 2$ solutions, and $r = 1, 2$ for $N = 4$ solutions.

This is because the constant norm condition $\langle \phi_+^r, \phi_+^s \rangle = \delta^{rs}$, combined with a judicious application of $\text{Spin}(9)$ gauge transformations, allows one to find very simple canonical forms for the ϕ_+^r spinors, and consequently the linear systems obtained from the KSE are rendered more tractable. Having solved the KSE for the positive chirality spinors, the global analysis guarantees that there must exist negative chirality spinors ϕ_-^r which are parallel with respect to $\nabla^{(-)}$.

3 $N = 2$ solutions

In this section we revisit near-horizon geometries preserving exactly $N = 2$ supersymmetries. This has already been discussed in [26] working in a particular gauge for the ϕ_-^1 spinor. However, our analysis here does not involve that gauge fixing procedure, as we wish to retain sufficient gauge freedom to simplify the additional positive chirality spinor which we will consider in the later analysis of $N = 4$ solutions. It is convenient to solve the KSE in the positive chirality sector

$$\nabla_i^{(+)} \phi_+^1 = 0, \quad (3.1)$$

since, as we have mentioned in the previous section, ϕ_+^1 has constant norm, which enables us to write this spinor in a particularly simple canonical form. In particular, using the $\text{Spin}(9)$ gauge invariance of the KSEs (3.1), namely

$$\phi_+^1 \rightarrow e^{f^{ij} \Gamma_{ij}} \phi_+^1, \quad (3.2)$$

where Γ_{ij} are the generators of $\text{Spin}(9)$, the spinor ϕ_+^1 can be written in the following particularly simple canonical form

$$\phi_+^1 = \frac{1}{\sqrt{2}}(1 + e_{1234}), \quad (3.3)$$

which is stabilized by a $\text{Spin}(7)$ isotropy subgroup of $\text{Spin}(9)$ [37].

It will be particularly useful to demonstrate explicitly how this can be achieved, as it will illustrate how a similar analysis can be done to determine the common isotropy group of two spinors $\{\phi_+^1, \phi_+^2\}$ in the next section, which is the primary focus of this work. So, in order to show (3.3), let us start by writing down the general expression for a positive (lightcone) chirality Majorana spinor [45]

$$\phi_+^1 = w1 + \bar{w}e_{1234} + \psi^\alpha e_\alpha - \frac{1}{3!}(\star\bar{\psi})^{\alpha_1\alpha_2\alpha_3}e_{\alpha_1\alpha_2\alpha_3} + \frac{1}{2}(B^{\alpha_1\alpha_2} - (\star\bar{B})^{\alpha_2\alpha_1})e_{\alpha_1\alpha_2}, \quad (3.4)$$

where in (3.4) $\alpha, \beta = 1, 2, 3, 4$, $\star$ is the Hodge dual on $\mathbb{R}^4$ and w, ψ and B are functions of the horizon coordinates y . Defining $A^{\alpha\beta} := B^{\alpha\beta} - (\star\bar{B})^{\alpha\beta}$, we can rewrite (3.4) as follows

$$\phi_+^1 = w1 + \bar{w}e_{1234} + \psi^\alpha e_\alpha - \frac{1}{3!}(\star\bar{\psi})^{\alpha_1\alpha_2\alpha_3}e_{\alpha_1\alpha_2\alpha_3} + \chi^p e_{1p} - \frac{1}{2}\bar{\chi}^r \epsilon_r^{mn} \epsilon_{mn}, \quad (3.5)$$

where $\chi^p := A^{1p}$, with $p = 2, 3, 4$. Consider first the $su(3)$ transformation $S^{p\bar{q}}\Gamma_{p\bar{q}}$. Then

$$(S^{p\bar{q}}\Gamma_{p\bar{q}})e_{1r} = 2S^p_r e_{1p}. \quad (3.6)$$

Since the action of $SU(3)$ on $S^5 \subset \mathbb{C}^3$ is simply transitive, without loss of generality we can impose $\chi^3 = \chi^4 = 0$. Thus (3.5) simplifies to

$$\phi_+^1 = w1 + \bar{w}e_{1234} + \psi^\alpha e_\alpha - \frac{1}{3!}(\star\bar{\psi})^{\alpha_1\alpha_2\alpha_3}e_{\alpha_1\alpha_2\alpha_3} + \chi e_{12} - \bar{\chi} e_{34}, \quad (3.7)$$

with $\chi := \chi^1$. Next, let us consider the gauge transformations

$$T_1 := \Gamma_{12} + \Gamma_{\bar{1}\bar{2}}, \quad T_2 := i(\Gamma_{12} - \Gamma_{\bar{1}\bar{2}}), \quad T_3 := \Gamma_{34} + \Gamma_{\bar{3}\bar{4}}, \quad T_4 := i(\Gamma_{34} - \Gamma_{\bar{3}\bar{4}}), \quad (3.8)$$

acting on

$$v_1 := 1 + e_{1234}, \quad v_2 := i(1 - e_{1234}), \quad v_3 := e_{12} - e_{34}, \quad v_4 := i(e_{12} + e_{34}). \quad (3.9)$$

It is easy to show that the representative matrices $M^{(a)}$ ($a = 1, \dots, 4$) of T_a with respect to v_a are elements of $so(4)$. In particular

$$M^{(1)} - M^{(3)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.10)$$

and

$$M^{(1)} + M^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad (3.11)$$

are two copies of $so(2)$ acting on $\{v_1, v_3\}$ and $\{v_2, v_4\}$ respectively. The transitivity of $SO(2)$ on $S^1 \subset \mathbb{R}^2$ allows us to set $\Re(\chi) = \Im(\chi) = 0$, that is, $\chi = 0$. Also, $i\Gamma_{1\bar{1}}$ is a $so(2)$ transformation acting on $\{v_1, v_2\}$. Thus, we can set $\Im(w) = 0$, that is $w \in \mathbb{R}$. Combining the previous results, (3.7) simplifies to

$$\phi_+^1 = w(1 + e_{1234}) + \psi^\alpha e_\alpha - \frac{1}{3!}(\star\bar{\psi})^{\alpha_1\alpha_2\alpha_3} e_{\alpha_1\alpha_2\alpha_3}. \quad (3.12)$$

Next, consider a $su(4)$ transformation $\Lambda^{\alpha\bar{\beta}}\Gamma_{\alpha\bar{\beta}}$, where $\alpha = 1, 2, 3, 4$. We have that

$$(\Lambda^{\alpha\bar{\beta}}\Gamma_{\alpha\bar{\beta}})e_\lambda = 2\Lambda^\alpha_\lambda e_\alpha. \quad (3.13)$$

Using the transitivity of $SU(4)$ on $S^7 \subset \mathbb{C}^4$, we can set $\psi^2 = \psi^3 = \psi^4 = 0$, $\psi := \psi^1 \in \mathbb{R}$. Thus (3.12) boils down to

$$\phi_+^1 = w(1 + e_{1234}) + \psi(e_1 + e_{234}). \quad (3.14)$$

Moreover, $\Gamma_{\sharp 1} + \Gamma_{\sharp\bar{1}}$ is a $so(2)$ transformation on $1 + e_{1234}$ and $e_1 + e_{234}$, thus, using the transitivity of $SO(2)$, we can set $\psi = 0$ in (3.14), obtaining

$$\phi_+^1 = w(1 + e_{1234}). \quad (3.15)$$

Eventually, using $\|\phi_+^1\|^2 = 1$, we can set $w = \frac{1}{\sqrt{2}}$ in (3.15), obtaining (3.3).

The stabilizer, or isotropy group, of ϕ_+^1 , with ϕ_+^1 given by (3.3), is $\text{Spin}(7)$. To see this, note that the stabilizer of ϕ_+^1 is the subgroup of $\text{Spin}(9)$ whose generators satisfy

$$f^{ij}\Gamma_{ij}\phi_+^1 = 0. \quad (3.16)$$

Using the oscillatory basis for the Gamma matrices

$$\Gamma_\alpha := \frac{1}{\sqrt{2}}(\Gamma_\alpha - i\Gamma_{\alpha+5}), \quad \Gamma_{\bar{\alpha}} := \frac{1}{\sqrt{2}}(\Gamma_\alpha + i\Gamma_{\alpha+5}), \quad (3.17)$$

equation (3.16) boils down to

$$2f^\alpha_\alpha(e_{1234} - 1) + (2f^{\alpha\beta} - \epsilon^{\alpha\beta\gamma\delta}f_{\gamma\delta})e_{\alpha\beta} + 2\sqrt{2}f^\sharp_\alpha e_\alpha = 0, \quad (3.18)$$

which, using the linear independence of the basis elements e_I , implies that

$$f_\alpha^\alpha = 0, \quad f^\sharp_\alpha = 0, \quad f_{\bar{\alpha}\bar{\beta}} = \frac{1}{2}\epsilon_{\bar{\alpha}\bar{\beta}}^{\gamma\delta}f_{\gamma\delta}, \quad (3.19)$$

which is indeed the Lie algebra $spin(7)$ in a Hermitian basis [46].

Using the oscillator basis (3.17), we have computed the $N = 2$ linear system associated to the KSE (3.1), with ϕ_+^1 given by (3.3). The linear system is presented in appendix A, and its solution is the following

$$\frac{1}{18}X_{\mu_1\mu_2\mu_3\mu_4}\epsilon^{\mu_1\mu_2\mu_3\mu_4} - \frac{1}{6}X_{\mu_1\mu_2}{}^{\mu_1\mu_2} = -\frac{1}{6}\Omega^\mu_{\mu,\sharp} - \frac{5}{6}\Omega_{\mu,\sharp}{}^\mu - \frac{2}{3}\Omega_{\sharp,\mu}{}^\mu, \quad (3.20)$$

$$X_{\alpha\bar{\beta}\mu}{}^\mu + \frac{1}{4}\delta_{\alpha\bar{\beta}}X_{\mu_1\mu_2}{}^{\mu_1\mu_2} = -2\Omega_{(\alpha,|\sharp|\bar{\beta})} + \frac{1}{4}\delta_{\alpha\bar{\beta}}(\Omega_{\mu,\sharp}{}^\mu + \Omega^\mu_{\sharp,\mu}), \quad (3.21)$$

$$X_{(\alpha|\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3}{}^{\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3}{}_{|\beta)} = 6\Omega_{(\alpha,|\sharp|\beta)}, \quad (3.22)$$

$$X_{\alpha\beta\mu}{}^\mu + \frac{1}{2}X_{\bar{\mu}_1\bar{\mu}_2\nu}{}^\nu\epsilon^{\bar{\mu}_1\bar{\mu}_2}{}_{\alpha\beta} = 2\Omega_{[\alpha,|\sharp|\beta]} + 2\Omega_{\sharp,\alpha\beta} - (\Omega_{\sharp,\bar{\mu}_1\bar{\mu}_2} + \Omega_{\bar{\mu}_1,\sharp\bar{\mu}_2})\epsilon^{\bar{\mu}_1\bar{\mu}_2}{}_{\alpha\beta}, \quad (3.23)$$

$$X_{\sharp\alpha\beta\gamma} = -2\Omega_{[\alpha,\beta\gamma]} + \frac{2}{3} \left(-\Omega_{\nu,\bar{\mu}}^{\nu} + \Omega_{\bar{\mu},\nu}^{\nu} - \Omega_{\sharp,\sharp\bar{\mu}} \right) \epsilon^{\bar{\mu}}_{\alpha\beta\gamma}, \quad (3.24)$$

$$X_{\sharp\alpha\bar{\beta}\bar{\gamma}} = \frac{2}{3} (\Omega_{\alpha,\mu_1\mu_2} + \Omega_{\mu_1,\alpha\mu_2}) \epsilon^{\mu_1\mu_2}_{\bar{\beta}\bar{\gamma}} - 2\Omega_{\alpha,\bar{\beta}\bar{\gamma}} + \frac{4}{3} \left(-\Omega_{\nu,\bar{\beta}}^{\nu} + \Omega_{\bar{\beta},\nu}^{\nu} + \frac{1}{2} \Omega_{\sharp,\sharp[\bar{\beta}]} \right) \delta_{\bar{\gamma}}]_{\alpha}, \quad (3.25)$$

$$Y_{\alpha\bar{\beta}} = 2\Omega_{[\alpha,\sharp]\bar{\beta}} + \delta_{\alpha\bar{\beta}} \left(\Omega_{\sharp,\nu}^{\nu} + \frac{1}{2} \Omega_{\sharp,\nu}^{\nu} - \frac{1}{2} \Omega_{\nu,\sharp}^{\nu} \right), \quad (3.26)$$

$$Y_{\sharp\alpha} = \frac{2}{3} \Omega_{\sharp,\sharp\alpha} + \frac{2}{3} \Omega_{\alpha,\nu}^{\nu} + \frac{2}{3} \Omega_{\bar{\mu}_1,\bar{\mu}_2\bar{\mu}_3} \epsilon^{\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3}_{\alpha} - \frac{4}{3} \Omega^{\mu}_{\mu,\alpha}, \quad (3.27)$$

$$Y_{\alpha\beta} = (\Omega_{\bar{\mu}_1,\sharp\bar{\mu}_2} - \Omega_{\sharp,\bar{\mu}_1\bar{\mu}_2}) \epsilon^{\bar{\mu}_1\bar{\mu}_2}_{\alpha\beta} + 2\Omega_{\sharp,\alpha\beta}, \quad (3.28)$$

$$h_{\sharp} = -\frac{1}{2} \left(\Omega^{\mu}_{\sharp,\sharp\mu} + \Omega_{\mu,\sharp}^{\mu} \right), \quad (3.29)$$

$$h_{\alpha} = \frac{1}{3} \Omega_{\sharp,\sharp\alpha} - \frac{2}{3} \Omega_{\alpha,\nu}^{\nu} - \frac{2}{3} \Omega_{\bar{\mu}_1,\bar{\mu}_2\bar{\mu}_3} \epsilon^{\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3}_{\alpha} + \frac{4}{3} \Omega^{\mu}_{\mu,\alpha}. \quad (3.30)$$

After an extensive computation, the solution of the linear system, written in equations (3.20)–(3.30) in a $SU(4)$ representation, can be repackaged in a compact $\text{Spin}(7)$ covariant form, using the $\text{Spin}(7)$ invariant differential forms

$$e^{\sharp}, \quad \tau := \Re(\chi) - \frac{1}{2} \omega \wedge \omega, \quad (3.31)$$

where τ is the $\text{Spin}(7)$ fundamental 4-form, χ and ω are a 4-form and a 2-form respectively, which in a local frame e^{α} read

$$\chi := 4e^{1234}, \quad \omega := -i\delta_{\alpha\bar{\beta}} e^{\alpha\bar{\beta}}. \quad (3.32)$$

The $\text{Spin}(7)$ invariant forms (3.31) can be computed via the $\text{Spin}(7)$ gauge-invariant bilinears

$$\langle \phi_+^1, \Gamma_{i_1 \dots i_n} \phi_+^1 \rangle. \quad (3.33)$$

The $\text{Spin}(7)$ covariant solution of near-horizon geometries in $D = 11$ supergravity preserving $N = 2$ supersymmetries is given by

$$h = -\frac{1}{3} \mathcal{L}_{e^{\sharp}} e^{\sharp} + \frac{1}{6} \theta_{\tau}^{(9)} - \frac{2}{3} \left(\star_9 d \star_9 e^{\sharp} \right) e^{\sharp}, \quad (3.34)$$

$$Y = -d_h e^{\sharp} - \frac{1}{48} (\mathcal{L}_{e^{\sharp}} \tau)_{i_1 j_1 j_2 j_3} \tau^{j_1 j_2 j_3}_{i_2} e^{i_1 i_2}, \quad (3.35)$$

$$X = -\frac{1}{7} \left(\star_9 d \star_9 e^{\sharp} \right) \tau - \frac{1}{4} \mathcal{L}_{e^{\sharp}} \tau + \frac{1}{4!} \left((de^{\sharp})_{i_1 j} - \frac{3}{2} (\mathcal{L}_{e^{\sharp}} g)_{i_1 j} \right) \tau^j_{i_2 i_3 i_4} e^{i_1 i_2 i_3 i_4} - e^{\sharp} \wedge \left\{ \star_9 d \star_9 \tau + \frac{1}{3!} \left(-\frac{17}{24} (\mathcal{L}_{e^{\sharp}} e^{\sharp})_j + \frac{1}{6} (\theta_{\tau}^{(9)})_j \right) \tau^j_{i_1 i_2 i_3} e^{i_1 i_2 i_3} \right\} + X^{27}, \quad (3.36)$$

where $\theta_{\tau}^{(9)}$ is the Lee form of τ on $\mathcal{S}$, i.e.

$$\theta_{\tau}^{(9)} := \star_9 (\tau \wedge \star_9 d\tau), \quad (3.37)$$

and $X^{27} \in \mathbf{27}$ of $\text{Spin}(7)$ is projected out by the KSEs. The definition of the $\mathbf{27}$ component of a 4-form on a 8-dimensional $\text{Spin}(7)$ manifold is given in (D.8). Furthermore, no geometric conditions arise.

4 $N = 4$ solutions

In this section we study near-horizon geometries preserving exactly $N = 4$ supersymmetries, by obtaining the additional conditions on the geometry and the 4-form flux obtained by solving the KSE

$$\nabla_i^{(+)} \phi_+^2 = 0. \quad (4.1)$$

The first step is to write the second (positive chirality) spinor ϕ_+^2 in the simplest possible canonical form. Using the residual $\text{Spin}(7)$ gauge invariance, which preserves the canonical form of the first spinor ϕ_+^1 , the second spinor ϕ_+^2 can be put in the following form

$$\phi_+^2 = \frac{i}{\sqrt{2}} \cos \theta (1 - e_{1234}) + \frac{1}{\sqrt{2}} \sin \theta (e_1 + e_{234}), \quad (4.2)$$

where θ is a real function on $\mathcal{S}$. In order to show (4.2), we proceed similarly as for ϕ_+^1 . In particular, since $su(3) \subset spin(7)$, we can still simplify ϕ_+^2 to

$$\phi_+^2 = w1 + \bar{w}e_{1234} + \psi^\alpha e_\alpha - \frac{1}{3!} (\star \bar{\psi})^{\alpha_1 \alpha_2 \alpha_3} e_{\alpha_1 \alpha_2 \alpha_3} + \chi e_{12} - \bar{\chi} e_{34}. \quad (4.3)$$

In order to simplify the spinor further, notice that $M^{(1)} + M^{(3)} \in spin(7)$, while $M^{(1)} - M^{(3)} \notin spin(7)$ (the matrices $M^{(a)}$ are defined in the previous section). Thus we can only use $M^{(1)} + M^{(3)}$ to set $\Im(\chi) = 0$. However

$$-M^{(2)} + M^{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.4)$$

is a $so(2)$ transformation acting on $\{v_2, v_3\}$, thus we can still set $\Re(\chi) = 0$ and then $\chi = 0$. However, differently from the first spinor, we cannot use $i\Gamma_{1\bar{1}}$ to set $w \in \mathbb{R}$ since $i\Gamma_{1\bar{1}} \notin spin(7)$. Thus

$$\phi_+^2 = w1 + \bar{w}e_{1234} + \psi^\alpha e_\alpha - \frac{1}{3!} (\star \bar{\psi})^{\alpha_1 \alpha_2 \alpha_3} e_{\alpha_1 \alpha_2 \alpha_3}. \quad (4.5)$$

Using the $su(4) \subset spin(7)$ gauge invariance, equation (4.5) reduces to

$$\phi_+^2 = w_1 (1 + e_{1234}) + iw_2 (1 - e_{1234}) + \psi(e_1 + e_{234}). \quad (4.6)$$

Eventually, using the orthonormality relation $\langle \phi_+^r, \phi_+^s \rangle = \delta^{rs}$, we can set $w_1 = 0$, $w_2 = \frac{1}{\sqrt{2}} \cos \theta$ and $\psi = \frac{1}{\sqrt{2}} \sin \theta$ in (4.6), with θ an arbitrary function of the horizon coordinates, obtaining (4.2).

The stabilizer of the two spinors ϕ_+^r ($r = 1, 2$), with ϕ_+^1 given by (3.3) and ϕ_+^2 given by (4.2) is

- $SU(3)$ if $\cos \theta, \sin \theta \neq 0$,
- G_2 if $\cos \theta = 0$,
- $SU(4)$ if $\sin \theta = 0$,

as we shall show now. The stabilizer of the two spinors is the subgroup of $\text{Spin}(7)$ whose generators satisfy

$$f^{ij}\Gamma_{ij}\phi_+^2 = 0, \quad (4.7)$$

with $f^{ij} \in \text{spin}(7)$. The direct computation shows that $(p, q, r = 2, 3, 4)$

$$\begin{aligned} f^{ij}\Gamma_{ij}\phi_+^2 &= \frac{i}{\sqrt{2}} \cos \theta \left(-2\epsilon^{pqr}(f_{r1}e_{pq} + f_{qr}e_{1p}) \right) \\ &\quad + \frac{1}{\sqrt{2}} \sin \theta \left(2\epsilon^{pqr}e_{1pq}(f_{1r} + f_{\bar{1}r}) - 4e_p(f_1^p + f_{\bar{1}}^p) + 4f_p^p(e_1 - e_{234}) \right) = 0, \end{aligned} \quad (4.8)$$

where we used (3.19). Let us assume now that both $\cos \theta$ and $\sin \theta$ are non-vanishing. Then, using the linear independence of the basis elements e_I , (4.8) imply

$$f_{1p} = f_{1\bar{p}} = f_{pq} = f_{1\bar{1}} = f_p^p = 0, \quad (4.9)$$

that is $f \in \mathfrak{su}(3)$. Now consider $\cos \theta = 0$. Equations (3.19) and (4.8) imply

$$f_{1\bar{1}} = f_p^p = 0, \quad f_{1p} + f_{\bar{1}p} = 0, \quad f_{1p} - f_{\bar{1}p} = \epsilon_p^{\bar{q}\bar{r}} f_{\bar{q}\bar{r}}, \quad (4.10)$$

that is $f \in \mathfrak{g}_2$, where $\mathfrak{g}_2$ is the Lie algebra of G_2 . Eventually, let us consider $\sin \theta = 0$. Equations (3.19) and (4.8) yield

$$f_{\alpha\beta} = 0, \quad f_\alpha^\alpha = 0, \quad (4.11)$$

that is $f \in \mathfrak{su}(4)$.

In solving the KSE (4.1), the three cases outlined above must be treated separately.

4.1 $\text{SU}(3)$ isotropy group

In the $\text{SU}(3)$ case, the second spinor is given by (4.2), with both $\cos \theta$ and $\sin \theta$ non-vanishing. The linear system associated to the second spinor is presented in appendix B. After an extensive computation, the solution of the linear system can be expressed in terms of the $\text{SU}(3)$ gauge-invariant bilinears

$$\langle \phi_+^r, \Gamma_{i_1 \dots i_n} \phi_+^s \rangle. \quad (4.12)$$

The non-trivial bilinears (4.12) are the following

$$e^1, \quad e^\sharp, \quad \hat{\omega}, \quad \hat{\chi}, \quad (4.13)$$

where e^1 is complex and

$$\hat{\omega} := -i\delta_{p\bar{q}}e^{p\bar{q}}, \quad \hat{\chi} := 2\sqrt{2}e^{234} \quad (4.14)$$

are the usual $\text{SU}(3)$ -invariant forms of 6-dimensional $\text{SU}(3)$ -structures. Investigating the linear system (B.1)–(B.20), it turns out that it is convenient to choose as 1-form bilinears the following orthonormal set

$$\begin{aligned} K &:= \frac{1}{\sqrt{2}}(e^1 + e^{\bar{1}}) \\ U &:= \frac{i}{\sqrt{2}} \sin \theta (e^1 - e^{\bar{1}}) + \cos \theta e^\sharp \\ W &:= \frac{i}{\sqrt{2}} \cos \theta (e^1 - e^{\bar{1}}) - \sin \theta e^\sharp, \end{aligned} \quad (4.15)$$

rather than $e^1, e^{\bar{1}}$ and $e^\sharp$.

We have found the following (covariant) geometric conditions (hatted indices are real 6-dimensional SU(3)-indices)

$$(dK)_{\hat{i}\hat{j}} = 0, \quad (4.16)$$

$$(dW)_{\hat{i}\hat{j}} = 0, \quad (4.17)$$

$$\sin \theta (dU)_{\hat{i}\hat{j}} - \frac{1}{2} \hat{\omega}^{\hat{l}_1 \hat{l}_2} \nabla_{[\hat{i}} \Re(\hat{\chi})_{\hat{j}]\hat{l}_1 \hat{l}_2} + \cos \theta \Re(\hat{\chi})_{\hat{i}\hat{j}}^{\hat{l}} ((\mathcal{L}_W K)_l + [K, W]_l) \quad (4.18)$$

$$+ \Im(\hat{\chi})_{\hat{i}\hat{j}}^{\hat{l}} \left[\frac{1}{2} \left(\theta_{\Re(\hat{\chi})}^{(9)} \right)_l + \cos^2 \theta ((\mathcal{L}_U U)_l + (\mathcal{L}_K K)_l) - \left(\cos^2 \theta + \frac{1}{2} \right) \left(\theta_{\hat{\omega}}^{(9)} \right)_l \right] = 0, \quad (4.19)$$

$$J_{(\hat{i}}^{\hat{l}} (\mathcal{L}_U g)_{\hat{j})l} + \frac{1}{2} \sin \theta J_{(\hat{i}}^{\hat{l}} M_{\hat{j})l} = 0, \quad (4.20)$$

$$(\mathcal{L}_U K)_{\hat{i}} = 0, \quad (4.21)$$

$$(\mathcal{L}_U W)_{\hat{i}} = 0, \quad (4.22)$$

$$(\mathcal{L}_W W)_{\hat{i}} - (\mathcal{L}_K K)_{\hat{i}} = 0, \quad (4.23)$$

$$(\mathcal{L}_K W)_{\hat{i}} + (\mathcal{L}_W K)_{\hat{i}} = 0, \quad (4.24)$$

$$-\frac{1}{2} \sin \theta \hat{\omega}^{\hat{j}_1 \hat{j}_2} (\mathcal{L}_W \Re(\hat{\chi}))_{\hat{j}_1 \hat{j}_2 \hat{i}} + \sin^2 \theta (\mathcal{L}_W U)_{\hat{i}} + [W, U]_{\hat{i}} + \cos \theta J_{\hat{i}}^{\hat{j}} [K, U]_{\hat{j}} = 0, \quad (4.25)$$

$$-\frac{1}{2} \sin \theta \hat{\omega}^{\hat{j}_1 \hat{j}_2} (\mathcal{L}_K \Re(\hat{\chi}))_{\hat{j}_1 \hat{j}_2 \hat{i}} + \sin^2 \theta (\mathcal{L}_K U)_{\hat{i}} + [K, U]_{\hat{i}} - \cos \theta J_{\hat{i}}^{\hat{j}} [W, U]_{\hat{j}} = 0, \quad (4.26)$$

$$\frac{1}{4} \hat{\omega}^{\hat{j}_1 \hat{j}_2} (\mathcal{L}_U \Re(\hat{\chi}))_{\hat{j}_1 \hat{j}_2 \hat{i}} + \sin \theta \left(-(\mathcal{L}_U U)_{\hat{i}} + \frac{1}{2} \left(\theta_{\hat{\omega}}^{(9)} \right)_{\hat{i}} \right) = 0, \quad (4.27)$$

$$2 \cot \theta \partial_{\hat{i}} \theta - \left(\theta_{\Re(\hat{\chi})}^{(9)} \right)_{\hat{i}} + \frac{1}{2} \sin \theta (dU)_{\hat{j}_1 \hat{j}_2} \Im(\hat{\chi})^{\hat{j}_1 \hat{j}_2}_{\hat{i}} = 0, \quad (4.28)$$

$$\sin \theta \star_9 (\Re(\hat{\chi}) \wedge \star_9 d\hat{\omega}) - i_U \Theta = 0, \quad (4.29)$$

$$(1 + \cos^2 \theta) \star_9 (\Im(\hat{\chi}) \wedge \star_9 d\hat{\omega}) + \sin \theta i_U \Theta_{\Re(\hat{\chi})}^{(9)} = 0, \quad (4.30)$$

$$i_U (\mathcal{L}_W W - \mathcal{L}_K K) = 0, \quad (4.31)$$

$$i_U (\mathcal{L}_K W + \mathcal{L}_W K) = 0, \quad (4.32)$$

$$(1 + \cos^2 \theta) \left(i_K \mathcal{L}_W K - \frac{1}{2} \star_9 d \star_9 W \right) + \frac{1}{4} \cos \theta \left(\cos \theta i_W \theta_{\Re(\hat{\chi})}^{(9)} - i_K \Theta \right) = 0, \quad (4.33)$$

$$(1 + \cos^2 \theta) \left(i_W \mathcal{L}_K W - \frac{1}{2} \star_9 d \star_9 K \right) + \frac{1}{4} \cos \theta \left(\cos \theta i_K \theta_{\Re(\hat{\chi})}^{(9)} + i_W \Theta \right) = 0, \quad (4.34)$$

$$(1 + \cos^2 \theta) i_K (-\mathcal{L}_W W + \mathcal{L}_U U) + \frac{1}{2} \cos \theta i_W \Theta - \frac{1}{2} i_K \theta_{\Re(\hat{\chi})}^{(9)} = 0, \quad (4.35)$$

$$(1 + \cos^2 \theta) \mathcal{L}_W \theta - \frac{1}{4} \sin(2\theta) i_W \theta_{\Re(\hat{\chi})}^{(9)} + \frac{1}{2} \sin \theta i_K \Theta = 0, \quad (4.36)$$

$$(1 + \cos^2 \theta) \mathcal{L}_K \theta - \frac{1}{4} \sin(2\theta) i_K \theta_{\Re(\hat{\chi})}^{(9)} - \frac{1}{2} \sin \theta i_W \Theta = 0, \quad (4.37)$$

$$(1 + \cos^2 \theta) \mathcal{L}_U \theta - \frac{1}{4} \sin(2\theta) i_U \theta_{\Re(\hat{\chi})}^{(9)} = 0, \quad (4.38)$$

where

$$\theta_{\hat{\omega}}^{(9)} := \star_9 (\hat{\omega} \wedge \star_9 d\hat{\omega}), \quad (4.39)$$

$$\theta_{\Re(\hat{\chi})}^{(9)} := \star_9 (\Re(\hat{\chi}) \wedge \star_9 d\Re(\hat{\chi})), \quad (4.40)$$

are the Lee forms of $\hat{\omega}$ and $\Re(\hat{\chi})$ on $\mathcal{S}$ respectively, $J_i^{\hat{j}} := g^{\hat{j}\hat{l}}\hat{\omega}_{\hat{l}i}$ and

$$\Theta := \star_9(\Re(\hat{\chi}) \wedge \star_9 d\Im(\hat{\chi})), \quad (4.41)$$

$$M_{i\hat{j}} := -\hat{\omega}^{l_1 l_2} \nabla_{(\hat{i}} \Re(\hat{\chi})_{\hat{j}) l_1 l_2}. \quad (4.42)$$

Also, we have found a fully 9-dimensional covariant expression for $d\theta$, which expresses in a compact way the conditions (4.28), (4.36)–(4.38):

$$\begin{aligned} (1 + \cos^2 \theta) \partial_i \theta &= \frac{1}{4} \sin(2\theta) \left(\theta_{\Re(\hat{\chi})}^{(9)} \right)_i + \frac{1}{2} \tan \theta J_i^j \Theta_j \\ &\quad - \frac{1}{4} \frac{\sin^2 \theta}{\cos \theta} (1 + \cos^2 \theta) (dU)_{j_1 j_2} \Im(\hat{\chi})^{j_1 j_2}_i \\ &\quad + \frac{1}{2} \sin \theta ((-i_K \Theta) W_i + (i_W \Theta) K_i), \end{aligned} \quad (4.43)$$

where $J_i^{\hat{j}} \equiv g^{\hat{j}l} \hat{\omega}_{il}$. After having dealt with the geometric conditions, let us consider the fluxes. h and Y are already fixed by the $N = 2$ solution, see (3.34) and (3.35), thus we have to analyse only X . It turns out that all of X is fixed but the totally traceless (2,2) part, that is

$$X_{\bar{p}\bar{q}rs}^{(2,2)\text{TT}} := X_{\bar{p}\bar{q}rs} - 2X_{rt}^t {}_{[\bar{q}} \delta_{\bar{p}]} s + 2X_{st}^t {}_{[\bar{q}} \delta_{\bar{p}]} r + X_{s_1 s_2}^{s_1 s_2} \delta_{r[\bar{p}} \delta_{\bar{q}]} s. \quad (4.44)$$

The 4-form X can be expressed in terms of the SU(3)-gauge invariant bilinears (4.12) as follows

$$\begin{aligned} X &= K \wedge U \wedge W \wedge C_1 + K \wedge U \wedge C_2 + K \wedge W \wedge C_3 + U \wedge W \wedge C_4 + K \wedge C_5 \\ &\quad + U \wedge C_6 + W \wedge C_7 + C_8, \end{aligned} \quad (4.45)$$

where K , U and W are defined in (4.15), $C_1 \in \Omega^1(\mathcal{S})$, $C_2, C_3, C_4 \in \Omega^2(\mathcal{S})$, $C_5, C_6, C_7 \in \Omega^3(\mathcal{S})$ and $C_8 \in \Omega^4(\mathcal{S})$ are fully expressed in terms of the bilinears (4.12), with the exception of C_8 , which contains the unfixed $X^{(2,2)\text{TT}}$. The explicit expression of the C_i ($i = 1, 2, \dots, 8$) is quite involved and we have preferred to list it in appendix C.

To sum up, the full SU(3)-solution is given by

- Geometric conditions: equations (4.16)–(4.38),
- X : equations (C.1)–(C.15),
- Y : equation (3.35),
- h : equation (3.34).

Equations (3.35) and (3.34) are written in Spin(7) notation, but they can be broken down to SU(3) straightforwardly; however, we did not perform explicitly this computation, since it does not add any particular further insight.

4.2 G_2 isotropy group

In this section we solve the KSE (4.1) assuming $\cos \theta = 0$ (thus $\sin \theta = 1$), that is the second spinor (4.2) boils down to

$$\phi_+^2 = \frac{1}{\sqrt{2}}(e_1 + e_{234}). \quad (4.46)$$

The linear system associated to the second spinor is presented in appendix B (set $\cos \theta = 0$, $\sin \theta = 1$ in equations (B.1)–(B.20)). The solution of the linear system, initially expressed in $SU(3)$ representations, after an extensive computation, can be written in terms of the G_2 gauge-invariant bilinears (4.12), which are given by

$$e^\sharp, \quad K, \quad \phi, \quad (4.47)$$

where K is given by the first equation in (4.15) and ϕ is the fundamental G_2 3-form, that is

$$\phi := \Re(\hat{\chi}) + e^7 \wedge \hat{\omega}, \quad (4.48)$$

where $\Re(\hat{\chi})$ and $\hat{\omega}$ are defined by (4.14) and $e^7 \equiv \sqrt{2} \Im(e^1)$ is the seventh direction of the G_2 manifold. We can also construct the 4-form

$$\star_7 \phi = -e^7 \wedge \Im(\hat{\chi}) - \frac{1}{2} \hat{\omega} \wedge \hat{\omega}. \quad (4.49)$$

We have found the following (covariant) geometric conditions ($\tilde{i}, \tilde{j}, \dots$ are 7-dimensional real G_2 indices)

$$(dK)_{\tilde{i}\tilde{j}} = 0, \quad (4.50)$$

$$(de^\sharp)_{\tilde{i}\tilde{j}} = 0, \quad (4.51)$$

$$\phi^{\tilde{l}_1 \tilde{l}_2 \tilde{l}_3} \nabla_{(\tilde{i}} \star_7 \phi_{\tilde{j}) \tilde{l}_1 \tilde{l}_2 \tilde{l}_3} = 0, \quad (4.52)$$

$$(\mathcal{L}_{e^\sharp} e^\sharp)_{\tilde{i}} - (\mathcal{L}_K K)_{\tilde{i}} = 0, \quad (4.53)$$

$$(\mathcal{L}_K e^\sharp)_{\tilde{i}} + (\mathcal{L}_{e^\sharp} K)_{\tilde{i}} = 0, \quad (4.54)$$

$$\phi^{\tilde{j}_1 \tilde{j}_2 \tilde{j}_3} (\mathcal{L}_{e^\sharp} \star_7 \phi)_{\tilde{j}_1 \tilde{j}_2 \tilde{j}_3 \tilde{i}} = 0, \quad (4.55)$$

$$\phi^{\tilde{j}_1 \tilde{j}_2 \tilde{j}_3} (\mathcal{L}_K \star_7 \phi)_{\tilde{j}_1 \tilde{j}_2 \tilde{j}_3 \tilde{i}} = 0, \quad (4.56)$$

$$(\theta_\phi^{(9)})_{\tilde{i}} = 0, \quad (4.57)$$

$$[K, e^\sharp]_{\tilde{i}} - (\mathcal{L}_K e^\sharp)_{\tilde{i}} = 0, \quad (4.58)$$

$$(\star_7 \phi)^{\tilde{j}_1 \tilde{j}_2 \tilde{j}_3 \tilde{j}_4} (d\phi)_{\tilde{j}_1 \tilde{j}_2 \tilde{j}_3 \tilde{j}_4} = 0, \quad (4.59)$$

$$3i_K \mathcal{L}_{e^\sharp} e^\sharp + i_K \theta_\phi^{(9)} = 0, \quad (4.60)$$

$$3i_{e^\sharp} \mathcal{L}_K K + i_{e^\sharp} \theta_\phi^{(9)} = 0, \quad (4.61)$$

where

$$\theta_\phi^{(9)} := \star_9(\phi \wedge \star_9 d\phi) \quad (4.62)$$

is the Lee form of ϕ on $\mathcal{S}$.

Equations (4.50), (4.51), (4.53), (4.54), (4.60) and (4.61) imply that the exterior derivatives of K and $e^\sharp$ on $\mathcal{S}$ can be written as follows

$$dK = \left(-\frac{1}{3}i_{e^\sharp}\theta_\phi^{(9)}\right) K \wedge e^\sharp + K \wedge \mathcal{L}_K K + e^\sharp \wedge \mathcal{L}_{e^\sharp} K, \quad (4.63)$$

$$de^\sharp = \left(\frac{1}{3}i_K\theta_\phi^{(9)}\right) K \wedge e^\sharp - K \wedge \mathcal{L}_{e^\sharp} K + e^\sharp \wedge \mathcal{L}_K K. \quad (4.64)$$

We can use (4.63) and (4.64) to introduce local coordinates on $\mathcal{S}$. In fact, equations (4.63) and (4.64) imply that K and $e^\sharp$ form a 2-dimensional differential ideal on $\mathcal{S}$. Frobenius' theorem then ensures that locally we can choose a coordinate system [47, 48]

$$y^{\tilde{i}}, \quad y^8 := v_1, \quad y^9 := v_2 \quad (4.65)$$

such that

$$K = g_1 dv_1 + g_2 dv_2, \quad e^\sharp = f_1 dv_1 + f_2 dv_2, \quad (4.66)$$

for some functions f_1, f_2, g_1 and g_2 . The slices $v_1 = \text{const}$, $v_2 = \text{const}$ are 2-dimensional integral manifolds, and every slice can be equipped with a G_2 structure with fundamental form given by

$$\phi = \frac{1}{3!}\phi_{l_1 l_2 l_3}(v_1, v_2, y^{\tilde{i}}) dy^{l_1} \wedge dy^{l_2} \wedge dy^{l_3}. \quad (4.67)$$

It is worth noticing that the (2+7)-splitting of $\mathcal{S}$ which emerges in the G_2 case, there exists also in the $\text{SU}(3)$ case, although it is less transparent. In fact, the conditions (4.16), (4.17), (4.21) and (4.22) imply that K and W form a 2-dimensional ideal on $\mathcal{S}$. However, in the $\text{SU}(3)$ case we also have the 1-form bilinear U , but there are no simple conditions on dU , the only condition on it being (4.19).

After having covariantized the geometric conditions, we are left to covariantize X . A somewhat long computation allows to express X in terms of the G_2 gauge invariant bilinears (4.47) as follows

$$\begin{aligned} X = & K \wedge e^\sharp \wedge \left(\frac{1}{4!}\phi^{j_1 j_2 j_3} \nabla_{i_1} \star_7 \phi_{i_2 j_1 j_2 j_3} e^{i_1 i_2}\right) \\ & + \frac{1}{4} \left(K \wedge (\mathcal{L}_{e^\sharp} g)_{i_1 j} - e^\sharp \wedge (\mathcal{L}_K g)_{i_1 j}\right) \phi^j_{i_2 i_3} e^{i_1 i_2 i_3} + X^{27}, \end{aligned} \quad (4.68)$$

where $X^{27} \in \mathbf{27}$ of G_2 is projected out by the KSEs. The definition of the $\mathbf{27}$ component of a 4-form on a 7-dimensional G_2 manifold is given in (D.24).

4.3 $\text{SU}(4)$ isotropy group

In this section we solve the KSE (4.1) assuming $\sin \theta = 0$, i.e. the second spinor (4.2) boils down to

$$\phi_+^2 = \frac{i}{\sqrt{2}}(1 - e_{1234}). \quad (4.69)$$

The linear system associated to the second spinor is given by

$$-\frac{1}{4}h_\alpha - \frac{1}{4}Y_{\sharp\alpha} + \frac{1}{2}\Omega_{\alpha,\mu}{}^\mu + \frac{1}{4}X_{\sharp\alpha\mu}{}^\mu = 0, \quad (4.70)$$

$$X_{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} = 0, \quad (4.71)$$

$$\frac{1}{4}Y_\alpha{}^\beta - \frac{1}{2}\Omega_{\alpha,\sharp}{}^\beta - \frac{1}{4}X_{\alpha\mu}{}^{\mu\beta} + \frac{1}{12}\delta_\alpha{}^\beta \left(-Y_\mu{}^\mu - \frac{1}{2}X_{\mu_1 \mu_2}{}^{\mu_1 \mu_2}\right) = 0, \quad (4.72)$$

$$\frac{1}{2}\Omega_{\alpha,\beta_1\beta_2} + \frac{1}{12}X_{\sharp\alpha\beta_1\beta_2} = 0, \quad (4.73)$$

$$\frac{1}{2}\Omega_{\alpha,\bar{\mu}_1\bar{\mu}_2}\epsilon^{\bar{\mu}_1\bar{\mu}_2}_{\beta_1\beta_2} + \frac{1}{4}X_{\sharp\alpha\bar{\mu}_1\bar{\mu}_2}\epsilon^{\bar{\mu}_1\bar{\mu}_2}_{\beta_1\beta_2} + \frac{1}{6}\epsilon_{\alpha\mu\beta_1\beta_2}\left(Y_{\sharp}^{\mu} - X_{\sharp\nu}^{\nu\mu}\right) = 0, \quad (4.74)$$

$$2\Omega_{\alpha,\beta\sharp} + \frac{1}{3}Y_{\alpha\beta} + \frac{1}{3}X_{\alpha\beta\mu}^{\mu} = 0, \quad (4.75)$$

$$-\frac{1}{3}Y_{\bar{\mu}_1\bar{\mu}_2}\epsilon^{\bar{\mu}_1\bar{\mu}_2}_{\alpha\beta} - \frac{1}{3}X_{\alpha\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3}\epsilon^{\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3}_{\beta} + \frac{1}{3}\epsilon_{\alpha\beta\mu_1\mu_2}X_{\nu}^{\nu\mu_1\mu_2} = 0, \quad (4.76)$$

$$-\frac{1}{2}h_{\alpha} - \frac{1}{6}Y_{\sharp\alpha} - \Omega_{\alpha,\mu}^{\mu} - \frac{1}{6}X_{\sharp\alpha\mu}^{\mu} = 0, \quad (4.77)$$

$$X_{\sharp\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3}\epsilon^{\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3}_{\alpha} = 0, \quad (4.78)$$

$$-\frac{1}{2}h_{\sharp} + \Omega_{\sharp,\mu}^{\mu} - \frac{1}{6}Y_{\mu}^{\mu} - \frac{1}{12}X_{\mu_1\mu_2}^{\mu_1\mu_2} = 0, \quad (4.79)$$

$$\frac{1}{3}Y_{\sharp\alpha} - \Omega_{\sharp,\alpha} + \frac{1}{3}X_{\sharp\alpha\mu}^{\mu} = 0, \quad (4.80)$$

$$\Omega_{\sharp,\alpha\beta} - \frac{1}{6}Y_{\alpha\beta} - \frac{1}{6}X_{\gamma}^{\gamma}{}_{\alpha\beta} = 0, \quad (4.81)$$

$$\frac{1}{2}\Omega_{\sharp,\bar{\mu}_1\bar{\mu}_2}\epsilon^{\bar{\mu}_1\bar{\mu}_2}_{\alpha\beta} + \frac{1}{12}\epsilon^{\bar{\mu}_1\bar{\mu}_2}_{\alpha\beta}\left(-Y_{\bar{\mu}_1\bar{\mu}_2} + X_{\nu}^{\nu}{}_{\bar{\mu}_1\bar{\mu}_2}\right) = 0. \quad (4.82)$$

Combining (4.70)–(4.82) with the linear system of ϕ_+^1 , namely equation (A.1)–(A.7), we can solve the linear system resulting from these two lots of equations and express the solution in terms of the SU(4) gauge-invariant bilinears (4.12), which are given by

$$e^{\sharp}, \quad \omega, \quad \chi, \quad (4.83)$$

where ω and χ are defined by (3.32). We have found the following geometric conditions ($\check{i}, \check{j}, \dots$ are 8-dimensional real SU(4) indices)

$$(d\omega)^{\check{j}_1\check{j}_2\check{j}_3}\Re(\chi)_{\check{j}_1\check{j}_2\check{j}_3\check{i}} = 0, \quad (4.84)$$

$$\frac{1}{2}(\star_9 d \star_9 \Re(\chi))_{\check{i}_1\check{i}_2\check{i}_3} + \left(\frac{1}{2}(\theta_{\omega}^{(9)})_{\check{j}} - (\mathcal{L}_{e^{\sharp}}e^{\sharp})_{\check{j}}\right)\Re(\chi)_{\check{i}_1\check{i}_2\check{i}_3}^{\check{j}} = 0, \quad (4.85)$$

$$\frac{1}{12}\mathcal{L}_{e^{\sharp}}\Re(\chi)_{\check{j}_1\check{j}_2\check{j}_3[\check{i}_1}\Re(\chi)_{\check{i}_2]}^{\check{j}_1\check{j}_2\check{j}_3} + \omega_{\check{i}_1\check{i}_2}(\mathcal{L}_{e^{\sharp}}e^{\sharp})_{\check{j}_1\check{j}_2}^{\check{j}_1\check{j}_2}\omega^{\check{j}_1\check{j}_2} = 0, \quad (4.86)$$

$$J_{(\check{i}_1}^{\check{j}}(\mathcal{L}_{e^{\sharp}}g)_{\check{i}_2)\check{j}} = 0. \quad (4.87)$$

Moreover, the fluxes can be covariantized as follows

$$h = \frac{1}{2}(\star_9 d \star_9 e^{\sharp})e^{\sharp} - \mathcal{L}_{e^{\sharp}}e^{\sharp} + \theta_{\omega}^{(9)}, \quad (4.88)$$

$$Y = -d_h e^{\sharp} + \frac{1}{2}(de^{\sharp})_{j_1 j_2} \omega^{j_1 j_2} \omega, \quad (4.89)$$

$$X = \frac{1}{2}e^{\sharp} \wedge \left\{ \star_9 d \star_9 (\omega \wedge \omega) + \frac{1}{3!} \left((\theta_{\omega}^{(9)})_j - \frac{5}{4}(\mathcal{L}_{e^{\sharp}}e^{\sharp})_j \right) (\omega \wedge \omega)^j{}_{i_1 i_2 i_3} e^{i_1 i_2 i_3} \right\} \\ - S \wedge \omega + X^{\text{TT}(2,2)}, \quad (4.90)$$

where

$$\theta_{\Re(\chi)}^{(9)} := \star_9 (\Re(\chi) \wedge \star_9 d \Re(\chi)), \quad (4.91)$$

$$\theta_{\omega}^{(9)} := \star_9 (\omega \wedge \star_9 d \omega), \quad (4.92)$$

are the Lee forms of $\Re(\chi)$ and ω on $\mathcal{S}$ respectively,

$$S_{ij} := J_{[i}{}^l \left((de^\sharp)_{j]l} + \frac{1}{2}(\mathcal{L}_{e^\sharp} g)_{j]l} \right), \quad (4.93)$$

and $X^{\text{TT}(2,2)}$ is the $\text{SU}(4)$ -traceless $(2,2)$ part of X , that is

$$X_{\bar{\alpha}\bar{\beta}\gamma\delta}^{\text{TT}(2,2)} := X_{\bar{\alpha}\bar{\beta}\gamma\delta} - X_{\gamma\lambda}{}^\lambda{}_{[\bar{\beta}}\delta_{\bar{\alpha}]\delta} + X_{\delta\lambda}{}^\lambda{}_{[\bar{\beta}}\delta_{\bar{\alpha}]\gamma} + \frac{1}{3}X_{\mu\nu}{}^{\mu\nu}\delta_{\gamma[\bar{\alpha}}\delta_{\bar{\beta}]\delta}. \quad (4.94)$$

5 Integrability conditions

In this section we shall show that all the components of the Einstein equation (2.8) are implied by the 11-dimensional KSE (2.11), the gauge field equation (2.5) and the Bianchi identities $dF = 0$. For a generic supersymmetric solution of $D = 11$ supergravity, this is not always true. In particular, as shown in [44], for a $N = 1$ supersymmetric solution which generates a null gauge-invariant isometry, it is known that not all of the components of the Einstein equation are implied — one component along the null-direction of the isometry must be imposed by hand.

However, for near-horizon solutions, we shall demonstrate that all the components of the Einstein equation are implied, irrespective of whether or not Δ vanishes. This analysis is purely local, and does not require any assumptions on global properties of the solution. To investigate the integrability conditions, we assume that the KSE (2.11) are satisfied. Then the integrability conditions

$$[\mathcal{D}_M, \mathcal{D}_N]\epsilon = 0 \quad (5.1)$$

yield [49]

$$\begin{aligned} 0 = & E_{MN}\Gamma^N\epsilon - \frac{1}{36}\star\left(d\star F + \frac{1}{2}F\wedge F\right)_{N_1N_2N_3}\left(\Gamma_M{}^{N_1N_2N_3} - 6\delta_M{}^{N_1}\Gamma^{N_2N_3}\right)\epsilon \\ & - \frac{1}{6!}(dF)_{N_1N_2N_3N_4N_5}\left(\Gamma_M{}^{N_1N_2N_3N_4N_5} - 10\delta_M{}^{N_1}\Gamma^{N_2N_3N_4N_5}\right)\epsilon \end{aligned} \quad (5.2)$$

where we have denoted

$$E_{MN} := R_{MN} - \frac{1}{12}F_{ML_1L_2L_3}F_N{}^{L_1L_2L_3} + \frac{1}{144}g_{MN}F^2. \quad (5.3)$$

Enforcing the gauge field equation (2.5) and the Bianchi identities, (5.2) boils down to

$$E_{MN}\Gamma^N\epsilon = 0. \quad (5.4)$$

In the analysis of the integrability conditions, we shall not assume any of the results which follow from the global assumptions made in the previous sections. In particular, on integrating up the KSE along the lightcone directions, we find the following Killing spinors

$$\epsilon_1 = \phi_- + u\Gamma_+\Theta_-\phi_- + ur\Gamma_-\Theta_+\Gamma_+\Theta_-\phi_-, \quad \epsilon_2 = \phi_+ + r\Gamma_-\Theta_+\phi_+. \quad (5.5)$$

Two separate cases must be considered. The first case corresponds to $\phi_- \neq 0$. Consider

$$E_{MN}\Gamma^N\epsilon_1 = 0, \quad (5.6)$$

where $M, N = +, -, i$. Setting $u = 0$ in (5.6), we obtain

$$E_{MN}\Gamma^N\phi_- = 0, \quad (5.7)$$

which is equivalent to

$$E_{M-}\Gamma_+\phi_- + E_{Mi}\Gamma^i\phi_- = 0. \quad (5.8)$$

The first term in (5.8) has positive (lightcone) chirality, while the second has negative (lightcone) chirality, thus (5.8) holds iff both terms vanish independently, which in turn imply that

$$E_{M-} = E_{Mi} = 0. \quad (5.9)$$

This means that all the components of E_{MN} must vanish, apart from E_{++} . However, as is shown in [26], in the near-horizon limit, $E_{++} = 0$ is a consequence of the other bosonic field equations, thus indeed $E_{MN} = 0$ for all M, N .

The second case corresponds to $\phi_- = 0$ and $\phi_+ \neq 0$. Consider

$$E_{MN}\Gamma^N\epsilon_2 = 0. \quad (5.10)$$

This equation boils down to

$$E_{M+}\Gamma_-\phi_+ + 2rE_{M-}\Theta_+\phi_+ + E_{Mi}\Gamma^i(\phi_+ + r\Gamma_-\Theta_+\phi_+) = 0. \quad (5.11)$$

The first and the fourth terms in (5.11) have negative (lightcone) chirality, while the second and the third ones have positive (lightcone) chirality, thus (5.8) holds iff both terms vanish independently, which in turn imply that

$$E_{M+}\phi_+ - rE_{Mi}\Gamma^i\Theta_+\phi_+ = 0 \quad (5.12)$$

and

$$2rE_{M-}\Theta_+\phi_+ + E_{Mi}\Gamma^i\phi_+ = 0. \quad (5.13)$$

Taking the component $M = -$ of (5.12), and exploiting the identities $E_{--} = 0, E_{-i} = 0$ (these expressions hold automatically for all near-horizon metrics), we get $E_{+-} = 0$. Taking the component $M = j$ of (5.13), we obtain $E_{ij} = 0$. Taking the component $M = +$ of (5.13), using $E_{+-} = 0$, we get $E_{+i} = 0$. Eventually, taking the $M = +$ component of (5.12), we get $E_{++} = 0$. Thus $E_{MN} = 0$ for all M, N .

Thus, we have shown that for near-horizon geometries all the components of the Einstein equation (2.8) are implied by the 11-dimensional KSE (2.11), the gauge field equation (2.5) and the Bianchi identities $dF = 0$.

6 Conclusion

We have classified the conditions imposed on the geometry and 4-form flux obtained from requiring that an extreme near-horizon geometry in $D = 11$ supergravity preserves $N = 4$ supersymmetry. This analysis is tractable because previous global analysis in [25] has been used to reduce the calculation to that of solving

$$\nabla_i^{(+)}\phi_+^r = 0, \quad r = 1, 2 \quad (6.1)$$

and for $N = 4$ solutions, there is sufficient $\text{Spin}(9)$ gauge freedom present to enable one to write $\{\phi_+^1, \phi_+^2\}$ in one of three simple canonical forms, on exploiting the condition $\langle \phi_+^r, \phi_+^s \rangle = \delta^{rs}$. In particular, the common isotropy group of $\{\phi_+^1, \phi_+^2\}$ is one of three classes; $\text{SU}(3)$, G_2 or $\text{SU}(4)$. For each class, the conditions on the geometry and fluxes have been expressed in a fully gauge-invariant fashion, in terms of the gauge-invariant spinor bilinears corresponding to each of these cases.

There are a number of additional issues relevant to these $N = 4$ solutions which would be interesting to explore further. First, although we have mentioned in section 2.3 that a generic $N = 4$ near-horizon solution admits two commuting rotational isometries on $\mathcal{S}$, $\check{V}$ and $Z = Z^{12}$, it remains to determine how these isometries relate to the geometric structures on $\mathcal{S}$ which we have derived in section 4. In practice, it is rather difficult to do this, because the isometries $\check{V}$ and Z are constructed from both ϕ_+ and ϕ_- spinors. We have used the majority of the gauge freedom to simplify the canonical forms of the ϕ_+ spinors. In general, the ϕ_- spinors are rather complicated in form, as in the absence of useful gauge transformations to simplify these, one must instead utilize (2.22) and (2.23) to relate ϕ_+ to ϕ_- , and as a result there is a non-trivial appearance of various flux terms in the explicit expressions for the isometries.

Furthermore, it would be useful to see if there are any further conditions on the $N = 4$ spinors from global analysis. In particular, we have seen that for the $\text{SU}(3)$ isotropy group case, there is a function θ which appears in the spinor ϕ_+^2 . Although the manner in which this function appears explicitly is dependent on the gauge choice we have made, one can straightforwardly write $\cos 2\theta$ in terms of gauge-invariant bilinears, and hence $\cos 2\theta$ is a globally well-defined and smooth function on $\mathcal{S}$. In addition, the analysis of the KSE produces an expression for $d(\cos 2\theta)$ in terms of various $\text{SU}(3)$ invariant bilinears (4.43). As we already have a constant norm condition, $\langle \phi_+^r, \phi_+^s \rangle = \delta^{rs}$ which is obtained via a (global) maximum principle argument, [25], it is natural to enquire if θ can be shown to be constant via an analogous analysis. However, on computing the Laplacian of $\cos 2\theta$, there does not appear, a priori, to be any way of controlling the sign of the resulting terms in a way which is compatible with a maximum principle argument. We remark that there is also no immediate contradiction obtained if one assumes that θ is constant, with $\sin \theta \neq 0$, $\cos \theta \neq 0$; this would appear to be simply a special case of the more general analysis. It may also be the case that other properties of the geometry in the $N = 4$ solutions could in principle be further constrained via similar global analysis, and it would be interesting to explore this further.

Acknowledgments

DF is partially supported by the STFC DTP Grant ST/S505742. JG is supported by the STFC Consolidated Grant ST/L000490/1. The authors would like to thank Martin Wolf for useful conversations.

A $N = 2$ SU(4) KSE: linear system

The linear system obtained from the spinorial geometry analysis of the KSE of the $N = 2$ supersymmetric near-horizon geometries is:

$$-\frac{1}{4}h_\alpha - \frac{1}{4}Y_{\sharp\alpha} + \frac{1}{2}\Omega_{\alpha,\mu}{}^\mu + \frac{1}{4}X_{\sharp\alpha\mu}{}^\mu = 0, \quad (\text{A.1})$$

$$Y_\alpha{}^\beta - 2\Omega_{\alpha,\sharp}{}^\beta - X_{\alpha\mu}{}^{\mu\beta} + \frac{1}{3}\delta_\alpha{}^\beta \left(-Y_\mu{}^\mu - \frac{1}{2}X_{\mu_1\mu_2}{}^{\mu_1\mu_2} - \frac{1}{12}X_{\mu_1\mu_2\mu_3\mu_4}\epsilon^{\mu_1\mu_2\mu_3\mu_4} \right) = 0, \quad (\text{A.2})$$

$$\begin{aligned} \frac{1}{2}\Omega_{\alpha,\beta_1\beta_2} - \frac{1}{4}\Omega_{\alpha,\bar{\mu}_1\bar{\mu}_2}\epsilon^{\bar{\mu}_1\bar{\mu}_2}{}_{\beta_1\beta_2} - \frac{1}{8}X_{\sharp\alpha\bar{\mu}_1\bar{\mu}_2}\epsilon^{\bar{\mu}_1\bar{\mu}_2}{}_{\beta_1\beta_2} + \frac{1}{12}X_{\sharp\alpha\beta_1\beta_2} \\ + \frac{1}{12}\epsilon_{\alpha\mu\beta_1\beta_2} \left(-Y_\sharp{}^\mu + X_{\sharp\nu}{}^{\nu\mu} \right) = 0, \end{aligned} \quad (\text{A.3})$$

$$Y_{\alpha\beta} + \Omega_{\alpha,\beta\sharp} + Y_{\bar{\mu}_1\bar{\mu}_2}\epsilon^{\bar{\mu}_1\bar{\mu}_2}{}_{\alpha\beta} + X_{\alpha\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3}\epsilon^{\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3}{}_\beta + X_{\alpha\beta\mu}{}^\mu - \epsilon_{\alpha\beta\mu_1\mu_2}X_{\nu}{}^{\nu\mu_1\mu_2} = 0, \quad (\text{A.4})$$

$$-\frac{1}{4}h_\alpha - \frac{1}{12}Y_{\sharp\alpha} - \frac{1}{2}\Omega_{\alpha,\mu}{}^\mu - \frac{1}{12}X_{\sharp\alpha\mu}{}^\mu + \frac{1}{18}X_{\sharp\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3}\epsilon^{\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3}{}_\alpha = 0, \quad (\text{A.5})$$

$$-\frac{1}{4}h_\sharp + \frac{1}{2}\Omega_{\sharp,\mu}{}^\mu - \frac{1}{12}Y_\mu{}^\mu - \frac{1}{24}X_{\mu_1\mu_2}{}^{\mu_1\mu_2} + \frac{1}{72}X_{\mu_1\mu_2\mu_3\mu_4}\epsilon^{\mu_1\mu_2\mu_3\mu_4} = 0, \quad (\text{A.6})$$

$$\frac{1}{6}Y_{\sharp\alpha} - \frac{1}{2}\Omega_{\sharp,\alpha\sharp} + \frac{1}{6}X_{\sharp\alpha\mu}{}^\mu + \frac{1}{18}X_{\sharp\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3}\epsilon^{\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3}{}_\alpha = 0, \quad (\text{A.7})$$

$$\frac{1}{2}\Omega_{\sharp,\alpha\beta} - \frac{1}{4}\Omega_{\sharp,\bar{\mu}_1\bar{\mu}_2}\epsilon^{\bar{\mu}_1\bar{\mu}_2}{}_{\alpha\beta} - \frac{1}{12}Y_{\alpha\beta} - \frac{1}{12}X_{\gamma}{}^\gamma{}_{\alpha\beta} + \frac{1}{24}\epsilon^{\bar{\mu}_1\bar{\mu}_2}{}_{\alpha\beta} \left(Y_{\bar{\mu}_1\bar{\mu}_2} - X_{\nu}{}^\nu{}_{\bar{\mu}_1\bar{\mu}_2} \right) = 0. \quad (\text{A.8})$$

B $N = 4$ SU(3) KSE: linear system

The $N = 4$ linear system associated to the KSE (4.1), with ϕ_+^2 given by (4.2), for both $\cos\theta$ and $\sin\theta$ non-vanishing, is given by ($p = 2, 3, 4$)

$$\begin{aligned} \frac{i}{\sqrt{2}}\partial_1 \cos\theta + \frac{i}{\sqrt{2}}\cos\theta \left(-\frac{1}{4}h_1 - \frac{1}{4}Y_{\sharp 1} + \frac{1}{2}\Omega_{1,1\bar{1}} + \frac{1}{2}\Omega_{1,p}{}^p + \frac{1}{4}X_{\sharp 1p}{}^p \right) \\ + \sin\theta \left(\frac{1}{2}\Omega_{1,\sharp 1} + \frac{1}{12}X_{1pqr}\epsilon^{pqr} \right) = 0, \end{aligned} \quad (\text{B.1})$$

$$\begin{aligned} i \cos\theta \left(\frac{1}{6}Y_{1\bar{1}} - \frac{1}{2}\Omega_{1,\sharp\bar{1}} - \frac{1}{12}Y_p{}^p - \frac{1}{6}X_{1\bar{1}p}{}^p - \frac{1}{24}X_{pq}{}^{pq} + \frac{1}{36}X_{1pqr}\epsilon^{pqr} \right) \\ + \frac{1}{\sqrt{2}}\partial_1 \sin\theta + \frac{1}{\sqrt{2}}\sin\theta \left(-\frac{1}{4}h_1 + \frac{1}{12}Y_{\sharp 1} + \frac{1}{2}\Omega_{1,p}{}^p \right. \\ \left. - \frac{1}{2}\Omega_{1,1\bar{1}} - \frac{1}{12}X_{\sharp 1p}{}^p - \frac{1}{18}X_{\sharp pqr}\epsilon^{pqr} \right) = 0, \end{aligned} \quad (\text{B.2})$$

$$\begin{aligned} i \cos\theta \left(\frac{1}{4}Y_1{}^p - \frac{1}{2}\Omega_{1,\sharp}{}^p - \frac{1}{4}X_{1t}{}^{tp} \right) + \frac{1}{\sqrt{2}}\sin\theta \left(-\Omega_{1,1}{}^p \right. \\ \left. - \frac{1}{2}\Omega_{1,qr}\epsilon^{pqr} + \frac{1}{4}X_{\sharp 1qr}\epsilon^{pqr} \right) = 0, \end{aligned} \quad (\text{B.3})$$

$$\begin{aligned} \frac{i}{\sqrt{2}}\cos\theta \left(\Omega_{1,\bar{1}}{}^p + \frac{1}{6}Y_{\sharp}{}^p + \frac{1}{3}X_{\sharp 1\bar{1}}{}^p - \frac{1}{6}X_{\sharp q}{}^{qp} + \frac{1}{2}\Omega_{1,qr}\epsilon^{pqr} \right. \\ \left. + \frac{1}{12}X_{\sharp 1qr}\epsilon^{pqr} \right) + \sin\theta \left(-\frac{1}{12}Y_1{}^p - \frac{1}{2}\Omega_{1,\sharp}{}^p + \frac{1}{12}Y_{qr}\epsilon^{pqr} \right) \end{aligned}$$

$$+\frac{1}{12}X_{1t}{}^{tp}+\frac{1}{6}X_{1\bar{1}qr}\epsilon^{pqr}+\frac{1}{36}X_{qrs}{}^p\epsilon^{qrs}\Big)=0, \quad (\text{B.4})$$

$$\begin{aligned} &\frac{i}{\sqrt{2}}\cos\theta\left(\frac{1}{2}\Omega_{1,}{}^{qr}\epsilon_{pqr}+\frac{1}{4}X_{\sharp 1}{}^{qr}\epsilon_{pqr}+\Omega_{1,1p}\right) \\ &+\sin\theta\left(\frac{1}{4}Y_{1p}+\frac{1}{2}\Omega_{1,\sharp p}+\frac{1}{4}X_{1pq}{}^q\right)=0, \end{aligned} \quad (\text{B.5})$$

$$\begin{aligned} &i\cos\theta\left(-\frac{1}{6}X_{1\bar{1}}{}^{qr}\epsilon_{pqr}-\frac{1}{12}Y^{qr}\epsilon_{pqr}+\frac{1}{12}X_s{}^{sqr}\epsilon_{pqr}+\frac{1}{12}Y_{1p}\right. \\ &\left.-\frac{1}{2}\Omega_{1,\sharp p}+\frac{1}{12}X_{1pq}{}^q\right)+\frac{1}{\sqrt{2}}\sin\theta\left(\frac{1}{2}\Omega_{1,}{}^{qr}\epsilon_{pqr}+\Omega_{1,\bar{1}p}-\frac{1}{6}Y_{\sharp p}\right. \\ &\left.-\frac{1}{12}X_{\sharp 1}{}^{qr}\epsilon_{pqr}-\frac{1}{3}X_{\sharp p\bar{1}}-\frac{1}{6}X_{\sharp pq}{}^q\right)=0. \end{aligned} \quad (\text{B.6})$$

$$\begin{aligned} &i\cos\theta\left(-\frac{1}{12}X_1{}^{pqr}\epsilon_{pqr}+\frac{1}{2}\Omega_{1,\sharp 1}\right)+\frac{1}{\sqrt{2}}\partial_1\sin\theta \\ &+\frac{1}{\sqrt{2}}\sin\theta\left(-\frac{1}{4}h_1+\frac{1}{4}Y_{\sharp 1}+\frac{1}{2}\Omega_{1,1\bar{1}}-\frac{1}{2}\Omega_{1,p}{}^p+\frac{1}{4}X_{\sharp 1p}{}^p\right)=0, \end{aligned} \quad (\text{B.7})$$

$$\begin{aligned} &-\frac{i}{\sqrt{2}}\partial_1\cos\theta+\frac{i}{\sqrt{2}}\cos\theta\left(-\frac{1}{18}X_{\sharp}{}^{pqr}\epsilon_{pqr}+\frac{1}{4}h_1+\frac{1}{12}Y_{\sharp 1}+\frac{1}{2}\Omega_{1,1\bar{1}}\right. \\ &+\frac{1}{2}\Omega_{1,p}{}^p+\frac{1}{12}X_{\sharp 1p}{}^p\Big)+\sin\theta\left(\frac{1}{6}Y_{1\bar{1}}+\frac{1}{2}\Omega_{1,\sharp\bar{1}}+\frac{1}{12}Y_p{}^p\right. \\ &+\frac{1}{6}X_{1\bar{1}p}{}^p+\frac{1}{36}X_1{}^{pqr}\epsilon_{pqr}-\frac{1}{24}X_{pq}{}^{pq}\Big)=0, \end{aligned} \quad (\text{B.8})$$

$$\begin{aligned} &\frac{i}{\sqrt{2}}\partial_p\cos\theta+\frac{i}{\sqrt{2}}\cos\theta\left(-\frac{1}{4}h_p-\frac{1}{4}Y_{\sharp p}+\frac{1}{2}\Omega_{p,1\bar{1}}+\frac{1}{2}\Omega_{p,q}{}^q\right. \\ &+\frac{1}{4}X_{\sharp p\bar{1}}+\frac{1}{4}X_{\sharp pq}{}^q\Big)+\sin\theta\left(\frac{1}{4}Y_{p1}+\frac{1}{2}\Omega_{p,\sharp 1}+\frac{1}{4}X_{1pq}{}^q\right)=0, \end{aligned} \quad (\text{B.9})$$

$$\begin{aligned} &i\cos\theta\left(\frac{1}{4}Y_{p\bar{1}}-\frac{1}{2}\Omega_{p,\sharp\bar{1}}+\frac{1}{4}X_{\bar{1}pq}{}^q\right)+\frac{1}{\sqrt{2}}\partial_p\sin\theta \\ &+\frac{1}{\sqrt{2}}\sin\theta\left(-\frac{1}{4}h_p+\frac{1}{4}Y_{\sharp p}+\frac{1}{2}\Omega_{p,q}{}^q-\frac{1}{2}\Omega_{p,1\bar{1}}+\frac{1}{4}X_{\sharp p\bar{1}}-\frac{1}{4}X_{\sharp pq}{}^q\right)=0, \end{aligned} \quad (\text{B.10})$$

$$\begin{aligned} &i\cos\theta\left(\frac{1}{4}Y_p{}^t-\frac{1}{2}\Omega_{p,\sharp}{}^t-\frac{1}{12}\delta_p{}^tY_{1\bar{1}}-\frac{1}{12}\delta_p{}^tY_s{}^s-\frac{1}{4}X_{1\bar{1}p}{}^t\right. \\ &+\frac{1}{4}X_{pq}{}^{tq}+\frac{1}{12}\delta_p{}^tX_{1\bar{1}q}{}^q-\frac{1}{24}\delta_p{}^tX_{mn}{}^{mn}+\frac{1}{36}\delta_p{}^tX_{1qrs}\epsilon^{qrs}\Big) \\ &+\frac{1}{\sqrt{2}}\sin\theta\left(-\Omega_{p,1}{}^t-\frac{1}{2}\Omega_{p,qr}\epsilon^{qrt}-\frac{1}{6}\delta_p{}^tY_{\sharp 1}-\frac{1}{2}X_{\sharp 1p}{}^t+\frac{1}{4}X_{\sharp pqr}\epsilon^{qrt}\right. \\ &+\frac{1}{6}\delta_p{}^tX_{\sharp 1q}{}^q-\frac{1}{18}\delta_p{}^tX_{\sharp qrs}\epsilon^{qrs}\Big)=0, \end{aligned} \quad (\text{B.11})$$

$$\begin{aligned} &\frac{i}{\sqrt{2}}\cos\theta\left(\Omega_{p,\bar{1}}{}^t-\frac{1}{6}\delta_p{}^tY_{\sharp\bar{1}}-\frac{1}{2}X_{\sharp\bar{1}p}{}^t+\frac{1}{6}\delta_p{}^tX_{\sharp\bar{1}q}{}^q+\frac{1}{2}\Omega_{p,qr}\epsilon^{qrt}\right. \\ &+\frac{1}{4}X_{\sharp pqr}\epsilon^{qrt}-\frac{1}{18}\delta_p{}^tX_{\sharp qrs}\epsilon^{qrs}\Big)+\sin\theta\left(-\frac{1}{4}Y_p{}^t-\frac{1}{2}\Omega_{p,\sharp}{}^t\right. \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{12} \delta_p^t Y_s^s - \frac{1}{12} \delta_p^t Y_{1\bar{1}} - \frac{1}{4} X_{1\bar{1}p}^t - \frac{1}{4} X_{pq}^{tq} - \frac{1}{4} X_{\bar{1}pqr} \epsilon^{qtr} \\
 & + \frac{1}{12} \delta_p^t X_{1\bar{1}q}^q + \frac{1}{24} \delta_p^t X_{mn}^{mn} + \frac{1}{18} \delta_p^t X_{\bar{1}pqr} \epsilon^{pqr} \Big) = 0, \quad (\text{B.12})
 \end{aligned}$$

$$\begin{aligned}
 & \frac{i}{\sqrt{2}} \cos \theta \left(\frac{1}{2} \Omega_{p,qr} \epsilon_{qrs} + \frac{1}{4} X_{\sharp p}^{qr} \epsilon_{qrs} + \frac{1}{6} \epsilon_{prs} Y_{\sharp}^r - \frac{1}{6} \epsilon_{prs} X_{\sharp 1\bar{1}}^r - \frac{1}{6} \epsilon_{prs} X_{\sharp t}^{tr} \right. \\
 & \left. - \Omega_{p,s1} - \frac{1}{6} X_{\sharp 1ps} \right) + \sin \theta \left(\frac{1}{12} Y_{ps} + \frac{1}{2} \Omega_{p,\sharp s} + \frac{1}{6} \epsilon_{prs} Y_1^r - \frac{1}{4} X_{p1}^{qr} \epsilon_{qrs} \right. \\
 & \left. - \frac{1}{12} X_{1\bar{1}ps} + \frac{1}{12} X_{psr}^r - \frac{1}{6} \epsilon_{prs} X_{1t}^{tr} \right) = 0, \quad (\text{B.13})
 \end{aligned}$$

$$\begin{aligned}
 & i \cos \theta \left(-\frac{1}{4} X_{p\bar{1}}^{qr} \epsilon_{qrs} + \frac{1}{6} \epsilon_{prs} Y_{\bar{1}}^r - \frac{1}{6} \epsilon_{prs} X_{\bar{1}t}^{tr} + \frac{1}{12} Y_{ps} - \frac{1}{2} \Omega_{p,\sharp s} \right. \\
 & \left. + \frac{1}{12} X_{psr}^r + \frac{1}{12} X_{1\bar{1}ps} \right) + \frac{1}{\sqrt{2}} \sin \theta \left(\frac{1}{2} \Omega_{p,qr} \epsilon_{qrs} - \Omega_{p,s\bar{1}} - \frac{1}{6} Y_{\sharp}^r \epsilon_{prs} \right. \\
 & \left. - \frac{1}{4} X_{\sharp p}^{qr} \epsilon_{qrs} + \frac{1}{6} X_{\sharp 1ps} - \frac{1}{6} X_{\sharp 1\bar{1}}^q \epsilon_{pqs} + \frac{1}{6} \epsilon_{pqs} X_{\sharp r}^{rq} \right) = 0, \quad (\text{B.14})
 \end{aligned}$$

$$\begin{aligned}
 & i \cos \theta \left(-\frac{1}{12} Y^{qr} \epsilon_{pqr} - \frac{1}{12} X_p^{qrs} \epsilon_{qrs} + \frac{1}{12} X_{1\bar{1}}^{qr} \epsilon_{pqr} + \frac{1}{12} X_s^{sqr} \epsilon_{pqr} \right. \\
 & \left. + \frac{1}{12} Y_{1p} + \frac{1}{2} \Omega_{p,\sharp 1} + \frac{1}{12} X_{1pq}^q \right) + \frac{1}{\sqrt{2}} \partial_p \sin \theta + \frac{1}{\sqrt{2}} \sin \theta \left(-\frac{1}{4} h_p \right. \\
 & \left. + \frac{1}{12} Y_{\sharp p} + \frac{1}{2} \Omega_{p,1\bar{1}} - \frac{1}{2} \Omega_{p,q}^q - \frac{1}{12} X_{\sharp p 1\bar{1}} + \frac{1}{12} X_{\sharp pq}^q + \frac{1}{6} X_{\sharp 1}^{qr} \epsilon_{pqr} \right) = 0, \quad (\text{B.15})
 \end{aligned}$$

$$\begin{aligned}
 & -\frac{i}{\sqrt{2}} \partial_p \cos \theta + \frac{i}{\sqrt{2}} \cos \theta \left(\frac{1}{6} X_{\sharp \bar{1}}^{qr} \epsilon_{pqr} + \frac{1}{4} h_p + \frac{1}{12} Y_{\sharp p} + \frac{1}{2} \Omega_{p,1\bar{1}} \right. \\
 & \left. + \frac{1}{2} \Omega_{p,q}^q + \frac{1}{12} X_{\sharp p 1\bar{1}} + \frac{1}{12} X_{\sharp pq}^q \right) + \sin \theta \left(-\frac{1}{12} Y_{\bar{1}p} + \frac{1}{2} \Omega_{p,\sharp \bar{1}} + \frac{1}{12} Y^{qr} \epsilon_{pqr} \right. \\
 & \left. - \frac{1}{12} X_{\bar{1}pq}^q + \frac{1}{12} X_p^{qrs} \epsilon_{qrs} + \frac{1}{12} X_{1\bar{1}}^{qr} \epsilon_{pqr} - \frac{1}{12} X_t^{tqr} \epsilon_{pqr} \right) = 0, \quad (\text{B.16})
 \end{aligned}$$

$$\begin{aligned}
 & \frac{i}{\sqrt{2}} \partial_{\sharp} \cos \theta + \frac{i}{\sqrt{2}} \cos \theta \left(-\frac{1}{4} h_{\sharp} + \frac{1}{2} \Omega_{\sharp,1\bar{1}} + \frac{1}{2} \Omega_{\sharp,p}^p - \frac{1}{12} Y_{1\bar{1}} \right. \\
 & \left. - \frac{1}{12} Y_p^p + \frac{1}{12} X_{1\bar{1}p}^p - \frac{1}{24} X_{pq}^{pq} - \frac{1}{18} X_{1pqr} \epsilon^{pqr} \right) \\
 & + \sin \theta \left(\frac{1}{6} Y_{\sharp 1} + \frac{1}{2} \Omega_{\sharp,\sharp 1} - \frac{1}{6} X_{\sharp 1p}^p + \frac{1}{18} X_{\sharp pqr} \epsilon^{pqr} \right) = 0, \quad (\text{B.17})
 \end{aligned}$$

$$\begin{aligned}
 & i \cos \theta \left(\frac{1}{6} Y_{\sharp \bar{1}} - \frac{1}{2} \Omega_{\sharp,\sharp \bar{1}} - \frac{1}{6} X_{\sharp \bar{1}p}^p + \frac{1}{18} X_{\sharp pqr} \epsilon^{pqr} \right) + \frac{1}{\sqrt{2}} \partial_{\sharp} \sin \theta \\
 & + \frac{1}{\sqrt{2}} \sin \theta \left(-\frac{1}{4} h_{\sharp} + \frac{1}{2} \Omega_{\sharp,p}^p - \frac{1}{2} \Omega_{\sharp,1\bar{1}} + \frac{1}{12} Y_p^p - \frac{1}{12} Y_{1\bar{1}} + \frac{1}{12} X_{1\bar{1}q}^q \right. \\
 & \left. + \frac{1}{24} X_{pq}^{pq} + \frac{1}{18} X_{\bar{1}pqr} \epsilon^{pqr} \right) = 0, \quad (\text{B.18})
 \end{aligned}$$

$$i \cos \theta \left(\frac{1}{6} Y_{\sharp}^p - \frac{1}{2} \Omega_{\sharp,\sharp}^p - \frac{1}{6} X_{\sharp 1\bar{1}}^p - \frac{1}{6} X_{\sharp q}^{qp} - \frac{1}{6} X_{\sharp 1qr} \epsilon^{pqr} \right)$$

$$\begin{aligned}
 & + \frac{1}{\sqrt{2}} \sin \theta \left(-\Omega_{\sharp, \bar{1}}{}^p - \frac{1}{2} \Omega_{\sharp, qr} \epsilon^{pqr} - \frac{1}{6} Y_1{}^p - \frac{1}{12} Y_{qr} \epsilon^{pqr} \right. \\
 & \quad \left. + \frac{1}{6} X_{1q}{}^{qp} + \frac{1}{12} X_{1\bar{1}qr} \epsilon^{pqr} - \frac{1}{36} X_{qrs}{}^p \epsilon^{qrs} \right) = 0, \quad (\text{B.19})
 \end{aligned}$$

$$\begin{aligned}
 & \frac{i}{\sqrt{2}} \cos \theta \left(\Omega_{\sharp, \bar{1}}{}^p - \frac{1}{6} Y_{\bar{1}}{}^p + \frac{1}{6} X_{\bar{1}q}{}^{qp} - \frac{1}{12} Y_{qr} \epsilon^{pqr} + \frac{1}{2} \Omega_{\sharp, qr} \epsilon^{pqr} \right. \\
 & - \frac{1}{36} X_{qrs}{}^p \epsilon^{qrs} - \frac{1}{12} X_{1\bar{1}qr} \epsilon^{pqr} \left. \right) + \sin \theta \left(-\frac{1}{6} Y_{\sharp}{}^p - \frac{1}{2} \Omega_{\sharp, \sharp}{}^p - \frac{1}{6} X_{\sharp 1\bar{1}}{}^p \right. \\
 & \quad \left. + \frac{1}{6} X_{\sharp t}{}^{tp} + \frac{1}{6} X_{\sharp \bar{1}qr} \epsilon^{pqr} \right) = 0. \quad (\text{B.20})
 \end{aligned}$$

C $N = 4$ SU(3) KSE: 4-form X

The covariantized expression of the 4-form X in the SU(3) case is given by

$$\begin{aligned}
 X = & K \wedge U \wedge W \wedge C_1 + K \wedge U \wedge C_2 + K \wedge W \wedge C_3 + U \wedge W \wedge C_4 + K \wedge C_5 \\
 & + U \wedge C_6 + W \wedge C_7 + C_8, \quad (\text{C.1})
 \end{aligned}$$

where $C_1 \in \Omega^1(\mathcal{S})$, $C_2, C_3, C_4 \in \Omega^2(\mathcal{S})$, $C_5, C_6, C_7 \in \Omega^3(\mathcal{S})$ and $C_8 \in \Omega^4(\mathcal{S})$ can be expressed in terms of the SU(3) gauge-invariant bilinears as follows

- C_8

$$C_8 = \mathcal{F}_8 \wedge \Re(\hat{\chi}) + G \wedge \hat{\omega} + X^{(2,2)\text{TT}}, \quad (\text{C.2})$$

where $X^{(2,2)\text{TT}}$ is the traceless (2, 2) part of X , given by (4.44),

$$(\mathcal{F}_8)_i := -\frac{1}{2} \tan \theta \Theta_i + \frac{1}{4 \cos \theta} (dU)_{l_1 l_2} \Re(\hat{\chi})^{l_1 l_2}{}_i, \quad (\text{C.3})$$

$$G_{ij} := \frac{1}{2} \cot \theta J_{[i}{}^l M_{j]l}, \quad (\text{C.4})$$

$$M_{ij} := -\hat{\omega}^{l_1 l_2} \nabla_{(i} \Re(\hat{\chi})_{j) l_1 l_2}, \quad (\text{C.5})$$

- C_7

$$\begin{aligned}
 C_7 = & \frac{1}{2(1 + \cos^2 \theta)} \left\{ \Re(\hat{\chi}) \left(\frac{\cos^2 \theta + 2}{2} \tan \theta i_W \Theta - \frac{\cos^2 \theta}{\sin \theta} i_K \theta_{\Re(\hat{\chi})}^{(9)} \right) \right. \\
 & \quad \left. + \Im(\hat{\chi}) \left(-\cot \theta i_W \theta_{\Re(\hat{\chi})}^{(9)} + \frac{\sin \theta}{2} i_K \Theta \right) \right\} + \mathcal{F}_7 \wedge \hat{\omega} \\
 & + \frac{1}{4} \left(\cot \theta (\mathcal{L}_W g)_{i_1 j} \Im(\hat{\chi})^j{}_{i_2 i_3} + \frac{1}{\sin \theta} (\mathcal{L}_K g)_{i_1 j} \Re(\hat{\chi})^j{}_{i_2 i_3} \right) e^{i_1 i_2 i_3}, \quad (\text{C.6})
 \end{aligned}$$

$$\begin{aligned}
 (\mathcal{F}_7)_i := & -\frac{1}{4 \sin \theta} (\mathcal{L}_K \hat{\omega})_{j_1 j_2} \Re(\hat{\chi})^{j_1 j_2}{}_i - (\mathcal{L}_K U)_i \\
 & - \frac{1}{8} \cot \theta (\mathcal{L}_W \hat{\omega})_{j_1 j_2} \Im(\hat{\chi})^{j_1 j_2}{}_i, \quad (\text{C.7})
 \end{aligned}$$

- C_6

$$C_6 = \Re(\hat{\chi}) \left(\frac{1 + \sin^2 \theta}{2 \sin(2\theta)} i_U \Theta \right) + \Im(\hat{\chi}) \left(- \frac{\cot \theta}{2(1 + \cos^2 \theta)} i_U \theta_{\Re(\hat{\chi})}^{(9)} \right) \\ + \mathcal{F}_6 \wedge \hat{\omega} + \frac{1}{4} \cot \theta (\mathcal{L}_U g)_{i_1 j} \Im(\hat{\chi})^j_{i_2 i_3} e^{i_1 i_2 i_3}, \quad (C.8)$$

$$(\mathcal{F}_6)_i := \frac{1}{8} \cot \theta \left((dU)_{j_1 j_2} \Re(\hat{\chi})^{j_1 j_2}_i - (\mathcal{L}_U \hat{\omega})_{j_1 j_2} \Im(\hat{\chi})^{j_1 j_2}_i \right), \quad (C.9)$$

- C_5

$$C_5 = \frac{1}{2(1 + \cos^2 \theta)} \left\{ \Re(\hat{\chi}) \left(\frac{\cos^2 \theta + 2}{2} \tan \theta i_K \Theta + \frac{\cos^2 \theta}{\sin \theta} i_W \theta_{\Re(\hat{\chi})}^{(9)} \right) \right. \\ \left. + \Im(\hat{\chi}) \left(- \cot \theta i_K \theta_{\Re(\hat{\chi})}^{(9)} - \frac{\sin \theta}{2} i_W \Theta \right) \right\} + \mathcal{F}_5 \wedge \hat{\omega} \\ + \frac{1}{4} \left(\cot \theta (\mathcal{L}_K g)_{i_1 j} \Im(\hat{\chi})^j_{i_2 i_3} - \frac{1}{\sin \theta} (\mathcal{L}_W g)_{i_1 j} \Re(\hat{\chi})^j_{i_2 i_3} \right) e^{i_1 i_2 i_3}, \quad (C.10)$$

$$(\mathcal{F}_5)_i := \frac{1}{4 \sin \theta} (\mathcal{L}_W \hat{\omega})_{j_1 j_2} \Re(\hat{\chi})^{j_1 j_2}_i + (\mathcal{L}_W U)_i - \frac{\cot \theta}{8} (\mathcal{L}_K \hat{\omega})_{j_1 j_2} \Im(\hat{\chi})^{j_1 j_2}_i, \quad (C.11)$$

- C_4

$$C_4 = -\mathcal{L}_K \hat{\omega} + \frac{1}{2} \left\{ 2J_{i_1}^j (\mathcal{L}_K g)_{j i_2} + \left(\frac{4 \cos^2 \theta - 3}{4 \sin \theta} (\mathcal{L}_K U)_j - \frac{1}{4 \sin \theta} [K, U]_j \right. \right. \\ \left. \left. + \frac{1}{2} \cot \theta J_j^l [W, U]_l \right) \Re(\hat{\chi})^j_{i_1 i_2} \right\} e^{i_1 i_2}, \quad (C.12)$$

- C_3

$$C_3 = \mathcal{L}_U \hat{\omega} + \frac{1}{2} \left\{ -J_{i_1}^j \left(2(\mathcal{L}_U g)_{j i_2} + (dU)_{i_2 j} + \frac{1}{2 \sin \theta} M_{i_2 j} \right) \right. \\ \left. + \left(\sin \theta (\mathcal{L}_V V)_j + \frac{\cos^2 \theta}{\sin \theta} (\mathcal{L}_K K)_j - \frac{1}{2} \cot \theta J_j^l [K, W]_l \right) \Re(\hat{\chi})^j_{i_1 i_2} \right\} e^{i_1 i_2} \\ + \frac{1}{2} \hat{\omega} \left(\frac{1}{2} i_U \theta_{\Re(\hat{\chi})}^{(9)} - \star_9 d \star_9 U \right), \quad (C.13)$$

- C_2

$$C_2 = -\mathcal{L}_W \hat{\omega} + \frac{1}{2} \left\{ 2J_{i_1}^j (\mathcal{L}_W g)_{j i_2} + \left(\frac{4 \cos^2 \theta - 3}{4 \sin \theta} (\mathcal{L}_W U)_j - \frac{1}{4 \sin \theta} [W, U]_j \right. \right. \\ \left. \left. - \frac{1}{2} \cot \theta J_j^l [K, U]_l \right) \Re(\hat{\chi})^j_{i_1 i_2} \right\} e^{i_1 i_2}, \quad (C.14)$$

- C_1

$$C_1 = \left\{ J_i^j \left(\frac{1}{2} (\theta_{\Re(\hat{\chi})}^{(9)})_j - \frac{3}{4} (\theta_{\hat{\omega}}^{(9)})_j + (\mathcal{L}_K K)_j - \frac{1}{2} (\mathcal{L}_U U)_j \right) \right. \\ \left. - \frac{1}{8} \sin \theta (dU)_{l_1 l_2} \Re(\hat{\chi})^{l_1 l_2}_i \right\} e^i - \cos \theta \mathcal{L}_K W. \quad (C.15)$$

D Useful Spin(7) and G_2 identities

In this appendix we list some useful identities we have used extensively during our work.

The fundamental Spin(7) 4-form τ , defined by (3.31) satisfies the following identities

$$\begin{aligned}\tau_{il_1l_2l_3}\tau^{jl_1l_2l_3} &= 42\delta_i^j \\ \tau_{i_1i_2l_1l_2}\tau^{j_1j_2l_1l_2} &= -4\tau_{i_1i_2}{}^{l_1l_2} + 12\delta_{[i_1}^{[j_1}\delta_{i_2]}^{j_2]}, \\ \tau_{i_1i_2i_3l}\tau^{j_1j_2j_3l} &= -9\delta_{[i_1}^{[j_1}\tau_{i_2i_3]}^{j_2j_3]} + 6\delta_{[i_1}^{[j_1}\delta_{i_2}^{j_2}\delta_{i_3]}^{j_3]},\end{aligned}\tag{D.1}$$

where $i, j, \dots$ are 8-dimensional. The existence of a Spin(7)-structure on a 8-dimensional manifold M^8 determines a decomposition of the space Ω^k of the k -differential forms on M^8 into irreducible Spin(7)-representations $\Omega_{\mathbf{n}}^k$ of dimension $\mathbf{n}$, see [46] and also [50]. Ω^1 is irreducible; also, for $k > 4$, $\Omega^k = \star_8 \Omega^{8-k}$, thus we are left to decompose

- 2-forms

$$\Omega^2 = \Omega_7^2 \oplus \Omega_{21}^2,\tag{D.2}$$

where, given a 2-form θ

$$\begin{aligned}\theta_{ij}^7 &= \frac{1}{4} \left(\theta_{ij} - \frac{1}{2} \tau_{ij}{}^{kl} \theta_{kl} \right), \\ \theta_{ij}^{21} &= \frac{3}{4} \left(\theta_{ij} + \frac{1}{6} \tau_{ij}{}^{kl} \theta_{kl} \right).\end{aligned}\tag{D.3}$$

- 3-forms

$$\Omega^3 = \Omega_8^3 \oplus \Omega_{48}^3,\tag{D.4}$$

where, given a 3-form λ

$$\begin{aligned}\lambda_{ijk}^8 &= \frac{1}{7} \left(\lambda_{ijk} - \frac{3}{2} \tau_{[ij}^{mn} \lambda_{k]mn} \right), \\ \lambda_{ijk}^{48} &= \frac{6}{7} \left(\lambda_{ijk} + \frac{1}{4} \tau_{[ij}^{mn} \lambda_{k]mn} \right).\end{aligned}\tag{D.5}$$

Notice that

$$\lambda_{j_1j_2j_3}\tau^{j_1j_2j_3}{}_i = \lambda_{j_1j_2j_3}^8 \tau^{j_1j_2j_3}{}_i.\tag{D.6}$$

- 4-forms

$$\Omega^4 = \Omega_1^4 \oplus \Omega_7^4 \oplus \Omega_{27}^4 \oplus \Omega_{35}^4,\tag{D.7}$$

where, given a 4-form ξ

$$\begin{aligned}\xi_{i_1i_2i_3i_4}^1 &= \frac{1}{336} (\Pi^1 \xi)_{i_1i_2i_3i_4}, \\ \xi_{i_1i_2i_3i_4}^7 &= \frac{1}{8} \xi_{i_1i_2i_3i_4} + \frac{1}{224} (\Pi^1 \xi)_{i_1i_2i_3i_4} - \frac{3}{224} (\Pi^2 \xi)_{i_1i_2i_3i_4} + \frac{5}{168} (\Pi^3 \xi)_{i_1i_2i_3i_4}, \\ \xi_{i_1i_2i_3i_4}^{27} &= \frac{3}{8} \xi_{i_1i_2i_3i_4} - \frac{1}{224} (\Pi^1 \xi)_{i_1i_2i_3i_4} + \frac{15}{224} (\Pi^2 \xi)_{i_1i_2i_3i_4} + \frac{1}{56} (\Pi^3 \xi)_{i_1i_2i_3i_4}, \\ \xi_{i_1i_2i_3i_4}^{35} &= \frac{1}{2} \xi_{i_1i_2i_3i_4} - \frac{1}{336} (\Pi^1 \xi)_{i_1i_2i_3i_4} - \frac{3}{56} (\Pi^2 \xi)_{i_1i_2i_3i_4} - \frac{1}{21} (\Pi^3 \xi)_{i_1i_2i_3i_4},\end{aligned}\tag{D.8}$$

with

$$\begin{aligned}
 (\Pi^1 \xi)_{i_1 i_2 i_3 i_4} &:= \tau_{i_1 i_2 i_3 i_4} \tau^{j_1 j_2 j_3 j_4} \xi_{j_1 j_2 j_3 j_4}, \\
 (\Pi^2 \xi)_{i_1 i_2 i_3 i_4} &:= \tau_{[i_1 i_2}^{j_1 j_2} \tau_{i_3 i_4]}^{j_3 j_4} \xi_{j_1 j_2 j_3 j_4}, \\
 (\Pi^3 \xi)_{i_1 i_2 i_3 i_4} &:= \tau_{[i_1 i_2 i_3}^{j_1 j_2 j_3} \tau_{i_4]}^{j_4} \xi_{j_1 j_2 j_3 j_4}.
 \end{aligned} \tag{D.9}$$

The fundamental G_2 3-form ϕ , defined by (4.48), and its Hodge dual $\star_7 \phi$ satisfy the following identities ($i, j, \dots$ are 7-dimensional real G_2 indices)

$$\phi_{i l_1 l_2} \phi^{j l_1 l_2} = 6 \delta_i^j, \tag{D.10}$$

$$\phi_{i_1 i_2 l} \phi^{j_1 j_2 l} = 2 \delta_{[i_1}^{j_1} \delta_{i_2]}^{j_2} - \star_7 \phi_{i_1 i_2}^{j_1 j_2}, \tag{D.11}$$

$$\phi_i^{l_1 l_2} (\star_7 \phi)_{j_1 j_2 l_1 l_2} = -4 \phi_{i j_1 j_2}, \tag{D.12}$$

$$\phi_{i_1 i_2 l} (\star_7 \phi)^{j_1 j_2 j_3 l} = 6 \delta_{[i_1}^{j_1} \phi_{i_2]}^{j_2 j_3}]. \tag{D.13}$$

$$(\star_7 \phi)_{i l_1 l_2 l_3} (\star_7 \phi)^{j l_1 l_2 l_3} = 24 \delta_i^j, \tag{D.14}$$

$$(\star_7 \phi)_{i_1 i_2 l_1 l_2} (\star_7 \phi)^{j_1 j_2 l_1 l_2} = 8 \delta_{[i_1}^{j_1} \delta_{i_2]}^{j_2} - 2 (\star_7 \phi)_{i_1 i_2}^{j_1 j_2}, \tag{D.15}$$

$$(\star_7 \phi)_{i_1 i_2 i_3 l} (\star_7 \phi)^{j_1 j_2 j_3 l} = 6 \delta_{[i_1}^{j_1} \delta_{i_2}^{j_2} \delta_{i_3]}^{j_3} - 9 \delta_{[i_1}^{j_1} (\star_7 \phi)_{i_2 i_3]}^{j_2 j_3} - \phi_{i_1 i_2 i_3} \phi^{j_1 j_2 j_3}. \tag{D.16}$$

The existence of a G_2 -structure on a 7-dimensional manifold M^7 determines a decomposition of the space Ω^k of the k -differential forms on M^7 into irreducible G_2 -representations $\Omega_{\mathbf{n}}^k$ of dimension $\mathbf{n}$. The space Ω^k is irreducible if $k = 1, 6, 7$ and $\Omega^k = \star_7 \Omega^{7-k}$, thus we are left to decompose (see [51]):

- 2-forms

$$\Omega^2 = \Omega_7^2 \oplus \Omega_{14}^2, \tag{D.17}$$

where, given a 2-form θ

$$\begin{aligned}
 \theta_{i_1 i_2}^7 &= \frac{1}{3} \theta_{i_1 i_2} - \frac{1}{6} \star_7 \phi_{i_1 i_2}^{j_1 j_2} \theta_{j_1 j_2}, \\
 \theta_{i_1 i_2}^{14} &= \frac{2}{3} \theta_{i_1 i_2} + \frac{1}{6} \star_7 \phi_{i_1 i_2}^{j_1 j_2} \theta_{j_1 j_2}.
 \end{aligned} \tag{D.18}$$

Notice that

$$\theta_{j_1 j_2} \phi^{j_1 j_2}_i = \theta_{j_1 j_2}^7 \phi^{j_1 j_2}_i. \tag{D.19}$$

- 3-forms

$$\Omega^3 = \Omega_1^3 \oplus \Omega_7^3 \oplus \Omega_{27}^3, \tag{D.20}$$

where, given a 3-form λ

$$\begin{aligned}
 \lambda_{i_1 i_2 i_3}^1 &= \frac{1}{42} \phi^{j_1 j_2 j_3} \lambda_{j_1 j_2 j_3} \phi_{i_1 i_2 i_3}, \\
 \lambda_{i_1 i_2 i_3}^7 &= \frac{1}{4} \lambda_{i_1 i_2 i_3} - \frac{1}{24} \phi^{j_1 j_2 j_3} \lambda_{j_1 j_2 j_3} \phi_{i_1 i_2 i_3} - \frac{3}{8} \lambda_{j_1 j_2 [i_1} \star_7 \phi_{i_2 i_3]}^{j_1 j_2}, \\
 \lambda_{i_1 i_2 i_3}^{27} &= \frac{3}{4} \lambda_{i_1 i_2 i_3} + \frac{1}{56} \phi^{j_1 j_2 j_3} \lambda_{j_1 j_2 j_3} \phi_{i_1 i_2 i_3} + \frac{3}{8} \lambda_{j_1 j_2 [i_1} \star_7 \phi_{i_2 i_3]}^{j_1 j_2}.
 \end{aligned} \tag{D.21}$$

Notice that

$$\lambda_{j_1 j_2 j_3} \star_7 \phi^{j_1 j_2 j_3}_i = \lambda_{j_1 j_2 j_3}^7 \star_7 \phi^{j_1 j_2 j_3}_i. \quad (\text{D.22})$$

The 4-forms are dual to the 3-forms; nonetheless, let us state explicitly the decomposition of the 4-forms

$$\Omega^4 = \Omega_1^4 \oplus \Omega_7^4 \oplus \Omega_{27}^4, \quad (\text{D.23})$$

where, given a 4-form ξ

$$\begin{aligned} \xi_{i_1 i_2 i_3 i_4}^1 &= \frac{1}{168} \star_7 \phi^{j_1 j_2 j_3 j_4} \xi_{j_1 j_2 j_3 j_4} \star_7 \phi_{i_1 i_2 i_3 i_4}, \\ \xi_{i_1 i_2 i_3 i_4}^7 &= \frac{1}{4} \xi_{i_1 i_2 i_3 i_4} - \frac{1}{96} \star_7 \phi^{j_1 j_2 j_3 j_4} \xi_{j_1 j_2 j_3 j_4} \star_7 \phi_{i_1 i_2 i_3 i_4} - \frac{3}{4} \star_7 \phi^{j_1 j_2}_{[i_1 i_2} \xi_{i_3 i_4] j_1 j_2}, \\ \xi_{i_1 i_2 i_3}^{27} &= \frac{3}{4} \xi_{i_1 i_2 i_3 i_4} + \frac{1}{224} \star_7 \phi^{j_1 j_2 j_3 j_4} \xi_{j_1 j_2 j_3 j_4} \star_7 \phi_{i_1 i_2 i_3 j_4} + \frac{3}{4} \star_7 \phi^{j_1 j_2}_{[i_1 i_2} \xi_{i_3 i_4] j_1 j_2}. \end{aligned} \quad (\text{D.24})$$

Notice that

$$\xi_{i j_1 j_2 j_3} \phi^{j_1 j_2 j_3} = \xi_{i j_1 j_2 j_3}^7 \phi^{j_1 j_2 j_3}. \quad (\text{D.25})$$

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