Published for SISSA by Springer

Received: May 20, 2021
Accepted: June 30, 2021
Published: July 21, 2021

Normal form of nilpotent vector field near the tip of the pure spinor cone

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Abstract: Pure spinor formalism implies that supergravity equations in space-time are equivalent to the requirement that the worldsheet sigma-model satisfies certain properties. Here we point out that one of these properties has a particularly transparent geometrical interpretation. Namely, there exists an odd nilpotent vector field on some singular supermanifold, naturally associated to space-time. Is it true that all supergravity fields are encoded in this vector field, as coefficients in its normal form, and the nilpotence is equivalent to the target space equations of motion? We show that this is approximately correct. The normal form is parametrized by some tensor fields, which satisfy hyperbolic equations. These equations are slightly weaker than the full supergravity equations.

Keywords: Superspaces, Superstrings and Heterotic Strings

ArXiv ePrint: 2103.15600

https://doi.org/10.1007/JHEP07(2021)150

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1 Introduction

In the low energy limit of superstring theory, spacetime fields satisfy supergravity (SUGRA) equations of motion, which are super-analogues of the Einstein equations. It is one of the main principles of string theory, that these target space equations of motion are equivalent to the BRST invariance of the string worldsheet theory. In the case of pure spinor string, the action of the BRST operator on matter fields and pure spinor ghosts does not contain worldsheet derivatives.[1] This means that, if we think of the pure spinor ghosts as part of target space, the BRST operator defines on the target space an odd nilpotent vector field, which we denote Q. In other words, the target space of the sigma-model becomes a Q-manifold. Moreover, in generic space-time (for example in $AdS_5 \times S^5$, but not in flat space-time) the energy-momentum tensor and the b-ghost can be interpreted as symmetric tensors on the target space (see [1]).

Generally speaking, the space of fields in BV formalism is a Q-manifold. It is, however, infinite-dimensional. Here we point out that to a pure spinor sigma-model corresponds a finite-dimensional Q-manifold (its target space). It is easier to study finite-dimensional objects than infinite-dimensional objects.

In particular, we can try to bring this Q to a normal form. Usually, an odd nilpotent vector field can be "simplified" by a clever choice of coordinates. This is called "normal form". If a vector field is non-vanishing, one can choose coordinates so that the it is $\frac{\partial}{\partial\theta}$ where θ is one of fermionic coordinates. If it vanishes at some point, then the normal form would be (in the notations of [2]) $\eta^a \frac{\partial}{\partial x^a}$. But in out case, the target space is not a smooth supermanifold, because pure spinor ghosts live on a cone. The vector Q vanishes precisely at the singular locus, and the problem of classification of normal forms is a nontrivial cohomological computation.

1.1 Definition of M

The particular singularity which we are interested in can be described as follows. Consider the space M with bosonic coordinates x^m (m running from 1 to 10) and λ_L^α, $\lambda_R^{\hat\alpha}$ (α and $\hat\alpha$ both running from 1 to 16), and fermionic θ_L^α and $\theta_R^{\hat\alpha}$, with the constraint:

$$\lambda_L^\alpha \Gamma^m_{\alpha\beta} \lambda_L^\beta = \lambda_R^{\hat\alpha} \Gamma^m_{\hat\alpha\hat\beta} \lambda_R^{\hat\beta} = 0 \tag{1.1}$$

where Γ^m are ten-dimensional gamma-matrices. These constraints are called "pure spinor constraints". We understand eqs. (1.1) as specifying the singular locus in M, from the point of view of differential geometry. All we need from these equations is to know how M deviates from being smooth. The singular locus is the tip of the cone (1.1):

$$\lambda_L = 0 \texttt{ or } \lambda_R = 0 \tag{1.2}$$

Pure spinor constraints (1.1) are invariant under the action of the group

$$G = \mathrm{Spin}(10)_L \times \mathbf{C}_L^\times \times \mathrm{Spin}(10)_R \times \mathbf{C}_R^\times \tag{1.3}$$

[1] The worldsheet derivatives will appear when we consider the action on the conjugate momenta to matter fields and pure spinor ghosts, but they can be considered separately. Their BRST transformations can be *derived* from the BRST transformations of matter fields and ghosts.

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The diagonal

$$\mathbf{C}^\times \subset \mathbf{C}^\times_L \times \mathbf{C}^\times_R \tag{1.4}$$

is called "ghost number symmetry". Infinitesimal ghost number symmetry is generated by $\lambda^\alpha_L \frac{\partial}{\partial \lambda^\alpha_L} + \lambda^{\hat{\alpha}}_R \frac{\partial}{\partial \lambda^{\hat{\alpha}}_R}$.

Consider an odd vector field Q satisfying the following properties:

- Q has ghost number 1, i.e. $\left[\lambda^\alpha_L \frac{\partial}{\partial \lambda^\alpha_L} + \lambda^{\hat{\alpha}}_R \frac{\partial}{\partial \lambda^{\hat{\alpha}}_R}\,,\ \ Q\right] = Q$

- $Q^2 = 0$

- Q is "smooth" in the sense that it can be obtained as a restriction to the cone (1.1) of a smooth (but not nilpotent) vector field in the space parametrized by unconstrained x, θ, λ

- Q is zero at $\lambda_L = \lambda_R = 0$

We want to classify such vector fields modulo coordinate transformations. Coordinate transformations are supermaps $(x, \lambda, \theta) \mapsto (\tilde{x}, \tilde{\lambda}, \tilde{\theta})$ such that $\tilde{\lambda}$ satisfy the same constraints (1.1).

Such a vector field is one of the geometrical structures associated to the pure spinor superstring worldsheet theory [3, 4]. In particular, flat background (empty ten-dimensional spacetime) corresponds to $Q = Q^{\text{flat}}$:

$$\begin{aligned}
Q^{\text{flat}} &= Q^{\text{flat}}_L + Q^{\text{flat}}_R \ \texttt{where:} \\
Q^{\text{flat}}_L &= \lambda^\alpha_L \frac{\partial}{\partial \theta^\alpha_L} + (\lambda^\alpha_L \Gamma^m_{\alpha\beta} \theta^\beta_L) \frac{\partial}{\partial x^m} \\
Q^{\text{flat}}_R &= \lambda^{\hat{\alpha}}_R \frac{\partial}{\partial \theta^{\hat{\alpha}}_R} + (\lambda^{\hat{\alpha}}_R \Gamma^m_{\hat{\alpha}\hat{\beta}} \theta^{\hat{\beta}}_R) \frac{\partial}{\partial x^m}
\end{aligned} \tag{1.5}$$

String worldsheet theory also has, besides Q, some other structures which are less geometrically transparent (various couplings in the string worldsheet sigma-model). All these structures should satisfy certain consistency conditions.

> *Question:* is it true, that just a nilpotent vector field Q already includes, as various coefficients in its normal form, all the supergravity fields, and the supergravity equations of motion are automatically satisfied (i.e. follow from $Q^2 = 0$)?

This may be false in two ways. First, it could be that some supergravity fields do not enter as coefficients in the normal form of Q (i.e. they would only appear as some couplings in the sigma-model, but would not enter in Q). Second, it could be that just $Q^2 = 0$ would not be enough to impose SUGRA equations of motion (i.e. one would have to also require the Q-invariance of the worldsheet sigma-model action).

1.2 Our results

In this paper we will derive the normal form of Q as a deformation of Q^{flat}:

$$Q = Q^{\text{flat}} + \epsilon Q_1 + \epsilon^2 Q_2 + \dots \tag{1.6}$$

Our analysis will be restricted to the terms linear in ϵ (i.e. Q_1). It turns out that Q_1 is parameterized by some tensor fields satisfying *hyperbolic* partial differential equations. (This means that the solutions are determined by Cauchy data on spacelike hypersurface.) These equations are similar to the linearized supergravity equations, but contain some additional components, section 5.4.

It is useful to compare to the pure spinor description of the super-Yang-Mills equations. The super-Yang-Mills equations are equivalent to having an odd nilpotent operator:

$$Q_{\text{SYM}} = \lambda^\alpha \left(\frac{\partial}{\partial \theta^\alpha} + \Gamma^m_{\alpha\beta}\theta^\beta \frac{\partial}{\partial x^m} + A^a_\alpha(x,\theta)\mathbf{t}_a \right) \tag{1.7}$$

where $\mathbf{t}_a$ are generators of the gauge group, and $A^a_\alpha(x,\theta)$ is vector potential. Zero solution corresponds to $A_\alpha = 0$. In this sense, the SYM solutions can be considered as deformations of the differential operator:

$$Q^{(0)}_{\text{SYM}} = \lambda^\alpha \left(\frac{\partial}{\partial \theta^\alpha} + \Gamma^m_{\alpha\beta}\theta^\beta \frac{\partial}{\partial x^m} \right) \tag{1.8}$$

where the leading symbol (i.e. the derivatives) remains undeformed. Here we consider, instead, the deformations of the leading symbol.

1.3 Relation to partial G-structures

The variables λ_L and λ_R parametrize the normal direction to the singularity locus $Z \subset M$:

$$i \,:\, Z \to M \tag{1.9}$$

The first infinitesimal neighborhood is a bundle over Z with the fiber $C_L \times C_R$ — the product of two cones. Filling the cones, we obtain a vector bundle over Z with the fiber $V = \mathbf{C}^{32}$. The vector field Q is power series in λ_L, λ_R, with zero at the tip of $C_L \times C_R$. The derivative of Q at the zero locus defines a linear map:

$$Q_* \,:\, V \to i^* TM \tag{1.10}$$

This map is not an isomorphism, since the image of Q_* only covers a $(0|32)$-dimensional subbundle of TZ. We can interpret M as $(C_L \times C_R) \times_G \widehat{Z}$ where $\widehat{Z}$ is a partial frame bundle of Z and G is given by eq. (1.3). It was shown in [5] that Q defines a connection in a partial G-structure on Z with some constraints on torsion, modulo some equivalence relation.

1.4 Open questions

At least at the linearized level, our conclusion is that $Q^2 = 0$ is actually a bit weaker than SUGRA equations of motion. In order to reproduce the SUGRA equations of motion, we have to require the existence of some additional objects. Do they correspond to some additional (besides the nilpotent vector field Q) geometrical structures on M? One additional requirement could be that the cohomology of Q in the ghost number 2 should be sufficiently rich. In the pure spinor formalism the cohomology of Q in the ghost number 2 corresponds to the deformations of the pure spinor sigma model. These deformations correspond to solutions of linearized (around the given background) SUGRA equations. Therefore, the cohomology of Q in the space of functions on M of ghost number 2 should also correspond to solutions of hyperbolic equations. Another structure, suggested by the considerations of [1, 6], might be a Q-invariant 2-form B satisfying $\iota_Q dB = 0$. This two-form is a rational function of λ_L, λ_R; it is not clear to us, at this point, which poles should be allowed in B.

It is necessary to extend this analysis to full nonlinear SUGRA equations, i.e. higher order terms in eq. (1.6). The potential obstacle to extending linearized solutions to the solution of the nonlinear equation $Q^2 = 0$ lies in $H^2\left(\left[Q_L^{\text{flat}} + Q_R^{\text{flat}}, \_\right]\right)$. We will not compute $H^2\left(\left[Q_L^{\text{flat}} + Q_R^{\text{flat}}, \_\right]\right)$ in this paper, but the results of [7] suggest that $H^2\left(\left[Q_L^{\text{flat}} + Q_R^{\text{flat}}, \_\right]\right)$ is actually nonzero. (We would expect it to be similar to $H^1\left(\left[Q_L^{\text{flat}} + Q_R^{\text{flat}}, \_\right]\right)$ which we compute here, perhaps isomorphic to it.) But we also know that the *actual* obstacle is zero, because of the consistency of the nonlinear supergravity equations of [3]. For some reason, $\{Q_1, Q_1\}$ is a coboundary.

2 Notations

To avoid the discussion of reality conditions, we consider complex vector fields. The notation:

$$\mathbf{C}\langle v_1, v_2, \ldots\rangle \tag{2.1}$$

means the space of all linear combinations of vectors $v_1, v_2, \ldots$ with complex coefficients.

We introduce the abbreviated notations:

$$\begin{aligned}
((\lambda\theta))^m &= \lambda^\alpha \Gamma^m_{\alpha\beta}\theta^\beta \\
((\lambda\theta\theta))_\gamma &= \lambda^\alpha \Gamma^m_{\alpha\beta}\theta^\beta\theta^\delta\Gamma^m_{\delta\gamma} \\
[v \otimes \psi]^{1/2}_\alpha &= \Gamma^m_{\alpha\beta} v_m \psi^\beta \\
[v \otimes \psi]^\alpha_{1/2} &= \Gamma^{m\alpha\beta} v_m \psi_\beta
\end{aligned}$$

3 Setup for cohomological perturbation theory

3.1 Definition of θ_L^α and $\theta_R^{\hat\alpha}$

We define odd coordinates θ so that:

$$Q_L\theta_L^\alpha = \lambda_L^\alpha + O(\theta^2), \quad Q_L\theta_R^{\hat\alpha} = O(\theta^2), \quad Q_R\theta_R^{\hat\alpha} = \lambda_R^{\hat\alpha} + O(\theta^2), \quad Q_R\theta_L^\alpha = O(\theta^2) \tag{3.1}$$

3.2 Flat Q and expansion around it

Flat spacetime corresonds to $Q = Q^{\text{flat}} = Q_L^{\text{flat}} + Q_R^{\text{flat}}$ where:

$$Q_L^{\text{flat}} = \lambda_L \frac{\partial}{\partial \theta_L} + (\!(\lambda_L \theta_L)\!) \frac{\partial}{\partial x}$$
$$Q_R^{\text{flat}} = \lambda_R \frac{\partial}{\partial \theta_R} + (\!(\lambda_R \theta_R)\!) \frac{\partial}{\partial x}$$

Let us consider Q as a small deformation of Q^{flat}:

$$Q = Q^{\text{flat}} + \epsilon Q_1 \tag{3.2}$$

to the first order in ϵ. Such deformations form a linear space. They correspond to odd vector fields Q_1 satisfying:

$$[Q^{\text{flat}}, Q_1] = 0 \tag{3.3}$$

modulo the equivalence relation, corresponding to the action of diffeomorphisms:

$$Q_1 \simeq Q_1 + [Q^{\text{flat}}, R] \tag{3.4}$$

where R is a ghost number zero vector field on M. Therefore, the classification of nilpotent vector fields of the form (3.2) is equivalent to the computation of the cohomology of the operator $[Q^{\text{flat}}, \_]$ on the space of vector fields.

In the rest of this paper we will compute the cohomology of $[Q^{\text{flat}}, \_]$ on the space of vector fields.

3.3 Spectral sequence

The grading operator:

$$N = \theta_L \frac{\partial}{\partial \theta_L} + \theta_R \frac{\partial}{\partial \theta_R} + \lambda_L \frac{\partial}{\partial \lambda_L} + \lambda_R \frac{\partial}{\partial \lambda_R} \tag{3.5}$$

defines a filtration on the algebra of functions on $\text{Fun}(M)$, and on the space of vector fields as a $\text{Fun}(M)$-module. Let $F^N\text{Vect}$ be the space of vector fields with grade at least N. This filtration defines a spectral sequence converging to the cohomology of $[Q^{\text{flat}}, \_]$.

3.4 First page

The first page of this spectral sequence is the cohomology of:

$$\left[Q_L^{(0)} + Q_R^{(0)}\, , \_\right] = \left[\lambda_L \frac{\partial}{\partial \theta_L} + \lambda_R \frac{\partial}{\partial \theta_R}\, , \_\right] \tag{3.6}$$

on the space of vector fields on M. For a set of coordinates $x, y, \ldots$ we denote $\text{Fun}(x, y, \ldots)$ the space of functions of $x, y, \ldots$ and $\text{Vect}(x, y, \ldots)$ the space of vector fields (i.e. differentiations of $\text{Fun}(x, y, \ldots)$). Let us introduce the following complexes:

$$C_L^{\text{vect}} = \text{Vect}(\theta_L, \lambda_L) \text{ with differential } \left[Q_L^{(0)}, \_\right]$$
$$C_R^{\text{vect}} = \text{Vect}(\theta_R, \lambda_R) \text{ with differential } \left[Q_R^{(0)}, \_\right]$$
$$C_L^{\text{fun}} = \text{Fun}(\theta_L, \lambda_L) \text{ with differential } Q_L^{(0)}$$
$$C_R^{\text{fun}} = \text{Fun}(\theta_R, \lambda_R) \text{ with differential } Q_R^{(0)}$$

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Then, Vect(M) with differential $Q_L^{(0)} + Q_R^{(0)}$ decomposes as follows:

$$\mathrm{Vect}(M) = \quad \mathrm{Fun}(x) \otimes C_R^{\mathrm{fun}} \otimes C_L^{\mathrm{vect}} \tag{3.7}$$

$$\oplus\ \mathrm{Fun}(x) \otimes C_L^{\mathrm{fun}} \otimes C_R^{\mathrm{vect}} \tag{3.8}$$

$$\oplus\ \mathrm{Fun}(x) \otimes C_L^{\mathrm{fun}} \otimes C_R^{\mathrm{fun}} \otimes \frac{\partial}{\partial x} \tag{3.9}$$

(We do not need to take care about the completions of the tensor products, since all our functions are polynomials in θ and λ.) The cohomology of C_L^{fun} and C_R^{fun} is well known, see e.g. the review part of [8]:

$$\begin{aligned}
H^0(C^{\mathrm{fun}}) &= \mathbf{C}\langle \mathbf{1} \rangle \\
H^1(C^{\mathrm{fun}}) &= \mathbf{C}\left\langle ((\lambda\theta)),\ [((\lambda\theta)) \otimes \theta]^{1/2} \right\rangle \\
H^2(C^{\mathrm{fun}}) &= \mathbf{C}\left\langle ((\lambda\theta))^m ((\lambda\theta))^n \Gamma_{mn}\theta,\ ((\lambda\theta))^m ((\lambda\theta))^n (\theta\Gamma_{lmn}\theta) \right\rangle \\
H^3(C^{\mathrm{fun}}) &= \mathbf{C}\left\langle ((\lambda\theta))^l ((\lambda\theta))^m ((\lambda\theta))^n (\theta\Gamma_{lmn}\theta) \right\rangle
\end{aligned}$$

Parts of the cohomology of C_L^{vect} and C_R^{vect} which are relevant to this work will be computed in section 4.

4 Cohomology of $Q^{(0)}$ in the space of vector fields

4.1 Notations

Let X denote the singular supermanifold parametrized by bosonic λ^α and fermionic θ^α satisfying the pure spinor constraint:

$$\lambda^\alpha \Gamma^m_{\alpha\beta} \lambda^\beta = 0 \tag{4.1}$$

(The space M introduced in section 1.1 is the direct product of two copies of X, and the space parametrized by x^m.) Let $\mathcal{O}(X)$ denote the algebra of polynomial functions on X, and Vect(X) = Der$(\mathcal{O}(X))$ the space of polynomial vector fields. Consider the odd nilpotent vector field $Q^{(0)}$:

$$Q^{(0)} = \lambda^\alpha \frac{\partial}{\partial \theta^\alpha} \tag{4.2}$$

The commutation $[Q^{(0)}, -]$ is a nilpotent operator on Vect(X). We will now compute the cohomology of this operator.

Any vector field $V \in \mathrm{Vec}(X)$ can be written as

$$V = \xi^\alpha(\lambda,\theta) \frac{\partial}{\partial \lambda^\alpha} + u^\alpha(\lambda,\theta) \frac{\partial}{\partial \theta^\alpha}$$

$$(\lambda\gamma_m)_\alpha \xi^\alpha = 0$$

The condition $(\lambda\gamma_m)_\alpha \xi^\alpha = 0$ is needed because λ^α is constrained to satisfy eq. (4.1).

Consider the subsheaf $U \subset TX$ consisting of vectors of the form $u^\alpha \frac{\partial}{\partial\theta^\alpha}$ (in other words, $\xi^\alpha = 0$). Its space of sections is:

$$\Gamma(U) = \{ v \in \mathrm{Vect}(X) \mid \mathcal{L}_v \lambda^\alpha = 0 \} \tag{4.3}$$

We observe that $\Gamma(U) \subset \mathrm{Vect}(X)$ is invariant under the action of $[Q^{(0)}, \_]$. Therefore, we can think of both $\Gamma(U)$ and $\Gamma(TX/U)$ as complexes with the differential $[Q^{(0)}, \_]$.

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4.2 Summary of results for $H^1(\mathrm{Vect}(X))$

Using the notations of section 2:

$$H^1(\mathrm{Vect}(X))_{\mathrm{odd}} = \mathbf{C}\left\langle [(\!(\lambda\theta)\!) \otimes \theta]^{1/2} \otimes \frac{\partial}{\partial\theta}, \right. \\ \left. (\!(\lambda\theta)\!) \left(\lambda^\alpha \frac{\partial}{\partial\lambda^\alpha} + \theta^\alpha \frac{\partial}{\partial\theta^\alpha}\right) \right\rangle \tag{4.4}$$

$$H^1(\mathrm{Vect}(X))_{\mathrm{even}} = \mathbf{C}\left\langle [(\!(\lambda\theta)\!) \otimes \theta]^{1/2} \otimes \left(\lambda\Gamma_{mn}\frac{\partial}{\partial\lambda} + \theta\Gamma_{mn}\frac{\partial}{\partial\theta}\right) \right\rangle \tag{4.5}$$

In the rest of this section we will explain the computation.

4.3 Exact sequences

Consider the short exact sequence of complexes:

$$0 \longrightarrow \Gamma(U) \longrightarrow \mathrm{Vect}(X) \longrightarrow \Gamma(TX/U) \longrightarrow 0 \tag{4.6}$$

The corresponding long exact sequence in cohomology of $[Q^{(0)}, \_]$ is:

$$\begin{aligned} &\longrightarrow H^{n-1}(\Gamma(U)) \longrightarrow H^{n-1}(\mathrm{Vect}(X)) \longrightarrow H^{n-1}(\Gamma(TX/U)) \longrightarrow \\ &\longrightarrow H^{n}(\Gamma(U)) \longrightarrow H^{n}(\mathrm{Vect}(X)) \longrightarrow H^{n}(\Gamma(TX/U)) \longrightarrow \\ &\longrightarrow H^{n+1}(\Gamma(U)) \longrightarrow \dots \end{aligned}$$

4.4 Computation of $H^1(\mathrm{Vect}(X))_{\mathrm{odd}}$

4.4.1 Summary of result

We use the following segment of the long exact sequence:

$$\begin{aligned} &H^0(\Gamma(TX/U))_{\mathrm{even}} \xrightarrow{\delta} \\ &\xrightarrow{\delta} H^1(\Gamma(U))_{\mathrm{odd}} \longrightarrow H^1(\mathrm{Vect}(X))_{\mathrm{odd}} \longrightarrow H^1(\Gamma(TX/U))_{\mathrm{odd}} \xrightarrow{\delta} \\ &\xrightarrow{\delta} H^2(\Gamma(U))_{\mathrm{even}} \end{aligned}$$

The cohomology groups participating in this segment have the following description:

$$H^0(\Gamma(TX/U))_{\mathrm{even}} = \mathbf{C}\,\langle D, M_{mn}\rangle \text{ of eq. (4.12)}$$

$$H^1(\Gamma(U))_{\mathrm{odd}} = \mathbf{C}\left\langle (\lambda\Gamma^m\theta)(\theta\Gamma_m)_\alpha \frac{\partial}{\partial\theta^\beta}\right\rangle \tag{4.7}$$

$$\left[\delta \;:\; H^0(\Gamma(TX/U))_{\mathrm{even}} \longrightarrow H^1(\Gamma(U))_{\mathrm{odd}}\right] = 0 \quad \text{Section 4.4.2} \tag{4.8}$$

$$H^1(\Gamma(TX/U))_{\mathrm{odd}} = \mathbf{C}\left\langle (\lambda\Gamma^m\theta)\left(\lambda^\alpha\frac{\partial}{\partial\lambda^\alpha}\right)\right\rangle \tag{4.9}$$

Section 4.4.2

$$\left[\delta \;:\; H^1(\Gamma(TX/U))_{\mathrm{odd}} \longrightarrow H^2(\Gamma(U))_{\mathrm{even}}\right] = 0 \tag{4.10}$$

This implies:

$$H^1(\mathrm{Vect}(X))_{\mathrm{odd}} = H^1(\Gamma(U))_{\mathrm{odd}} \oplus H^1(\Gamma(TX/U))_{\mathrm{odd}} = \\ = \mathbf{C}\left\langle\; (\lambda\Gamma^m\theta)(\theta\Gamma_m)_\alpha\frac{\partial}{\partial\theta^\beta}\,,\; (\lambda\Gamma^m\theta)\left(\lambda^\alpha\frac{\partial}{\partial\lambda^\alpha} + \theta^\alpha\frac{\partial}{\partial\theta^\alpha}\right)\;\right\rangle \tag{4.11}$$

We will now explain the computation.

4.4.2 Computation

$\Gamma(TX/U)$. The space $\Gamma(TX/U)$ is generated as an $\mathcal{O}(X)$-module, by the following vector fields:

$$D = \lambda^\alpha \frac{\partial}{\partial\lambda^\alpha}\,, \quad M_{mn} = (\lambda\gamma_{mn})^\alpha \frac{\partial}{\partial\lambda^\alpha} \tag{4.12}$$

However $\Gamma(TX/U)$ is *not* a free $\mathcal{O}(X)$ module, because there is a relation:

$$\frac{1}{10}(\lambda\gamma^{mn})^\alpha M_{mn} + \lambda^\alpha D = 0 \tag{4.13}$$

$\delta\ :\ H^0(\Gamma(TX/U))_{\rm even} \longrightarrow H^1(\Gamma(U))_{\rm odd}$. It is zero because both D and M_{mn} can be extended to elements of $\mathrm{Vect}(X)$ commuting with $Q^{(0)}$:

$$D \mapsto \lambda^\alpha \frac{\partial}{\partial\lambda^\alpha} + \theta^\alpha \frac{\partial}{\partial\theta^\alpha} \tag{4.14}$$

$$M_{mn} \mapsto \left(\lambda\Gamma_{mn}\frac{\partial}{\partial\lambda}\right) + \left(\theta\Gamma_{mn}\frac{\partial}{\partial\theta}\right) \tag{4.15}$$

$H^1(\Gamma(TX/U))_{\rm odd}$ and $\delta\ :\ H^1(\Gamma(TX/U))_{\rm odd} \to H^2(\Gamma(U))_{\rm even}$. For any tensor A^{lmn}, consider vector fields of the form:

$$A^{l,mn}(\lambda\Gamma_l\theta)\left(\lambda\Gamma_m\Gamma_n\frac{\partial}{\partial\lambda}\right) \tag{4.16}$$

Such vector fields generate $Z^1(\Gamma(TX/U))_{\rm odd}$. But some of them are $Q^{(0)}$-exact:

$$(\lambda\Gamma_{[l}\theta)\left(\lambda\Gamma_{m]}\Gamma_n\frac{\partial}{\partial\lambda}\right) = \frac{1}{4}\left[Q^{(0)}, (\theta\Gamma_{lm}\Gamma_p\theta)\left(\lambda\Gamma_p\Gamma_n\frac{\partial}{\partial\lambda}\right)\right] \ \texttt{mod}\ \Gamma(U) \tag{4.17}$$

Therefore the vector fields of the form eq. (4.16) with A^{lmn} of the form:

$$A^{l,mn} = X^{[lm]n} + Y^{l(mn)}\,, \quad Y^{lmm} = 0 \tag{4.18}$$

are zero in $H^1(\Gamma(TX/U))_{\rm odd}$. This implies that $H^1(\Gamma(TX/U))_{\rm odd}$ is generated by the vector fields of the form:

$$(\lambda\Gamma^i\theta)\left(\lambda^\alpha\frac{\partial}{\partial\lambda^\alpha}\right) \tag{4.19}$$

A vector field of eq. (4.16) is zero in cohomology iff:

$$A^{l,lm} - A^{l,ml} - A^{m,ll} = 0 \tag{4.20}$$

Vector fields of the form (4.19) correspond to:

$$\begin{aligned} A^{l,mn} &= \frac{1}{10}\delta_{il}\delta_{mn} \\ A^{l,lm} - A^{l,ml} - A^{m,ll} &= -\delta_{im} \end{aligned}$$

Notice that the section of $\Gamma(TX/U)$ defined by eq. (4.19) can be extended to a $[Q^{(0)}, \_]$-closed section of TX:

$$(\lambda\Gamma^m\theta)\left(\left(\lambda^\alpha\frac{\partial}{\partial\lambda^\alpha}\right) + \left(\theta^\alpha\frac{\partial}{\partial\theta^\alpha}\right)\right) \tag{4.21}$$

This means that $\delta\ :\ H^1(\Gamma(TX/U))_{\rm odd} \to H^2(\Gamma(U))_{\rm even}$ is zero.

4.5 Computation of $H^1(\Gamma(TX))_{\text{even}}$

4.5.1 Summary of result

$$H^1(\text{Vect}(X))_{\text{even}} = \mathbf{C}\left\langle (\lambda\Gamma^m\theta)\Gamma_{m\alpha\beta}\theta^\beta \left(\lambda\Gamma_{kl}\frac{\partial}{\partial\lambda} + \theta\Gamma_{kl}\frac{\partial}{\partial\theta}\right)\right\rangle \tag{4.22}$$

4.5.2 Computation

We use the following segment of the long exact sequence:

$$\begin{aligned}
&H^0(\Gamma(TX/U))_{\text{odd}} \xrightarrow{\delta} \\
\xrightarrow{\delta} &H^1(\Gamma(U))_{\text{even}} \longrightarrow H^1(\text{Vect}(X))_{\text{even}} \longrightarrow H^1(\Gamma(TX/U))_{\text{even}} \xrightarrow{\delta} \\
\xrightarrow{\delta} &H^2(\Gamma(U))_{\text{odd}}
\end{aligned}$$

$H^0(\Gamma(TX/U))_{\text{odd}}$. Is generated by:

$$Z^q_\alpha = (\Gamma_p\theta)_\alpha \left(\lambda\Gamma^p\Gamma^q\frac{\partial}{\partial\lambda}\right) \tag{4.23}$$

$H^1(\Gamma(U))_{\text{even}}$. Is generated by:

$$Y^m_\alpha = (\theta\Gamma^m\lambda)\frac{\partial}{\partial\theta^\alpha} \tag{4.24}$$

$\delta \,:\, H^0(\Gamma(TX/U))_{\text{odd}} \longrightarrow H^1(\Gamma(U))_{\text{even}}$.

$$\begin{aligned}
\delta Z^q_\alpha &= (\Gamma_p\theta)_\alpha \left(\lambda\Gamma^p\Gamma^q\frac{\partial}{\partial\theta}\right) = \\
&= (\theta\Gamma^p\lambda)\left(\Gamma_p\Gamma^q\frac{\partial}{\partial\theta}\right)_\alpha \mod [Q^{(0)}, _] = (\Gamma_m\Gamma^q)^\beta_\alpha Y^m_\beta \mod [Q^{(0)}, _]
\end{aligned} \tag{4.25}$$

The linear map $Y^q_\alpha \mapsto (\Gamma_m\Gamma^q)^\beta_\alpha Y^m_\beta$ is a bijection. Therefore $H^0(\Gamma(TX/U))_{\text{odd}}$ cancels with $H^1(\Gamma(U))_{\text{even}}$.

$H^1(\Gamma(TX/U))_{\text{even}}$ and $\delta \,:\, H^1(\Gamma(TX/U))_{\text{even}} \to H^2(\Gamma(U))_{\text{odd}}$. The space of cocycles $Z^1(\Gamma(TX/U))_{\text{even}}$ is generated by:

$$\begin{gathered}
[((\lambda\theta)) \otimes \theta]^{1/2} D \\
[((\lambda\theta)) \otimes \theta]^{1/2} M_{mn}
\end{gathered}$$

where D and M_{mn} are from eq. (4.12). Since both D and M_{mn} extend to $[Q^{(0)}, \_]$-closed sections of TX by eqs. (4.14) and (4.15), the coboundary operator $\delta \,:\, H^1(\Gamma(TX/U))_{\text{even}} \to H^2(\Gamma(U))_{\text{odd}}$ is zero.

But some cocycles are exact. Indeed, as sections of TX/U:

$$
\begin{aligned}
&Q^{(0)}\left(\Gamma_{pq}\Gamma_m\theta\,(\theta\Gamma_{kpq}\theta)\left(\lambda\Gamma^k\Gamma^m\frac{\partial}{\partial\lambda}\right)\right)=\\
&=2\Gamma_{pq}\Gamma^k\lambda(\theta\Gamma_{kpq}\theta)\left(\lambda\frac{\partial}{\partial\lambda}\right)+4\Gamma_{nq}\Gamma_m\theta(\theta\Gamma^q\lambda)\left(\lambda\Gamma^n\Gamma^m\frac{\partial}{\partial\lambda}\right)=\\
&=2\Gamma_{kpq}\lambda(\theta\Gamma_{kpq}\theta)\left(\lambda\frac{\partial}{\partial\lambda}\right)+4\Gamma_n\Gamma_q\Gamma_m\theta(\theta\Gamma^q\lambda)\left(\lambda\Gamma^n\Gamma^m\frac{\partial}{\partial\lambda}\right)=\\
&=2\Gamma_{kpq}\lambda(\theta\Gamma_{kpq}\theta)\left(\lambda\frac{\partial}{\partial\lambda}\right)-4\Gamma_n\Gamma_m\Gamma_q\theta(\theta\Gamma^q\lambda)\left(\lambda\Gamma^n\Gamma^m\frac{\partial}{\partial\lambda}\right)+8\Gamma_n\theta(\theta\Gamma^m\lambda)\left(\lambda\Gamma^n\Gamma^m\frac{\partial}{\partial\lambda}\right)=\\
&=2\Gamma_{kpq}\lambda(\theta\Gamma_{kpq}\theta)\left(\lambda\frac{\partial}{\partial\lambda}\right)-4\Gamma_n\Gamma_m\Gamma_q\theta(\theta\Gamma^q\lambda)\left(\lambda\Gamma^n\Gamma^m\frac{\partial}{\partial\lambda}\right)+16\Gamma_m\theta(\theta\Gamma^m\lambda)\left(\lambda\frac{\partial}{\partial\lambda}\right)=\\
&=-32\Gamma_m\theta(\theta\Gamma^m\lambda)\left(\lambda\frac{\partial}{\partial\lambda}\right)-4\Gamma_n\Gamma_m\Gamma_q\theta(\theta\Gamma^q\lambda)\left(\lambda\Gamma^n\Gamma^m\frac{\partial}{\partial\lambda}\right)
\end{aligned}
$$

5 Coefficients of normal form satisfy wave equations

Modulo F^4Vect we can choose the coordinates so that:

$$Q_L=\lambda_L\frac{\partial}{\partial\theta_L}+((\lambda_L\theta_L))^m E^{L\mu}_m\frac{\partial}{\partial x^\mu}+ \tag{5.1}$$

$$+((\lambda_L\theta_L))^m\left(\lambda_L\Omega_L{}^L_{Lm}\partial_{\lambda_L}+\theta_L\Omega_L{}^L_{Lm}\partial_{\theta_L}+\lambda_R\Omega_L{}^R_{Rm}\partial_{\lambda_R}+\theta_R\Omega_L{}^R_{Rm}\partial_{\theta_R}\right)+ \tag{5.2}$$

$$+((\lambda_L\theta_L\theta_L))\left(P_{LL}\frac{\partial}{\partial\theta_L}+P_{LR}\frac{\partial}{\partial\theta_R}\right)\mod F^4 \tag{5.3}$$

$$
\begin{aligned}
Q_R=&\lambda_R\frac{\partial}{\partial\theta_R}+((\lambda_R\theta_R))^m E^{R\mu}_m\frac{\partial}{\partial x^\mu}+\\
&+((\lambda_R\theta_R))^m\left(\lambda_R\Omega_R{}^R_{Rm}\partial_{\lambda_R}+\theta_R\Omega_R{}^R_{Rm}\partial_{\theta_R}+\lambda_L\Omega_R{}^L_{Lm}\partial_{\lambda_L}+\theta_L\Omega_R{}^L_{Lm}\partial_{\theta_L}\right)+\\
&+((\lambda_R\theta_R\theta_R))\left(P_{RL}\frac{\partial}{\partial\theta_L}+P_{RR}\frac{\partial}{\partial\theta_R}\right)\mod F^4
\end{aligned}
\tag{5.4}
$$

where E,Ω,P are some functions of x. Indeed, using section 3.4:

- $H^1(C^{\rm fun}_L)_{\rm odd}\otimes\frac{\partial}{\partial x}$ enters on Line (5.1),
- Second part of $H^1(C^{\rm vect}_L)_{\rm odd}$ (see eq. (4.11)) and $H^1(C^{\rm fun}_L)_{\rm odd}\otimes H^0(C^{\rm vect}_R)_{\rm even}$ on Line (5.2),
- First part of $H^1(C^{\rm vect}_L)_{\rm odd}$ (see eq. (4.11)) and $H^1(C^{\rm fun}_L)_{\rm even}\otimes H^0(C^{\rm vect}_R)_{\rm odd}$ on Line (5.3)

5.1 Equations for tetrad and spin connection

5.1.1 Fixing $(so(10)\oplus\mathbf{C})_L$ and $(so(10)\oplus\mathbf{C})_R$

Let us study the linearized order in deviations from flat space-time. In flat space-time $E^{L\mu}_m=E^{R\mu}_m=\delta^\mu_m$. The deviation from flatness can be written as:

$$E^{L\mu}_m=\delta^\mu_m+\delta^\mu_n e^L_{n,m}\quad\texttt{and}\quad E^{R\mu}_m=\delta^\mu_m+\delta^\mu_n e^R_{n,m} \tag{5.5}$$

where e^L and e^R are infinitesimal. We assume summation over repeated indices. We can choose a freedom of $so(10) \oplus \mathbf{C}$ redefinitions of both (λ_L, θ_R) and (λ_R, θ_R) to fix:

$$\begin{aligned} e^L_{[m,n]} &= e^R_{[m,n]} = 0 \\ e^L_{m,m} &= e^R_{m,m} \end{aligned} \tag{5.6}$$

At this point, the only remaining freedom in redefinition of λ and θ is overall rescaling of $(\lambda_L, \lambda_R, \theta_L, \theta_R)$. We fixed both $(so(10) \oplus \mathbf{C})_L \oplus (so(10) \oplus \mathbf{C})_R$ down to the diagonal $\mathbf{C}$.

5.1.2 Fixing $\Omega_L{}^L_L$ and $\Omega_R{}^R_R$

According to section 4.4.2, we can choose:

$$\Omega_L{}^L_{Lm,lk} = \frac{1}{10} \Omega_L{}^{L(s)}_{Lm} \delta_{lk} \tag{5.7}$$

From $\{Q_L, Q_R\} = 0$, the coefficient of $((\lambda_L \theta_L))^l ((\lambda_R \theta_R))^n \left(\lambda_L \frac{\partial}{\partial \lambda_L} + \theta_L \frac{\partial}{\partial \theta_L}\right)$, projected to $H^1(\Gamma(TX))$ (see eq. (4.20)):

$$2\partial_m \Omega_R{}^L_{Ln,[ml]} - \partial_l \Omega_R{}^L_{Ln,mm} + \partial_n \Omega_L{}^{L(s)}_{Ll} = 0 \tag{5.8}$$

Similarly with $L \leftrightarrow R$:

$$2\partial_m \Omega_L{}^R_{Rn,[ml]} - \partial_l \Omega_L{}^R_{Rn,mm} + \partial_n \Omega_R{}^{R(s)}_{Rl} = 0 \tag{5.9}$$

Eqs. (5.7) and (5.8) and similar equations with $L \leftrightarrow R$ determine $\Omega_L{}^L_L$ and $\Omega_R{}^R_R$ in terms of $\Omega_R{}^L_L$ and $\Omega_L{}^R_R$ Let us denote:

$$\Omega^L_{m,nk} = -\Omega_R{}^L_{Lm,[nk]} + \frac{1}{2} \Omega_R{}^L_{Lm,pp} \delta_{nk} \tag{5.10}$$

and similar definition for $\Omega^R_{m,nk}$ in terms of $\Omega_L{}^R_R$.

This notation is useful, because for any vector V_l:

$$\Omega_R{}^L_{Lm,nk} V_l \left(\Gamma_n \Gamma_k \Gamma_l + \Gamma_l (\Gamma_n \Gamma_k)^T\right) = 4 V_p \Omega^L_{m,pq} \Gamma_q \tag{5.11}$$

From $\{Q_L, Q_R\} = 0$, the coefficient of $((\lambda_L \theta_L))^m ((\lambda_R \theta_R))^n \frac{\partial}{\partial x}$:

$$\begin{aligned} \frac{\partial}{\partial x^m} E^R_n + E^R_k \Omega^R_{m,kn} &= \frac{\partial}{\partial x^n} E^L_m + E^L_k \Omega^L_{n,km} \\ \partial_m e_{Rn,k} + \Omega^R_{m,nk} &= \partial_n e_{Lm,k} + \Omega^L_{n,mk} \end{aligned}$$

This implies:

$$\partial_{[m} (e_L + e_R)_{n],k} + \left(\Omega^L + \Omega^R\right)_{[m,n]k} = 0 \tag{5.12}$$

$$\partial_{(m} (e_L - e_R)_{n),k} + \left(\Omega^L - \Omega^R\right)_{(m,n)k} = 0 \tag{5.13}$$

Eq. (5.12) is zero torsion of the "average" (i.e. left plus right) connection.

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Let us denote:

$$g_{mn} = e_{L(m,n)} + e_{R(m,n)}$$
$$\Omega_{m,nk} = \frac{1}{2}\left(\Omega^L + \Omega^R\right)_{m,nk}$$

Then eq. (5.12) implies the existence of a_m such that:

$$\Omega_{m,nk} = -g_{m[n} \overleftarrow{\partial}_{k]} + a_m\delta_{nk} - 2\delta_{m[n}a_{k]} \tag{5.14}$$

Infinitesimal coordinate redefinition $\tilde{x}^\mu = x^\mu + \varepsilon v^\mu$ corresponds to:

$$\delta_v e_{Lm,k} = \delta_v e_{Rm,k} = \partial_{(m}v_{k)}$$
$$\delta_v\Omega_{m,nk} = \partial_m(\partial_{[n}v_{k]})$$
$$\delta\Omega_{L}{}^{L(s)}_{Lm} = 2\partial_p\partial_{[p}v_{n]}$$

The overall rescaling

$$\delta_\gamma(\lambda_L, \lambda_R, \theta_L, \theta_R) = (\gamma\lambda_L, \gamma\lambda_R, \gamma\theta_L, \gamma\theta_R) \tag{5.15}$$

corresponds to:

$$\delta_\gamma g_{m,n} = 2\gamma\delta_{mn} \tag{5.16}$$
$$\delta_\gamma a_m = -\partial_m\gamma \tag{5.17}$$
$$\delta_\gamma\Omega_{m,nk} = -\partial_m\gamma\delta_{nk} \tag{5.18}$$

From $\{Q_L, Q_L\} = 0$ and $\{Q_R, Q_R\} = 0$ follows that $\Omega_L{}^R_{Rm}$ and $\Omega_R{}^L_{Lm}$ both satisfy Maxwell equations:

$$\frac{\partial}{\partial x^n}\frac{\partial}{\partial x^{[m}}\Omega_L{}^R_{Rn]pq} = \frac{\partial}{\partial x^n}\frac{\partial}{\partial x^{[m}}\Omega_R{}^L_{Ln]pq} = 0 \tag{5.19}$$

Considering the scalar part, we conclude that a_m satisfies the Maxwell equations:

$$\partial_m\partial_{[m}a_{n]} = 0 \tag{5.20}$$

and g_{mn} satisfies:

$$\begin{aligned} &\partial_p\partial_{[p}g_{m][n} \overleftarrow{\partial}_{k]} + 2\partial_p\partial_{[p}\delta_{m][n}a_{k]} = 0 \\ \Rightarrow\ &\partial_k\left(2\partial_{[p}g_{n][m} \overleftarrow{\partial}_{p]} + \partial_m a_n + \delta_{mn}\partial^p a_p\right) - (k \leftrightarrow n) = 0 \\ \Rightarrow\ &\exists b_m\ :\ 2\partial_{[p}g_{n][m} \overleftarrow{\partial}_{p]} + \partial_m a_n + \delta_{mn}\partial^p a_p = -\partial_n b_m \end{aligned} \tag{5.21}$$

It follows from the symmetry $m \leftrightarrow n$ that exists ϕ such that $b_m = a_m - \partial_m\phi$. Therefore:

$$2\partial_{[p}g_{n][m} \overleftarrow{\partial}_{p]} + \delta_{mn}\partial^p a_p + 2\partial_{(m}a_{n)} = \partial_m\partial_n\phi \tag{5.22}$$

The rescaling eqs. (5.16), (5.17) and (5.18) are accompanied by:

$$\delta_\gamma\phi = (10-4)\gamma \tag{5.23}$$

Eq. (5.21) is actually the consistency of the sum of eq. (5.8) and eq. (5.9).

We can fix the δ_γ gauge transformations by requiring $\partial_p a_p = 0$. In this gauge eq. (5.22) implies that the Riemann-Christoffel tensor is harmonic:

$$\partial_p \partial_p \, \partial_{[i} g_{n][m} \overleftarrow{\partial}_{j]} = 0$$

and therefore, up to a finite-dimesional space, that ϕ is harmonic.

Eq. (5.13) implies, after total symmetrization:

$$\partial_{(m}(e_L - e_R)_{n,k)} + \Omega^{L(s)}_{(m} \delta_{nk)} - \Omega^{R(s)}_{(m} \delta_{nk)} = 0 \tag{5.24}$$

Modulo finite dimensional spaces, eqs. (5.24), (5.13) and (5.6) imply that (cf. eq. (A.9)):

$$\begin{aligned} e_L - e_R &= 0 \\ \Omega^{L(s)}_m - \Omega^{R(s)}_m &= 0 \\ \left(\Omega^L - \Omega^R\right)_{(m,n)k} &= 0 \end{aligned} \tag{5.25}$$

Therefore $\Omega^L - \Omega^R$ is antisymmetric:

$$\left(\Omega^L - \Omega^R\right)_{k,lm} = H_{klm} = H_{[klm]} \tag{5.26}$$

Eqs. (5.19) imply:

$$\partial^p \partial_{[p} H_{q]mn} = 0 \tag{5.27}$$

The consistency of the difference of eq. (5.8) and eq. (5.9) implies that H_{lmn} is harmonic:

$$\partial_p \partial^p H_{lmn} = 0 \tag{5.28}$$

and, modulo a constant, divergenceless:

$$\partial^p H_{pmn} = 0 \tag{5.29}$$

The antisymmetric tensor field H_{lmn} should be identified with the field strength of the NSNS B-field: $H = dB$. However, our considerations do not imlpy that $dH = 0$. All we can say is, the space of solutions to eqs. (5.27), (5.29) has *a subspace* consisting of H_{lmn} satisfying the equations:

$$\partial_{[k} H_{lmn]} = 0 \quad \texttt{and} \quad \partial^l H_{lmn} = 0 \tag{5.30}$$

5.2 Equations for Ramond-Ramond bispinor

To get $\{Q_L, Q_R\} = 0$ we need to require (see section 4.5):

$$\begin{aligned} \Gamma^m_{\beta\alpha} \nabla^L_m P^{\hat\alpha\alpha}_{RL} &= 0 \\ \nabla^L_m P^{\hat\alpha\hat\beta}_{RR} &= 0 \\ \Gamma^m_{\hat\beta\hat\alpha} \nabla^R_m P^{\alpha\hat\alpha}_{LR} &= 0 \\ \nabla^R_m P^{\alpha\beta}_{LL} &= 0 \end{aligned}$$

To get $\{Q_L, Q_L\} = 0$ and $\{Q_R, Q_R\} = 0$ we need, in addition:

$$\Gamma^m_{\beta\alpha} \nabla^L_m P^{\alpha\hat{\alpha}}_{LR} = 0$$
$$\Gamma^m_{\hat{\beta}\hat{\alpha}} \nabla^R_m P^{\alpha\hat{\alpha}}_{RL} = 0$$

For generic E and Ω, there are no covariantly constant tensors. Therefore:

$$P_{LL} = P_{RR} = 0 \tag{5.31}$$

and the bispinors P_{LR} and P_{RL} both satisfy Dirac equatins in both spinorial indices.

5.3 Conclusion: Q is defined by solutions of hyperbolic partial differential equations

How many solutions does the equation $Q^2 = 0$ have? In principle, it could be that the solutions are parametrized by arbitrary functions of x. However, it so happens, that Q is comletely determined by a set of solutions of hyperbolic equations (such as Maxwell or Einstein equations). Indeed, the structure of the cohomology of $\left[\lambda_L \frac{\partial}{\partial\theta_L}, \_\right] + \left[\lambda_R \frac{\partial}{\partial\theta_R}, \_\right]$ on vector fields (section 4.4.2) imlplies that the Q is completely determined by the first few terms in the θ-expansion listed in eqs. (5.3) and (5.4). The coefficients $\Omega_L{}^R_{Rm}$ and $\Omega_R{}^L_{Lm}$ satisfy Maxwell equations, and then $\Omega_L{}^L_{Lm}$ is determined by eq. (5.8), and $\Omega_R{}^R_{Rm}$ by a similar equation with $L \leftrightarrow R$. The coefficients P_{LL} and P_{RR} are zero modes. The coefficients P_{LR} and P_{RL} satisfy the Dirac equations.

5.4 Difference with SUGRA equations

Nevertheless, $Q^2 = 0$ is *weaker* (less constraining) than supergravity equations of motion. We have ingnored several finite-dimensional spaces of solutios, somewhat similar to the "nonphysical vertices" of [9]. But besides that, even the "main" spaces of solutions those which satisfy hyperbolic equations, are larger than the spaces of supergravity solutions:

5.4.1 Vector field a_m

The field a_m defined in eq. (5.14) is absent in Type II supergravity. If $a_m = 0$, the field ϕ corresponds to the dilaton and $g_{ij} - \frac{1}{2}\delta_{ij}\phi$ to the graviton in the Einstein frame.

5.4.2 Extra components of H_{lmn}

Eqs. (5.27), (5.29) are weaker than the SUGRA eqs. (5.30).

5.4.3 Doubling of the RR bispinor

Instead of one RR field P we have two: P_{LR} and P_{RL}.

6 Fermionic fields

In section 4.4.2 we restricted ourselves with Q_L and Q_R parameterized by *even* functions $E^L, E^R, \ldots$. We will now add the terms parameterized by *odd* functions. According to section 4.5 these terms are:

$$Q'_L = ((\lambda_L\theta_L\theta_L))_\alpha \psi_L^{\alpha\mu}(x)\frac{\partial}{\partial x^\mu} \; + \xi^{\hat\beta}_{LRm}(x)((\lambda_L\theta_L))^m \frac{\partial}{\partial \theta_R^{\hat\beta}} \; + \\ + \; \Xi_L{}_L^{L\alpha[mn]}(x)((\lambda_L\theta_L\theta_L))_\alpha \left(\lambda_L\Gamma_{mn}\frac{\partial}{\partial\lambda_L} + \theta_L\Gamma_{mn}\frac{\partial}{\partial\theta_L}\right) \; + \\ + \; \Xi_L{}_R^{R\alpha mn}(x)((\lambda_L\theta_L\theta_L))_\alpha \left(\lambda_R\Gamma_m\Gamma_n\frac{\partial}{\partial\lambda_R} + \theta_R\Gamma_m\Gamma_n\frac{\partial}{\partial\theta_R}\right) \; +$$

$$Q'_R = ((\lambda_R\theta_R\theta_R))_{\hat\alpha} \psi_R^{\hat\alpha\mu}(x)\frac{\partial}{\partial x^\mu} \; + \xi^{\beta}_{RLm}(x)((\lambda_R\theta_R))^m \frac{\partial}{\partial \theta_L^{\beta}} \; + \\ + \; \Xi_R{}_R^{R\hat\alpha[mn]}(x)((\lambda_R\theta_R\theta_R))_{\hat\alpha} \left(\lambda_R\Gamma_{mn}\frac{\partial}{\partial\lambda_R} + \theta_R\Gamma_{mn}\frac{\partial}{\partial\theta_R}\right) \; + \\ + \; \Xi_R{}_L^{L\hat\alpha mn}(x)((\lambda_R\theta_R\theta_R))_{\hat\alpha} \left(\lambda_L\Gamma_m\Gamma_n\frac{\partial}{\partial\lambda_L} + \theta_L\Gamma_m\Gamma_n\frac{\partial}{\partial\theta_L}\right) \; +$$

Considering the coefficient of $((\lambda_L\theta_L\theta_L))((\lambda_R\theta_R))\frac{\partial}{\partial x}$, we deduce that $\psi_L^{\alpha\mu}$ satisfies:

$$\partial^\nu \psi_L^{\alpha\mu} + 4\Xi_L{}_R^{R\alpha[\nu\mu]} - 2\Xi_L{}_R^{R\alpha mm} g^{\mu\nu} = 0 \tag{6.1}$$

and a similar equation for $\psi_R^{\hat\alpha\mu}$. This implies (see section A) that modulo finite dimensional subspaces (which we ignore):

$$\begin{aligned} \psi_L^{\alpha\mu} &= 0 \\ \Xi_L{}_R^{R\alpha\nu\mu} &= 0 \\ \psi_R^{\hat\alpha\mu} &= 0 \\ \Xi_R{}_L^{L\hat\alpha\nu\mu} &= 0 \end{aligned}$$

The coefficients $\xi^{\hat\alpha}_{LRm}$ and ξ^{α}_{RLm} come with gauge transformations:

$$\begin{aligned} \delta_{\phi_L}\xi^{\hat\alpha}_{LRm} &= \partial_m \phi_L^{\hat\alpha} \\ \delta_{\phi_R}\xi^{\alpha}_{RLm} &= \partial_m \phi_R^{\alpha} \end{aligned}$$

Considering the coefficient of $((\lambda_L\theta_L\theta_L))((\lambda_R\theta_R))\left(\lambda_L\frac{\partial}{\partial\lambda_L} + \theta_L\frac{\partial}{\partial\theta_L}\right)$, we conclude that $\Xi_L{}_L^{L\alpha[mn]}$ (and similarly $\Xi_R{}_R^{R\hat\alpha[mn]}$) are constants, and we ignore them.

Requiring $Q_L^2 = 0$, the "Maxwell bishop move":

$$\xi^{\hat\beta}_{LRm}(x)((\lambda_L\theta_L))^m \xrightarrow{((\lambda_L\theta_L))\partial_x} \partial_n\xi^{\hat\beta}_{LRm}(x)((\lambda_L\theta_L))^m((\lambda_L\theta_L))^n \xrightarrow{(\lambda_L\partial_{\theta_L})^{-1}} \ldots \xrightarrow{((\lambda_L\theta_L))\partial_x} \\ \longrightarrow \; \partial^n\partial_{[n}\xi^{\hat\beta}_{LRm]}(x)((\lambda_L\theta_L))^p((\lambda_L\theta_L))^q((\theta_L\theta_L))^{pqm}$$

we conclude that ξ_{LR} (and similarly ξ_{RL}) should satisfy the Maxwell equations:

$$\partial_m\partial_{[m}\xi_{LRn]} = 0 \tag{6.2}$$

To summarize, the part of Q which involves fermionic fields is very simple:

$$Q'_L = \xi^{\hat\beta}_{LRm}(x)((\lambda_L\theta_L))^m \frac{\partial}{\partial\theta^{\hat\beta}_R}$$
$$Q'_R = \xi^{\beta}_{RLm}(x)((\lambda_R\theta_R))^m \frac{\partial}{\partial\theta^{\beta}_L}$$

The only fermionic superfields of [3] are $C^{\alpha\hat\gamma}_{\beta}$ and $C^{\hat\alpha\gamma}_{\hat\beta}$. The top component of $C^{\alpha\hat\gamma}_{\beta}$ corresponds to $\partial_{[m}\xi^{\hat\gamma}_{LRn]}(\Gamma^{mn})^{\alpha}_{\beta}$, and the top component of $\hat C^{\hat\alpha\gamma}_{\hat\beta}$ to $\partial_{[m}\xi^{\gamma}_{RLn]}(\Gamma^{mn})^{\hat\alpha}_{\hat\beta}$.

7 Supersymmetries and dilatation

The vector field Q^{flat} of eq. (1.5) is manifestly supersymmetry-invariant. In other words, it commutes with the super-Poincare algebra, which is generated by $\frac{\partial}{\partial\theta^\alpha_L} - \Gamma^m_{\alpha\beta}\theta^\beta_L\frac{\partial}{\partial x^m}$ and $\frac{\partial}{\partial\theta^{\hat\alpha}_R} - \Gamma^m_{\hat\alpha\hat\beta}\theta^{\hat\beta}_R\frac{\partial}{\partial x^m}$. It is also invariant under dilatations, if we define the weight of x to be twice the weight of θ_L, θ_R. It is perhaps less straightforward to see that there are no other symmetries. For example, there are no conformal symmetries. (But the dilatation symmetry is present.) We will now prove that there are no other symmetries.

We have to compute the cohomology of Q^{flat} in the space of vector fields of ghost number 0. The cohomology of $\lambda^\alpha_L\frac{\partial}{\partial\theta^\alpha_L} + \lambda^{\hat\alpha}_R\frac{\partial}{\partial\theta^{\hat\alpha}_L}$ at the ghost number 0 is (see section 4):

$$T_m = \frac{\partial}{\partial x^m}$$
$$S^L_\alpha = \frac{\partial}{\partial\theta^\alpha_L}$$
$$S^R_{\hat\alpha} = \frac{\partial}{\partial\theta^{\hat\alpha}_R}$$
$$D^L = \lambda^\alpha_L\frac{\partial}{\partial\lambda^\alpha_L} + \theta^\alpha_L\frac{\partial}{\partial\theta^\alpha_L}$$
$$M^L_{mn} = \left(\lambda_L\Gamma_{mn}\frac{\partial}{\partial\lambda_L}\right) + \left(\theta_L\Gamma_{mn}\frac{\partial}{\partial\theta_L}\right)$$
$$D^R = \lambda^{\hat\alpha}_R\frac{\partial}{\partial\lambda^{\hat\alpha}_R} + \theta^{\hat\alpha}_R\frac{\partial}{\partial\theta^{\hat\alpha}_R}$$
$$M^R_{mn} = \left(\lambda_R\Gamma_{mn}\frac{\partial}{\partial\lambda_R}\right) + \left(\theta_R\Gamma_{mn}\frac{\partial}{\partial\theta_R}\right)$$

This means that any infinitesimal symmetry can be brought to the form:

$$\begin{aligned}
v = {}& T^m(x)\frac{\partial}{\partial x^m} + \\
&+ D_L(x)\left(\lambda^\alpha_L\frac{\partial}{\partial\lambda^\alpha_L} + \theta^\alpha_L\frac{\partial}{\partial\theta^\alpha_L}\right) + M^{mn}_L(x)\left(\left(\lambda_L\Gamma_{mn}\frac{\partial}{\partial\lambda_L}\right) + \left(\theta_L\Gamma_{mn}\frac{\partial}{\partial\theta_L}\right)\right) + \\
&+ D_R(x)\left(\lambda^{\hat\alpha}_R\frac{\partial}{\partial\lambda^{\hat\alpha}_R} + \theta^{\hat\alpha}_R\frac{\partial}{\partial\theta^{\hat\alpha}_R}\right) + M^{mn}_R(x)\left(\left(\lambda_R\Gamma_{mn}\frac{\partial}{\partial\lambda_R}\right) + \left(\theta_R\Gamma_{mn}\frac{\partial}{\partial\theta_R}\right)\right) + \\
&+ S^\alpha_L(x)\frac{\partial}{\partial\theta^\alpha_L} + S^{\hat\alpha}_R(x)\frac{\partial}{\partial\theta^{\hat\alpha}_R} + \dots
\end{aligned}$$

where ... stand for terms of the higher order in the grading defined by eq. (3.5). Commuting v with $((\!(\lambda_L\theta_L)\!)^m + (\!(\lambda_R\theta_R)\!)^m)\frac{\partial}{\partial x^m}$, we have to cancel the coefficients of all generators of $[Q_L^{(0)} + Q_R^{(0)}, \_]$ (see section 3.4). The vanishing of the coefficient of $(\!(\lambda_R\theta_R)\!)^m$ $\left(\lambda_L^\alpha\frac{\partial}{\partial\lambda_L^\alpha} + \theta_L^\alpha\frac{\partial}{\partial\theta_L^\alpha}\right)$ implies that $D_L(x) = D_{L0}$ (constant in x). Similarly, $M_L^{mn}(x) = M_{L0}^{mn}$, $D_R(x) = D_{R0}$, $M_R^{mn}(x) = M_{R0}^{mn}$. The vanishing of the coefficient of $(\!(\lambda_L\theta_L)\!)^m\frac{\partial}{\partial x^n}$ and $(\!(\lambda_R\theta_R)\!)^m\frac{\partial}{\partial x^n}$ imply:

$$
\begin{aligned}
D_{L0} &= D_{R0} =: D_0\\
M_{L0}^{mn} &= M_{R0}^{mn} =: M_0^{mn}\\
T^m(x) &= T_0^m + 2D_0x^m + M_0^{mn}x^n
\end{aligned}
$$

The vanishing of the coefficients of $(\!(\lambda_R\theta_R)\!)\frac{\partial}{\partial\theta_L}$ and $(\!(\lambda_L\theta_L)\!)\frac{\partial}{\partial\theta_R}$ imply $S_L^\alpha(x) = S_{L0}^\alpha$ and $S_R^{\hat\alpha}(x) = S_{R0}^{\hat\alpha}$ (do not depend on x).

Acknowledgments

This work was supported in part by ICTP-SAIFR FAPESP grant 2016/01343-7, and in part by FAPESP grant 2019/21281-4. We want to thank Nathan Berkovits and Andrey Losev for useful discussions.

A Higher spin conformal Killing tensors

Consider tensor fields on the flat N-dimensional space $\mathbf{R}^N$ with coordinates:

$$x^m\,, \quad m \in \{1, \ldots, N\} \tag{A.1}$$

They are functions with indices: $f_{m_1,\ldots m_r}(x)$, where r is the rank of the tensor. There are some differential equations which only have finite-dimensional spaces of solutions. For example:

$$\frac{\partial}{\partial x^{(m}} f_{n)}(x) = 0 \tag{A.2}$$

The solutions of this equation are parameterized by constant antisymmetric tensors b_{mn}:

$$f_m = b_{mn}x^n \tag{A.3}$$

More generally, consider the equation:

$$\frac{\partial}{\partial x^{(m_0}} f_{m_1\ldots m_n)}(x) = 0 \tag{A.4}$$

We want to classify the solutions of this equation. Consider the Taylor expansion of $f_{m_1\ldots m_n}$ near $x = 0$. Since eq. (A.4) is homogeneous in x, we can consider each order of the Taylor expansion separately. In other words, it is enough to consider $f_{m_1\ldots m_n}(x)$ a homogeneous polynomial of x. Let us introduce auxiliary variable y^m and consider the generating function:

$$\hat f(x, y) = y^{m_1}\cdots y^{m_n} f_{m_1\ldots m_n}(x) \tag{A.5}$$

Homogeneous polynomials $\hat{f}(x,y)$ of x,y of the order N form a finite-dimensional representation of $sl(2,\mathbf{R})$, with the generators defined as follows:

$$E = y^m \frac{\partial}{\partial x^m}\ ,\ F = x^m \frac{\partial}{\partial y^m}\ ,\ H = y^m \frac{\partial}{\partial y^m} - x^m \frac{\partial}{\partial x^m} \tag{A.6}$$

Eq. (A.4) implies that $\hat{f}(x,y)$ is a highest weigh vector:

$$E\hat{f} = 0 \tag{A.7}$$

On the other hand, $\hat{f}$ being a polynomial of the order n in y implies:

$$F^{n+1}\hat{f} = 0 \tag{A.8}$$

Therefore, the space of polynomial solutions of eq. (A.4) decomposes into the direct sum of representations of dimension $0,1,2,\ldots,n$. They correspond to polynomials of degree $0,1,2,\ldots,n$ in x. We conclude that all solutions of eq. (A.4) are polynomials of order n in x (not necessarily homogeneous).

Let us now consider a weaker equation. Instead of requiring $\partial_{(m_0} f_{m_1\ldots m_n)}$ be zero, we require the existence of $g_{m_2,\ldots m_n}(x)$ such that:

$$\frac{\partial}{\partial x^{(m_0}} f_{m_1\ldots m_n)}(x) = \delta_{(m_0 m_1} g_{m_2\ldots m_n)}(x) \tag{A.9}$$

$$\delta^{m_1 m_2} f_{m_1\ldots m_n} = 0 \tag{A.10}$$

(We can think of eq. (A.9) as having a gauge symmetry $\delta f_{m_1\ldots m_n} = \delta_{(m_1 m_2} h_{m_3\ldots m_n)}$, $\delta g_{m_2\ldots m_n} = \partial_{(m_2} h_{m_3\ldots m_n)}$, and eq. (A.10) as fixing the gauge.) The solutions of eq. (A.9) are higher spin conformal Killing tensors. They correspond to traceless Killing tensors in AdS [10]. Given a traceless Killing tensor in AdS, we can consider the leading Taylor coefficient of its expansion around a point in AdS. It will satisfy eq. (A.4) (with an additional condition $\delta^{m_1 m_2} f_{m_1 m_2\ldots m_n} = 0$) implying that the space of solutions is finite-dimensional.

References

[1] A. Mikhailov, *A minimalistic pure spinor sigma-model in AdS*, *JHEP* **07** (2018) 155 [arXiv:1706.08158] [INSPIRE].

[2] M. Alexandrov, A. Schwarz, O. Zaboronsky and M. Kontsevich, *The Geometry of the master equation and topological quantum field theory*, *Int. J. Mod. Phys. A* **12** (1997) 1405 [hep-th/9502010] [INSPIRE].

[3] N. Berkovits and P.S. Howe, *Ten-dimensional supergravity constraints from the pure spinor formalism for the superstring*, *Nucl. Phys. B* **635** (2002) 75 [hep-th/0112160] [INSPIRE].

[4] S. Guttenberg, *Superstrings in General Backgrounds*, Ph.D. thesis, Vienna, Tech. U., 2007. arXiv:0807.4968 [INSPIRE].

[5] A. Mikhailov, *On the construction of integrated vertex in the pure spinor formalism in curved background*, Nucl. Phys. B **907** (2016) 542 [arXiv:1503.01005] [INSPIRE].

[6] N. Berkovits, *Simplifying and Extending the $AdS_5 \times S^5$ Pure Spinor Formalism*, JHEP **09** (2009) 051 [arXiv:0812.5074] [INSPIRE].

[7] A. Mikhailov, *Vertex operators of ghost number three in Type IIB supergravity*, Nucl. Phys. B **907** (2016) 509 [arXiv:1401.3783] [INSPIRE].

[8] C.R. Mafra, *Superstring Scattering Amplitudes with the Pure Spinor Formalism*, Ph.D. thesis, Sao Paulo, IFT, 2008. arXiv:0902.1552 [INSPIRE].

[9] A. Mikhailov, *Cornering the unphysical vertex*, JHEP **11** (2012) 082 [arXiv:1203.0677] [INSPIRE].

[10] A. Mikhailov, *Notes on higher spin symmetries*, hep-th/0201019 [INSPIRE].