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BPS Wilson loop in $\mathcal{N}=2$ superconformal SU(N) "orientifold" gauge theory and weak-strong coupling interpolation

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Abstract: We consider the expectation value $\langle \mathcal{W} \rangle$ of the circular BPS Wilson loop in $\mathcal{N}=2$ superconformal SU(N) gauge theory containing a vector multiplet coupled to two hypermultiplets in rank-2 symmetric and antisymmetric representations. This theory admits a regular large N expansion, is planar-equivalent to $\mathcal{N}=4$ SYM theory and is expected to be dual to a certain orbifold/orientifold projection of $AdS_5 \times S^5$ superstring theory. On the string theory side $\langle \mathcal{W} \rangle$ is represented by the path integral expanded near the same AdS_2 minimal surface as in the maximally supersymmetric case. Following the string theory argument in [5], we suggest that as in the $\mathcal{N}=4$ SYM case and in the $\mathcal{N}=2$ SU(N) $\times$ SU(N) superconformal quiver theory discussed in [19], the coefficient of the leading non-planar $1/N^2$ correction in $\langle \mathcal{W} \rangle$ should have the universal $\lambda^{3/2}$ scaling at large 't Hooft coupling. We confirm this prediction by starting with the localization matrix model representation for $\langle \mathcal{W} \rangle$. We complement the analytic derivation of the $\lambda^{3/2}$ scaling by a numerical high-precision resummation and extrapolation of the weak-coupling expansion using conformal mapping improved Padé analysis.

Keywords: AdS-CFT Correspondence, $1/N$ Expansion, Extended Supersymmetry

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1 Introduction

Wilson loops are an important class of observables in gauge and string theory that, in particular, help clarifying the interpolation between the weak and strong coupling regimes from the AdS/CFT perspective. In several supersymmetric gauge theories it is possible to compute expectation values of BPS Wilson loops using localization in terms of matrix model integrals (see, e.g., [1]).

In the maximally supersymmetric $\mathcal{N} = 4$ $SU(N)$ gauge theory the corresponding matrix model can be solved for any gauge coupling and gauge group rank [2].[1] This allows, in particular, to study the strong coupling limit of the coefficients in the $1/N$ expansion,

[1]The same is true also for the $\mathcal{N} = 4$ theory with $SO(N)$ and $USp(N)$ gauge groups [3, 4].

providing a possibility to compare to the large tension limit of the coefficients in the expansion in powers of string coupling (genus) on the dual $\mathrm{AdS}_5 \times S^5$ string theory side, and thus leading to highly non-trivial checks of AdS/CFT duality [5–7].

Localization method applies also to a large class of gauge theories with reduced $\mathcal{N}=2$ supersymmetry, but the associated matrix models have non-polynomial potentials and are not directly solvable. While developing the small λ expansion is straightforward, extracting the strong coupling limit of the gauge theory observables is a non-trivial problem. At leading order in the large N expansion this requires a Wiener-Hopf analysis of the matrix model as first exploited in [8] for $\mathcal{N}=2$ SU(N) SYM with $N_F=2N$ fundamental hypermultiplets, and later generalized to other superconformal Lagrangian models admitting a large N limit [9–15]. The methods used at leading planar level are not, however, applicable to the analysis of the higher $1/N$ corrections.

An interesting class of models where $1/N$ corrections happen to be more tractable is that of $\mathcal{N}=2$ superconformal gauge theories with SU(N) gauge group whose leading large N limit is equivalent (in a particular "common" sector) to that of the $\mathcal{N}=4$ SYM theory. One such example is the SU(N)$\times$SU(N) quiver gauge theory with bi-fundamental hypermultiplets and equal gauge couplings. This $\mathcal{N}=2$ theory may be interpreted as a $\mathbb{Z}_2$ orbifold of the $\mathcal{N}=4$ SU($2N$) SYM and is dual to superstring theory on $\mathrm{AdS}_5\times(S^5/\mathbb{Z}_2)$ [16].

Let us first review some basic results about the expectation value $\langle\mathcal{W}\rangle$ of circular $\frac{1}{2}$-BPS Wilson loop in $\mathcal{N}=4$ SYM theory. Its planar limit is given by is [2, 17, 18]

$$\langle\mathcal{W}\rangle_0 = \frac{2\,N}{\sqrt{\lambda}}\,I_1(\sqrt{\lambda}) = \sqrt{\frac{2}{\pi}}\,N\,\lambda^{-3/4}\,e^{\sqrt{\lambda}}\left[1+\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right], \qquad \lambda = g^2_{\mathrm{YM}}\,N\,. \tag{1.1}$$

The relative weight of the leading $1/N$ correction to $\langle\mathcal{W}\rangle$ with respect to the planar result may be represented as

$$\frac{\langle\mathcal{W}\rangle}{\langle\mathcal{W}\rangle_0} = 1+\frac{1}{N^2}\,q(\lambda)+\mathcal{O}\left(\frac{1}{N^4}\right), \tag{1.2}$$

where the function $q(\lambda)$ (and its analogs in the $\mathcal{N}=2$ models with planar equivalence to $\mathcal{N}=4$ SYM) will be our main interest below. Starting with the general (Laguerre polynomial) expression for $\langle\mathcal{W}\rangle$ in $\mathcal{N}=4$ SU(N) SYM [2] one finds for the leading terms in $q(\lambda)$ at weak and strong coupling[2]

$$q^{\mathcal{N}=4}(\lambda) = \frac{\lambda}{96}\left[\frac{\sqrt{\lambda}\,I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})}-12\right] = \begin{cases} -\frac{1}{8}\lambda+\frac{1}{384}\lambda^2-\frac{1}{9216}\,\lambda^3+\mathcal{O}(\lambda^4), & \lambda\to 0,\\ \frac{1}{96}\lambda^{3/2}-\frac{9}{64}\lambda+\frac{1}{256}\lambda^{1/2}+\mathcal{O}(1), & \lambda\to\infty\,. \end{cases} \tag{1.3}$$

The $\lambda^{3/2}$ scaling of $q^{\mathcal{N}=4}$ at $\lambda\gg 1$ has a string interpretation. The string coupling and tension for the dual string theory on $\mathrm{AdS}_5\times S^5$ are defined as

$$g_{\mathrm{s}} = \frac{\lambda}{4\pi N}, \qquad T = \frac{L^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}. \tag{1.4}$$

As was argued in [5], the leading large T dependence of the string theory expectation value for $\langle\mathcal{W}\rangle$ at each order in g_{s} is also controlled by the Euler number, $\chi=1-2p$, (or genus

[2]In what follows the label "$\mathcal{N}=4$" will always refer to the $\mathcal{N}=4$ SU(N) SYM expression.

p) of the string world sheet, i.e.

$$\langle \mathcal{W} \rangle = \sum_{p=0}^{\infty} \langle \mathcal{W} \rangle_p = e^{2\pi T} \sum_{p=0}^{\infty} c_p \left(\frac{g_s}{\sqrt{T}} \right)^{2p-1} \Big[1 + \mathcal{O}\big(T^{-1}\big) \Big] . \tag{1.5}$$

Written in terms of N and λ in (1.4) this reads ($c'_p = \frac{c_p}{(8\pi)^{p-1/2}}$)

$$\langle \mathcal{W} \rangle = N\, e^{\sqrt{\lambda}} \sum_{p=0}^{\infty} c'_p \frac{\lambda^{\frac{6p-3}{4}}}{N^{2p}} \left[1 + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}} \right) \right] , \tag{1.6}$$

and thus matches the structure of the $1/N$ expansion of the exact $\mathcal{N} = 4$ SYM result [2]. In particular, comparing to (1.2), we have, in agreement with (1.3),

$$\frac{1}{N^2}\, q^{\mathcal{N}=4}(\lambda) \overset{\lambda \gg 1}{\approx} \frac{g_s^2}{T} \propto \frac{\lambda^{3/2}}{N^2} . \tag{1.7}$$

The discussion in [5] leading to (1.5) relied only on the fact that one expands near the AdS_2 minimal surface embedded into the AdS_3 part of AdS_5 space; thus it should apply not only to $AdS_5 \times S^5$ superstring but also to its closely related orbifold and orientifold modifications based on $AdS_5 \times S'^5$ where S'^5 is locally a 5-sphere. Indeed, since the fluctuations of string world sheet fields related to S'^5 remain "massless", the reasoning [5] determining the tension dependence from the way how the AdS_5 radius appears in the 1-loop (leading large T) string partition function should not change.

In [19] it was argued that this should apply, in particular, to the orbifold $AdS_5 \times (S^5/\mathbb{Z}_2)$ theory and evidence for the validity of (1.5), (1.6) was provided at the first non-trivial $1/N^2$ order. To recall, in the $SU(N) \times SU(N)$ orbifold model, for each of the two $SU(N)$ factors, it is possible to define the $\frac{1}{2}$-BPS circular Wilson loops coupled to the associated gauge and scalar fields and $\langle \mathcal{W}_1 \rangle = \langle \mathcal{W}_2 \rangle \equiv \langle \mathcal{W} \rangle^{\rm orb}$. At the leading planar level one has $\langle \mathcal{W} \rangle^{\rm orb}_{N\to\infty} = \langle \mathcal{W} \rangle^{\mathcal{N}=4}_{N\to\infty} = \langle \mathcal{W} \rangle_0$ [20, 21].[3] Starting from the corresponding localization matrix model representation for $\langle \mathcal{W} \rangle^{\rm orb}$, a numerical analysis of the $q^{\rm orb}(\lambda)$ function, defined as in (1.3), extrapolated to large λ values gave the following estimate [19]

$$q^{\rm orb}(\lambda) \overset{\lambda \gg 1}{=} C\, \lambda^{\eta}, \qquad \eta = 1.49(2), \qquad C \simeq -0.0049(5) . \tag{1.8}$$

The value of the asymptotic exponent η is thus quite consistent with the string theory expectation $3/2$ in (1.7).

In this paper we shall consider another $\mathcal{N} = 2$ superconformal model where the structure of the large N, strong coupling expansion should be of the same universal form as in (1.5). We shall confirm this expectation with an analytic argument for the strong-coupling scaling in (1.7), in addition to numerical evidence based on high-precision extrapolation from the weak to strong coupling regimes.

This theory is the $SU(N)$ gauge theory with $\mathcal{N} = 2$ vector multiplet coupled to two hypermultiplets — in rank-2 symmetric and antisymmetric $SU(N)$ representations.[4] This

[3]The strong coupling limit of the planar expectation value of a similar Wilson loop in quiver gauge theory with unequal gauge couplings was solved in [21], see also [22–24].

[4]It is one of the five cases of 4d $\mathcal{N} = 2$ superconformal theories with gauge group $SU(N)$ defined for an arbitrary value of N [25, 26].

model admits a regular 't Hooft large N expansion and its string theory dual is expected to be a particular orientifold of $\mathrm{AdS}_5 \times S^5$ type IIB superstring theory [27, 28]. For that reason in what follows we shall refer to this $\mathcal{N}=2$ gauge theory as the "orientifold theory".

To recall, in $\mathcal{N}=2$ gauge theories the β-function for the gauge coupling has only the 1-loop contribution: in a model with N_F hypermultiplets in the fundamental, N_S in rank-2 symmetric, and N_A in rank-2 antisymmetric representations one has $\beta_{1-\text{loop}} = 2N - N_F - N_S(N+2) - N_A(N-2)$. It thus vanishes for the orientifold theory where $N_F = 0$, $N_A = N_S = 1$. Let us also mention for completeness that the 4d Weyl anomaly coefficients a and c for an $\mathcal{N}=2$ theory with n_V vector and n_H hyper multiplets are given by $\mathrm{a} = \frac{5}{24}\, n_V + \frac{1}{24}\, n_H$, $\mathrm{c} = \frac{1}{6}\, n_V + \frac{1}{12}\, n_H$, so that in the present case with $n_V = N^2 - 1$ and $n_H = \frac{1}{2}N(N+1) + \frac{1}{2}N(N-1) = N^2$, we get $\mathrm{a} = \frac{1}{4}N^2 - \frac{5}{24}$, $\mathrm{c} = \frac{1}{4}N^2 - \frac{1}{6}$. Thus a and c are equal at the leading N^2 order which is consistent with the existence of a well defined holographic dual.[5]

Our aim will be to consider the expectation value of the $\frac{1}{2}$-BPS circular Wilson loop in the orientifold theory. At the leading large N order it is the same as in the $\mathcal{N}=4$ SYM theory in (1.1).[6] The main focus will be on the leading non-planar correction represented by the function $q^{\text{orient}}(\lambda)$ defined as in (1.2).

The string dual of this $\mathcal{N}=2$ model is the type IIB superstring theory defined on the orientifold $\mathrm{AdS}_5 \times S^5/G_{\text{orient}}$ [28]. Here $G_{\text{orient}} = \mathbb{Z}_2^{\text{orb}} \times \mathbb{Z}_2^{\text{orient}}$, where Z_2^{orient} in addition to the target space coordinate inversions (in directions transverse to the original D3-branes) involves the product world-sheet parity operator Ω and $(-1)^{F_L}$. The compact part of the 10d space $S'^5 = S^5/G_{\text{orient}}$ is different from S^5 only by special identifications of the angular coordinates [28]:

$$
\begin{aligned}
&ds_5'^2 = d\theta_1^2 + \sin^2\theta_1\, d\phi_3^2 + \cos^2\theta_1\, ds_3'^2, \qquad ds_3'^2 = d\theta_2^2 + \sin^2\theta_2\, d\phi_2^2 + \cos^2\theta_2\, d\phi_1^2,\\
&\theta_1 \equiv \theta_1 + \frac{\pi}{2}, \;\; \theta_2 \equiv \theta_2 + \frac{\pi}{2}, \quad \phi_1 \equiv \phi_1 + \frac{\pi}{2}, \quad \phi_2 \equiv \phi_2 - \frac{\pi}{2}, \quad \phi_3 \equiv \phi_3 + \pi\,.
\end{aligned}
$$

The dual string theory description of the circular Wilson loop is based again on the string partition function expanded near the AdS_2 minimal surface embedded in AdS_5. As the UV divergent part of the 1-loop fluctuation determinants [32] near this minimal surface should not be sensitive to the global identifications in the S'^5 part of the orientifold geometry, the argument in [5] leading to the universal structure of strong-coupling expansion (1.5), (1.6) should apply not only to the original $\mathrm{AdS}_5 \times S^5$ or orbifold theory considered in [19] but also to this orientifold theory as well.

Below we shall provide evidence for this, i.e. for the validity of (1.5), (1.6), on the dual orientifold gauge theory side by showing that the localization matrix model representation for the circular BPS Wilson loop implies that the $1/N^2$ term in (1.2) indeed scales as

[5] Let us also note that it should be possible to reproduce the subleading terms in a and c on the dual orientifold string theory side by summing up the 1-loop contributions of the "massless" $D=10$ supergravity fields (corresponding to short multiplets represented by towers of Kaluza-Klein modes) similarly to how that was done in the case of the $\mathcal{N}=4$ SU(N) SYM theory (where a=c= $\frac{1}{4}N^2 - \frac{1}{4}$) [29] and some orbifold theories [30].

[6] This is a manifestation of the planar equivalence between the orientifold theory and the $\mathcal{N}=4$ SYM in the "untwisted" sector. For a detailed discussion of planar equivalence violations in "odd" sectors see [31].

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in (1.7) at the leading order at strong coupling, i.e.

$$q^{\text{orient}}(\lambda) \overset{\lambda \gg 1}{\approx} \lambda^{3/2} \,. \tag{1.9}$$

Let us briefly summarize our main results. The aim will be to present a detailed study of the $1/N^2$ coefficient in (1.2) in the orientifold theory, i.e. of $q^{\text{orient}}(\lambda)$. As in the orbifold theory discussed in [19], from the matrix model representation for the Wilson loop in the orientifold theory one can relate the difference between $q^{\text{orient}}(\lambda)$ and $q^{\mathcal{N}=4}(\lambda)$

$$\Delta q(\lambda) \equiv q^{\text{orient}}(\lambda) - q^{\mathcal{N}=4}(\lambda) \,, \tag{1.10}$$

to the $N \to \infty$ limit of the difference of the corresponding free energies[7]

$$\Delta q(\lambda) = -\frac{\lambda^2}{4}\frac{d}{d\lambda}\,\Delta F(\lambda) \,, \qquad\qquad \Delta F(\lambda) \equiv \lim_{N\to\infty} \Delta F(\lambda; N) \,, \tag{1.11}$$

$$\Delta F(\lambda; N) \equiv F^{\text{orient}}(\lambda; N) - F^{\mathcal{N}=4}(\lambda; N) = -\log \frac{Z^{\text{orient}}(\lambda; N)}{Z^{\mathcal{N}=4}(\lambda; N)} \,. \tag{1.12}$$

Here Z^{orient} and $Z^{\mathcal{N}=4}$ are the corresponding partition functions on S^4. The function $\Delta F(\lambda)$ turns out to have the following weak coupling expansion

$$\begin{aligned} \Delta F(\lambda) &= 5\zeta_5 \hat\lambda^3 - \frac{105}{2}\zeta_7 \hat\lambda^4 + 441\zeta_9 \hat\lambda^5 - (25\zeta_5^2 + 3465\zeta_{11})\hat\lambda^6 \\ &\quad + \left(525\zeta_5\zeta_7 + \frac{3.6355}{8}\zeta_{13}\right)\hat\lambda^7 + \cdots \,, \qquad\qquad \hat\lambda = \frac{\lambda}{8\pi^2} \,. \end{aligned} \tag{1.13}$$

Extracting the strong coupling expansion is much harder. Since in the $\mathcal{N} = 4$ SYM theory the matrix model representation implies that $F^{\mathcal{N}=4} = -\frac{1}{2}(N^2 - 1)\log\lambda$ [34], combining (1.11), (1.12) and (1.9) we get, as in the orbifold theory case [19], the following prediction for F^{orient}

$$F^{\text{orient}}(\lambda; N) \overset{\lambda \gg 1}{=} -\frac{1}{2}\,N^2 \log\lambda + \left[c_1\sqrt{\lambda} + \mathcal{O}(\log\lambda)\right] + \mathcal{O}\left(\frac{1}{N^2}\right) . \tag{1.14}$$

The leading $\mathcal{O}(N^2)$ term in (1.14) is implied by the planar equivalence to the SU(N) SYM theory and should follow from the leading type IIB supergravity term evaluated on $\text{AdS}_5 \times (S^5/G_{\text{orient}})$.[8]

Below we will analytically derive the $\sqrt{\lambda}$ term in (1.14) and thus in ΔF finding that

$$\Delta F(\lambda) \overset{\lambda \gg 1}{=} \frac{\sqrt{\lambda}}{2\pi} \quad \to \quad \Delta q(\lambda) \overset{\lambda \gg 1}{=} -\frac{\lambda^{3/2}}{16\pi} + \cdots \,, \tag{1.15}$$

where we used (1.11). We will also confirm this result by a high-precision resummation and extrapolation analysis of the weak-coupling expansion by numerical methods including a conformal-mapping improved Padé analysis.

[7] Note that while the individual free energies on S^4 are, in general, scheme-dependent, their difference ΔF is scheme-independent. Earlier discussion of leading terms in perturbative expansion in Wilson loop and free energy in this theory was in [33].

[8] Planar equivalence also implies that like in the $\mathcal{N} = 4$ SYM theory this leading N^2 term should not get string $\frac{1}{\sqrt{\lambda}}$ corrections: they should still vanish on $\text{AdS}_5 \times (S^5/G_{\text{orient}})$.

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Let us note that while the coefficient in strong-coupling limit of q in $\mathcal{N}=4$ SYM case is positive, $q^{\mathcal{N}=4} = \frac{1}{96}\lambda^{3/2} + \ldots$ (see (1.3)), it was found [19] to be negative in the $\mathcal{N}=2$ orbifold theory (1.8). The result in (1.15) implies that it is also negative in the orientifold theory, $q^{\text{orient}} = q^{\mathcal{N}=4} + \Delta q = (\frac{1}{96} - \frac{1}{16\pi})\lambda^{3/2} + \ldots \approx -0.00948\lambda^{3/2} + \ldots$. It would be interesting to understand the reason for this sign change on the dual string theory side where q should be expressed in terms of the string partition function on the disc with one handle.

The rest of the paper is organised as follows. In section 2 we shall describe the localization matrix model representation for the expectation value of the $\frac{1}{2}$-BPS Wilson loop in the orientifold gauge theory. We shall then present the derivation of the relation (1.11) between the coefficient Δq of the $1/N^2$ term in the ratio of the orientifold and $\mathcal{N}=4$ SYM Wilson loops and the large N limit $\Delta F(\lambda)$ of the difference of the corresponding free energies. This reduces the problem of determining the strong coupling limit of Δq to finding that of ΔF.

In section 3 we shall first study $\Delta F(\lambda)$ at weak coupling and then find its explicit representation (3.25), i.e. $\Delta F = \frac{1}{2}\log\det(1+M) = -\sum_{n=1}^{\infty}\frac{1}{n}(-1)^n \text{tr}\, M^n$, in terms of an infinite-dimensional matrix M (3.26). Each $\text{tr}\, M^n$ term in ΔF turns out to be of fixed order n in products of ζ-function values when written in the weak-coupling expansion. In section 4 we shall study the first two $\text{tr}\, M$ and $\text{tr}\, M^2$ terms finding that at large λ one has $\text{tr}\, M^n \sim \lambda^n$.

The derivation of the strong coupling asymptotics (1.15) of the total ΔF implying $\Delta q \sim \lambda^{3/2}$ is given in section 5. In section 6 we shall independently test this $\Delta F \sim \lambda^{1/2}$ scaling by two different numerical methods. Some technical details are delegated to appendices.

The methods used here may be applicable to other similar $\mathcal{N}=2$ models. One candidate is the $\mathcal{N}=2$ superconformal $\text{SU}(N)$ gauge theory with $N_F=4$ fundamental and $N_A=2$ rank-2 antisymmetric hypermultiplets. In this case the dual string theory is expected to be again a IIB orientifold of $\text{AdS}_5 \times S^5$ where S^5 is modded out by a $\mathbb{Z}_4$ that mixes non-trivially the orbifold and orientifold twists [28].[9] However, the presence of fundamentals means that here the large N expansion will go in powers of $1/N$ rather than $1/N^2$ and thus will be different in structure from (1.5), (1.6).

2 Matrix model representation and $1/N^2$ correction to Wilson loop

The field content of the orientifold theory is represented by the adjoint $\mathcal{N}=2$ vector multiplet (gauge vector A_μ, a complex scalar φ, and two Weyl fermions) and rank-2 symmetric and antisymmetric hypermultiplets (each containing two complex scalars and two Weyl fermions). The $\frac{1}{2}$-BPS Wilson loop is defined in terms of the fields of the vector multiplet as

$$\mathcal{W} = \text{tr}\,\mathcal{P}\,\exp\left\{g_{\text{YM}}\oint\left[i\,A_\mu(x)dx^\mu + \frac{1}{\sqrt{2}}(\varphi(x)+\varphi^+(x))\,ds\right]\right\}, \tag{2.1}$$

where the contour $x^\mu(s)$ represents a circle of unit radius.

[9] In this model the Weyl anomaly coefficients are $\mathrm{a} = \frac{1}{4}N^2 + \frac{1}{8}N - \frac{5}{24}$ and $\mathrm{c} = \frac{1}{4}N^2 + \frac{1}{4}N - \frac{1}{6}$. The $\mathcal{O}(N)$ terms in a and c should be possible to derive on the dual string theory side as in [35] (see also [36, 37]) using that here the background involves D7-branes wrapping AdS_5 and S^3 of S'^5 with R^2terms in the effective 8-dimensional world-volume theory.

The supersymmetric localization implies that the partition function of this gauge theory on a sphere S^4 of unit radius admits a representation in terms of an integral over the eigenvalues $\{m_i\}_{i=1}^N$ of a traceless hermitian $N\times N$ matrix m [18]

$$Z^{\rm orient} \equiv e^{-F^{\rm orient}} = \int \mathcal{D}m\, e^{-S(m)}\,, \tag{2.2}$$

$$S(m) = S_0(m) + S_{\rm int}(m)\,, \qquad\qquad S_0 = \frac{8\pi^2 N}{\lambda}{\rm tr}\, m^2\,, \qquad \lambda = g^2_{\rm YM}N\,, \tag{2.3}$$

$$\mathcal{D}m \equiv \prod_{i=1}^N dm_i\, \delta\Big(\sum_j m_j\Big)\,\big[\Delta(m)\big]^2\,, \qquad \Delta(m) = \prod_{i<j}(m_i - m_j)\,. \tag{2.4}$$

In the case of $\mathcal{N}=4$ SYM theory $S_{\rm int}=0$ and the matrix model is Gaussian. As we shall discuss the $1/N$ expansion, we can neglect the instanton contribution term in $S_{\rm int}$ so that[10]

$$S_0(m) = \frac{8\pi^2 N}{\lambda}\sum_i m_i^2\,, \qquad S_{\rm int}(m) = \sum_{i,j}\Big[\log H(m_i+m_j) - \log H(m_i - m_j)\Big]\,, \tag{2.5}$$

where H is expressed in terms of the Barnes G-function

$$H(x) = \prod_{n=1}^\infty \left(1+\frac{x^2}{n^2}\right)^n e^{-\frac{x^2}{n}} = e^{-(1+\gamma_{\rm E})\,x^2}\,{\rm G}(1+ix)\,{\rm G}(1-ix)\,, \tag{2.6}$$

$$\log H(x) = \sum_{n=1}^\infty \frac{(-1)^n}{n+1}\zeta_{2n+1}\,x^{2n+2}\,. \tag{2.7}$$

Here and below the constants $\zeta_{2n+1}\equiv\zeta(2n+1)$ are the Riemann ζ-function values.

The normalized expectation value of the Wilson loop (2.1) can be computed as the matrix model average of ${\rm tr}\, e^{2\pi m}$, i.e.

$$\langle \mathcal{W}\rangle^{\rm orient} = \frac{\int \mathcal{D}m\, e^{-S(m)}\,{\rm tr}\, e^{2\pi m}}{\int \mathcal{D}m\, e^{-S(m)}} = \frac{1}{Z^{\rm orient}}\int \mathcal{D}m\, e^{-S(m)}\sum_i e^{2\pi m_i}\,. \tag{2.8}$$

In the leading planar approximation $S_{\rm int}$ is effectively suppressed and thus we get the same result as in the $\mathcal{N}=4$ SYM: $\langle\mathcal{W}\rangle^{\rm orient} = \langle\mathcal{W}\rangle^{\mathcal{N}=4} + \mathcal{O}(\frac{1}{N})$. Here we will be interested in the leading non-planar correction in the ratio (cf. (1.2), (1.10))

$$\frac{\langle\mathcal{W}\rangle^{\rm orient}}{\langle\mathcal{W}\rangle^{\mathcal{N}=4}} = 1 + \frac{1}{N^2}\,\Delta q(\lambda) + \mathcal{O}\left(\frac{1}{N^4}\right)\,. \tag{2.9}$$

Let us show that $\Delta q(\lambda)$ can be represented as a λ derivative (1.11) of the difference of the free energies (1.12). The Wilson loop ratio (2.9) may be represented in general as

$$\frac{\langle\mathcal{W}\rangle^{\rm orient}}{\langle\mathcal{W}\rangle^{\mathcal{N}=4}} = \frac{\left\langle e^{-S_{\rm int}}\,{\rm tr}\, e^{2\pi m}\right\rangle_0}{\langle e^{-S_{\rm int}}\rangle_0\,\langle {\rm tr}\, e^{2\pi m}\rangle_0}\,, \qquad \langle\ldots\rangle_0 \equiv \int Dm\, e^{-S_0(m)}\ldots\,, \quad \langle 1\rangle_0 = 1\,, \tag{2.10}$$

[10]This is a specialization of the general analysis in [18]. See also [38, 39] for applications to other $\mathcal{N}=2$ gauge theories.

where Dm is the normalized measure, i.e. $\int Dm\, e^{-S_0(m)} \ldots = \frac{\int \mathcal{D}m\, e^{-S_0(m)} \ldots}{\int \mathcal{D}m\, e^{-S_0(m)}}$. Then $\langle \mathcal{W} \rangle^{\mathcal{N}=4} = \langle \operatorname{tr} e^{2\pi m} \rangle_0$ and

$$\left\langle e^{-S_{\rm int}} \right\rangle_0 = \frac{Z^{\rm orient}}{Z^{\mathcal{N}=4}} = e^{-\Delta F}, \qquad \Delta F(\lambda; N) = F^{\rm orient} - F^{\mathcal{N}=4}. \tag{2.11}$$

At large N the correlators in (2.10) factorize and the ratio goes to 1. The non-planar correction is given by the large N limit of the "connected" part of $\left\langle e^{-S_{\rm int}} \operatorname{tr} e^{2\pi m} \right\rangle_0$. The leading $N \to \infty$ contribution should come from the first non-trivial term in the expansion of $\operatorname{tr} e^{2\pi m}$ in powers of m $(\operatorname{tr} 1 = N,\ \operatorname{tr} m = 0)$:

$$\left\langle e^{-S_{\rm int}} \operatorname{tr} e^{2\pi m} \right\rangle_0 = N \left\langle e^{-S_{\rm int}} \right\rangle_0 + 2\pi^2 \left\langle e^{-S_{\rm int}} \operatorname{tr} m^2 \right\rangle_0 + \ldots . \tag{2.12}$$

The insertion of a factor of $\operatorname{tr} m^2$ is the same as the insertion of the free action in (2.3) and thus it can be obtained by differentiating the partition function (2.2) over λ. As shown in appendix A, taking the large N limit we then find that Δq in (2.9) can be represented as in (1.11), i.e.

$$\Delta q = -\frac{\lambda^2}{4} \lim_{N\to\infty} \frac{\partial}{\partial \lambda} \Delta F(\lambda; N) = -\frac{\lambda^2}{4} \frac{d}{d\lambda} \Delta F(\lambda), \qquad \Delta F(\lambda) = \lim_{N\to\infty} \Delta F(\lambda; N). \tag{2.13}$$

This is essentially the same relation as was observed to hold in the $\mathcal{N} = 2$ orbifold model in [19] (up to factor of 2 due to the $SU(N) \times SU(N)$ instead of $SU(N)$ gauge group).

3 Large N limit of free energy difference $\Delta F = F^{\rm orient} - F^{\mathcal{N}=4}$

Since the leading non-planar correction to the Wilson loop can be expressed (2.13) in terms of $\Delta F(\lambda)$, in what follows we shall concentrate on the study of its structure both at weak and strong coupling. Redefining the matrix model variable $m \to a$ as

$$a = \sqrt{\frac{8\pi^2 N}{\lambda}}\, m, \tag{3.1}$$

we can represent $\Delta F(\lambda; N)$ in (2.11) as

$$e^{-\Delta F(\lambda;N)} = \int Da\, e^{-S_{\rm int}(a)}\, e^{-\operatorname{tr} a^2}, \tag{3.2}$$

where Da is the standard integration measure for the traceless matrix a, normalized so that $\int Da\, e^{-\operatorname{tr} a^2} = 1$. This measure is same as in (2.4) when written in terms of the eigenvalues and dropping the "angular" part that cancels in expectation values of relevant correlators (functions of traces of matrix a).

Using (2.5), (2.7) we can write $S_{\rm int}$ in (2.5) as a weak coupling expansion

$$S_{\rm int}(a) = 2 \sum_{n=1}^{\infty} \left(\frac{\hat\lambda}{N}\right)^{n+1} \frac{(-1)^n}{n+1} \zeta_{2n+1} \sum_{p=0}^{n} \binom{2n+2}{2p+1} \operatorname{tr} a^{2p+1} \operatorname{tr} a^{2n-2p+1}, \qquad \hat\lambda \equiv \frac{\lambda}{8\pi^2}. \tag{3.3}$$

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3.1 Weak coupling expansion

The weak coupling ($\lambda \ll 1$) expansion of ΔF in (3.2) is easily worked out by expanding $e^{-S_{\text{int}}}$ and doing the Gaussian integrations. The result has a finite limit for $N \to \infty$ since the leading N^2 terms present in both the $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2$ partition functions cancel out in ΔF as a manifestation of the planar equivalence of the two models. For the leading large N contribution $\Delta F(\lambda)$ defined in (2.13) we obtain the following expansion (cf. (1.13))

$$\begin{aligned}\Delta F(\lambda) = &5\zeta_5\hat{\lambda}^3 - \frac{105}{2}\zeta_7\hat{\lambda}^4 + 441\zeta_9\hat{\lambda}^5 - (25\zeta_5^2 + 3465\zeta_{11})\hat{\lambda}^6 + \left(525\zeta_5\zeta_7 + \frac{3.6355}{8}\zeta_{13}\right)\hat{\lambda}^7 \\ &- \left(\frac{22785}{8}\zeta_7^2 + \frac{8505}{2}\zeta_5\zeta_9 + \frac{6441435}{32}\zeta_{15}\right)\hat{\lambda}^8 \\ &+ \left(\frac{500}{3}\zeta_5^3 + \frac{94815}{2}\zeta_7\zeta_9 + \frac{63525}{2}\zeta_5\zeta_{11} + \frac{12167155}{8}\zeta_{17}\right)\hat{\lambda}^9 \\ &- \left(5250\zeta_5^2\zeta_7 + 201852\zeta_9^2 + \frac{724185}{2}\zeta_7\zeta_{11} + \frac{920205}{4}\zeta_5\zeta_{13} + \frac{91869921}{8}\zeta_{19}\right)\hat{\lambda}^{10} + \cdots .\end{aligned} \tag{3.4}$$

A check of the general relation (2.13) may be given by the direct comparison of the independent weak-coupling expansions for ΔF and Δq (see appendix A). The weak coupling expansion of $\Delta q(\lambda) = q^{\text{orient}}(\lambda) - q^{\mathcal{N}=4}(\lambda)$ is found to be

$$\begin{aligned}\frac{1}{4\pi^2}\Delta q(\lambda) = &-\frac{15}{2}\zeta_5\hat{\lambda}^4 + 105\zeta_7\hat{\lambda}^5 - \frac{2205}{2}\zeta_9\hat{\lambda}^6 + (75\zeta_5^2 + 10395\zeta_{11})\hat{\lambda}^7 \\ &- \left(\frac{3675}{2}\zeta_5\zeta_7 + \frac{1486485}{16}\zeta_{13}\right)\hat{\lambda}^8 + \left(\frac{22785\zeta_7^2}{2} + 17010\zeta_5\zeta_9 + \frac{6441435\zeta_{15}}{8}\right)\hat{\lambda}^9 + \cdots ,\end{aligned} \tag{3.5}$$

which is indeed consistent with (3.4) and (2.13).

3.2 Explicit representation for ΔF

It is possible to derive a remarkable closed expression for $\Delta F(\lambda)$ in (2.13) as a $\log\det$ of an infinite-dimensional matrix.

Let us start with representing $S_{\text{int}}(a)$ in (3.3) as an infinite double sum of $\frac{1}{\sqrt{N}}$-normalized traces of odd powers of the matrix a with coefficients C_{ij} that depend only on λ

$$S_{\text{int}}(a) = \sum_{i,j=1}^{\infty} C_{ij}(\lambda)\, \text{tr}\left(\frac{a}{\sqrt{N}}\right)^{2i+1} \text{tr}\left(\frac{a}{\sqrt{N}}\right)^{2j+1}, \tag{3.6}$$

$$C_{ij}(\lambda) = 4\,\hat{\lambda}^{i+j+1}\,(-1)^{i+j}\,\zeta_{2i+2j+1}\frac{\Gamma(2i+2j+2)}{\Gamma(2i+2)\,\Gamma(2j+2)}\,. \tag{3.7}$$

Next, let us define the generating function

$$X(\eta) = \int Da\, e^{-\text{tr}\, a^2}\, e^{V(\eta,a)}\,, \qquad V(\eta,a) = \sum_{k=1}^{\infty}\eta_k\, \text{tr}\left(\frac{a}{\sqrt{N}}\right)^{2k+1}. \tag{3.8}$$

Using (3.6) can then represent $e^{-S_{\rm int}(a)}$ and thus the integral over a in (3.2) as[11]

$$e^{-S_{\rm int}(a)} = e^{-\mathscr{D}(\lambda)}\, e^{V(\eta,a)}\Big|_{\eta=0}\,, \qquad e^{-\Delta F(\lambda;N)} = e^{-\mathscr{D}(\lambda)}\, X(\eta)\Big|_{\eta=0}\,, \tag{3.9}$$

$$\mathscr{D}(\lambda) \equiv C_{ij}(\lambda)\frac{\partial}{\partial\eta_i}\frac{\partial}{\partial\eta_j}\,. \tag{3.10}$$

The large N limit of $\Delta F(\lambda;N)$ is thus directly related to that of $X(\eta)$.

Expanding e^V in (3.8) in powers of a, computing the Gaussian integrals over a, and then rearranging the result back into the exponential form gives the following expression for the leading large N part of $X(\eta)$

$$X(\eta) = e^{Q(\eta)}\left[1+\mathcal{O}\left(\frac{1}{N}\right)\right], \tag{3.11}$$

where $Q(\eta)$ is the following quadratic form

$$Q(\eta) \equiv Q_{ij}\eta_i\eta_j = \frac{3}{16}\eta_1^2 + \frac{15}{16}\eta_1\eta_2 + \frac{5}{4}\eta_2^2 + \frac{63}{32}\eta_1\eta_3 + \frac{175}{32}\eta_2\eta_3 + \frac{1575}{256}\eta_3^2 + \cdots\,. \tag{3.12}$$

The closed form of the infinite matrix Q_{ij} can be found from the results in [31]

$$Q_{ij} = \frac{1}{\pi}\frac{2^{i+j}\,i\,j\,\Gamma(i+\frac{3}{2})\,\Gamma(j+\frac{3}{2})}{(i+j+1)\,\Gamma(i+2)\,\Gamma(j+2)}\,. \tag{3.13}$$

As a result, we get from (3.9), (3.11)

$$e^{-\Delta F(\lambda)} = \lim_{N\to\infty} e^{-\Delta F(\lambda;N)} = e^{-\mathscr{D}(\lambda)}\, e^{Q(\eta)}\Big|_{\eta=0}\,. \tag{3.14}$$

To evaluate (3.14), let us first change the variables as $\eta = Q^{-1/2}x$ so that $Q_{ij}\eta_i\eta_j = x_i x_i$ and

$$\mathscr{D} = C_{ij}\frac{\partial}{\partial\eta_i}\frac{\partial}{\partial\eta_j} = (Q^{1/2}CQ^{1/2})_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}. \tag{3.15}$$

Introducing the notation

$$\widetilde{C} = 4Q^{1/2}\,C\,Q^{1/2}\,, \qquad \widetilde{M} = 4CQ = Q^{-1/2}\,\widetilde{C}\,Q^{1/2}\,, \tag{3.16}$$

we then have (here $\partial_i = \frac{\partial}{\partial x_i}$, $\mathcal{N} = \text{const}$)

$$\begin{aligned} e^{-\Delta F} &= e^{-\frac{1}{4}\widetilde{C}_{ij}\partial_i\partial_j}\, e^{x^2}\Big|_{x=0} = e^{-\frac{1}{4}\widetilde{C}_{ij}\partial_i\partial_j}\,\mathcal{N}\int dy\, e^{x\cdot y-\frac{1}{4}y^2}\Big|_{x=0} = \mathcal{N}\int dy\, e^{-\frac{1}{4}y\widetilde{C}y-\frac{1}{4}y^2} \\ &= [\det(1+\widetilde{C})]^{-1/2} = [\det(1+\widetilde{M})]^{-1/2}\,. \end{aligned} \tag{3.17}$$

Here we used an auxiliary Gaussian integral over y_i and that $e^{-A_{ij}\partial_i\partial_j}\, e^{x\cdot y}\big|_{x=0} = e^{-A_{ij}y_iy_j}$.

Thus we find the following exact representation for $\Delta F(\lambda)$ in terms of the infinite matrix $\widetilde{M}$

$$\Delta F = \frac{1}{2}\log\det(1+\widetilde{M}) = \frac{1}{2}\text{tr}\,\log(1+\widetilde{M}) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\,\text{tr}\,\widetilde{M}^n. \tag{3.18}$$

[11] Here and below we assume summation over repeated indices i, j.

Note that the last equality in (3.18) and the explicit form of C_{ij} in (3.7) imply that each power of $\widetilde{M}$ defined in (3.16) brings in one extra factor of the ζ_k constants. Explicitly, we have

$$\begin{aligned}
\frac{1}{2}\mathrm{tr}\,\widetilde{M} &= 5\zeta_5\hat{\lambda}^3 - \frac{105}{2}\zeta_7\hat{\lambda}^4 + 441\zeta_9\hat{\lambda}^5 - 3465\zeta_{11}\hat{\lambda}^6 + \frac{3.6355}{8}\zeta_{13}\hat{\lambda}^7 \\
&\quad - \frac{6441435}{32}\zeta_{15}\hat{\lambda}^8 + \frac{12167155}{8}\zeta_{17}\hat{\lambda}^9 - \frac{91869921}{8}\zeta_{19}\hat{\lambda}^{10} + \cdots, \\
-\frac{1}{2\times 2}\mathrm{tr}\,\widetilde{M}^2 &= -25\zeta_5^2\hat{\lambda}^6 + 525\zeta_5\zeta_7\hat{\lambda}^7 - \left(\frac{22785}{8}\zeta_7^2 + \frac{8505}{2}\zeta_5\zeta_9\right)\hat{\lambda}^8 + \left(\frac{94815}{2}\zeta_7\zeta_9 + \frac{63525}{2}\zeta_5\zeta_{11}\right)\hat{\lambda}^9 \\
&\quad - \left(201852\zeta_9^2 + \frac{724185}{2}\zeta_7\zeta_{11} + \frac{920205}{4}\zeta_5\zeta_{13}\right)\hat{\lambda}^{10} + \cdots, \\
\frac{1}{2\times 3}\mathrm{tr}\,\widetilde{M}^3 &= \frac{500}{3}\zeta_5^3\hat{\lambda}^9 - 5250(\zeta_5^2\zeta_7)\hat{\lambda}^{10} + \cdots.
\end{aligned} \tag{3.19}$$

That way the weak-coupling expansion in (3.18) reproduces the ζ_k, $\zeta_k\zeta_n$, $\zeta_k\zeta_n\zeta_m, \ldots$ terms in the expansion of ΔF in (3.4).[12]

The explicit form of the matrix $\widetilde{M}$ in (3.16) appearing in (3.18) is

$$\widetilde{M}_{ij} = \frac{16}{\pi}\sum_{k=1}^{\infty}\hat{\lambda}^{k+j+1}\,(-1)^{k+j}\,\zeta_{2k+2j+1}\,\frac{2^{i+k}\,i\,k\,\Gamma(i+\frac{3}{2})\,\Gamma(k+\frac{3}{2})}{(i+k+1)\,\Gamma(i+2)\,\Gamma(k+2)}\,\frac{\Gamma(2k+2j+2)}{\Gamma(2k+2)\;\Gamma(2j+2)}. \tag{3.20}$$

Motivated by the analysis in [31], let us introduce the following matrix

$$M_{ij} = 8\sqrt{2i+1}\sqrt{2j+1}\sum_{k=0}^{\infty}\left(\frac{\hat{\lambda}}{2}\right)^{i+j+k+1}(-1)^k\,c_{i,j,k}\,\zeta_{2i+2j+2k+1}\,, \tag{3.21}$$

$$c_{i,j,k} = \sum_{m=0}^{k}\frac{\Gamma(2i+2j+2k+2)}{\Gamma(m+1)\,\Gamma(2i+m+2)\,\Gamma(k-m+1)\,\Gamma(2j+k-m+2)}\,. \tag{3.22}$$

Remarkably, $\widetilde{M}$ in (3.20) and M in (3.21) happen to be related by a similarity transformation,

$$M = U^{-1}\widetilde{M}U\,, \tag{3.23}$$

where[13]

$$U_{ij} = \frac{(-1)^{1-j}2^{1-i}\sqrt{1+2j}\,\Gamma(2+2i)}{\sqrt{3}\,\Gamma(1+i-j)\;\Gamma(2+i+j)}\,. \tag{3.24}$$

One can then replace $\widetilde{M}$ in (3.18) by M, getting

$$\Delta F(\lambda) = \frac{1}{2}\mathrm{tr}\,\log(1+M) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\,\mathrm{tr}\,M^n. \tag{3.25}$$

[12] Keeping only a finite number of ζ_k constants in the matrix $\widetilde{M}_{ij}$ and using the first equality in (3.18) gives immediately the resummation of all monomials involving those ζ_k. For example, the terms with only ζ_5 and ζ_7 come from the expansion of the exact expression $\Delta F_{\zeta_5,\zeta_7} = \frac{1}{2}\log(1+10\zeta_5\hat{\lambda}^3 - 105\zeta_7\hat{\lambda}^4 - \frac{735}{4}\zeta_7^2\hat{\lambda}^8)$, and so on.

[13] Notice that U is a lower triangular matrix due to the argument of the first Γ function in the denominator being non-positive integer for $j > i$.

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The advantage of this form of ΔF is that the matrix M in (3.21) admits the following Bessel function representation

$$M_{ij} = 8\,(-1)^{i+j}\,\sqrt{2i+1}\sqrt{2j+1}\,\int_0^\infty \frac{dt}{t}\,\frac{e^{2\pi t}}{(e^{2\pi t}-1)^2}\,J_{2i+1}(t\sqrt{\lambda})\,J_{2j+1}(t\sqrt{\lambda})\,, \tag{3.26}$$

which will prove to be useful in the analysis of the strong-coupling limit.

4 Contributions to ΔF of finite degree in ζ-function values

The weak coupling expansion of ΔF in (3.4) can be represented as

$$\Delta F = \sum_{n=1}^{\infty} \Delta F^{(n)}\,, \qquad \Delta F^{(n)} = \sum_{k_1,\ldots,k_n} c_{k_1\ldots k_n}(\lambda)\,\zeta_{k_1}\ldots\zeta_{k_n} = \frac{(-1)^{n+1}}{n}\mathrm{tr}\,M^n\,, \tag{4.1}$$

where $\Delta F^{(n)}$ is the total contribution of terms that are products of a fixed number n of the Riemann ζ-function values. Equivalently, $\Delta F^{(n)}$ represents the contribution of the $\mathrm{tr}\,M^n$ term in (3.25) (cf. (3.19)).

Here we will study $\Delta F^{(n)}$, computing, in particular, its leading strong coupling asymptotic expansion. We shall focus in detail on the $n=1$ term and then discuss the $n=2$ one. We will find that

$$\Delta F^{(n)}(\lambda) \overset{\lambda\gg 1}{=} C_n\lambda^n + \cdots\,. \tag{4.2}$$

In the next section 5 we will compute all the coefficients C_n in (4.2) and then evaluate the sum of all $\Delta F^{(n)}$ thus determining the strong coupling asymptotics of ΔF.

Defining

$$G(t,t') \equiv 8\sum_{i=1}^{\infty}(2i+1)\,J_{2i+1}(t)\,J_{2i+1}(t') = -\frac{4tt'}{t^2-t'^2}\Big[t\,J_1(t)\,J_2(t') - t'\,J_2(t)\,J_1(t')\Big]\,, \tag{4.3}$$

which has also the following integral form [40]

$$G(t,t') = 2\,t\,t'\int_0^1 du\,J_2(t\sqrt{u})\,J_2(t'\sqrt{u})\,, \tag{4.4}$$

we may use the Bessel function representation of the matrix M (3.26) to represent the traces of M^n as the iterated integrals

$$\mathrm{tr}\,M = \int_0^\infty \widehat{dt}\,G(t\sqrt{\lambda},t\sqrt{\lambda}), \qquad\qquad \widehat{dt} = \frac{dt}{t}\,\frac{e^{2\pi t}}{(e^{2\pi t}-1)^2}, \tag{4.5}$$

$$\mathrm{tr}\,M^2 = \int_0^\infty \widehat{dt}\int_0^\infty \widehat{dt'}\,G(t\sqrt{\lambda},t'\sqrt{\lambda})\,G(t'\sqrt{\lambda},t\sqrt{\lambda}), \tag{4.6}$$

$$\mathrm{tr}\,M^3 = \int_0^\infty \widehat{dt}\int_0^\infty \widehat{dt'}\int_0^\infty \widehat{dt''}\,G(t\sqrt{\lambda},t'\sqrt{\lambda})\,G(t'\sqrt{\lambda},t''\sqrt{\lambda})\,G(t''\sqrt{\lambda},t'\sqrt{\lambda})\,,\quad \text{etc.} \tag{4.7}$$

We remark that (4.3) coincides with the Tracy-Widom kernel [40] upon the change of variables $t=\sqrt{x}$. It remains to be clarified whether this is a coincidence or there is some deeper relation to eigenvalue statistics. This Bessel kernel also appears in the BES equation [41] and seems prevalent in integrable equations/models.

4.1 Term linear in ζ_n

$\Delta F^{(1)}$ or tr M is just a single integral (4.5) and may be treated exactly. Using (4.3) we have

$$\Delta F^{(1)} = \frac{1}{2}\text{tr}\, M = 4 \int_0^\infty \frac{dt}{t} \frac{e^{2\pi t}}{(e^{2\pi t}-1)^2} \sum_{i=1}^\infty (2i+1) \left[J_{2i+1}(t\sqrt{\lambda})\right]^2. \tag{4.8}$$

From the identity

$$4 \sum_{i=1}^\infty (2i+1) \left[J_{2i+1}(x)\right]^2 = x^2 \left[J_0(x)\right]^2 + (x^2-4) \left[J_1(x)\right]^2, \tag{4.9}$$

we obtain

$$\begin{aligned} \Delta F^{(1)} &= \int_0^\infty \frac{dt\, e^{2\pi t}}{t(e^{2\pi t}-1)^2} \Big(t^2\lambda \left[J_0(t\sqrt{\lambda})\right]^2 + (t^2\lambda - 4) \left[J_1(t\sqrt{\lambda})\right]^2\Big) \\ &= \frac{\lambda}{2\pi} \int_0^\infty \frac{dt}{e^{2\pi t}-1} \left(\left[J_0(t\sqrt{\lambda})\right]^2 - \frac{8\, J_0(t\sqrt{\lambda}) J_1(t\sqrt{\lambda})}{t\sqrt{\lambda}} + \frac{(12-t^2\lambda)\left[J_1(t\sqrt{\lambda})\right]^2}{t^2\lambda}\right), \end{aligned} \tag{4.10}$$

where we used integration by parts. This expression is exact and may be expanded at weak or strong coupling.

Weak coupling expansion. Using

$$\int_0^\infty dt\, \frac{e^{2\pi t}}{(e^{2\pi t}-1)^2} t^{2p+1} = \frac{(2p+1)!}{(2\pi)^{2p+2}}\, \zeta_{2p+1}\,, \tag{4.11}$$

and expanding in λ we recover from (4.10) the first line in (3.19). One can find the following all-order result[14]

$$\Delta F^{(1)} = \frac{4}{\pi} \sum_{k=2}^\infty (-8)^k \frac{(k-1)\, k\, (k+2)\, \Gamma(k+\frac{1}{2}) \Gamma(k+\frac{3}{2})}{\left[\Gamma(k+3)\right]^2} \zeta_{2k+1} \hat{\lambda}^{k+1}. \tag{4.12}$$

By the standard ratio test this shows that the radius of convergence is π^2, as could be expected. Indeed, $\lambda = \pi^2$ is the radius of convergence of perturbative expansion in $\mathcal{N}=4$ SYM theory in the planar limit (as suggested by the single-magnon dispersion relation, fixed by the superconformal symmetry [41, 42], or by the quantum algebraic curve approach [43]). The same is expected to apply also to the $\mathcal{N}=2$ superconformal theories (as was first observed in the mass-deformed $\mathcal{N}=2^*$ theory [44], and recently found also in the orbifold theory case [19]).

Expanding the exponentials in the integral in (4.10) gives an alternative representation in terms of a sum of elliptic integrals

$$\Delta F^{(1)} = 2 \sum_{n=1}^\infty n \left[-1 - \frac{8\pi^2 n^2 - \lambda}{2\pi\lambda}\, \mathbb{E}\left(-\frac{\lambda}{\pi^2 n^2}\right) + \frac{8\pi^2 n^2 + 7\lambda}{2\pi\lambda}\, \mathbb{K}\left(-\frac{\lambda}{\pi^2 n^2}\right)\right]. \tag{4.13}$$

Here one sees explicit singularities at $\lambda = -\pi^2 n^2$ where the argument of $\mathbb{K}$ becomes unity.

[14] This remarkably simple form of the coefficients follows from the relation

$$\left[J_0(t)\right]^2 - \frac{8\, J_0(t) J_1(t)}{t} + \frac{(12-t^2)\left[J_1(t)\right]^2}{t^2} = \frac{5}{192}\, t^4\, {}_1F_2\left(\frac{7}{2}; 4, 5; -t^2\right).$$

Strong coupling expansion. The strong coupling (asymptotic) expansion of $\Delta F^{(1)}$ may be computed by Mellin transform methods [45, 46]. Defining the Mellin transform $\widetilde{f}(s) = \int_0^\infty dx\, x^{s-1}\, f(x)$ and considering the convolution

$$(f \star g)(x) = \int_0^\infty dt\, f(t\, x)\, g(t), \tag{4.14}$$

we have $(\widetilde{f \star g})(s) = \widetilde{f}(s)\, \widetilde{g}(1-s)$. Let $\alpha < s < \beta$ be the fundamental strip of analyticity of $\widetilde{f}(s)$. The asymptotic expansion of $f(x)$ for $x \to \infty$ is obtained by looking at the poles of $\widetilde{f}(s)$ in the region $s \geq \beta$. Then the pole $\frac{1}{(s-s_0)^N}$ in the Mellin transform leads to the term $\frac{(-1)^N}{(N-1)!}\frac{1}{x^{s_0}} \log^{N-1} x$ in the original function. In our case, we can compare the right hand side of (4.10) with (4.14) as

$$x = \sqrt{\lambda}, \quad g(t) = \frac{1}{4\pi}\frac{1}{e^{2\pi t}-1}, \quad f(t) = \big[J_0(t)\big]^2 - \frac{8\, J_0(t) J_1(t)}{t} + \frac{(12-t^2)\, \big[J_1(t)\big]^2}{t^2}. \tag{4.15}$$

The Mellin transform is then

$$(\widetilde{f \star g})(s) = \frac{2^{-6+s} s (2+s) \csc^2(\frac{\pi s}{2}) \Gamma(2-s)\, \zeta(s)}{\big[\Gamma(1-\frac{s}{2})\big]^2\, \Gamma(2-\frac{s}{2})\, \Gamma(3-\frac{s}{2})}, \tag{4.16}$$

and the asymptotic expansion at strong coupling can be extracted from the poles at $s = 0, 1, 2, \ldots$. This gives

$$\Delta F^{(1)} = \frac{\lambda}{16\pi^2} - \frac{\sqrt{\lambda}}{2\pi^2} + \frac{1}{6} + \frac{\sqrt{\lambda}}{2\pi^{7/2}} \sum_{p=1}^{\infty} \frac{\Gamma(\frac{5}{2}+p)\Gamma(p-\frac{1}{2})\Gamma(p-\frac{3}{2})}{\Gamma(p)} \frac{\zeta_{2p+1}}{\lambda^p}. \tag{4.17}$$

The infinite sum in (4.17) has zero radius of convergence, with factorially divergent coefficients.[15] The leading order λ term corresponds to the $n = 1$ case of the general pattern (4.2).

The leading term in (4.17) can be derived more directly. We can expand the integrand in (4.5) at large λ and read off the coefficient of a suitable power of λ from a convergent integral[16]

$$\begin{aligned} \operatorname{tr} M &= \int_0^\infty \frac{dt}{t} \frac{e^{2\pi t}}{(e^{2\pi t}-1)^2}\, G(t\sqrt{\lambda}, t\sqrt{\lambda}) = \int_0^\infty \frac{dt}{t} \frac{e^{2\pi t/\sqrt{\lambda}}}{(e^{2\pi t/\sqrt{\lambda}}-1)^2}\, G(t,t) \\ &= \frac{\lambda}{2\pi^2} \int_0^\infty \frac{dt}{t} \big[J_2(t)^2 - J_1(t) J_3(t)\big] + \cdots = \frac{\lambda}{8\pi^2} + \ldots . \end{aligned} \tag{4.18}$$

As $\Delta F^{(1)} = \frac{1}{2} \operatorname{tr} M$ (cf. (4.8)), this result is thus in agreement with (4.17).

[15] Let us note that replacing the ζ-values by the integral using (4.11) and doing the sum, we obtain another representation

$$\Delta F^{(1)} = \frac{\lambda}{16\pi^2} - \frac{\sqrt{\lambda}}{2\pi^2} + \frac{1}{6} + \frac{2\lambda}{\pi^3} \int_0^\infty \frac{dt}{e^{2t\sqrt{\lambda}}-1} \Big[\mathbb{K}(t^2) - (1+8t^2)\, \mathbb{E}(t^2)\Big].$$

This integral has a logarithmic singularity at $t = 1$ on the t integration contour, and so should be understood as an average above and below the cut.

[16] This procedure works for the leading order; at subleading orders one gets divergent integrals requiring a more careful treatment.

4.2 Term quadratic in ζ_n

In the case of $\Delta F^{(2)} = -\frac{1}{4}\text{tr}\, M^2$ in (4.1) we can obtain an all-order weak coupling expansion in almost-closed form. Although it is not as explicit as (4.12) for $\Delta F^{(1)}$, it may be used to generate a very large number of terms. Here we will present the final result, with details given in appendix B. Let us define the polynomials

$$d_\ell(x) = (-1)^\ell \sum_{p=0}^{\ell} \frac{P_p^{(2,-2p-5)}(1-2x^2)\; P_{\ell-p}^{(2,-2\ell+2p-5)}(1-2x^2)}{4^{p+2}4^{\ell-p+2}\Gamma(p+3)\Gamma(p+4)\Gamma(\ell-p+3)\Gamma(\ell-p+4)}, \tag{4.19}$$

where $P_n^{(\alpha,\beta)}(x)$ are Jacobi polynomials. We may write d_ℓ in the form

$$d_\ell(x) = x^\ell \sum_{\substack{m=m_0\\ \Delta m=2}}^{\ell} a_m^{(\ell)}(x^m + x^{-m}), \tag{4.20}$$

where $m_0 = 0/1$ if ℓ is even/odd and m varies in steps of 2. The weak coupling expansion of $\text{tr}\, M^2$ can then be written in terms of sums with coefficients $a_m^{(\ell)}$ that are easily computed from (4.19), (4.20)

$$\text{tr}\, M^2 = 8\sum_{\ell=0}^{\infty}(2\pi)^{-12-2\ell}\,\lambda^{\ell+6}\sum_{m}^{\ell} a_m^{(\ell)}\Gamma(\ell+6+m)\Gamma(\ell+6-m)\,\zeta_{\ell+5+m}\,\zeta_{\ell+5-m}. \tag{4.21}$$

Leading term at strong coupling. The expansion (4.21) may not be used directly at strong coupling. Nevertheless, we succeed in applying the manipulation we exploited in (4.18). Indeed, we have

$$\text{tr}\, M^2 = \lambda^2\int_0^\infty\int_0^\infty dt dt' \,\frac{[t' J_1(t')J_2(t) - tJ_1(t)J_2(t')]^2}{\pi^4 tt'(t^2-t'^2)^2} + \dots \tag{4.22}$$

The integrand is symmetric so we write

$$\int_0^\infty\int_0^\infty dtdt' f(t,t') = 2\int_0^\infty dt\int_0^t dt' f(t,t') = 2\int_0^\infty dt\, t\int_0^1 dx\, f(t,tx). \tag{4.23}$$

Doing first the integral over t, we get

$$\begin{aligned}\text{tr}\, M^2 &= \lambda^2\int_0^1 dx\,\frac{x(15-7x^2-7x^4+15x^6)-3(1-x^2)^2(5+6x^2+5x^4)\,\text{arctanh}\, x}{144\pi^6 x^5} + \dots\\ &= \frac{\lambda^2}{192\pi^4} + \dots\,.\end{aligned} \tag{4.24}$$

This strong-coupling asymptotics follows again the general pattern (4.2). A numerical test of this prediction will be discussed in section 6.1.

5 Strong coupling limit of ΔF: analytic derivation

Let us now generalize the derivation of strong-coupling limit to the full ΔF. The starting point will be the explicit form of the large λ expansion of the matrix M in (3.26). It can be found as in (4.14)–(4.17) using the Mellin transform. We have

$$M_{ij} = 8(-1)^{i+j}\sqrt{(2i+1)(2j+1)}\, N_{ij}\,, \tag{5.1}$$

$$N_{ij} \equiv \sqrt{\lambda}\,(f\star g_{ij})(\sqrt{\lambda})\,, \qquad f(t)=\frac{e^{2\pi t}}{(e^{2\pi t}-1)^2}\,, \qquad g_{ij}(t)=\frac{1}{t}J_{2i+1}(t)J_{2j+1}(t). \tag{5.2}$$

Evaluating the Mellin transforms and taking residues, we get the asymptotic expansion of N_{ij}

$$N_{ij} \overset{\lambda\gg 1}{=} \left[\frac{\delta_{ij}}{i(i+1)(2i+1)}+\frac{\delta_{i+1,j}}{(i+1)(2i+1)(2i+3)}+\frac{\delta_{i,j+1}}{i(2i-1)(2i+1)}\right]\frac{\lambda}{64\pi^2} - \frac{\delta_{ij}}{24\,(2i+1)}+\frac{\zeta_3}{2\pi^2}\cos\left(\pi(i-j)\right)\frac{1}{\sqrt{\lambda}}+\cdots. \tag{5.3}$$

Then the leading strong-coupling part of M may be written as

$$M \overset{\lambda\gg 1}{=} \frac{\lambda}{2\pi^2}\mathrm{S}+\ldots\,, \tag{5.4}$$

$$\mathrm{S}_{ij} = \frac{1}{4}(-1)^{i+j}\sqrt{\frac{2j+1}{2i+1}}\left[\frac{\delta_{ij}}{i(i+1)}+\frac{\delta_{i+1,j}}{(i+1)(2i+3)}+\frac{\delta_{i,j+1}}{i(2i-1)}\right], \tag{5.5}$$

where S is a symmetric three-diagonal infinite-dimensional matrix. As a result, we get

$$\mathrm{tr}\, M^n \overset{\lambda\gg 1}{=} b_n\left(\frac{\lambda}{2\pi^2}\right)^n+\cdots\,, \qquad b_n=\mathrm{tr}\,\mathrm{S}^n\,. \tag{5.6}$$

The explicit values of the coefficients b_n (related to C_n in (4.2) as $C_n=\frac{(-1)^{n+1}}{n(2\pi^2)^n}b_n$) are given in appendix C.

Remarkably, S in (5.5) is essentially the same (up to 1/2) as the matrix appearing in eq. (2.7) of [47]. It follows from the analysis in [47] that in the infinite matrix limit the eigenvalues $\{\mathrm{s}_1,\mathrm{s}_2,\ldots\}$ of S are

$$\mathrm{s}_k=\frac{2}{\mathrm{j}_{1,k}^2}\,, \qquad k=1,2,\ldots, \tag{5.7}$$

where $\mathrm{j}_{1,k}$ are the zeroes of the Bessel function $J_1(x)$. Hence, we get the following remarkable relation[17]

$$\det\left(1+\frac{\lambda}{2\pi^2}\mathrm{S}\right)=\prod_{k=1}^{\infty}\left(1+\frac{\lambda}{\pi^2}\frac{1}{\mathrm{j}_{1,k}^2}\right)=\frac{2\pi}{i\sqrt{\lambda}}J_1\left(\frac{i\sqrt{\lambda}}{\pi}\right)=\frac{2\pi}{\sqrt{\lambda}}I_1\left(\frac{\sqrt{\lambda}}{\pi}\right). \tag{5.8}$$

As a result, we get for ΔF in (3.25)

$$\Delta F=\frac{1}{2}\log\det(1+M)\overset{\lambda\gg 1}{=}\frac{1}{2}\log\det\left(1+\frac{\lambda}{2\pi^2}\mathrm{S}\right)=\frac{1}{2}\log\left[\frac{2\pi}{\sqrt{\lambda}}I_1\left(\frac{\sqrt{\lambda}}{\pi}\right)\right]\overset{\lambda\gg 1}{=}\frac{\sqrt{\lambda}}{2\pi}+\cdots. \tag{5.9}$$

[17] This follows from the Weierstrass infinite product representation of the Bessel function in terms of its zeroes:
$J_\nu(z)=\frac{(z/2)^\nu}{\Gamma(\nu+1)}\prod_{n=1}^{\infty}(1-\frac{z^2}{\mathrm{j}_{\nu,n}^2})$, see for instance section 15.41 in [48].

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Eq. (5.9) implies that c_1 in (1.14) is equal to $\frac{1}{2\pi}$. Then using (1.11) we obtain the following expression (1.15) for the strong-coupling limit of Δq

$$\Delta q(\lambda) \overset{\lambda\gg 1}{=} -\frac{\lambda^{3/2}}{16\pi} + \cdots . \tag{5.10}$$

6 Numerical evaluation of ΔF: interpolation from small to large λ

In this final section we present various approaches to test the analytical result (5.9) for the strong coupling limit of ΔF by numerical methods. We will first consider the approach based on Padé approximants using as an input many terms in the weak coupling expansion of ΔF. Then, we will discuss a method based on a direct evaluation of $\Delta F = \frac{1}{2}\text{tr}\,\log(1+M)$ where the large λ limit of the infinite matrix M is first replaced by its finite-size truncation.

6.1 Padé-conformal method

We begin with the small λ expansion of ΔF:

$$\Delta F(\tilde{\lambda}) = \sum_k c_k \tilde{\lambda}^k\,, \qquad\qquad \tilde{\lambda} \equiv 8\hat{\lambda} = \frac{\lambda}{\pi^2}\,. \tag{6.1}$$

The particular definition of $\tilde{\lambda}$ is chosen so that the radius of convergence of the series in (6.1) is as close as possible to 1. This is helpful for the numerical analysis, as it avoids the appearance of very large or very small coefficients at high order.

The technical goal is to extrapolate from small to large λ, starting from a *finite* number of terms in the weak coupling expansion. Optimal and near-optimal methods for such an extrapolation have been analyzed recently in [49–51]. The key information is some knowledge, either analytic or numerical, of the singularity structure of the function $\Delta F(\tilde{\lambda})$. This information can be extracted numerically by suitable combinations of ratio tests, Padé approximants, and conformal maps.

The magnitude of the leading singularity is equal to the radius of convergence R, which can be found by a simple ratio test:

$$\left|\frac{c_{k+1}}{c_k}\right| \to \frac{1}{R}, \qquad k\to\infty. \tag{6.2}$$

The convergence of this ratio of successive coefficients to the inverse radius can be accelerated using Richardson acceleration [52] (for example, for the $\text{tr}\, M$ case see the left hand panel of figure 1 below).

This permits an extremely precise numerical estimate of the radius of convergence, if it is not known analytically. For $\Delta F = \frac{1}{2}\text{tr}\log(1+M)$, we will see that the leading singularity is at $\tilde{\lambda} \approx -1$, i.e. $\lambda \approx -\pi^2$. By studying the subleading corrections to this ratio test limit one can determine the nature of the leading singularity, using Darboux's theorem, see appendix D. For this orientifold model the small λ expansion indicates that the leading singularity is logarithmic (see the right hand panel of figure 1 below). This is consistent with the exact analytical structure of individual $\text{tr}\, M^n$ terms for finite n, see section 5.

A closely related method, which also yields information about the singularity structure is based on the use of a Padé approximant [52, 53]. Here one matches the finite number K of terms of the expansion to the expansion of a ratio of polynomials R_L and Q_M:

$$\mathcal{P}_{[L,M]}\Big[\sum_k^K c_k\tilde{\lambda}^k\Big] = \frac{R_L(\tilde{\lambda})}{Q_M(\tilde{\lambda})} + O(\tilde{\lambda}^{K+1})\,. \tag{6.3}$$

Since it is an approximation in terms of rational functions, Padé only has poles as singularities, which are the zeros of the denominator polynomial Q_M. If the truncated series is that of a function with branch point singularities, then Padé produces arcs of poles accumulating at the branch points.[18] The practical implication of this is that if one has enough expansion terms one can frequently distinguish between an isolated pole and a branch point simply by looking at the poles of a Padé approximant. Indeed, the left panel of figure 2 shows a line of Padé poles accumulating to the branch point at $\tilde{\lambda} = -1$.

However, this reveals a fundamental problem with Padé, because these accumulating poles, which are trying to represent a branch cut, obscure possible higher singularities which may be physical. This problem can be resolved by making a conformal map before making the Padé approximation [49–51]. Based on the leading branch cut $(\infty, -1]$ on the negative real $\tilde{\lambda}$ axis, as suggested by the Padé approximation in this case (see the left hand panel of figure 2), one maps the expansion into the unit disk $|z| \leq 1$:

$$z = \frac{\sqrt{1+\tilde{\lambda}}-1}{\sqrt{1+\tilde{\lambda}}+1}, \qquad\qquad \tilde{\lambda} = \frac{4z}{(1-z)^2}. \tag{6.4}$$

We re-expand $\Delta F\left(\frac{4z}{(1-z)^2}\right)$ in powers of z to the same order K, and **then** construct a Padé approximant in terms of z.[19] Inside the unit disk this expansion is convergent by construction, but further singularities along the line $\tilde{\lambda} \in (\infty, -1]$ will appear as singularities on the unit circle. If these are branch points they will appear as the accumulation points of arcs of Padé poles.

The advantage of the conformal map is that collinear singularities in the $\tilde{\lambda}$ plane (which may be hidden under a line of accumulating poles) are separated to different points on the unit circle. See for example the right panel of figure 2, which shows the leading singularity at $z = -1$, the conformal map image of $\tilde{\lambda} = -1$, but also clearly shows further singularities at the conformal map images of $\tilde{\lambda} = -4$, at $\tilde{\lambda} = -9$, and so on. This numerical evidence suggests that the singularities are:

$$\text{singularites}\big(\Delta F(\lambda)\big) = -l^2\,\pi^2\,, \qquad\qquad l = 1, 2, 3, \ldots\,. \tag{6.5}$$

[18] There is a deep connection to electrostatics and potential theory, whereby (in this interpretation it is easiest to consider an expansion about infinity instead of about zero) in the $K \to \infty$ limit a Padé approximation produces lines of poles that form a capacitor having minimal capacitance [51, 54].

[19] As a technical comment: when dealing with high order Padé approximants, numerical instabilities can arise due to close zeros and poles, also associated with large coefficients of the Padé polynomials. This instability can be ameliorated by converting the Padé representation to a partial fraction expansion, which in principle is equivalent but in practice is more stable numerically.

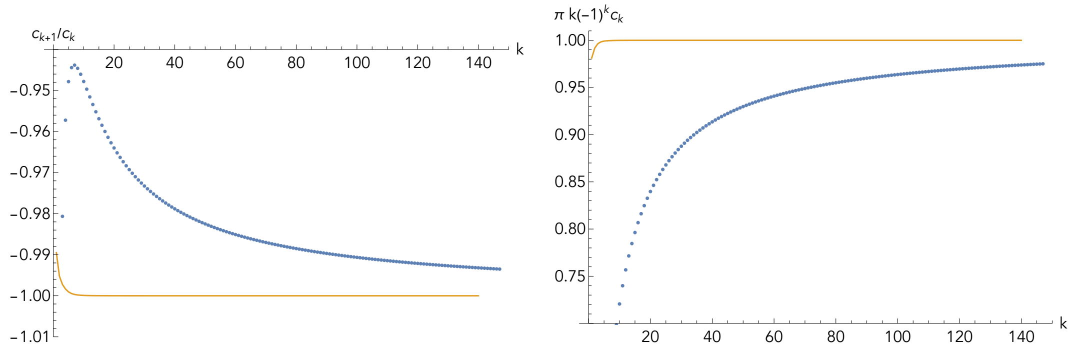

Figure 1. $\mathrm{tr}\, M$ ratio test. On the left, we show in blue the ratio c_{k+1}/c_k that tends to -1. The orange line is obtained after applying a 5th order Richardson acceleration. On the right, we present the same analysis for the Darboux indicator $\pi k(-1)^k c_k$.

The source of these singularities can be understood analytically from the study of $\mathrm{tr}\, M^n$ for finite n, and the singularity structure appears to be inherited by $\mathrm{tr}\, \log(1 + M)$.

A further advantage of the conformal map is that it enhances the precision of the subsequent Padé extrapolation. To construct the Padé-conformal extrapolation[20] we make a Padé approximant in terms of z and then evaluate it on the inverse map in (6.4). This introduces square roots; thus we are representing the function not just by rational approximations, but in a much wider class of functions. For branch point singularities the increase in precision can be quantified precisely using the asymptotics of orthogonal polynomials [50] and is quite dramatic, as is illustrated in figures 3 and 4 below.

6.1.1 Example: tr M

To illustrate this Padé-conformal extrapolation technique, we first consider the expansions of $\mathrm{tr}\, M$ and $\mathrm{tr}\, M^2$, for which we can compare with analytic results found in section 4. But we stress that the power of this method is in cases when such analytic comparisons are not available, and one is only presented with a truncated series, and possibly some physical intuition about the singularity structure. For $\mathrm{tr}\, M$ we have the exact expansion (cf. (4.12))

$$\mathrm{tr}\, M = \sum_{k=2}^{\infty} \frac{(-1)^k (k-1)k(k+2)\, \zeta_{2k+1}\, \Gamma\left(k+\frac{1}{2}\right) \Gamma\left(k+\frac{3}{2}\right)}{\pi \left[\Gamma(k+3)\right]^2} \tilde{\lambda}^{k+1} . \tag{6.6}$$

The ratio c_{k+1}/c_k is plotted in the left panel of figure 1 based on the first 150 terms, indicating an alternating series with radius of convergence 1. The fact that the leading singularity is logarithmic is shown by the fact that $c_k \sim \frac{(-1)^k}{k} \times$ constant as $k \to \infty$. See the right panel in figure 1. The fact that the leading singularity is a branch point is also indicated by the Padé poles, which are shown in the left panel of figure 2, accumulating along the negative real axis to the branch point at $\tilde{\lambda} = -1$.

After the conformal map (6.4), followed by re-expansion to 150 terms in z, the poles of the resulting diagonal Padé approximant are shown in the right panel of the same figure.

[20]This was applied to the Borel transform function in [49, 50], but it can also be applied to any series with a finite radius of convergence [51].

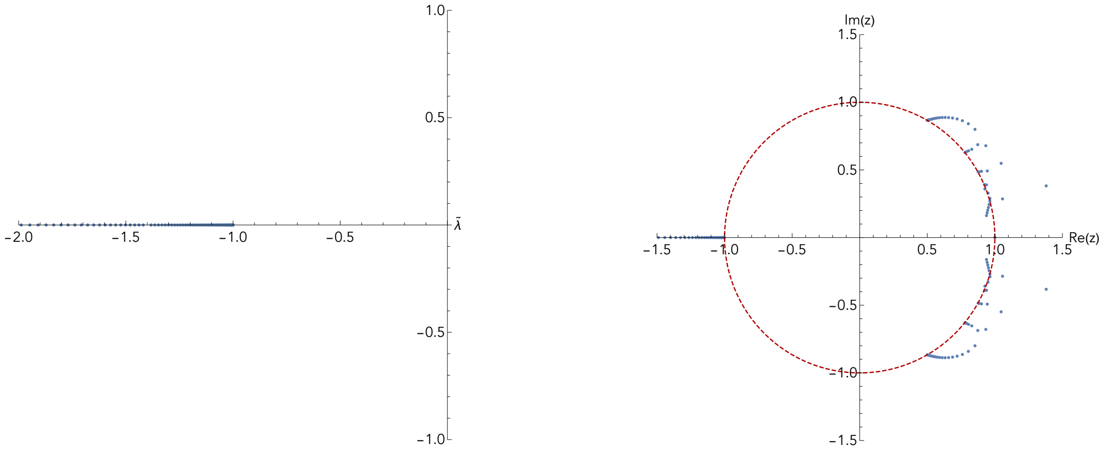

Figure 2. Padé poles of $\operatorname{tr} M$ from 150 terms. On the left, we show the poles of the direct approximants. These poles lie on the negative real axis and accumulate to $\tilde{\lambda} = -1$. On the right we show the poles in z-plane after application of the conformal transformation (6.4) followed by Padé. In this case collinear singularities on the line $\tilde{\lambda} \in (-\infty, -1]$ are separated and made visible as arcs converging to points on the unit circle in the z plane. These agree with a similar analysis for the whole ΔF, see (6.5).

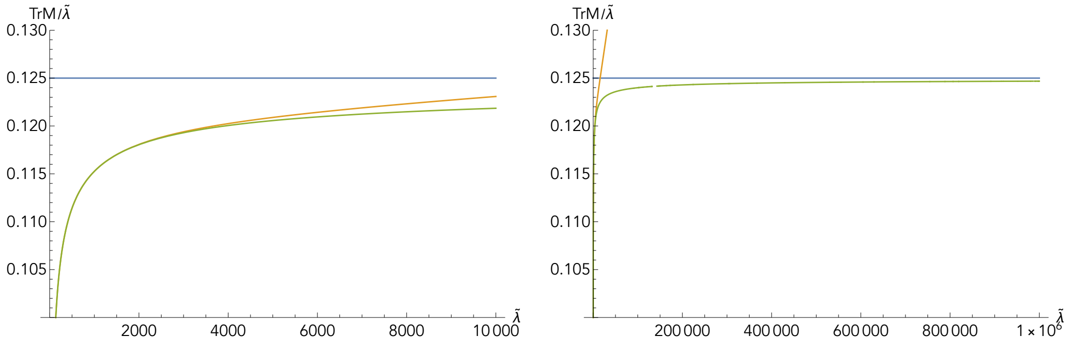

Figure 3. Extrapolations of $\operatorname{tr} M(\tilde{\lambda})/\tilde{\lambda}$ compared to the analytic value $1/8$ (blue line). The left and right plot differ only in the range of $\tilde{\lambda}$ values, 10^4 on the left and 10^6 on the right. The orange line is diagonal Padé of order 75, applied to the first 150 terms in the weak coupling expansion (6.6). The green line is the Padé-conformal extrapolation based on the transformation (6.4).

This figure indicates the existence of branch point singularities at the z plane images of $\tilde{\lambda} = -1, -4, -9, -16$. The data becomes noisy at the conformal image of -25, with unphysical poles appearing inside the unit disk. These can be resolved by taking more terms in the original expansion.

We now map this Padé approximant back to the physical $\tilde{\lambda}$ plane using the inverse conformal map in (6.4), and plot to large λ. Figure 3 compares the diagonal Padé extrapolation (orange curve), divided by $\tilde{\lambda}$, with the analytic large $\tilde{\lambda}$ limit of $\frac{1}{8}$ (blue curve) and the Padé-conformal extrapolation (green curve). The first plot extends out to $\tilde{\lambda} = 10^4$, while the second plot extends out to $\tilde{\lambda} = 10^6$. Note that the Padé approximant eventually breaks down at $\tilde{\lambda} \approx 1.5 \cdot 10^4$, while the Padé-conformal approximant extends much further to very

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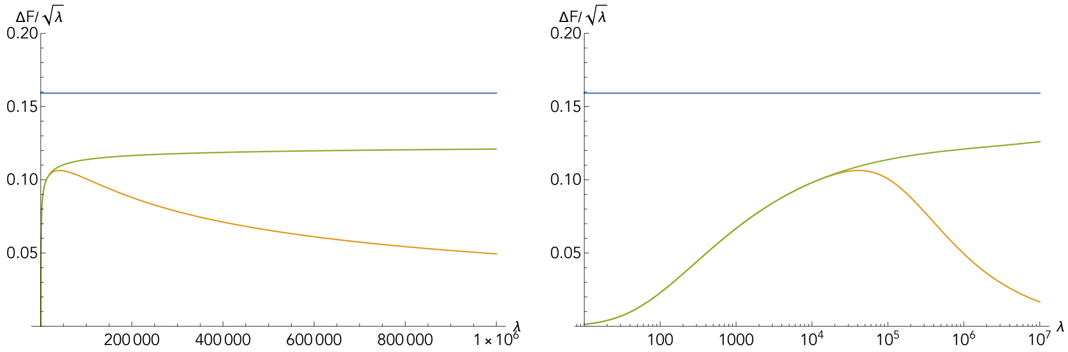

Figure 4. Extrapolations of $\Delta F/\sqrt{\lambda}$ (plotted here as functions of λ, not $\tilde{\lambda}$) compared to the asymptotic value $c_1 = \frac{1}{2\pi}$ (blue line). In the left plot, in linear scale, the orange line is the diagonal Padé approximant based on 150 terms of the full weak-coupling expansion, i.e. the extension of (3.4) to the order λ^{150}. This Padé approximant breaks down shortly after $\lambda = 10^4$. The green line is the Padé conformal result and it extends to much higher values of the coupling λ. The left plot strongly supports the functional form $\Delta F \sim \sqrt{\lambda}$ at large λ. The convergence of the coefficient to the asymptotic value is steady but slow, as illustrated in the right panel on a logarithmic scale. See section 6.2 for a more refined estimate of the overall coefficient.

large $\tilde{\lambda}$. We stress that exactly the same input coefficient data was used in producing these two extrapolations, illustrating the dramatic effect of the conformal map.

A similar analysis can be applied to $\operatorname{tr} M^2$ where we do not have a simple closed form expression for the expansion coefficients, but there is a systematic way to expand to very high order (multiple hundreds of terms, see (4.21)). The resulting structure is very similar to that for the $\operatorname{tr} M$ case discussed above, so we do not repeat the analogous plots.

6.1.2 ΔF

Let us now consider the large λ extrapolation of the full ΔF. We begin with the small λ expansion discussed in section 3.1. We generated 150 terms of this expansion, with 450 digit precision for the coefficients. The coefficients are sums of products of odd ζ_{2k+1}-values, but it is faster to work with finite but high precision coefficients. The ratio test and Padé analysis again indicate a leading singularity at $\lambda = -\pi^2$, so we make the same conformal map (6.4) and subsequent Padé approximant and inverse map back to the physical λ plane.

Figure 4 shows the result, and we again see that the Padé-conformal extrapolation extends to a much larger value of λ. This extrapolation shows that the functional form of the large λ behavior is (left panel of the figure)

$$\Delta F(\lambda) = \frac{1}{2}\operatorname{tr}\log(1+M) \overset{\lambda \to +\infty}{=} c_1\sqrt{\lambda}\,. \tag{6.7}$$

This functional form matches the result of resumming the leading large λ terms of $\operatorname{tr} M^n$ in (5.9), and the coefficients approximately agree.

We stress that the only input information used for this extrapolation from small λ to large λ was the list of 150 perturbative coefficients. To get a better estimate of the result requires fitting the ratio $\Delta F/\sqrt{\lambda}$ and it is hard to support a specific functional form.

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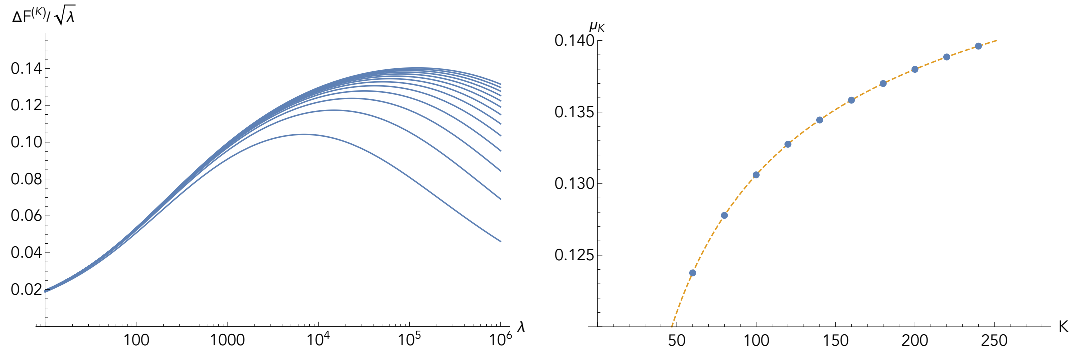

Figure 5. Analysis of ΔF by considering a truncated leading order approximation of the matrix M. In the left panel we plot the ratio $\Delta F_K/\sqrt{\lambda}$ where F_K is defined in (6.8), and $K = 20, 40, \dots, 260$ from bottom to top. For each K, there is a maximal value μ_K. In the right panel we plot μ_K vs. K (blue dots) and compute its best fit (orange dashed line) with a constant plus a leading $\sim K^{-1/2}$ and subleading $\sim K^{-1}$ terms. The best fit parameters are in (6.10).

The slow convergence shown in the right panel of figure 4 should be due to the expected logarithmic corrections in (1.14) if they do not happen to cancel in ΔF.

6.2 Evaluation of ΔF at large λ using truncation method

In this subsection we use a complementary numerical method in order to extract the precise large λ behaviour of ΔF. Starting from the expansion (5.3), let us denote by M_K the $K\times K$ matrix which is the linear in λ part of M, truncated to the first K rows and columns. Then

$$\Delta F(\lambda) = \lim_{K\to\infty} \Delta F_K(\lambda)\,, \qquad \Delta F_K(\lambda) = \frac{1}{2}\text{tr}\,\log(1+M_K). \tag{6.8}$$

To determine the large λ behaviour of $\Delta F(\lambda)$, we need to take first $K \to \infty$, and then $\lambda \to \infty$.

To bypass this double limit procedure, we will fix K, increase λ until the ratio $\frac{\Delta F_K(\lambda)}{\sqrt{\lambda}}$ reaches a maximum

$$\mu_K = \max_{\lambda} \frac{\Delta F_K(\lambda)}{\sqrt{\lambda}}\,, \tag{6.9}$$

and, finally, extrapolate μ_K to $K \to \infty$. According to (5.10), the expected value is $c_1 = \frac{1}{2\pi}$. The explicit numerical results are collected in figure 5. In the left panel we show the curves $\frac{\Delta F_K(\lambda)}{\sqrt{\lambda}}$ for $K = 20, 40, 60, \dots 260$. For each K a maximum in (6.9) is reached at a value of λ that increases with K. The maximum value μ_K is shown in the right panel of the figure and fitted by the dashed curve

$$\mu_K^{\text{fit}} = 0.158 - \frac{0.301}{\sqrt{K}} + \frac{0.290}{K}, \tag{6.10}$$

that empirically works very well. The estimated value of the coefficient of $\sqrt{\lambda}$ in (5.10) is thus 0.158, which differs by less then 1% from the analytical prediction $\frac{1}{2\pi} \simeq 0.159$.

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A Direct computation of Δq at weak coupling

The expectation value of the Wilson loop (2.8) at finite N reads

$$\langle \mathcal{W} \rangle^{\text{orient}} = \langle \mathcal{W} \rangle_0 + \frac{1}{N^2}\Big(\langle \mathcal{W} \rangle_1^{\mathcal{N}=4} + \langle \mathcal{W} \rangle_1^{\mathcal{N}=2} \Big) + \mathcal{O}\Big(\frac{1}{N^4}\Big), \tag{A.1}$$

where $\langle \mathcal{W} \rangle_0 = \langle \mathcal{W} \rangle_0^{\mathcal{N}=4}$ is the planar $\mathcal{N} = 4$ SYM expression (1.1) and $\langle \mathcal{W} \rangle_1^{\mathcal{N}=4}$ is [2] (cf. (1.3))

$$\langle \mathcal{W} \rangle_1^{\mathcal{N}=4} = \frac{1}{48}\Big[-12\sqrt{\lambda} I_1(\sqrt{\lambda}) + \lambda I_2(\sqrt{\lambda}) \Big]. \tag{A.2}$$

We use the label "$\mathcal{N} = 2$" to separate the genuine correction to the $\mathcal{N} = 4$ result. The explicit calculation starting with the matrix model expression (2.8), (3.3) gives

$$\begin{aligned}\langle \mathcal{W} \rangle_1^{\mathcal{N}=2} =& -\frac{15\zeta_5}{4\,(8\pi^2)^3}\,\lambda^4 + \Big(-\frac{15\zeta_5}{32\,(8\pi^2)^3} + \frac{105\zeta_7}{2\,(8\pi^2)^4} \Big)\lambda^5 + \Big(-\frac{5\zeta_5}{256\,(8\pi^2)^3} + \frac{105\zeta_7}{16\,(8\pi^2)^4} - \frac{2205\zeta_9}{4\,(8\pi^2)^5} \Big)\lambda^6 \\ & + \Big(-\frac{5\zeta_5}{12288\,(8\pi^2)^3} + \frac{75\zeta_5^2}{2\,(8\pi^2)^6} + \frac{35\zeta_7}{128\,(8\pi^2)^4} - \frac{2205\zeta_9}{32\,(8\pi^2)^5} + \frac{10395\zeta_{11}}{2\,(8\pi^2)^6} \Big)\lambda^7 + \mathcal{O}(\lambda^8).\end{aligned} \tag{A.3}$$

The function $\Delta q(\lambda)$ in (1.2), (1.10) is obtained dividing by $\langle \mathcal{W} \rangle_0$ and this gives precisely (3.5), i.e. the result consistent with (2.13). We checked the relation (2.13) to order $\mathcal{O}(\lambda^{20})$ by an independent computation of both ΔF and Δq.

Let us note that each of the monomials in the ζ_n-values appears in the weak-coupling expansion with a simple single-power dependence on λ. The corresponding leading $1/N^2$ corrections in $\langle \mathcal{W} \rangle^{\text{orient}}$ happen to have its non-trivial dependence on λ via the Bessel function factor $\sim \lambda^{-1/2} I_1(\sqrt{\lambda})$. This property can be proved for specific monomials in ζ_n-values by the methods described in [19]. It may be made explicit by collecting terms in (A.3) as

$$\begin{aligned}\langle \mathcal{W} \rangle_1^{\mathcal{N}=2} =& -\frac{15\zeta_5}{4\,(8\pi^2)^3}\,\lambda^4 \Big(1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \frac{\lambda^3}{9216} + \cdots \Big) \\ & + \frac{105\zeta_7}{2\,(8\pi^2)^4}\,\lambda^5 \Big(1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \cdots \Big) - \frac{2205\zeta_9}{4\,(8\pi^2)^5}\,\lambda^6 \Big(1 + \frac{\lambda}{8} + \cdots \Big) + \cdots.\end{aligned} \tag{A.4}$$

Here, one can see that each monomial in the ζ_n-values is multiplied by the expansion of the factor $\langle \mathcal{W} \rangle_0 = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$. As discussed in [19], this property is important for the relation (2.13) to hold.

B Weak coupling expansion of tr M^2

Here we shall provide the proof of the result (4.21) for the weak-coupling expansion of tr M^2. We begin with the expansion of the product of two Bessel functions as a series of Jacobi polynomials

$$2tt'J_2(t\sqrt{u})J_2(t'\sqrt{u}) = 2\,(tt')^3 \sum_{m=0}^{\infty}(-1)^m \frac{u^{m+2}}{4^{m+2}\big[\Gamma(m+3)\big]^2} t^{2m} P_m^{(2,-2m-5)}\left(1-2\frac{t'^2}{t^2}\right). \tag{B.1}$$

From (4.4), the kernel in (4.3) admits the representation

$$G(t\sqrt{\lambda},t'\sqrt{\lambda}) = 2\,(tt')^3 \sum_{m=0}^{\infty}\lambda^{m+3} \frac{(-1)^m}{4^{m+2}(m+3)\big[\Gamma(m+3)\big]^2} t^{2m} P_m^{(2,-2m-5)}\left(1-2\frac{t'^2}{t^2}\right). \tag{B.2}$$

Plugging this into (4.6) gives

$$\operatorname{tr} M^2 = 8\sum_{\ell=0}^{\infty}\lambda^{\ell+6}\int_0^1 dx x^5 d_\ell(x)\int_0^\infty dt \frac{e^{2\pi t}}{(e^{2\pi t}-1)^2}\frac{e^{2\pi x t}}{(e^{2\pi x t}-1)^2}t^{2\ell+11}, \tag{B.3}$$

where the polynomials $d_\ell(x)$ were defined in (4.19) and (4.20). Using $d_\ell(1/x) = x^{-2\ell}d_\ell(x)$, we get

$$\begin{aligned}
\operatorname{tr} M^2 &= 8\sum_{\ell=0}^{\infty} f_\ell\,\lambda^{\ell+6},\\
f_\ell &= \int_0^1 dx x^5 d_\ell(x)\int_0^\infty dt \frac{e^{2\pi t}}{(e^{2\pi t}-1)^2}\frac{e^{2\pi x t}}{(e^{2\pi x t}-1)^2}t^{2\ell+11}\\
&= \int_1^\infty \frac{dx}{x^2}x^{-5}x^{-2\ell}d_\ell(x)\int_0^\infty dt \frac{e^{2\pi t}}{(e^{2\pi t}-1)^2}\frac{e^{2\pi t/x}}{(e^{2\pi t/x}-1)^2}t^{2\ell+11}\\
&= \int_1^\infty dx x^5 d_\ell(x)\int_0^\infty dt\, \frac{e^{2\pi x t}}{(e^{2\pi x t}-1)^2}\frac{e^{2\pi t}}{(e^{2\pi t}-1)^2}t^{2\ell+11}.
\end{aligned} \tag{B.4}$$

f_ℓ may be written as an integral over the whole half-line $[0,\infty]$ and have (cf. (4.20))

$$\begin{aligned}
f_\ell &= \frac{1}{2}\int_0^\infty dx x^5 d_\ell(x)\int_0^\infty dt \frac{e^{2\pi t}}{(e^{2\pi t}-1)^2}\frac{e^{2\pi x t}}{(e^{2\pi x t}-1)^2}t^{2\ell+11}\\
&= \sum_m a_m^{(\ell)}\frac{1}{2}\int_0^\infty dx x^{5+\ell}(x^m+x^{-m})\int_0^\infty dt \frac{e^{2\pi t}}{(e^{2\pi t}-1)^2}\frac{e^{2\pi x t}}{(e^{2\pi x t}-1)^2}t^{2\ell+11}\\
&= \frac{1}{2}\sum_m a_m^{(\ell)}(I_{\ell+m+5}I_{\ell+5-m}+I_{\ell-m+5}I_{\ell+5+m}) = \sum_m a_m^{(\ell)} I_{\ell+m+5}I_{\ell+5-m},
\end{aligned} \tag{B.5}$$

where

$$I_n = \int_0^\infty dt \frac{e^{2\pi t}}{(e^{2\pi t}-1)^2}t^n = (2\pi)^{-n-1}\Gamma(n+1)\zeta_n\,. \tag{B.6}$$

Hence,

$$f_\ell = (2\pi)^{-12-2\ell}\sum_m a_m^{(\ell)}\Gamma(\ell+6+m)\Gamma(\ell+6-m)\,\zeta_{\ell+5+m}\,\zeta_{\ell+5-m}\,. \tag{B.7}$$

Combined with (B.4), this proves the relation (4.21).

C Coefficients b_n in strong coupling limit of tr M^n

Here we shall discuss the explicit form of the coefficients b_n in (5.6). They can be computed by explicit evaluation of the traces $\operatorname{tr} \mathrm{S}^n$ of S in (5.5), since the infinite sums $\sum_{i=1}^{\infty} \mathrm{S}_{ii}$, $\sum_{i,j=1}^{\infty} \mathrm{S}_{ij}\mathrm{S}_{ji}$, etc. are all convergent. For instance,

$$b_1 = \operatorname{tr} \mathrm{S} = \sum_{i=1}^{\infty} \frac{1}{4i(i+1)} = \frac{1}{4}, \qquad b_2 = \operatorname{tr} \mathrm{S}^2 = \frac{1}{60} + \frac{3}{8} \sum_{i=2}^{\infty} \frac{1}{i(i+1)(2i-1)(2i+3)} = \frac{1}{48}, \; \ldots . \tag{C.1}$$

An alternative representation for b_n that avoids infinite summations is found using $\frac{1}{t}\frac{e^{2\pi t/\sqrt{\lambda}}}{(e^{2\pi t/\sqrt{\lambda}}-1)^2} = \frac{\lambda}{4\pi^2 t^3} - \frac{1}{12\, t} + \cdots$ and the integral representation (4.4):

$$b_n = \int_0^1 dx_1 \cdots \int_0^1 dx_n \; f(x_1, x_2)\, f(x_2, x_3) \cdots f(x_{n-1}, x_n)\, f(x_n, x_1), \tag{C.2}$$

$$f(x, y) = \int_0^{\infty} \frac{dt}{t} J_2(t\sqrt{x})\, J_2(t\sqrt{y}) = \begin{cases} \frac{y}{4x}, & x \geq y, \\ \frac{x}{4y}, & x < y\,. \end{cases} \tag{C.3}$$

The explicit values of the coefficients b_n can be easily computed from (C.2)

$$\{b_n\}_{n=1,2,\ldots} = \left\{\frac{1}{4}, \frac{1}{48}, \frac{1}{384}, \frac{1}{2880}, \frac{13}{276480}, \frac{11}{1720320}, \frac{647}{743178240}, \frac{1133}{9555148800}, \frac{43213}{2675441664000}, \ldots \right\}. \tag{C.4}$$

A much more efficient way to determine b_n is based on using (5.8) since it implies the following explicit expression for their generating function

$$b(x) = \sum_{n=1}^{\infty} b_n x^{n-1} = \frac{1}{\sqrt{2x}} \frac{J_2(\sqrt{2x})}{J_1(\sqrt{2x})}. \tag{C.5}$$

Expanding (C.5) at small x reproduces the values (C.4).

To prove (C.5) let us first note that (5.9) implies that

$$b_n = \operatorname{tr} \mathrm{S}^n = 2^n\, \sigma_n, \qquad \sigma_n = \sum_{k=1}^{\infty} \frac{1}{\mathrm{j}_{1,k}^{2n}}\,, \tag{C.6}$$

where $\{\mathrm{j}_{1,k}\}$ are the (positive) zeroes of the Bessel function $J_1(x)$. The generating function (C.5) may then be obtained as a corollary of the results in the recent paper [55] that proved that

$$\sigma_n = \frac{(-1)^{n+1}}{2^{2n} n! (2)_{n-1}} \mathsf{B}_{2n,0}(1)\,, \tag{C.7}$$

where $\mathsf{B}_{2n,0}(x)$ are a special case of the Bernoulli-Dunkl polynomials. They are generated by

$$\frac{t}{2} + \frac{t}{2}\frac{I_0(t)}{I_1(t)} - 1 = \sum_{m=1}^{\infty} \mathsf{B}_{m,0}(1) \frac{t^m}{\gamma_{n,0}}, \qquad \gamma_{m,0} = \begin{cases} 2^{2n}\, k!\, (1)_n, & m = 2n, \\ 2^{2n+1}\, n!\, (1)_{n+1}, & m = 2n+1. \end{cases} \tag{C.8}$$

Taking the even in t part of (C.8) gives

$$\frac{t}{2}\frac{I_0(t)}{I_1(t)} - 1 = \sum_{n=1}^{\infty} \mathsf{B}_{2n,0}(1)\frac{t^{2n}}{\gamma_{2n,0}} = \sum_{n=1}^{\infty}(-1)^{n+1}\sigma_n t^{2n}. \tag{C.9}$$

Finally, comparing with (C.5) and (C.6), we get the proof of (C.5)

$$b(x) = \sum_{n=1}^{\infty} 2^n\,\sigma_n\,x^{n-1} = -\frac{1}{x}\left[i\sqrt{\frac{x}{2}}\,\frac{I_0(i\sqrt{2x})}{I_1(i\sqrt{2x})} - 1\right] = \frac{1}{\sqrt{2x}}\frac{J_2(\sqrt{2x})}{J_1(\sqrt{2x})}\,. \tag{C.10}$$

By the same methods, the results in [55] can be used to construct the generating function for the sums $\sum_{k=1}^{\infty}\frac{1}{\mathrm{j}_{a,k}^{2n}}$ of inverse negative even powers of zeroes of $J_a(x)$.

D Darboux theorem

Darboux's theorem states that for a convergent series expansion, the large-order growth of the expansion coefficients about a point (say $t=0$) is directly related to the behaviour of the expansion in the vicinity of a nearby singularity. For example, suppose

$$f(t)\Big|_{t\to t_0} \sim \phi(t)\left(1-\frac{t}{t_0}\right)^{-g} + \psi(t)\,, \tag{D.1}$$

where $\phi(t)$ and $\psi(t)$ are analytic near t_0. Then the Taylor expansion coefficients of $f(t) = \sum_k a_k t^k$ near the origin have large-order ($k\to\infty$) growth

$$a_k \sim \frac{1}{t_0^k}\binom{k+g-1}{k}\left[\phi(t_0) - \frac{(g-1)\,t_0\,\phi'(t_0)}{(k+g-1)} + \frac{(g-1)(g-2)\,t_0^2\,\phi''(t_0)}{2!(k+g-1)(k+g-2)} - \dots\right]. \tag{D.2}$$

Thus, leading and subleading large-order behaviour terms determine the Taylor expansion of the analytic function $\phi(t)$ which multiplies the branch-cut factor in (D.1). The function $\psi(t)$ can be extracted similarly by multiplying $f(t)$ through by $\left(1-\frac{t}{t_0}\right)^g$, and applying the same procedure. If the singularity is logarithmic,

$$f(t)\Big|_{t\to t_0} \sim \phi(t)\,\ln\left(1-\frac{t}{t_0}\right) + \psi(t)\,, \tag{D.3}$$

where $\phi(t)$ and $\psi(t)$ are analytic near t_0, then the Taylor expansion coefficients of $f(t)$ near the origin have large-order ($k\to\infty$) growth

$$a_k \sim \frac{1}{t_0^k}\cdot\frac{1}{k}\left[\phi(t_0) - \frac{t_0\,\phi'(t_0)}{(k-1)} + \frac{t_0^2\,\phi''(t_0)}{(k-1)(k-2)} - \dots\right]. \tag{D.4}$$

This logarithmic behaviour is found for the expansion coefficients of $\Delta F(\tilde{\lambda})$, as shown in the right hand panel of figure 1.

References

[1] V. Pestun et al., *Localization techniques in quantum field theories*, J. Phys. A **50** (2017) 440301 [arXiv:1608.02952] [INSPIRE].

[2] N. Drukker and D.J. Gross, *An Exact prediction of $N = 4$ SUSYM theory for string theory*, J. Math. Phys. **42** (2001) 2896 [hep-th/0010274] [INSPIRE].

[3] B. Fiol, B. Garolera and G. Torrents, *Exact probes of orientifolds*, JHEP **09** (2014) 169 [arXiv:1406.5129] [INSPIRE].

[4] S. Giombi and B. Offertaler, *Wilson loops in $\mathcal{N} = 4$ SO(N) SYM and D-branes in $AdS_5 \times \mathbb{RP}^5$*, arXiv:2006.10852 [INSPIRE].

[5] S. Giombi and A.A. Tseytlin, *Strong coupling expansion of circular Wilson loops and string theories in $AdS_5 \times S^5$ and $AdS_4 \times CP^3$*, JHEP **10** (2020) 130 [arXiv:2007.08512] [INSPIRE].

[6] M. Beccaria and A.A. Tseytlin, *On the structure of non-planar strong coupling corrections to correlators of BPS Wilson loops and chiral primary operators*, JHEP **01** (2021) 149 [arXiv:2011.02885] [INSPIRE].

[7] M. Beccaria and A. Hasan, *On topological recursion for Wilson loops in $\mathcal{N} = 4$ SYM at strong coupling*, JHEP **04** (2021) 194 [arXiv:2102.12322] [INSPIRE].

[8] F. Passerini and K. Zarembo, *Wilson Loops in $N = 2$ Super-Yang-Mills from Matrix Model*, JHEP **09** (2011) 102 [*Erratum ibid.* **10** (2011) 065] [arXiv:1106.5763] [INSPIRE].

[9] J.G. Russo and K. Zarembo, *Massive $\mathcal{N} = 2$ Gauge Theories at Large N*, JHEP **11** (2013) 130 [arXiv:1309.1004] [INSPIRE].

[10] K. Zarembo, *Strong-Coupling Phases of Planar $\mathcal{N} = 2^*$ Super-Yang-Mills Theory*, Theor. Math. Phys. **181** (2014) 1522 [arXiv:1410.6114] [INSPIRE].

[11] M. Baggio, V. Niarchos and K. Papadodimas, *Exact correlation functions in* SU(2) *$\mathcal{N} = 2$ superconformal QCD*, Phys. Rev. Lett. **113** (2014) 251601 [arXiv:1409.4217] [INSPIRE].

[12] M. Baggio, V. Niarchos and K. Papadodimas, *On exact correlation functions in* SU(N) *$\mathcal{N} = 2$ superconformal QCD*, JHEP **11** (2015) 198 [arXiv:1508.03077] [INSPIRE].

[13] M. Baggio, V. Niarchos, K. Papadodimas and G. Vos, *Large-N correlation functions in $\mathcal{N} = 2$ superconformal QCD*, JHEP **01** (2017) 101 [arXiv:1610.07612] [INSPIRE].

[14] B. Fiol, B. Garolera and G. Torrents, *Probing $\mathcal{N} = 2$ superconformal field theories with localization*, JHEP **01** (2016) 168 [arXiv:1511.00616] [INSPIRE].

[15] K. Zarembo, *Localization and AdS/CFT Correspondence*, J. Phys. A **50** (2017) 443011 [arXiv:1608.02963] [INSPIRE].

[16] S. Kachru and E. Silverstein, *4-D conformal theories and strings on orbifolds*, Phys. Rev. Lett. **80** (1998) 4855 [hep-th/9802183] [INSPIRE].

[17] J.K. Erickson, G.W. Semenoff and K. Zarembo, *Wilson loops in $N = 4$ supersymmetric Yang-Mills theory*, Nucl. Phys. B **582** (2000) 155 [hep-th/0003055] [INSPIRE].

[18] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, Commun. Math. Phys. **313** (2012) 71 [arXiv:0712.2824] [INSPIRE].

[19] M. Beccaria and A.A. Tseytlin, *$1/N$ expansion of circular Wilson loop in $\mathcal{N} = 2$ superconformal* SU(N) $\times$ SU(N) *quiver*, JHEP **04** (2021) 265 [arXiv:2102.07696] [INSPIRE].

JHEP07(2021)085

[20] S.-J. Rey and T. Suyama, *Exact Results and Holography of Wilson Loops in $N = 2$ Superconformal (Quiver) Gauge Theories*, JHEP **01** (2011) 136 [arXiv:1001.0016] [INSPIRE].

[21] K. Zarembo, *Quiver CFT at strong coupling*, JHEP **06** (2020) 055 [arXiv:2003.00993] [INSPIRE].

[22] V. Mitev and E. Pomoni, *Exact effective couplings of four dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry*, Phys. Rev. D **92** (2015) 125034 [arXiv:1406.3629] [INSPIRE].

[23] V. Mitev and E. Pomoni, *Exact Bremsstrahlung and Effective Couplings*, JHEP **06** (2016) 078 [arXiv:1511.02217] [INSPIRE].

[24] H. Ouyang, *Wilson loops in circular quiver SCFTs at strong coupling*, JHEP **02** (2021) 178 [arXiv:2011.03531] [INSPIRE].

[25] P.S. Howe, K.S. Stelle and P.C. West, *A Class of Finite Four-Dimensional Supersymmetric Field Theories*, Phys. Lett. B **124** (1983) 55 [INSPIRE].

[26] I.G. Koh and S. Rajpoot, *Finite $\mathcal{N} = 2$ Extended Supersymmetric Field Theories*, Phys. Lett. B **135** (1984) 397 [INSPIRE].

[27] J. Park, R. Rabadán and A.M. Uranga, *Orientifolding the conifold*, Nucl. Phys. B **570** (2000) 38 [hep-th/9907086] [INSPIRE].

[28] I.P. Ennes, C. Lozano, S.G. Naculich and H.J. Schnitzer, *Elliptic models, type IIB orientifolds and the AdS/CFT correspondence*, Nucl. Phys. B **591** (2000) 195 [hep-th/0006140] [INSPIRE].

[29] M. Beccaria and A.A. Tseytlin, *Higher spins in AdS_5 at one loop: vacuum energy, boundary conformal anomalies and AdS/CFT*, JHEP **11** (2014) 114 [arXiv:1410.3273] [INSPIRE].

[30] A. Arabi Ardehali, J.T. Liu and P. Szepietowski, *$1/N^2$ corrections to the holographic Weyl anomaly*, JHEP **01** (2014) 002 [arXiv:1310.2611] [INSPIRE].

[31] M. Beccaria, M. Billò, F. Galvagno, A. Hasan and A. Lerda, *$\mathcal{N} = 2$ Conformal SYM theories at large $\mathcal{N}$*, JHEP **09** (2020) 116 [arXiv:2007.02840] [INSPIRE].

[32] N. Drukker, D.J. Gross and A.A. Tseytlin, *Green-Schwarz string in $AdS_5 \times S^5$: Semiclassical partition function*, JHEP **04** (2000) 021 [hep-th/0001204] [INSPIRE].

[33] B. Fiol, J. Martínez-Montoya and A. Rios Fukelman, *The planar limit of $\mathcal{N} = 2$ superconformal field theories*, JHEP **05** (2020) 136 [arXiv:2003.02879] [INSPIRE].

[34] J.G. Russo and K. Zarembo, *Large N Limit of $\mathcal{N} = 2$ SU(N) Gauge Theories from Localization*, JHEP **10** (2012) 082 [arXiv:1207.3806] [INSPIRE].

[35] M. Blau, K.S. Narain and E. Gava, *On subleading contributions to the AdS /CFT trace anomaly*, JHEP **09** (1999) 018 [hep-th/9904179] [INSPIRE].

[36] O. Aharony, J. Pawelczyk, S. Theisen and S. Yankielowicz, *A Note on anomalies in the AdS/CFT correspondence*, Phys. Rev. D **60** (1999) 066001 [hep-th/9901134] [INSPIRE].

[37] S.G. Naculich, H.J. Schnitzer and N. Wyllard, *1/N corrections to anomalies and the AdS/CFT correspondence for orientifolded $N = 2$ orbifold and $N = 1$ conifold models*, Int. J. Mod. Phys. A **17** (2002) 2567 [hep-th/0106020] [INSPIRE].

[38] A. Bourget, D. Rodriguez-Gomez and J.G. Russo, *Universality of Toda equation in $\mathcal{N} = 2$ superconformal field theories*, JHEP **02** (2019) 011 [arXiv:1810.00840] [INSPIRE].

JHEP07(2021)085

[39] M. Billò, F. Galvagno and A. Lerda, *BPS Wilson loops in generic conformal* $\mathcal{N} = 2$ SU(N) *SYM theories*, *JHEP* **08** (2019) 108 [arXiv:1906.07085] [INSPIRE].

[40] C.A. Tracy and H. Widom, *Level spacing distributions and the Bessel kernel*, *Commun. Math. Phys.* **161** (1994) 289 [hep-th/9304063] [INSPIRE].

[41] N. Beisert, B. Eden and M. Staudacher, *Transcendentality and Crossing*, *J. Stat. Mech.* **0701** (2007) P01021 [hep-th/0610251] [INSPIRE].

[42] N. Beisert, V. Dippel and M. Staudacher, *A Novel long range spin chain and planar* $\mathcal{N} = 4$ *super Yang-Mills*, *JHEP* **07** (2004) 075 [hep-th/0405001] [INSPIRE].

[43] N. Gromov, *Introduction to the Spectrum of* $N = 4$ *SYM and the Quantum Spectral Curve*, arXiv:1708.03648 [INSPIRE].

[44] J.G. Russo and K. Zarembo, *Evidence for Large-*N *Phase Transitions in* $\mathcal{N} = 2^*$ *Theory*, *JHEP* **04** (2013) 065 [arXiv:1302.6968] [INSPIRE].

[45] D. Zagier, *The Mellin transformation and other useful analytic techniques*, in *Quantum Field Theory I: Basics in Mathematics and Physics*, pp. 307–323, Springer (2006) [DOI].

[46] P. Flajolet, X. Gourdon and P. Dumas, *Mellin transforms and asymptotics: Harmonic sums*, *Theor. Comput. Sci.* **144** (1995) 3.

[47] Y. Ikebe, Y. Kikuchi and I. Fujishiro, *Computing zeros and orders of Bessel functions*, *J. Comput. Appl. Math.* **38** (1991) 169.

[48] G.N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press (1995).

[49] O. Costin and G.V. Dunne, *Resurgent extrapolation: rebuilding a function from asymptotic data. Painlevé I*, *J. Phys. A* **52** (2019) 445205 [arXiv:1904.11593] [INSPIRE].

[50] O. Costin and G.V. Dunne, *Physical Resurgent Extrapolation*, *Phys. Lett. B* **808** (2020) 135627 [arXiv:2003.07451] [INSPIRE].

[51] O. Costin and G.V. Dunne, *Uniformization and Constructive Analytic Continuation of Taylor Series*, arXiv:2009.01962 [INSPIRE].

[52] C.M. Bender and S.A. Orszag, *Advanced mathematical methods for scientists and engineers I: Asymptotic methods and perturbation theory*, Springer Science & Business Media (2013).

[53] G.A. Baker and P. Graves-Morris, *Padé Approximants*, Cambridge University Press (1996) [DOI].

[54] H. Stahl, *The convergence of Padé approximants to functions with branch points*, *J. Approx. Theor.* **91** (1997) 139.

[55] O. Ciaurri, A.J. Durán and M. Pérez and J.L. Varona, *Bernoulli-Dunkl and Apostol-Euler-Dunkl polynomials with applications to series involving zeros of Bessel functions*, *J. Approx. Theor.* **235** (2018) 20.

JHEP07(2021)085