Topological term, QCD anomaly, and the η' chiral soliton lattice in rotating baryonic matter

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ABSTRACT: We study the ground states of low-density hadronic matter and high-density color-flavor locked color superconducting phase in three-flavor QCD at finite baryon chemical potential under rotation. We find that, in both cases under sufficiently fast rotation, the combination of the rotation-induced topological term for the η' meson and the QCD anomaly leads to an inhomogeneous condensate of the η' meson, known as the chiral soliton lattice (CSL). We find that, when baryon chemical potential is much larger than isospin chemical potential, the critical angular velocity for the realization of the η' CSL is much smaller than that for the π_0 CSL found previously. We also argue that the η' CSL states in flavor-symmetric QCD at low density and high density should be continuously connected, extending the quark-hadron continuity conjecture in the presence of the rotation.

KEYWORDS: Effective Field Theories, Phase Diagram of QCD, Topological States of Matter

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1 Introduction

Investigating the phase diagram of quantum chromodynamics (QCD) at finite temperature T and/or baryon chemical potential μ_B is one of the important problems in the Standard Model of particle physics. In recent years, QCD matter not only at finite T and μ_B , but also under rotation Ω has attracted much attention. Experimentally, it has been reported that quark-gluon plasmas produced in noncentral heavy ion collision experiments at Relativistic Heavy Ion collider (RHIC) have the largest vorticity observed so far, of order $10^{22}/s$ [1]. There have also been theoretical studies on the phases of QCD matter under rotation mostly using the effective models of QCD [2–10].

Recently, it has been shown in ref. [11], based on a systematic low-energy effective theory, that the ground state of two-flavor QCD at finite μ_B and isospin chemical potential μ_I under sufficiently fast rotation is an inhomogeneous condensate of the π_0 meson, called the chiral soliton lattice (CSL). Generally, the CSL is a periodic array of topological solitons that spontaneously breaks parity and translational symmetries. This CSL is a universal state of matter in that it appears in various systems from condensed matter physics to

Physical system	NG boson	Total derivative term	Explicit symmetry breaking
Chiral magnet	Magnon φ	DM interaction $\mathbf{D} \cdot \nabla \varphi$	Magnetic field
QCD ($\mathbf{B}, \mu_B$)	π_0	WZW-type term $\mu_B \mathbf{B} \cdot \nabla \pi_0$	Quark masses
QCD ($\mathbf{\Omega}, \mu_B, \mu_I$)	π_0	WZW-type term $\mu_B \mu_I \mathbf{\Omega} \cdot \nabla \pi_0$	Quark masses
QCD ($\mathbf{\Omega}, \mu_B$)	η'	WZW-type term $\mu_B^2 \mathbf{\Omega} \cdot \nabla \eta'$	QCD anomaly and quark masses

Table 1. Examples of the CSL in chiral magnets and QCD matter.

high-energy physics, such as chiral magnets [12, 13], cholesteric liquid crystals [14], and QCD at finite μ_B in an external magnetic field [15]; see also refs. [16, 17] for the CSL in QCD-like theories without the fermion sign problem.

As summarized in table 1, the realization of the CSL typically requires three essential ingredients: a Nambu-Goldstone (NG) boson field (say θ) associated with some symmetry G , a total derivative term for θ , and explicit breaking of the symmetry G that provides a mass term for θ . In chiral magnets, the so-called Dzyaloshinskii-Moriya (DM) interaction gives a total derivative term for the magnon field φ of the form $\mathbf{D} \cdot \nabla \varphi$, where $\mathbf{D}$ is the DM vector [18], and an external magnetic field gives a mass term for φ due to the explicit breaking of the spin rotational symmetry. In QCD matter, the Wess-Zumino-Witten (WZW) type terms in a magnetic field [19, 20] or under rotation [11] lead to total derivative terms for π_0 , and the quark mass provides a mass term for π_0 due to the explicit breaking of chiral symmetry. Note that the total derivative term for π_0 under rotation is present only at *both* finite μ_B and μ_I [11] (see also section 2.1 below), and as a result, the CSL of the π_0 meson is not realized in baryonic matter under rotation.

In this paper, we show that, in three-flavor QCD at finite μ_B under sufficiently fast rotation, another CSL-type ground state is realized — the CSL of the η' meson, or simply the η' CSL. The ingredients for the existence of the η' CSL are a new WZW-type topological term for η' at finite μ_B under rotation (see eqs. (2.12) and (3.20)) and the QCD anomaly (or the $U(1)_A$ anomaly), which provides an additional mass term for η' . In both cases of low-density hadronic matter and high-density color-flavor locked (CFL) color superconducting phase, we analytically derive the critical angular velocities above which the η' CSL states are realized and the CSL-type ground-state configurations. In particular, we find that, in the regime $\mu_B \gg \mu_I$, the critical angular velocity for the realization of the η' CSL is much smaller than that for the π_0 CSL found in ref. [11]. We also argue that these η' CSL states at low density and high density should be continuously connected in flavor-symmetric QCD, extending the quark-hadron continuity conjecture [21–23] in the presence of the rotation.

This paper is organized as follows. In sections 2 and 3, we construct the low-energy effective theories for low-density hadronic matter and high-density CFL phase under rotation, respectively, and show that their ground states under sufficiently fast rotation are the η' CSL states. Section 4 is devoted to discussion and conclusion.

Throughout this paper, we consider QCD at finite μ_B and at zero temperature. We set the angular velocity along the z direction, $\mathbf{\Omega} \equiv \Omega \hat{z}$, without loss of generality. The effect of this rotation can be expressed by the metric

$$ds^2 = (1 - \Omega^2 r_\perp^2) dt^2 - 2g_{0i} dt dx^i, \quad (1.1)$$

where $r_\perp \equiv \sqrt{x^2 + y^2}$ is the distance from the z axis and g_{0i} satisfies $\Omega = \epsilon^{zjk} \partial_j g_{0k}$, or explicitly,

$$g_{\mu\nu} = \begin{pmatrix} 1 - \Omega^2(x^2 + y^2) & \Omega y & -\Omega x & 0 \\ \Omega y & -1 & 0 & 0 \\ -\Omega x & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (1.2)$$

We also assume that $r_\perp \Omega < 1$ so that the velocity of the boundary does not exceed the speed of light.

2 Low-density hadronic matter under rotation

In this section, we consider the low-energy effective theory — the chiral perturbation theory (ChPT) — for low-density hadronic matter under rotation.

We start from massless three-flavor QCD and we ignore the $U(1)_A$ anomaly for a moment. (We will consider the effects of quark masses and the $U(1)_A$ anomaly later.) In this case, QCD has the $U(3)_R \times U(3)_L$ chiral symmetry:

$$q_R \rightarrow e^{-i\lambda_0 \theta_0^R} V_R q_R, \quad q_L \rightarrow e^{-i\lambda_0 \theta_0^L} V_L q_L, \quad (2.1)$$

where $V_{R,L} \equiv \exp(-i\lambda_A \theta_A^{R,L})$ are the $SU(3)_{R,L}$ transformations for right- and left-handed quarks $q_{R,L}$ and λ_a are the $U(3)$ generator with the normalization $\text{tr}(\lambda_a \lambda_b) = 2\delta_{ab}$. Here and below, the indices A and a, b, c stand for $A = 1, 2, \dots, 8$ and $a, b, c = 0, 1, \dots, 8$, respectively. We assume that this chiral symmetry is spontaneously broken down to the vector $U(3)_V$ symmetry in the vacuum and low-density hadronic matter. As a result, the nonet of NG bosons appears. We can parametrize the field of the nonet mesons by the $U(3)$ matrix U ,

$$U = \Sigma \exp\left(\frac{i\lambda_0 \eta'}{f_{\eta'}}\right), \quad \Sigma = \exp\left(\frac{i\lambda_A \pi_A}{f_\pi}\right), \quad (2.2)$$

where $f_{\pi, \eta'}$ are the decay constants of the octet and singlet mesons. The field Σ transforms under $SU(3)_R \times SU(3)_L$ as

$$\Sigma \rightarrow V_L \Sigma V_R^\dagger, \quad (2.3)$$

while η' transforms under $U(1)_A$ as

$$\eta' \rightarrow \eta' + 2f_{\eta'} \theta_0, \quad (2.4)$$

where $\theta_0 \equiv \theta_0^R = -\theta_0^L$.

The kinetic terms invariant under eqs. (2.3) and (2.4) up to $\mathcal{O}(\partial^2)$ are written as

$$\mathcal{L}_{\text{kin}}^{\text{ChPT}} = \frac{f_\pi^2}{4} g^{\mu\nu} \text{tr}(\partial_\mu \Sigma \partial_\nu \Sigma^\dagger) + \frac{1}{2} g^{\mu\nu} \partial_\mu \eta' \partial_\nu \eta', \quad (2.5)$$

where $g^{\mu\nu}$ is an inverse matrix of the metric $g_{\mu\nu}$ in eq. (1.1). This effective theory is based on the expansion in terms of the small parameter $p/(4\pi f_{\pi, \eta'}) \ll 1$ with p being the momentum. In the following, we will also take $\Omega/(4\pi f_{\pi, \eta'})$ as a small expansion parameter and we will consider the leading-order contributions of Ω in the effective theory.

2.1 Chiral vortical effect and topological term

We review the topological terms in QCD matter under the global rotation derived in ref. [11].¹ The idea is based on the matching of the anomaly-induced transport phenomenon called the chiral vortical effect (CVE) [25–30], which is a current along the direction of a vorticity or rotation in relativistic matter, between the microscopic theory (QCD) and low-energy effective theory (ChPT).

We consider QCD with finite chemical potentials μ_a ($a = 0, 1, \dots, 8$) associated with conserved charges $\bar{q}\gamma^0\lambda_a q$. Under the global rotation $\mathbf{\Omega}$, the system exhibits the axial vector currents $\mathbf{j}_a^5$ in the direction of the rotation axis:

$$\mathbf{j}_a^5 = N_c \frac{d_{abc}}{2\pi^2} \mu_b \mu_c \mathbf{\Omega}, \quad d_{abc} \equiv \frac{1}{2} \text{tr}[\lambda_a \{\lambda_b, \lambda_c\}], \quad (2.6)$$

where N_c is the number of colors and the transport coefficient d_{abc} is the same as the chiral anomaly coefficient [28–30]. Because this expression of the CVE is exact independently of the energy scale similarly to the chiral anomaly, it must also be reproduced in terms of the NG bosons in the ChPT.

To derive the effective Lagrangian that reproduces the CVE, we consider a local axial rotation,

$$q \rightarrow e^{-i\lambda_a \theta_a \gamma_5} q, \quad (2.7)$$

where the parameter θ_a depends on x^μ . Under the infinitesimal transformation, the action of QCD changes as

$$\delta S_{\text{QCD}} = \int d^4x \partial_\mu \theta_a \cdot j_a^{5\mu}, \quad (2.8)$$

where we used $\partial_\mu j_a^{5\mu} = 0$ in the chiral limit. In terms of the NG bosons, the chiral rotation is expressed by

$$\pi_A \rightarrow \pi_A + 2f_\pi \theta_A, \quad \eta' \rightarrow \eta' + 2f_{\eta'} \theta_0. \quad (2.9)$$

Thus, the change of the effective action under the transformation is

$$\delta S_{\text{EFT}} = \int d^4x \left[\partial_\mu \left(\frac{\delta \pi_A}{2f_\pi} \right) \cdot j_A^{5\mu} + \partial_\mu \left(\frac{\delta \eta'}{2f_{\eta'}} \right) \cdot j_{a=0}^{5\mu} \right], \quad (2.10)$$

where we define $\delta \pi_A \equiv 2f_\pi \theta_A$ and $\delta \eta' \equiv 2f_{\eta'} \theta_0$. The exactness of the CVE leads to the matching condition $\delta S_{\text{QCD}} = \delta S_{\text{EFT}}$, from which we arrive at the topological term [11],

$$\mathcal{L}_{\text{CVE}} = N_c \frac{d_{Abc}}{4\pi^2 f_\pi} \mu_b \mu_c \nabla \pi_A \cdot \mathbf{\Omega} + N_c \frac{d_{0bc}}{4\pi^2 f_{\eta'}} \mu_b \mu_c \nabla \eta' \cdot \mathbf{\Omega}. \quad (2.11)$$

This derivation is similar to the anomaly matching condition used to derive the WZW term in the ChPT [31, 32], which is responsible for, e.g., the $\pi_0 \rightarrow 2\gamma$ decay. Moreover,

¹See also ref. [24] for a related recent work.

in the external magnetic field at finite μ_B , there is an additional WZW-type term of the form, $\sim \mu_B \mathbf{B} \cdot \nabla \pi_0$ [19, 20], which is responsible for the chiral magnetic effect [33–35].

In the case of QCD at finite μ_B , the only nonvanishing component of eq. (2.11) is

$$\mathcal{L}_{\text{CVE}}^{\eta'} = \frac{\mu_B^2}{4\pi^2 f_{\eta'}} \sqrt{\frac{2}{3}} \nabla \eta' \cdot \Omega, \quad (2.12)$$

where we used $\sqrt{\frac{2}{3}}\mu_0 = \frac{1}{N_c}\mu_B$ and $N_c = 3$. On the other hand, the topological term for the π_0 meson appears only at finite μ_B and μ_I as [11]

$$\mathcal{L}_{\text{CVE}}^{\pi_0} = \frac{\mu_B \mu_I}{2\pi^2 f_\pi} \nabla \pi_0 \cdot \Omega. \quad (2.13)$$

Therefore, the topological term (2.12) is dominant compared with eq. (2.13) in the regime $\mu_B \gg |\mu_I|$.

2.2 $U(1)_A$ anomaly and quark mass terms

We next include the effects of explicit chiral symmetry breaking in the ChPT: finite quark masses and the $U(1)_A$ anomaly.

We first consider the corrections in the ChPT due to the quark masses. We introduce the quark mass term in the original QCD Lagrangian,

$$\mathcal{L}_{\text{mass}}^{\text{QCD}} = -\bar{q}_L M q_R - \bar{q}_R M^\dagger q_L, \quad (2.14)$$

$$M = \text{diag}(m_u, m_d, m_s). \quad (2.15)$$

In order to determine the form of the corresponding effective Lagrangian at low energy, we promote the quark mass matrix M into a spurion field and require that M transforms under the chiral rotation (2.1) as

$$M \rightarrow e^{-2i\lambda_0 \theta_0} V_L M V_R^\dagger, \quad (2.16)$$

such that eq. (2.14) is invariant under this transformation. The effective Lagrangian involving M and U that is invariant under the extended symmetry in eqs. (2.3), (2.4) and (2.16) up to $\mathcal{O}(M)$ reads

$$\mathcal{L}_{\text{mass}}^{\text{ChPT}} = B \text{tr}(MU + \text{h.c.}), \quad (2.17)$$

where the parameter B is not determined by the symmetry alone.

It is also known that the $U(1)_A$ part of the chiral symmetry $U(3)_R \times U(3)_L$ is explicitly broken by the QCD anomaly. This results in the symmetry breaking $U(1)_A \rightarrow \mathbb{Z}_6$ in three-flavor QCD. Let us incorporate this $U(1)_A$ anomaly into the ChPT. Such an anomalous term breaks $U(3)_R \times U(3)_L$ symmetry, but preserves $SU(3)_R \times SU(3)_L \times U(1)_B \times \mathbb{Z}_6$. Then, one of the options is (see, e.g., [36])

$$\mathcal{L}_{\text{anom}} = \frac{a}{2} \left(\det U + \det U^\dagger \right), \quad (2.18)$$

where a represents the strength of the $U(1)_A$ anomaly. As $\det \Sigma = 1$, eq. (2.18) is

$$\mathcal{L}_{\text{anom}} = a \cos \left(3 \sqrt{\frac{2}{3}} \frac{\eta'}{f_{\eta'}} \right) \quad (2.19)$$

which gives the additional mass term of the η' meson as

$$\delta m_{\eta'}^2 = \frac{6a}{f_{\eta'}^2}. \quad (2.20)$$

An important remark is in order here. It is known that the Lagrangian (2.18) does not satisfy the proper $1/N_c$ counting rules in the large- N_c expansion, e.g., for the quartic η' interaction [37]. One could instead write down the Lagrangian with the proper counting rules as

$$\mathcal{L}'_{\text{anom}} = -\frac{f_{\pi}^2 a}{4N_c} (\text{i} \ln \det U)^2, \quad (2.21)$$

where $f_{\pi} = \mathcal{O}(N_c^{1/2})$ and a is some constant with $a = \mathcal{O}(N_c^0)$, and the normalization of the prefactor is chosen following ref. [37]. However, it turns out that eq. (2.21) explicitly breaks spatial translational symmetry due to the cusp singularities in the potential (see appendix A for details), and so it does not work out for our purpose to study its spontaneous breaking by the CSL state. For this reason, we will adopt the Lagrangian (2.18) below. In fact, the detailed form of the Lagrangian for the $U(1)_A$ anomaly will be irrelevant to our discussions, but the only essential piece will be just the strength of the $U(1)_A$ anomaly — the coefficient a in the case of the Lagrangian (2.18). In passing, we also note that mathematically the same form of the Lagrangian as eq. (2.18) appears in CFL phase at high density, as we will see in section 3.1.

2.3 Ground state in the chiral limit

In the following, we will concentrate on the η' meson and set $U = e^{i\phi}$ with $\phi \equiv \sqrt{6}\eta'/(3f_{\eta'})$. For convenience, we use the cylindrical coordinate (r, θ, z) .

We first consider the case where the $U(1)_A$ anomaly is sufficiently large so that the quark mass term is negligible. (We will consider the effects of the quark mass in section 2.4). In this case, adding eqs. (2.5), (2.17), (2.18) and (2.11) together, we obtain the effective Hamiltonian in the static limit as

$$\mathcal{H}_{\text{ChPT}} = \frac{3}{4} f_{\eta'}^2 (\partial_r \phi)^2 + \frac{3}{4r^2} f_{\eta'}^2 (\partial_{\theta} \phi)^2 + \frac{3}{4} f_{\eta'}^2 (\partial_z \phi)^2 + a(1 - \cos 3\phi) - \frac{\mu_B^2}{4\pi^2} \Omega \partial_z \phi, \quad (2.22)$$

where we ignored the terms of order $\mathcal{O}(\Omega^2)$ and we set $\mathcal{H}_{\text{ChPT}} = 0$ in the QCD vacuum ($\phi = 0$). To minimize the Hamiltonian (2.22), we need to set $\partial_{\theta} \phi = 0$ and $\partial_r \phi = 0$, and so the ground-state configuration is independent of r and θ . On the other hand, the effective Hamiltonian in the z direction is

$$\mathcal{H}_{\text{ChPT}} = \frac{3}{4} f_{\eta'}^2 (\partial_z \phi)^2 + a(1 - \cos 3\phi) - \frac{\mu_B^2}{4\pi^2} \Omega \partial_z \phi, \quad (2.23)$$

whose ground-state configuration can have a nontrivial z dependence. As we will look for a configuration where the first term is comparable to the third term in eq. (2.23), for which $p \sim \mu_B^2 \Omega / (4\pi^2 f_{\eta'}^2)$, the effective theory is valid when $\mu_B^2 \Omega / (16\pi^3 f_{\eta'}^3) \ll 1$.

We note that the Hamiltonian (2.23), which corresponds to the forth line of table 1, is mathematically equivalent to that in QCD at finite μ_B and μ_I under rotation in ref. [11] (see eq. (3.7)), which corresponds to the third line of table 1. More concretely, the former is obtained from the latter by the following replacement:

$$\phi \rightarrow 3\phi, \quad f_\pi^2 \rightarrow \frac{f_{\eta'}^2}{6}, \quad m_\pi^2 \rightarrow \frac{6a}{f_{\eta'}^2}, \quad \mu_I \Omega \rightarrow \frac{1}{6} \mu_B \Omega, \quad (2.24)$$

see also eq. (2.2) in ref. [15], which corresponds to the second line of table 1. Therefore, by making use of the results, e.g., in ref. [11], one can obtain the ground state in the present case — the η' CSL. It should be emphasized, however, that the physics is somewhat different: while the π_0 meson becomes massive by the quark mass in ref. [11], the η' meson becomes massive by the $U(1)_A$ anomaly here.

To make this paper self-contained, we will briefly summarize the derivation of the ground state for the Hamiltonian (2.23). The equation of motion is

$$\partial_z^2(3\phi) = \frac{6a}{f_{\eta'}^2} \sin(3\phi), \quad (2.25)$$

which can be analytically solved by using the Jacobi's elliptic function as

$$\cos \frac{3\phi(\bar{z})}{2} = \text{sn}(\bar{z}, k), \quad (2.26)$$

where $\bar{z} \equiv z\sqrt{6a}/(f_{\eta'}k)$ is a dimensionless coordinate and k ($0 \leq k \leq 1$) is the elliptic modulus. This solution has a period

$$\ell = \frac{2f_{\eta'}kK(k)}{\sqrt{6a}}, \quad (2.27)$$

where $K(k)$ is the complete elliptic integral of the first kind.

The free parameter k is determined by minimizing the total energy of the system at fixed volume V with respect to k as

$$\frac{E(k)}{k} = \frac{\mu_B^2 \Omega}{8\pi f_{\eta'} \sqrt{6a}}, \quad (2.28)$$

where $E(k)$ is the complete elliptic integral of the second kind. The inequality $E(k)/k \geq 1$ leads to the critical angular velocity

$$\Omega_{\eta'}^0 = \frac{8\pi f_{\eta'} \sqrt{6a}}{\mu_B^2}. \quad (2.29)$$

We can also show that, when $\Omega > \Omega_{\eta'}^0$, the η' CSL is energetically more stable than the QCD vacuum for $\mu_B < m_N$ and nuclear matter for $\mu_B \approx m_N$ with m_N the nucleon

mass. One can show the former statement by writing the energy of each lattice per unit area, satisfying the minimization condition (2.28), as

$$\frac{\mathcal{E}}{S} = \int_0^\ell dz \mathcal{H}_{\text{ChPT}} = \frac{2f_{\eta'}\sqrt{6a}}{3} \left(k - \frac{1}{k}\right) K(k) < 0. \quad (2.30)$$

One can also show the latter by computing the baryon number of each lattice per unit area as

$$\frac{N_B}{S} = - \int_0^\ell dz \frac{\partial \mathcal{H}_{\text{ChPT}}}{\partial \mu_B} = \frac{\mu_B \Omega}{3\pi}, \quad (2.31)$$

which, combined with eq. (2.30), indicates that the energy of the η' CSL per unit baryon number is smaller than that of nuclear matter, $m_N - \mu_B \approx 0$.

2.4 Ground state with finite quark masses

So far, we have ignored the explicit chiral symmetry breaking by the finite quark masses. We now discuss the quark mass effects on the η' CSL. We assume that

$$m_u = m_d \equiv m_{\text{ud}}, \quad m_{\text{ud}} < m_s, \quad (2.32)$$

and that the quark mass term (2.17) is sufficiently small compared with the $U(1)_A$ anomaly (2.18) so that we can treat eq. (2.17) as a perturbation to eq. (2.18). Under this assumption, the off-diagonal parts of the mass matrix for η and η' are negligible. In this case, we can determine the ground state analytically even away from the chiral limit.

From the effective Hamiltonian

$$\mathcal{H}_{\text{ChPT}} = \frac{3}{4} f_{\eta'}^2 (\partial_z \phi)^2 + a(1 - \cos 3\phi) + 2B \text{tr} M (1 - \cos \phi) - \frac{\mu_B^2}{4\pi^2} \Omega \partial_z \phi, \quad (2.33)$$

we get the equation of motion

$$\partial_z^2 \phi = A_1 \sin \phi + 3A_2 \sin 3\phi, \quad (2.34)$$

where we set

$$A_1 \equiv \frac{4B \text{tr} M}{3f_{\eta'}^2}, \quad A_2 \equiv \frac{2a}{3f_{\eta'}^2}. \quad (2.35)$$

One can solve this equation analytically in a similar way to solving the equation of motion for a single pendulum. This equation has a conserved quantity C ,²

$$C = \frac{1}{2} (\partial_z \phi)^2 + A_1 \cos \phi + A_2 (\cos 3\phi - 1). \quad (2.36)$$

We then get

$$\frac{d\phi}{dz} = \pm \frac{2\sqrt{A_1}}{\bar{k}} \left[1 + \bar{k}^2 \frac{A_2}{2A_1} (1 - \cos 3\phi) - \bar{k}^2 \cos^2 \frac{\phi}{2} \right]^{\frac{1}{2}}, \quad (2.37)$$

²There is an ambiguity on the choice of C : e.g., one could also choose $C' \equiv C - A_1$ as a conserved quantity. The present choice will be convenient, as C satisfies $C \geq A_1$ and so $\bar{k}$ defined below satisfies $\bar{k} \leq 1$ similarly to the elliptic modulus k satisfying $k \leq 1$.

where $\bar{k} \equiv \sqrt{2A_1/(C+A_1)}$ is a counterpart of the elliptic modulus k in eq. (2.25) and satisfies $0 < \bar{k} \leq 1$. Without loss of generality, we can choose $\frac{d\phi}{dz} > 0$. Integrating eq. (2.37) and taking $\phi(0) = 0$, we have

$$\frac{2\sqrt{A_1}}{\bar{k}}z = \int_0^\phi d\theta \left[1 + \bar{k}^2 \frac{A_2}{2A_1} (1 - \cos 3\theta) - \bar{k}^2 \cos^2 \frac{\theta}{2} \right]^{-\frac{1}{2}}. \quad (2.38)$$

Unlike the conventional CSL state [18], this solution cannot be expressed by the Jacobi elliptic function. Still, one can show that this solution has the following three properties. First, it is periodic in ϕ with the period

$$\bar{\ell} \equiv \frac{\bar{k}\bar{K}(\bar{k})}{2\sqrt{A_1}}, \quad \bar{K}(\bar{k}) \equiv \int_0^{2\pi} d\theta \left[1 + \bar{k}^2 \frac{A_2}{2A_1} (1 - \cos 3\theta) - \bar{k}^2 \cos^2 \frac{\theta}{2} \right]^{-\frac{1}{2}}. \quad (2.39)$$

Second, it has a topological charge defined as

$$\int_0^{\bar{\ell}} \frac{dz}{2\pi} \partial_z \phi(z) = \frac{1}{2\pi} [\phi(\bar{\ell}) - \phi(0)] = 1, \quad (2.40)$$

which depends only on the boundary values of each lattice. Third, it breaks parity symmetry because η' is a pseudoscalar.

In summary, the ground state (2.38) is a periodic array of topological solitons which spontaneously breaks parity and continuous translational symmetries. Since these properties are characteristic of the CSL, this ground state can be regarded as the CSL, although the ground-state configuration is mathematically different from the conventional CSL solution [18]. One can further show that this inhomogeneous state is energetically more favorable than the QCD vacuum ($\phi = 0$) for $\Omega > \Omega_{\eta'}$ (see appendix B):

$$\Omega_{\eta'} = \frac{6\pi f_{\eta'}^2}{\mu_B^2} \int_0^{2\pi} d\theta \left(A_1 \sin^2 \frac{\theta}{2} + A_2 \sin^2 \frac{3\theta}{2} \right)^{\frac{1}{2}}. \quad (2.41)$$

Let us compare this critical angular velocity for the η' CSL with that of the π_0 CSL at finite μ_B and μ_I previously derived in ref. [11]:

$$\Omega_{\pi_0} = \frac{8\pi m_\pi f_\pi^2}{\mu_B |\mu_I|}. \quad (2.42)$$

One should notice the difference of the chemical potential dependences between the two: $\Omega_{\pi_0} \propto 1/(\mu_B |\mu_I|)$ while $\Omega_{\eta'} \propto 1/\mu_B^2$. In particular, when $\mu_B \gg |\mu_I|$, we have $\Omega_{\pi_0} \gg \Omega_{\eta'}$, showing that the η' CSL is realized much earlier than the π_0 CSL as Ω is increased.

When $\Omega > \Omega_{\pi_0}$, one needs to compare the total energy of the η' CSL with that of the π_0 CSL to determine which state is realized as a ground state. Such a comparison is feasible when $\Omega \approx \Omega_{\pi_0}$: the total energy of the former is given by eq. (B.9) and satisfies $\bar{\mathcal{E}}_{\text{tot}} < 0$. On the other hand, the total energy of the latter is $\mathcal{E}_{\text{tot}}^{\pi_0} \approx 0$ [11]. Therefore, the ground state in this case is the η' CSL.

3 High-density color-flavor locked (CFL) phase under rotation

In this section, we show that the η' CSL also appears in the high-density CFL phase under rotation. Below we will consider the regime $\mu_B \gg \Omega$.

3.1 Effective theory of the CFL

We first construct the low-energy effective theory for the CFL phase under rotation. The CFL phase is characterized by the diquark condensates [38],

$$\langle q_{L\beta}^j C q_{L\gamma}^k \rangle = \epsilon^{ijk} \epsilon_{\alpha\beta\gamma} [d_L^\dagger]_{\alpha i}, \quad \langle q_{R\beta}^j C q_{R\gamma}^k \rangle = \epsilon^{ijk} \epsilon_{\alpha\beta\gamma} [d_R^\dagger]_{\alpha i}, \quad (3.1)$$

where (i, j, k) and (α, β, γ) indicate flavor and color indices, respectively, and C is the charge conjugation operator. These condensates lead to 10 NG bosons: the octet of mesons $(\tilde{\pi}, \tilde{K}, \tilde{\eta})$ associated with the spontaneous chiral symmetry breaking $SU(3)_c \times SU(3)_R \times SU(3)_L \rightarrow SU(3)_{c+R+L}$, and the H boson and $\tilde{\eta}'$ meson associated with the spontaneous breaking of the $U(1)_B$ and $U(1)_A$ symmetries, respectively. (Here and below, the NG bosons in the CFL phase are denoted by $\tilde{\pi}, \tilde{K}, \tilde{\eta}$, etc. to distinguish from the mesons π, K, η , etc. in the hadronic phase.) Note that, due to the suppression of the instanton effect at large μ_B , this system has the approximate $U(1)_A$ symmetry and we can also regard the $\tilde{\eta}'$ meson as the NG boson [39]. Since the dynamics of the H boson is decoupled from other NG modes that we are interested in, we will ignore the H boson in the following discussion.

The nonet of the mesons corresponds to phase fluctuations defined by

$$d_L d_R^\dagger = |d_L d_R^\dagger| \tilde{U}, \quad (3.2)$$

where the $U(3)$ matrix $\tilde{U}$ is the field of the nonet of the mesons. We split $\tilde{U}$ into the octet and singlet parts as

$$\tilde{U} = \tilde{\Sigma} \exp\left(\frac{i\lambda_0 \tilde{\eta}'}{f_{\tilde{\eta}'}}\right), \quad \tilde{\Sigma} = \exp\left(\frac{i\lambda_A \tilde{\pi}_A}{f_{\tilde{\pi}}}\right). \quad (3.3)$$

The field $\tilde{\Sigma}$ transforms under $SU(3)_R \times SU(3)_L \times SU(3)_c$ as

$$\tilde{\Sigma} \rightarrow V_L \tilde{\Sigma} V_R^\dagger, \quad (3.4)$$

while the $\tilde{\eta}'$ meson transforms under the $U(1)_A$ as

$$\tilde{\eta}' \rightarrow \tilde{\eta}' + 4f_{\tilde{\eta}'}\theta_0, \quad (3.5)$$

where the factor 4 originates from the fact that the $\tilde{\eta}'$ meson in the CFL phase is a $\bar{q}q$ state rather than a $\bar{q}q$ state as in the hadronic phase.

Setting $\tilde{\phi} \equiv \sqrt{6}\tilde{\eta}'/(3f_{\tilde{\eta}'}),$ the kinetic terms for these NG modes up to $\mathcal{O}(\partial^2)$ are [40]

$$\mathcal{L}_{\text{kin}} = \frac{1}{4} f_{\tilde{\pi}}^2 v_{\tilde{\pi}}^2 g_{\tilde{\pi}}^{\mu\nu} \text{tr} \left(\partial_\mu \tilde{\Sigma} \partial_\nu \tilde{\Sigma}^\dagger \right) + \frac{3}{4} f_{\tilde{\eta}'}^2 v_{\tilde{\eta}'}^2 g_{\tilde{\eta}'}^{\mu\nu} \partial_\mu \tilde{\eta}' \partial_\nu \tilde{\eta}', \quad (3.6)$$

where $f_{\tilde{\pi}, \tilde{\eta}'}$ and $v_{\tilde{\pi}, \tilde{\eta}'}$ are the decay constants and velocities of the octet and singlet mesons, respectively, and we defined $\partial_\mu^{\tilde{\pi}}$ and $\partial_\mu^{\tilde{\eta}'}$, such that $\partial_0^{\tilde{\pi}, \tilde{\eta}'} \equiv (1/v_{\tilde{\pi}, \tilde{\eta}'})\partial_0$ and $\partial_i^{\tilde{\pi}, \tilde{\eta}'} \equiv \partial_i$. We also defined $g_{\tilde{\pi}, \tilde{\eta}'}^{\mu\nu}$ as the inverses of the “effective metrics” $g_{\mu\nu}^{\tilde{\pi}, \tilde{\eta}'}$ given by eq. (1.1) with the replacement $\Omega \rightarrow \Omega/v_{\tilde{\pi}, \tilde{\eta}'}$. At sufficiently high density, $f_{\tilde{\pi}, \tilde{\eta}'}$ and $v_{\tilde{\pi}, \tilde{\eta}'}$ can be computed by the weak-coupling analysis as [40]

$$f_{\tilde{\pi}}^2 = \frac{21 - 8\ln 2}{18} \frac{\mu_B^2}{18\pi^2}, \quad f_{\tilde{\eta}'}^2 = \frac{\mu_B^2}{24\pi^2}, \quad (3.7)$$

$$v_{\tilde{\pi}}^2 = v_{\tilde{\eta}'}^2 = \frac{1}{3}. \quad (3.8)$$

As the energy scale of this effective theory must be much smaller than the mass gap of quarks (i.e., the superconducting gap Δ), the expansion parameter of the effective theory is $p/\Delta \ll 1$ with p being the momentum. We note that, due to the relation $\Delta \ll \mu_B$ at sufficiently high density, the condition $p/(4\pi f_{\tilde{\pi}, \tilde{\eta}'}) \sim p/\mu_B \ll 1$ is then automatically satisfied.

Let us then turn to the potential terms for the NG modes which generally consist of three parts: the quark mass term, the instanton-induced term related to the $U(1)_A$ anomaly, and the topological term induced by rotation.

First, we consider the quark mass term. Similarly to the derivation of eq. (2.17), this term can be written down based on the symmetries up to $\mathcal{O}(M^2)$ as [40, 41]

$$\mathcal{L}_{\text{mass}}^{\text{CFL}} = c \left[\det M \text{tr}(M^{-1} \tilde{U}) + \text{h.c.} \right]. \quad (3.9)$$

The form of this term is different from eq. (2.17) because the transformation law (3.5) is different from eq. (2.9). The parameter c can be determined at sufficiently high density by the weak-coupling calculation as [40, 42]

$$c = \frac{3\Delta^2}{2\pi^2}. \quad (3.10)$$

Next, we construct the potential term due to the $U(1)_A$ anomaly or the instanton effect. For this purpose, it is important to recall that the $U(1)_A$ anomaly induces a nonvanishing chiral condensate in the CFL phase even in the chiral limit. We define the chiral condensate as $\Phi_{ij} = \langle \bar{q}_R^j q_L^i \rangle$. Considering the transformation of Φ under $SU(3)_R \times SU(3)_L \times SU(3)_c \times U(1)_A$,

$$\Phi \rightarrow e^{-2i\theta_0 \lambda_0} V_L \Phi V_R^\dagger, \quad (3.11)$$

the effective potential at lowest order in Φ is given by [22, 23]

$$\mathcal{L}_{1\text{-inst}} = -\gamma \text{tr} \left(\Phi^\dagger \tilde{U} + \text{h.c.} \right). \quad (3.12)$$

This term explicitly breaks $U(1)_A$ down to $\mathbb{Z}_6$ and it stems from the $U(1)_A$ anomaly. In fact, the parameter γ can be explicitly computed from the instanton-induced interaction as (see appendix C for the detail)

$$\gamma = \frac{18\pi^2 C_{N_c, N_f}}{(N_c + 1)^2} N_c^{b+1} N_f^{-\frac{b+5}{2}} \Gamma\left(\frac{b+5}{2}\right) \left(\frac{8\pi^2}{g^2}\right)^{2N_c+1} \left(\frac{\Lambda_{\text{QCD}}}{\mu_B}\right)^b \frac{\Delta^2}{\mu_B}, \quad (3.13)$$

where $N_f = 3$ is the number of flavors, Λ_{QCD} is the QCD scale, and

$$C_{N_c, N_f} = \frac{0.466 \exp(-1.679 N_c) 1.34^{N_f}}{(N_c - 1)!(N_c - 2)!}, \quad (3.14)$$

$$b = \frac{11}{3} N_c - \frac{2}{3} N_f. \quad (3.15)$$

Here we ignored the effects of rotation on the instantons, which is a reasonable approximation in the regime $\mu_B \gg \Omega$. We also ignored the multi-instanton contributions because instantons are dilute at sufficiently large μ_B .

To calculate the chiral condensate Φ_{ij} in the chiral limit, we turn on a small quark mass for a moment, which will be turned off at the end of the computation. The effective Lagrangian induced by the single instanton is [39, 43, 44]

$$\mathcal{L}_{1\text{-inst}}^{\text{CFL}} = \frac{\tilde{a}}{2} \left[\text{tr}(M^\dagger \tilde{U}) + \text{h.c.} \right] + \mathcal{O}(M^2), \quad (3.16)$$

where $\tilde{a}$ denotes the strength of the $U(1)_A$ anomaly in the CFL phase, which can be determined through the instanton-induced six-fermion interaction as [43, 44]

$$\tilde{a} = \frac{24C_{N_c, N_f}}{N_c^2 - 1} N_c^{b-1} N_f^{-\frac{b+3}{2}} \Gamma\left(\frac{b+3}{2}\right) \left(\frac{8\pi^2}{g^2}\right)^{2N_c+1} \left(\frac{\Lambda_{\text{QCD}}}{\mu_B}\right)^b \mu_B \Delta^2. \quad (3.17)$$

From the effective potential for the Lagrangian (3.16), $V_{1\text{-inst}}^{\text{CFL}} = -\mathcal{L}_{1\text{-inst}}^{\text{CFL}}|_{\Sigma \rightarrow 1}$, the chiral condensate reads [44]

$$\Phi_{ij} = \frac{\partial V_{1\text{-inst}}^{\text{CFL}}}{\partial (M^\dagger)_{ji}} \Big|_{m \rightarrow 0} = -\frac{\tilde{a}}{2} \delta_{ij}, \quad (3.18)$$

which is nonvanishing even in the chiral limit. Then, eq. (3.12) becomes

$$\mathcal{L}_{1\text{-inst}} = \frac{1}{2} \gamma \tilde{a} \text{tr}(\tilde{U} + \tilde{U}^\dagger). \quad (3.19)$$

Finally, let us write down the topological term in the CFL phase under rotation. Since the transport coefficient of the CVE is again topologically protected independently of symmetry breaking patterns, we can also use the formula (2.11) in this case (but with a minor modification as we will immediately explain below), which yields

$$\mathcal{L}_{\text{CVE}} = \frac{\mu_B^2}{8\pi^2 f_{\tilde{\eta}'}} \sqrt{\frac{2}{3}} \nabla \tilde{\eta}' \cdot \Omega. \quad (3.20)$$

Note that the coefficient here is one half of that of eq. (2.12) because of the difference of the transformation laws (2.9) and (3.5) under the $U(1)_A$ rotation.

In summary, the effective theory of the CFL phase under rotation is given by eqs. (3.6), (3.9), (3.12), and (3.20). We are now ready to consider the ground state of this system.

3.2 Ground state in the chiral limit

We first consider the case in the chiral limit. Similarly to the previous discussion on the low-density hadronic matter, we will focus on $\tilde{\eta}'$. Adding the eqs. (3.6), (3.19) and (3.20) together, we obtain the effective Hamiltonian for $\tilde{\eta}'$,

$$\mathcal{H}_{\text{CFL}} = \frac{\mu_B^2}{96\pi^2} (\partial_z \tilde{\phi})^2 + 3\gamma \tilde{a} (1 - \cos \tilde{\phi}) - \frac{\mu_B^2}{8\pi^2} \Omega \partial_z \tilde{\phi}. \quad (3.21)$$

As in the case of low-density hadronic matter in section 2.3, we will look for a ground-state solution where the first term is comparable to the third term, which requires that $\Omega/\Delta \ll 1$ for the effective theory to be valid. This Hamiltonian is again mathematically equivalent

to that of QCD at finite μ_B and μ_I under rotation in ref. [11]: the latter is mapped to the former by the replacement

$$f_\pi^2 \rightarrow \frac{\mu_B^2}{48\pi^2}, \quad m_\pi^2 \rightarrow \frac{144\pi^2 \tilde{a}\gamma}{\mu_B^2}, \quad \mu_I \Omega \rightarrow \frac{1}{4}\mu_B \Omega. \quad (3.22)$$

By repeating the discussion in section 2.3, we find that the ground state is the CSL of the $\tilde{\eta}'$ meson, whose configuration is given by

$$\cos \frac{\tilde{\phi}(\bar{z})}{2} = \text{sn}(\bar{z}, k), \quad \bar{z} \equiv \frac{12\pi\sqrt{\gamma\tilde{a}}}{\mu_B k} z. \quad (3.23)$$

This solution has a period

$$\tilde{\ell} = \frac{\mu_B k K(k)}{6\pi\sqrt{\gamma\tilde{a}}}, \quad (3.24)$$

and the critical angular velocity is

$$\Omega_{\tilde{\eta}'}^0 = \frac{8\sqrt{\gamma\tilde{a}}}{\mu_B}, \quad (3.25)$$

above which the CSL state is energetically favorable than the CFL state ($\tilde{\phi} = 0$). The μ_B dependence of $\Omega_{\tilde{\eta}'}^0$ is determined by substituting eqs. (3.13) and (3.17) into eq. (3.25) as

$$\Omega_{\tilde{\eta}'}^0 = 2^5 3^{\frac{11}{2}} 5\pi C_{N_c, N_f} \left(\frac{8\pi^2}{g^2} \right)^7 \left(\frac{\Lambda_{\text{QCD}}}{\mu_B} \right)^{10} \frac{\Delta^2}{\Lambda_{\text{QCD}}}. \quad (3.26)$$

In particular, $\Omega_{\tilde{\eta}'}^0 \rightarrow 0$ at asymptotically large μ_B .

3.3 Ground state with finite quark masses

Let us now turn on finite quark masses in flavor-symmetric QCD. In the case with flavor asymmetry, the equation of motion of the system is not analytically solvable even without the instanton-induced term and the CSL-type solution does not exist (see appendix D). Below we consider sufficiently large μ_B for simplicity, where the instanton-induced term (3.16) is suppressed compared with the quark mass term in eq. (3.9).

Let us consider eq. (3.9) with the flavor symmetric masses:

$$M = \text{diag}(m, m, m). \quad (3.27)$$

In this case, there is no mixing between $\tilde{\eta}'$ meson and the other mesons in the mass matrix. Then, we obtain the effective Hamiltonian

$$\mathcal{H}_{\text{CFL}} = \frac{\mu_B^2}{96\pi^2} (\partial_z \tilde{\phi})^2 + \frac{9m^2 \Delta^2}{\pi^2} (1 - \cos \tilde{\phi}) - \frac{\mu_B^2}{8\pi^2} \Omega \partial_z \tilde{\phi}, \quad (3.28)$$

Similarly to the discussions in previous sections, the ground state of this Hamiltonian is obtained from eq. (3.9) in ref. [11] by the following replacement:

$$f_\pi^2 \rightarrow \frac{\mu_B^2}{48\pi^2}, \quad m_\pi^2 \rightarrow \frac{432m^2 \Delta^2}{\mu_B^2}, \quad \mu_I \Omega \rightarrow \frac{1}{4}\mu_B \Omega. \quad (3.29)$$

In particular, the critical angular velocity for the CSL state is found as

$$\Omega_{\tilde{\eta}'} = \frac{8\sqrt{3}m\Delta}{\pi\mu_B}. \quad (3.30)$$

4 Discussion and conclusion

We have shown that the ground state of (nearly) three-flavor symmetric QCD under sufficiently fast rotation is the η' CSL both in low-density hadronic matter and high-density CFL phase. The critical angular velocity for the η' CSL is given in eq. (2.41) at low density and eq. (3.30) at high density. In both regions, the parity symmetry and the continuous translational symmetry in the direction of the angular velocity are spontaneously broken, leading to a phonon as the additional low-energy excitation. Since the symmetry breaking patterns of the ground states and excitations around them are the same between the two regions, it is plausible that the η' CSLs on both sides are continuously connected. This can be regarded as an extension of the quark-hadron continuity conjecture [21–23] in the presence of rotation. On the other hand, in the case away from the flavor symmetry where the quark mass difference can no longer be treated as a perturbation, we cannot analytically solve the equation of motion for η' , and as a result, the ground state can be different from the η' CSL.

In our analysis, we have studied the leading-order low-energy effective theory, where the interactions between η' and other mesons (including the superfluid phonon associated with the spontaneous $U(1)_B$ symmetry breaking) are higher order in derivatives and are negligible. In this context, it should be remarked that, in the case of the π_0 CSL in baryonic matter with an external magnetic field $\mathbf{B}$, the term $\sim \partial^\mu \pi_0 (\pi_1 \partial_\mu \pi_2 - \pi_2 \partial_\mu \pi_1)$ in the presence of the π_0 CSL background $\langle \nabla \pi_0 \rangle \sim \mu_B \mathbf{B}$ can lead to Bose-Einstein condensation of charged pions when the magnitude of the magnetic field is increased [15]. However, such a phenomenon will not occur for the η' CSL in baryonic matter when Ω is increased further, since the kinetic term for η' is decoupled from those of other mesons at the leading order. We hence expect that the η' CSL is always realized above the critical angular velocity within the applicability of our effective theory at leading order. It would be interesting to study the possible modifications of our results due to the next-to-leading order corrections and the flavor asymmetry.

Finally, let us discuss a possible realization of the rotation-induced η' CSL in physical systems. One possible candidate is noncentral heavy ion collision experiments. There, we may roughly estimate $\Omega_{\eta'}$ and Ω_{π_0} for a rotating nuclear matter made up from $^{197}_{79}\text{Au}$ with saturation density $n \approx 0.16/\text{fm}^3$ (corresponding to $\mu_B \approx 1 \text{ GeV}$ and $\mu_I \approx -10 \text{ MeV}$) as³

$$\Omega_{\eta'} \approx 4 \times 10^2 \text{ MeV}, \quad \Omega_{\pi_0} \approx 6 \times 10^3 \text{ MeV}, \quad (4.1)$$

respectively, where we used the vacuum values of the quantities $f_\pi \approx 93 \text{ MeV}$, $f_{\eta'}/f_\pi \approx 1.1$, $m_\pi \approx 140 \text{ MeV}$, $m_K \approx 500 \text{ MeV}$ and $m_{\eta'} \approx 960 \text{ MeV}$ [45], together with the Gell-Mann-Oakes-Renner relation

$$f_\pi^2 m_\pi^2 = 4B m_{\text{ud}}, \quad f_\pi^2 m_K^2 = 2B(m_{\text{ud}} + m_s), \quad (4.2)$$

³Although the validity of the low-energy effective field theory for such large μ_B is subtle, we here simply extrapolate the parametric dependence of $\Omega_{\eta'}$ and Ω_{π_0} in eqs. (2.41) and (2.42) into such a regime to provide semi-quantitative estimates.

and

$$f_{\eta'}^2 m_{\eta'}^2 = \frac{4B}{3}(2m_{\text{ud}} + m_s) + 6a. \quad (4.3)$$

This result shows that $\Omega_{\eta'}$ is an order of magnitude smaller than Ω_{π_0} previously found in ref. [11], although it does not still reach the experimentally measured angular velocity in noncentral heavy ion collisions, $\Omega_{\text{exp}} \sim 10 \text{ MeV}$ [1]. Nonetheless, because $\Omega_{\eta'}$ decreases with increasing μ_B as eqs. (2.41) and (3.30), the η' CSL could be potentially realized in high-density matter to be produced in low-energy heavy ion collision experiments in the future. (Even in such a case, it might be difficult to realize the π_0 CSL since $\Omega_{\pi_0} \gg \Lambda_{\text{QCD}}$.) To understand its possible realization in heavy ion collisions more realistically, it is necessary to take into account the finite-temperature effects on the η' CSL; see also ref. [46] for the study on the low-temperature effects on the π_0 CSL in a magnetic field. Further studies in this direction will be reported elsewhere.

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A The η' potential in large- N_c QCD

In this appendix, we show that the Lagrangian (2.21) explicitly breaks the continuous translational symmetry in space.

Using the equality, $\det \exp(i\lambda_a \pi_a / f_\pi) = \exp \text{tr}(i\lambda_a \pi_a / f_\pi)$, eq. (2.21) becomes

$$\mathcal{L}'_{\text{anom}} = -\frac{f_\pi^2 a}{4N_c} (3\phi - 2\pi n)^2, \quad (A.1)$$

where $\phi \equiv \sqrt{6}\eta'/(3f_\pi)$ as defined in the main text and n is integer. The corresponding potential energy is

$$V_{\text{anom}} = \frac{f_\pi^2 a}{4N_c} (3\phi - 2\pi n)^2, \quad (2n-1)\pi \leq 3\phi < (2n+1)\pi, \quad (A.2)$$

as sketched in figure 1. Note that this potential has cusp singularities at $3\phi = (2n-1)\pi$ with integer n .

To demonstrate the explicit breaking of spacial translational symmetry, consider an infinitesimal spacetime translation $x^\mu \rightarrow x^\mu - \epsilon^\mu$, under which the variation of ϕ is $\delta\phi(x) = \epsilon^\mu \partial_\mu \phi(x) + \mathcal{O}(\epsilon^2)$. When $\delta\phi$ does not cross a cusp singularity, i.e., when $(2n-1)\pi/3 \leq$

Figure 1. V_{anom} as a function of ϕ .

$\phi \leq (2n+1)\pi/3$ and $(2n-1)\pi/3 \leq \phi + \delta\phi \leq (2n+1)\pi/3$ (see figure 2), the variation of eq. (2.21) is written as a total derivative:

$$\delta_L \mathcal{L}'_{\text{anom}} = \epsilon^\nu \partial_\nu \mathcal{L}'_{\text{anom}}, \quad (\text{A.3})$$

and thus, it is invariant under the spacetime translation. However, when $\delta\phi$ crosses a cusp singularity, i.e., when $(2n-1)\pi/3 \leq \phi \leq (2n+1)\pi/3$ and $(2n+1)\pi/3 \leq \phi + \delta\phi \leq (2n+3)\pi/3$ (see figure 2), the variation of eq. (2.21) is

$$\delta_L \mathcal{L}'_{\text{anom}} = \epsilon^\nu \partial_\nu \mathcal{L}'_{\text{anom}} + \frac{3\pi f_\pi^2 a}{N_c} \left(\phi - \frac{2n+1}{3} \pi \right), \quad (\text{A.4})$$

where the second term is not a total derivative. Therefore, the Lagrangian (2.21) is not invariant under the spatial translational symmetry.

B Derivation of the critical angular velocity $\Omega_{\eta'}$

In this appendix, we give the derivation of $\Omega_{\eta'}$ in eq. (2.41). The energy of each lattice per unit area is

$$\frac{\bar{\mathcal{E}}}{S} = \frac{3}{2} f_{\eta'}^2 \sqrt{A_1} \left[\frac{2}{\bar{k}} \bar{E}(\bar{k}) + \left(\bar{k} - \frac{1}{\bar{k}} \right) \bar{K}(\bar{k}) \right] - \frac{\mu_B^2 \Omega}{2\pi}, \quad (\text{B.1})$$

where $\bar{K}(\bar{k})$ is defined in eq. (2.39) and $\bar{E}(\bar{k})$ is defined as

$$\bar{E}(\bar{k}) \equiv \int_0^{2\pi} d\theta \left[1 + \bar{k}^2 \frac{A_2}{2A_1} (1 - \cos 3\theta) - \bar{k}^2 \cos^2 \frac{\theta}{2} \right]^{\frac{1}{2}}, \quad (\text{B.2})$$

both of which should not be confused with the complete elliptic integrals of the first and second kinds, $K(k)$ and $E(k)$. The total energy of the system of length L is given by

$$\bar{\mathcal{E}}_{\text{tot}} \equiv \frac{L}{\ell} \bar{\mathcal{E}} = 3V f_{\eta'}^2 A_1 \left[\frac{2\bar{E}(\bar{k})}{\bar{k}^2 \bar{K}(\bar{k})} + 1 - \frac{1}{\bar{k}^2} - \frac{\mu_B^2 \Omega}{3\pi f_{\eta'}^2 \sqrt{A_1}} \frac{1}{\bar{k} \bar{K}(\bar{k})} \right], \quad (\text{B.3})$$

where $\bar{\ell}$ is defined in eq. (2.39) and $V \equiv LS$.

Figure 2. (a) Both ϕ and $\phi + \delta\phi$ are between $(2n - 1)\pi/3$ and $(2n + 1)\pi/3$. (b) ϕ is between $(2n - 1)\pi/3$ and $(2n + 1)\pi/3$, while $\phi + \delta\phi$ is between $(2n + 1)\pi/3$ and $(2n + 3)\pi/3$.

To minimize $\bar{\mathcal{E}}_{\text{tot}}$ with respect to $\bar{k}$, we calculate its derivative as

$$\frac{d\bar{\mathcal{E}}_{\text{tot}}}{d\bar{k}} = 3V f_{\eta'}^2 A_1 \frac{\bar{H}(\bar{k})}{\bar{k}^2 \bar{K}(\bar{k})^2} \left[\frac{\mu_B^2 \Omega}{3\pi f_{\eta'}^2 \sqrt{A_1}} - \frac{2\bar{E}(\bar{k})}{\bar{k}} \right], \quad (\text{B.4})$$

where we used the relations

$$\frac{d\bar{E}(\bar{k})}{d\bar{k}} = \frac{\bar{E}(\bar{k}) - \bar{K}(\bar{k})}{\bar{k}}, \quad (\text{B.5})$$

$$\frac{d\bar{K}(\bar{k})}{d\bar{k}} = \frac{\bar{H}(\bar{k}) - \bar{K}(\bar{k})}{\bar{k}}, \quad (\text{B.6})$$

and we defined

$$\bar{H}(\bar{k}) \equiv \int_0^{2\pi} d\theta \left[1 + \bar{k}^2 \frac{A_2}{2A_1} (1 - \cos 3\theta) - \bar{k}^2 \cos^2 \frac{\theta}{2} \right]^{-\frac{3}{2}}. \quad (\text{B.7})$$

Because the factor in front of the bracket in eq. (B.4) is positive, we can focus on the factor inside the bracket,

$$g(\bar{k}) \equiv \frac{\mu_B^2 \Omega}{3\pi f_{\eta'}^2 \sqrt{A_1}} - \frac{2\bar{E}(\bar{k})}{\bar{k}}. \quad (\text{B.8})$$

Since $\bar{E}(\bar{k})/\bar{k}$ monotonically decreases as a function of $\bar{k}$ and it is bounded from below as $\bar{E}(\bar{k})/\bar{k} \geq \bar{E}(1)$ for $0 \leq \bar{k} \leq 1$, the CSL solution exists if and only if $\Omega \geq \Omega_{\eta'}$, where $\Omega_{\eta'}$ is given by eq. (2.41). We can also calculate $\bar{\mathcal{E}}_{\text{tot}}$ satisfying the minimization condition $d\bar{\mathcal{E}}_{\text{tot}}/d\bar{k} = 0$ as

$$\bar{\mathcal{E}}_{\text{tot}} = 3V f_{\eta'}^2 A_1 \left(1 - \frac{1}{\bar{k}^2} \right) < 0. \quad (\text{B.9})$$

Therefore, the CSL solution is energetically more favorable than the QCD vacuum ($\phi = 0$).

C Calculation of the instanton-induced potential

We here provide the derivation of eqs. (3.12) and (3.13). We start from the six-fermion instanton-induced vertex [47–49]

$$\begin{aligned} \mathcal{L}_{\text{inst}} = \int_0^\infty d\rho n(\rho) \frac{(2\pi\rho)^6 \rho^3}{6N_c(N_c^2 - 1)} \epsilon_{i_1 i_2 i_3} \epsilon_{j_1 j_2 j_3} \left[\frac{2N_c + 1}{2N_c + 4} (\bar{q}_R^{i_1} q_L^{j_1}) (\bar{q}_R^{i_2} q_L^{j_2}) (\bar{q}_R^{i_3} q_L^{j_3}) \right. \\ \left. - \frac{3}{8(N_c + 2)} (\bar{q}_R^{i_1} q_L^{j_1}) (\bar{q}_R^{i_2} \sigma_{\mu\nu} q_L^{j_2}) (\bar{q}_R^{i_3} \sigma^{\mu\nu} q_L^{j_3}) + (\text{R} \leftrightarrow \text{L}) \right], \end{aligned} \quad (\text{C.1})$$

where ρ is the instanton size, $i_{1,2,3}$ and $j_{1,2,3}$ are flavor indices, and $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$. The instanton size distribution $n(\rho)$ is given by [49, 50]

$$n(\rho) = C_{N_c, N_f} \left(\frac{8\pi^2}{g^2} \right)^{2N_c} \rho^{-5} \exp\left(-\frac{8\pi^2}{g(\rho)^2}\right) e^{-N_f \mu^2 \rho^2}, \quad (\text{C.2})$$

$$\frac{8\pi^2}{g(\rho)^2} = -b \ln(\rho \Lambda_{\text{QCD}}), \quad (\text{C.3})$$

where C_{N_c, N_f} and b are defined in eq. (3.15).

We evaluate the expectation value of eq. (C.1) in the presence of the diquark and chiral condensate in the mean-field approximation. Replacing $\bar{q}_R q_L$ with the chiral condensate Φ , we get

$$\begin{aligned} \mathcal{L}_{\text{inst}} \simeq \int_0^\infty d\rho n(\rho) \frac{(2\pi\rho)^6 \rho^3}{6N_c(N_c^2 - 1)} \epsilon_{i_1 i_2 i_3} \epsilon_{j_1 j_2 j_3} \left[\frac{2N_c + 1}{2N_c + 4} 3(\bar{q}_R^{i_1} q_L^{j_1}) (\bar{q}_R^{i_2} q_L^{j_2}) \Phi_{j_3 i_3} \right. \\ \left. - \frac{3}{8(N_c + 2)} \Phi_{j_1 i_1} (\bar{q}_R^{i_2} \sigma_{\mu\nu} q_L^{j_2}) (\bar{q}_R^{i_3} \sigma^{\mu\nu} q_L^{j_3}) + (\text{R} \leftrightarrow \text{L}) \right]. \end{aligned} \quad (\text{C.4})$$

To replace the remaining part with the diquark condensate, we use the Fierz transformations

$$(1-\gamma_5)_{\sigma\sigma'} (1-\gamma_5)_{\tau\tau'} = -\frac{1}{2} [(1-\gamma_5)C]_{\sigma\tau} [C(1-\gamma_5)]_{\tau'\sigma'} - \frac{1}{4} [(1-\gamma_5)\sigma_{\mu\nu}C]_{\sigma\tau} (C\sigma^{\mu\nu})_{\tau'\sigma'}, \quad (\text{C.5})$$

$$[\sigma^{\mu\nu}(1-\gamma_5)]_{\sigma\sigma'} [\sigma_{\mu\nu}(1-\gamma_5)]_{\tau\tau'} = 6[(1-\gamma_5)C]_{\sigma\tau} [C(1-\gamma_5)]_{\tau'\sigma'} - [(1-\gamma_5)\sigma_{\mu\nu}C]_{\sigma\tau} (C\sigma^{\mu\nu})_{\tau'\sigma'}, \quad (\text{C.6})$$

and the diquark condensate (3.1), which is related to the superconducting gap Δ by [51]

$$|d_R| = |d_L| = \frac{3\sqrt{2}\mu_B^2 \Delta}{2\pi g N_c^2}. \quad (\text{C.7})$$

Then, we arrive at the instanton-induced potential in eqs. (3.12) and (3.13).

D CFL phase with flavor asymmetry

We here consider the CFL phase with flavor-asymmetric quark masses given in eq. (2.32). Expanding eq. (3.9) to the second order in meson fields, we have a 9×9 real-symmetric mass matrix for them, which can be decomposed in to 7×7 matrix for $\tilde{\pi}$ and $\tilde{K}$ and a nondiagonal 2×2 matrix for $\tilde{\eta}$ and $\tilde{\eta}'$. As we are interested in the low-energy dynamics and the lightest meson in the CFL phase is a mixed state of $\tilde{\eta}$ and $\tilde{\eta}'$ due to the inverse meson mass ordering [40], we will focus on the nondiagonal 2×2 matrix for $\tilde{\eta}$ and $\tilde{\eta}'$ in the following.⁴

The elements of the mass matrix in the $\tilde{\eta}$ - $\tilde{\eta}'$ sector in eq. (3.9) are

$$\begin{aligned} (M_{\tilde{\eta}-\tilde{\eta}'})_{00} &= \frac{4cm_{\text{ud}}(m_{\text{ud}} + 2m_{\text{s}})}{3f_{\tilde{\eta}'}^2}, & (M_{\tilde{\eta}-\tilde{\eta}'})_{88} &= \frac{4cm_{\text{ud}}(2m_{\text{ud}} + m_{\text{s}})}{3f_{\tilde{\pi}}^2}, \\ (M_{\tilde{\eta}-\tilde{\eta}'})_{08} &= (M_{\tilde{\eta}-\tilde{\eta}'})_{80} = \frac{4\sqrt{2}cm_{\text{ud}}(m_{\text{s}} - m_{\text{ud}})}{3f_{\tilde{\eta}'}f_{\tilde{\pi}}}. \end{aligned} \quad (\text{D.1})$$

We can diagonalize it by the real-orthogonal matrix T as

$$\begin{aligned} {}^t T M_{\tilde{\eta}-\tilde{\eta}'} T &= \frac{1}{2} \text{diag} \left((M_{\tilde{\eta}-\tilde{\eta}'})_{00} + (M_{\tilde{\eta}-\tilde{\eta}'})_{88} - \sqrt{[(M_{\tilde{\eta}-\tilde{\eta}'})_{00} - (M_{\tilde{\eta}-\tilde{\eta}'})_{88}]^2 + 4(M_{\tilde{\eta}-\tilde{\eta}'})_{08}^2}, \right. \\ &\quad \left. (M_{\tilde{\eta}-\tilde{\eta}'})_{00} + (M_{\tilde{\eta}-\tilde{\eta}'})_{88} + \sqrt{[(M_{\tilde{\eta}-\tilde{\eta}'})_{00} - (M_{\tilde{\eta}-\tilde{\eta}'})_{88}]^2 + 4(M_{\tilde{\eta}-\tilde{\eta}'})_{08}^2} \right), \end{aligned} \quad (\text{D.2})$$

where the mixing angle θ satisfies

$$\tan \theta = \frac{2(M_{\tilde{\eta}-\tilde{\eta}'})_{08}}{-(M_{\tilde{\eta}-\tilde{\eta}'})_{00} + (M_{\tilde{\eta}-\tilde{\eta}'})_{88} + \sqrt{[(M_{\tilde{\eta}-\tilde{\eta}'})_{00} - (M_{\tilde{\eta}-\tilde{\eta}'})_{88}]^2 + 4(M_{\tilde{\eta}-\tilde{\eta}'})_{08}^2}}. \quad (\text{D.3})$$

The eigenstates of the mass matrix (D.1) denoted as φ_1 and φ_2 are related to $\tilde{\eta}'$ and $\tilde{\eta}$ by⁵

$$\begin{pmatrix} \tilde{\eta}' \\ \tilde{\eta} \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \quad (\text{D.4})$$

In particular, one can see from eq. (D.2) that the mass eigenvalue for φ_2 is larger than that for φ_1 .

We now focus on φ_1 . Then, Σ can be approximated as

$$\Sigma \approx \text{diag} \left(e^{i\lambda_1 \varphi_1}, e^{i\lambda_1 \varphi_1}, e^{i\lambda_2 \varphi_1} \right), \quad (\text{D.5})$$

$$\lambda_1 \equiv \sqrt{\frac{2}{3}} \frac{\cos \theta}{f_{\tilde{\eta}'}} + \frac{1}{\sqrt{3}} \frac{\sin \theta}{f_{\tilde{\pi}}}, \quad \lambda_2 \equiv \sqrt{\frac{2}{3}} \frac{\cos \theta}{f_{\tilde{\eta}'}} - \frac{2}{\sqrt{3}} \frac{\sin \theta}{f_{\tilde{\pi}}}.$$

⁴See also ref. [52] for a related work on the meson mass spectra in the CFL phase. Note that we here set $m_{\text{u}} = m_{\text{d}}$ (see eq. (2.32)), and in this case, $\tilde{\eta}$ and $\tilde{\eta}'$ do not mix with $\tilde{\pi}_0$ in the mass matrix unlike refs. [40, 52].

⁵In the literature, η' and η in our notation here are often denoted by η_0 and η_8 , and φ_1 and φ_2 by η' and η , respectively.

Accordingly, we arrive at the low-energy effective Hamiltonian

$$\tilde{\mathcal{H}}_{\text{CFL}} = \frac{1}{2}v_{\tilde{\eta}'}^2(\partial_z\varphi_1)^2 + 4cm_{\text{ud}}m_s \cos(\lambda_1\varphi_1) + 2cm_{\text{ud}}^2 \cos(\lambda_2\varphi_1) - \frac{\mu_B^2 \cos\theta}{8\pi^2 f_{\tilde{\eta}'}} \sqrt{\frac{2}{3}}\Omega\partial_z\varphi_1, \quad (\text{D.6})$$

where $\lambda_{1,2}$ are functions of m_{ud} , m_s , $f_{\tilde{\pi}}$, and $f_{\tilde{\eta}'}$. The equation of motion for φ_1 is given by

$$v_{\tilde{\eta}'}^2\partial_z^2\varphi_1 = 4cm_{\text{ud}}m_s\lambda_1 \sin(\lambda_1\varphi_1) + 2cm_{\text{ud}}^2\lambda_2 \sin(\lambda_2\varphi_1). \quad (\text{D.7})$$

To the best of our knowledge, this differential equation cannot be analytically solved unlike the flavor symmetric case, and we cannot repeat the previous discussion. In fact, the solution of eq. (D.7) has a periodicity in φ_1 only if the ratio λ_1/λ_2 is a rational number. As λ_1/λ_2 is generically an irrational number, the periodicity is lost and the solution is no longer CSL in this case.⁶

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⁶Note however that this does not mean that the η' CSL is absent for *any* nonzero $m_s - m_{\text{ud}}$. When $m_{\text{ud}} \simeq m_s$, $\tilde{\eta}$ and $\tilde{\eta}'$ are almost degenerate and integrating out φ_2 above is not well justified. Physically, one expects that the η' CSL persists as long as $m_s - m_{\text{ud}}$ is so small that it can be treated as a perturbation.

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