

Witten anomaly in 4d heterotic compactifications with $\mathcal{N} = 2$ supersymmetry

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ABSTRACT: We showed that there is no $SU(2)$ Witten anomaly in a large class of 4d $\mathcal{N} = 2$ supersymmetric Heterotic string compactifications. The consistency conditions we consider are the modularity of the new supersymmetric index, the integrality of BPS indices, and the discrete Peccei-Quinn shift symmetries. We also found an example where these conditions are not sufficient to show that the theory is anomaly free. This suggests that there are more conditions in the worldsheet SCFT essential for consistent string compactifications.

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1 Introduction

Theoretically consistent string vacua are believed to yield low-energy effective field theories without a theoretical inconsistency. If we find a low-energy inconsistency in a string vacuum, that is taken as a hint of some more theoretical consistency conditions of string theory. Gauge and gravitational anomalies in the effective theories in closed string vacua are known to cancel due to the modular invariance of the CFT [1, 2]. The same class of anomalies in the effective theories in open string vacua cancel locally because of the anomaly inflow mechanism [3–6], and they cancel globally because of the Bianchi identity of Ramond-Ramond fields.

In this article, we address the SU(2) Witten anomalies [7] of the 4d $\mathcal{N} = 2$ supersymmetric effective theories given by Heterotic compactifications. The organization of this article is as follows.

Section 2 is a short review of basic facts about the SU(2) Witten anomaly. Section 3 explains our setup (including the choice of SU(2) gauge symmetry) and some assumptions for the Heterotic compactifications.

In section 4, we show that the Witten anomaly indeed vanishes for the lattice¹ Λ_S and the charge vector v_0 , if Λ_S has a decomposition $U \oplus W$ for some even negative definite

¹In our setup, $\tilde{\Lambda}_S = U[-1] \oplus \Lambda_S$ is the charge lattice of chiral free bosons in the internal worldsheet CFT. The charge vector $v_0 \in \tilde{\Lambda}_S^\vee$ specifies the choice of the SU(2) gauge symmetry. See section 3 for more details.

lattice W and the charge vector v_0 stays within W^\vee . The derivation described in section 4.1 exploits

- the modularity of the generating functions² Φ and Ψ for BPS indices, defined by the internal worldsheet CFT language,
- the integrality of BPS indices, and
- the integrality of some coefficients of the leading terms in the prepotential for the effective theory, which follows from the discrete Peccei-Quinn shift symmetries.

Section 4.2 shows that for some choices of the lattice Λ_S and the charge vector v_0 , the cancellation of the Witten anomaly follows only by the modularity (the first condition).

Section 5 treats an example of Λ_S and v_0 for which one cannot show that the Witten anomaly always vanishes only from these conditions listed above. This means that we are missing some other important consistency conditions for the string compactifications.

In the appendix, we review what the discrete Peccei-Quinn shift symmetries imply for the leading coefficients of the prepotential.

2 SU(2) Witten anomaly

Since 4d $\mathcal{N} = 2$ supersymmetric theories are non-chiral, they are free from perturbative triangle anomalies. However there can be nonperturbative anomalies, called Witten anomalies [7]. In $\mathcal{N} = 2$ Sp(k) gauge theories, a half-hypermultiplet massless matter in the representation R with half-integral Dynkin index $T(R)$ produces Witten anomaly. In the case of SU(2) gauge theory, the possibly anomalous representations are those with the spin

$$j = \frac{1}{2} + 2l, \quad l = 0, 1, 2, \dots \tag{2.1}$$

The theory is free from Witten anomaly if and only if the total number of massless SU(2) half-hypermultiplets in such representations is even, i.e.

$$\text{WA} := \sum_{j \in \frac{1}{2} + 2\mathbb{Z}_{\geq 0}} N(R_j) \equiv 0 \pmod{2}, \tag{2.2}$$

where $N(R_j)$ is the number of massless half-hypermultiplets in the spin- j representation R_j . This quantity can also be written by N_j , the number of massless half-hypermultiplets with the $2j$ units of the fundamental weight (= SU(2) Cartan charge j):

$$\text{WA} = \sum_{j \in \frac{1}{2} + 2\mathbb{Z}_{\geq 0}} (N_j - N_{j+1}) \equiv \sum_{j \in \frac{1}{2} + \mathbb{Z}_{\geq 0}} N_j \pmod{2}. \tag{2.3}$$

Note that this summation may run over those states in SU(2) full-hypermultiplets because they appear in pairs and have no contributions modulo two. Moreover, if the theory has other U(1) gauge symmetries, only neutral states under those U(1)'s contribute to (2.3) because U(1)-charged matter has its conjugate and this pair contributes 2.

²See section 3 for the definitions for the modular form Φ . See also section 5.1 for the modular form Ψ .

3 Heterotic compactification and SU(2) enhancement

Let us consider a Heterotic compactification with 4d $\mathcal{N} = 2$ supersymmetry; we assume that it is without an $\mathbb{R}^{3,1}$ -filling NS5-brane and other solitons of similar kinds. The internal $(c, \tilde{c}) = (22, 9)$ CFT in the NSR formalism with $(0, 2)$ worldsheet supersymmetry for such a compactification is characterized by the conditions in [8–10]. Let $\tilde{\Lambda}_S$ denote the charge lattice of chiral free bosons that appear in the CFT [11, 12], $G_S := \tilde{\Lambda}_S^\vee / \tilde{\Lambda}_S$ its discriminant group, and $\rho := \text{rank}(\Lambda_S)$. We assume that the lattice $\tilde{\Lambda}_S$ has a primitive embedding into $U^{\oplus 4} \oplus E_8[-1]^{\oplus 2}$, where U is the even unimodular lattice of signature $(1, 1)$, and a structure³ $\tilde{\Lambda}_S = U[-1] \oplus \Lambda_S$.

When a charge vector $v_0 \in \tilde{\Lambda}_S^\vee$ satisfies the following conditions,

$$\frac{(v_0, v_0)}{2} = -\frac{1}{k}, \quad kv_0 = 0 \in G_S, \quad \exists k \in \mathbb{Z}_{>0}, \quad (3.1)$$

and⁴ $-2 \leq v_0^2 < 0$, a level- k SU(2) current algebra $\{J^\pm, J_3\}$ appears in the left-mover of the CFT at a complex codimension-1 subspace in the Coulomb branch moduli space; J^+ has charge v_0 . There is an enhanced SU(2) gauge symmetry in the 4d effective theory (e.g. [13, 14]). The SU(2) Cartan charge is

$$q_v := \frac{(v_0, v)}{(v_0, v_0)} \in \frac{1}{2}\mathbb{Z} \quad (3.2)$$

for states whose U(1) charge is $v \in \tilde{\Lambda}_S^\vee$.

Since only massless matters contribute to the anomaly, we can focus our attention to the BPS states of the 4d $\mathcal{N} = 2$ supersymmetry algebra. The multiplicities of purely electrically charged BPS states are captured by the new supersymmetric index in the Heterotic worldsheet language:

$$Z_{\text{new}}(\tau, \bar{\tau}) = \frac{-i}{\eta(\tau)^{24}} \text{Tr}_{\text{R-sector}} \left[e^{\pi i F_R} F_R q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \tilde{c}/24} \right] \quad (3.3)$$

$$=: \sum_{\gamma \in G_S} \theta_{\tilde{\Lambda}_S[-1] + \gamma}(\tau, \bar{\tau}) \frac{\Phi_\gamma(\tau)}{\eta(\tau)^{24}}, \quad (3.4)$$

where the trace is over the internal $(c, \tilde{c}) = (22, 9)$ worldsheet CFT with the right-mover in the Ramond sector, and F_R is the zero mode of total U(1) current in the right-mover [8–10]. The function $\theta_{\tilde{\Lambda}_S[-1]}$ is the Siegel theta function for the lattice $\tilde{\Lambda}_S[-1]$. (See [14] for more details.) The integral coefficients $c_\gamma(\nu)$ defined by

$$\frac{\Phi_\gamma(\tau)}{\eta(\tau)^{24}} =: \sum_{\gamma \in \gamma^2/2 + \mathbb{Z}} c_\gamma(\nu) q^\nu, \quad c_\gamma(\nu) = 0 \text{ for } \nu < -1, \quad (3.5)$$

³In the language of Type IIA dual, this structure corresponds to the 0-form and 4-form cohomology classes of the K3-fiber of the internal Calabi-Yau threefold staying distinct from the 2-form cohomology classes of the fiber.

⁴ (v_0, v_0) or v_0^2 denotes the square in the intersection form of $\tilde{\Lambda}_S$.

is relevant to BPS counting. In particular, when $v \in \tilde{\Lambda}_S^\vee$ satisfies $[v] \neq 0 \in G_S$ and $-2 < v^2 < 0$, there are $n_\gamma^{V/H}$ BPS vector/half-hyper multiplets of $U(1)$ -charge v , and the following equality holds:

$$c_{[v]}(v^2/2) = -2n_{[v]}^V + n_{[v]}^H \equiv n_{[v]}^H \pmod{2}. \quad (3.6)$$

The leading Fourier coefficients $c_\gamma(\nu)$ for $\gamma \in G_S$, where $-1 \leq \nu < 0$, are denoted by n_γ .⁵ The $\mathcal{N} = 2$ supersymmetry in $\mathbb{R}^{3,1}$ — not more, not less — implies $n_\gamma = -2$ for $\gamma = 0 \in G_S$. In the following arguments, we implicitly use the integrality of BPS indices $c_\gamma(\nu)$.

4 $\Lambda_S = U \oplus W$ cases

We show that the $SU(2)$ Witten anomaly vanishes if Λ_S has a decomposition $\Lambda_S = U \oplus W$ and $v_0 \in W^\vee \subset \tilde{\Lambda}_S^\vee$; here, the lattice W is even and negative definite and is assumed to have a primitive embedding into $U^{\oplus 2} \oplus E_8[-1]^{\oplus 2}$.

4.1 Derivation using also the discrete Peccei-Quinn symmetries

In the 4d effective theory, the action of the vector multiplets and the mass of the BPS states of the $\mathcal{N} = 2$ supersymmetry are determined by the prepotential

$$\mathcal{F} = \frac{s}{2}(t, t) + \frac{d_{abc}}{3!}t^a t^b t^c - \frac{a_{ab}}{2}t^a t^b - b_a t^a - \frac{\zeta(3)}{(2\pi i)^3} \frac{\chi}{2} + \mathcal{O}(e^{2\pi i s}, e^{2\pi i t}), \quad (4.1)$$

where s is the 4d axion-dilaton complex scalar, and $t \in \Lambda_S \otimes \mathbb{C}$ collectively denotes the Narain moduli; t is a set of local coordinates of $D(\tilde{\Lambda}_S) := \mathbb{P}\{\bar{U} \in \tilde{\Lambda}_S \otimes \mathbb{C} \mid \bar{U}^2 = 0, (\bar{U}, \bar{U}) > 0\}$, and $t = e_a t^a$ is the component description for an integral basis $\{e_a\}$ of Λ_S . Parameters d_{abc} , a_{ab} , b_a , and χ reflect more information of the internal space than must the lattice $\tilde{\Lambda}_S$.

Two points in the Coulomb branch moduli space $D(\tilde{\Lambda}_S)$ should describe the same lattice vertex operator algebra and an identical string vacuum, if they are in a common orbit of

$$\Gamma_S := \left\{ g \in \text{Isom}(\tilde{\Lambda}_S) \mid g \text{ acts trivially on } G_S \right\}. \quad (4.2)$$

As we will review in appendix A, it follows from the unphysical nature of the Γ_S action⁶ that

$$d_{abc} \in \mathbb{Z}, \quad a, b, c \in 1, \dots, \rho. \quad (4.3)$$

⁵The discriminant group G_S may contain an element $\gamma \neq 0$ with $(\gamma, \gamma) \in 2\mathbb{Z}$. In this article we do not assume that $n_\gamma = 0$ for such a non-trivial isotropic element γ although there is no type IIA dual in the geometric phase when $n_\gamma \neq 0$ for such a γ .

⁶In the language of Heterotic string in the geometric phase, with the structure $\Lambda_S = U \oplus W$ and $W \subset E_8[-1]^{\oplus 2}$, the Γ_S action on the Coulomb branch moduli (Narain moduli) includes +1 shift of the complex structure of T^2 , $+(2\pi)^2 \alpha'$ shift of $\int_{T^2} B$, and winding gauge transformation of W . In the dual Type IIA language, d_{abc} 's are the intersection numbers of the internal Calabi-Yau threefold. We call these $t^a \rightarrow t^a + \delta_a^a$ shifts discrete Peccei-Quinn symmetries in this article.

The vector-valued modular form Φ/η^{24} determines a part of d_{abc} [11, 12, 15–17]; 1-loop threshold corrections to a probe gauge group and a gravitational coupling depend on Φ/η^{24} , and hence d_{abc} 's on some of the coefficients $c_\gamma(\nu)$'s. It is known that d_{abc} 's would not be integers automatically, if Φ were a generic vector-valued modular form of weight $11 - \rho/2$ and in the Weil representation of $\text{Mp}_2\mathbb{Z}$ of Λ_S with integer-valued $c_\gamma(\nu)$ for small ν 's. So, the property (4.3) imposes non-trivial conditions on the BPS state multiplicities.

Appendix B.1.3 of [14] has worked out for a general W which part of the parameters d_{abc} 's are determined by Φ and how. Integrality of some of those d_{abc} 's is translated into [14]

$$\sum_{v \in W^\vee}^{(v,a) > 0} c_{[v]}(v^2/2) (v, r_1)(v, r_2)(v, r_3) \in 2\mathbb{Z}, \quad \forall r_{1,2,3} \in W, \forall a \in W \otimes \mathbb{R}. \quad (4.4)$$

We can use this condition to show that the Witten anomaly $\text{WA} \in \mathbb{Z}_2$ vanishes; to do this, let us use $a = -v_0$ for convenience, and set $r_1 = r_2 = r_3 = kv_0$. Then the condition reads

$$\sum_{v \in W^\vee}^{q_v > 0} n_{[v]}^H (2q_v)^3 \in 2\mathbb{Z}. \quad (4.5)$$

Since $2q_v$ is integral and $(2q_v)^3 \equiv 2q_v \pmod{2}$, this condition is nothing but the condition (2.3) which means the theory is free from Witten anomaly; only neutral states under $U(1)$'s contribute to (4.5) as stated earlier, which are states corresponding to v 's that are parallel to v_0 . They are indeed massless when the $SU(2)$ is enhanced.

4.2 An approach just using modularity of Φ

It is instructive to see how far one can go by using just the property that Φ is an integer-coefficient vector-valued modular form, without using the integrality of d_{abc} 's. We consider W and v_0 such that

$$W = \langle -2n \rangle, \quad n = 1, 2, \dots, \quad \text{and} \quad v_0 = 2/(2n)e, \quad (4.6)$$

where e is the generator of the rank-1 lattice W . The charge vector $v_0 = \frac{2}{2n}e$ satisfies the condition (3.1) with $k = n$ [14]. In this setup we will see that $\text{WA} \equiv 0 \pmod{2}$ if $n \not\equiv 1 \pmod{4}$. The integrality of d_{abc} 's turned out to be necessary, however, at least for some cases with $n \equiv 1 \pmod{4}$.

The coefficients of Φ/η^{24} have various linear relations among them. One way to obtain such a relation is to use the fact⁷

$$\sum_{\gamma \in G_S} \phi_\gamma^{(1/2)} \Phi_\gamma = -2[\phi^{(1/2)}]_{q^0} E_4 E_6, \quad \sum_{\gamma \in G_S} \phi_\gamma^{(9/2)} \Phi_\gamma = -2[\phi^{(9/2)}]_{q^0} E_4^2 E_6 \quad (4.7)$$

for a holomorphic vector-valued modular form $\phi^{(w)}$ of weight $w = 1/2$ or $9/2$ associated with the dual representation of $\text{Mp}_2\mathbb{Z}$. Here, E_4 and E_6 are Eisenstein series of weight-4 and 6, respectively. We assume that $\phi^{(w)}$ has a non-zero q^0 term only in the $\gamma = 0 \in G_S$

⁷For references and more explanations, see section 2.4 of [14], for example.

component, and the coefficient is denoted by $[\phi^{(w)}]_{q^0}$. For any given $\phi^{(w)}$, one linear relation on $c_\gamma(\nu)$'s is obtained by comparing coefficients of each term in the Fourier series expansion.

Here, we use $\phi^{(1/2)} = \theta_{W[-1]}$ and $\phi^{(9/2)} = (\partial^S)^2 \theta_{W[-1]}$, where ∂^S is Ramanujan-Serre derivative defined by

$$\partial^S F = \left(q \frac{\partial}{\partial q} - \frac{w_F}{12} E_2 \right) F, \quad (4.8)$$

where w_F is the weight of a modular form F . The linear relations (4.7) for the q^1 term are combined to yield⁸

$$\sum_{v=\ell e/(2n)} \left[\left(\frac{\ell^2}{4n} \right)^2 - \frac{1}{4} \left(\frac{\ell^2}{4n} \right) \right] c_{[\ell e/(2n)]}(-\ell^2/4n) - 3\delta_{n,1} = -3 \quad (4.9)$$

Only BPS states with $v \parallel v_0$ contribute to WA, as stated earlier. So

$$\text{WA} = \frac{1}{2} \sum_{\substack{-2 < -\ell^2/2n < 0 \\ -n \leq \ell \leq n, \ell \text{ odd}}} n_\ell \bmod 2, \quad (4.10)$$

where $n_\ell = c_{[v]}(v^2/2)$ with $v = (\ell/2n)e \in W^\vee$. We will use this omitted notation from now on. When n is even [resp. $n \equiv 3 \pmod{4}$], the relation (4.9) multiplied by $(8n^2)$ [resp. $(4n^2)$] and reduced modulo 2 implies that $\text{WA} = 0 \pmod{2}$. When $n \equiv 1 \pmod{4}$, the relation (4.9) does not yield useful information on whether $\text{WA} \equiv 0 \pmod{2}$ or not.

For example, when $n = 1$, there is no linear relation among n_γ 's, and any $n_1 \in \mathbb{Z}_{\geq -2}$ is allowed when $c_\gamma(\nu)$'s with small ν are required to be integers. In the cases with $n = 5$ and $n = 9$, on the other hand, there are two and three linear relations among n_γ 's, respectively. But they cannot be combined to yield $\sum_{0 < \ell, \text{odd}}^{\ell^2 \leq 4n} n_\ell \equiv 0 \pmod{2}$.

5 The case $\Lambda_S = \langle +28 \rangle$

Next we consider the case⁹ $\Lambda_S = \langle +28 \rangle$, because this turns out to be minimum degree rank 1 lattice where the theory is possibly anomalous. In this case

$$v_0 = 2(e'_0 + e'_4) + 14e/28 \in \widetilde{\Lambda}_S^\vee = (U[-1] \oplus \langle +28 \rangle)^\vee \quad (5.1)$$

satisfies

$$(v_0, v_0)/2 = -1/2, \quad 2v_0 = 0 \in G_S \quad (5.2)$$

and there can be an $SU(2)$ current algebra. Here e'_0, e'_4 and e are generators of $U[-1]$ and Λ_S^\vee , respectively. These satisfy $(e'_0, e'_4) = (e'_4, e'_0) = -1$, $(e, e) = 1/28$, and other pairs are 0. As for charged matters,

$$v_1 = v_0/2 = e'_0 + e'_4 + 7e/28 \quad (5.3)$$

⁸The $-3\delta_{n,1}$ term should be replaced by a little different expression when G_S has a non-trivial isotropic element γ with $n_\gamma \neq 0$. The arguments in the rest of this section remain valid even in that case.

⁹For any $n \in \mathbb{N}_{>0}$, a primitive embedding $\Lambda_S \hookrightarrow U^{\oplus 3} \oplus E_8[-1]^{\oplus 2}$ exists.

has Cartan charge $1/2$. This is the only relevant charged matter.¹⁰ At subspace in the Coulomb branch moduli space where these vectors and matters become massless, the effective field theory is an $SU(2)$ (and $U(1)$ graviphoton) gauge theory with n_7^H half-hyper doublets. Anomaly in this theory is equal to $n_7 \bmod 2$.

It is not known if there exists a Heterotic compactification with $\Lambda_S = \langle +28 \rangle$; no consistency condition is known to rule out such a Heterotic compactification, however. So we assume that there is one, and end up with a conclusion that modularity of some invariants of this compactification and the conditions (4.3), (A.8), (A.10) combined fail to predict that the $SU(2)$ Witten anomaly vanishes automatically. This is a clear indication that there are more theoretical consistency conditions in the worldsheet CFT than those properties we use in this analysis.

5.1 Modular forms Φ and Ψ

Suppose that there is a Heterotic compactification with $\Lambda_S = \langle +28 \rangle$. Then two vector-valued modular forms Φ and Ψ extract invariants of this string vacuum; Φ has appeared already in this article; the other one, Ψ , is of weight $25/2$, and also in the Weil representation of $Mp_2\mathbb{Z}$ associated with the lattice $\Lambda_S = \langle +28 \rangle$. The modular form Ψ as an invariant of a Heterotic-Type IIA dual vacuum appears manifestly already in [12]; see [14] for references and treatment of Ψ for a general Λ_S . The parameters d_{abc} are determined by the Fourier coefficients of both Φ and Ψ , and hence we need to deal with both¹¹ to exploit the condition (4.3).

The vector space of vector-valued modular forms of weight $21/2$ and Weil representation for $\langle +28 \rangle$ is of 14 dimensions, which we find by using the Riemann-Roch theorem [18]. There is one linear relation among the 15 parameters $n_\gamma = n_{-\gamma}$ with $\pm\gamma \in G_S/(-1)$; $G_S \cong \mathbb{Z}_{28}$, and the relation can be used to solve one of them in terms of the others:

$$\begin{aligned}
 n_9 &= 6n_0 + 3n_1 - 13n_2 - 3n_3 + 10n_4 - n_5 - 3n_6 + n_7 + 10n_8 \\
 &\quad - n_{10} - 3n_{11} + 4n_{12} + 3n_{13} - 3n_{14}.
 \end{aligned}
 \tag{5.4}$$

All the Fourier coefficients $c_\gamma(\nu)$ with $\nu \leq 1$ turn out¹² to be integers as long as all the 14 n_γ 's on the right-hand side are integers. The allowed region of n_γ 's is [14]

$$\begin{aligned}
 n_\gamma &\geq 0 \quad (\text{for } \gamma = 0, \dots, 13), \quad n_{14} \geq -2, \\
 \chi &\leq 2(\rho + 1) = 4,
 \end{aligned}
 \tag{5.5}$$

¹⁰Charged matter $v \in \widetilde{\Lambda}_S^\vee$ that contributes to Witten anomaly satisfies

$$v \parallel v_0, \quad -2 < (v, v) < 0, \quad q_v \in 1/2 + \mathbb{Z}.$$

v_1 and $-v_1$ are the only elements satisfying these conditions.

¹¹In the case of $\Lambda_S = U \oplus W$, some of d_{abc} 's depend only on the Fourier coefficients of Φ , not both. It just happens that the condition (4.3) for such d_{abc} 's is enough to guarantee that $WA \equiv 0 \bmod 2$.

¹²To work out those Fourier coefficients with $\nu \leq 1$ in terms of the 14 independent n_γ 's, and also to derive (5.4), (5.6), we used holomorphic Jacobi forms of index-14 and weight 4, 6, \dots , 12, and 16. See sections 2.4 and 3.1.2 of [14].

where χ is the following combination (the Euler number of the dual Type IIA Calabi-Yau)

$$\begin{aligned} \chi = & -300 + 112n_1 - 154n_2 + 182n_4 - 42n_6 + 16n_7 + 210n_8 \\ & - 14n_{10} + 70n_{12} + 112n_{13} - 40n_{14}. \end{aligned} \tag{5.6}$$

This allowed range is non-empty. For example, $n_0 = -2$, $n_{12} = 3$, $n_7 = n_{10} = n_*$ with $0 \leq n_* \leq 47$, and $n_\gamma = 0$ for all other γ 's.

Similarly, the vector space of vector-valued modular forms of weight $25/2$ and Weil representation for $\langle +28 \rangle$ is of 17 dimensions, which we compute by using the Riemann-Roch theorem. It turns out that the Fourier coefficients $c_\gamma^\Psi(\nu)$'s in

$$\frac{\Psi}{\eta^{24}} = \sum_{\gamma \in G_S} c_\gamma^\Psi(\nu) q^\nu \tag{5.7}$$

are parametrized by the 15 $c_\gamma^\Psi(\nu) = c_{-\gamma}^\Psi(\nu)$'s with $-1 \leq \nu < 0$, and two more, $c_{\gamma=0}(0)$ and $c_{\gamma=1}(1/56)$; we have seen that¹³ all of $c_\gamma^\Psi(\nu)$ with $\nu \leq 1$ are integers as long as all the 17 independent parameters are integers. Those 17 extra parameters of the Heterotic vacuum in question cannot be arbitrary integers, however. They are subject to the conditions [14]

$$d_\gamma(\nu) \in 12\mathbb{Z}, \quad m_0 = m_1 = m_2 = m_3 = m_8 = m_{11} = m_{13} = 0, \tag{5.8}$$

where $d_\gamma(\nu)$'s are the Fourier coefficients of $(\Phi E_2 - \Psi)/\eta^{24}$, and the leading coefficients ($d_\gamma(\nu)$'s with $-1 \leq \nu < 0$) are denoted by m_γ . $d_0(0)$ and $d_1(1/56)$ are simply denoted by d_0 and d_1 in the following.

5.2 The analysis

The prepotential \mathcal{F} (4.1) and the gravitational coupling F_1 in 4d

$$F_1 = 24s + (c_2)_a t^a + \mathcal{O}(e^{2\pi i s}, e^{2\pi i t}) \tag{5.9}$$

are determined by Φ and Ψ ; a survey of the procedure of computation applicable to the $\Lambda_S = \langle +28 \rangle$ case is available in [14]. The result in an appropriate basis is

$$\begin{aligned} (c_2)_1 = & (-134n_0 - 118n_1 + 46n_2 - 30n_3 - 126n_4 - 8n_5 + 22n_6 - 10n_7 \\ & - 128n_8 + 8n_{10} + 14n_{11} - 38n_{12} - 36n_{13} + 26n_{14} + 6l) \\ d_{111} = & (-233n_0 - 157n_1 + 97n_2 - 57n_3 - 237n_4 - 20n_5 + 37n_6 - 19n_7 \\ & - 236n_8 + 14n_{10} + 41n_{11} - 65n_{12} - 30n_{13} + 50n_{14} + 21l) \end{aligned} \tag{5.10}$$

¹³Similarly to the case of Φ , now we used holomorphic Jacobi forms of index-14 and weight 4, 6, \dots , 10, and 14.

where¹⁴

$$l = (-3d_0/2 - d_1 + 7m_4 + 5m_5 + 3m_6 + m_7 + 2m_9 + m_{10} - m_{12})/12. \quad (5.11)$$

The property (4.3) implies¹⁵ that $d_0 \in 24\mathbb{Z}$, not just $d_0 \in 12\mathbb{Z}$ as in (5.8). This implies that the probe gauge group is free from the SU(2) Witten anomaly (see discussions in [14]), but n_7 can be an arbitrary integer, so the SU(2) gauge group for the v_0 in (5.1) may have non-vanishing Witten anomaly.

If this Heterotic compactification has a Type IIA dual, and the dual is in a geometric phase, then it has a property that (see discussion around (A.10))

$$(c_2)_1 + 2d_{111} \in 12\mathbb{Z}. \quad (5.12)$$

This only leads to

$$n_{14} \in 2\mathbb{Z}. \quad (5.13)$$

The SU(2) gauge group associated with v_0 in (5.1) may have only an even number of half-hypermultiplets in the adjoint representation. Consequently there is no restriction on n_7 other than (5.5).

The modular nature of the elliptic genus was enough in proving that the low-energy gauge theory of a Heterotic compactification is free from perturbative anomalies [1, 2]. By examining the SU(2) Witten anomaly in the low-energy effective theory, however, we found that there is more consistency conditions in the internal space worldsheet SCFT other than the modularity and integrality¹⁶ of the invariants Φ and Ψ and the unphysical discrete Peccei-Quinn shift symmetries $t^a \rightarrow t^a + \delta_d^a$.

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¹⁴By following the arguments in [14], one can see that the extra parameters $d_\gamma(\nu)$'s of Ψ determine $(c_2)_a$ and d_{abc} only through at most ρ independent linear combinations. In the case of a lattice Λ_S with a given rank ρ and larger G_S , there tends to be more independent $d_\gamma(\nu)$'s than ρ , as in (5.10). We have multiple (mutually non-exclusive) interpretations for what is happening. In the language of dual Type IIA string theory, (a) there may be multiple Calabi-Yau three-folds in a common diffeomorphism class that are distinct in their symplectic/complex structure, (b) there may be multiple different hypermultiplet-moduli tunings of a Calabi-Yau three-fold X so that a singularity develops along the same curve class, and (c) there are more theoretical constraints on $d_\gamma(\nu)$'s than those discussed in [14].

¹⁵The parameters $(c_2)_1$ and d_{111} have also been worked out in terms of the coefficients of Φ and Ψ in the case of $\Lambda_S = \langle +2 \rangle$ [14] and $\Lambda_S = U$. The property (4.3) implies $d_0/24 \in \mathbb{Z}$, and hence the absence of the SU(2) Witten anomaly in the probe gauge group also in the case of $\Lambda_S = \langle +2 \rangle$ and U .

¹⁶To be rigorous, we have only exploited the integrality of $c_\gamma(\nu)$'s and $c_\gamma^\Psi(\nu)$'s with small ν 's in this article. Note also, the modular invariance of a Heterotic string compactification may be more than the modularity of that of Φ and Ψ .

A Consequences of the discrete Peccei-Quinn shift symmetries

Here, we think of a Heterotic compactification described in section 3. The lattice Λ_S is not necessarily of the form $U \oplus W$. We neither assume that the $(c, \tilde{c}) = (22, 9)$ CFT has a Lagrangian-based description in the Heterotic language, nor assume that the Type IIA dual is in a geometric phase. By exploiting the fact that the Γ_S action on $D(\tilde{\Lambda}_S)$ should be unphysical, we will derive (4.3) (cf. [19–22]).

Consider changing the Narain moduli t^a continuously to $(t')^a = t^a + \delta_d^a$ for some $d = 1, \dots, \rho$. There is an isometry $g_{(d)} \in \text{Isom}(\tilde{\Lambda}_S)$ so that $\mathcal{U}(t') = g_{(d)} \cdot \mathcal{U}(t)$, so $\langle v, \mathcal{U}(t) \rangle = \langle v', \mathcal{U}(t') \rangle$ for $v' := (g_{(d)}^{-1})^T \cdot v$ for any $v \in \tilde{\Lambda}_S^\vee$; it is straightforward to verify that this $g_{(d)}$ is in the subgroup $\Gamma_S \subset \text{Isom}(\tilde{\Lambda}_S)$. Thus, we should have arrived at the original vacuum at the end. The BPS mass spectrum should be the same before and after this deformation.

States from F1 Heterotic string have purely electric charges under the $(2 + \rho)$ U(1) vector fields in the 4d effective theory; the electric charges (simply referred to as charges outside of this appendix) v fill the free abelian group $\Lambda_{\text{el}} = \tilde{\Lambda}_S^\vee$. States with magnetic charges under the $(2 + \rho)$ U(1) vector fields arise from NS5-branes and objects of similar kinds; the magnetic charges m fill the free abelian group $\Lambda_{\text{mg}} = \Lambda_S$. The Dirac-Zwanziger quantization condition is satisfied, with a canonical symplectic form introduced on the free abelian group $\Lambda_{\text{el}} \oplus \Lambda_{\text{mg}} = \tilde{\Lambda}_S^\vee \oplus \Lambda_S$ [11]. So, the transformation $g_{(d)}$ on $\tilde{\Lambda}_S^\vee = \Lambda_{\text{el}}$ should be lifted to a consistent action $\tilde{g}_{(d)}$ on $\Lambda_{\text{el}} \oplus \Lambda_{\text{mg}}$, and the spectrum of the central charges remain the same before and after the deformations.

The central charge¹⁷ of the 4d $\mathcal{N} = 2$ supersymmetry algebra is given by (e.g., [20, 21])

$$Z = e^{K/2}(v_I X^I + m^I F_I), \quad K = -\ln(i(X^I \bar{F}_I - \bar{X}^I F_I)), \quad (\text{A.1})$$

$$X^I = X^0(1, t^a, (t, t)/2 + \mathcal{O}(e^{2\pi i s})), \quad (\text{A.2})$$

$$F_I = X^0(2\mathcal{F} - (s\partial_s + t\partial_t)\mathcal{F}, \partial_{t^a}\mathcal{F}, -s), \quad (\text{A.3})$$

where I, J range from $0, 1, \dots, \rho, \rho + 1$; $a, b = 1, \dots, \rho$. We use a prepotential of the form

$$\mathcal{F} = \frac{s}{2}(t, t) + \frac{d_{abc}}{6}t^a t^b t^c - \frac{a_{ab}}{2}t^a t^b - b_a t^a - \frac{\zeta(3)}{(2\pi i)^3} \frac{\chi}{2} + \mathcal{O}(e^{2\pi i s}, e^{2\pi i t}).$$

A rationale for assuming this form of \mathcal{F} even when the Type IIA dual is not necessarily associated with a geometric phase is found toward the end of this appendix.

The lift $\tilde{g}_{(d)}$ acting on $(X^A, F_A)^T$ should be

$$\tilde{g}_{(d)} = \begin{pmatrix} [g_{(d)}]_B^A & 0^{AB} \\ W_{AB} & [(g_{(d)}^{-1})^T]_A^B \end{pmatrix}, \quad (\text{A.4})$$

with

$$W_{AB} = \begin{pmatrix} -\left(2b_d + \frac{d_{ddd}}{6}\right) - \left(a_{db} + \frac{d_{ddb}}{2}\right) & 0 \\ -a_{ad} + \frac{d_{add}}{2} & d_{adb} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.5})$$

¹⁷The mass of BPS states of charge $Q = (v_I, m^I)$ is given by $m = |Z|/\sqrt{G_N}$, where G_N is the 4d Newton constant.

For $(\tilde{g}_{(d)}^{-1})^T$ to be interpreted as relabeling of electric and magnetic charges $(v_A, m^A) \in \Lambda_{\text{el}} \oplus \Lambda_{\text{mg}}$,

$$d_{abd} \in \mathbb{Z} \quad \forall a, b, d \in 1, \dots, \rho, \quad (\text{A.6})$$

and the parameters a_{ab} and b_a should be chosen so that¹⁸

$$a_{ab} \in \left(\frac{d_{abb}}{2} + \mathbb{Z} \right) \cap \left(\frac{d_{bba}}{2} + \mathbb{Z} \right), \quad 24b_a + 2d_{aaa} \in 12\mathbb{Z}. \quad (\text{A.7})$$

For the fractional part of a_{ab} to be determined consistently, parameters of the prepotential should also satisfy

$$d_{abb} + d_{aab} \equiv 0 \pmod{2}, \quad \forall a, b \in 1, \dots, \rho. \quad (\text{A.8})$$

Therefore, both (4.3) and (A.8) must be satisfied in a Heterotic compactification under consideration.

In the matching calculation using the Heterotic 1-loop threshold corrections to the gauge coupling of a probe level-1 gauge group,¹⁹ the parameters d_{abc} and the 1-loop corrected gauge kinetic function $f_{(\mathcal{R})} = (s + d'_a t^a)$ of the probe gauge group are determined only modulo shift $d_{abc} \rightarrow d_{abc} + [(\delta n_a)C_{bc} + \text{cyclic}]$ and $f_{(\mathcal{R})} \rightarrow f_{(\mathcal{R})} + (\delta n_a)t^a$ (cf [23]).²⁰ Because the parameters d'_a are regarded as part of the cubic term coefficients of the prepotential in the Coulomb branch of the probe gauge group, it follows that $d'_a \in \mathbb{Z}$. This indicates — based on the presentation in section 3.1.3 of [14] — that the property (4.3) is equivalent to the integrality of $d_{abc}^{(P)}$ in the cubic polynomial $P_3(t) =: d_{abc}^{(P)} t^a t^b t^c$ obtained in a Heterotic 1-loop computation. This subtle chain of logic is implicit in sections 4.1 and section 5.

A rationale for the form of prepotential (4.1) is the following. Think of the dual Type IIA compactification, and then the third derivative of the prepotential \mathcal{F} should be equal to the three-point function of the corresponding A-model, regardless of whether the Type IIA compactification is associated with a geometric phase. Due to the unphysical nature of Peccei-Quinn shift $t_X^i \rightarrow t_X^i + \delta_j^i$, where $(t_X^{i=1, \dots, \rho+1}) = (t^{i=a=1, \dots, \rho}, s)$ are the special (flat) coordinates, the three point functions should be of the form of

$$\kappa_{ijk} + \sum_{\beta \neq 0 \in L} K_{\beta,ijk} e^{2\pi i \langle \beta, t_X \rangle} \quad (\text{A.9})$$

for some rank- $(\rho+1)$ free abelian group L ; κ_{ijk} and $K_{\beta,ijk}$ are constant parameters. Thus, \mathcal{F} is a polynomial at most cubic in the special coordinates t_X^i , besides the exponential terms as in (4.1). Presumably it is possible to translate this argument into the language of Heterotic string.

¹⁸Difference $\Delta a_{ab} \in \mathbb{Z}$ and $\Delta b_a \in \mathbb{Z}$ is absorbed by a relabeling of the charges of the form $(v'_I, m'^I) = (v_I + w_{IJ} m^J, m^I)$ with $w_{IJ} \in \mathbb{Z}$ and $w_{IJ} = w_{JI}$.

¹⁹In this paragraph, we use notations, jargons, and detailed discussions in [14] without due amount of explanations. Curious readers are referred to [14].

²⁰ C_{ab} is the intersection matrix of Λ_S presented in an integral basis $\{e_a\}$ of Λ_S .

The prepotential (4.1) is not in the most general form one expects from the argument above. The structure of the s -linear term in the cubic part $(\kappa_{ijk}/6)t_X^i t_X^j t_X^k$ in \mathcal{F} is due to that of the tree-level gauge kinetic function of the $(\rho + 2)$ 4d vector fields. The absence of s^2 terms and s^3 terms in the cubic part, as well as the s -independence of the quadratic part $-(a_{ij}/2)t_X^i t_X^j$ and the linear part $-b_i t_X^i$ in (4.1) is explained by the fact that the combination $v_I X^I$ in an appropriate frame should be proportional to the right-moving momenta in the Heterotic string. When a Type IIA dual exists, and is in a geometric phase, all of those structures are translated into the following properties: (i) the fiber K3 class divisor D_s of a Calabi-Yau three-fold with a regular K3-fibration satisfies $D_s^2 = 0$, and (ii) $D_s \cdot D_a \cdot D_b = C_{ab}$ [24]; (iii) we know in a geometric phase [19, 22, 25] that $b_{(\rho+1)} + \mathbb{Z} = (24)^{-1} \int_X c_2(TX) D_s + \mathbb{Z} = \mathbb{Z}$ and $a_{a(\rho+1)} + \mathbb{Z} = 2^{-1} D_a \cdot D_s \cdot D_s + \mathbb{Z} = \mathbb{Z}$ [24].

If a Type IIA dual exists, and is in a geometric phase, the property (4.3) is a trivial statement that all the triple divisor intersection numbers are integers. The property (A.8) is also known as one of the necessary conditions for a real 6-manifold [26]. In a geometric phase, it is known [19, 27] that the parameters $b_a + \mathbb{Z}$ are equal to $(c_2)_a/24 + \mathbb{Z}$ appearing in the gravitational coupling (5.9). So the second property in (A.7) is read as

$$(c_2)_a + 2d_{aaa} \in 12\mathbb{Z}, \tag{A.10}$$

which is also one of the necessary conditions for a real 6-manifold [26]. The authors have not been able to build an argument relating the fractional part of the parameter b_a in \mathcal{F} and $(c_2)_a/24$ for the coefficient $(c_2)_a$ in (5.9) without assuming that a Type IIA compactification is in a geometric phase.^{21,22} Despite this caveat, it is tempting to include (A.10) as one of the properties of a Heterotic compactification introduced in section 3.

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²¹We mean to use only the following (i–iii) as assumptions: (i) the internal “space” is described by $N = (2, 2)$ compact unitary SCFT with $(c, \tilde{c}) = (9, 9)$, (ii) all the NS-NS sector states have integral charges under the left-mover $U(1)$ and the right-mover $U(1)$, and (iii) there exists one spectral flow in the left mover, and also one in the right mover.

²²The set of conditions (4.3), (A.8), (A.10) is not just a necessary condition, but also a sufficient condition for a real 6-manifold to exist [26]. So, filling this logical gap is also vital in claiming that all the Type IIA compactifications in the previous footnote are associated with some Calabi-Yau threefold.

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