Elliptic blowup equations for 6d SCFTs. Part III. E-strings, M-strings and chains

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ABSTRACT: We establish the elliptic blowup equations for E-strings and M-strings and solve elliptic genera and refined BPS invariants from them. Such elliptic blowup equations can be derived from a path integral interpretation. We provide toric hypersurface construction for the Calabi-Yau geometries of M-strings and those of E-strings with up to three mass parameters turned on, as well as an approach to derive the perturbative prepotential directly from the local description of the Calabi-Yau threefolds. We also demonstrate how to systematically obtain blowup equations for all rank one 5d SCFTs from E-string by blow-down operations. Finally, we present blowup equations for E–M and M string chains.

Contents

1 Introduction

Non-Lagrangian superconformal theories without gravity in various dimensions are among the most remarkable discoveries made in the framework of string theory. Four dimensional examples have been realised in limits of the Coulomb branch of $N = 2$ supersymmetric theories called Argyres-Douglas loci where electric — and magnetic charged states become

simultaneously massless [1–3]. An interesting sequence of superconformal rank one theories with E_n, $n = 6, 7, 8$, extended flavour symmetries has been described in [4, 5] that arises in conformal limits of Seiberg-Witten geometries from theories which do have a Lagrangian description [3]. In [6–8] it was understood how Seiberg-Witten geometries emerge in Type II String compactifications in non-compact limits of Calabi-Yau 3-folds M that decouple gravity. The local Calabi-Yau geometry from which the E_n theories of [4, 5] derive was discovered in [9, 10] in the context of F-theory. It is a local elliptic fibred surface with 12 Kodaira I_1 singularities over a base $\mathbb{P}^1$ with self intersection (-1). Shrinking this surface in an F-theory compactification gives rise to the six dimensional E-string theory with $(1, 0)$ super conformal symmetry and an affine E_8 symmetry. The sequence of 5d E_n theories[1] is associated to M-theory on non-compact Calabi-Yau constructed as the canonical line bundle over del Pezzo surfaces. The latter as well as their 4d limit can all be obtained by blowing down the E-string geometry. A non-critical string with $(2, 0)$ super conformal symmetry arises if the elliptic fibre is smooth over $\mathbb{P}^1$ with self intersection (-2).

In the absence of Lagrangian descriptions the BPS spectrum of these theories is of particular interest and certain BPS indices for the E-string compactified on S^1 have been obtained in [10], using mirror symmetry and the M-theory interpretation of 5d BPS invariants [11]. Due to an additional $U(1)_R$ symmetry in the local limit, the BPS indices can be refined to the actual count of BPS states whose quantum numbers are described by the central charge, in the geometric context identified with the class $\beta \in H_2(M, \mathbb{Z})$, their mass as well as their representation w.r.t. the 5d little group $su(2)_l \times su(2)_r$.

Four methods have been developed to calculate the refined BPS spectrum for the S^1 compactifications of E- and M-string as well as a class of interacting E- and M-string theories which emerge in elliptic fibrations over chains of intersecting base curves with self intersection (-1) and (-2), which was described in [12, 13] and referred to as E–M string chains.

The B-model topological string approach calculates the BPS indices by solving the all genus topological string amplitude using the holomorphic anomaly equations [14]. For elliptic fibered Calabi-Yau manifolds a more powerful version of the holomorphic anomaly equations can be solved using a modular bootstrap ansatz in terms of meromorphic Jacobi forms [15]. In the local models this can be refined and generalised to Jacobi forms with many elliptic elements [16–18] and the ambiguities in the ansatz can be fixed by geometric vanishing conditions on the BPS invariants [16–18]. Another method solves the E-string by a modification of the elliptic genera of auxiliary two dimensional SO(16) quiver models [19] for each winding of the base $\mathbb{P}^1$. Different quiver descriptions whose elliptic genera solve the E-string have been found in [20]. The third method is to use the topological vertex with identified legs. It was applied to the M-string in [21] and to the E-string in [22].

The purpose of this work is to extend the fourth method namely the elliptic blowup equations [16, 23, 24] (based on [25–27]) to the E- and M-string and the E–M string chains. The elliptic blowup equations extend a method developed by Nakajima and Yoshioka [28–

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[1]This sequence can be enlarged to include all the other rank one SCFTs with lower rank flavor symmetries which correspond to lower degree del Pezzo surfaces.

30] (see later development [31, 32]) to solve the $N = 2$ supersymmetric gauge theory partitions function by blowup equations,[2] that emerge from a relation between localisation results before and after blowing up a $\mathbb{P}^1$ in space time.

The main results of the paper are the explicit form of the unity elliptic blowup equations for the M-string (3.26) the E-string (3.34) as well as the E–M string chains (3.52). Only for the E-string one has vanishing blowup equations given in (3.37). For the E- and M-string we discuss their solutions and show that the elliptic blowup equations can be solved from the set of all possible $\underline{r}$-vectors using for example genus zero BPS invariants at base degree one as input. The information about the input data that are sufficient to solve the blowup equations is encoded in (4.13). For the M- and E-string the minimal inputs for given $\underline{r}$-vectors are summarised in (4.19) and (4.26) respectively. We also outline a new method to solve the blowup equations using the Weyl orbit expansion which respects the Weyl symmetries of the E-string flavour group in section 4.2.

The elliptic generalisation of the blowup equations of Nakajima and Yoshioka have not been derived so far. We give a physical derivation of the elliptic blowup equations using the path integral approach to the partition function of the 6d SCFT on $M_6 = T^2 \times_{\epsilon_1, \epsilon_2} \widehat{\mathbb{C}^2}$. Here $\widehat{\mathbb{C}^2}$ is the blown up target space used in [28] to obtain the original $N = 2$ 4d gauge theoretical blowup equations. A virtue of our derivation presented in section 6 is that it gives a natural explanation of how the Θ-functions in the elliptic blowup equations arise in the path integral. As mentioned above the solutions for the BPS states of the E-string encodes the solutions for the BPS states of the 5d E_n theories that are geometrically engineered on local del Pezzo surfaces by successive blow down limits. We follow this approach to derive blowup equations for refined invariants on the del Pezzo surfaces.

This paper is organized as follows: in section 2, we give the geometric construction of elliptic non-compact Calabi-Yau threefolds associated to E-string and M-string theories. In section 3, we present the elliptic blowup equations for E-strings, M-strings and their higher rank theories known as E–M string chain and M string chain. We also demonstrate how the blowup equations for the chains reduce to those of E-strings and M-strings themselves. In section 4, we show how to solve elliptic genera of E-, M-strings from blowup equations. We use two methods which are refined BPS expansion in section 4.1 and the Weyl orbit expansion in section 4.2. In section 5, we derive the blowup equations for local del Pezzo surfaces from those of E-strings. In section 6, we give a derivation of the elliptic blowup equations of E-strings and M-strings from path integral interpretation. The readers only interested in the form of the blowup equations for E-, M-strings can directly go to equations (3.26), (3.34) and (3.37).

2 Geometry of E-, M-string theories

Here we construct the embeddings of the non-compact Calabi-Yau threefolds associated to E-, M-string theories on T^2 into compact Calabi-Yau threefolds, which are hypersurfaces in toric varieties. There are two purposes of this construction. The first purpose is the

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[2]It was already noticed in [33, 34] that the blowup formulas of Donaldson invariants [35–37] could be used to solve the prepotential of Seiberg-Witten theory.

semiclassical partition function encoding classical geometric data like triple intersection numbers of divisors, which are needed for the formulation of the blowup equations, can be readily computed from such a compact construction. Secondly and more importantly, the construction of compact Calabi-Yau threefolds as hypersurfaces in toric varieties allows us to compute the instanton corrected prepotential, which as we will see in section 4 is needed as extra input in order to extract refined BPS invariants of E-, M-string theories on T^2 from blowup equations. We would like to point out that in the case of E-string, both purposes can be realised by using the mirror curve constructed by Sakai [38–40]. But the method of toric hypersurface construction presented here can be applied to generic rank one 6d SCFTs [41] as well.

The first purpose needs a little more explanation. The semiclassical partition function has the form

$$Z^{\text{cls}} = \exp\left(\frac{F_{(0,0)}}{\epsilon_1\epsilon_2} + F_{(1,0)} + \frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1\epsilon_2} F_{(0,1)}\right) \tag{2.1}$$

with[3]

$$F_{(0,0)} = \frac{1}{6}\sum_{i,j,k}\kappa_{ijk}t_it_jt_k, \quad F_{(1,0)} = \sum_i b_i^{\text{GV}}t_i, \quad F_{(0,1)} = \sum_i b_i^{\text{NS}}t_i. \tag{2.2}$$

Here t_i are Kähler moduli of curve classes Σ_i. The coefficients κ_{ijk} of the semiclassical prepotential $F_{(0,0)}$ are the triple intersection numbers of divisors Γ_i Poincaré dual to the curves Σ_i. They are usually defined rigorously and computed on a compact threefold $\widetilde{X}$ as integration of $(1,1)$-forms ω_i dual to Γ_i

$$\kappa_{ijk} = \int_{\widetilde{X}} \omega_i \wedge \omega_j \wedge \omega_k, \tag{2.3}$$

which is the reason that for the non-compact CY3 associated to E-, M-string theories on T^2 we would like to construct a compact embedding, compute $F_{(0,0)}$ there and then do a proper decompactification limit. We hasten to clarify that the $F_{(0,0)}$ of a 6d SCFT, at least its relevant components, does not depend on the way the associated non-compact CY3 is embedded into a compact one, which may not be unique. The non-compact CY3 can locally be seen as the neighborhood of a union of connected compact surfaces $S = \cup_i S_i$. It is important to distinguish between curves with non-trivial and vanishing intersection numbers with S. The curves in the first category are either fixed components of some surfaces or intersect with them, while the curves in the second category, which we will call *free curves*, can be moved freely away from S. We call the corresponding Kähler moduli the "true" moduli and the mass parameters respectively, as the latter usually correspond to flavor masses in the field theory engineered by the geometry. The *relevant* components of $F_{(0,0)}$ are those involving at least one true modulus, and the terms with only mass parameters decouple in all types of calculations. We will be only concerned with the relevant $F_{(0,0)}$, and we will give an argument in section 2.3 that it is inherently the property of the non-compact CY3 and does not depend on the way of its compact embedding. Finally we comment that the linear coefficients $b_i^{\text{GV}}, b_i^{\text{NS}}$ in (2.2) also encode

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[3]In general the perturbative $F_{(0,0)}$ can also have quadratic and linear terms. However, the quadratic terms are ambiguous for local geometries, and the linear terms do not contribute to blowup equations.

some topological information. Although they can be computed once a compact embedding is constructed, they are most easily fixed by consistency condition of blowup equations as we will see in section 3.

The idea of constructing the compact embedding is very simple. The E-, M-string theories are 6d SCFTs with no gauge symmetry but non-trivial flavor symmetry E_8, $SU(2)$, respectively. They also have a tensor multiplet with the labelling $\mathfrak{n} = 1, 2$, which is the coefficient of the Dirac pairing of the non-critical strings coupled to the tensor multiplet. Each of the theories can be constructed by F-theory compactified on a non-compact Calabi-Yau threefold, which are roughly speaking elliptic fibrations over $\mathcal{O}(-1)$ and $\mathcal{O}(-2)$ bundles of $\mathbb{P}^1$ respectively. It is natural to regard the two non-compact Calabi-Yau threefolds as the decompactification limit of the familiar compact Calab-Yau threefolds, namely the ellipic fibration over $\mathbb{F}_1, \mathbb{F}_2$, along the direction of the (0)-curve in the base [42]. The flavor symmetry can then be realised as weakly coupled gauge symmetry corresponding to singularity over the (0)-curve, which becomes ungauged in the decompactification limit. We will illustrate this idea in the following subsections, and write down the compact Calabi-Yau as hypersurfaces in toric varieties.

2.1 M-string geometry

The geometry X associated to the M-string theory was constructed concretely in [43–45]. It is locally the neighborhood of a complex surface S, which can be described as follows. We start with the Hirzebruch surface $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ blown up at two generic points and denote the resulting surface by $\mathbb{F}_0^{1+1}$. We take the independent curves in $\mathbb{F}_0^{1+1}$ to be the $\mathbb{P}^1$ base e and the $\mathbb{P}^1$ fiber f of the Hirzebruch surface as well as the two exceptional curves x, y. Their mutual intersection numbers inside the surface are

$$e^2 = f^2 = 0, \ x^2 = y^2 = -1, \ e.f = 1, \ e.x = e.y = f.x = f.y = x.y = 0. \qquad (2.4)$$

The surface S is obtained by gluing two (-1)-curves $e - x$ and $e - y$ together. It can be graphically represented as
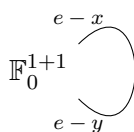
The canonical class of the self-glued surface S is [43]

$$K_S = K_{\mathbb{F}_0^{1+1}} + (e - x) + (e - y) = K_{\mathbb{F}_0} + x + y + (e - x) + (e - y) = -2f. \qquad (2.5)$$

One can then use the adjunction formula

$$2g(C) - 2 = C.K_S + C.C \qquad (2.6)$$

to compute the genera of curves. After the gluing, the curve class f becomes a genus one curve, and it is identified with the elliptic fiber of X, while the (0)-curve e remains rational

	$C.S$	$(C.C)_S$	$g(C)$
e	-2	0	0
f	0	0	1
$x+y$	0	-2	0

Table 1. Curve classes in the M-string geometry.

and it is the base of the elliptic fibration. On the other hand, neither of the two exceptional curves x, y alone is irreducible as they have fractional genus and they merge into a single irreducible rational (-2)-curve $x + y$. We tabulate the independent curve classes of S in table 1. We denote the Kähler moduli of these curve classes by

$$t_b = \mathrm{iVol}(e), \quad \tau = \mathrm{iVol}(f), \quad 2m = \mathrm{iVol}(x+y). \tag{2.7}$$

Inside the Calabi-Yau X, the intersection numbers of these curves with the surface S can be computed with the formula

$$C.S = (C.K_S)_S, \tag{2.8}$$

and the results are also given in table 1. We find that only t_b is a true Kähler modulus, while both τ and m are mass parameters. We will sometimes call m the flavor mass, as it is the holonomy of the SU(2) flavor symmetry on a circle.

2.1.1 Toric hypersurface construction

The M-string geometry X can be embedded into a compact CY3 $\widetilde{X}$, which is the anti-canonical divisor of a toric variety $\widetilde{Y}$. The toric data of $\widetilde{Y}$ including toric divisors, Mori cone generators, and their intersection numbers are given in table 2. This is based on the well-known toric variety whose anti-canonical divisor is the elliptic fibration over $\mathbb{F}_2$ [42], where in addition, we insert the exceptional divisor D'_v that results from blowing up the intersection point of the three divisors D_x, D_y, D_v (as well as the anti-canonical divisor). This operation effectively creates a resolved A_1 singularity over the (0)-curve in the base [46], and it is equivalent to constructing an A_1 type toric top over the (0)-curve à la [47, 48]. The non-compact CY3 X is obtained by decompactifying along the $l_{(\mathrm{de})} = l^{(2)}$ curve, which is the (0)-curve in the base. We identify the base curve l_b, the elliptic fiber l_f, and the curve associated to the mass paramter m to be

$$l_b = l^{(3)}, l_f = l^{(1)} + 3l^{(3)} + 3l^{(4)}, \; l_m = l^{(3)} + l^{(4)}. \tag{2.9}$$

The intersection ring of the compact CY3 $\widetilde{X}$ can be computed to be

$$
\begin{aligned}
\mathcal{R} =\ & 8J_1^3 + 4J_2J_1^2 + 26J_3J_1^2 + 24J_4J_1^2 + 2J_2^2J_1 + 80J_3^2J_1 + 68J_4^2J_1 + 13J_2J_3J_1 \\
& + 12J_2J_4J_1 + 74J_3J_4J_1 + 242J_3^3 + 189J_4^3 + 40J_2J_3^2 + 34J_2J_4^2 + 206J_3J_4^2 + 6J_2^2J_3 \\
& + 6J_2^2J_4 + 224J_3^2J_4 + 37J_2J_3J_4 \,,
\end{aligned} \tag{2.10}
$$

from which we can write down the perturbative prepotential. The genus 0 Gopakumar-Vafa invariants $n_0^{d=(d_b,d_f,d_m,d_{\mathrm{de}})}$ of $\widetilde{X}$ can also be computed with techniques of mirror symmetry,

	ν_i^*				$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$
D_0	0	0	0	0	0	0	0	-2
D_x	-1	0	0	0	-1	0	0	1
D_y	0	-1	0	0	0	0	0	1
D_z	2	3	0	0	1	-2	0	0
D_u	2	3	1	0	0	0	1	-1
S	2	3	0	-1	0	1	-2	2
D_v	2	3	-1	-2	-3	0	1	0
D'_v	1	2	-1	-2	3	0	0	-1
D_t	2	3	0	1	0	1	0	0

Table 2. Toric divisors and Mori cone generators of the toric variety whose anti-canonical divisor is the compact embedding of massive M-string geometry.

and the GV invariants of X are those of $\widetilde{X}$ with $d_{\text{de}} = 0$. They agree with the results of refined topological vertex, with one exception. The invariant[4] $n_0^{(1,-1,0)} = 1$ is missing, while we have a new invariant $n_0^{(-1,1,0)} = 1$. It means that the non-compact geometry we constructed through the toric method is slightly off in the Mori cone and we have to bring it to the correct Mori cone chamber by flopping the curve $-l_b + l_m$.

In addition, when we perform the decompactification limit $t_2 \to i\infty$, the only B-period which remains finite is

$$\Pi_B = \frac{\partial F_{(0,0)}}{\partial t_2} - 2\frac{\partial F_{(0,0)}}{\partial t_3} + 2\frac{\partial F_{(0,0)}}{\partial t_4} = t_1 t_3 + 2t_3^2 + t_3 t_4. \tag{2.11}$$

In terms of the Kähler moduli of the non-compact geometry it reads

$$\Pi_B = -2\frac{\partial F_{(0,0)}}{\partial t_b} + \frac{\partial F_{(0,0)}}{\partial t_{\text{de}}} = t_b^2 + t_b(\tau - 2m). \tag{2.12}$$

The relevant perturbative prepotential of the massive M-string geometry can be computed by integrating this B-period. After taking into account the flop of $-l_b + l_m$ by adding $(t_b - m)^3$, we find

$$F_{(0,0)} = -\frac{1}{4}t_b^2\tau + \frac{1}{2}t_b m^2. \tag{2.13}$$

2.2 E-string geometry

The non-compact CY3 associated to the E-string theory is locally the neighborhood of a compact surface S, which is $\mathbb{P}^2$ blown up at nine points, also known as the half K3 surface. The independent curve classes include the hyperplane class h of $\mathbb{P}^2$, and the nine exceptional curves x_i ($i = 1, \ldots, 9$). Their intersection number within S are

$$h^2 = 1, \quad x_i^2 = -1, \quad h.x_i = x_i.x_j = 0, \quad i \neq j, \tag{2.14}$$

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[4]The apperance of negative curve degree is because l_b, l_f, l_m do not constitute the Mori cone basis.

while they intersect with S by

$$h.S = -3, \quad x_i.S = -1. \tag{2.15}$$

The half K3 surface is an elliptic rational surface. The base of the elliptic fibration can be chosen as $b = x_9$, while the elliptic fiber is the anti-canonical class

$$f = -K_S = 3h - x_1 - \ldots - x_9. \tag{2.16}$$

This is a free curve with trivial intersection of $\mathcal{S}$. There are eight other linearly independent free curves, which we choose to be

$$\alpha_i = x_i - x_{i+1}, \ i = 1, \ldots, 7, \ \text{and} \ \alpha_8 = h - x_1 - x_2 - x_3. \tag{2.17}$$

They are all (-2) rational curves. They actually correspond to the simple roots of the Lie algebra E_8, as their mutual intersections within S form the negative Cartan matrix of E_8

$$(\alpha_i.\alpha_j)_S = -A_{ij}^{E_8}, \quad i, j = 1, \ldots, 8. \tag{2.18}$$

Therefore they generate the E_8 lattice Λ_{E_8} in $H_2(S, \mathbb{Z})$. The E_8 lattice can be embedded into $\mathbb{R}^8$, which has a standard basis e_i $(i = 1, \ldots 8)$ with orthonormal inner product. In terms of this basis, we can write the E_8 simple roots as

$$\alpha_1 = \frac{e_1 + e_8}{2} - \frac{e_2 + e_3 + e_4 + e_5 + e_6 + e_7}{2},$$
$$\alpha_i = -e_{i-1} + e_i, \quad i = 2, \ldots, 7, \quad \text{and} \ \alpha_8 = e_1 + e_2. \tag{2.19}$$

We denote the Kähler moduli of these curves by

$$t_b = \mathrm{iVol}(b), \quad \tau = \mathrm{iVol}(f), \quad m_i = \mathrm{iVol}(e_i), \ i = 1, \ldots, 8. \tag{2.20}$$

Here m_i parameterise a vector $\underline{m}$ in $\Lambda_{E_8} \otimes \mathbb{R}$, where one can define a Weyl invariant bilinear form

$$(\underline{m}, \underline{m})_{E_8} = \sum_{i=1}^{8} m_i^2. \tag{2.21}$$

2.2.1 Toric hypersurface construction

Instead of the fully massive E-string geometry, we consider here the embedding of the non-compact CY3 $X^{(n)}$ associated to the E-string theory with $n = 1, 2, 3$ flavor masses turned on. It corresponds to breaking a subgroup $SU(2), SU(3), SO(7)$ of the E_8 flavor group. The non-compact geometry can also be embedded into a compact CY3 $\widetilde{X}^{(n)}$, which is the anti-canonical divisor of a toric variety $\widetilde{Y}^{(n)}$, a blow-up of the well-known toric variety whose anti-canonical divisor is the elliptic fiberation over $\mathbb{F}_1$. The $\widetilde{Y}^{(1)}$ is obtained by blowing up along the intersection of D_x, D_y, D_v with exceptional divisor D_v', while $\widetilde{Y}^{(2)}$ is obtained by further blowing up along the intersection of D_y, D_v' with exceptional divisor D_v'', $\widetilde{Y}^{(3)}$ by blowing up in addition along the intersection of D_x, D_v'' with exceptional divisor D_v'''. Similar to the M-string geometry construction, the result here is to create a

	ν_i^*				$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$
D_0	0	0	0	0	0	0	-2	0
D_x	-1	0	0	0	-1	0	1	0
D_y	0	-1	0	0	0	0	1	0
D_z	2	3	0	0	1	-1	0	-2
D_u	2	3	1	0	0	1	0	0
S	2	3	0	-1	0	-1	0	1
D_v	2	3	-1	-1	-3	1	1	0
D_v'	1	2	-1	-1	3	0	-1	0
D_t	2	3	0	1	0	0	0	1

Table 3. Toric divisors and Mori cone generators of the toric variety whose anti-canonical divisor is the compact embedding of the E-string geometry with one flavor mass turned on breaking $SU(2) \subset E_8$.

	ν_i^*				$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(5)}$
D_0	0	0	0	0	0	0	0	-3	0
D_x	-1	0	0	0	0	0	0	1	0
D_y	0	-1	0	0	-1	0	1	0	0
D_z	2	3	0	0	1	-1	0	0	-2
D_u	2	3	1	0	0	1	0	0	0
S	2	3	0	-1	0	-1	0	0	1
D_v	2	3	-1	-1	-2	1	0	1	0
$\widetilde{D}_v'$	2	3	-2	-2	0	0	1	-2	0
D_v''	1	1	-1	-1	2	0	-2	3	0
D_t	2	3	0	1	0	0	0	0	1

Table 4. Toric divisors and Mori cone generators of the toric variety whose anti-canonical divisor is the compact embedding of the E-string geometry with two flavor masses turned on breaking $SU(3) \subset E_8$.

resolve A_1, A_2, B_3 singularity over the (0)-curve in the base [46] or a A_1, A_2, B_3 type toric top [47, 48]. See tables 3, 4, 5 for details of the toric construction.[5]

In the case of one flavor mass, the non-compact CY3 $X^{(1)}$ is obtained from $\widetilde{X}^{(1)}$ by decompactifying along the $l_{(\mathrm{de})} = l^{(4)}$ curve. The base curve l_b, the elliptic fiber l_f, and the curve l_{m_1} associated to the one flavor mass are identified to be

$$l_b = l^{(2)}, \quad l_f = l^{(1)} + 3l^{(3)}, \quad l_{m_1} = 2l^{(3)}, \tag{2.22}$$

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[5]In tables 4, 5 we have defined $\widetilde{D}_v' = D_v' + D_v''$.

	ν_i^*				$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(5)}$	$l^{(6)}$
D_0	0	0	0	0	0	0	0	0	0	−1
D_x	−1	0	0	0	−2	0	0	0	0	1
D_y	0	−1	0	0	0	−1	0	0	1	0
D_z	2	3	0	0	0	1	−2	−1	0	0
D_u	2	3	1	0	0	0	0	1	0	0
S	2	3	0	−1	0	0	1	−1	0	0
D_v	2	3	−1	−1	1	−2	0	1	0	0
$\widetilde{D}'_v$	2	3	−2	−2	−2	0	0	0	1	0
D''_v	1	1	−1	−1	0	2	0	0	−2	1
D'''_v	0	1	−1	−1	3	0	0	0	0	−1
D_t	2	3	0	1	0	0	1	0	0	0

Table 5. Toric divisors and Mori cone generators of the toric variety whose anti-canonical divisor is the compact embedding of the E-string geometry with three flavor masses turned on breaking $SO(7) \subset E_8$.

and their volumes are t_b, t_f, t_{m_1}. From the intersection of l_{m_1} with D_v, D'_v, we find that it correponds to the simple root α_1 of A_1. Any curve corresponding to the weight $\omega = d_{m_1}\alpha_1$ has volume $d_{m_1}t_{m_1}$. Similarly with two flavor masses, the non-compact CY3 $X^{(2)}$ is obtained from $\widetilde{X}^{(2)}$ by decompactifying along the $l_{(\mathrm{de})} = l^{(5)}$ curve. We have the identification

$$l_b = l^{(2)}, \quad l_f = l^{(1)} + 4l^{(3)} + 2l^{(4)}, \quad l_{m_1} = l^{(4)}, \quad l_{m_2} = 3l^{(3)} + l^{(4)}, \tag{2.23}$$

whose volumes are $t_b, t_f, t_{m_1}, t_{m_2}$. From the intersection of l_{m_1}, l_{m_2} with D_v, D'_v, D''_v, we find they correpond to the simple roots α_1, α_2 of A_2, and thus the volume of any curve corresponding to the weight $\omega = d_{m_1}\alpha_1 + d_{m_2}\alpha_2$ is $d_{m_1}t_{m_1} + d_{m_2}t_{m_2}$. Finally with three flavor masses turned on, the non-compact CY3 $X^{(3)}$ is obtained from $\widetilde{X}^{(3)}$ by decompactifying along $l_{(\mathrm{de})} = l^{(3)}$, and we have

$$l_b = l^{(4)}, \quad l_f = 2l^{(1)} + l^{(2)} + 4l^{(5)} + 6l^{(6)}, \quad l_{m_1} = l^{(5)} + 2l^{(6)}, \quad l_{m_2} = l^{(1)} + 2l^{(6)}, \quad l_{m_3} = l^{(5)}. \tag{2.24}$$

We use mirror symmetry techniques to compute the genus 0 GV invariants of $\widetilde{X}^{(n)}$ and extract the GV invariants of $X^{(n)}$ by choosing those with degree zero along $l_{(\mathrm{de})}$. They agree well with the genus 0 GV invariants of the E-string theory [49]. For instance, the massless E-string theory at $(d_b, d_f) = (1, 1)$ has the invariant

$$n_0^{(1,1)} = 252 = 240 + 12 \cdot 1, \tag{2.25}$$

which becomes the characters of the Weyl orbit $\mathcal{O}_{2,240}$ and $\mathcal{O}_{0,1}$ when all flavor masses are turned on. Here we denote by $\mathcal{O}_{n,p}$ the Weyl orbit whose size is p and the norm square of whose elements is n and its character by $\chi_{n,p}^{\mathfrak{g}}(\underline{m})$. We expect that when only one flavor

mass is turned on, given the branching rule of $E_8 \supset E_7 \oplus \mathfrak{su}(2)$

$$\mathcal{O}_{2,240} = (\mathcal{O}_{2,126}, 1) \oplus \left(\mathcal{O}_{\frac{3}{2},56}, \mathcal{O}_{\frac{1}{2},2} \right) \oplus (1, \mathcal{O}_{2,2}), \tag{2.26}$$

the $(d_b, d_f) = (1, 1)$ invariant should be broken to

$$n_0^{(1,1)} : 252 \to 138 + 56\chi_{\frac{1}{2},2}^{\mathfrak{su}_2}(\underline{m}) + \chi_{2,2}^{\mathfrak{su}(2)}(\underline{m}) = 138 + 56(Q_{m_1}^{1/2} + Q_{m_1}^{-1/2}) + (Q_{m_1} + Q_{m_1}^{-1}), \tag{2.27}$$

where $Q_{m_i} = \mathbf{e}[t_{m_i}]$ with the notation $\mathbf{e}[x] = \exp(2\pi i x)$. When two flavor masses are turned on, with the branching rule of $E_8 \supset E_6 \oplus \mathfrak{su}(3)$,[6]

$$\mathcal{O}_{2,240} = (\mathcal{O}_{2,72}, 1) \oplus \left(\mathcal{O}_{\frac{4}{3},27}, \mathcal{O}_{\frac{2}{3},\overline{3}} \right) \oplus \left(\mathcal{O}_{\frac{4}{3},\overline{27}}, \mathcal{O}_{\frac{2}{3},3} \right) \oplus (1, \mathcal{O}_{2,6}), \tag{2.28}$$

the $(d_b, d_f) = (1, 1)$ invariant should be broken to

$$n_0^{(1,1)} : 252 \to 84 + 27 \left(\chi_{\frac{2}{3},\overline{3}}^{\mathfrak{su}_3}(\underline{m}) + \chi_{\frac{2}{3},3}^{\mathfrak{su}(3)}(\underline{m}) \right) + \chi_{2,8}^{\mathfrak{su}(3)}(\underline{m})$$
$$= 84 + 27(Q_{m_1}^{2/3}Q_{m_2}^{1/3} + Q_{m_1}^{1/3}Q_{m_2}^{2/3} + Q_{m_1}^{1/3}Q_{m_2}^{-1/3} + \{Q_{m_i} \to 1/Q_{m_i}\})$$
$$+ (Q_{m_1} + Q_{m_2} + Q_{m_1}Q_{m_2} + \{Q_{m_i} \to 1/Q_{m_i}\}). \tag{2.29}$$

When three flavor masses are turned on, with the branching rule of $E_8 \supset SO(16) \supset SO(9) \oplus SO(7)$

$$\mathcal{O}_{2,240} = (\mathcal{O}_{2,24}, 1) \oplus 2 \left(\mathcal{O}_{\frac{3}{4},8}, 1 \right) \oplus (\mathcal{O}_{1,8} + 2*1, \mathcal{O}_{1,6}) \oplus \left(\mathcal{O}_{1,16}, \mathcal{O}_{\frac{3}{4},8} \right) \oplus (1, \mathcal{O}_{2,12}), \tag{2.30}$$

the $(d_b, d_f) = (1, 1)$ invariant should be broken to

$$n_0^{(1,1)} : 252 \to 52 + 10\chi_{1,6}^{\mathfrak{so}(7)}(\underline{m}) + 16\chi_{\frac{4}{3},8}^{\mathfrak{so}(7)}(\underline{m}) + \chi_{2,12}^{\mathfrak{so}(7)}(\underline{m}). \tag{2.31}$$

This is precisely what we find.

In addition, we computed the perturbative prepotential of $\widetilde{X}^{(n)}$, and obtain that of $X^{(n)}$ by integrating over the B-periods that remain finite in the limit $\mathrm{Vol}(l_{(\mathrm{de})}) \to \infty$, as in section 2.1.1. We find that for $X^{(1)}$

$$F_{(0,0)} = -\frac{1}{2}t_b^2\tau - \frac{1}{2}t_b\tau^2 + \frac{1}{3}t_b t_{m_1}^2, \tag{2.32}$$

for $X^{(2)}$

$$F_{(0,0)} = -\frac{1}{2}t_b^2\tau - \frac{1}{2}t_b\tau^2 + \frac{1}{4}t_b(t_{m_1}^2 + t_{m_2}^2 + t_{m_1}t_{m_2}). \tag{2.33}$$

and for $X^{(3)}$

$$F_{(0,0)} = -\frac{1}{2}t_b^2\tau - \frac{1}{2}t_b\tau^2 + \frac{1}{4}t_b(t_{m_1}^2 + 2t_{m_2}^2 + 3t_{m_2}^2 + 2t_{m_1}t_{m_2} + 2t_{m_1}t_{m_3} + 4t_{m_2}t_{m_3}). \tag{2.34}$$

All of them can be put in the form

$$F_{(0,0)} = -\frac{1}{2}t_b^2\tau - \frac{1}{2}t_b\tau^2 + \frac{1}{2}t_b(\underline{m}, \underline{m})_{\mathfrak{g}}, \tag{2.35}$$

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[6]The weights in $\mathcal{O}_{\frac{2}{3},\overline{3}}$ are opposite of those in $\mathcal{O}_{\frac{2}{3},3}$; it is the same as the weight space of the irrep $\overline{\mathbf{3}}$.

where

$$(\underline{m}, \underline{m})_{\mathfrak{g}} = \sum_{i,j=1}^{r_{\mathfrak{g}}} t_{m_i} t_{m_j} (\omega_i, \omega_j)_{\mathfrak{g}}, \tag{2.36}$$

with ω_i being the fundamental weights and $(\bullet, \bullet)_{\mathfrak{g}}$ the Weyl invariant bilinear form on the complexified weight lattice. Generalising this, we conclude that the perturbative prepotential of the massive E-string theory with all eight flavor masses turned on should be

$$F_{(0,0)} = -\frac{1}{2} t_b^2 \tau - \frac{1}{2} t_b \tau^2 + \frac{1}{2} t_b (\underline{m}, \underline{m})_{E_8}, \tag{2.37}$$

up to irrelevant terms.

2.3 Direct computation of intersection numbers

We argue here that the relevant perturbative prepotential is inherently well-defined for the non-compact CY3 associated to our 6d SCFT on the torus and it does not depend on the compact embedding of the non-compact CY3.

A non-compact CY3 X is locally the neighborhood of a union of r connected compact surfaces $S = \cup_i S_i$ [43, 44]. Let the number of independent compact curve classes in S be n, which is necessarily greater than or equal to r. Among these curves we can find r independent curves u_i which have non-trivial intersection numbers with S_i, while the remaining $n - r$ curves v_k are free curves and have vanishing intersection numbers with any S_i. We denote their Kähler moduli by[7]

$$t_i = i\text{Vol}(u_i), \quad m_k = i\text{Vol}(v_k). \tag{2.38}$$

The $r \times r$ matrix of intersection numbers between u_i and S_j is of full rank since they generate dual lattices. We would like to enlarge the intersection matrix $(u_i.S_j)$ to an $n \times n$ full rank matrix which includes the $n-r$ free curves as well. This requires carefully choosing $n-r$ non-compact surfaces, and it can be done as follows. Among the $n-r$ free curves there is a unique genus one curve $v_0 = f$, the elliptic fiber, and the others v_k $(k = 1, \ldots, n-r-1)$ are all rational curves. Their self-intersection inside S can be determined by the adjunction formula to be

$$(f^2)_S = 0, \quad (v_k^2)_S = -2, \quad k = 1, \ldots, n-r-1. \tag{2.39}$$

We choose the first non-compact surface N_0 to be the base of elliptic fibration. It only has non-trivial intersection numbers with the base curve b (not a free curve) and the elliptic fiber f

$$b.N_0 = \mathfrak{n} - 2, \quad f.N_0 = 1. \tag{2.40}$$

We define additional $n - r - 1$ non-compact surfaces N_k by their gluing curves with S

$$N_k.S = v_k, \quad k = 1, \ldots, n-r-1. \tag{2.41}$$

The intersection matrix $(v_k.N_l)_{k,l=1,\ldots,n-r-1}$ is of full rank because

$$v_k.N_l = S.N_k.N_l = (v_k.v_l)_S. \tag{2.42}$$

---

[7]Here the mass parameters m_k include not only flavor masses but also the volume of elliptic fiber τ.

The total curve-surface intersection matrix

$$\begin{pmatrix} u_i.S_j & u_i.N_l \\ 0 & v_k.N_l \end{pmatrix}_{\substack{i,j=1,\ldots,r \\ k,l=0,1,\ldots,n-r-1}} \tag{2.43}$$

is block upper triangular, and its determinant factorises

$$\det \begin{pmatrix} u_i.S_j & u_i.N_l \\ 0 & v_k.N_l \end{pmatrix}_{\substack{i,j=1,\ldots,r \\ k,l=0,1,\ldots,n-r-1}} = \det \left(u_i.S_j \right)_{i,j=1,\ldots,r} \det \left(v_k.v_l \right)_{k,l=1,\ldots,n-r-1} \neq 0, \tag{2.44}$$

where we have used (2.40). Therefore it is also of full rank.

The Kähler class of X can be decomposed as

$$J = \sum_{i=1}^{r} \phi_i S_i + \sum_{k=0}^{n-r-1} \varphi_k N_k, \tag{2.45}$$

where the coefficients ϕ_i, φ_k are related to t_i, m_k by

$$t_i = \sum_{j=1}^{r} (u_i.S_j)\phi_j + \sum_{l=0}^{n-r-1} (u_i.N_l)\varphi_l,$$
$$m_k = \sum_{l=0}^{n-r-1} (v_k.N_l)\varphi_l. \tag{2.46}$$

The perturbative prepotential $F_{(0,0)}$ can be computed by integrating B-periods. B-periods measure in the semiclassical limit volumes of surfaces. Therefore in a local CY3, the only well-defined B-periods are those associated to compact surfaces S_j

$$\Pi_j := \frac{\partial F_{(0,0)}}{\partial \phi_j} = \sum_{i=1}^{r} \frac{\partial F_{(0,0)}}{\partial t_i}(u_i.S_j). \tag{2.47}$$

Given that the matrix $(u_i.S_\alpha)$ is of full rank, if $\frac{\partial F_{(0,0)}}{\partial \phi_j}$ are known, $\frac{\partial F_{(0,0)}}{\partial t_i}$ can be solved from (2.47) and further be integrated to produce all relevant terms in $F_{(0,0)}$. To compute $\frac{\partial F_{(0,0)}}{\partial \phi_j}$, recall that $F_{(0,0)}$ can also be written as

$$F_{(0,0)} = \frac{1}{6}J^3 = \frac{1}{6}\left(\sum_{i=1}^{r} \phi_i S_i + \sum_{k=0}^{n-r-1} \varphi_k N_k \right)^3, \tag{2.48}$$

which leads to

$$\frac{\partial F_{(0,0)}}{\partial \phi_j} = \frac{1}{2} D_j.\left(\sum_{i=1}^{r} \phi_i S_i + \sum_{k=0}^{n-r-1} \varphi_k N_k \right)^2. \tag{2.49}$$

The triple intersection numbers on the r.h.s. of (2.49) involve at least one compact surface, and they can all be converted to intersection of curves in S and thus are computable (see section 2.6 of [44]). We then substitute t_i, m_k for ϕ_i, φ_k by inverting (2.46).

To summarise, we provide here a prescription to compute $\frac{\partial F_{(0,0)}}{\partial t_i}$ and therefore all relevant terms in $F_{(0,0)}$ from the local description of a non-compact CY3, thus proving that the former does not depend on the compact embedding of the non-compact CY3. We illustrate this idea with examples.

2.3.1 M-string

Following the prescription in the previous section, we choose the non-compact surface N_0 to be the base of the elliptic fibration, and define an additional non-compact surface N_1 by its gluing

$$S.N_1 = x + y. \tag{2.50}$$

The full curve-surface intersection matrix is given in table 6, which is indeed of full rank. We define the Kähler class of the M-string geomtry to be

$$J = \phi S + \varphi_0 N_0 + \varphi_1 N_1 \,, \tag{2.51}$$

whose coefficients are related to the volumes of $e, f, x+y$ by

$$\begin{pmatrix} t_b \\ \tau \\ 2m \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} \phi \\ \varphi_1 \\ \varphi_2 \end{pmatrix}. \tag{2.52}$$

The only well-defined B-period measures semiclassically the volume of S, and it reads

$$\frac{\partial F_{(0,0)}}{\partial \phi} = -2 \frac{\partial F_{(0,0)}}{\partial t_b}. \tag{2.53}$$

Using the Kähler form representation of $F_{(0,0)}$ we can compute this B-period

$$\frac{\partial F_{(0,0)}}{\partial \phi} = -\frac{1}{2} D. \left(\phi D + \varphi_1 N_1 + \varphi_2 N_2 \right)^2 \tag{2.54}$$

$$= -2\phi\varphi_1 - \varphi_2^2, \tag{2.55}$$

where we have use the following triple intersection numbers

$$\begin{aligned}
S^3 &= (K_S.K_S)_S = 0, \\
S^2.N_0 &= (K_S.e)_S = -2, \\
S^2.N_1 &= (K_S.(x+y))_S = 0, \\
S.N_0^2 &= (e.e)_S = 0, \\
S.N_1^2 &= ((x+y).(x+y))_S = -2, \\
S.N_0.N_1 &= (e.(x+y))_S = 0.
\end{aligned} \tag{2.56}$$

Substituting t_b, τ, m for $\phi, \varphi_0, \varphi_1$, we find

$$\frac{\partial F_{(0,0)}}{\partial \phi} = t_b \tau - m^2, \tag{2.57}$$

which, together with (2.53), integrates to (2.13) up to irrelevant terms.

	S	N_0	N_1
e	-2	0	0
f	0	1	0
$x+y$	0	0	-2

Table 6. The full curve-divisor intersection matrix in the M-string geometry.

2.3.2 E-string

There are nine linearly independent free curves in the E-string geometry, the elliptic fiber f and the eight (-2) curves α_i $(i = 1, \ldots, 8)$. Following the prescription in the beginning of the section, we need to first choose nine non-compact surfaces.

We choose the base of the elliptic fibration to be the first non-compact surface N_0 and define the remaining eight non-compact surfaces N_i by

$$S.N_i = \alpha_i, \quad i = 1, \ldots, 8. \tag{2.58}$$

N_0 only intersects non-trivially with the base curve and the elliptic fiber of the elliptic fibration by

$$b.N_0 = -1, \quad f.N_0 = 1. \tag{2.59}$$

The other eight surfaces N_i only intersect non-trivally with α_i and the intersection numbers form the negative Cartan E_8 lattice according to (2.42). We display the full curve-surface intersection matrix in table 7.

We introduce the Kähler class of the E-string geometry

$$J = \phi S + \sum_{k=0}^{8} \varphi_i N_i, \tag{2.60}$$

whose coefficients are related to the Kähler moduli t_b, τ, m_i by (2.46). The only well-defined B-period is the one associated to S and it reads

$$\frac{\partial F_{(0,0)}}{\partial \phi} = -\frac{\partial F_{(0,0)}}{\partial t_b}. \tag{2.61}$$

Plugging in the Kähler class form of $F_{(0,0)}$ and using the following triple intersection numbers

$$\begin{aligned}
S^3 &= (K_S.K_S)_S = 0, \\
S^2.N_0 &= (K_S.x_9)_S = 1, \\
S^2.N_i &= (K_S.\alpha_i)_S = 0, \quad i = 1, \ldots, 8, \\
S.N_0^2 &= (x_9.x_9)_S = -1, \\
S.N_i.N_j &= (\alpha_i.\alpha_j)_S = -A_{ij}^{E_8}, \quad i,j = 1, \ldots, 8, \\
S.N_0.N_i &= (x_9.\alpha_i)_S = 0, \quad i = 1, \ldots, 8.
\end{aligned} \tag{2.62}$$

and substituting t_b, τ, m_i for ϕ, φ_i, we find

$$\frac{\partial F_{(0,0)}}{\partial \phi} = t_b \tau + \frac{1}{2}\tau^2 - \frac{1}{2}(\underline{m}, \underline{m})_{E_8}, \tag{2.63}$$

which integrates to (2.37) up to irrelevant terms.

	S	N_0	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8
b	-1	-1	0	0	0	0	0	0	0	0
f	0	1	0	0	0	0	0	0	0	0
α_1	0	0	-2	1	0	0	0	0	0	0
α_2	0	0	1	-2	1	0	0	0	0	0
α_3	0	0	0	1	-2	1	0	0	0	1
α_4	0	0	0	0	1	-2	1	0	0	0
α_5	0	0	0	0	0	1	-2	1	0	0
α_6	0	0	0	0	0	0	1	-2	1	0
α_7	0	0	0	0	0	0	0	1	-2	0
α_8	0	0	0	0	1	0	0	0	0	-2

Table 7. The full curve-surface intersection matrix of the E-string geometry.

2.3.3 Higher rank E-, M-string theories

We further illustrate the power of the direct computation to derive the perturbative pre-potentials for two higher rank 6d SCFTs, the higher rank E-, M-string theories.

A higher rank 6d SCFT corresponds to in the F-theory compactification multiple $\mathbb{P}^1$'s in the base of elliptic fibration, and it is characterised by the negative-definite intersection matrix $-\Omega$ of the base curves as well as the singular elliptic fibers over the base curves. A higher rank 6d SCFT can be obtained by properly gluing rank one 6d SCFTs corresponding to a single base curve following certain consistency rules. We consider in this paper two simple higher rank theories, higher rank M-string theory and higher rank E-string theory. The former corresponds to a chain of (-2) curves in the base, and the latter corresponds to in addition a (-1) curve glued to one end of the chain [50]. Both of these two theories have no gauge symmetry. Together with rank one M-string and E-string theories, they constitute the full list of relatively simple 6d SCFTs with no gauge symmetry. Since the geometry of higher rank M-string can be obtained from that of higher rank E-string by decompactifying the (-1) curve in the base, we will first compute the perturbative prepotential of the higher rank E-string, and then deduce the prepotential of higher rank M-string as a special limit.

Consider a rank r E-string theory with one (-1) curve u_0 attached to a chain of $r-1$ (-2) curves u_i for $i = 1, \ldots, r-1$ in the base with the intersection matrix

$$-\Omega = -\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \tag{2.64}$$

Let the Kähler parameters of the these base curves be

$$t_0 = \mathrm{iVol}(u_0), \quad t_i = \mathrm{iVol}(u_i), \quad i = 1, \ldots, r-1. \tag{2.65}$$

There are in addition ten mass parameters: the Kähler parameter of the elliptic fiber τ, the flavor mass $\underline{m}$ of rank eight for the flavor symmetry E_8 associated to the (-1) curve, and the flavor mass m for the $SU(2)$ flavor symmetry associated to the (-2) chain. There are r connected compact surfaces S_0, S_i for $i = 1, \ldots, r-1$, which are pull-backs of the elliptic fibration from the base curves u_0, u_i $(i = 1, \ldots, r-1)$. S_0 is the half K3 surface associated to the rank one E-string, and each of S_i $(i = 1, \ldots, r-1)$ is a self-glued $\mathbb{F}_0^{1+1}$ associated to the rank one M-string. Therefore, using (2.63) and (2.57), we can write down the B-periods that measure the volume of each compact surface

$$\Pi_0 = \text{Vol}(S_0) = t_0\tau + \frac{1}{2}\tau^2 - \frac{1}{2}(\underline{m}, \underline{m})_{E_8}, \tag{2.66}$$

$$\Pi_i = \text{Vol}(S_i) = t_i\tau - m^2, \quad i = 1, \ldots, r-1. \tag{2.67}$$

On the other hand, following (2.47) the B-periods should be related to the perturbative prepotenital by

$$\Pi_j = -\sum_{i=0}^{r-1} \frac{\partial F_{(0,0)}}{\partial t_i}\Omega_{ij}. \tag{2.68}$$

It follows that up to irrelevant terms the perturbative prepotential of the rank r E-string should be

$$F_{(0,0)} = -\frac{1}{2}\sum_{i,j=0}^{r-1}\tau t_i t_j(\Omega^{-1})_{ij} - \frac{1}{2}\sum_{j=0}^{r-1}\tau^2 t_j(\Omega^{-1})_{0j} + \frac{1}{2}\sum_{i=0}^{r-1}t_i(\underline{m},\underline{m})_{E_8}(\Omega^{-1})_{0i} + m^2\sum_{j=0}^{r-1}\sum_{i=1}^{r-1}t_j(\Omega^{-1})_{ij}. \tag{2.69}$$

To obtain the perturbative prepotential of the rank r M-string, we remove everything related to the index 0 in (2.69) and increase the summation upper bound to r:

$$F_{(0,0)} = -\frac{1}{2}\sum_{i,j=1}^{r}\tau t_i t_j(\Omega^{-1})_{ij} + m^2\sum_{i,j=1}^{r}t_j(\Omega^{-1})_{ij}. \tag{2.70}$$

Here Ω is the negative intersection matrix of the (-2) curve chain, and it coincides with the Cartan matrix of the Lie algebra A_r.

3 Blowup equations

Let us quickly review the formalism of the generalised blowup equations [16, 23, 24, 27] (see also [25, 26]). Given a non-compact Calabi-Yau threefold X, we denote by $C = (C_{ij})$ the matrix of intersections between compact curve classes $[\Sigma_i]$, $i = 1, \ldots, b_2^c$ and compact divisor classes $[\mathfrak{D}_j]$, $j = 1, \ldots, b_4^c$ of X. We also define the vector

$$\underline{R}(\underline{n}) = C \cdot \underline{n} + \underline{r}/2, \tag{3.1}$$

which parameterises the shift of Kähler parameters with $\underline{n} \in \mathbb{Z}^{b_4^c}$ and $\underline{r} \in \mathbb{Z}^{b_4^c}$. The integral vector $\underline{r}$, which we call the $\underline{r}$-field, is consistent with the checkerboard pattern of non-vanishing refined BPS invariants N_{j_l,j_r}^{β}, in other words, they satisfy

$$2j_l + 2j_r + 1 \equiv \underline{r} \cdot \beta \mod 2. \tag{3.2}$$

The claim of the generalised blowup equations is that there exists a non-empty set $\mathcal{S}$ of $\underline{r}$-fields such that the twisted partition function of topological string defined by

$$\widehat{Z}(\underline{t}, \epsilon_1, \epsilon_2) = Z^{\mathrm{cls}}(\underline{t}, \epsilon_1, \epsilon_2) Z^{\mathrm{inst}}(\underline{t} + \underline{r}, \epsilon_1, \epsilon_2), \tag{3.3}$$

satisfies the following identity[8]

$$\sum_{\underline{n} \in \mathbb{Z}^{b_4^c}} (-1)^{|\underline{n}|} \widehat{Z}(\underline{t} + \epsilon_1 \underline{R}(\underline{n}), \epsilon_1, \epsilon_2 - \epsilon_1) \widehat{Z}(\underline{t} + \epsilon_2 \underline{R}(\underline{n}), \epsilon_1 - \epsilon_2, \epsilon_2) = \Lambda(\epsilon_1, \epsilon_2, \underline{m}, \underline{r}) \widehat{Z}(\underline{t}, \epsilon_1, \epsilon_2),$$
$$\tag{3.4}$$

with $|\underline{n}| = \sum_{i=1}^{b_4^c} n_i$. Here we have separated the Kähler parameters $\underline{m}$ from the Kähler parameters $\underline{t}$ to denote those curve classes that do not intersect with compact divisors $[\mathfrak{D}_k]$, $k = 1, \ldots, b_4^c$, and these $\underline{m}$ correspond to mass parameters in the gauge theory context, while the other Kähler parameters correspond to Coulomb branch parameters, thus are also called "true" parameters. It is important here that the coefficient $\Lambda(\epsilon_1, \epsilon_2, \underline{m}, \underline{r})$ on the r.h.s. of (3.4) depends in addition to $\epsilon_{1,2}$ only on mass parameters but not true Kähler parameters. We will also make the distinction between the equations where Λ vanishes identically and those where Λ is non-trivial. We call the former the *vanishing* blowup equations, and the latter the *unity* blowup equations.

For any non-compact Calabi-Yau threefold, the first interesting question is whether such $\underline{r}$-fields exist so that (3.4) holds, and if they do, how to find all of them and write down the corresponding blowup equations. This is the goal of the current section for the geometries of E-, M-string theories and their higher rank brethrens. In the next section, we discuss the next question: the computation of refined BPS invariants from these equations.

The first thing one immediately notices is that one only has to consider the set of $\underline{r}$-fields in (3.4) modulo

$$\underline{r} \sim \underline{r}' = \underline{r} + 2C \cdot \underline{n}', \quad \underline{n}' \in \mathbb{Z}^{b_4^c}, \tag{3.5}$$

as any such a shift can be absorbed into the summation index vector $\underline{n}$. Next, we can constrain the $\underline{r}$-fields by looking at the contribution of Z^{cls} to the blowup equations. If we divide both hand sides of (3.4) by $\widehat{Z}(\underline{t}, \epsilon_1, \epsilon_2)$, the leading contribution of each summand is

$$(-1)^{|\underline{n}|} \frac{Z^{\mathrm{cls}}(\epsilon_1, \epsilon_2 - \epsilon_1) Z^{\mathrm{cls}}(\epsilon_1 - \epsilon_2, \epsilon_2)}{Z^{\mathrm{cls}}(\epsilon_1, \epsilon_2)} = (-1)^{|\underline{n}|} \exp\left(f_0(\underline{n})(\epsilon_1 + \epsilon_2) + \sum_{k=1}^{b_2^c} f_k(\underline{n}) t_k \right), \tag{3.6}$$

where

$$f_0(\underline{n}) = -\frac{1}{6} \sum_{i,j,k=1}^{b_2^c} \kappa_{ijk} R_i R_j R_k + \sum_{i=1}^{b_2^c} (b_i^{\mathrm{GV}} + b_i^{\mathrm{NS}}) R_i, \tag{3.7}$$

$$f_k(\underline{n}) = b_k^{\mathrm{GV}} - b_k^{\mathrm{NS}} - \frac{1}{2} \sum_{i,j=1}^{b_2^c} \kappa_{ijk} R_i R_j, \quad k = 1, \ldots, b_2^c. \tag{3.8}$$

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[8]Practically one can also shift the $\underline{r}$ field in the polynomial part but keep the instanton part unshifted.

Let us assume that we have chosen a basis of t_k such that the first b_4^c of them are true Kähler parameters, while the remaining $b_2^c - b_4^c$ are mass parameters. The function $\Lambda(\epsilon_1, \epsilon_2, \underline{m})$ on the r.h.s. should be

$$\Lambda(\epsilon_1, \epsilon_2, \underline{m}, \underline{r}) = \sum_{\underline{n} \in \mathcal{I}} (-1)^{|n|} \exp\left(f_0(\underline{n})(\epsilon_1 + \epsilon_2) + \sum_{k=1}^{b_2^c} f_k(\underline{n}) t_k \right). \tag{3.9}$$

Here $\mathcal{I}$ is the set of $\underline{n}$ that "locally" minimize $f_k(\underline{n})$ for $k = 1, \ldots, b_4^c$, which means that one cannot find any $\underline{n}'$ such that

$$f_k(\underline{n}') \leq f_k(\underline{n}), \quad k = 1, \ldots, b_4^c; \text{ at least one inequality is not saturated.} \tag{3.10}$$

One necessary condition is that every $f_k(\underline{n})$ for $k = 1, \ldots, b_4^c$ is positive semi-definite. In addition, for vanishing blowup equations, the r.h.s. of (3.9) should cancel. For unity blowup equations, the set $\mathcal{I}$ should contain a single element that minimizes $f_k(\underline{n})$ for $k = 1, \ldots, b_4^c$ simultaneously, and the minimal values should all be zero. Using the equivalence relation (3.5), we can assume $\mathcal{I} = \{\underline{0}\}$. We find that these conditions constrain the possible $\underline{r}$-fields to a finite set $\mathcal{S}'$. For the theories considered here, we verify that the corresponding blowup equations are valid up to very high orders of exponentiated Kähler parameters; in other words, the set $\mathcal{S}'$ determined in this way is correct.

One problem of this approach of fixing the $\underline{r}$-fields is that it depends on the values of the topological data $b_k^{\mathrm{GV}}, b_k^{\mathrm{NS}}$ for $k = 1, \ldots, b_4^c$ associated to true Kähler parameters. Let us first write down the components of the $\underline{r}$-field: $\underline{r} = (r_b, r_\tau, \underline{r}_m)$, which correspond to the base curve b, the fiber curve f, and the flavor curves. Recall that the component of the $\underline{r}$-field associated to a rational curve C with normal bundle $\mathcal{O}(-n) \oplus \mathcal{O}(-2+n)$ is n modulo 2 [23]. Then r_b, r_τ must be $-\mathfrak{n}, 0$ modulo 2, and $\underline{r}_m$ twice a weight vector of SU(2) or E_8 for M-string and E-string theories respectively.[9] The component r_b can be further reduced by the equivalence condition (3.5) to within the range

$$r_b = -\mathfrak{n} + 2j, \quad j = 0, \ldots, \mathfrak{n} - 1. \tag{3.11}$$

In the case of E-, M-string theories, only t_b is a true Kähler parameter. The positive semi-definite condition then implies that

$$r_\tau = 0. \tag{3.12}$$

In addition, for unity equations, the condition that the minimum of f_b is 0 is equivalent to the solution of $\underline{r}_m$ in terms of the data $b_k^{\mathrm{GV}}, b_k^{\mathrm{NS}}$

$$\left(\frac{1}{2}\underline{r}_m, \frac{1}{2}\underline{r}_m \right)_{\mathfrak{g}} = b_b^-. \tag{3.13}$$

Here $(\bullet, \bullet)_{\mathfrak{g}}$ is the Weyl invariant bilinear form on the weight lattice of the Lie algebra $\mathfrak{g}$ of the flavor symmetry G, and we have used the notation

$$b_i^\pm = b_i^{\mathrm{GV}} \pm b_i^{\mathrm{NS}}. \tag{3.14}$$

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[9]See sections 3.1, 3.2 for detailed explanation for the last statement.

The coefficients b_k^{GV} are the second Chern class evaluated at divisors and they can be computed once we have a toric compact embedding of the non-compact CY3. b_k^{NS}, on the other hand, are more elusive as they are only defined in the refined holomorphic anomaly equations [51]. For the E-string theory, the refined holomorphic anomaly equations can be formulated using the mirror curve of Sakai [40] and b_i^{NS} can be computed. But for a generic 6d SCFT this is difficult to do, and we solve this problem from a different angle.

Let us expand the instanton partition function in terms of the exponentiated tensor modulus $Q_b = \mathbf{e}[t_b]$, which is identified with the volume of the base curve[10]

$$Z^{\text{inst}}(\underline{t}, \epsilon_1, \epsilon_2) = 1 + \sum_{k=1}^{\infty} Q_b^k Z_k(\tau, \underline{m}, \epsilon_1, \epsilon_2) = 1 + \sum_{k=1}^{\infty} Q_{\text{ell}}^k \mathbb{E}_k(\tau, \underline{m}, \epsilon_1, \epsilon_2). \tag{3.15}$$

In the last equality we use the fact that the instanton partition function Z_k can be identified with the k-string elliptic genus $\mathbb{E}_k$ with

$$Q_{\text{ell}} = Q_b Q_\tau^{-\frac{\mathfrak{n}-2}{2}}. \tag{3.16}$$

We can expand the blowup equations in terms of Q_{ell} and obtain the following equations of elliptic genera of the E-, M-string theories, which we call the elliptic blowup equations

$$\sum_{k_1+k_2=k} \theta_{3,4}^{[a]} \left(\mathfrak{n}\tau, \frac{1}{2}\underline{r}_m \cdot \underline{m} + y(\epsilon_1 + \epsilon_2) - \mathfrak{n}(k_1\epsilon_1 + k_2\epsilon_2) \right)$$

$$\times \mathbb{E}_{k_1} \left(\tau, \underline{m} + \frac{1}{2}\underline{r}_m\epsilon_1, \epsilon_1, \epsilon_2 - \epsilon_1 \right) \mathbb{E}_{k_2} \left(\tau, \underline{m} + \frac{1}{2}\underline{r}_m\epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2 \right)$$

$$= \theta_{3,4}^{[a]} \left(\mathfrak{n}\tau, \frac{1}{2}\underline{r}_m \cdot \underline{m} + y(\epsilon_1 + \epsilon_2) \right) \mathbb{E}_k(\tau, \underline{m}, \epsilon_1, \epsilon_2). \tag{3.17}$$

Here the subscript of theta function is 3 if $\mathfrak{n}$ is even and 4 if $\mathfrak{n}$ is odd. The characteristic a is given by the r_b component

$$a = -\frac{r_b}{2\mathfrak{n}} = \frac{1}{2} - \frac{j}{\mathfrak{n}}, \quad j = 0, \ldots, \mathfrak{n} - 1. \tag{3.18}$$

We also have

$$y = \frac{1}{2}(\underline{r}_m/2, \underline{r}_m/2)_\mathfrak{g} - \mathfrak{n}b_b^+. \tag{3.19}$$

One important property of these equations is that every component is a Jacobi form. In particular the elliptic genus $\mathbb{E}_k$ is a meromorphic Jacobi form with modular weight 0 and modular index polynomial [16, 21]

$$\text{ind}_k = -\frac{3 - \mathfrak{n}}{4} k(\epsilon_1 + \epsilon_2)^2 + \frac{k(\mathfrak{n}k + 2 - n)}{2}\epsilon_1\epsilon_2 + \frac{k}{2}(\underline{m}, \underline{m})_\mathfrak{g}. \tag{3.20}$$

---

[10]Usually the instanton partition function also includes a nontrivial "1-loop" contribution coming from BPS states wrapping only fibral curves. For theories with no gauge symmetry, this contribution is either absent or can be factored out of the blowup equations.

One natural consistency condition for (3.17) is that every term has the same modular index polynomial depending only on k but not on k_1, k_2 individually, which we will call the *modularity* condition. This imposes the constraint that

$$y = \frac{\mathfrak{n} - 1}{4} + \frac{1}{2}(\underline{r}_m/2, \underline{r}_m/2)_{\mathfrak{g}}, \tag{3.21}$$

from which b_b^+ can be read off. Combined with b_b^{GV} computed using the toric embedding we constructed in section 2, one can write down b_b^- and proceed to constrain $\underline{r}_m$ of unity blowup equations with (3.13). Alternatively, the $\underline{r}$-fields of unity equations can be directly constrained by the modularity condition to be

$$(\underline{r}_m/2, \underline{r}_m/2)_{\mathfrak{g}} = \frac{7 - 3\mathfrak{n}}{2}. \tag{3.22}$$

We point out that vanishing blowup equations can arise if the theta function on the r.h.s. of (3.17) vanishes identically for certain value of $\underline{r}_m$, as we will see in section 3.2 for the E-string theory. We write down the elliptic blowup equations for the E-, M-string theories explicitly in the following subsections.

3.1 M-strings

We first argue that $\frac{1}{2}\underline{r}_m$ is a $\mathfrak{su}(2)$ weight vector. The fact that $2m$ is associated to a rational (-2) curve $(x + y)$, cf. section 2.1, has two implications. First the associated $\underline{r}$-field component should satisfies

$$2r_m \equiv 0 \mod 2, \tag{3.23}$$

in other words, r_m is an integer. Second, it is natural to treat $2m$ as a component of a one dimensional vector $\underline{m} \in P(\mathfrak{su}(2)) \otimes_{\mathbb{Z}} \mathbb{R}$ through

$$2m = (\alpha_1, \underline{m})_{\mathfrak{su}(2)} \iff \underline{m} = m\,\alpha_1, \tag{3.24}$$

where α_1 is the simple root of $\mathfrak{su}(2)$. Likewise, we can promote r_m to a one-dimensional vector $\underline{r}_m = r_m \alpha_1$. Consequently, $\frac{1}{2}\underline{r}_m$ must be a weight vector of $\mathfrak{su}(2)$.

Using the modularity constraint (3.22), we conclude that the $\underline{r}_m$ for unity blowup equations can only be

$$\frac{1}{2}\underline{r}_m = \pm\omega_1, \tag{3.25}$$

where ω_1 is the fundamental weight of $\mathfrak{su}(2)$. The corresponding unity blowup equations read

$$\sum_{k_1+k_2=k} \theta_3^{[a]}\left(2\tau, 2\left(\pm\frac{m}{2}+\frac{\epsilon_1+\epsilon_2}{4}-k_1\epsilon_1-k_2\epsilon_2\right)\right) \mathbb{E}_{k_1}\left(\tau, m\pm\frac{\epsilon_1}{2}, \epsilon_1, \epsilon_2-\epsilon_1\right) \mathbb{E}_{k_2}\left(\tau, m\pm\frac{\epsilon_2}{2}, \epsilon_1-\epsilon_2, \epsilon_2\right)$$

$$= \theta_3^{[a]}\left(2\tau, 2\left(\pm\frac{m}{2}+\frac{\epsilon_1+\epsilon_2}{4}\right)\right) \mathbb{E}_k(\tau, m, \epsilon_1, \epsilon_2), \tag{3.26}$$

where $a = 0, -1/2$.

These equations can be checked very explicitly. From the domain wall picture, we know that the elliptic genera of the M-string theory are [21][11]

$$\mathbb{E}_k(\tau, m, \epsilon_1, \epsilon_2) = \sum_{|\nu|=k} \prod_{(i,j)\in\nu} \frac{\theta_1(z_{ij})\theta_1(v_{ij})}{\theta_1(w_{ij})\theta_1(u_{ij})}, \tag{3.27}$$

where

$$z_{ij} = -m + (\nu_i - j + 1/2)\epsilon_1 + (i - 1/2)\epsilon_2, \quad v_{ij} = -m - (\nu_i - j + 1/2)\epsilon_1 - (i - 1/2)\epsilon_2,$$
$$w_{ij} = (\nu_i - j + 1)\epsilon_1 - (\nu_j^t - i)\epsilon_2, \qquad u_{ij} = (\nu_i - j)\epsilon_1 - (\nu_j^t - i + 1)\epsilon_2. \tag{3.28}$$

In particular, the one-string elliptic genus is [21]

$$\mathbb{E}_1(\tau, m, \epsilon_1, \epsilon_2) = \frac{\theta_1\left(\frac{1}{2}(\epsilon_1 + \epsilon_2) + m\right)\theta_1\left(\frac{1}{2}(\epsilon_1 + \epsilon_2) - m\right)}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)}. \tag{3.29}$$

Substituting (3.29) into (3.26), we find the unity blowup equations at base degree one is equivalent to

$$\frac{\theta_3^{[a]}(2\tau, \mp m + (3\epsilon_1 - \epsilon_2)/2)\theta_1(\epsilon_2/2 + (m \pm \epsilon_1/2))\theta_1(\epsilon_2/2 - (m \pm \epsilon_1/2))}{\theta_1(\epsilon_1)\theta_1(\epsilon_2 - \epsilon_1)}$$

$$+\frac{\theta_3^{[a]}(2\tau, \mp m + (-\epsilon_1 + 3\epsilon_2)/2)\theta_1(\epsilon_1/2 + (m \pm \epsilon_2/2))\theta_1(\epsilon_1/2 - (m \pm \epsilon_2/2))}{\theta_1(\epsilon_1 - \epsilon_2)\theta_1(\epsilon_2)}$$

$$=\frac{\theta_3^{[a]}(2\tau, \mp m - (\epsilon_1 + \epsilon_2)/2)\theta_1((\epsilon_1 + \epsilon_2)/2 + m)\theta_1((\epsilon_1 + \epsilon_2)/2 - m)}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)}, \tag{3.30}$$

which we have checked up to Q_τ^{10}. Using (3.27), we have also checked the unity blowup equations of base degrees two and three up to Q_τ^{10}.

We note that M-string theory has no vanishing blowup equations, as there is no value of $\underline{r}_m$ with which the theta function on the r.h.s. of (3.26) vanishes identically.

3.2 E-strings

We start by arguing that the sub-vector $\underline{r}_m$ of the $\underline{r}$-field associated to the flavor mass $\underline{m}$ is twice a weight vector of E_8. Let r_{m_i} $(i = 1, \ldots, r)$ be the components of $\underline{r}_m$ associated to e_i just like m_i, cf. (2.20). Since the curves α_i $(i = 1, \ldots)$ given by (2.19) are rational (-2)-curves, the components r_{m_i} must satisfy

$$r_{m_1} + r_{m_2} \equiv 0 \mod 2,$$
$$r_{m_i} - r_{m_{i+1}} \equiv 0 \mod 2, \quad i = 1, \ldots, 7, \tag{3.31}$$
$$\frac{r_{m_1} + r_{m_8}}{2} - \frac{r_{m_2} + \ldots + r_{m_7}}{2} \equiv 0 \mod 2.$$

These conditions are equivalent to that $\frac{1}{2}r_{m_i}$ are either all integers or all half integers and that $\sum_{i=1}^8 \frac{1}{2}r_{m_i}$ is an even number; in other words

$$\frac{1}{2}\underline{r}_m = \left(\frac{1}{2}r_{m_1}, \ldots, \frac{1}{2}r_{m_8}\right) \tag{3.32}$$

is a vector in the E_8 weight lattice.

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[11]We suppress the modular parameter of theta function if it is τ.

Using the modularity condition (3.22), we find that $\frac{1}{2}\underline{r}_m$ for unity blowup equations can be any of the 240 E_8 weight vectors whose norm square is 2; in other words, we have

$$\frac{1}{2}\underline{r}_m \in \mathcal{O}_{2,240} \tag{3.33}$$

for unity blowup equations of E-string theory. The equation itself reads

$$\sum_{k_1+k_2=k} \theta_1\left(\tau, \frac{1}{2}\underline{r}_m \cdot \underline{m} + \epsilon_1 + \epsilon_2 - k_1\epsilon_1 - k_2\epsilon_2\right) \mathbb{E}_{k_1}\left(\tau, \underline{m} + \frac{\epsilon_1}{2}\underline{r}_m, \epsilon_1, \epsilon_2 - \epsilon_1\right) \mathbb{E}_{k_2}\left(\tau, \underline{m} + \frac{\epsilon_2}{2}\underline{r}_m, \epsilon_1 - \epsilon_2, \epsilon_2\right)$$

$$= \theta_1\left(\tau, \frac{1}{2}\underline{r}_m \cdot \underline{m} + \epsilon_1 + \epsilon_2\right) \mathbb{E}_k(\tau, \underline{m}, \epsilon_1, \epsilon_2). \tag{3.34}$$

These equations can be checked explicitly. Using the expression of the one-string elliptic genus [16]

$$\mathbb{E}_1(\tau, \underline{m}, \epsilon_1, \epsilon_2) = -\left(\frac{A_1(\underline{m})}{\eta^8}\right)\frac{\eta^2}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)}, \tag{3.35}$$

where $A_1(\underline{m}) = \Theta_{E_8}(\tau, \underline{m})$ is the E_8 theta function, the unity blowup equation at base degree one reads

$$\frac{\theta_1\left(\frac{1}{2}\underline{r}_m \cdot \underline{m} + \epsilon_2\right) A_1\left(\underline{m} + \frac{1}{2}\underline{r}_m\epsilon_1\right)}{\theta_1(\epsilon_1)\theta_1(\epsilon_2 - \epsilon_1)} + \frac{\theta_1\left(\frac{1}{2}\underline{r}_m \cdot \underline{m} + \epsilon_1\right) A_1\left(\underline{m} + \frac{1}{2}\underline{r}_m\epsilon_2\right)}{\theta_1(\epsilon_1 - \epsilon_2)\theta_1(\epsilon_2)} = \frac{\theta_1\left(\frac{1}{2}\underline{r}_m \cdot \underline{m} + \epsilon_1 + \epsilon_2\right) A_1(\underline{m})}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)}, \tag{3.36}$$

which we have verified to very high orders of Q_τ. We have also verified the unity blowup equation at base degree two.

On the other hand, if we choose $\underline{r}_m = 0$, the parameter y and then the entire r.h.s. of (3.17) vanishes. The resulting vanishing blowup equation has been presented and verified in [16, 27]. We include it here as well for completeness

$$\sum_{k_1+k_2=k} \theta_1(\tau, k_1\epsilon_1 + k_2\epsilon_2)\mathbb{E}_{k_1}(\tau, \underline{m}, \epsilon_1, \epsilon_2 - \epsilon_1)\mathbb{E}_{k_2}(\tau, \underline{m}, \epsilon_1 - \epsilon_2, \epsilon_2) = 0. \tag{3.37}$$

It has been verified up to base degree $k = 3$ for high orders of Q_τ. Since there is no shift for the E_8 parameters, it is easy to see the above equation is also the vanishing blowup equations for massless E-string theory. In fact, it is the unique blowup equation for massless E-string.

3.3 Higher rank E- and M-strings

Here we construct the first instances of blowup equations for higher rank 6d SCFTs. The blowup equations of a higher rank 6d SCFT can in principle be obtained by gluing those of rank one theories.

We construct blowup equations for the simplest higher rank 6d SCFTs, the higher rank M-, E-strings, whose perpurbative prepotentials have been computed in section 2.3.3, using the blowup equations of M-, E-string theories as constituents. We will report the progress on constructing blowup equations for generic higher rank 6d SCFTs through gluing in the near future [52].

We first introduce a central ingredient of our construction. Given the lattice Γ of r base curves and a symmetric positive definite bilinear form on Γ defined by matrix Ω, we can naturally define a family of generalised theta functions with $\underline{z} \in \mathbb{C}^r$

$$\Theta_\Omega^{[\underline{a}]}(\tau, \underline{z}) = \sum_{\underline{n} \in \underline{a} + \mathbb{Z}^r} (-1)^{\underline{n} \cdot \text{diag}(\Omega)} \mathbf{e}\left[\frac{1}{2}\underline{n} \cdot \Omega \cdot \underline{n}\, \tau + \underline{n} \cdot \Omega \cdot \underline{z}\right], \qquad (3.38)$$

where $\text{diag}(\Omega)$ is the diagonal vector, and

$$\underline{a} \in \Omega^{-1} \cdot \left(\frac{1}{2}\text{diag}(\Omega) + \mathbb{Z}^r\right). \qquad (3.39)$$

It is clear that any two $\underline{a}$ differ by an integer vector are equivalent. The number of inequivalent $\underline{a}$-vectors is $\det(\Omega)$. The generalised theta function has the modular index polynomial

$$\text{ind}_\Omega = \frac{1}{2}\underline{z} \cdot \Omega \cdot \underline{z}. \qquad (3.40)$$

Consider the rank $r > 1$ M-string theory whose matrix Ω is the Cartan matrix of $\text{SU}(r+1)$. Let m be the SU(2) flavor symmetry, and $\mathbb{E}_{\underline{k}}(\tau, m, \epsilon_1, \epsilon_2)$ be the elliptic genus with wrapping numbers $\underline{k} = (k_1, \ldots, k_r)$ of the base curves. The idea to construct blowup equations of this theory is to "glue" the blowup equations for each individual (-2) base curves by merging the theta functions $\theta_3^{[a]}$ in those equations into Θ_Ω. The type of the resulting new equations can be determined by the following simple rule. We obtain a unity blowup equation if all the constituent blowup equations are of the unity type, and a vanishing blowup equation if one of the constituent blowup equations is of the vanishing type. Schematically we have

$$\text{U} \star \text{U} = \text{U}, \quad \text{U} \star \text{V} = \text{V}, \quad \text{V} \star \text{V} = \text{V}. \qquad (3.41)$$

Since the rank one M-string theory has only unity blowup equations, the higher rank M-string has also only unity blowup equations, whose exact form can be derived from the semiclassical data (2.70) to be

$$\sum_{\underline{k}'+\underline{k}''=\underline{k}} \Theta_\Omega^{[\underline{a}]}(\tau, \underline{M}_u - \underline{k}'\epsilon_1 - \underline{k}''\epsilon_2)\mathbb{E}_{\underline{k}'}\left(\tau, m + \frac{s}{2}\epsilon_1, \epsilon_1, \epsilon_2 - \epsilon_1\right)\mathbb{E}_{\underline{k}''}\left(\tau, m + \frac{s}{2}\epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2\right)$$

$$= \Theta_\Omega^{[\underline{a}]}(\tau, \underline{M}_u - \underline{k}'\epsilon_1 - \underline{k}''\epsilon_2)\mathbb{E}_{\underline{k}}(\tau, m, \epsilon_1, \epsilon_2), \qquad (3.42)$$

where

$$\underline{M}_u = \Omega^{-1} \cdot \left(sm + \frac{\epsilon_1 + \epsilon_2}{2}, \ldots, sm + \frac{\epsilon_1 + \epsilon_2}{2}\right), \qquad (3.43)$$

with $s = \pm 1$. The characteristic $\underline{a}$ takes the value in (3.39), and their total number is $\det(\Omega) = r + 1$.

These blowup equations can be checked in various ways. Using the modular index polynomial of $\mathbb{E}_{\underline{k}}$ of the higher rank M-string [16, 21, 53]

$$\text{ind}_{\underline{k}}^{\text{M}^r} = -\frac{(\epsilon_1 + \epsilon_2)^2}{4}\sum_{i=1}^r k_i + \frac{\epsilon_1\epsilon_2}{2}\underline{k} \cdot \Omega \cdot \underline{k} + m^2\sum_{i=1}^r k_i \qquad (3.44)$$

and (3.40), one can find easily that the modularity condition is satisfied. Furthermore, we have verified these equations at $\underline{k} = (1, 1)$ to high degrees of Q_τ with the explicit expressions of $\mathbb{E}_{\underline{k}}$ in [21]. Finally, it is possible to demonstrate that these equations reduce properly to the blowup equations of rank one M-string theory. We will use the shorthand notation that for a theory T,

$$\mathrm{V}_T^{[\underline{a}]} = 0, \quad \mathrm{U}_T^{[\underline{a}]} = 0 \tag{3.45}$$

denote the vanishing and the unity blowup equations with characteristic $\underline{a}$ respectively, where in the latter case we have moved the two sides of the equation together. Let us consider the M-M chain and decompactify the (-2) curve on the right. We can choose the inequivalent characteristics $\underline{a}$ of the unity blowup equations to be $\underline{a} = (0, 0)$, $(1/3, 2/3)$, $(2/3, 1/3)$, with the corresponding equations denoted by

$$\mathrm{U}_{\mathrm{MM}}^{[0,0]} = 0, \quad \mathrm{U}_{\mathrm{MM}}^{\left[\frac{1}{3}, \frac{2}{3}\right]} = 0, \quad \mathrm{U}_{\mathrm{MM}}^{\left[\frac{2}{3}, \frac{1}{3}\right]} = 0. \tag{3.46}$$

We can decompactify the (-2) curve on the right by setting k_2, k_2', k_2'' to zero. Then the two dimensions in the summation in $\Theta_\Omega^{[\underline{a}]}$ decouple. It is easy to deduce that in this limit

$$0 = \mathrm{U}_{\mathrm{MM}}^{[0,0]} = \theta_3^{[0]}(6\tau, 3z)\mathrm{U}_{\mathrm{M}}^{[0]} + \theta_3^{\left[-\frac{1}{2}\right]}(6\tau, 3z)\mathrm{U}_{\mathrm{M}}^{\left[\frac{1}{2}\right]},$$

$$0 = \mathrm{U}_{\mathrm{MM}}^{\left[\frac{1}{3}, \frac{2}{3}\right]} = \theta_3^{\left[\frac{1}{3}\right]}(6\tau, 3z)\mathrm{U}_{\mathrm{M}}^{[0]} + \theta_3^{\left[-\frac{1}{6}\right]}(6\tau, 3z)\mathrm{U}_{\mathrm{M}}^{\left[\frac{1}{2}\right]}, \tag{3.47}$$

$$0 = \mathrm{U}_{\mathrm{MM}}^{\left[\frac{2}{3}, \frac{1}{3}\right]} = \theta_3^{\left[\frac{2}{3}\right]}(6\tau, 3z)\mathrm{U}_{\mathrm{M}}^{[0]} + \theta_3^{\left[\frac{1}{6}\right]}(6\tau, 3z)\mathrm{U}_{\mathrm{M}}^{\left[\frac{1}{2}\right]},$$

where $z = sm + (\epsilon_1 + \epsilon_2)/2$, $s = \pm 1$. Since this is clearly a full-rank system for $\mathrm{U}_{\mathrm{M}}^{[0]}$ and $\mathrm{U}_{\mathrm{M}}^{[1/2]}$, we conclude

$$\mathrm{U}_{\mathrm{M}}^{[0]} = 0, \qquad \mathrm{U}_{\mathrm{M}}^{\left[\frac{1}{2}\right]} = 0. \tag{3.48}$$

These are exactly the unity blowup equations for M-string, as we already know. Similar situation happens when M^r chain reduces to M^{r-1} chain.

Let us move onto the rank $r > 1$ E-string theory. The matrix Ω is

$$\Omega = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \tag{3.49}$$

where the lower right $(r - 1) \times (r - 1)$ submatrix is the Cartan matrix of $\mathrm{SU}(r)$, which will be denoted by $\hat{\Omega}$. Let $\underline{m}$ and m be the E_8 and $\mathrm{SU}(2)$ flavor masses respectively, and $\mathbb{E}_{\underline{k}}(\tau, \underline{m}, m, \epsilon_1, \epsilon_2)$ with $\underline{k} = (k_0, k_1, \ldots, k_{r-1})$ be the elliptic genus with wrapping number k_0 on the (-1) base curve and wrapping numbers $\hat{\underline{k}} = (k_1, \ldots, k_{r-1})$ on the (-2) curves. The blowup equations of this theory is again constructed by merging the theta functions in

the constituent blowup equations of rank one E-, M-string theories to Θ_Ω. Following the rule (3.41), we expect vanishing blowup equations constructed from vanishing equations of the E-string theory and unity equations of the M-string theory, and unity blowup equations constructed from unity equations of both the E-, M-string theories. The exact forms of these blowup equations can be derived from the semiclassical data (2.69). The vanishing blowup equations of the rank r E-string theory read

$$\sum_{\underline{k}'+\underline{k}''=\underline{k}} \Theta_\Omega^{[\underline{a}]}(\tau, \underline{M}_v - \underline{k}'\epsilon_1 - \underline{k}''\epsilon_2)\mathbb{E}_{\underline{k}'}\left(\tau, \underline{m}, m+\frac{s}{2}\epsilon_1, \epsilon_1, \epsilon_2-\epsilon_1\right)\mathbb{E}_{\underline{k}''}\left(\tau, \underline{m}, m+\frac{s}{2}\epsilon_2, \epsilon_1-\epsilon_2, \epsilon_2\right) = 0\,, \tag{3.50}$$

where $s = \pm 1$ and

$$\underline{M}_v = \Omega^{-1} \cdot \left(0, sm + \frac{\epsilon_1 + \epsilon_2}{2}, \ldots, sm + \frac{\epsilon_1 + \epsilon_2}{2}\right). \tag{3.51}$$

The unity blowup equations read

$$\sum_{\underline{k}'+\underline{k}''=\underline{k}} \Theta_\Omega^{[\underline{a}]}(\tau, \underline{M}_u - \underline{k}'\epsilon_1 - \underline{k}''\epsilon_2)\mathbb{E}_{\underline{k}'}\left(\tau, \underline{m}+\alpha\epsilon_1, m+\frac{s}{2}\epsilon_1, \epsilon_1, \epsilon_2-\epsilon_1\right)\mathbb{E}_{\underline{k}''}\left(\tau, \underline{m}+\alpha\epsilon_2, m+\frac{s}{2}\epsilon_2, \epsilon_1-\epsilon_2, \epsilon_2\right)$$
$$= \Theta_\Omega^{[\underline{a}]}(\tau, \underline{M}_u)\mathbb{E}_{\underline{k}'}(\tau, \underline{m}, m\epsilon_1, \epsilon_2), \tag{3.52}$$

where $s = \pm 1$ and α is one of the 240 roots of E_8, and

$$\underline{M}_v = \Omega^{-1} \cdot \left(\alpha \cdot \underline{m} + \epsilon_1 + \epsilon_2, sm + \frac{\epsilon_1 + \epsilon_2}{2}, \ldots, sm + \frac{\epsilon_1 + \epsilon_2}{2}\right). \tag{3.53}$$

In both equations, $\underline{a}$ is unique and it can be writte as

$$\underline{a} = \Omega^{-1} \cdot \left(\frac{1}{2}, 0, \ldots, 0\right). \tag{3.54}$$

We verify these blowup equations in the following ways. First of all, using the modular index polynomial of $\mathbb{E}_{\underline{k}}$

$$\text{ind}_{\underline{k}}^{\text{EM}^{r-1}} = -\frac{(\epsilon_1 + \epsilon_2)^2}{4}\left(2k_0 + \sum_{i=1}^{r-1} k_i\right) + \frac{\epsilon_1\epsilon_2}{2}(\hat{\underline{k}}\cdot\hat{\Omega}\cdot\hat{\underline{k}}+k_0) + \frac{k_0}{2}(\underline{m}, \underline{m})_{E_8} + m^2\sum_{i=1}^{r-1} k_i. \tag{3.55}$$

and (3.40), we find (3.50), (3.52) satisfy the modularity condition. Furthermore, we verified these equations at $\underline{k} = (1, 1)$ up to high orders of Q_τ with the explicit expressions of $\mathbb{E}_{\underline{k}}$ given in [50]. Finally, we demonstrate that the blowup equations of the rank two E-string, or the E–M chain, can be reduced to the blowup equations of E-, M-string theories by decompactifying base curves. The blowup equations of the E–M chain all have a unique characteristic which we choose to be $\underline{a} = (0, 1/2)$. Let us first decompactify the (-1) curve by setting $k_0 = 0$. The vanishing blowup equations of the E–M chain become in this limit

$$0 = \text{V}_{\text{EM}}^{[0,\frac{1}{2}]} = \theta_3(2\tau, sm + (\epsilon_1 + \epsilon_2)/2) \cdot \text{U}_{\text{M}}^{[-\frac{1}{2}]} - \theta_3(2\tau, sm + (\epsilon_1 + \epsilon_2)/2) \cdot \text{U}_{\text{M}}^{[0]}, \tag{3.56}$$

while the unity blowup equations become

$$0 = \text{U}_{\text{EM}}^{[0,\frac{1}{2}]} = \theta_2(2\tau, 2\underline{m}\cdot\alpha + sm + 5(\epsilon_1 + \epsilon_2)/2) \cdot \text{U}_{\text{M}}^{[-\frac{1}{2}]} - \theta_3(2\tau, 2\underline{m}\cdot\alpha + sm + 5(\epsilon_1 + \epsilon_2)/2) \cdot \text{U}_{\text{M}}^{[0]}. \tag{3.57}$$

Since $s = \pm 1$ and $\alpha \in \Delta(E_8)$, we have a full rank system for $\mathrm{U}_{\mathrm{M}}^{\left[-\frac{1}{2}\right]}$ and $\mathrm{U}_{\mathrm{M}}^{[0]}$, and therefore $\mathrm{U}_{\mathrm{M}}^{\left[-\frac{1}{2}\right]} = 0$ and $\mathrm{U}_{\mathrm{M}}^{[0]} = 0$, which are the unity blowup equations of the M-string as we know. Next we decompactify the (-2) curve by setting $k_1 = 1$. The vanishing blowup equations of the E–M chain become in this limit

$$0 = \mathrm{V}_{\mathrm{EM}}^{\left[0,\frac{1}{2}\right]} = \theta_2(\tau, sm + (\epsilon_1 + \epsilon_2)/2) \cdot \mathrm{V}_{\mathrm{E}}^{\left[-\frac{1}{2}\right]}. \tag{3.58}$$

Thus $\mathrm{V}_{\mathrm{E}}^{\left[-\frac{1}{2}\right]} = 0$, which is the vanishing elliptic blowup equation for E-string as we know. The unity blowup equations of E–M chain become

$$0 = \mathrm{U}_{\mathrm{EM}}^{\left[0,\frac{1}{2}\right]} = \theta_2(\tau, \underline{m} \cdot \alpha + sm + 3(\epsilon_1 + \epsilon_2)/2) \cdot \mathrm{U}_{\mathrm{E}}^{\left[-\frac{1}{2}\right]}. \tag{3.59}$$

Thus $\mathrm{U}_{\mathrm{E}}^{\left[-\frac{1}{2}\right]} = 0$, which are the 240 unity elliptic blowup equations for E-string.

We comment that in general, the blowup equations of a higher rank theory do not necessarily reduce to all the blowup equations of the blowup equations of lower rank blowup equations due to the gluing rules. We will give a more detailed discussion in our future work [52].

4 Solution of blowup equations

Here we would like to argue that the elliptic genera of the E-, M-string theories, or equivalently the refined BPS invariants of the associated geometries, can be solved completely from their blowup equations.

To see this quickly, we quote the following statement from [27], obtained by counting the number of equations satisfied by the refined topological string free energies $F_{(n,g)}$ extracted from blowup equations through expansion in terms of ϵ_1, ϵ_2. Given a generic local Calaib-Yau threefold X with b_2^c Kähler parameters, and let w_u, w_v be the numbers of inequivalent unity and vanishing blowup equations respectively.[12] Then all the refined BPS invariants on X can be completely solved from the blowup equations with the input of $F_{(0,0)}$ if

$$w_u \geq 1 \quad \text{and} \quad w_u + w_v \geq b_2^c. \tag{4.1}$$

For the E-string theory with $b_2^c = 10$ and

$$w_u = 240, \quad w_v = 1, \tag{4.2}$$

and the M-string theory with $b_2^c = 3$ and

$$w_u = 4, \quad w_v = 0, \tag{4.3}$$

the condition (4.1) is clearly satisfied.

Next we corroborate the solvability of the E-, M-string theories by presenting two explicit algorithms to solve the two theories from the blowup equations.

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[12]In [27] w_u, w_v are defined to be the numbers of inequivalent $\underline{r}$-fields which are in addition *not* reflective to each other. This additional condition is actually not necessary.

4.1 Extraction of refined BPS invarianys

The first method is to extract relations of refined BPS invariants from unity blowup equations. The instanton partition function Z^{inst} encodes the BPS invariants by [51, 54]

$$Z^{\text{inst}} = \exp\left[\sum_{j_L,j_R=0}^{\infty}\sum_{\beta}\sum_{w=1}^{\infty}(-1)^{2(j_L+j_R)}\frac{N_{j_L,j_R}^{\beta}}{w}f_{(j_L,j_R)}(q_1^w,q_2^w)Q^{w\beta}\right]. \quad (4.4)$$

Here we define

$$f_{(j_L,j_R)}(q_1,q_2) = \frac{\chi_{j_L}(q_L)\chi_{j_R}(q_R)}{(q_1^{1/2}-q_1^{-1/2})(q_2^{1/2}-q_2^{-1/2})}, \quad (4.5)$$

where $\chi_j(q)$ is the SU(2) character given by

$$\chi_j(q) = \frac{q^{2j+1}-q^{-2j-1}}{q-q^{-1}}. \quad (4.6)$$

In addition, we define

$$Q^{\beta} = \mathbf{e}[\underline{d}(\beta)\cdot\underline{t}] \quad (4.7)$$

with $\underline{d}(\beta)$ the degree of the curve class β in certain basis of $H_2(X,\mathbb{R})$.[13] We choose a basis which we call *positive* basis so the $\underline{d}(\beta)$ is a non-negative vector[14] for any effective curve β in X. The contribution of a spin (j_L,j_R) BPS state wrapping curve class β to the blowup equation is then

$$Bl_{(j_L,j_R,R)}(q_1,q_2) = f_{(j_L,j_R)}(q_1,q_2/q_1)q_1^R + f_{(j_L,j_R)}(q_1/q_2,q_2)q_2^R - f_{(j_L,j_R)}(q_1,q_2), \quad (4.8)$$

where $R = R(\beta,\underline{n})$ is the component of the $\underline{R}$-vector (3.1) associated to β, and the blowup equations (3.4) can be written in the following form

$$\sum_{\underline{n}\in\mathbb{Z}^{b_4^c}}(-1)^{|\underline{n}|}e^{f_0(\underline{n})(\epsilon_1+\epsilon_2)+\sum_{i=1}^{b_2^c}f_i(\underline{n})t_i}\exp\left[-\sum_{j_L,j_R,\beta}\sum_{w=1}^{\infty}N_{j_L,j_R}^{\beta}\frac{Q^{w\beta}}{w}Bl_{(j_L,j_R,R)}(q_1^w,q_2^w)\right] = \Lambda(\epsilon_1,\epsilon_2,\underline{m}). \quad (4.9)$$

Then if there exists a unity $\underline{r}$-field[15] so that for a positive basis of $H_2(X,\mathbb{R})$

$$* : \text{all } f_i(\underline{n}) \text{ can be minimised at the same time with a unique solution } \underline{n} = \underline{n}_0 \quad (4.10)$$

the coefficients of Q^{β} can be written as

$$\sum_{j_L,j_R}N_{j_L,j_R}^{\beta}Bl_{(j_L,j_R,R(\beta,\underline{n}_0))}(q_1,q_2) = I^{\beta}(q_1,q_2), \quad (4.11)$$

---

[13]Sometimes the basis we choose is not integral and the curve degrees can be fractional numbers.

[14]In the case that the Mori cone of X is simplicial, one can choose the basis to consist of Mori cone generators. When the Mori cone is not simplicial, one can choose the generators of a larger but simplicial cone that covers the Mori cone.

[15]For a vanishing $\underline{r}$-field, the solutions $\underline{n}$ that minimize $f_i(\underline{n})$ are all in pairs.

where $I^\beta(q_1, q_2)$ collects contributions of multi-wrapping $w > 1$, summands with $\underline{n} \neq \underline{n}_0$, and higher order expansions of the exponential, and it only contains BPS invariants of curve degrees $\underline{d} < \underline{d}(\beta)$, which means

$$\underline{d} < \underline{d}(\beta): \quad d_i \leq d_i(\beta), \quad i = 1, \ldots, b_2^c; \quad \text{at least one inequality is not saturated.} \quad (4.12)$$

Since the unique minimal solution $\underline{n}_0$, if exists, can be shifted to $\underline{0}$ using the equivalence relation (3.5), we always assume that $\underline{n}_0 = \underline{0}$ and thus $R(\beta, \underline{n}_0) = \underline{r} \cdot \underline{d}(\beta)/2$. Let us now assume that the r.h.s. of (4.11) has already be computed. Then given the lemma proved in appendix A that, with a few exceptions, all $Bl_{(j_L, j_R, R)}$ with fixed R and different j_L, j_R (which are finite in number) are linearly independent to each other and non-vanishing, the BPS invariants $N^\beta_{j_L, j_R}$ can be extracted from the l.h.s. of (4.11) and thus recursively computed. The only exceptions are the spin $(0,0)$ or $\left(0, \frac{1}{2}\right)$ BPS invariants with curve classes β satisfying

$$|\underline{r} \cdot \underline{d}(\beta)| \leq 1. \quad (4.13)$$

These invariants cannot be solved from the l.h.s. of (4.11) due to the fact that

$$Bl_{\left(0,0,\pm\frac{1}{2}\right)} = Bl_{\left(0,\frac{1}{2},0\right)} = 0. \quad (4.14)$$

In practice, we can use all the unity blowup equations whose $\underline{r}$-fields satisfy the condition (4.10) to solve all the refined BPS invariants, provided that we have some extra data which can help determine the spin $(0,0)$, $\left(0, \frac{1}{2}\right)$ BPS invariants whose curve class β satisfy (4.13) for all the unity $\underline{r}$-fields that we have used. In an elliptic non-compact CY3, the number of such curve classes can be infinite. Nevertheless in the examples where the toric hypersurface construction or the mirror curve is known, the genus 0 GV invariants of β can be computed and used as extra input data, from which, together with the BPS invariants of β with spins other than $(0,0)$ or $\left(0, \frac{1}{2}\right)$ recursively extracted from (4.11), the unknown invariants $N^\beta_{0,0}$ or $N^\beta_{0,\frac{1}{2}}$ can be deduced.[16] We have successfully used this approach to compute the refined BPS invariants of the massive M-string and the E-string with up to three flavor masses turned on, where the genus 0 GV invariants as extra input data are computed using the toric hypersurface construction in sections 2.1.1, 2.2.1.

In the following, we illustrate for the examples of E-, M-string theories the finding of unity $\underline{r}$-fields satisfying (4.10) and all the ambiguous BPS invariants with β of the type (4.13).

4.1.1 M-string

We choose the positive basis of $H_2(X, \mathbb{R})$ to be the generators of the Mori cone, whose Kähler parameters are [21]

$$t_b - m, \quad \tau - m, \quad m. \quad (4.15)$$

---

[16]We remind the readers that due to the checkerboard pattern BPS states wrapping a fixed curve classs can have either spin $(0,0)$ or $\left(0, \frac{1}{2}\right)$ but not both.

The corresponding $f_i(n)$ functions are

$$f_m(n) = \left(n - \frac{r_b}{4} + \frac{r_m}{2}\right)^2 + \text{const.},$$

$$f_{\tau-m}(n) = \left(n - \frac{r_b}{4}\right)^2 + \text{const.}, \tag{4.16}$$

$$f_{b-m}(n) = \text{const.}.$$

The unity $\underline{r}$-fields of the M-string theory associated to the elliptic blowup equations (3.26) are[17]

$$\underline{r} = (r_b, r_m, r_\tau) = (2k, \pm 1, 0), \quad k \in \mathbb{Z}, \tag{4.17}$$

among which, the ones which minimise (4.16) with the unique solution $\underline{n} = \underline{0}$ are

$$\underline{r} = (0, \pm 1, 0), (2, 1, 0), (-2, -1, 0). \tag{4.18}$$

The only BPS invariants which cannot be determined from the unity blowup equations with these $\underline{r}$-fields are the spin (0,0) invariants with

$$\underline{d} = (d_b, d_m, d_\tau) = (1, -1, d_\tau), \quad d_\tau = 0, 1, 2, \ldots. \tag{4.19}$$

4.1.2 E-string

In the case of E-string theory, we cannot find a positive basis of $H_2(X, \mathbb{R})$.[18] However we find that the following basis

$$t_b, \quad \tau, \quad \hat{m}_i = (\alpha_i, \underline{m}), \ i = 1, \ldots, 8, \tag{4.20}$$

where α_i are the simple roots of E_8, is good enough with which a similar statement concerning the solvability of BPS invariants from unity blowup equations can be made. Using the perturbative prepotential (2.37), we find the corresponding $f_i(n)$ functions are

$$f_b(n) = \text{const.},$$

$$f_\tau(n) = \frac{1}{2}\left(n - \frac{r_b}{2}\right)^2 + \text{const.}, \tag{4.21}$$

$$f_{\hat{m},i}(n) = \frac{1}{2}\left(n - \frac{r_b}{2}\right) r_{\hat{m},i} + \text{consts.}, \ i = 1, \ldots, 8,$$

as well as

$$f_0(n) = \frac{1}{2}\left(n - \frac{r_b}{2}\right)(\underline{r}_m/2, \underline{r}_m/2)_{E_8} + \text{const.}. \tag{4.22}$$

Let us recall that in a unity $\underline{r}$-field the $\underline{r}_m$ vector when divided by two is an element in $\mathcal{O}_{2,240}$ so that when expanded in terms of α_i the coefficients $r_{\hat{m},i}$ are either all non-negative or all non-positive. We denote these two cases by $\underline{r}_m > 0$ and $\underline{r}_m < 0$ respectively. We then choose 240 unity $\underline{r} = (r_b, \underline{r}_m, r_\tau)$ fields to be

$$\underline{r} = (+1, \underline{r}_m > 0, 0), (-1, \underline{r}_m < 0, 0), \quad \frac{1}{2}\underline{r}_m \in \mathcal{O}_{2,240}, \tag{4.23}$$

---

[17]We do not mod out the shift $2C \cdot n$ equivalence here.

[18]This is due to the fact that the half K3 surface associated to the E-string theory has a highly non-simplicial Mori cone whose generators are infinitely many.

so that

$$r_b r_{\hat{m},i} \geq 0, \quad i = 1, \ldots, 8, \quad \text{and } r_b \sum_i r_{\hat{m},i} > 0. \tag{4.24}$$

The BPS invariants with $d_b = 0$ decouple from the true Kähler modulus t_b and are thus factored out of the blowup equations. For the remaining BPS invariants, using the fact that $d_b > 0, d_\tau \geq 0$ and that $f_\tau(n)$ is minimised by $n \in \{0, r_b\}$, we can mimic (4.11) and by expanding (4.9) find

$$\sum_{j_L, j_R} N^{d_b, d_\tau, d_{\hat{m},i}}_{j_L, j_R} Bl_{(j_L, j_R, R(\beta, 0))}(q_1, q_2)$$
$$- (q_1 q_2)^{\frac{r_b}{2}\left(\frac{r_m}{2}, \frac{r_m}{2}\right)_{E_8}} \sum_{j_L, j_R} N^{d_b, d_\tau, d_{\hat{m},i} - \frac{1}{2} r_b r_{\hat{m},i}}_{j_L, j_R} Bl_{\left(j_L, j_R, R\left(\beta, \frac{r_b}{2}\right)\right)} = I^{d_b, d_\tau, d_{\hat{m},i}}(q_1, q_2), \tag{4.25}$$

where again $I^{d_b, d_\tau, d_{\hat{m},i}}(q_1, q_2)$ collects contributions of multi-wrapping $w > 1$, summands with $n \neq 0, r_b$, and etc. so that it only constrains BPS invariants with $(d'_b, d'_\tau, d'_{\hat{m},i}) < (d_b, d_\tau, d_{\hat{m},i})$ in the sense of (4.12). On the other hand, since $\left(d_b, d_\tau, d_{\hat{m},i} - \frac{1}{2} r_b r_{\hat{m},i}\right) < (d_b, d_\tau, d_{\hat{m},i})$, we can move the second term on the l.h.s. to the r.h.s. and absorb it into $I^{d_b, d_\tau, d_{\hat{m},i}}(q_1, q_2)$. We find the same recursive expression as (4.11) and the same procedures following (4.11) apply.

We also find that the only BPS invariants which cannot be determined from the unity blowup equations associated to (4.23) are the spin (0,0) BPS invariants with

$$\underline{d} = (d_b, d_{\hat{m},i}, d_\tau) = (1, 0, d_\tau), \quad d_\tau = 0, 1, 2, \ldots . \tag{4.26}$$

4.2 Weyl orbit expansion

In this section, we show how to directly solve elliptic genera from blowup equations rather than computing the refined BPS invariants. The basic idea is to express an elliptic genus as an expansion with respect to $q = Q_\tau$ and $v = \sqrt{q_1 q_2}$ with coefficients as Weyl orbits of the flavor symmetry F and $\mathrm{SU}(2)_x$ where $x = \sqrt{q_1/q_2}$. It is well-known the reduced one-string elliptic genus is independent from $\mathrm{SU}(2)_x$. This makes solving the one-string elliptic genus from blowup equations particularly simple. In the following we mainly demonstrate how this method works for E-strings. As a byproduct, we obtain some interesting functional equations for the E_8 theta function. The efficiency of this method will be further illustrated in [41].

The reduced one E-string elliptic genus is well-known to be

$$\mathbb{E}_1^{\mathrm{red}} = \frac{\theta_1(\epsilon_1)\theta_1(\epsilon_2)}{\eta^2}\mathbb{E}_1 = \eta^{-8}\Theta_{E_8}(\underline{m}) = \eta^{-8}\sum_{\mathcal{O}_{p,k}} q^{p/2} \cdot \mathcal{O}_{p,k}$$
$$= q^{-\frac{1}{3}}\left(1 + \mathbf{248}q + (\mathbf{3875} + \mathbf{248} + 1)q^2 + (\mathbf{30380} + \mathbf{3875} + 2 \times \mathbf{248} + 1)q^3 \right.$$
$$\left. + (\mathbf{147250} + 2 \times \mathbf{30380} + \mathbf{3875} + 5 \times \mathbf{248} + 1)q^4 + \ldots \right), \tag{4.27}$$

where

$$\Theta_{E_8}(\tau, \underline{m}) = \sum_{\underline{k} \in \Gamma_{E_8}} \exp(\pi i \tau \underline{k} \cdot \underline{k} + 2\pi i \underline{m} \cdot \underline{k}) = \frac{1}{2}\sum_{k=1}^4 \prod_{\ell=1}^8 \theta_k(\tau, m_\ell). \tag{4.28}$$

$R = \alpha \cdot w$	-5	-4	-3	-2	-1	0	1	2	3	4	5
$\mathcal{O}_{0,1}$						1					
$\mathcal{O}_{2,240}$			1	56	126	56	1				
$\mathcal{O}_{4,2160}$			126	576	756	576	126				
$\mathcal{O}_{6,6720}$		56	756	1512	2072	1512	756	56			
$\mathcal{O}_{8,240}$		1		56		126		56		1	
$\mathcal{O}_{8,17280}$			576	2016	4032	4032	4032	2016	576		
$\mathcal{O}_{10,30240}$		126	1512	4158	5544	7560	5544	4158	1512	126	
$\mathcal{O}_{12,60480}$		756	4032	7560	12096	11592	12096	7560	4032	756	
$\mathcal{O}_{14,13440}$	56	56	1512	1512	1568	4032	1568	1512	1512	56	56
$\mathcal{O}_{14,69120}$		2016	4032	10080	12096	12672	12096	10080	4032	2016	
$\mathcal{O}_{16,2160}$		126		576		756		576		126	
$\mathcal{O}_{16,138240}$	576	4032	12096	16128	24192	24192	24192	16128	12096	4032	576

Table 8. Intersection numbers between roots and elements of E_8 Weyl orbits.

The first few E_8 Weyl orbits are as follows:

$$\mathcal{O}_{0,1}, \quad \mathcal{O}_{2,240}, \quad \mathcal{O}_{4,2160}, \quad \mathcal{O}_{6,6720}, \quad \mathcal{O}_{8,240}, \quad \mathcal{O}_{8,17280}, \quad \mathcal{O}_{10,30240}$$
$$\mathcal{O}_{12,60480}, \quad \mathcal{O}_{14,13440}, \quad \mathcal{O}_{14,69120}, \quad \mathcal{O}_{16,2160}, \quad \mathcal{O}_{16,138240}, \quad \mathcal{O}_{18,240}, \quad \mathcal{O}_{18,181440},$$
$$\mathcal{O}_{20,30240}, \quad \mathcal{O}_{20,241920}, \quad \mathcal{O}_{22,138240}, \quad \mathcal{O}_{22,181440}, \quad \mathcal{O}_{24,6720}, \quad \mathcal{O}_{24,483840},$$
$$\mathcal{O}_{26,13440}, \quad \mathcal{O}_{26,30240}, \quad \mathcal{O}_{26,483840}, \quad \cdots . \tag{4.29}$$

In unity blowup equations, each Weyl orbit breaks down due to the shifts proportional to a root. For example, for $\mathcal{O}_{2,240}$,

$$\sum_{w \in \mathcal{O}_{2,240}} \mathrm{e}^{w \cdot (m + \epsilon_1 \alpha)} = q_1^{-2} \mathrm{e}^{-\alpha \cdot m} + \sum_{\alpha \cdot w = -1} q_1^{-1} \mathrm{e}^{w \cdot m} + \sum_{\alpha \cdot w = 0} \mathrm{e}^{w \cdot m} + \sum_{\alpha \cdot w = 1} q_1 \mathrm{e}^{w \cdot m} + q_1^2 \mathrm{e}^{\alpha \cdot m}.$$
$$\tag{4.30}$$

This forces us to look into how every Weyl orbit splits under the shift of a root. Due to the Weyl symmetry, all the elements in one Weyl orbit intersect with any of the roots in the same way, i.e. for any root the distribution of intersection numbers $R = \alpha \cdot w$ between the root and all Weyl orbit elements is the same. For example, for any positive root α, we list the distribution for some Weyl orbits in table 8. Note the elements are all Weyl orbits of E_7. Knowing (4.27), it is easy to check the unity blowup equations (3.34) are correct.

Conversely it is possible to solve (4.27) from the blowup equations. Let us first write $\mathbb{E}_1^{\mathrm{red}} = f(q, v, \underline{m})/\eta^8$.[19] The vanishing blowup equation (3.37) gives

$$f\left(q, \frac{\epsilon_1}{2}, \underline{m}\right) = f\left(q, \frac{\epsilon_2}{2}, \underline{m}\right). \tag{4.31}$$

---

[19]The denominator η^8 can be later determined by requiring that $\mathbb{E}_1^{\mathrm{red}}$ be decomposed as representations of E_8, rather than just Weyl orbits. Besides, there is an overall constant in front of the whole elliptic genus $\mathbb{E}_1$ that can not determined by blowup equations due to the lack of gauge symmetry. This is of course not surprising. Here we assume the overall constant is 1.

Thus $f(q, v, \underline{m})$ is independent from v. We can simply write it as

$$f(q, \underline{m}) = \sum_{n=0}^{\infty} q^n \sum_{\mathcal{O}_{p,k}} x_{n,p,k} \mathcal{O}_{p,k}. \tag{4.32}$$

The task is to determine all $x_{n,p,k}$. It is convenient to write the unity blowup equations as

$$\theta_1(\epsilon_2)\theta_1(\alpha \cdot m + \epsilon_2) f(q, m + \epsilon_1 \alpha) - \theta_1(\epsilon_1)\theta_1(\alpha \cdot m + \epsilon_1) f(q, m + \epsilon_2 \alpha) = \theta_1(\epsilon_2 - \epsilon_1)\theta_1(\alpha \cdot m + \epsilon_1 + \epsilon_2) f(q, m), \tag{4.33}$$

where $\alpha \in \Delta(E_8)$. We conjecture the solution is uniquely $f(\tau, \underline{m}) = \eta^{-8}\Theta_{E_8}(\tau, \underline{m})$ under the conditions:

- The q expansion coefficients of $\mathbb{E}_1$ can be decomposed as sums of irreducible representations of E_8;

- The leading q expansion coefficient is 1, i.e. the trivial E_8 orbit $\mathcal{O}_{0,1}$.

Note the blowup equations themselves only determine $f(q, \underline{m})$ up to a free function of τ. The two assumptions assure the prefactor is η^{-8}.[20] In fact, it is proved by Don Zagier that (4.33) has a unique solution which is the E_8 theta function up to a free function of τ, and similar statements can be made for arbitrary positive definite even unimodular lattices generated by roots, such as the $E_8 \times E_8$ lattice and the Barnes-Wall lattice Λ_{16} in dimension 16 and the 23 Niemeier lattices in dimension 24. We give the proof in appendix B.

Now we briefly show how the Weyl orbit recursion works. Given that we have assumed the leading q order of $f(q, \underline{m})$ is the trivial orbit $\mathcal{O}_{0,1}$, we find that in order for the subleading order of (4.33) to be satisfied, the subleading order of $f(q, \underline{m})$ should have two $\mathcal{O}_{0,1}$ with $R = \pm 2$. Looking up in table 8, one finds that in order to store the E_8 symmetry, one has to add two E_7 orbits of length 56 at $R = \pm 1$ and one E_7 orbit of length 126 at $R = 0$. Thus in the subleading order, $f(q, \underline{m})$ has the E_8 orbit $\mathcal{O}_{2,240}$. Next, for the subsubleading order of (4.33) to be satisfied, one needs to add two E_7 orbits of length 126 at $R = \pm 2$ in the subsubleading order of $f(q, \underline{m})$ to cancel the effect of the previous E_7 orbit of length 56. Then to restore the E_8 symmetry, one needs to add two E_7 orbits of length 576 at $R = \pm 1$ and one E_7 orbit of length 756 at $R = 0$. Repeating this process, we find each sub E_7 Weyl orbit in table 8 is in an infinite series of the ones in the larger Weyl orbits. Moreover, the contributions from each infinite series can be organized into one of the following two identities:

$$\theta_1(\epsilon_2)\theta_1(\lambda + \epsilon_2)\theta_3(2\tau, \lambda + 2\epsilon_1) - \theta_1(\epsilon_1)\theta_1(\lambda + \epsilon_1)\theta_3(2\tau, \lambda + 2\epsilon_2) = \theta_1(\epsilon_2 - \epsilon_1)\theta_1(\lambda + \epsilon_1 + \epsilon_2)\theta_3(2\tau, \lambda)$$

$$\theta_1(\epsilon_2)\theta_1(\lambda + \epsilon_2)\theta_2(2\tau, \lambda + 2\epsilon_1) - \theta_1(\epsilon_1)\theta_1(\lambda + \epsilon_1)\theta_2(2\tau, \lambda + 2\epsilon_2) = \theta_1(\epsilon_2 - \epsilon_1)\theta_1(\lambda + \epsilon_1 + \epsilon_2)\theta_2(2\tau, \lambda). \tag{4.34}$$

With this in mind, one can directly write down the Weyl orbit expansion satisfying the unity blowup equations by the following rule: each sub E_7 orbit in an E_8 Weyl orbit $\mathcal{O}_{p,k}$ with intersection number R generates an infinite series of sub E_7 orbits in E_8 Weyl orbits $\mathcal{O}_{p',k'}$ with intersection numbers R' where p' grows quadratically and $|R'|$ grows linearly. To be more explicit,

---

[20] This agrees with the degrees of refined BPS invariants that need to be input for the E-string blowup equations in (4.26).

- If R is even, the growth is based on $\theta_3(2\tau, 2z)$, i.e. p' increases by $2n^2$ and $|R'|$ increases by $2n$;

- If R is odd, the growth is based on $\theta_2(2\tau, 2z)$, i.e. p' increases by $2n(n+1)$ and $|R'|$ increases by $2n$.

We have marked some sub E_7 orbits in the same series with the same color in table 8. It turns out all E_8 Weyl orbits appear and just appear once in $f(q,\underline{m})$, which means it is indeed the E_8 theta function.

The same procedure can also be used to determine the two E-string elliptic genus, where the flavor symmetry is effectively $E_8 \times \mathrm{SU}(2)_x$. In this case, it is convenient to work with the following reduced version of two E-strings elliptic genus

$$f(q,v,x,\underline{m}) = Z_2^{\mathrm{E\text{-}str}}(\tau,\epsilon_1,\epsilon_2,\underline{m})\theta_1(\epsilon_1)\theta_1(2\epsilon_1)\theta_1(\epsilon_2)\theta_1(2\epsilon_2)\eta^{-2}. \tag{4.35}$$

From the unity blowup equations, one can solve it as

$$\begin{aligned}
f(q,v,x,\underline{m}) =& (v^{-1}+v)+(\mathbf{248}(x^{-1}+x)+\mathbf{248}(v^{-1}+v)-(v^{-3}+v^3)(x^{-2}+x^2))q \\
&+(-(v^{-5}+v^5)(x^{-2}+x^2)-\mathbf{248}(v^{-4}+v^4)(x^{-1}+x)+(v^{-3}+v^3)(-\mathbf{248}(x^{-2}+x^2)+\mathbf{3875}) \\
&+(v^{-1}+v)((x^{-4}+x^4)+\mathbf{3875}(x^{-2}+x^2)+\mathbf{30380}+3\times\mathbf{248})+\mathbf{248}(x^{-3}+x^3) \\
&+(\mathbf{30380}+\mathbf{3875}+\mathbf{248}+1)(x^{-3}+x^3))q^2+\dots.
\end{aligned}$$
$$\tag{4.36}$$

For M-strings, it is completely parallel and actually easier to solve elliptic genera from the unity blowup equations, as the flavor SU(2) is much simpler than E_8.

5 From E-strings to del Pezzo surfaces

Rank one[21] 5d SCFTs have been classified [55–60]. They can be constructed through M-theory compactified on the canonical bundle over del Pezzo surfaces, which include $\mathbb{P}^2, \mathbb{F}_0$ or d ($d = 1,\dots,8$) points blow-ups of $\mathbb{P}^2$ denoted by dP_d, and their BPS spectra have been computed in [49, 60]. The superconformal limit is reached when the entire complex surface is shrunk to a point. In particular the 5d SCFTs associated to dP_d ($d = 6,7,8$) are the famous Minaham-Nemeschansky theories with E_d flavor symmetry [4, 5]. All these 5d SCFTs can be obtained from the E-string theory on S^1. Indeed the del Pezzo surfaces can be obtained from the half K3 surface by successively blowing down the exceptional divisors. From the physics point of view, rank one 5d SCFTs are effectively 5d SU(2) gauge theory in the IR[22] with up to 7 hypermultiplets transforming in the fundamental representation of SU(2), while the E-string theory on S^1 has an effective description of a 5d SU(2) gauge theory with 8 hypermultiplets. The blow-down operation in geometry corresponds to successively decoupling hypermultiplets by giving them infinitely large mass.[23] The relation between E-string theory and the family of rank one 5d SCFTs can be summarised in figure 1.

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[21]The rank of a 5d SCFT is the dimension of its Coulomb branch.

[22]The 5d SCFT associated to $\mathbb{P}^2$ does not have a low energy effective gauge theory description. See the diagram in the main text.

[23]To be exact, the mass of the hypermultiplet to be decoupled should be sent to negative infinity.
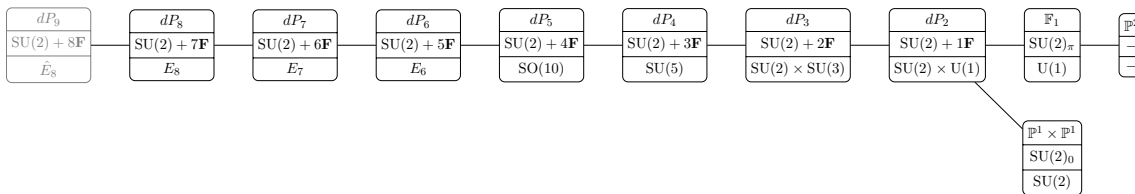
Figure 1. Rank one 5d SCFTs and E-string theory on S^1: the associated geometry, the IR effective gauge theory, and the global symmetry at the conformal point.

We demonstrate here that blowup equations for rank one 5d SCFTs in the Coulomb branch can be obtained from the blowup equations of the E-string theory by carefully carrying out the transformations that correspond to the blow-down operations.

Our starting point is the unity and vanishing blowup equations of the E-string theory (3.34), (3.37), which we can uniformly write as

$$
\sum_{n\in\mathbb{Z}}(-1)^n\mathbf{e}\left[\frac{1}{2}\left(n+\frac{1}{2}\right)^2\tau+\left(n+\frac{1}{2}\right)\left(\frac{1}{2}\underline{r}_m\cdot\underline{m}+\frac{1}{8}\underline{r}_m\cdot\underline{r}_m(\epsilon_1+\epsilon_2)\right)\right]\widehat{Z}_{dP_9}^{\mathrm{inst}}(\tau,\underline{m},t_b,\epsilon_1,\epsilon_2)^{-1}
$$

$$
\times\widehat{Z}_{dP_9}^{\mathrm{inst}}\left(\tau,\underline{m}+\frac{\epsilon_1}{2}\underline{r}_m,t_b-\left(n+\frac{1}{2}\right)\epsilon_1,\epsilon_1,\epsilon_2-\epsilon_1\right)\widehat{Z}_{dP_9}^{\mathrm{inst}}\left(\tau,\underline{m}+\frac{\epsilon_2}{2}\underline{r}_m,t_b-\left(n+\frac{1}{2}\right)\epsilon_2,\epsilon_1-\epsilon_2,\epsilon_2\right)
$$

$$
=\sum_{n\in\mathbb{Z}}(-1)^n\mathbf{e}\left[\frac{1}{2}\left(n+\frac{1}{2}\right)^2\tau+\left(n+\frac{1}{2}\right)\left(\frac{1}{2}\underline{r}_m\cdot\underline{m}+\frac{1}{8}\underline{r}_m\cdot\underline{r}_m(\epsilon_1+\epsilon_2)\right)\right]. \tag{5.1}
$$

Here we have massaged the equations a little bit by summing up elliptic genera to the total instanton partition function, or alternatively by refraining from expanding the instanton partition function in terms of t_b in (3.4). We use the hatted notation of the instanton partition function to emphasize that the Kähler moduli have been twisted in the sense of (3.3), and have chosen $r_b=-1$ following (3.11). Let us first derive the blowup equation for the E_8 Minaham-Nemeschansky theory. To this end, we blow down the exceptional curve x_9 which also serves as the base of the elliptic fibration and take the limit

$$
\lim_{dP_8}: \quad t_b\to-\mathrm{i}\infty, \quad \tau\to+\mathrm{i}\infty, \quad t_{B_8}=t_b+\tau \text{ finite.} \tag{5.2}
$$

Here we have defined t_{B_8} which measures the volume of the anti-canonical class of dP_8

$$
-K_{dP_8}=3h-\sum_{i=1}^8 x_i \tag{5.3}
$$

and its corresponding $\underline{r}$-component is

$$
r_{B_8}=r_b+r_\tau=-1. \tag{5.4}
$$

In this process, the terms in the instanton partition function of the form $Q_b^{d_b}Q_\tau^{d_\tau}=\mathbf{e}[d_bt_b+d_\tau\tau]$ with $d_b>d_\tau$ become divergent, which need to resummed and analytically continued. It turns out that the only terms which diverge in $\widehat{Z}_{dP_9}^{\mathrm{inst}}$ are those that come with the BPS

invariant $N_{(0,0)}^{(d_b,d_\tau,\underline{d}_m)=(1,0,\underline{0})} = 1$, which corresponds to the base curve b, and the instanton partition function factorises to the product of the contribution of this BPS state $Z_{(0,0)}^{(1,0,\underline{0})}$ and the partition function of the daughter E_8 MN theory

$$\lim_{dP_8} Z_{dP_9}^{\text{inst}}(\tau, \underline{m}, t_b, \epsilon_1, \epsilon_2) = \left(\lim_{dP_8} Z_{(0,0)}^{(1,0,\underline{0})}(t_b, \epsilon_1, \epsilon_2) \right) Z_{dP_8}^{\text{inst}}(t_{B_8}, \underline{m}, \epsilon_1, \epsilon_2). \qquad (5.5)$$

In general, a spin $(0,0)$ BPS state accompanying the Kähler parameter t contributes to the instanton partition function by

$$Z_{(0,0)}(t, \epsilon_1, \epsilon_2) = \text{PE}\left[f_{(0,0)}(q_1, q_2) Q_t \right] := \text{PE}\left[\frac{Q_t}{(q_1^{1/2} - q_1^{-1/2})(q_2^{1/2} - q_2^{-1/2})} \right], \qquad (5.6)$$

with $\text{PE}[g(\bullet)] = \exp\left(\sum_{n=1}^{\infty} \frac{g(\bullet^n)}{n} \right)$. In each summand on the l.h.s. of (5.1), the contribution of such a BPS state to the blowup equation before turning on the twisting is then

$$Bl_{(0,0)}(t, \epsilon_1, \epsilon_2) := \frac{Z_{(0,0)}(t+R\epsilon_1, \epsilon_1-\epsilon_2, \epsilon_2) Z_{(0,0)}(t+R\epsilon_2, \epsilon_1, \epsilon_2-\epsilon_1)}{Z_{(0,0)}(t, \epsilon_1, \epsilon_2)}$$

$$= \text{PE}\left[\left(f_{(0,0)}(q_1, q_2/q_1) q_1^R + f_{(0,0)}(q_1/q_2, q_2) q_2^R - f_{(0,0)}(q_1, q_2) \right) Q_t \right]$$

$$= \text{PE}\left[- \sum_{\substack{m,n \geq 0 \\ m+n \leq R-3/2}} q_1^{m+1/2} q_2^{n+1/2} Q_t \right] = \prod_{\substack{m,n \geq 0 \\ m+n \leq R-3/2}} (1 - q_1^{m+1/2} q_2^{n+1/2} Q_t),$$
$$(5.7)$$

where R is the component of the $\underline{R}$-vector defined in (3.1). If in addition the corresponding curve is a rational (-1) curve, after turning on the twisting $Q_t \to -Q_t$ and taking the limit $Q_t \to \infty$ or $t \to -\text{i}\infty$, we find the divergent contribution

$$\widehat{Bl}_{(0,0)}(t, \epsilon_1, \epsilon_2) = \mathbf{e}\left[\frac{(2R-1)(2R+1)}{8} t + \frac{(2R-1)R(2R+1)}{24}(\epsilon_1 + \epsilon_2) \right], \qquad (5.8)$$

which should cancel with vanishing terms in the semiclassical contributions. Applying this formula to the base curve b with Kähler parameter t_b and $R = -n - 1/2$, and inserting it into (5.1), we obtain the following unity blowup equations for the E_8 MN theory or the dP_8 geometry

$$\sum_{n \in \mathbb{Z}} (-1)^n \mathbf{e}\left[\frac{n(n+1)}{2} t_{B_8} + \frac{1}{2} n \underline{r}_m \cdot \underline{m} + P_{dP_8}(n)(\epsilon_1 + \epsilon_2) \right] \widehat{Z}_{dP_8}^{\text{inst}}(\underline{m}, t_{B_8}, \epsilon_1, \epsilon_2)^{-1}$$

$$\times \widehat{Z}_{dP_8}^{\text{inst}}\left(\underline{m} + \frac{\epsilon_1}{2} \underline{r}_m, t_{B_8} - \left(n + \frac{1}{2} \right) \epsilon_1, \epsilon_1, \epsilon_2 - \epsilon_1 \right) \widehat{Z}_{dP_8}^{\text{inst}}\left(\underline{m} + \frac{\epsilon_2}{2} \underline{r}_m, t_{B_8} - \left(n + \frac{1}{2} \right) \epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2 \right)$$

$$= \sum_{n \in \{0, -1\}} (-1)^n \mathbf{e}\left[\frac{n(n+1)}{2} t_{B_8} + \frac{1}{2} n \underline{r}_m \cdot \underline{m} + P_{dP_8}(\underline{n})(\epsilon_1 + \epsilon_2) \right], \qquad (5.9)$$

with

$$P_{dP_8}(n) = n \delta_u - \frac{n(n+1)(2n+1)}{12}, \qquad (5.10)$$

where δ_u means 1 in unity equations and 0 in vanishing equations. Here because the E_8 symmetry is unbroken, all the 240 unity $\underline{r}_m$-fields forming the Weyl orbit $\mathcal{O}_{2,240}$ as well as the lone vanishing $\underline{r}_m = \underline{0}$ survive the limit. Note that when we apply the blow-down limit to the blowup equations, we have the liberty of focusing exclusively on the transformation of the l.h.s., factoring out any terms which are independent of the summation index n along the way. The r.h.s. of the resulting blowup equations can be obtained by collecting all the terms of the l.h.s. which do not vanish when t_{B_8} (or its analogue in further blow-downs) is sent to $i\infty$. This is the strategy we will follow in the following.

To further perform blow-downs, it is convenient to change the basis of flavor parameters from m_i to the following basis associated to the exceptional curves

$$\widetilde{m}_i = \text{Vol}(x_i) - \frac{1}{3}\text{Vol}(h), \quad i = 1, \dots, 8. \tag{5.11}$$

It is easy to check that the corresponding homology classes $[x]_i - 1/3[h]$ have trivial intersection numbers with the complex surface itself. They are related to m_i by equating the r.h.s. of (2.17), (2.19), and we find

$$\widetilde{m}_1 = -\frac{1}{3}\sum_{j=1}^{7} m_j + \frac{1}{3}m_8, \quad \widetilde{m}_i = -m_i - \frac{1}{6}m_8 + \frac{1}{6}\sum_{j=1}^{7} m_j, \quad i = 2, \dots, 8. \tag{5.12}$$

The corresponding $\underline{r}$-components are given by

$$r_{\widetilde{m},1} = -\frac{1}{3}\sum_{j=1}^{7} r_{m,j} + \frac{1}{3}r_{m,8}, \quad r_{\widetilde{m},i} = -r_{m,i} - \frac{1}{6}r_{m,8} + \frac{1}{6}\sum_{j=1}^{7} r_{m,j}, \quad i = 2, \dots, 8. \tag{5.13}$$

To flow from dP_8 surface to dP_d ($0 \le d \le 7$), we blow down $8 - d$ exceptional curves $x_8, \dots, x_{d+1}$ corresponding to taking the limit

$$\lim_{dP_d}: \ \widetilde{m}_i \to -i\infty, \ \ i = d+1, \dots, 8, \quad t_{B_8} + \sum_{i=d+1}^{8} \widetilde{m}_i \ \text{finite.} \tag{5.14}$$

The first thing we notice is that among the $\underline{r}$-components of mass parameters only $r_{\widetilde{m},i}$ ($i = 1, \dots, d$) survive, and they organise themselves into a weight vector of E_d with Dynkin labels[24]

$$\left(\frac{r_{\widetilde{m},1} - r_{\widetilde{m},2}}{2}, \ \dots, \ \frac{r_{\widetilde{m},d-1} - r_{\widetilde{m},d}}{2}, \ -\frac{r_{\widetilde{m},1} + r_{\widetilde{m},2} + r_{\widetilde{m},3}}{2} \right) \tag{5.15}$$

if $d \ge 3$ (here E_3, E_4, E_5 denote respectively $\text{SU}(3) \times \text{SU}(2), \text{SU}(5), \text{SO}(10)$) and a weight vector of $\text{SU}(2)$ with Dynkin label

$$\left(\frac{r_{\widetilde{m},1} - r_{\widetilde{m},2}}{2} \right) \tag{5.16}$$

---

[24]Recall that the first homology group of dP_d ($d \ge 3$) contains the root lattice of E_d generated by $x_i - x_{i+1}$ and $h - x_1 - x_2 - x_3$.

dP_d	8	7	6	5	4
G	E_8	E_7	E_6	$SO(10)$	$SU(5)$
$\frac{1}{2}\underline{r}_m$	$\mathcal{O}_{2,240}$	$\mathcal{O}_{2,126}, \mathcal{O}_{\frac{3}{2},56}$	$\mathcal{O}_{2,72}, \pm\mathcal{O}_{\frac{4}{3},27}$	$\mathcal{O}_{2,40}, \pm\mathcal{O}_{\frac{5}{4},16}, \pm\mathcal{O}_{1,10}$	$\mathcal{O}_{2,20}, \pm\mathcal{O}_{\frac{6}{5},10}, \pm\mathcal{O}_{\frac{4}{5},5}$

dP_d	3	2
G	$SU(3) \times SU(2)$	$SU(2)$
$\frac{1}{2}\underline{r}_m$	$(\mathcal{O}_{2,6},\mathcal{O}_{0,1}), \pm\left(\mathcal{O}_{\frac{2}{3},3},\mathcal{O}_{\frac{1}{2},2}\right), (\mathcal{O}_{0,1},\mathcal{O}_{2,2}), \pm\left(\mathcal{O}_{\frac{2}{3},3},\mathcal{O}_{0,1}\right), \left(\mathcal{O}_{0,1},\mathcal{O}_{\frac{1}{2},2}\right)$	$\mathcal{O}_{2,2},\mathcal{O}_{\frac{1}{2},2},\mathcal{O}_{0,1}$

Table 9. $\underline{r}_m$ components of mass parameters for dP_d ($d \geq 2$).

if $d = 2$. We will denote this weight vector also by $\frac{1}{2}\underline{r}_m$. The blow-up operation (5.14) can therefore be interpreted as the decomposition of the Weyl orbit of E_8 to those of E_d ($d < 8$). In vanishing blowup equations $\frac{1}{2}\underline{r}_m$ is always the zero vector, while in unity blowup equations $\frac{1}{2}\underline{r}_m$ must be weight vectors in the Weyl sub-orbits of $\mathcal{O}_{2,240}$, which we summarize in table 9.

In the instanton partition function, all the terms whose power of $Q_{\widetilde{m},j} = \mathbf{e}[\widetilde{m}_j]$ is greater than that of $Q_{B_8} = \mathbf{e}[t_{B_8}]$ for any $j \in \{d+1,\ldots,8\}$ diverge, which need to be resummed and analytically continued. All such terms are generated by the spin (0,0) BPS states wrapping curve classes of $(d_B, d_{\widetilde{m},i}) = (1,1,\ldots,1,2)$ and all permutations of $d_{\widetilde{m},i}$ for $i = d+1,\ldots,8$. There are $8-d$ such curve classes and they correspond to the exceptional curves

$$x_{d+1},\ldots,x_8. \tag{5.17}$$

All of these curves have -1 intersection with dP_8. We denote their Kähler moduli by

$$t_{x,i} = t_{B_8} + \widetilde{m}_i + \sum_{j=1}^{8} \widetilde{m}_j. \tag{5.18}$$

The divergent contributions of these BPS states to the summand on the l.h.s. of (5.9) in the limit (5.14) can be written down following (5.8)

$$\prod_{i=d+1}^{8} \widehat{Bl}_{(0,0)}(t_{x,i},\epsilon_1,\epsilon_2) = \prod_{i=d+1}^{8} \mathbf{e}\left[\frac{(2R_{x,i}-1)(2R_{x,i}+1)}{8}t_{x,i} + \frac{(2R_{x,i}-1)R_{x,i}(2R_{x,i}+1)}{24}(\epsilon_1+\epsilon_2)\right], \tag{5.19}$$

where

$$R_{x,i} = -n + \frac{r_{x,i}}{2} = -n + \frac{1}{2}\left(-1 + r_{\widetilde{m},i} + \sum_{j=1}^{8} r_{\widetilde{m},j}\right), \tag{5.20}$$

and it should cancel with vanishing terms in the semiclassical contribution in (5.9). On the other hand, all the other terms which do not diverge sum up to the instanton partition function of the dP_d surface [49]. Therefore, we have the factorisation of the twisted partition function

$$\lim_{dP_d} \widehat{Z}^{\mathrm{inst}}_{dP_8}(t_{B_8}, \underline{m}, \epsilon_1, \epsilon_2) = \left[\lim_{dP_d} \prod_{i=d+1}^{8} \widehat{Bl}_{(0,0)}(t_{x,i}, \epsilon_1, \epsilon_2)\right] \widehat{Z}^{\mathrm{inst}}_{dP_d}(t_{B_d}, \underline{m}, \epsilon_1, \epsilon_2). \tag{5.21}$$

The daughter theory associated to the dP_d surface is parameterised by the mass parameters $\widetilde{m}_i$ for $i = 1, \ldots, d$ as well as the volume of the anti-canonical class

$$- K_{dP_d} = 3h - x_1 - \ldots - x_d \tag{5.22}$$

given by

$$t_{B_d} = t_{B_8} + \sum_{i=d+1}^{8} t_{x,i} = (9-d)t_{B_8} + (8-d)\sum_{i=1}^{d} \widetilde{m}_i + (9-d)\sum_{j=d+1}^{8} \widetilde{m}_j, \tag{5.23}$$

with the associated $\underline{r}$-component

$$r_{B_d} = -(9-d) + (8-d)\sum_{i=1}^{d} r_{\widetilde{m},i} + (9-d)\sum_{j=d+1}^{8} r_{\widetilde{m},j}. \tag{5.24}$$

Since $-K_{dP_d}$ is the only true Kähler modulus in dP_d and its intersection with the complex surface is $-(9-d)$, we have the freedom to bring r_{B_d} within the range

$$- (9-d) \leq r_{B_d} < 9-d \tag{5.25}$$

using the equivalence relation (3.5) without affecting $\underline{r}_m$. We will always assume that r_{B_d} falls inside this range. Furthermore, since the curve class $-K_{dP_d}$ is invariant under Weyl reflections of E_d as well as the Weyl transformations of the E_8 lattice which are perpendicular to the root lattice of E_d, its associated $\underline{r}$ component r_{B_d} should only depend on the Weyl orbit of E_d. In the case of dP_2 and dP_1 which have an additional U(1) factor, r_{B_d} depends on the U(1) charge as well. We summarize the full $\underline{r}$-fields of unity blowup equations for the dP_d theory, given by

$$\underline{r} = (r_{B_d}, \underline{r}_m), \tag{5.26}$$

in table 10. On the other hand, the vanishing $\underline{r}$-field is always

$$\underline{r} = (r_B, \underline{r}_m) = (-(9-d), \underline{0}). \tag{5.27}$$

Taking the limit of (5.14) on (5.9) and using (5.21), after some algebraic gymnastics in the end we arrive at the following blowup equations of dP_d

$$\sum_{n \in \mathbb{Z}} (-1)^n \mathbf{e}\left[\frac{1}{2}n\left(n + \frac{r_{B_d}}{9-d}\right)t_B + \frac{1}{2}n\sum_{i=1}^{d}\widetilde{m}_i\left(r_{\widetilde{m},i} + \frac{|r_{\widetilde{m}}|}{9-d}\right) + P_{dP_d}(n)(\epsilon_1 + \epsilon_2)\right]\widehat{Z}_{dP_d}^{\text{inst}}(t_{B_d}, \underline{m}, \epsilon_1, \epsilon_2)^{-1}$$

$$\times \widehat{Z}_{dP_d}^{\text{inst}}\left(t_{B_d} + \frac{\epsilon_1}{2}r_{B_d}, \underline{m} + \frac{\epsilon_1}{2}\underline{r}_m, \epsilon_1, \epsilon_2 - \epsilon_1\right)\widehat{Z}_{dP_d}^{\text{inst}}\left(t_{B_d} + \frac{\epsilon_2}{2}r_{B_d}, \underline{m} + \frac{\epsilon_2}{2}\underline{r}_m, \epsilon_1 - \epsilon_2, \epsilon_2\right)$$

$$= \begin{cases} 0, & r_{B_d} = -(9-d), \underline{r}_m = \underline{0} \\ 1 - \mathbf{e}\left[\frac{1}{2}\sum_{i=1}^{d}\widetilde{m}_i\left(r_{\widetilde{m}_i} + \frac{|r_{\widetilde{m}}|}{9-d}\right) + (\epsilon_1 + \epsilon_2)\right], & r_{B_d} = -(9-d), \underline{r}_m \neq \underline{0} \\ 1, & \text{otherwise}, \end{cases} \tag{5.28}$$

	G	#vanishing	#unity	unity $\frac{1}{2}\underline{r} = \left(\frac{1}{2}r_{B_d}, \frac{1}{2}\underline{r}_m\right)$
dP_8	E_8	1	240	$\left(-\frac{1}{2}, \mathcal{O}_{2,240}\right)$
dP_7	E_7	1	182	$\left(-1, \mathcal{O}_{2,126}\right), \left(0, \mathcal{O}_{\frac{3}{2},56}\right)$
dP_6	E_6	1	126	$\left(-\frac{3}{2}, \mathcal{O}_{2,72}\right), \pm\left(-\frac{1}{2}, \mathcal{O}_{\frac{4}{3},27}\right)$
dP_5	$SO(10)$	1	82	$\left(-2, \mathcal{O}_{2,40}\right), \pm\left(-1, \mathcal{O}_{\frac{5}{4},16}\right), \pm(0, \mathcal{O}_{1,10})$
dP_4	$SU(5)$	1	50	$\left(-\frac{5}{2}, \mathcal{O}_{2,20}\right), \pm\left(-\frac{3}{2}, \mathcal{O}_{\frac{6}{5},10}\right), \pm\left(-\frac{1}{2}, \mathcal{O}_{\frac{4}{5},5}\right)$
dP_3	$SU(3) \times SU(2)$	1	28	$\left(-3, \mathcal{O}_{2,6}, \mathcal{O}_{0,1}\right), \pm\left(-2, \mathcal{O}_{\frac{2}{3},3}, \mathcal{O}_{\frac{1}{2},2}\right), \left(-3, \mathcal{O}_{0,1}, \mathcal{O}_{2,2}\right), \pm\left(-1, \mathcal{O}_{\frac{2}{3},3}, \mathcal{O}_{0,1}\right), \left(0, \mathcal{O}_{0,1}, \mathcal{O}_{\frac{1}{2},2}\right)$
dP_2	$SU(2) \times U(1)$	1	14	$\left(-\frac{7}{2}, \mathcal{O}_{2,2;0}\right), \pm\left(-\frac{5}{2}, \mathcal{O}_{0,1;4}\right), \pm\left(-\frac{5}{2}, \mathcal{O}_{\frac{1}{2},2;-3}\right), \pm\left(-\frac{3}{2}, \mathcal{O}_{\frac{1}{2},2;+3}\right), \pm\left(-\frac{1}{2}, \mathcal{O}_{0,1;-2}\right)$
dP_1	$U(1)$	1	6	$\pm(-3_{-3}), \pm(-2_2), \pm(-1_{-1})$
$\mathbb{P}^1 \times \mathbb{P}^1$	$SU(2)$	1	7	$\left(-4, \mathcal{O}_{2,2}\right), \pm\left(-2, \mathcal{O}_{\frac{1}{2},2}\right), \left(0, \mathcal{O}_{0,1}\right)$
$\mathbb{P}^2$	–	1	2	$\pm\left(\frac{3}{2}\right)$

Table 10. $\underline{r}$ fields for del Pezzo surfaces. We denote $\frac{1}{2}\underline{r}_m$ as an element of a Weyl orbit $\mathcal{O}_{n,p}$. In the case of dP_2 and dP_1, we use an extra subscript to denote the U(1) charge.

where $|r_{\widetilde{m}}| = \sum_{i=1}^{d} r_{\widetilde{m},i}$ and[25]

$$P_{dP_d}(n) = \left(\delta_u - \frac{9-d}{12}\right) n - \frac{r_{B_d}}{4} n^2 - \frac{9-d}{6} n^3. \qquad (5.29)$$

Furthermore, by taking the limit (5.14) on (2.37), we also find the universal form of the perpurbative prepotential of dP_d, which reads[26]

$$F_{(0,0)}^{dP_d} = \frac{9-d}{6} t_h^3 - \frac{1}{2} t_h^2 \sum_{i=1}^{d} \widetilde{m}_i - \frac{1}{2} t_h \sum_{i=1}^{d} \widetilde{m}_i^2 \qquad (5.30)$$

as well as

$$b_h^{\text{GV}} = -\frac{3+d}{24}, \quad b_h^{\text{NS}} = -\frac{3-d}{12}, \qquad (5.31)$$

where we have defined the Kähler parameter

$$t_h = \frac{1}{3}\text{Vol}(h) = \frac{t_{B_d} + \sum_{i=1}^{d} \widetilde{m}_i}{9-d}. \qquad (5.32)$$

These results as well as the $\underline{r}$-fields given in table 10 have been successfully compared with [27].[27]

There is an additional del Pezzo surface $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, which does not fall into the family of dP_d. On the other hand, it is known that dP_2 is isomorphic to the one-point blowup of $\mathbb{F}_0$. The $\mathbb{P}^1$ base e, the $\mathbb{P}^1$ fiber f, and exceptional curve x of the latter are identified as

$$e = h - x_1, \quad f = h - x_2, \quad h - x_1 - x_2. \qquad (5.33)$$

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[25]This formula is derived from an identity of the orbit $9r_h^2 - \sum_{i=1}^{d}(r_h + r_{\tilde{m}_i})^2 = 1 - d$. Here $r_h = \frac{r_{B_d} + \sum_{i=1}^{d} r_{\widetilde{m},i}}{9-d}$.

[26]The prepotential of the half K3 surface (2.37) can be seen as a special case of this universal formula with $d = 9$ if we identify $\widetilde{m}_9 = -\tau - \sum_{i=1}^{8} \widetilde{m}_i$.

[27]In [27] the last $\underline{r}$-field component other than $\underline{r}_m$ is r_h associated to t_h, which is related to r_{B_d} by $r_h = \frac{r_{B_d} + \sum_{i=1}^{d} r_{\widetilde{m},i}}{9-d}$.

Therefore the surface $\mathbb{F}_0$ can be obtained by blowing down the exceptional curve $h - x_1 - x_2$ of dP_2, corresponding to taking the limit

$$\widetilde{m}_{1,2} \to \widetilde{m}_{1,2} - 2\varepsilon, \quad t_{B_2} \to t_{B_2} - 3\varepsilon, \quad \varepsilon \to -\mathrm{i}\infty. \tag{5.34}$$

We identify the Kähler parameter of the anti-canonical class $t_{B_{\mathbb{F}_0}}$ and the mass parameter $\widetilde{m}$ of $\mathbb{F}_0$ as

$$t_{B_{\mathbb{F}_0}} = \frac{8}{7} t_{B_2} - \frac{6}{7}(\widetilde{m}_1 + \widetilde{m}_2), \quad \widetilde{m} = \widetilde{m}_1 - \widetilde{m}_2. \tag{5.35}$$

Another commonly used set of Kähler parameters for $\mathbb{F}_0$ consists of volumes of e and f denoted by $t_{1,2}$, which are related to $t_{B_{\mathbb{F}_0}}$ and $\widetilde{m}$ through

$$t_1 = \frac{1}{4} t_{B_{\mathbb{F}_0}} - \frac{1}{2} m, \quad t_2 = \frac{1}{4} t_{B_{\mathbb{F}_0}} + \frac{1}{2} m, \tag{5.36}$$

with which the prepotential up to irrelevant terms reads

$$F_{(0,0)} = -\frac{1}{24}(t_1^3 + t_2^3) + \frac{1}{8}(t_1^2 t_2 + t_1 t_2^2). \tag{5.37}$$

The transformation of the blowup equations are analogous. Only the spin $(0,0)$ BPS state wrapping the exceptional curve $h - x_1 - x_2$ contribute diverge terms in the blowup equation and they can be computed using (5.8). The $\underline{r}$-fields can be computed by decomposing those of dP_2. We list the unity $\underline{r}$-fields in table 10, and the only vanishing $\underline{r}$-field is

$$\underline{r} = (-8, 0). \tag{5.38}$$

They also agree with [27].[28] Note that the theory of $\mathbb{F}_0$ is a bit special. Unlike dP_d theories, there is a unity equation with $\underline{r}_m = 0$ in the theory of $\mathbb{F}_0$. In addition, $r_{B_{\mathbb{F}_0}}$ of the anti-canonical class is not completely determined by $\underline{r}_m$: for $\frac{1}{2}\underline{r}_m \in \mathcal{O}_{\frac{1}{2},2}$, which is real, $r_{B_{\mathbb{F}_0}}$ has two possible values not equivalent to each other.

6 From path integral to blowup equations

Here we try to understand the form of the elliptic blowup equations (3.17) for the E-, M-string theory on torus from the path integral point of view.

6d SCFTs can be constructed by compactifying F-theory on elliptic Calabi-Yau three-efolds with the following geometric properties of base and fiber. The base B is a non-compact, complex two-dimensional space. As such, it contains 2-cycles C^i which are $\mathbb{P}^1$'s with negative intersection matrix $-\Omega^{ij} = C^i \cdot C^j$. Furthermore, in general above each C^i the elliptic fiber can degenerate according to a Kodaira singularity. In the resulting 6d field theory, each 2-cycle C^i in the base gives rise to a tensor multiplet. The bosonic components of the tensor multiplets are denoted by (ϕ_i, B_i), where ϕ_i are real scalars and B_i are 2-forms with field strengths H_i. The volume of the 2-cycle C^i is proportional to $\phi^i = \Omega^{ij} \phi_j$. Singular elliptic fibers over C^i signal the existence of gauge symmetry. Since

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[28]In [27] the $\underline{r}$-field is presented by the components r_z, r_m associated to curves $e, f - e$. They are related to $r_{B_{\mathbb{F}_0}}, r_m$ via $r_{B_{\mathbb{F}_0}} = 4r_z + 2r_m$.

we only consider theories with no gauge symmetry, the Calabi-Yau has no singular elliptic fiber. Finally there are hypermultiplets corresponding to isolated (-1) curves in the fibral direction. We will also use the fact that there can be string-like objects charged under the tensor multiplets. A string with worldsheet Σ_2 and tensor charges n_i sources a flux of H_i

$$dH_i = n_i \delta_{\Sigma_2} \quad \Longleftrightarrow \quad n_i = \int_{\Sigma_3} H_i, \tag{6.1}$$

where δ_{Σ_2} is the unit delta function localised on Σ_2, which is linked by the three-cycle Σ_3.

When we put the 6d theory on T^2, the hypermultiplets become hypermultiplets in the 4d field theory, and the tensor multiplets are reduced to vector multiplets by

$$\phi_i \to \phi_i, \quad H_i \to F_i \wedge \mathrm{d}x + \star^{(4)} F_i \wedge \mathrm{d}y + a_i \mathrm{d}x \wedge \mathrm{d}y, \tag{6.2}$$

which induce a non-Abelian gauge symmetry. Here $\star^{(4)}$ is the 4d Hodge star so that $H_i = \star H_i$ in 6d. Each vector multiplet includes as bosonic components a complex scalar $\varphi_i = \phi_i + \mathrm{i}a_i$ and a 1-form gauge connection A_i whose field strength is F_i. Their kinetic terms in the Lagrangian are

$$\mathcal{L}_{\mathrm{Coulomb}}^{(4)} = -\tau \Omega^{ij} \left(F_i \wedge \star^{(4)} F_j + \partial_\mu \varphi_i \partial^\mu \overline{\varphi}_j \right) + \dots. \tag{6.3}$$

Here the bare gauge coupling τ is identified with the complex structure parameter of the torus on which the 6d SCFT is compactified.

We would like to compute the partition function of the 6d SCFT on $M_6 = T^2 \times_{\epsilon_1, \epsilon_2} M_4$ with M_4 being $\widehat{\mathbb{C}}^2$, namely the blowup of $\mathbb{C}^2$ at the origin. The four manifold M_4 is the same as the total space of the $\mathcal{O}(-1)$ bundle over $\mathbb{P}^1$, which can be parameterised by $z_0, z_1, z_2 \in \mathbb{C}$ with the equivalence relation

$$(z_0, z_1, z_2) \sim (\lambda^{-1} z_0, \lambda^1 z_1, \lambda^1 z_2), \quad \lambda \in \mathbb{C}^*. \tag{6.4}$$

We also turn on the 6d Omega background [61] which has the effect of rotating the M_4 when one goes around 1-cycles in T^2 with the $\mathrm{U}(1)_{\epsilon_1} \times \mathrm{U}(1)_{\epsilon_2}$ action as follows:

$$(z_0, z_1, z_2) \mapsto (z_0, \mathrm{e}^{\epsilon_1} z_1, \mathrm{e}^{\epsilon_2} z_2). \tag{6.5}$$

The $\mathrm{U}(1)_{\epsilon_1} \times \mathrm{U}(1)_{\epsilon_2}$ action has two localised points at the north pole and the south pole of the exceptional $\mathbb{P}^1 = S^2$ at $(z_0, z_1, z_2) = (0, 1, 0), (0, 0, 1)$.

$$\begin{aligned}
(z_0 z_1, z_2/z_1) &\mapsto (\mathrm{e}^{\epsilon_1} z_0 z_1, \mathrm{e}^{\epsilon_2 - \epsilon_1} z_2/z_1) \\
(z_0 z_2, z_1/z_2) &\mapsto (\mathrm{e}^{\epsilon_2} z_0 z_2, \mathrm{e}^{\epsilon_1 - \epsilon_2} z_1/z_2).
\end{aligned} \tag{6.6}$$

Note that the $\mathrm{U}(1)_+$ part of the Omega background with fugacity $\epsilon_+ = (\epsilon_1 + \epsilon_2)/2$ can be identified with the $\mathrm{U}(1)$ component of the 6d $\mathrm{SU}(2)_R$-symmetry, which acts non-trivially on the Wilson lines of the 6d flavor symmetry. This is due to the fact that upon compactifying a 6d SCFT on T^2 with non-trivial Wilson lines the conformal symmetry present in 6d is broken and the 6d conformal stress-tensor multiplet splits into a 4d stress-tensor multiplet and a flavor current multiplet associated with the Kaluza-Klein symmetry, i.e. the

momentum in the reduced direction. This KK current then gets identified with a linear combination of the flavor currents of the original 6d flavor symmetry [62]. This identification can be thought of as an embedding of the $\mathrm{SU}(2)_R$-symmetry into the 6d flavor symmetry group. As a result the twisted flavor mass

$$\widehat{\underline{m}} := \underline{m} + \widehat{\underline{r}}\epsilon_+ \tag{6.7}$$

with $\widehat{\underline{r}}$ some root vector of the flavor group, should be invariant throughout M_4. We will identify $\widehat{\underline{r}}$ with $\underline{r}_m$ later.

There are essentially two types of configurations in the 6d field theory with finite energy. The first type corresponds to the string worldsheets wrapping the torus. They appear as point-like instantons in M_4 and because of the action of $\mathrm{U}(1)_{\epsilon_1}$ and $\mathrm{U}(1)_{\epsilon_2}$ are localised at the north pole and the south pole of the exceptional $\mathbb{P}^1$. The path integral receives contributions from the one-loop determinant corresponding to the elliptic genera of the strings. In the sector of $\underline{k}'$ wrapped strings localised at the north pole and $\underline{k}''$ wrapped strings localised at the south pole of $\mathbb{P}^1$ we expect the contribution of

$$\mathbb{E}_{\underline{k}'}\left(\tau, \underline{m} + \frac{1}{2}\widehat{\underline{r}}\epsilon_1, \epsilon_1, \epsilon_2 - \epsilon_1\right)\mathbb{E}_{\underline{k}''}\left(\tau, \underline{m} + \frac{1}{2}\widehat{\underline{r}}\epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2\right). \tag{6.8}$$

The shifts on the flavor mass are to make sure that the twisted flavor mass (6.7) is invariant.

The other type of finite energy configuration is the flux of the self-dual 3-form through $S^1 \times \mathbb{P}^1$. We provide two ways to evaluate its contribution to the partition function. The first method is to consider the 4d field theory resulting from the torus compactification. Let us first consider a 6d SCFT with a single tensor multiplet which is reduced on T^2 to a $\mathrm{U}(1)$ gauge theory in 4d with gauge connection A. The partition function of such a theory on a four manifold M_4 is [63]

$$Z(M_4) = \sum_{\mathcal{L}} \int DA \exp(-I(A)), \tag{6.9}$$

where the sum runs over all the line bundles on which A is a connection. To describe the path integral more explicitly, one decomposes $A = A' + A_h^{\mathcal{L}}$, where A' is a connection on a trivial line bundle $\mathcal{O}$, and $A_h^{\mathcal{L}}$ is a connection on $\mathcal{L}$ of harmonic curvature $F_h^{\mathcal{L}}$. The path integral then becomes

$$\sum_{\mathcal{L}} \int DA \exp(-I(A)) = \int DA' \exp(-I(A')) \sum_{\mathcal{L}} \exp(-I(A_h^{\mathcal{L}})). \tag{6.10}$$

Let us now look at the sum over $\mathcal{L}$. On the lattice $H^2(M; \mathbb{Z})$, there is a natural, generally indefinite quadratic form given, for x an integral harmonic two-form, by $(x, x) = \int_M x \wedge x$. There is also a positive-definite but metric-dependent form $\langle x, x \rangle = \int_M x \wedge \star^{(4)} x$. The indefinite form (x, x) has signature $(b_{2,+}, b_{2,-})$, where $b_{2,\pm}$ are the dimensions of the spaces of self-dual and anti-self-dual harmonic two-forms. Setting $x = F_h^{\mathcal{L}}/2\pi$, the sum over line bundles becomes

$$\Theta = \sum_{x \in H^2(M; \mathbb{Z})} \exp\left(-\pi \mathrm{Im}(\tau)\langle x, x \rangle + i\pi \mathrm{Re}(\tau)(x, x)\right). \tag{6.11}$$

This is a Siegel-Narain theta function with modular parameter τ. In the case where the 4d theory comes from a 6d parent theory as in our case, the big theta function splits into a product of holomorphic and anti-holomorphic theta functions:

$$\Theta = \sum_a \Theta^{[a]} \overline{\Theta}^{[a]}, \tag{6.12}$$

where the sum over a is a sum over the so-called "quadratic refinements" and $\Theta^{[a]}$ arise from the path-integral over self-dual parts of the 3-form H while $\overline{\Theta}^{[a]}$ correspond to the anti-self-dual parts. In the case of $M = \widehat{\mathbb{C}}^2$, $(b_{2,+}, b_{2,-}) = (1,0)$, the two quadratic forms (x,x) and $\langle x,x \rangle$ are identical and $\overline{\Theta}^{[a]}$ become trivial.

Relaxing now the condition that the tensor branch of the 6d SCFT is only one-dimensional and using (6.3), we deduce that in the general case the path integral of the 4d theory produces the theta function

$$\Theta_\Omega^{[a]}(\tau, z) \equiv \sum_{n_i \in \mathbb{Z}} \mathbf{e}\left[\frac{1}{2}\Omega^{ij}(n_i + a_i)(n_j + a_j)\tau + \Omega^{ij}(n_i + a_i)z_j\right]. \tag{6.13}$$

which is essentially the generalised theta function we defined in (3.38) up to the sign factor. The $\Theta_\Omega^{[a]}$ can be seen as sections of a line bundle over the torus

$$\mathbb{T} \equiv \mathbb{C}^r / \left(\Omega\mathbb{Z}^r \oplus \tau\Omega\mathbb{Z}^r\right). \tag{6.14}$$

We remark at this point that the number of such sections is equal to the determinant of Ω. We also include elliptic parameters z_j which are possible source terms coupled to F_i. These can be the instanton strings localised at the north and the south poles of $\mathbb{P}^1$ or they can result from the following Green-Schwarz counter-terms in the 6d theory

$$2\pi \int_{M_6} \Omega^{ij} B_i \wedge I_j. \tag{6.15}$$

where I_j are the 't Hooft anomaly four-forms on the worldsheet of strings. We will write down explicit expressions of the elliptic parameters with the second method of evaluation.

The second method of evaluation is through a holographic argument which we outline below. Following [64–67], there exists a holographic action for the tensor branch of 6d SCFTs on a seven-manifold Y_7 with boundary $\partial Y_7 = M_6$, whose topological part is:

$$S_7^{\text{top}} = 2\pi \int_{Y_7} \Omega^{ij} \left(\frac{1}{2}dC_i \wedge C_j + C_i \wedge I_j\right), \tag{6.16}$$

where C_i are 3-forms. Variation with respect to C_i gives

$$
\begin{aligned}
0 &= \int_{Y_7} (dC_i - I_i) \wedge \delta C_j \Omega^{ji} \\
&= \int_M (C_i - \omega_i) \wedge \delta C_j \Omega^{ji} \quad \text{giving } C_i = \omega_i \text{ on } M_6,
\end{aligned} \tag{6.17}
$$

where $d\omega_i = I_i$. We want to compute the partition function arising from the path integral

$$\int \left[\prod_i \mathcal{D}C_i\right]_{C_i = \omega_i \text{ on } M_6} \exp\left(iS_7^{\text{top}}\right). \tag{6.18}$$

The result will be a state in the Hilbert space $\mathcal{H}_{M_6}$ arising from the quantisation of the CS-action. Let us see what this Hilbert space is. From the action (6.16) we get upon quantisation the following commutation relations:

$$[C_i(x), C_j(y)] = -2\pi\mathsf{i}\,\Omega_{ij}^{-1}\,\text{Vol}_{M_6}\,\delta^6(x-y), \tag{6.19}$$

Taking again M_6 to be of the form $S_A^1 \times S_B^1 \times \widehat{\mathbb{C}^2}$, we can define operators

$$\Phi_i^A = \exp\left(\mathsf{i}\int_{S_A^1 \times \mathbb{P}^1} C_i\right), \quad \Phi_j^B = \exp\left(\mathsf{i}\int_{S_B^1 \times \mathbb{P}^1} C_j\right), \tag{6.20}$$

These satisfy commutation relations

$$\Phi_A^i \Phi_B^j = \Phi_B^j \Phi_A^i \exp\left(2\pi\mathsf{i}\,\Omega_{ij}^{-1}\right). \tag{6.21}$$

This defines a Heisenberg group extension of

$$K \times K \equiv H^2(\widehat{\mathbb{C}^2}, \mathbb{Z}_{\det\Omega}) \times H^2(\widehat{\mathbb{C}^2}, \mathbb{Z}_{\det\Omega}) = \mathbb{Z}_{\det\Omega} \times \mathbb{Z}_{\det\Omega} :$$
$$0 \to \mathbb{Z}_{\det\Omega} \to F \to K \times K \to 0. \tag{6.22}$$

Let us now for simplicity restrict to the case where there is only one tensor multiplet and $\Omega^{11} = \mathfrak{n}$. Heisenberg groups have a unique irreducible representation $\mathcal{R}$ such that there exists a state $|\Omega\rangle \in \mathcal{R}$ with the property

$$\Phi^A(a)|\Omega\rangle = |\Omega\rangle, \quad a \in \mathbb{Z}_\mathfrak{n}. \tag{6.23}$$

From this state we obtain a basis for $\mathcal{H}$ as follows

$$\Psi_a = \Phi^B(a)|\Omega\rangle, \quad \text{for } a \in \mathbb{Z}_\mathfrak{n}. \tag{6.24}$$

We want to identity $\mathcal{R}$ with $\mathcal{H}_{M_6}$. There is a natural $\text{SL}(2,\mathbb{Z})$ action on $\mathcal{H}_{M_6}$:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad : \quad \tau \mapsto -\frac{1}{\tau} \tag{6.25}$$

implying that Φ^A maps to Φ^{B-1} and Φ^B maps to Φ^A. Under this maps $|\Omega\rangle$ maps to

$$|\widetilde{\Omega}\rangle \sim \sum_{b \in \mathbb{Z}_\mathfrak{n}} \Phi^B(b)|\Omega\rangle, \tag{6.26}$$

while Ψ_a maps to

$$\Phi^A(a)|\widetilde{\Omega}\rangle \sim \sum_{b \in \mathbb{Z}_\mathfrak{n}} \exp(2\pi\mathsf{i}ab/\mathfrak{n})\Phi^B(b)|\Omega\rangle. \tag{6.27}$$

This is precisely the transformation law for a theta function. Another way to see this is to define

$$\alpha_i^A \equiv \int_{S_A^1 \times \mathbb{P}^1} C_i, \quad \alpha_j^B \equiv \int_{S_B^1 \times \mathbb{P}^1} C_j, \tag{6.28}$$

which give phase space coordinates of our 7d Chern-Simons theory. α_i^A and α_j^B take values in V/Γ where

$$V = H^2(\mathbb{P}^1, \mathbb{R}), \quad \text{and } \Gamma = H^2(\mathbb{P}^1, \mathbb{Z}). \tag{6.29}$$

Thus we see that the phase space is a torus

$$V/\Gamma \times V/\Gamma = H^3(M_6, \mathbb{R})/H^3(M_6, \mathbb{Z}) = J_{M_6}, \tag{6.30}$$

with J_{M_6} being the intermediate Jacobian of M_6. Quantisation amounts to finding the appropriate line bundle $\mathcal{L}$ over J_{M_6}. Since the symplectic form carries a factor of Ω^{ij}, the sections are theta functions of the form (6.13). This perspective also helps resolve the apparent discrepancy of the sign factor between (6.13) and (3.38). In the case of $\mathfrak{n} = 1$ (E-string), one has to choose characteristic $a = 1/2$ and make the shift $z \to z + \frac{1}{2}$ so that $\Theta_\Omega^{[a]}$ reduces to $\theta_1(\tau, z)$ as it is the unique $\mathrm{SL}(2, \mathbb{Z})$-invariant section for a line bundle with principal polarisation, which precisely means $\mathfrak{n} = 1$ here, over the elliptic curve (see [64]). This issue does not arise for $\mathfrak{n} = 2$ (M-string) and we can make the canonical choice of the characteristic $a = i/\mathfrak{n}$ for $i = 0, 1$ without shifting z, whereupon $\Theta_\Omega^{[a]}$ reduces to $\theta_2(2\tau, z), \theta_3(2\tau, z)$ that furnish an irreducible represetnation of $\mathrm{SL}(2, \mathbb{Z})$.

A glance at (6.18) and (6.28) shows that the elliptic parameters of the theta function are given by

$$z_k = \int_{S_A^1 \times \mathbb{P}^1} \omega_k + \mathsf{i} \int_{S_B^1 \times \mathbb{P}^1} \omega_k. \tag{6.31}$$

From now on, let us restrict to theories with only one tensor multiplet and no gauge symmetry in 6d. In this case $\Omega^{ii} = \mathfrak{n}$ and the anomaly four-form reads

$$\Omega^{ii} I_i = \frac{1}{4} \operatorname{Tr} F_a^2 + c_2(R) - \frac{2 - \mathfrak{n}}{4} p_1(M_6). \tag{6.32}$$

Therefore we expect

$$z = \int_{S^1 \times \mathbb{P}^1} \frac{1}{4} \omega_F + \omega_R - \frac{2 - \mathfrak{n}}{4} p_1^{-1}(M_6), \tag{6.33}$$

where

$$\omega_F = \operatorname{Tr} \left(\frac{2}{3} A_a^3 + A_a \wedge \mathrm{d} A_a \right), \quad \omega_R = \operatorname{Tr} \left(\frac{2}{3} A_R^3 + A_R \wedge \mathrm{d} A_R \right). \tag{6.34}$$

and p_1^{-1} is the anti-derivative of the Pontryagin class. The first term gives

$$\frac{1}{4} \int_{S^1 \times \mathbb{P}^1} \omega_F \to \frac{1}{4} \left(\int_{\mathbb{P}^1} \underline{F} \right) \cdot \underline{m}. \tag{6.35}$$

In order to compute the flux vector $\frac{1}{4} \int_{\mathbb{P}^1} \underline{F}$, we make use of the fact that our partition function should be invariant up to a sign under large gauge transformations [64] (we treat the flavor symmetry as a weakly coupled gauge symmetry)

$$\underline{m} \mapsto \underline{m} + \alpha^\vee \tag{6.36}$$

where $\alpha^\vee$ is an arbitrary coroot of the flavor group. This implies that the flux vector $\frac{1}{4}\int_{\mathbb{P}^1}\underline{F}$ has to be a weight vector of the flavor group, which we also denote by $\frac{1}{2}\underline{r}_m$. We also do the replacement

$$\int_{S^1\times\mathbb{P}^1}\omega_R \mapsto \epsilon_+/2 \tag{6.37}$$

as well as

$$\int_{S^1\times\mathbb{P}^1}p_1^{-1}(M_6) \mapsto \epsilon_1 + \epsilon_2 = 2\epsilon_+. \tag{6.38}$$

Finally, given the embedding of the $SU(2)_R$-symmetry in the flavor symmetry, we can do the shift on flavor mass

$$\underline{m} \to \underline{m} + \frac{1}{2}\underline{r}_m\epsilon_+. \tag{6.39}$$

Putting all these pieces together, we find the following elliptic parameter

$$z = \frac{1}{2}\underline{r}_m\cdot\underline{m} + \left(\frac{\mathfrak{n}-1}{4} + \frac{1}{8}\underline{r}_m\cdot\underline{r}_m\right)(\epsilon_1+\epsilon_2), \tag{6.40}$$

which has the same form as the elliptic parameter of the theta function on the l.h.s. of (3.17) up to the last two terms corresponding to coupling with strings. We notice that in the case of unity equations, with the constraint (3.22), we find the elliptic parameter (up to the last two terms) can also be written as

$$z = 2(\mathfrak{n}-2)\epsilon_+ + \frac{1}{2}\underline{r}_m\cdot(\underline{m}+\underline{r}_m\epsilon_+), \tag{6.41}$$

which shows that the mass $\underline{m}$ is corretly replaced by the twisted mass $\underline{m}+\underline{r}_m\epsilon_+$.

The coupling with string sources can be added to the Chern-Simons action (6.16) as follows (see [67]):

$$\begin{aligned}
S_7^{\text{top}} &= 2\pi\int_{Y_7}\Omega^{ij}\left(\frac{1}{2}\mathrm{d}C_i\wedge C_j + C_i\wedge(I_j + d_{j,1}\chi_4(N) + d_{j,2}\chi_4(N))\right)\\
&= 2\pi\int_{Y_7}\Omega^{ij}\left(\frac{1}{2}\mathrm{d}C_i\wedge C_j + C_i\wedge I_j\right) + 2\pi\Omega^{ij}\left(d_{j,1}\int_{T^2\times\mathbb{R}_+}C_i + d_{j,2}\int_{T^2\times\mathbb{R}_+}C_i\right),
\end{aligned} \tag{6.42}$$

where $\chi_4(N)$ is the Euler class of the normal bundle of the string. In the case of $\mathrm{rk}\,\Omega = 1$, using

$$\chi_4(N) = de_3^{(0)}, \tag{6.43}$$

where $e_3^{(0)}$ is the global angular form of the S^3 bundle of the tubular neighborhood of the string, we see that the elliptic parameter of our theta function gets shifted by

$$\mathfrak{n}\left(k_1\int_{S^1\times\mathbb{P}^1}e_3^{(0_N)} + k_2\int_{S^1\times\mathbb{P}^1}e_3^{(0_S)}\right) = \mathfrak{n}(\epsilon_1 k_1 + \epsilon_2 k_2), \tag{6.44}$$

where we have inserted string sources at the north and south pole of the exceptional $\mathbb{P}^1$. We have thus completely reproduced the theta function, and together with (6.8), the entire l.h.s. of (3.17).

In order to obtain the r.h.s. of (3.17) we simply choose the S^3 to surround the entire exceptional $\mathbb{P}^1$ fully. The terms in the theta function corresponding to coupling with strings disappear because

$$\mathfrak{n}k \int_{S^1 \times \mathbb{P}^1} e_3^{(0)} = 0 \,, \tag{6.45}$$

and we are left only with the elliptic parameter (6.40). The contributions of string worldsheet wrapping T^2 merge to

$$\mathbb{E}_k(\tau, \underline{m}, \epsilon_1, \epsilon_2) \quad \text{with } k = k_1 + k_2. \tag{6.46}$$

Finally we can invoke the modularity consistency condition to uniquely fix $\underline{r}_m$ by (3.22) through (3.21). Vanishing equations can arise if there exist values of $\underline{r}_m$ and c so that the theta function on the r.h.s. vanishes identically, which happens in the case of E-string.

7 Conclusion

In this paper we propose the blowup equations for E-string and M-string theories compactified on T^2, and solve their elliptic genera and equivalently the refined BPS invariants of the associated Calabi-Yau threefolds from these equations. Although the elliptic genera of these two theories have been studied via many other approaches, for instance localization in 2d quiver gauge theories [19, 21] or modular bootstrap [16] where one can write down explicit expressions for the elliptic genera, the blowup method still sheds some new light on this subject and gives inspiration on how to deal with more general 6d theories. As is well-known, the E-string and M-string are the two simplest rank-one 6d SCFTs. Both have no gauge symmetry but only flavor symmetry. In the previous papers of this series, we have established the elliptic blowup equations for rank-one pure gauge 6d SCFTs on torus [23, 24]. With the inspiration obtained in the current paper with regard to the E- and M-string theories, we indeed find the elliptic blowup equations for all rank one SCFTs with both gauge and flavor symmetry [41] and even higher rank ones [52], thus exhausting all possibilities of untwisted torus reductions of 6d SCFTs.

We also present the elliptic blowup equations for M-string chains and E-M string chains. Note that the M-string chains are special cases of the much more general ADE string chains [50]. Indeed, the N M-string chain is in fact the A_{N-1} type chain without gauge symmetry. Although the elliptic genera for all ADE string chains can be computed from localization, it is still interesting to consider their elliptic blowup equations and whether their elliptic genera can be calculated from these equations. We would like to leave the discussion of this subject to the higher rank paper [52].

In addition, we provide a simple procedure to compute the full perturbative prepotential with all mass parameters turned on of a non-compact Calabi-Yau threefold from its local description as a connected union of compact surfaces. This paves the way for deriving blowup equations for twisted circle reductions of 6d SCFTs from the local description of their associated geometries given in [68]. Relatedly, all rank one 5d SCFTs can be obtained from circle reductions of the E-string theory by decoupling a mass deformed hypermultiplet

or from the geometric point of view by blowing down an exceptional curve. We demonstrate in this paper how to perform this operation on the blowup equations of E-strings on the torus to obtain the blowup equations of rank one 5d SCFTs. Recently there has been much progress towards the classification of 5d SCFTs [43–45, 58–60, 68–70], and it is conjectured that all 5d SCFTs can be obtained from untwisted or twisted circle reductions of 6d SCFTs [58, 71]. Therefore, we can write down a recipe to derive blowup equations for all 5d SCFTs compactified on a circle from the blowup equations of untwisted and twisted torus reductions of 6d SCFTs.

As a byproduct of the unity blowup equations for the one E-string elliptic genus, we obtain some novel functional equations for the E_8 theta function. The unity blowup equations for more E-strings result in more identities among E_8 Weyl invariant Jacobi forms. It is desirable to know whether one can prove from the viewpoint of Jacobi forms that our system of elliptic blowup equations only allows for a one-dimensional solution space.

Besides, the blowup equations have connections with the bilinear relations of isomonodromic systems [72]. We hope the blowup equations we find in this paper can shed some new light on this subject. In particular, the E-string blowup equations are expected to produce the bilinear relations of the elliptic Painlevé equations [73], while the M-string blowup equations are expected to produce those of the isomonodromic system on the one-punctured torus [74].

Finally, a long-standing problem is a possible proof of the blowup equations for 6d SCFTs compactified on the torus. In this paper we make the first attempt and derive from the path integral point of view the elliptic blowup equations for the E-, M-string theories. This method can in principle be applied to derive the blowup equations for the M-string chains and E–M string chains as well, which also have no gauge symmetry. We hope our derivation can inspire proofs for the blowup equations for a generic 6d SCFT.

Acknowledgments

A   A lemma

To demonstrate that the blowup equations can determine all refined BPS invariants in section 4, we need the following lemma. Recall in section 4.1, we defined

$$f_{(j_L, j_R)}(q_1, q_2) = \frac{\chi_{j_L}(q_L)\chi_{j_R}(q_R)}{(q_1^{1/2} - q_1^{-1/2})(q_2^{1/2} - q_2^{-1/2})} \tag{A.1}$$

and

$$Bl_{(j_L, j_R, R)}(q_1, q_2) = f_{(j_L, j_R)}(q_1, q_2/q_1)q_1^R + f_{(j_L, j_R)}(q_1/q_2, q_2)q_2^R - f_{(j_L, j_R)}(q_1, q_2), \tag{A.2}$$

where $\chi_j(q)$ is the SU(2) character.

Lemma 1. $\forall R \in \mathbb{Z}/2$, $Bl_{(j_L, j_R, R)}(q_1, q_2)$ are linearly independent with only exceptions at $Bl_{(0,0,1/2)}(q_1, q_2) = Bl_{(0,0,-1/2)}(q_1, q_2) = Bl_{(0,1/2,0)}(q_1, q_2) = 0$.

Proof. For a generic fixed R and a finite set J of spin (j_L, j_R) which satisfy $2j_L + 2j_R + 1 \equiv 2R \mod 2$, we need to prove if

$$\sum_{(j_L, j_R) \in J} x_{(j_L, j_R)} Bl_{(j_L, j_R, R)}(q_1, q_2) = 0, \tag{A.3}$$

then all coefficients $x_{(j_L, j_R)}$ must vanish. Since J is finite, there exist maximum for j_L and j_R, denoted as $j_L^{\max}$ and $j_R^{\max}$. We expand J to the set of all spins on the rectangle from $(0,0)$ to $(j_L^{\max}, j_R^{\max})$. On such spin rectangle, we can define a strict total order of (j_L, j_R). Then one can use descending method to prove the coefficients $x_{(j_L, j_R)}$ vanish one by one. Such procedure was actually already given in section 6.2 in [27]. For $R = 1/2$ or $-1/2$, the lowest spin in such order is $(0,0)$ and the value of Bl function is 0, and for $R = 0$, the lowest spin in such order is $(0, 1/2)$ and the value of Bl function is 0 too. These are the only exceptions for linear independence.

B   Functional equations for theta functions of even unimodular lattices

The unity blowup equations for one E-string elliptic genus (4.33) give a set of interesting functional equations for E_8 theta function. Here we prove E_8 theta function is the unique solution for such equations up to a free function of τ. This statement can be generalized to the theta function associated to any positive definite even unimodular lattice that is generated by roots. The generalization and proof were shown to us by Don Zagier.

Proposition 1. *Let Λ be a positive definite even unimodular lattice that is generated by its roots, and let f be a holomorphic function on $\mathfrak{H} \times \Lambda_{\mathbb{C}}$ satisfying the functional equation*

$$\theta_1(\epsilon_2)\theta_1(\alpha \cdot m + \epsilon_2)f(\tau, m + \epsilon_1 \alpha) - \theta_1(\epsilon_1)\theta_1(\alpha \cdot m + \epsilon_1)f(\tau, m + \epsilon_2 \alpha)$$
$$= \theta_1(\epsilon_2 - \epsilon_1)\theta_1(\alpha \cdot m + \epsilon_1 + \epsilon_2)f(\tau, m), \tag{B.1}$$

for all roots α of Λ and all $\epsilon_1, \epsilon_2 \in \mathbb{C}$. Then f is a multiple (depending only on τ) of the theta series

$$\theta_\Lambda(\tau, m) = \sum_{w \in \Lambda} e^{2\pi i (w \cdot w/2 + m \cdot w)}. \tag{B.2}$$

Proof. Fix τ and also a root α and a vector $m_0 \in \Lambda$ with $m_0 \cdot \alpha = 0$, and set $F(\tau, \lambda) = f(m_0 + \lambda\alpha)$, $\lambda \in \mathbb{C}$. Using $\alpha \cdot \alpha = 2$ and setting $h_1 = \lambda + \epsilon_1$, $h_2 = \lambda + \epsilon_2$, $h_3 = \lambda$, we can write (B.1) in a symmetric form,

$$\sum_{i \,(\text{mod}\,3)} \theta_1(h_{i+1} - h_{i-1})\theta_1(h_{i+1} + h_{i-1})F(h_i) = 0, \quad (\text{any } \{h_i\}_{i\,(\text{mod}\,3)} \in \mathbb{C}^3). \qquad \text{(B.3)}$$

Here the τ dependence is implicit. Changing h_1 to $h_1 + 1$ and $h_1 + \tau$ with h_2 and h_3 fixed, we find $F(h+1) = F(h)$ and $F(h+\tau) = q^{-1}\xi^{-2}F(h)$, where $q = e^{2\pi i\tau}$, $\xi = e^{2\pi ih}$. Thus,

$$f(m + \alpha) = f(m), \qquad f(m + \alpha\tau) = e^{-2\pi i(\tau + \alpha \cdot m)}f(m). \qquad \text{(B.4)}$$

Since Λ is even unimodular and generated by all roots α, the first equation of (B.4) implies that we can write $f(\tau, m)$ as Fourier expansion $\sum_{w \in \Lambda} c_w(q)q^{w \cdot w/2}e^{2\pi im \cdot w}$ for some coefficients $c_w(q)$. The second equation of (B.4) implies that $c_{w+\alpha}(q) = c_w(q)$ for all w and α, so $c_w(q) = c_0(q)$ and $f(\tau, m) = c_0(q)\theta_\Lambda(\tau, m)$.

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