Holographic entanglement entropy in anisotropic background with confinement-deconfinement phase transition

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ABSTRACT: We discuss a general five-dimensional completely anisotropic holographic model with three different spatial scale factors, characterized by a Van der Waals-like phase transition between small and large black holes. A peculiar feature of the model is the relation between anisotropy of the background and anisotropy of the colliding heavy ions geometry. We calculate the holographic entanglement entropy (HEE) of the slab-shaped region, the orientation of which relatively to the beams line and the impact parameter is characterized by the Euler angles. We study the dependences of the HEE and its density on the thermodynamic (temperature, chemical potential) and geometric (parameters of anisotropy, thickness, and orientation of entangled regions) parameters. As a particular case the model with two equal transversal scaling factors is considered. This model is supported by the dilaton and two Maxwell fields. In this case we discuss the HEE and its density in detail: interesting features of this model are jumps of the entanglement entropy and its density near the line of the small/large black hole phase transition. These jumps depend on the anisotropy parameter, chemical potential, and orientation. We also discuss different definitions and behavior of c -functions in this model. The c -function calculated in the Einstein frame decreases while ℓ is increasing for all ℓ in the isotropic case (in regions of (μ, T) -plane far away from the line of the phase transition). We find the non-monotonicity of the c -functions for several anisotropic configurations, which however does not contradict with any of the existing c -theorems since they all are based on Lorentz invariance.

KEYWORDS: Holography and quark-gluon plasmas, AdS-CFT Correspondence, Gauge-gravity correspondence

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1 Introduction

Fundamental questions addressed in studies of high energy heavy ions collisions (HIC) at RHIC and LHC, and future experiments NICA and FAIR, concern understanding of quark-gluon plasma (QGP) formation, i.e. thermalization of media produced in HIC, thermodynamic entropy production, and its characteristics such as quantum entanglement, decoherence etc. Most of our knowledge on the formation and properties of QGP resulting from HIC is obtained from measurements of the yields and spectra of particles in the final state of colliding heavy ions and their thermo/hydrodynamic interpretation. According to our common understanding, within a short time of order $1 - 2$ fm/c collision systems reach a state that can be approximated by a thermal medium located in an expanding ball. This medium is characterized by local thermodynamic parameters including temperature and entropy density.

HIC experiments at RHIC and LHC have provided strong evidence that this medium is a QGP at large temperatures and densities. There is also a strong evidence for the existence of the confinement/deconfinement phase transition in the (μ, T) -plane, i.e. temperature on chemical potential. The experimental search for the QCD phase transition is nowadays one of the central goals for current and future collider facilities [1]. The experimental search is mainly related to the measurement of fluctuations of net-proton or net-charge multiplicity [2–5] which are expected to exhibit non-monotonic behavior near the phase transition. The proper understanding of the experimental results requires careful theoretical analysis of the dynamical processes taking place near the phase transition lines, especially near the critical endpoint (CEP)¹ [7]. There are several theoretical approaches to searches of the QCD phase transitions. One of the promising theoretical approaches is based on lattice calculations, but this approach encounters difficulties with non-zero chemical potential [8, 9].

Holographic duality provides an alternative approach to the study of the QCD phase transitions [10–14] (and for review see [15–21] and refs therein). Holography allows to define thermal entropy and free energy as functions of temperature and chemical potential of the gravitational background. With this approach, the thermalization process is dual to the black hole formation, linking the study of thermalization in the QCD with the study of the black hole formation in 5-dimensional gravity. The thermalization process is conveniently monitored, at least theoretically, by tracking the evolution of entanglement entropy of a selected region.² The entanglement entropy of the selected domain A is defined as the von Neumann entropy of the reduced density matrix obtained by tracing out the degrees of freedom located out of domain A . The entanglement entropy is hard to compute in QCD, but it is suitable to compute it in the lattice [24–26], see also [27]. The entanglement entropy can be also evaluated at the gravity side. It is called the holographic entanglement entropy (HEE) and is defined as the area of a minimal surface extending

¹There are indications that CEP’s location on the phase diagram is determined solely through chiral symmetry breaking, see for example [6].

²Note that in phenomenological discussions the entropy production in HIC is used as a signal for quark gluon plasma phase transition [22, 23].

Figure 1. The orientation of the coordinate system (x^1, x^2, x^3) in respect to colliding ions.

from some predefined surface A on the boundary into the bulk [28–30]. The HEE during thermalization usually evolves to the thermal entanglement entropy [31–42]. With this approach, there is a natural possibility of studying the evolution of entropy in HIC (thermalization) and phase transitions for the obtained thermal media in the framework of the same holographic model.

Starting from the Landau thermodynamic approach to high energy collision [43–45] one connects the entropy of the ball of QGP produced in HIC with the total multiplicity of particle production with HIC. The ideal hydrodynamics preserves the total entropy, but dissipative one does not. One can relate the entanglement entropy, associated with a given region of the ball, to the multiplicity of particles produced in this region during the HIC [46]. After thermalization the entanglement entropy of the area depends on whether this area belongs to Glauber’s participant area of ions collisions or does not. It occurs that the dependence of the entanglement entropy on geometrical size of entangled areas is closely related to energy loss and jet quenching.

The HEE has been also used to study phase transitions in equilibrium. In particular, in [47] it was proposed to use the HEE as a probe of confinement. The HEE has been extensively studied and applied in the investigation of the phase transitions in various holographic QCD (HQCD) models [48–68].

We start with the most general anisotropic holographic model and consider the general orientation of the slab-shaped entangled region with respect to the geometry of HIC. The natural coordinate system defined by the HIC is such that the first axis (the longitudinal axis) is directed along the line of collision and the second (the transversal axis) is determined by the direction of the impact parameter (figure 1). The orientation of the entangled region can be set by Euler angles (figure 2). Then, we calculate the HEE for an anisotropic model with symmetry in transversal directions studied in [69, 70]. The choice of this anisotropic model is motivated by its previous use for the holographic description of HIC [69]. It is this model that for special parameter of anisotropy $\nu = 4.5$ gives the

Figure 2. The entangling subsystem is presented as a green slab. Rotating the green slab by the Euler angles (ϕ, θ, ψ) we get the pink slab that is oriented along the axes (x^1, x^2, x^3) associated with the HIC geometry and shown in figure 1.

dependence of the produced entropy on energy in accordance with the experimental data for the energy dependence of the total multiplicity of particles produced in heavy ion collisions [71]. Isotropic holographic models had not been able to reproduce the experimental multiplicity dependence on energy [72–84]. As shown in [85], the model [70] describes smeared confinement/deconfinement phase transitions. As we will see, the model indicates the relations of the fluctuations of the multiplicity, i.e. the entanglement entropy, with the phase transitions.

There are more reasons to use the anisotropic models in context of HIC [86] and QCD itself [87]. Other anisotropic holographic models have been actively studied in recent years [88–93], and the main motivation for these studies was also the anisotropic nature of HIC.

We select an entangled slab-shaped area that has a finite extent in one direction and infinite extent in the other two directions³ (figure 3). We implicitly assume that the slab of the interest is located inside the overlapping area of two ions. In this case of the spacetimes with the fully anisotropic metric (see (2.1) below) the problem of finding the extremal area functional effectively reduces to finding the geodesics in some auxiliary 2-dimensional Euclidean space. For the model [70] we find that varying the angle between the axis of collisions and the direction of the smallest side of the slab-shaped entanglement area changes the slab HEE. Increasing this angle we enhance the HEE, and according to conjecture [46], this means the enhancement of the multiplicity of particle production from this area. This enhancement depends on geometrical (length), and thermodynamical (temperature T and chemical potential μ) parameters. The HEE density [96–99] is a

³See few refs with calculations of the HEE for more complicated entanglement region [94, 95].

Figure 3. The schematic picture of two ions collisions. (a) Each ion is presented as a disk with radius $D/2$ (blue and red disks). The trajectories of centers of two ions depicted by the points are shown by dashed blue and red lines and the directions of their movement are indicated by thick arrows. The overlapping area of two ions considered as two disks has the shape of a region bounded by two arcs of two circles (in the cross section perpendicular to the line of the ions collision). (b) An almost central collision. On the left side of each graph, we show the orientation of the slabs that we are considering.

more convenient object for study since it does not suffer from ultraviolet divergencies. The HEE and its density undergo jumps and these jumps increase while the angle is increasing. Moreover, the values of the HEE density and its jumps increase with the anisotropy increasing.

We also discuss various definitions and behavior of c-functions in this model. In the isotropic case we use different frames, the Einstein frame or the string frame, as well as different types of renormalizations. Here we mainly use the geometric renormalization scheme, which consists in subtracting the disconnected configuration from the connected one [47, 51]. For the isotropic case in regions of (μ, T) -plane far away from the phase transition line, the c-function calculated in the Einstein frame decreases while ℓ is increasing for all ℓ . We find the non-monotonicity of c-functions in the string frame, that is related

to dynamics of the dilaton field in UV in our model. There are several proposals in the literature how to define the holographic c-function in the anisotropic backgrounds [100–107]. Here we use prescription [105] adapted to our renormalization schemes. We find the non-monotonicity of c-functions for several anisotropic configurations and discuss their origin. Note, that generally speaking, the non-monotonicity of c-functions does not contradict any of the existing c-theorems [108–117] since they all base on Lorentz invariance.

The paper is organized as follows. In section 2.1 we briefly describe anisotropic holographic models. In section 2.1.1 we present the most general anisotropic holographic model. We present the action and the ansatz that solves the EOM for the anisotropic model with symmetry in transversal directions in section 2.1.2 and thermodynamics of the background in section 2.2. In section 3 we present an expression for the HEE for a slab-shaped entangling region oriented differently with respect to the HIC axes. In section 3.4.1 and section 3.4.2 we present its special forms corresponding to transversal and longitudinal orientations in respect to the collision axis, respectively, and discuss the regularization procedure used for the transversal and longitudinal orientations. Section 4 is devoted to entanglement entropy density and definitions of c-functions. In section 5 we display and discuss our main numerical results. In section 5.1 we demonstrate the dependence of the HEE on the geometrical parameters — length ℓ , anisotropy ν and orientations, and on thermodynamical parameters — temperature T and chemical potential μ . In section 5.2 we show dependencies of the HEE density on the thickness, anisotropy and orientation of the slab as well as on temperature and chemical potential. In section 5.3 we discuss the scaling behavior of the modified c-function. In section 5.4 we discuss possible origins of a non-monotonic behavior of c-functions. In section 5.5 we present behavior of different c-functions near the background phase transition. In section 5.6 we present a table of the dependencies of various c-functions on ℓ . In section 5.7 we compare the position of the phase transition for HEE with the positions of phase transitions related to the background instability.

Finally, we end the paper with the conclusion and discussion of future directions of research on the subject.

2 Setup

2.1 Anisotropic holographic models

2.1.1 General anisotropic model

We start with a general anisotropic holographic model

$$ds^2 = \frac{L^2 b_s(z)}{z^2} \sum_{M=0}^4 G_M(z) (dx^M)^2, \quad (2.1)$$

$$G_0 = -g(z), \quad G_i = \mathfrak{g}_i(z), \quad i = 1, 2, 3, \quad G_4 = \frac{1}{g(z)}. \quad (2.2)$$

Here $b_s(z) = b(z)e^{\sqrt{\frac{2}{3}}\phi(z)}$ is the AdS deformation factor (in the string frame in the presence of the dilaton), $g(z)$ is a blackening function and $\mathfrak{g}_i(z)$ are anisotropy factors.

2.1.2 Anisotropic model with symmetry in transversal directions

Taking in the previous formula $\mathfrak{g}_1(z) = 1$ and $\mathfrak{g}_2(z) = \mathfrak{g}_3(z) = \mathfrak{g}(z)$ we get

$$ds^2 = \frac{L^2 b_s(z)}{z^2} \left[-g(z)dt^2 + dx^2 + \mathfrak{g}(z) ((dy^1)^2 + (dy^2)^2) + \frac{dz^2}{g(z)} \right]. \quad (2.3)$$

Metric (2.3) with a special form of anisotropy factor $\mathfrak{g}(z)$ as in [36] and a particular case of $b(z)$ describing the holographic model for heavy quarks [70] present a special interest for us:

$$ds^2 = \frac{L^2 b(z)}{z^2} \left[-g(z)dt^2 + dx^2 + \left(\frac{z}{L}\right)^{2-\frac{2}{\nu}} ((dy^1)^2 + (dy^2)^2) + \frac{dz^2}{g(z)} \right], \quad (2.4)$$

where L is the characteristic length scale of the geometry and $b(z) = e^{cz^2/2}$. In all our calculations we set $c = -1$. Here the metric is in the Einstein frame. In the next section to perform calculation of the HEE we also switch to the string frame adding an extra dilaton-dependent exponential prefactor [28, 47]. The metric (2.4) is supported by the Einstein-Dilaton-two-Maxwell action with special potential V for the dilaton field ϕ and strength potentials f_1 and f_2 for two Maxwell fields:

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left[R - \frac{f_1(\phi)}{4} F^{(1)2} - \frac{f_2(\phi)}{4} F^{(2)2} - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right], \quad (2.5)$$

where $F^{(1)2}$ and $F^{(2)2}$ are the squares of the Maxwell fields.

The ansatz with the Maxwell fields $F_{\mu\nu}^{(1)} = \partial_\mu A_\nu^{(1)} - \partial_\nu A_\mu^{(1)}$, $A_\mu^{(1)} = A_t(z)\delta_\mu^0$, and $F^{(2)} = q dy^1 \wedge dy^2$, $\phi = \phi(z)$ and metric (2.4) satisfies the equations of motion under relations between the warp factor $b(z)$, dilaton and Maxwell potentials $V(\phi)$, $f_1(\phi)$, $f_2(\phi)$ [70].

The ansatz (2.4) breaks isotropy while preserving translation and (t, x) -boost invariances. The advantage of this model is that it allows to find potentials V , f_1 and f_2 and the blackening function g explicitly. Note that in this case the dilaton potential can be approximated by the sum of two exponents:

$$V(\phi, \mu, \nu) = V_0(\nu) + C_1(\mu, \nu)e^{k_1(\nu)\phi} + C_2(\mu, \nu)e^{k_2(\nu)\phi}. \quad (2.6)$$

Note, that in [118] an explicit isotropic solution for the dilaton potential as a sum of two exponents and zero chemical potential has been constructed. The isotropic version of the model has been considered earlier in [119].

There is another holographic anisotropic model that has symmetry in transversal directions, but breaks the boost invariance in (t, x^1) plane, keeping the (t, x^2) and (t, x^3) planes. It is based on the Mateos-Trancanelli metric [90, 91]

$$ds^2 = \frac{L^2 b(z)}{z^2} \left[-g(z)dt^2 + e^{2h(z)}(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + \frac{dz^2}{g(z)} \right]. \quad (2.7)$$

This metric is supported by the Einstein-Axion-Dilaton action.

Figure 4. (a) The temperature $T(z_h, \mu, c, \nu)$ dependence on the horizon size z_h for $\nu = 4.5$ and various values of the chemical potential μ . (b) The black hole free energy $F(T)$ for various values of μ in anisotropic case, $\nu = 4.5$. The intersection with the horizontal axis gives the value of the Hawking-Page horizon $z_{h,HP}(\nu)$.

2.2 Thermodynamics of the background (2.4)

The thermodynamical properties of the anisotropic holographic model were studied in [70]. The temperature of the model is given as $T = |g'(z_h)|/4\pi$, where $g(z_h)$ is given by the incomplete gamma function (see (2.37) in [70]).

The expressions for thermal entropy and the free energy are the following:

$$s(z_h, c, \nu) = \frac{e^{\frac{3}{4}cz_h^2}}{4z_h^{1+2/\nu}}, \quad F(z_h, c, \nu, \mu) = \int_{z_h}^{z_{h_2}} s(z_h, c, \nu) T'(z_h, c, \nu, \mu) dz_h, \quad (2.8)$$

where z_{h_2} is the second horizon, i.e. at this point $T(z_{h_2}) = 0$. For the considered blackening function the function $T(z_h)$ is three-valued i.e. has the Van der Waals type of behavior at $0 < \mu < \mu_{cr}(\nu)$ (figure 4(a)) and free energy shows the swallow-tailed dependence on the temperature in this range of chemical potentials (figure 4(b)). The loop of the swallow-tailed shape disappears at $(\mu, T) = (\mu_{cr}(\nu), T_{cr}(\nu))$. For $\mu \geq \mu_{cr}$, the function of free energy increases smoothly while temperature is decreasing. The lowest values of free energy correspond to the thermodynamically stable phases.

The line of free energy intersects itself at $T = T_{BB}(\nu, \mu)$, and here a small black hole transits to a large one. Note that for the non-zero values of chemical potential, the Hawking-Page (HP) transition occurs at the temperatures larger than the temperature of black hole to black hole transition (BB) (figure 4(b)). So the black hole solution is always dominant with respect to thermal AdS for $\mu \neq 0$. The position of BB phase transition line determines the phase diagram of the model [70, 85]. The entropy function $s(T)$ is multivalued for $\mu < \mu_{cr}$ and becomes one-to-one for chemical potentials $\mu \geq \mu_{cr}$. The value of the μ_{cr} depends on the anisotropic parameter ν , $\mu_{cr}(\nu = 1) = 0.117$, $\mu_{cr}(\nu = 1.5) = 0.189$, $\mu_{cr}(\nu = 2) = 0.270$, $\mu_{cr}(\nu = 3) = 0.299$, $\mu_{cr}(\nu = 4.5) = 0.349$.

In figure 5(a) the BB phase transitions in the (μ, T) -plane are presented for the isotropic and anisotropic backgrounds with different ν . At the temperature values $T = T_{BB}(\nu, \mu)$ the thermal entropy undergoes a significant jump (see figure 5(b)), where the entropy is presented in the logarithmic scale. On these plots we see that the jumps disappear for $\mu \geq \mu_{cr}$ ($\mu_{cr} = 0.117$ for $\nu = 1$ and $\mu_{cr} = 0.349$ for $\nu = 4.5$).

Figure 5. (a) The BB phase transition in the (μ, T) -plane for the isotropic background (green curve) and for the anisotropic backgrounds for various $\nu = 1.5, 2, 3, 4.5$ (green-gray, khaki, blue-gray and blue curves, respectively). $\mu_{\text{cr}}(\nu = 1) = 0.117, T_{\text{cr}}(\nu = 1) = 0.334$, $\mu_{\text{cr}}(\nu = 1.5) = 0.189, T_{\text{cr}}(\nu = 1.5) = 0.294$, $\mu_{\text{cr}}(\nu = 2) = 0.270, T_{\text{cr}}(\nu = 2) = 0.237$, $\mu_{\text{cr}}(\nu = 3) = 0.299, T_{\text{cr}}(\nu = 3) = 0.245$, $\mu_{\text{cr}}(\nu = 4.5) = 0.349, T_{\text{cr}}(\nu = 4.5) = 0.226$. Dots indicate the critical points. (b) The thermal entropy dependence on temperature for various values of chemical potential for $\nu = 1$ (green curves) and $\nu = 4.5$ (blue curves). Dashed lines with arrows show the thermal entropy jumps at the HP transition point (for zero chemical potential) and at the BB phase transition points (for non-zero chemical potential).

3 Entanglement entropy

3.1 General framework

The entanglement entropy is used to probe correlations in quantum systems. If the system is divided into two spatially disjoint parts A and $\bar{A}$, the entanglement entropy $S(A)$ gives an estimation of the amount of information loss corresponding to the restriction of an A . It is not simple to calculate the entanglement entropy from the strongly coupled system side, in particular in QCD. However, one can compute its holographic dual. For some boundary region A the HEE is obtained by extremizing the 3-surface functional

$$\mathcal{A} = \int d^3\xi \sqrt{|\det g_{MN} \partial_\alpha X^M \partial_\beta X^N|}, \quad (3.1)$$

that ends on the boundary surface A . In the dual field theory the entanglement entropy of a subsystem A is given by the formula [28–30]

$$S_{\text{EE}} = \frac{\mathcal{A}}{4G_5}. \quad (3.2)$$

In what follows we set $G_5 = 1$.

From (3.1) we see that the entanglement entropy depends on the geometry of the area A . It is difficult to do calculations for arbitrary A , compare, for example, with [94, 95]. We will do the calculations for the areas having the shape of parallelepipeds, two sides of which are long and one short, i.e. for parallelepipeds which look as slabs.

The orientation of these parallelepipeds can be specified by the Euler angles (ϕ, θ, ψ) relative to the axes x^1, x^2, x^3 , see figure 2. The axis x^1 is chosen along the collision line, the axis x^2 is chosen in the transversal direction along the direction of the impact parameter b , and the axis x^3 is chosen along the emerging magnetic field, see figure 1. In figure 2 the initial slab is oriented along the axes specified by the HIC geometry and is shown in green. The rotated slab shown in pink defines the entanglement area. We are going to calculate the HEE for the rotated pink parallelepiped.

The entangled slabs are supposed to be in the 3-dimensional overlap of two nuclei regions, so called Glauber regions, see figure 3. The overlapping region of two nuclei depends on time. At a fixed point of time it is a three-dimensional body in which the cross section, perpendicular to the axis of collisions, has a shape bounded by arcs of two circles shifted relative to each other according to an impact parameter. The area and the shape of this area depend on the impact parameter b . The overlapping area for a peripheral collision is approximated by the parallelepiped with sizes L_1, L_2 and $\ell = D - b$. We assume $L_1, L_2 \gg \ell$, each ion is presented as a disk with radius $D/2$ and the impact parameter b is essentially less than D , $\ell \ll D$. The overlapping area of two ions is almost cylindrical and we can consider the case of very short time after collision, $\ell \ll D$, $L_1, L_2 \approx D$. There are only two specific cases: peripheral collision and central one, see figure 3.

Let us show that for the spacetimes with the metric (2.1) the problem of finding the extremal area functional (3.1) for the slab with an arbitrary orientation effectively reduces to finding geodesics in some auxiliary 2-dimensional Euclidean space. To show this we consider the embedding in the static gauge and assume that orientation of the slab in respect to the HIC axes is given by the Euler angles, see figure 2.

We have

$$\begin{aligned} x^0(\xi) &= \text{const.}, \\ x^i(\xi) &= \sum_{j=1,2,3} a_{ij}(\phi, \theta, \psi) \xi^j, \quad i = 1, 2, 3, \\ x^4(\xi) &= z(\xi^1), \end{aligned} \tag{3.3}$$

x^i are spatial coordinates in (2.1), $a_{ij}(\phi, \theta, \psi)$ are entries of the rotation matrix

$$M(\phi, \theta, \psi) = \begin{pmatrix} a_{11}(\phi, \theta, \psi) & a_{12}(\phi, \theta, \psi) & a_{13}(\phi, \theta, \psi) \\ a_{21}(\phi, \theta, \psi) & a_{22}(\phi, \theta, \psi) & a_{23}(\phi, \theta, \psi) \\ a_{31}(\phi, \theta, \psi) & a_{32}(\phi, \theta, \psi) & a_{33}(\phi, \theta, \psi) \end{pmatrix} \tag{3.4}$$

and

$$\begin{aligned} a_{11}(\phi, \theta, \psi) &= \cos \phi \cos \psi - \cos \theta \sin \phi \sin \psi, \\ a_{12}(\phi, \theta, \psi) &= -\cos \psi \sin \phi - \cos \phi \cos \theta \sin \psi, \end{aligned}$$

$$\begin{aligned}
 a_{13}(\phi, \theta, \psi) &= \sin \theta \sin \psi, \\
 a_{21}(\phi, \theta, \psi) &= \cos \theta \cos \psi \sin \phi + \cos \phi \sin \psi, \\
 a_{22}(\phi, \theta, \psi) &= \cos \phi \cos \theta \cos \psi - \sin \phi \sin \psi, \\
 a_{23}(\phi, \theta, \psi) &= -\cos \psi \sin \theta, \\
 a_{31}(\phi, \theta, \psi) &= \sin \phi \sin \theta, \\
 a_{32}(\phi, \theta, \psi) &= \cos \phi \sin \theta, \\
 a_{33}(\phi, \theta, \psi) &= \cos \theta.
 \end{aligned} \tag{3.5}$$

Here ϕ is the angle between the ξ^1 axis and the node line (N), shown in figure 2 in black, θ is the angle between the ξ^3 and x^3 axis, ψ is the angle between the node line N and the x^1 axis.

We write the line element for the induced metric as

$$ds^2 = g_{\alpha\beta} d\xi^\alpha d\xi^\beta, \quad \alpha, \beta = 1, 2, 3 \tag{3.6}$$

and substitute the differentials dx^M from the embedding relations (3.3) in the r.h.s. of (2.1):

$$\begin{aligned}
 ds^2 &= \frac{L^2 b_s(z)}{z^2} \left(\sum_i \mathfrak{g}_i(z) dx^{i2} + \frac{dx^{42}}{g} \right) \\
 &= \frac{L^2 b_s(z)}{z^2} \left(\sum_i \mathfrak{g}_i(z) \left(\sum_j a_{ij}(\phi, \theta, \psi) d\xi^j \right)^2 + z'^2 \frac{d(\xi^1)^2}{g(z)} \right).
 \end{aligned} \tag{3.7}$$

We have

$$g_{\alpha\beta} = \frac{L^2 b_s(z)}{z^2} \bar{g}_{\alpha\beta} \tag{3.8}$$

and

$$\begin{aligned}
 \bar{g}_{11}(z, \phi, \theta, \psi) &= \mathfrak{g}_1 a_{11}^2 + \mathfrak{g}_2 a_{21}^2 + \mathfrak{g}_3 a_{31}^2 + \frac{z'^2}{g}, \\
 \bar{g}_{22}(z, \phi, \theta, \psi) &= \mathfrak{g}_1 a_{12}^2 + \mathfrak{g}_2 a_{22}^2 + \mathfrak{g}_3 a_{32}^2, \\
 \bar{g}_{33}(z, \phi, \theta, \psi) &= \mathfrak{g}_1 a_{13}^2 + \mathfrak{g}_2 a_{23}^2 + \mathfrak{g}_3 a_{33}^2, \\
 \bar{g}_{12}(z, \phi, \theta, \psi) &= \mathfrak{g}_1 a_{11} a_{12} + \mathfrak{g}_2 a_{21} a_{22} + \mathfrak{g}_3 a_{31} a_{32}, \\
 \bar{g}_{13}(z, \phi, \theta, \psi) &= \mathfrak{g}_1 a_{11} a_{13} + \mathfrak{g}_2 a_{21} a_{23} + \mathfrak{g}_3 a_{31} a_{33}, \\
 \bar{g}_{23}(z, \phi, \theta, \psi) &= \mathfrak{g}_1 a_{12} a_{13} + \mathfrak{g}_2 a_{22} a_{23} + \mathfrak{g}_3 a_{32} a_{33}, \\
 \bar{g}_{21} &= \bar{g}_{12}, \quad \bar{g}_{32} = \bar{g}_{23}, \quad \bar{g}_{31} = \bar{g}_{13}.
 \end{aligned} \tag{3.9}$$

The determinant of the induced metric is

$$\det g_{\alpha\beta} = \left(\frac{L^2 b_s}{z^2} \right)^3 \left(\mathfrak{g}_1 \mathfrak{g}_2 \mathfrak{g}_3 + \frac{z'^2}{g} (\bar{g}_{22} \bar{g}_{33} - \bar{g}_{23}^2) \right) \tag{3.10}$$

and the Nambu-Goto action is

$$\mathcal{S} = \int_{\mathcal{P}} \left(\frac{L^2 b_s}{z^2} \right)^{3/2} \sqrt{\left(\mathfrak{g}_1 \mathfrak{g}_2 \mathfrak{g}_3 + \frac{z'^2}{g} (\bar{g}_{22} \bar{g}_{33} - \bar{g}_{23}^2) \right)} d\xi^1 d\xi^2 d\xi^3, \quad (3.11)$$

where $g, \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3$ are functions of z and $\bar{g}_{22}, \bar{g}_{33}, \bar{g}_{23}$ are functions of z and the Euler angles. Since there is no dependence of the integrand on ξ^2, ξ^3 we can perform the integration in these variables that gives the sizes of the parallelepiped in second and third directions:

$$\frac{\mathcal{S}}{L_1 L_2} = \int_{-\ell/2}^{\ell/2} \left(\frac{L^2 b_s}{z^2} \right)^{3/2} \sqrt{\left(\mathfrak{g}_1 \mathfrak{g}_2 \mathfrak{g}_3 + \frac{z'^2}{g} (\bar{g}_{22} \bar{g}_{33} - \bar{g}_{23}^2) \right)} d\xi^1. \quad (3.12)$$

Here ℓ, L_1, L_2 are the lengths of the parallelepiped in the first, second and third directions. ℓ can be fixed by boundary conditions, see below (3.17). In what follows we assume $L_1 = L_2 = 1$.

The action (3.12) is a particular case of the BI action

$$\mathcal{S} = \int_{-\ell/2}^{\ell/2} M(z(\xi)) \sqrt{\mathcal{F}(z(\xi)) + (z'(\xi))^2} d\xi. \quad (3.13)$$

This action defines the dynamical system with dynamical variable $z = z(\xi)$ and time ξ . An effective potential is

$$\mathcal{V}(z(\xi)) \equiv M(z(\xi)) \sqrt{\mathcal{F}(z(\xi))}. \quad (3.14)$$

This system has the first integral:

$$\frac{M(z(\xi)) \mathcal{F}(z(\xi))}{\sqrt{\mathcal{F}(z(\xi)) + (z'(\xi))^2}} = \mathcal{I}. \quad (3.15)$$

From (3.15) we can find the “top” point z_* (the closest position of the minimal surface to the horizon), where $z'(\xi) = 0$:

$$M(z_*) \sqrt{\mathcal{F}(z_*)} = \mathcal{I}. \quad (3.16)$$

Finding z' from (3.15) one gets representations for the length ℓ and the action $\mathcal{S}$ (3.13), that defines the HEE S (3.2) up to the factor $1/4$:

$$\frac{\ell}{2} = \int_0^{z_*} \frac{1}{\sqrt{\mathcal{F}(z)}} \frac{dz}{\sqrt{\frac{\mathcal{V}^2(z)}{\mathcal{V}^2(z_*)} - 1}}, \quad (3.17)$$

$$\frac{\mathcal{S}}{2} = \int_{\epsilon}^{z_*} \frac{M(z) dz}{\sqrt{1 - \frac{\mathcal{V}^2(z)}{\mathcal{V}^2(z_*)}}}. \quad (3.18)$$

For action (3.12) we have

$$M(z) = \left(\frac{L^2 b_s}{z^2} \right)^{3/2} \sqrt{\frac{(\bar{g}_{22} \bar{g}_{33} - \bar{g}_{23}^2)}{g}}, \quad (3.19)$$

$$\mathcal{F}(z) = \frac{\mathfrak{g}_1 \mathfrak{g}_2 \mathfrak{g}_3 g}{(\bar{g}_{22} \bar{g}_{33} - \bar{g}_{23}^2)}, \quad (3.20)$$

$$\mathcal{V}(z) = \left(\frac{L^2 b_s}{z^2} \right)^{3/2} \sqrt{\mathfrak{g}_1 \mathfrak{g}_2 \mathfrak{g}_3}. \quad (3.21)$$

Few remarks concerning (3.17) and (3.12) are in order.

- The form of the effective potential $\mathcal{V}(z)$ does not depend on the slab orientation. Therefore, the location of the dynamical wall, defined by location of the minimum of the effective potential $\mathcal{V}(z)$, is the same for all orientations for fixed z_h and ν .
- The expressions for $M(z)$ and $\mathcal{F}(z)$ depend only on the combination

$$\bar{g}_{22}\bar{g}_{33} - \bar{g}_{23}^2, \quad (3.22)$$

that in its turn depends on 3 angles ϕ, ψ and θ :

$$\bar{g}_{22}\bar{g}_{33} - \bar{g}_{23}^2 = \mathfrak{g}_1\mathfrak{g}_2a_{31}^2(\phi, \theta, \psi) + \mathfrak{g}_1\mathfrak{g}_3a_{21}^2(\phi, \theta, \psi) + \mathfrak{g}_2\mathfrak{g}_3a_{11}^2(\phi, \theta, \psi). \quad (3.23)$$

3.2 Geometric renormalization

It is convenient to perform renormalization of the entanglement entropy by subtraction the “disconnected” surface contribution from the “connected” one [47, 51, 59]. The contribution of the “disconnected” surfaces is given by the doubled area of the surface hanging from the boundary to the horizon $z = z_h$ or to the dynamical wall z_{DW} and the area of the surface along $z = z_h$ or $z = z_{\text{DW}}$ (if the dynamical wall exists for the considered parameters).

The difference between the “connected” and “disconnected” parts $S_{CD} \equiv S_{\text{conn}} - S_{\text{discon}}$ for arbitrary oriented slab is

$$\begin{aligned} S_{CD} = & 2 \int_{\epsilon}^{z_*} \frac{L^3 b_s^{3/2}(z)}{z^3} \sqrt{\frac{\mathfrak{g}_1\mathfrak{g}_2a_{31}^2 + \mathfrak{g}_1\mathfrak{g}_3a_{21}^2 + \mathfrak{g}_2\mathfrak{g}_3a_{11}^2}{g(z)}} \frac{dz}{\sqrt{1 - \frac{\nu^2(z_*)}{\nu^2(z)}}} \\ & - 2 \int_{\epsilon}^{z_D} \left(\frac{L^2 b_s(z)}{z^2} \right)^{3/2} \sqrt{\frac{\mathfrak{g}_1\mathfrak{g}_2a_{31}^2 + \mathfrak{g}_1\mathfrak{g}_3a_{21}^2 + \mathfrak{g}_2\mathfrak{g}_3a_{11}^2}{g(z)}} dz \\ & - \left(\frac{L^2 b_s(z_D)}{z_D^2} \right)^{3/2} \sqrt{\mathfrak{g}_1\mathfrak{g}_2\mathfrak{g}_3} \Big|_{z=z_D} \ell, \end{aligned} \quad (3.24)$$

where z_D is the minimum of the two values z_{DW} and z_h (z_{DW} is the position of the dynamical wall, at this point $\mathcal{V}'(z_{\text{DW}}) = 0$).

Here we used the fact that the determinant of 3×3 induced matrix corresponding to the 3-dim body hanging along the z -axis is

$$\begin{aligned} \det \bar{M} &= \frac{(a_{13}a_{22} - a_{12}a_{23})^2 \mathfrak{g}_1\mathfrak{g}_2 + (a_{13}a_{32} - a_{12}a_{33})^2 \mathfrak{g}_1\mathfrak{g}_3 + (a_{23}a_{32} - a_{22}a_{33})^2 \mathfrak{g}_2\mathfrak{g}_3}{g} \\ &= \frac{\mathfrak{g}_1\mathfrak{g}_2a_{31}^2 + \mathfrak{g}_1\mathfrak{g}_3a_{21}^2 + \mathfrak{g}_2\mathfrak{g}_3a_{11}^2}{g}, \end{aligned} \quad (3.25)$$

In (3.24) $z = z_D$ is the position of the horizon or the dynamical wall. Here we consider the dynamical wall for HEE, that is defined as the position of the minimum of the effective potential $\mathcal{V}(z)$,

$$\mathcal{V}(z) = \left(\frac{L^2 b_s(z)}{z^2} \right)^{3/2} \sqrt{\mathfrak{g}_1(z)\mathfrak{g}_2(z)\mathfrak{g}_3(z)}. \quad (3.26)$$

It is interesting to note that $\mathcal{V}$ does not depend on the Euler angles. The HEE dynamical wall does not coincide with Wilson loop dynamical walls, which depend on the orientation of the loop in respect to the collision axes [85].

3.3 HEE for $\mathbf{g}_1 = 1$, $\mathbf{g}_2 = \mathbf{g}_3 = \mathbf{g}$ and arbitrary orientation

In this section we consider the metric (2.3). From the general formula (3.12) we get

$$\frac{\mathcal{S}}{2} = \int_{-\ell/2}^{\ell/2} \left(\frac{L^2 b_s(z)}{z^2} \right)^{3/2} \sqrt{s(\mathbf{g}, \phi, \theta, \psi)} dz, \quad (3.27)$$

where

$$s(\mathbf{g}, \phi, \theta, \psi) = \mathbf{g}^2(z) + \left[a_{21}^2(\phi, \theta, \psi) + a_{31}^2(\phi, \theta, \psi) + a_{11}^2(\phi, \theta, \psi) \mathbf{g}(z) \right] \frac{\mathbf{g}(z) z'^2}{g(z)},$$

and a_{ij} are given by (3.5) and satisfy the relation

$$a_{21}^2(\phi, \theta, \psi) + a_{31}^2(\phi, \theta, \psi) + a_{11}^2(\phi, \theta, \psi) = 1. \quad (3.28)$$

So we can use a new parametrization

$$a_{11}(\phi, \theta, \psi) = \cos \varphi, \quad (3.29)$$

and in this case

$$s(\mathbf{g}, \phi, \theta, \psi) = \mathbf{g}^2(z) + \left[\sin^2 \varphi + \mathbf{g}(z) \cos^2 \varphi \right] \frac{\mathbf{g}(z) z'^2}{g(z)}. \quad (3.30)$$

Note that φ is nothing but the angle between the ξ^1 and x^1 axes. In this parametrization we have

$$\mathcal{V}(\mathbf{g}, \phi, \theta, \psi) = \left(\frac{L^2 b_s(z)}{z^2} \right)^{3/2} \mathbf{g}(z), \quad (3.31)$$

$$M(\mathbf{g}, \phi, \theta, \psi) = \left(\frac{L^2 b_s(z)}{z^2} \right)^{3/2} \sqrt{\frac{\mathbf{g}(z)}{g(z)} (\sin^2 \varphi + \cos^2 \varphi \mathbf{g}(z))}, \quad (3.32)$$

$$\mathcal{F}(\mathbf{g}, \phi, \theta, \psi) = \frac{\mathbf{g}(z) g(z)}{\sin^2 \varphi + \mathbf{g}(z) \cos^2 \varphi}. \quad (3.33)$$

This answer is equivalent to the HEE of the slab that is obtained by rotation of the initial slab, oriented along the natural HIC axes, around the ξ^3 axis by the angle φ , see figure 6.

3.3.1 Application of geometric renormalization

According to (3.24) we have

$$\begin{aligned} \mathcal{S}_{CD, \varphi} = & 2 \int_{\epsilon}^{z_*} \left(\frac{L^2 b_s(z)}{z^2} \right)^{3/2} \sqrt{\frac{\mathbf{g}(z) (\sin^2 \varphi + \cos^2 \varphi \mathbf{g}(z))}{g(z)}} \frac{dz}{\sqrt{1 - \frac{\mathcal{V}^2(z_*)}{\mathcal{V}^2(z)}}} \\ & - 2 \int_{\epsilon}^{z_D} \left(\frac{L^2 b_s(z)}{z^2} \right)^{3/2} \sqrt{\frac{\mathbf{g}(z) (\sin^2 \varphi + \cos^2 \varphi \mathbf{g}(z))}{g(z)}} dz \end{aligned}$$

Figure 6. The entangling subsystem defined by the Euler angles $(\varphi, 0, 0)$. The HEE in the geometry (2.3) is invariant with respect to rotations of the slab around the x^1 -axis.

$$-\left(\frac{L^2 b_s(z)}{z^2}\right)^{3/2} \mathfrak{g}(z) \Big|_{z=z_D} \ell(\mathfrak{g}, \varphi), \quad (3.34)$$

where

$$\frac{\ell(\mathbf{g}, \varphi)}{2} = \int_0^{z_*} \frac{\sqrt{\sin^2 \varphi + \mathbf{g}(z) \cos^2 \varphi}}{\sqrt{\mathbf{g}(z)g(z)}} \frac{\mathcal{V}(z_*)}{\mathcal{V}(z)} \frac{dz}{\sqrt{1 - \frac{\mathcal{V}^2(z_*)}{\mathcal{V}^2(z)}}}. \quad (3.35)$$

3.4 The holographic entanglement entropy for $\mathfrak{g}_1 = \mathfrak{g}_2 = (z/L)^{2-2/\nu}$

We get the following representations for the character length and the action. For (2.4) we have

$$\frac{\ell_\varphi}{2} = \int_0^{z_*} \frac{\mathcal{V}(z_*)}{\mathcal{V}(z)} \sqrt{\frac{(\frac{z}{L})^{-2+2/\nu} \sin^2 \varphi + \cos^2 \varphi}{g(z)(1 - \frac{\mathcal{V}^2(z_*)}{\mathcal{V}^2(z)})}} dz, \quad \mathcal{V}(z) = \left(\frac{L}{z}\right)^{1+2/\nu} b_s^{3/2}(z), \quad (3.36)$$

$$\frac{\mathcal{S}_\varphi}{2} = \int_\epsilon^{z_*} \mathcal{V}(z) \sqrt{\frac{(\frac{z}{L})^{-2+2/\nu} \sin^2 \varphi + \cos^2 \varphi}{g(z)(1 - \frac{\mathcal{V}^2(z_*)}{\mathcal{V}^2(z)})}} dz. \quad (3.37)$$

$M_\phi(z)$ defined as

$$M_\varphi(z) = \mathcal{V}(z) \sqrt{\frac{\left(\frac{z}{L}\right)^{-2+2/\nu} \sin^2 \varphi + \cos^2 \varphi}{q(z)}} \quad (3.38)$$

determines the degree of divergence of $\mathcal{S}_\varphi$. Near $z = 0$ it has a different behavior for $\varphi = 0$ and $\varphi \neq 0$. However, we can use the universal renormalization (3.24).

For the particular case of the longitudinal ($\varphi = 0$, subscript xYY) orientation, the difference between “connected” and “disconnected” parts according to (3.24) is

$$S_{CD,xYY} = \frac{1}{2} \int_{\epsilon}^{z_*} \frac{\mathcal{V}(z) dz}{\sqrt{g(z)}} \left[\frac{1}{\sqrt{1 - \frac{\mathcal{V}^2(z_*)}{\mathcal{V}^2(z)}}} - 1 \right] - \frac{1}{2} \int_{z_*}^{z_D} \frac{\mathcal{V}(z) dz}{\sqrt{g(z)}} - \frac{\mathcal{V}(z_D)}{4} \ell_{xYY}, \quad (3.39)$$

where

$$\frac{\ell_{xYY}}{2} = \int_0^{z_*} \frac{\mathcal{V}(z_*)}{\mathcal{V}(z)} \frac{dz}{\sqrt{g(z)(1 - \frac{\mathcal{V}^2(z_*)}{\mathcal{V}^2(z)})}}. \quad (3.40)$$

The difference between the “connected” and “disconnected” parts for the transversal orientation ($\varphi = \pi/2$, subscript yXY) is

$$\begin{aligned} S_{CD,yXY} = & \frac{1}{2} \int_\epsilon^{z_*} \frac{L^{1-1/\nu} \mathcal{V}(z)}{z^{1-1/\nu}} \frac{dz}{\sqrt{g(z)}} \left[\frac{1}{\sqrt{1 - \frac{\mathcal{V}^2(z_*)}{\mathcal{V}^2(z)}}} - 1 \right] \\ & - \frac{1}{2} \int_{z_*}^{z_D} \frac{L^{1-1/\nu} \mathcal{V}(z)}{z^{1-1/\nu}} \frac{dz}{\sqrt{g(z)}} - \frac{\mathcal{V}(z_D)}{4} \ell_{yXY}, \end{aligned} \quad (3.41)$$

where

$$\frac{\ell_{yXY}}{2} = \int_0^{z_*} \frac{\mathcal{V}(z_*)}{\mathcal{V}(z)} \frac{L^{1-1/\nu}}{z^{1-1/\nu}} \frac{dz}{\sqrt{g(z)(1 - \frac{\mathcal{V}^2(z_*)}{\mathcal{V}^2(z)})}}. \quad (3.42)$$

The HEE is a UV divergent quantity, so the HEE needs renormalization for $z \sim 0$. Note that we can analyze the behavior of the integral at the upper limit z_* . If $z_* \neq z_{\text{DW}}$ (z_{DW} is the point, where $\mathcal{V}'(z_{\text{DW}}) = 0$) we have an integrable singularity (because $\mathcal{V}'(z) \neq 0$ for $0 < z < z_*$). If $z_* = z_{\text{DW}}$ we have a logarithmic singularity [85].

Now we can determine the power of the integrand singularity at $z = 0$. It is defined by M behavior near $z = 0$:

$$M(z) \left[\frac{1}{\sqrt{(1 - \frac{\mathcal{V}^2(z_*)}{\mathcal{V}^2(z)})}} - 1 \right] \underset{z \sim 0}{\sim} \frac{M(z)}{(\mathcal{V}(z))^2} \underset{z \sim 0}{\sim} z^{\kappa_0}. \quad (3.43)$$

Here κ_0 is defined by the asymptotic of M and $\mathcal{V}$ at $z = 0$ and depends on the orientation, so we use subscripts to specify the different orientations.

Taking into account the dilaton field $\phi(z, z_h, c, \nu)$ the asymptotic is given by (eq. (2.58) in [70]):

$$\phi(z, z_h, c, \nu) \sim -k(z_h, \nu, c) + \frac{2\sqrt{\nu-1}}{\nu} \log\left(\frac{z}{z_h}\right), \quad (3.44)$$

where $k(z_h, \nu, c)$ does not depend on z . Therefore we have the following asymptotic of the functions $b_s(z, \nu)$ and $M(z)$ at $z \rightarrow 0$:

$$b_s(z) \underset{z \sim 0}{\sim} B_s(\nu, c, z_h) z^{\frac{\sqrt{\frac{8}{3}(\nu-1)}}{\nu}}, \quad (3.45)$$

$$M_{xYY}(z) \underset{z \sim 0}{\sim} z^{\kappa_{xYY}(\nu)} B_s^{3/2}(\nu, c, z_h), \quad (3.46)$$

$$M_{yXY}(z) \underset{z \sim 0}{\sim} z^{\kappa_{yXY}(\nu)} B_s^{3/2}(\nu, c, z_h), \quad (3.47)$$

where

$$B_s(\nu, c, z_h) \equiv e^{-\sqrt{\frac{2}{3}} \left(\frac{2\sqrt{\nu-1}}{\nu} \log z_h + k(z_h, \nu, c) \right)}, \quad (3.48)$$

$$\kappa_{xYY}(\nu) \equiv \frac{\sqrt{6(\nu-1)}}{\nu} - 1 - \frac{2}{\nu}, \quad (3.49)$$

$$\kappa_{yXY}(\nu) \equiv \frac{\sqrt{6(\nu-1)}}{\nu} - 2 - 1/\nu. \quad (3.50)$$

The expression for $B_s(\nu, c, z_h)$:

$$\begin{aligned} B_s(\nu, c, z_h)^{3/2} &= 2^{-\frac{\sqrt{6}\sqrt{\nu-1}}{\nu}-1} 3^{-\frac{\sqrt{\frac{3}{2}}\sqrt{\nu-1}}{\nu}} \left(\frac{\sqrt{6}\sqrt{\nu-1}}{\nu} - \frac{1}{\nu} - 1 \right) z_*^{-\frac{\sqrt{6}\sqrt{\nu-1}}{\nu} + \frac{1}{\nu} + 2} \\ &\times \left(\frac{\mathcal{N}(\nu, c, z_h) + \sqrt{3}\nu (cz_h^2 - 3)}{2\sqrt{2}\sqrt{\nu-1} - 3\sqrt{3}\nu} \right)^{9/4} \\ &\times \left(\frac{-9c\nu^2 z_h^2 + \sqrt{2(\nu-1)}\mathcal{N}(\nu, c, z_h) + 8\nu - 8}{\nu - 1} \right)^{\frac{\sqrt{\frac{3}{2}}\sqrt{\nu-1}}{\nu}} \\ &\times \exp \left(\frac{\sqrt{\frac{3}{2}}\sqrt{\nu-1}}{\nu} - \frac{\sqrt{3}\mathcal{N}(\nu, c, z_h)}{4\nu} \right) \left(\frac{z}{z_h} \right)^{\frac{\sqrt{6}\sqrt{\nu-1}}{\nu}}, \end{aligned} \quad (3.51)$$

where

$$\mathcal{N}(\nu, c, z_h) = \sqrt{3c\nu^2 z_h^2 (cz_h^2 - 6) + 8\nu - 8}. \quad (3.52)$$

The potential $\mathcal{V}$ asymptotic at $z \sim 0$:

$$\mathcal{V}(z) = \frac{b_s(z)^{3/2}}{z^{2+1/\nu}} \underset{z \sim 0}{\sim} M_{xYY}(z). \quad (3.53)$$

Comparing with $\kappa_{xYY}(\nu)$ we obtain

$$\kappa_{yXY}(\nu) = \kappa_{xYY}(\nu) - 1 + 1/\nu. \quad (3.54)$$

The integrand (3.43) for the yXY case has a larger degree of divergence than for xYY case, so

$$\kappa_0 = - \left(2 - 2/\nu + \frac{\sqrt{6(\nu-1)}}{\nu} - 2 - 1/\nu \right) = 3/\nu - \frac{\sqrt{6(\nu-1)}}{\nu}. \quad (3.55)$$

Note that $\kappa_0 > -1$ for $\nu \geq 1$ (see figure 7), which means that the HEE is finite after the geometric renormalization for $\nu \geq 1$.

3.4.1 Minimal renormalization for $\varphi = 0$

Here we consider (3.37) in a particular case $\varphi = 0$, i.e. the smallest size of the slab is oriented along the longitudinal direction. In notation (3.13) we have:

$$\mathcal{V}_{xYY}(z) = \frac{b_s^{3/2}(z)}{z^{1+2/\nu}}, \quad M_{xYY}(z) = \frac{b_s^{3/2}(z)}{z^{1+2/\nu}\sqrt{g(z)}}, \quad \mathcal{F}_{xYY}(z) = g(z). \quad (3.56)$$

The UV divergencies are defined by behavior of $M_{xYY}(z)$ at $z \sim 0$ (3.47). We see that $M_{xYY}(z)$ has an integrable singularity at $z = 0$ for $\nu \geq 1.67$.

Figure 7. Functions $\kappa_{yXY}(\nu)$, $\kappa_{xYY}(\nu)$ and κ_0 are shown by blue, magenta and gray lines.

For $\nu = 1$ we have to perform a renormalization and for the renormalized HEE we get:

$$S_{\text{ren}} = \frac{1}{2} \int_{\epsilon}^{z_*} dz \left[\frac{M(z)}{\sqrt{1 - \frac{\mathcal{V}^2(z_*)}{\mathcal{V}^2(z)}}} - M_{as}(z) \right] + \frac{1}{2} \int^{z_*} M_{as}(z) dz, \quad (3.57)$$

where the last term means the indefinite integral of $M_{as}(z)$ at $z = z_*$.

For xYY case and $\nu > 1.67$ we have an integrable singularity. For xYY case and $1 < \nu < 1.67$ we have to perform renormalization:

$$S_{xYY, \text{ren}} = \frac{1}{2} \int_{\epsilon}^{z_*} dz \left[\frac{M_{xYY}(z)}{\sqrt{1 - \frac{\mathcal{V}_{xYY}^2(z_*)}{\mathcal{V}_{xYY}^2(z)}}} - M_{xYY, as}(z) \right] + \frac{1}{2} \int^{z_*} M_{xYY, as}(z) dz, \quad (3.58)$$

where we use the expression for the indefinite integral of M :

$$\int^{z_*} M_{xYY, as}(z) = B_s(\nu, c, z_h)^{3/2} \frac{z_*^{\kappa_{xYY}(\nu)+1}}{\kappa_{xYY}(\nu) + 1}. \quad (3.59)$$

3.4.2 Minimal renormalization for $\varphi \neq 0$

Here we consider a configuration for $\varphi \neq 0$. As we can see from (3.37), the type of UV asymptotic is the same for all $\varphi \neq 0$ and further we consider the case $\varphi = \pi/2$ for simplicity. Due to the invariance in the transversal directions we can choose $\theta = 0$, $\psi = 0$. This corresponds to parametrization (3.3) with $\phi = \pi/2$, $\theta = 0$, $\psi = 0$ and in this configuration the small subsystem orientation is delineated along the transversal direction y_1 . We call this configuration the transversal and denote with the subscript yXY . In notation (3.13) we have:

$$\mathcal{V}_{yXY}(z) = \frac{b_s^{3/2}(z)}{z^{1+2/\nu}}, \quad M_{yXY}(z) = \frac{b_s^{3/2}(z)}{z^{2+1/\nu} \sqrt{g(z)}}, \quad \mathcal{F}_{yXY}(z) = g(z) z^{2-2/\nu}, \quad (3.60)$$

UV divergences are now defined by the asymptotic of $M_{yXY}(z)$ at $z \sim 0$ (3.47).

The plots of functions $\kappa_{yXY}(\nu)$ and $\kappa_{xYY}(\nu)$ are presented in figure 7. We see that $M_{yXY}(z)$ has a nonintegrable singularity for all ν values at $z = 0$.

For yXY case and $\nu > 1$ there is a nonintegrable singularity and we have to perform a renormalization:

$$S_{yXY,\text{ren}} = \frac{1}{2} \int_{\epsilon}^{z_*} dz \left[\frac{M_{yXY}(z)}{\sqrt{(1 - \frac{\mathcal{V}_{yXY}^2(z_*)}{\mathcal{V}_{yXY}^2(z)})}} - M_{yXY,as}(z) \right] + \frac{1}{2} B_s(\nu, c, z_h)^{3/2} \frac{z_*^{\kappa_{yXY}(\nu)+1}}{\kappa_{yXY}(\nu) + 1}, \quad (3.61)$$

where

$$\int M_{yXY,as} dz = B_s(\nu, c, z_h)^{3/2} \frac{z^{\kappa_{yXY}(\nu)+1}}{\kappa_{yXY}(\nu) + 1}.$$

4 Entanglement entropy density and c-functions

It is also instructive to consider the entanglement entropy density that is defined as [96, 97]

$$\eta = \frac{dS(\ell)}{d\ell} \quad (4.1)$$

(compare with [98, 99]). The advantage of dealing with the HEE density is that it has no divergences. For fixed temperature T (i.e. fixed z_h) the entanglement entropy density can be obtained from equation (3.18). We see that it can be expressed in terms of the value of the effective potential $\mathcal{V}$,

$$\eta(z_*) = \frac{dS(z_*)}{d\ell(z_*)} = \frac{\frac{dS(z_*)}{dz_*}}{\frac{d\ell(z_*)}{dz_*}} = \frac{\mathcal{V}(z_*)}{4}. \quad (4.2)$$

The potential $\mathcal{V}(z)$ in the Born-Infeld action (3.12) is the same for different orientations of the entangling domain. From (3.21) we get

$$\eta(z_*) = \frac{L^3 b_s^{3/2}(z_*)}{4z_*^3} (\mathfrak{g}_1(z_*) \mathfrak{g}_2(z_*) \mathfrak{g}_3(z_*))^{1/2}. \quad (4.3)$$

However, the angular dependence of the density exists because the length ℓ , given by (3.36), depends on the orientation.

Definition (4.3) requires a comment. Despite the fact that the density of entropy does not contain divergences, its definition may depend on a finite renormalization. In particular, adopting the geometric renormalization (3.24) we have

$$\begin{aligned} \eta_{CD} &= \frac{1}{4} (\mathcal{V}(z_*) - \mathcal{V}(z_D)) \\ &= \frac{L^3}{4} \left(\frac{b_s^{3/2}(z_*)}{z_*^3} (\mathfrak{g}_1(z_*) \mathfrak{g}_2(z_*) \mathfrak{g}_3(z_*))^{1/2} - \frac{b_s^{3/2}(z_D)}{z_D^3} (\mathfrak{g}_1(z_D) \mathfrak{g}_2(z_D) \mathfrak{g}_3(z_D))^{1/2} \right), \end{aligned} \quad (4.4)$$

where $z_D = z_{\text{DW}}$ or $z_D = z_h$.

	SF				EF			
	ISO	ANIS			ISO	ANIS		
		TRAN	φ	LONG		TRAN	φ	LONG
COLOR	Green	Blue	various	Magenta	Brown	Cyan	various	Gray

Table 1. The color scheme of the lines used in the plots in this article.

The anisotropic c-function can be defined as

$$c_\varphi = \ell_\varphi^{m_\varphi} \frac{dS(\ell_\varphi)}{d\ell_\varphi}, \quad (4.5)$$

Here m is the scaling power. It depends on the model and we make few comments about m in the next subsection 5.3. Generally speaking, unlike conformal theories [108–110] and especially the holographic theories with conformal invariance [29, 111–115], m_φ is different in UV and IR, can depend on the orientation of the slab as well (see considerations of candidates for c-functions in theories with Lorentz violation [100–103], and especially, [105–107]).

We can also consider

$$c_{CD,\varphi} = \frac{\ell_\varphi^m L^3}{4} \left(\frac{b_s^{3/2}(z_*)}{z_*^3} (\mathfrak{g}_1(z_*)\mathfrak{g}_2(z_*)\mathfrak{g}_3(z_*))^{1/2} - \frac{b_s^{3/2}(z_D)}{z_D^3} (\mathfrak{g}_1(z_D)\mathfrak{g}_2(z_D)\mathfrak{g}_3(z_D))^{1/2} \right). \quad (4.6)$$

5 Numerical results

In this section we display and discuss our main results using numerical calculations. In what follows we set $L = 1$. We present all the plots according to the color scheme of the lines indicated in table 1.

5.1 Entanglement entropy near the background phase transition

In this section we present plots of the entanglement entropy dependence on the geometric characteristics of the entangling region (orientation and thickness of the slab) and the thermodynamic characteristics of the medium (temperature and chemical potential) for the model (2.4). We find the slab HEE dependence on the smallest length ℓ numerically performing integration of (3.36) and (3.37), and then excluding the dependence on z_* by solving equation (3.36) for a given ℓ .

In this construction the location of the dynamical walls and the horizon plays a special role. In figure 8 we show the appearance of the dynamical wall. The dynamical wall appears when the effective potential $\mathcal{V}$ gets a saddle point.

In figure 8 we also illustrate the location of z_{DW} for various choices of the horizon size z_h for $\nu = 1$ (a) and $\nu = 4.5$ (b). Solid lines present $\mathcal{V}$ and dashed lines present $\mathcal{V}'$. We see that the location of the dynamical wall does not depend on the horizon. In figure 8(c) we show the location of the dynamical wall for arbitrary $1 \leq \nu \leq 4.5$.

In figure 9 we see that ℓ increases when z_* approaches z_{DW} or the horizon. The location of the dynamical wall does not depend on the orientation of slab and also does not depend

Figure 8. Locations of the dynamical wall for HEE for (a) $\nu = 1$ and (b) $\nu = 4.5$. Solid lines present $0.15 \cdot \mathcal{V}$, dashed lines present $\mathcal{V}'$. We see that the location of the dynamical wall does not depend on z_h . Note that only for $z_h < z_{\text{HP}}(\nu)$ we obtain the stable solutions, $z_{\text{HP}}(1) = 1.505$, $z_{\text{HP}}(4.5) = 1.138$.

on the horizon, see figure 8 below. In figure 8 the plots are presented for $\nu = 4.5$, $T = 0.25$ and $\mu = 0.2$.

We present in figure 9(a) the ℓ dependence on z_* for $\nu = 1$, $z_h = 0.5, 1, 1.5$ and $\mu = 0$. The red dashed line shows the position of the dynamical wall. The plots in figure 9(b) show the ℓ dependence on z_* for $\nu = 4.5$, $T = 0.25$ and $\mu = 0.2$ for different angles $\varphi = 0, \pi/6, \pi/4, \pi/2$. The value of z_* does not exceed $z_{\text{DW}}(\mu, T)$, the position of the dynamical wall for given μ and T . Here $z_{\text{DW}} = 0.349$.

We tried to avoid divergences in (3.37) and used the minimum subtraction of the UV asymptotic in accordance with the formulas above in 3.4.1 and section 3.4.2.

5.1.1 Entanglement entropy dependence on ℓ

In figure 10 we present the dependence of the slab HEE on its smallest length ℓ at various values of the temperature. These temperatures are chosen equidistantly around the temperature of the BB phase transition at $\mu = 0.05$ in the isotropic case. These temperatures are indicated on the (z_h, T) -curve by the points (A,B,C) above the T_{BB} and the points (D,E,F) below T_{BB} for $\mu = 0.05$ (see figure 10(a)). For these temperatures and $\mu = 0.05$ we depict the dependence of the HEE on ℓ . Here ℓ is chosen to be less than ℓ_c , the length

Figure 9. (a) The ℓ dependence on z_* for $\nu = 1$, $z_h = 0.5, 1, 1.5$ and $\mu = 0$. The red dashed line show the position of the dynamical wall, $z_{\text{DW}} = 1.41$. (b) The plot shows the ℓ dependence on z_* for $\nu = 4.5$, $T = 0.25$ and $\mu = 0.2$ for different angles $\varphi = 0, \pi/6, \pi/4, \pi/3, \pi/2$. Note that z_* does not exceed $z_{\text{DW}}(T, \mu)$, the position of the dynamical wall for given T and μ . Here $z_{\text{DW}} = 0.349$.

at which the turning point of the connected entangling surface moves closer towards the dynamical wall z_{DW} (see figure 9). We see that the HEE undergoes a jump when the temperature crosses the phase transition line at the point $T_{BB}(\mu)$. We also see that these jumps depend smoothly on ℓ at least for $\ell_{\text{UV}} < \ell < \ell_c$.

The similar picture takes place for the anisotropic background. The HEE dependencies on ℓ for the transversal and longitudinal orientations at equidistant values of the temperature around the BB phase transition for $\mu = 0.2$ and $\nu = 4.5$ are presented in figure 11(a) and figure 11(b), respectively. We see that with decreasing of the angle φ from $\varphi = \pi/2$ to $\varphi = 0$ we increase the HEE at fixed ℓ and fixed thermodynamic parameters (with our adopted method to eliminate divergences).

Therefore, we have demonstrated that the HEE depends on the regularization scheme.

- For the minimal regularisation scheme both the longitudinal and the transversal HEE
 - increase linearly at large ℓ

$$S_{yXY} \underset{\ell \rightarrow \infty}{\sim} C_{yXY}(T, \mu)\ell, \quad S_{xYY} \underset{\ell \rightarrow \infty}{\sim} C_{xYY}(T, \mu)\ell, \quad (5.1)$$

and $C_{yXY}(T, \mu) > C_{xYY}(T, \mu)$. The linearity region for longitudinal entanglement entropy begins at lower ℓ compared to the transverse entanglement entropy.

- they have different behavior for $\ell \rightarrow 0$

$$S_{xYY} \underset{\ell \rightarrow 0}{=} 0, \quad S_{yXY} \underset{\ell \rightarrow 0}{\sim} \ell^{-\xi_{yXY}(T, \mu)}, \quad \xi_{yXY}(T, \mu) > 0, \quad (5.2)$$

(compared with results of [36]).

- For the CD regularisation scheme both the longitudinal and the transversal HEE

Figure 10. (a) T as a function of z_h for various values of the chemical potential μ . Here curves with different thicknesses correspond to $\mu = 0, 0.05, 0.117$ and 0.2 . Plots (b) and (c) show the dependence of HEE for a slab on its smallest length ℓ for various temperatures near the temperature of the BB phase transition, $T_{BB} = 0.3445$ ($z_h = 1.532$) corresponding to $\mu = 0.05$ and $\nu = 1$. In plot (b) we use the minimal renormalization scheme, in (c) we use the geometric renormalization.

- tend to constants at large ℓ

$$S_{yXY} \underset{\ell \rightarrow \infty}{\sim} a_{yXY}(T, \mu), \quad S_{xYY} \underset{\ell \rightarrow \infty}{\sim} a_{xYY}(T, \mu) \quad (5.3)$$

and $a_{yXY}(T, \mu) > a_{xYY}(T, \mu)$;

- at small ℓ they have different behavior for $\ell \rightarrow 0$

$$S_{xYY,CD} \underset{\ell \rightarrow 0}{\sim} \ell^{-\xi_{CD,xYY}(T,\mu)}, \quad S_{yXY,CD} \underset{\ell \rightarrow 0}{\sim} \ell^{-\xi_{CD,yXY}(T,\mu)}. \quad (5.4)$$

5.1.2 Entanglement entropy dependence on temperature

It is instructive to depict the HEE dependence on the temperature for a fixed value of ℓ near the phase transition. For this purpose we first find the dependence of z_* on the length ℓ (at fixed horizon) and then calculate the HEE from (3.57) and (3.58). The dependence

Figure 11. The HEE dependencies on the smallest length of the entangling slab ℓ for $\varphi = \pi/2$, the transversal orientation, and $\varphi = 0$, the longitudinal orientation, at equidistant values of the temperature around the BB phase transition for $\mu = 0.2$ and $\nu = 4.5$ are presented in (a) and (b), respectively. Here the renormalizations of the HEE are performed according to (3.58) and (3.61). In (c) and (d) we show the HEE for renormalizations performed according to (3.41) and (3.39), (c) for $\varphi = \pi/2$ and (d) for $\varphi = 0$

of z_* on z_h for the fixed value of $\ell = 1$ and different values of the angle φ at the chemical potentials $\mu = 0$ are shown by blue lines in figure 12. For comparison, in this plot the dependence of z_* on z_h is shown for the isotropic case by green lines also for $\mu = 0$. We notice a significant dependence on z_h only for small values of z_h (large BHs).

Substituting the dependence of z_* on z_h to equations (3.57) and (3.58) we get the HEE

Figure 12. The dependence of z_* on z_h at fixed $\ell = 1$ and different angles for the anisotropic case (blue lines for angles $\varphi = 0, \pi/4, \pi/3, \pi/2$ from bottom to top) and for isotropic case (green lines) at $\mu = 0$.

Figure 13. The HEE of the slab with the fixed value of $\ell = 1$ dependences on the temperature for (a) longitudinal and (b) transversal cases in the anisotropic ($\nu = 4.5$) case for chemical potential below and above the critical chemical potential $\mu_{cr} = 0.349$ are shown by the blue lines with different thickness. For comparison, the dependences of HEE of the same slab in isotropic background are shown by green lines in both plots .

dependence on the temperature at the fixed value of $\ell = 1$ for different chemical potentials μ for (a) longitudinal and (b) transversal cases. These dependences are shown by blue lines in figure 13 (a) and figure 13 (b) for various chemical potentials μ near the critical value $\mu_{cr} = 0.349$ for the longitudinal and transversal cases, respectively. For comparison, the dependences of the HEE on temperature for the same slab are shown by green lines for different chemical potentials μ near $\mu_{cr} = 0.117$ for the isotropic case, $\nu = 1$ in both graphs.

These plots figure 13 show that in the vicinity of the phase transition temperature

Figure 14. The HEE of the slab with the fixed value of $\ell = 1$ depends on the temperature for (a) longitudinal and (b) transversal cases in the anisotropic ($\nu = 4.5$) case for chemical potential below and above the critical chemical potential $\mu_{cr} = 0.349$ are shown by the blue and magenta lines with different thicknesses.

Figure 15. Comparison of the HEE of the slab with the fixed value of $\ell = 1$ calculated according to two different renormalization schemes near the background phase transition line.

$T = T_{BB}(\nu, \mu)$ the HEE, like the thermal entropy (see figure 5(b)) undergoes significant jumps. From these plots we also see that for $\mu > \mu_{cr}$ ($\mu_{cr} = 0.117$ for $\nu = 1$ and $\mu_{cr} = 0.349$ for $\nu = 4.5$) the jumps disappear, similar to the thermal entropy behavior.

To summarise this subsection, we note that,

- the HEE of the slab with constant thickness ℓ increases with increasing temperature and fixed chemical potential;
- the HEE jumps near the phase transition do not depend significantly on the type of renormalization (geometric and minimal renormalization), see figure 15.

Figure 16. The dependence of the slab entropy density on ℓ for various temperatures around the critical temperature at $\mu = 0.2$ for (a) transversal and (b) longitudinal orientations. The density exhibits a jump near the BB phase transition ($T_{BB} = 0.2457$ for $\mu = 0.2$, we can compare $T = 0.235, 0.240, 0.245, 0.250, 0.255, 0.260$).

5.2 Entanglement entropy density

The entanglement entropy density, defined in the previous section 4 as (4.3), in our case takes the form

$$\eta(z_*) = \frac{1}{4} \frac{b_s^{3/2}(z_*)}{z_*^{1+2/\nu}}, \quad (5.5)$$

but if defined according to 4.4, it takes the form

$$\eta_{CD}(z_*) = \frac{1}{4} \left(\frac{b_s^{3/2}(z_*)}{z_*^{1+2/\nu}} - \frac{b_s^{3/2}(z_D)}{z_D^{1+2/\nu}} \right). \quad (5.6)$$

In this section we study the dependence of the entanglement entropy density (5.5) on temperature and chemical potential near background phase transition. Also we compare the results in minimal and geometric renormalizations, (5.5) and (5.6).

5.2.1 Entanglement entropy density dependence on ℓ

In figure 16 we show the dependence of the HEE density (5.5) on ℓ at fixed values of the temperature and chemical potential near the background phase transition line for the model (2.4). We see that the density function decreases monotonically depending on the length. Here, similarly to the considerations in the previous section 5.1, we consider the density of the entanglement entropy for different T and μ near the phase transition point ($T_{BB} = 0.2457, \mu_{BB} = 0.2$), namely for the points A, B, C and D, E, F indicated in figure 4.A. We see that the densities change significantly when we cross the phase transition point (T_{BB}, μ_{BB}).

Figure 17. The dependence of the slab entropy density defined by (5.6) on ℓ for various temperatures around the critical temperature at $\mu = 0.2$ for (a) transversal and (b) longitudinal orientations. Considered temperatures correspond to the point indicated in figure 4. The density exhibits a jump near the BB phase transition point ($T_{BB} = 0.2457, \mu = 0.2$).

Figure 18. The dependence of the slab entropy density (5.6) on ℓ for various temperatures around the critical temperature at $\mu = 0.05$ for isotropic case. Considered temperatures correspond to the point indicated in figure 10(a). The density exhibits a jump near the BB phase transition ($T_{BB} = 0.3445, \mu = 0.05$).

In figure 17 we show the dependence of the HEE density (5.6) on ℓ at the same temperatures and chemical potential as in figure 16. We get the similar behavior in both cases. For comparison, we present the dependence of the densities (5.5) and (5.6) on ℓ in the isotropic case in figure 18.

Figure 19. The HEE density dependence on the temperature for fixed value of $\ell = 1$. Blue lines on (a) and (b) show the HEE density dependence for different values of the chemical potential ($\mu = 0, 0.2, 0.349, 0.45$) for longitudinal and transversal orientations, respectively, $\nu = 4.5$. Green lines show the same dependencies in the isotropic case for the chemical potential ($\mu = 0, 0.05, 0.117, 0.2$). Arrows indicate the HEE density jumps at the points of HP and BB phase transitions ($T_{HP}(\mu = 0)$ and $T_{BB}(\mu > 0)$).

5.2.2 Entanglement entropy density dependence on temperature

Now, we fix the length $l = 1$ and present in figure 19 the slab HEE density dependence on the temperature for different angles, $\varphi = 0$ and $\varphi = \pi/2$, and different chemical potentials μ , below and above the critical values $\mu_{cr}^{(anis)} = 0.349$ corresponding to $\nu = 4.5$. For comparison, we present here the similar plots for isotropic cases where $\mu_{cr}^{(iso)} = 0.117$. We see a dependence on orientation.

In figure 21 the angular dependence of the logarithm of entanglement entropy density for angle values $\varphi = 0, \pi/6, \pi/4, \pi/2$ (thickness increases with increasing angle) at the chemical potential $\mu = 0$ and $\mu = 0.2$ are shown in (a) and (b), respectively. We see that the shapes of the curves shown in (a) are the same for different orientation angles, and the curves only shift with increasing angle φ . The same applies to the curves shown in (b).

In figure 22 we show the angular dependence of the entanglement entropy density for different values of ℓ (thickness increases with slab width increase) at the chemical potential $\mu = 0.2$ and $T = 0.25$ in (a) minimal and (b) geometrical regularizations.

Comparing the plots in figure 21 and figure 22 we see that the transversal HEE density depends weakly on the regularization type while there is a substantial difference for the longitudinal density in different regularization schemes.

In figure 23 we show the angular dependence of the entanglement entropy density for different values of ℓ (thickness increases with slab width increase) at the chemical potential $\mu = 0.2$ and $T = 0.25$ in minimal (a) and (b) geometrical regularization. Since the magnitude of the HEE density above the critical temperatures is substantially larger than the values of HEE density below the phase transition, the plot of the jumps of HEE density figure 24 is similar to the figure 23.

Figure 20. The HEE density dependence on the temperature for fixed value of $\ell = 1$. The lines in (a) and (b) show the HEE density dependence for different values of the chemical potential, $\mu = 0, 0.2, 0.349, 0.45$ (thickness increases with increasing μ) for longitudinal and transversal orientations, respectively, $\nu = 4.5$.

Figure 21. The temperature dependence of the logarithm of the entanglement entropy density (5.5) for angle values $\varphi = 0$ (magenta), $\pi/6$ (magenta blue), $\pi/4$ (blue magenta) and $\pi/2$ (blue) at the chemical potential (a) $\mu = 0$ and (b) $\mu = 0.2$.

We see that the HEE density undergoes jumps near the BB phase transition and these jumps depend on the anisotropy, slab orientation and chemical potential.

To summarize this subsection, note that similarly to the plots presented for the entanglement entropy in the previous subsection 5.1, in figure 16 - figure 23, we present dependence of the HEE densities (5.5) and (5.6) on the slab width l for equidistant values of temperature near background phase transition. We observe that the HEE density undergoes a jump when the temperature crosses the phase transition line. We also observe a non-substantial dependence on the regularization scheme.

Figure 22. The temperature dependence of the logarithm of entanglement entropy density (5.6) for angle values $\varphi = 0$ (magenta) , $\pi/6, \pi/4, \pi/2$ (blue) at the chemical potential (a) $\mu = 0$ and (b) $\mu = 0.2$.

Figure 23. The angular dependence of entanglement entropy density (a) η_φ and (b) $\eta_{CD,\varphi}$ for $T = 0.25$, $\mu = 0.2$, $\nu = 4.5$, $\ell = 1, 2, 3$ (thickness increases with increasing length).

5.3 c-functions

Now we consider the behavior of c-function numerically

$$c_{\nu,\varphi,F} = \frac{\ell_\varphi^{m_{\varphi,F}}}{4} \left(\frac{b_F^{3/2}(z_*)}{z_*^{1+2/\nu}} - \frac{b_F^{3/2}(z_D)}{z_D^{1+2/\nu}} \right), \quad (5.7)$$

where $m_{\varphi,F}$ is a scaling power. It depends on the model, in our particular case on the anisotropy parameter ν , on the orientation (index φ) and frame. The frame is denoted by index F , $F = \text{EF}$ (Einstein frame), $F = \text{SF}$ (string frame). In isotropic conformal invariant case in 5 dimensions, $m_{\text{AdS}_5} = 3$.

Figure 24. The angular dependence of entanglement entropy density jumps (thickness increases with increasing length, $\ell = 1, 2, 3$), (a) $\Delta\eta_\phi$ and (b) $\Delta\eta_{CD,\phi}$, at the chemical potential $\mu = 0.2$.

In our model in the EF we have

$$ds_{\text{UV}}^{\text{EF}^2} \sim \frac{1}{z^2} \left(-f(z)dt^2 + dx^2 + z^{2-2/\nu}(dy_1^2 + dy_2^2) + \frac{dz^2}{f(z)} \right), \quad (5.8)$$

This metric is invariant under the rescaling

$$t \rightarrow \Lambda t, \quad x_i \rightarrow \Lambda x_i, \quad y_j \rightarrow \Lambda^{1/\nu} y_j, \quad i = 1, j = 1, 2, z \rightarrow \Lambda z. \quad (5.9)$$

Using the suggestion of [105], see also [106, 107], we get

$$m_{x,\text{EF}} = d_1 + d_2 \frac{n_2}{n_1} = 1 + \frac{2}{\nu}, \quad (5.10)$$

$$m_{y,\text{EF}} = d_2 - 1 + d_1 \frac{n_1}{n_2} = 2 + \nu, \quad (5.11)$$

where $n_1 = 1, d_1 = 1$ and $n_2 = 1/\nu, d_2 = 2$. Taking into account the behavior of the dilaton in UV [70] we get in the SF

$$ds_{\text{UV}}^{\text{SF}^2} \sim \frac{1}{z^{K_{\text{UV}}}} \left(-f(z)dt^2 + dx^2 + z^{2-2/\nu}(dy_1^2 + dy_2^2) + \frac{dz^2}{f(z)} \right), \quad (5.12)$$

where

$$K_{\text{UV}}(\nu) = 2 - \frac{\sqrt{\frac{8}{3}(\nu - 1)}}{\nu} \quad (5.13)$$

therefore,

$$t \rightarrow \Lambda^{K_{\text{UV}}/2} t, \quad x_i \rightarrow \Lambda^{K_{\text{UV}}/2} x_i, \quad z \rightarrow \Lambda^{K_{\text{UV}}/2} z \quad (5.14)$$

$$y_j \rightarrow \Lambda^{K_{\text{UV}}/2 - 1 + 1/\nu} y_j, \quad i = 1, j = 1, 2 \quad (5.15)$$

and

$$n_1 = K_{\text{UV}}/2, \quad (5.16)$$

$$n_2 = K_{\text{UV}}/2 - 1 + 1/\nu. \quad (5.17)$$

Figure 25. The “scaling factors” $m_{x,y}^{\text{SF}}(z, z_h, \nu)$ and $m_{x,y}^{\text{EF}}(z, z_h, \nu)$ defined by (5.22) and (5.23) and their UV approximations (5.20) and (5.21) shown by the dashed lines.

In the SF we take

$$m_{x,\text{UV}}^{\text{SF}} = 1 + 2 \frac{K_{\text{UV}}/2 - 1 + 1/\nu}{K_{\text{UV}}/2}, \quad (5.18)$$

$$m_{y,\text{UV}}^{\text{SF}} = 2 + \frac{K_{\text{UV}}/2}{K_{\text{UV}}/2 - 1 + 1/\nu}. \quad (5.19)$$

In the particular case, $\nu = 4.5$, we have

$$m_x^{\text{EF}} \Big|_{\nu=4.5} = 1.444, \quad m_x^{\text{SF}} \Big|_{\nu=4.5} = 0.645, \quad (5.20)$$

$$m_y^{\text{EF}} \Big|_{\nu=4.5} = 6.5, \quad m_y^{\text{SF}} \Big|_{\nu=4.5} = -3.635. \quad (5.21)$$

In (5.7) in the SF z_D means the minimum of two values, the position of the horizon and the position of the corresponding dynamical wall. In the EF there is no dynamical wall and D means the position of the horizon. Notice that position of the dynamical wall depends on the anisotropy parameter and does not depend on the horizon position, see figure 8.

We have to mention, that here we use UV asymptotics of the warp factors $b(z)$ and $b_s(z)$. The scaling exponents are changed already in the intermittent region. The regions of the validity of the scaling (5.20) and (5.21) depend on the size of the horizon, see figure 25.

The scaling functions $m_{x,y}^{\text{SF}}$ and $m_{x,y}^{\text{EF}}$ shown in (25) are defined by the formula similar to (5.18) and (5.19)

$$m_x^F(z, z_h, \nu) = 1 + 2 \frac{K^F(z, z_h, \nu)/2 - 1 + 1/\nu}{K^F(z, z_h, \nu)/2}, \quad (5.22)$$

$$m_y^F(z, z_h, \nu) = 2 + \frac{K^F(z, z_h, \nu)/2}{K^F(z, z_h, \nu)/2 - 1 + 1/\nu}, \quad (5.23)$$

where F is the index indicated the frame, $F = \text{EF}, \text{SF}$ and

$$ds^{\text{SF}2} = \frac{1}{z K^{\text{SF}}(z, z_h, \nu)} \left(-f(z) dt^2 + dx^2 + z^{2-2/\nu} (dy_1^2 + dy_2^2) + \frac{dz^2}{f(z)} \right), \quad (5.24)$$

$$K^{\text{SF}}(z, z_h, \nu) = \frac{\log \left(\frac{b_s(z, c, \nu, z_h, \mu)}{z^2} \right)}{\log \left(\frac{1}{z} \right)} \quad K^{\text{EF}}(z, z_h, \nu) = \frac{\log \left(\frac{b(z, c, \nu, z_h, \mu)}{z^2} \right)}{\log \left(\frac{1}{z} \right)}. \quad (5.25)$$

Figure 26. The dependence of T on z_h for $\nu = 1$ (green line) and $\nu = 4.5$ (blue line) for $\mu = 0$. The dots show location of z_{HP} points, $z_{\text{HP}}(1) = 1.505$, $z_{\text{HP}}(4.5) = 1.138$.

We see that the UV scaling factor $m_{\text{UV},x,y}^{\text{EF}}$ approximates the function $m_{x,y}^{\text{EF}}(z, z_h, \nu)$ relatively well up to holographic coordinates about a half of the horizon position. The same concerns also the approximation $m_{\text{UV},y}^{\text{SF}}$ to $m_y^{\text{SF}}(z, z_h, \nu)$, meanwhile the UV scaling factor $m_{\text{UV},x}^{\text{SF}}$ cannot be used as an approximation for $m_y^{\text{EF}}(z, z_h, \nu)$. Moreover, all $m_{x,y}^F(z, z_h, \nu)$ exhibit the singular behavior at large ℓ .

Let us first consider the case of zero chemical potential. The dependence of temperature on the horizon for $\mu = 0$ is presented in figure 26. For large BH's the temperature increases when $z_h \rightarrow 0$, and for small BH's the temperature increases also for $z_h \rightarrow \infty$. At $z_h = z_{\text{HP}}$ the HP phase transition takes place and all small BH's are unstable.

5.3.1 The c-function in the isotropic case

For the isotropic case in the EF, i.e. with b -factor without a dilaton, the c-function is defined as in the conformal invariant case

$$c_{\nu=1,\text{EF}} = \frac{\ell_{\nu=1}^3}{4} \left(\frac{b^{3/2}(z_*)}{z_*^{1+2/\nu}} - \frac{b^{3/2}(z_D)}{z_D^{1+2/\nu}} \right). \quad (5.26)$$

In figure 27 we depict the $c_{\nu=1,F}$ vs ℓ_{iso} by brown lines for $z_h = 0.8, 1, 1.5$ (thickness increases with decreasing z_h). In all these cases the disconnected parts hang up to the horizon (there is no dynamic wall in the EF) therefore in (5.26) $z_D = z_h$.

For comparison, in figure 27 the dependences of $c_{\nu=1,\text{SF}}$ on ℓ , i.e. for calculations that are done in the SF with $b_S(z)$,

$$c_{\nu=1,\text{SF}} = \frac{\ell_{\nu=1}^{m_{\nu=1,\text{SF}}}}{4} \left(\frac{b_s^{3/2}(z_*)}{z_*^{1+2/\nu}} - \frac{b_s^{3/2}(z_D)}{z_D^{1+2/\nu}} \right) \quad (5.27)$$

are shown by green lines. In this case there is the dynamical wall and if it is not covered by the horizon, $z_{\text{DW}} = 1.413 < z_h$, the disconnected parts hang up to the dynamical wall. For the cases depicted in figure 27 this corresponds to the cases $z_h = 0.8$ and $z_h = 1$. Note that $z_h = 1.3$ corresponds to the thermodynamically unstable phase.

We observe a completely different behavior of green and brown curves. There are saddle points on the green lines, but not on the brown ones. These saddle points are

Figure 27. Comparison of dependences of c on ℓ for $z_h = 0.8, 1, 1.5$ in the isotropic case (line thickness increases with decreasing horizon) for calculations that are done in the EF (brown lines) and SF (green lines). The dynamical wall in the SF is at $z_{\text{DW}} = 1.41$ and there is no dynamical wall in the EF.

related with existence of solution of the equation

$$c'_{\nu,\varphi,F}(z_0) = 0 = \ell \mathcal{V}' + m_F \ell' (\mathcal{V}(z_0) - \mathcal{V}(z_D)). \quad (5.28)$$

Notice, that by definition the DW corresponds to $\mathcal{V}'|_{z=z_{\text{DW}}} = 0$, but $\ell \rightarrow \infty$ at $z \rightarrow z_{\text{DW}}$. In the second term the difference is zero at $z = z_{\text{DW}}$, but $\ell' \rightarrow \infty$ and we have uncertainty at $z \rightarrow z_{\text{DW}}$. Calculations show that $c'_{\text{SF}} \rightarrow 0$ for $z \rightarrow z_{\text{DW}}$.

5.3.2 The c-function in the anisotropic case

In anisotropic cases the definition of the c-function is modified [105–107]

$$c_\varphi = \frac{\ell_\varphi^{m_\varphi}}{4} \left(\frac{b_s^{3/2}(z_*)}{z_*^{1+2/\nu}} - \frac{b_s^{3/2}(z_D)}{z_D^{1+2/\nu}} \right). \quad (5.29)$$

Here the power m_φ depends on the orientation. For the particular cases of the transversal and longitudinal orientations these powers, m_y and m_x , are defined according to different scaling in transversal and longitudinal directions. We use here the scaling factors m_φ defined in accordance with the behavior of the metric in the UV region. Note that since the behavior of $b_s(z)$ and $b(z)$ are different in the UV region, we get different factors m_φ at different frames, and we put a subscript $m_{\varphi,F}$ to indicate this; $F = \text{EF}$ for the Einstein frame and $F = \text{SF}$ for the string frame (we have the same for the isotropic case). The factor $m_{\varphi,F}$ is defined according to the formula (5.18) and (5.19) and for $\nu = 4.5$ we have (5.20) and (5.21).

Figure 28. The dependences of c , defined by (5.29), on ℓ for $\nu = 4.5$, and various values of z_h for calculations done in the EF. *Top line:* (a) transversal orientation and EF, $z_h = 1, 1.13$ and $z_h = z_{\text{HP}} = 1.138$ (red lines), here we also insert the lines with an unstable value of $z_h = 1.5$ (unstable point, orange line) and a stable one $z_h = 0.8$ (the inset shows the multi-valued behavior of $c = c(\ell)$ in the region near $z = z_{\text{HP}}$, indicated by a light blue rectangle); $\mu = 0$. (b) the same for $\mu = 0.5$. *Bottom line:* longitudinal orientation and EF for (c) $\mu = 0$ and (d) $\mu = 0.5$ (the insets show the multi-valued behavior of $c = c(\ell)$; here $\ell_{\text{left},L}$ and $\ell_{\text{right},R}$ are indicated).

In figure 28 and figure 29 we present dependences of c on ℓ for $\nu = 4.5$, various z_h and $\mu = 0, 0.5$ calculated in the EF and SF for transversal and longitudinal orientations. We see that we get rather different behavior in EF and SF.

- We see in figure 28(a) that in the transversal case the c -function has rather nontrivial behavior in the EF for $\mu = 0$ already. Namely we see that there is a region of z_h , here we see multivalued c -function as function of ℓ . This takes place not only at nonstable points (one of them is shown by orange), but also at $z_{h,\text{HP}} = 1.138$, and this multivalued phenomena disappears at $z_h = 1.13$. We also see that c -function has

Figure 29. The dependences of c , defined by (5.29), on ℓ for $\nu = 4.5$, and various values of z_h for calculations done in the SF. *Top line:* transversal orientation for (a) $\mu = 0$ and (b) $\mu = 0.5$. *Bottom line:* longitudinal orientation for (c) $\mu = 0$ and (d) $\mu = 0.5$. In the SF the dynamical wall is at $z_{\text{DW}} = 0.34$.

local minimum and maximum at the points $\ell_{\min}$ and $\ell_{\max}$ correspondingly, and there is a region of $\ell_{\min} < \ell < \ell_{\max}$ where c increases when ℓ increases.

- We see in figure 28(b) that this multi-validity is preserved at $\mu = 0.5$ and the region of z_h where this multi-validity takes place, is wider in comparison with $\mu = 0$ case.
- The c -function for the longitudinal orientation for $\mu = 0$ monotonically decreases.
- While at $\mu = 0.5$ and rather large z_h (small BH), the c -function exhibits a non-monotonic behavior around some ℓ_{cr} (see the plot in the inset of figure 29(d)).
- In the SF the c -function for the transversal orientation decreases monotonically.
- For the longitudinal orientation there is a saddle point ℓ_0 (which depends on z_h and μ) and the c -function starts to decrease only for $\ell > \ell_0$.

Figure 30. *Top line:* $\ell_\phi^{m_F(\nu)} \eta_{CD}(\nu)$ vs z_* for F=EF and $\varphi = \pi/2$ (transversal case) at (a) $z_h = 1$, $\mu = 0$, (b) $z_h = 1$, $\mu = 0.2$ and (c) $z_h = 2.5$, $\mu = 0.2$. *Bottom line:* $\ell_\phi^{m_F(\nu)} \eta_{CD}(\nu)$ vs z_* for F=EF and $\varphi = 0$ (longitudinal case) at (d) $z_h = 1$, $\mu = 0$, (e) $z_h = 1$, $\mu = 0.2$ and (f) $z_h = 2.5$, $\mu = 0.2$.

5.4 Origin of non-monotonic behavior of c-functions

Let us make a few comments about the origin of the behavior of the c-functions presented in figure 28 and figure 29. To understand the origin of such an untypical behavior we first consider the dependence of c-function on z_* for the cases presented here, and then incorporate the dependence of ℓ on z_* .

5.4.1 The c-function as a function of z_*

Figure 30 shows contour plots of $\ell_\phi^{m_F(\nu)} \eta_{CD}(\nu)$ over the (z_*, ν) plane for the EF at different z_h and μ , and different orientations (*top line* for the transversal case and *bottom line* for the longitudinal one). All these plots demonstrate that the c-function as a function of z_* monotonically decreases while increasing z_* . One can also see that the c-function essentially depends on the orientation, c-function is larger for the transversal case for the same thermodynamical parameters and the same z_* . Also the c-function falls with temperature faster in the transversal case. Note that the $c(z_*)$ does not change substantially when we change the chemical potential. This is due to the fact that the c-function depends mainly on the effective potential, which does not depend on the blackening function. There

is a weak dependence of the form of the function $c = c(z_*)$ on z_h and μ , compare the graphs on the same lines in figure 30.

In figure 31 contour plots for $\ell_\varphi^{m_F(\nu)} \eta_{CD}(\nu)$ over (z_*, ν) -plane for the SF at $z_h = 1$ and $\varphi = \pi/2$ (transversal case), and for different values of the chemical potential are shown: figure 31(a) $\mu = 0.2$ and figure 31(b) $\mu = 0.5$. We see that two contour plots are almost identical. On the *bottom line* of figure 31(c) the plots similar to plots in figure 31(a) for various discrete values of ν are presented; also the similar plots are presented in figure 31(d) for various discrete values of ν . We see that there is a small quantitative difference in the right inserts of both plots: the coordinates of the saddle points for figure 31(a) $\mu = 0.2$ are $z_*|_{\nu=1.5} = 0.7146$, $c|_{\nu=1.5} = 0.4590$ and $z_*|_{\nu=1} = 0.9361$, $c|_{\nu=1} = 0.1170$, and for figure 31(b) $\mu = 0.5$ are $z_*|_{\nu=1.5} = 0.71748$, $c|_{\nu=1.5} = 0.46135$ and $z_*|_{\nu=1} = 0.8489$, $c|_{\nu=1} = 0.14450$.

In the *top line* of figure 32 contour plots of $\ell_\phi^{m_F(\nu)} \eta_{CD}(\nu)$ over (z_*, ν) -plane for the SF at $z_h = 1$ and $\varphi = 0$ (longitudinal case), and for different values of the chemical potential are presented. Here in: figure 32(a) $\mu = 0$ and in figure 32(b) $\mu = 0.5$. We see that two contour plots are practically identical. In the *bottom line* of figure 32 $\ell_\phi^{m_F(\nu)} \eta_{CD}(\nu)$ as functions of z_* for discrete values of ν are presented. For comparison, c-functions for $\mu = 0$ (magenta lines) and $\mu = 0.5$ (darker magenta lines) for discrete values of ν are presented on the same plot in figure 32(d). We see that two sets of lines almost coincide.

5.4.2 ℓ as a function of z_*

The length ℓ as a function of z_* has a rather nontrivial form for some values of thermodynamic parameters, especially in the longitudinal case. In figure 33(a) the functions $\ell = \ell(z_*, z_h, \mu)$ for F=EF for transversal case at $z_h = 1, 1.06, 1.138, 1.2$ and $\mu = 0$ are presented. It is interesting to note that even on stable backgrounds, i.e. $z_{h,ip} = 1.108 < z_h < z_{HP} = 1.138$, the functions $\ell = \ell(z_*, z_h, \mu)$ are non-monotonic. Here $z_{h,ip} = 1.108$ corresponds to the line, that has an inflection point (see the inset of figure 33(a)). This means that the same ℓ can be realized at different z_* , or 3 different values of entanglement entropy, entropy density and c-function corresponds to the same length.

Figure 33(b)-figure 33(d) show what happens when we change the chemical potential. In figure 33(b) $\mu = 0.2$ and $z_h = 1.285$ (an unstable point, the corresponding curves are displayed in orange), $z_h = 1.1266$ (the point of the BB phase transition depicted in red), $z_h = 1.1$ (the corresponding curve has an inflection point) and $z_h = 1$. In figure 33(c) $\mu = 0.2$ and points with $z_h > 2.76$ correspond to the stable backgrounds. An unstable point with $z_h > 2.6$ is in orange. In figure 33(d) $\mu = 0.5$ and all curves correspond to the stable backgrounds.

In figure 34 on the *top line* ℓ vs z_* for the EF and $\varphi = 0$ (longitudinal case) is shown: figure 34(a) at $z_h = 1.2, 1.138, 1.108, 1$ and $\mu = 0$; figure 34(b) at $z_h = 2, 2.5, 3.0, 3.5$ and $\mu = 0.5$. On the *bottom line* ℓ vs z_* for the SF is shown: figure 34(c) for $\varphi = \pi/2$ (transversal case) at $z_h = 0.3, 0.325, 0.35, 0.375$ and 0.4 . The right curve corresponds to surfaces touching a dynamic wall located at $z_{DW} = 0.354$ and for the rest, z_* is located near the horizons. In figure 34(d) the plot for $\varphi = 0$ (longitudinal case) is shown.

Figure 31. *Top line:* contour plots for $\ell_\phi^{m_F(\nu)} \eta_{CD}(\nu)$ vs z_* (horizontal axis) and ν (vertical axis) for $F=SF$ at $z_h = 1$ and $\varphi = \pi/2$ (transversal case), and for different values of the chemical potential: (a) $\mu = 0.2$, and (b) $\mu = 0.5$. We see that two contour plots are almost identical. *Bottom line:* (c) the same as in (a) for various discrete values of ν ; (d) the same as in (b) for various discrete values for ν . We see that there is a small quantitative difference in the right inserts of both plots: the coordinates of the saddle points for (a) $\mu = 0.2$ are $z_*|_{\nu=1.5} = 0.7146$, $c|_{\nu=1.5} = 0.4590$ and $z_*|_{\nu=1} = 0.9361$, $c|_{\nu=1} = 0.1170$, and for (b) $\mu = 0.5$ are $z_*|_{\nu=1.5} = 0.71748$, $c|_{\nu=1.5} = 0.46135$ and $z_*|_{\nu=1} = 0.8489$, $c|_{\nu=1} = 0.14450$.

The multi-valued dependency of ℓ on z_* in holographic models was previously observed in [60]. The authors established a new type of the phase transition associated with the swallow-tail like structure for S_{HEE} as the function of ℓ . For completeness, we present the dependence of the entanglement entropy in the EF on ℓ in figure 36 and figure 37. For the isotropic case the Van-der-Waals behavior of the entanglement entropy depends on a slab width takes place only for small black holes. For the anisotropic there is a small region of temperatures where S_{HEE} is multi-valued also for large black holes. The c-function undergoes a jump at ℓ_{crit} obtained from the self-intersection of (S_{HEE}, ℓ) diagram. The lengths $\ell_{left,L}$, $\ell_{right,L}$ in figure 28(d) correspond to the turning points.

Figure 32. *Top line:* contour plots for $\ell_\phi^{m_F(\nu)} \eta_{CD}(\nu)$ vs z_* (horizontal axis) and ν (vertical axis) for F=SF at $z_h = 1$ and $\varphi = 0$ (longitudinal case), and for different values of the chemical potential: (a) $\mu = 0$ and (b) $\mu = 0.5$. We see that two contour plots are practically identical. *Bottom line:* (c) and (d) are the same as (a) and (b), but for discrete values of ν ; comparison of c-functions for (c) $\mu = 0$ (darker magenta lines) and (d) $\mu = 0.5$ (magenta lines) for discrete values of ν . We see that two sets of lines almost coincide.

5.5 The c-function near the background phase transition

In this subsection we present dependences of c on ℓ in proximity of the background phase transitions. First we consider c-functions defined by (5.29) in the SF. In figure 38 we present this c-function for $\nu = 1$ and $\nu = 4.5$ (see figure 38(a)) for transversal (see figure 38(b)) and longitudinal orientations (see figure 38(c)). In the main panels plots for the temperature below the phase transitions are shown, while the insets show graphs at temperatures above phase transitions. We see that the c-functions below the phase transition line are negligibly small as compared with c-functions above this line. Note that for isotropic case c-functions have saddle points, but there are no saddle points in the anisotropic case for transversal configuration.

In figure 39 c-fuctions versus ℓ in the EF are shown. For $\nu = 1$ we calculate the c-function with η given by (5.5) (see figure 39(a)) and with (5.6) (see figure 39(b)). Here

Figure 33. ℓ vs z_* for $F=EF$ and $\varphi = \pi/2$ (transversal case). (a) Here $z_h = 1.2, 1.138, 1.108, 1$ and $\mu = 0$ (in vicinity of the HP phase transition). Here $z_{HP} = 1.138$ is displayed in red and $z_h = 1.12$, corresponding to the unstable background, in orange. (Inset: the lines with $z_{HP} = 1.138$ and $z_h = 1.2$ have local maxima and the line with $z_h = 1.108$ has an inflection point). (b) Here $\mu = 0.2$ and $z_h = 1.285$ (an unstable point, the corresponding curves are displayed in orange), $z_h = 1.1266$ (the point of the BB PT, depicted in red), $z_h = 1.1$ (the corresponding curve has an inflection point) and $z_h = 1$. (c) Here $\mu = 0.2$ and points with $z_h > 2.76$ correspond to the stable backgrounds. An unstable point with $z_h > 2.6$ is in orange. (d) Here $\mu = 0.5$ and all curves correspond to stable background.

one could see curves corresponding to various values of the temperature below and above the phase transition point $T = 0.3445$, $\mu = 0.05$: $T = 0.33, 0.335, 0.34$ (below) and $T = 0.345, 0.35, 0.355$ (above) (with increasing thickness with increasing temperature) and $\mu = 0.05$. We also present the plots of c-functions for $\nu = 4.5$ and transversal and longitudinal orientation and values of temperature below and above the phase transition (see figure 39(c) and figure 39(d)). Here we see the prints of the length phase transition similar to [58]. This phase transition is observed particularly clearly for transversal configurations and temperatures above the phase transition line.

5.6 Various c-functions as functions of ℓ

Let us summarize and comment on the results obtained in the previous section 5.3 and section 5.5. The summary of these results is presented also in table 2.

- We have found, that c-functions in the EF decrease with increasing ℓ for not overly

Figure 34. *Top line:* ℓ vs z_* for F=EF for $\varphi = 0$ (longitudinal case) at (a) $z_h = 1.2, 1.138, 1.108, 1$ and $\mu = 0$, at (b) $z_h = 2, 2.5, 3.0, 3.5$ and $\mu = 0.5$. *Bottom line:* ℓ vs z_* for F=SF for (c) $\varphi = \pi/2$ (transversal case) at $z_h = 0.3, 0.325, 0.35, 0.375$ and 0.4 . The right curve corresponds to surfaces touching a dynamic wall located at $z_{DW} = 0.354$ and for the rest, z_* is located near the horizons. The plot (d) is the same as (c) for $\varphi = 0$ (longitudinal case).

Figure 35. ℓ vs z_* for F=EF at $\mu = 0.2$, different orientations $\varphi = 0, \pi/6, \pi/4, \pi/3, \pi/2$, $\mu = 0.2$, and $z_h = 0.5, 1.1388, 2.80463$ (corresponding temperatures are $T = 0.418, 0.25, 0.235$), are presented in (a), (b) and (c), respectively.

Figure 36. The entanglement entropy S_{HEE} vs ℓ in the EF for $\nu = 1$ and various values of the temperature below and above the phase transition point ($T = 0.3445$, $\mu = 0.05$): $T = 0.335, 0.34$ (below) and $T = 0.345, 0.35$ (above) (thickness increases with increasing temperature) and $\mu = 0.05$. The inset shows a zoom of the yellow domains.

Figure 37. The entanglement entropy S_{HEE} vs ℓ in the EF for $\nu = 4.5$, transversal and longitudinal cases are presented in (a) and (b), respectively. Here values of temperature are below and above the phase transition point ($T = 0.2457$, $\mu = 0.2$). In all panels the insets show zooms of the yellow domains.

large ℓ (see the left part of the table 2).

- For the isotropic case in regions of (μ, T) -plane far away from the line of the phase transition the c-function decreases with increasing ℓ for all ℓ .
- For the anisotropic case the c-function decreases while increasing ℓ for all transversal and longitudinal orientations for small ℓ in all regions of (μ, T) -plane.
 - * For the transversal orientation different behaviors exhibits in different regions. In the region of small μ in the transversal case the c-function has a

Figure 38. The dependencies of c , defined by (5.29), on ℓ in proximity of the phase transitions (background shown in figure 41) in SF: (a) $\nu = 1$; (b) $\nu = 4.5$ and transversal orientation; (c) $\nu = 4.5$ and transversal orientation. *Insets:* $c(\ell)$ above the corresponding phase transitions.

local minimum at $\ell_{\min}$ after which it increases up to a local maximum at $\ell_{\max}$ and only then decreases. We get even more interesting behavior with increasing μ . The c -function as a function of ℓ becomes multivalued in some interval of ℓ , $\ell_{left,T} < \ell < \ell_{right,T}$. For $\ell > \ell_{right,T}$ crossing the line of the phase transition, the c -function has a jump.

- * For longitudinal orientation the c -function for small μ only decreases with increasing ℓ . But if we increase μ we see that the c -function also exhibits multivalued behavior for $\ell_{left,L} < \ell < \ell_{right,L}$ and for $\ell > \ell_{right,T}$ it has jumps as it crosses the phase transition line.

Figure 39. Dependence of c on ℓ in EF. *Top line:* $\nu = 1$ and the c -function calculated with η given (a) by (5.5) and (b) by (5.6). Various values of the temperature above and below the phase transition point $T = 0.3445$, $\mu = 0.05$; $T = 0.33, 0.335, 0.34$ (below) and $T = 0.345, 0.35, 0.355$ (above) (thickness increases with increasing temperature) and $\mu = 0.05$. *Bottom line:* (c) $\nu = 4.5$, transversal orientation and values of temperature above and below the phase transition; (d) $\nu = 4.5$, longitudinal orientation and values of temperature below and above the phase transition.

μ	EF			SF		
	ISO	T	L	ISO	T	L
0
	Figure 27	Figure 28(a)	Figure 28(c)	Figure 27	Figure 29(a)	Figure 29(c)
0.5
	Figure 27	Figure 28(b)	Figure 28(d)	Figure 27	Figure 29(b)	Figure 29(d)
PT
	Figure 39(b)	Figure 39(c)	Figure 39(d)	Figure 38(a)	Figure 38(b)	Figure 38(c)

Table 2. A summary of the c-function behavior considered in the present work in the EF and the SF based on definition (5.7). “ISO” denotes isotropic solution. “L” and “T” denote the longitudinal and transversal orientations of the slab in the anisotropic metric with $\nu = 4.5$. “PT” corresponds to configurations near the background phase transition.

- The c-functions calculated in the SF also exhibit non-monotonic behavior in some cases, but there is no multi-valued behavior here (see the right part of table 2).
 - In isotropic case in the SF c-function increases while increasing ℓ in UV up to $\ell = \ell_{\max}(T, \mu)$. This non-monotony behavior is related to dilaton behavior near $z = 0$.
 - There is different behavior in anisotropic case in the SF.
 - * The c-function exhibits monotonic behavior for the transversal orientation. It decreases with increasing ℓ for all ℓ .
 - * The c-function increases with increasing ℓ in UV up to $\ell = \ell_{\max, \text{SF}}(T, \mu)$ and when decreases for the longitudinal orientation.

There are two reasons why we do not need to worry about all these.

- First of all, as has been mentioned in the text, a non-monotonicity in the anisotropic case is not in contradiction with any of the existing c-theorems as all of them are based on Lorentz invariance.
- The saddle points $\ell_{\max, \text{SF}}(T, \mu)$ as well as the regions of multi-validity of the c-functions, $\ell_{\text{left}, L} < \ell < \ell_{\text{right}, L}$ or $\ell_{\text{left}, T} < \ell < \ell_{\text{right}, T}$ are located in the regions with large enough values of ℓ , where the definition of the c-function using the UV asymptotics of the solutions can be violated.

5.7 Entanglement entropy phase transition

Let us remind that the criterion for the confinement/deconfinement phase transition in QCD is the behavior of the potential between quarks or the behavior of the temporal Wilson loops. The Wilson loops can also be computed in HQCD. It turns out that location of the confinement/deconfinement line in the (μ, T) plane can be close to the background phase transition [60, 70, 85], but not necessary coincide with it. For special models the phase transition of the HEE can be used as an indication of the HQCD phase transition [47, 62].

The position of the background phase transition depends on the particular holographic model, see [120, 121] and refs therein. It can be also located in the right-bottom part of the (μ, T) plane starting from a point $(\mu_{\text{cr}}, T_{\text{cr}})$ and going down with increasing μ till the zero temperature. It also can be located in the left part of the plane, starting from T_0 at $\mu = 0$ and going down with increasing μ till the point $(\mu_{\text{cr}}, T_{\text{cr}})$.

To find the location of the HEE phase transition on the (μ, T) plane one has to find the location of points where the free energy F_A corresponding to the reduced ρ_A -matrix has a multi-valued behavior. It turns out that the effective free energy corresponding to the entangled region A

$$dF_A = -S_A dT, \quad (5.30)$$

has a behavior similar to the behavior of the free thermal energy. We can define the density of the entanglement effective free energy as

$$df_A = -\eta_A dT. \quad (5.31)$$

The density of the effective free energy as a function of temperature also has the swallow-tail behavior.

This is due to the fact that, on the one hand, slabs that extend infinitely in two directions and being thick enough can accumulate enough degrees of freedom to repeat the characteristic form of thermal entropy, and on the other hand, the three-valued behavior of the function $z_h = z_h(T)$ is an intrinsic feature of the background and inevitably leads to swallow-tail behavior for both ordinary and effective free energy even for thin slabs. However the slab does not include all degrees of freedom, so the thermal entropy and the effective entanglement entropy do not coincide exactly, as well as the lines of the thermal and entanglement phase transitions.

Let us summarize what we found studying behavior of the effective entanglement free energy in different schemes of regularizations and frames.

- We checked the behavior of the effective entanglement free energy density as a function of temperature in the SF using different renormalization schemes, see figure 40(a). We see that there is no essential dependence on used regularization schemes in the SF. The transition points for the entanglement free energy in both regularization schemes almost coincide, also these points are very close to the transition point obtained from the thermal free energy.
- We also compared behavior of the effective entanglement free energy and thermal free energy densities as the functions of temperature in the EF, see figure 40(b). Here we

Figure 40. (a) Swallow-tailed temperature dependences of the thermal free energy density divided by factor 10 (solid blue line) and the entanglement effective free energy density in the SF (magenta dashed line based on η and magenta solid line based on η_{CD}). (b) The same in the EF (the entanglement free energy density with η_{CD} in the EF is shown by gray solid line). (c) The entanglement free energy density based on η in the EF. The inset in (c) shows that the form of temperature dependence of the entanglement free energy density presented in the main panel is in fact swallow-tailed. All these plots are done for the anisotropic case with $\nu = 4.5$ and $\mu = 0.2$.

see that the temperatures of the phase transition points almost coincide, while the values of the free energy densities at these points do not coincide.

- Moreover, we analysed behavior of the effective entanglement free energy defined with the minimal renormalization scheme in EF and found the swallow-tail looking more flattened, see figure 40(c).

Figure 41. Phase diagrams for the thermal entropy (solid lines) and entanglement entropy densities (dashed lines) for various anisotropy parameters ν , $\nu = 1$ (green line), $\nu = 3$ (khaki lines) and $\nu = 4.5$ (blue lines).

- We checked that the locations of the critical points extracted from the HEE density and the HEE itself almost coincide (numerical calculations were done for different ℓ and orientations).

In figure 41 phase diagrams for the thermal entropy (solid lines) and entanglement entropy densities (dashed lines) for various anisotropy parameters ν , $\nu = 1$ (green lines), $\nu = 3$ (khaki lines) and $\nu = 4.5$ (blue lines) obtained by numerical calculations based on (3.36), (3.37) and (5.5) are presented.

Generally speaking, the location of the phase transition for the HEE is rather close to the background phase transition, see figure 41. Note that in contrast to Wilson loops behavior in anisotropic background [70], there is no visible orientation dependence of HEE phase transition line in anisotropic cases.

6 Conclusion

We have considered the most general anisotropic holographic model and found the expression for the HEE in terms of the Euler angles defined by the orientation of the slab-shape area in respect to HIC axes. In a particular case, we have considered the HEE for the model invariant in the transversal directions with the unique anisotropy scaling factor supported by the Einstein-Dilaton-two-Maxwell action [70]. The choice of the model [69] is motivated by agreement of the energy dependence of the produced entropy with the experimental data for the energy dependence of the total multiplicity of particles produced in HIC [71] in the anisotropic metric. This model describes multiplicity and quark confinement (for heavy quarks), predicts crossover transition line between confinement/deconfinement phases, anisotropy in hadron spectrum (for a short time after collisions) [85, 121] and phase transition for the spatial Wilson loops [20].

We have calculated the HEE and its density in the holographic anisotropic model [70]. We have shown that the HEE and its density have significant fluctuations near the BB phase transition line in (μ, T) -plane for all values of the anisotropy parameter. The lines of thermal and entanglement entropies phase transition in (μ, T) -plane are different in the anisotropic cases but do not depend on the orientation of the entangling area. Note that for isotropic case $\nu = 1$ the differences between these lines in the phase diagram plane are not visible. We have discussed an application of enormous increasing of the HEE as an indicator of the background phase transition.

We have studied the dependence of the c-function on the thickness ℓ of the entanglement slab. We have found saddle points of c-function as a function of ℓ as well as its multivalued behaviour. The obtained results are schematically presented in table 2. The c-functions in the EF decrease while increasing ℓ for not too large ℓ . Moreover, in the isotropic case in regions of (μ, T) -plane remote from the line of the phase transition, the c-function decreases while increasing ℓ for all ℓ . In the anisotropic case, the c-function decreases with increasing ℓ for all transversal and longitudinal orientations for small ℓ in all regions of (μ, T) -plane. The c-functions calculated in the SF in some cases exhibit non-monotonic behavior, but there is no multivalued behavior here. In isotropic case in the SF c-function increases with increasing ℓ in UV up to $\ell = \ell_{\max}(T, \mu)$. This non-monotonic behavior is related with dilaton behavior near $z = 0$. There is different behavior in anisotropic case in the SF. It has been mentioned in the text, a non-monotonicity in an anisotropic case is not in contradiction with any of the existing c-theorems because all of them are based on Lorentz invariance. The saddle points as well as the regions of multi-validity of the c-functions are located in the regions where the definition of the c-function using the UV asymptotics of the solutions can be violated.

As to further development, we suppose to study modifications of the model [70] to include the light quarks following [122], incorporate the chiral phase transition [123], and also perform the numerical calculations in full anisotropic case to incorporate the magnetic field, as has been done in [124].

We hope that the results presented in this paper, their interpretations and their further possible adjustment to the phenomenology data can be of interest for experiments at the future facilities of FAIR [125], NICA [126], for RHIC's BES II program and CERN, III run.

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