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Classifying 5d SCFTs via 6d SCFTs: rank one

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ABSTRACT: Following a recent proposal, we delineate a general procedure to classify 5d SCFTs via compactifications of 6d SCFTs on a circle (possibly with a twist by a discrete global symmetry). The path from 6d SCFTs to 5d SCFTs can be divided into two steps. The first step involves computing the Coulomb branch data of the 5d KK theory obtained by compactifying a 6d SCFT on a circle of finite radius. The second step involves computing the limit of the KK theory when the inverse radius along with some other mass parameters is sent to infinity. Under this RG flow, the KK theory reduces to a 5d SCFT. We illustrate these ideas in the case of untwisted compactifications of rank one 6d SCFTs that can be constructed in F-theory without frozen singularities. The data of the corresponding KK theory can be packaged in the geometry of a Calabi-Yau threefold that we explicitly compute for every case. The RG flows correspond to flopping a collection of curves in the threefold and we formulate a concrete set of criteria which can be used to determine which collection of curves can induce the relevant RG flows, and, in principle, to determine the Calabi-Yau geometries describing the endpoints of these flows. We also comment on how to generalize our results to arbitrary rank.

KEYWORDS: F-Theory, Field Theories in Higher Dimensions, M-Theory, Supersymmetric Gauge Theory

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1 Introduction

There is a long standing dream that it will be possible to obtain all lower dimensional quantum field theories via compactifications of higher dimensional quantum field theories. The supersymmetric version of this dream states that it will be possible to obtain all supersymmetric quantum field theories in spacetime dimension $d \leq 5$ by compactifying $6d$ SCFTs since it is believed that all UV complete QFTs can be obtained by deforming CFTs and $d = 6$ is the maximum dimension permitting the existence of an SCFT¹ [2]. If we assume that the dream is correct, then we can hope to obtain all $5d$ SCFTs by compactifying $6d$ SCFTs on a circle.²

A way to make this hope concrete was recently proposed in [3] which classified all the possible rank³ two $5d$ SCFTs that can be obtained by compactifying M-theory on a smooth Calabi-Yau threefold. It was found that there are a huge number of such $5d$ SCFTs, and the Coulomb branch of each of these is described by the data of the corresponding Calabi-Yau threefold used for compactifying M-theory. However, it was also noticed that all of these Calabi-Yau threefolds can be obtained by blowing down some curves inside a small number of “parent” Calabi-Yau threefolds. The parent Calabi-Yau threefolds also give rise to rank two $5d$ theories but these $5d$ theories are not $5d$ SCFTs. Rather they can be thought of as circle compactified $6d$ SCFTs viewed as $5d$ theories with KK modes. Thus these parent theories were dubbed as $5d$ KK theories in [3]. Now, the process of blowing down curves physically corresponds to performing RG flows where some BPS particles are integrated out of the theory. Based on this observation, the following conjecture was proposed in [3], which we reiterate:

Every rank n $5d$ SCFT can be obtained by a rank preserving RG flow starting from a $6d$ SCFT compactified on a circle with/without a discrete twist.

The conjecture can also be justified by the studies of $5d$ gauge theories where it has been found that for low number of flavors, the gauge theory has a $5d$ UV completion; but, if we keep adding flavors then we reach a gauge theory which has a $6d$ UV completion rather than a $5d$ one [4, 5]. Similarly, we expect that if we keep adding BPS particles consistently to a $5d$ SCFT (while keeping the number of BPS strings constant), then at some point we reach a $5d$ KK theory of the same rank.

¹It is known that $6d$ SCFTs do not admit relevant deformations [1].

²Here we should also include twists by discrete global symmetry transformations as one traverses the compactification circle. The twisted compactifications can lead to $5d$ SCFTs which cannot be obtained from untwisted compactifications of $6d$ SCFTs.

³We say that a $5d$ theory has rank n if its Coulomb branch is of dimension n . The number of BPS strings in the $5d$ spectrum is also n .

In this paper, we start a systematic study of circle compactifications of $6d$ SCFTs, which according to the above conjecture can be used to classify $5d$ SCFTs. We will focus our attention on the untwisted compactifications of rank⁴ one $6d$ SCFTs which can be constructed in the unfrozen phase⁵ [9, 10] of F-theory. In particular, we will associate a smooth Calabi-Yau threefold $X_{\mathfrak{T}}$ to each such rank one $6d$ SCFT $\mathfrak{T}$. We will determine $X_{\mathfrak{T}}$ by resolving the elliptically fibered singular Calabi-Yau threefold $Y_{\mathfrak{T}}$ appearing in F-theory construction for $\mathfrak{T}$.

Compactifying M-theory on $X_{\mathfrak{T}}$ leads to the $5d$ KK theory $\mathfrak{T}_{\text{KK}}$ obtained by compactifying $\mathfrak{T}$ on S^1 without any twist. The Coulomb branch prepotential for $\mathfrak{T}_{\text{KK}}$ can be recovered from the data of intersection numbers of holomorphic cycles in $X_{\mathfrak{T}}$. The spectrum of BPS particles relevant for rank preserving RG flows to $5d$ SCFTs can be identified with rational curves of self-intersection -1 in $X_{\mathfrak{T}}$. The KK mode of $\mathfrak{T}_{\text{KK}}$ can be identified with the elliptic fiber in $X_{\mathfrak{T}}$ whose volume is identified with the inverse radius of compactification R^{-1} . We note that some special cases of our results were already obtained by [12] who studied the Calabi-Yau threefolds corresponding to very special rank one $6d$ SCFTs that are completely Higgsed in the sense that they cannot be Higgsed to obtain some other $6d$ theory.

Let us close this introduction with a justification for capturing the data of $5d$ KK theories in terms of Calabi-Yau geometries rather than proceeding field theoretically. As emphasized in [13], various important physical processes (e.g. integrating out BPS particles or phase transitions) require us to know the precise functional dependence of the masses of BPS particles in terms of Coulomb branch moduli and mass parameters of the $5d$ theory. This dependence is only known field theoretically for particles that can be seen perturbatively in the $6d$ SCFT on its tensor branch. But the reduction on a circle generates new non-perturbative particles whose masses cannot be determined using present field theoretic methods. The Calabi-Yau geometry makes all these particles manifest and the calculation of their masses a straightforward task.

This paper is organized as follows. In section 2, we review aspects of Coulomb branches in $5d$ supersymmetric theories and review how a Calabi-Yau threefold can describe a $5d$ Coulomb branch. In section 3, we review how a $6d$ SCFT compactified on a circle can be viewed as a $5d$ KK theory, and describe the general structure expected of Calabi-Yau threefolds describing the Coulomb branch of $5d$ KK theories. We also introduce our graphical notation which packages the relevant data of a Calabi-Yau in terms of a graph. In section 4, we compile the main results of this paper, which is the association of a Calabi-Yau threefold to every⁶ rank one $6d$ SCFT. In section 5, we formulate criteria which, in

⁴Here we are referring to the $6d$ rank which is different from the $5d$ rank used above. The $6d$ rank counts the number of tensor multiplets on the tensor branch of the $6d$ SCFT.

⁵Frozen singularities in F-theory can construct $6d$ SCFTs which do not admit an F-theory construction without frozen singularities [6, 7]. See also [8]. The incompleteness of the classification of [9, 10] can also be noticed by comparing their classification with the gauge-theoretic classification of [11].

⁶Actually, due to technical reasons, our methods do not allow us to associate a Calabi Yau threefold to a particular rank one $6d$ SCFT. The tensor branch of this theory is described by $\text{SO}(13)$ with a half-hyper in spinor representation and 7 hypers in vector. The F-theory construction of this SCFT involves a non-split I_3^* fiber over a -2 curve.

principle, allow one to determine the Calabi-Yaus describing the $5d$ SCFTs arising the end points of RG flows that start from KK theories. In section 6, we comment on how to generalize our results to higher rank SCFTs. In appendix A, we collect some mathematical facts and notions that we use throughout the paper. In appendix B, we provide sample computations of the Calabi-Yau threefolds for some hand-picked KK theories that illustrate some key features of our results.

2 Coulomb branches in five dimensions: a review

2.1 Field theoretic aspects

The minimal supersymmetry algebra in $5d$ has eight supercharges, denoted as $\mathcal{N} = 1$. The $\mathcal{N} = 1$ vector multiplet contains a real scalar, which parametrizes a Coulomb branch of vacua $\mathcal{C}$ on which the IR physics is described by an $\mathcal{N} = 1$ abelian gauge theory. The kinetic term for the scalars $\phi_{5d,i}$ in the low energy theory provides a natural metric on $\mathcal{C}$.

A $5d$ $\mathcal{N} = 1$ SCFT has a space of relevant deformations $\mathcal{M}$ parametrized by mass parameters m_α . Each point p in $\mathcal{M}$ corresponds to a $5d$ $\mathcal{N} = 1$ QFT and we can associate its Coulomb branch to p leading to a fibration $\mathcal{P}$ of $\mathcal{C}$ over $\mathcal{M}$. The spectrum over each point in $\mathcal{P}$ contains massive BPS particles and strings. The central charge for BPS particles can be written as

$$Z = n_e^i a_i + f^\alpha m_\alpha \quad (2.1)$$

where n_e^i denotes the electric charge under a low energy gauge group $U(1)_i$, $a_i := \phi_{5d,i}$, and f_α denotes the charge under a flavor $U(1)_\alpha$ associated to m_α . As a gauge $U(1)$ can be redefined by a linear combination flavor $U(1)$ s, we have the freedom of shifting a_i by a linear combination of m_α . The central charge for BPS strings can be written as

$$\hat{Z} = n_{m,i} a_D^i \quad (2.2)$$

where $n_{m,i}$ are the magnetic charges under $U(1)_i$ and $a_D^i = \frac{\partial \mathcal{F}}{\partial a_i}$ where $\mathcal{F}$ is the prepotential for the low energy abelian gauge theory.

$\mathcal{F}$ is in general a cubic polynomial⁷ [14, 15] in a_i and m_α . As we have discussed above, its first derivatives w.r.t. a_i control the tensions of the BPS strings. Its second derivatives w.r.t. a_i determine the kinetic terms for $\phi_{5d,i}$ and hence control the metric on $\mathcal{C}$. Its third derivatives w.r.t. a_i and m_α determine Chern-Simons terms.

The Chern-Simons levels of the low-energy abelian gauge theory can jump across some walls in $\mathcal{P}$ leading to phase transitions [14, 16]. The locations of these walls are parametrized by some particles becoming massless.

2.2 Geometric aspects

A five dimensional $\mathcal{N} = 1$ field theory can be realized as the low energy effective description of M-theory compactified on a local Calabi-Yau threefold X' [16]. The hallmark of

⁷ $\mathcal{F}$ can contain some absolute values which means that it is not smooth. More precisely, its third derivatives are not continuous.

this correspondence is the identification between the parameter space $\mathcal{P}$ of the 5d theory and the Kähler moduli space of X' . The BPS particles arise from M2 branes wrapping compact holomorphic curves and the BPS strings arise from M5 branes wrapping compact holomorphic surfaces in X' . Their masses and tensions are proportional to the volumes of the corresponding curves and surfaces. To reach the conformal point, we want to be able to shrink all compact curves and surfaces inside X' at a finite distance in the moduli space. Such a threefold is called shrinkable and the Kähler moduli space takes the form of a cone in such a case.

We can determine the geometry and intersection structure of the holomorphic 4-cycles in terms of a basis of (compact and non-compact) divisors S_i . To do this, we begin by expanding a Kähler class J in terms of this basis:

$$J = \phi_i S_i, \quad \phi_i \in \mathbb{R}_{\geq 0}. \quad (2.3)$$

In order to ensure positive volumes of $\text{vol}(C_p) = \frac{1}{p!}(J^p \cdot C_p)_{X'}$ of holomorphic p -cycles C_p , J is required to be effective:

$$(J \cdot C)_{X'} = \text{vol}(C) \geq 0 \quad (2.4)$$

for all holomorphic curves C in X' ; by a theorem when the above partial inequality is strict, this property holds for all C_p . Assuming transverse intersections $S_i \cap S_j$ which locally satisfy the Calabi-Yau condition 3.3, the intersection structure of the divisors S_i is determined in its entirety by the triple intersection numbers⁸

$$k_{ijk} = (S_i \cdot S_j \cdot S_k)_{X'} \quad (2.5)$$

which appear naturally in the expression for the relative volume of X' , or equivalently the geometric realization of the prepotential $\mathcal{F}$:

$$\text{vol}(X') = \frac{1}{3!}(J^3)_{X'} = \frac{1}{3!}k_{ijk}\phi_i\phi_j\phi_k = \mathcal{F}. \quad (2.6)$$

The following formula is useful to keep in mind when computing triple intersections:

$$(S_i \cdot S_j \cdot S_k)_{X'} = (S_i|_{S_k} \cdot S_j|_{S_k})_{S_k}, \quad (2.7)$$

where $S_i|_{S_j}$ indicates the restriction of the complex surface S_i to the surface S_j , and the above formula holds for any permutation of the indices i, j, k .

3 5d KK theories and associated Calabi-Yau geometries

We start out in section 3.1 by sketching what one expects the general structure of a smooth Calabi-Yau associated to a 5d KK theory to look like. We then introduce in section 3.2 a notation which packages all the relevant information about the Calabi-Yau into a graph. Section 3.3 describes the tools to compute the resolved Calabi-Yau $\tilde{X}$ starting from a Weierstrass model for an elliptically fibered singular Calabi-Yau X defining a 6d SCFT.

⁸Mathematically, the triple intersection of three non-compact divisors is not well-defined. Field theoretically, such triple intersections appear as monomials in $\mathcal{F}$ depending only on mass parameters, and hence can be eliminated.

3.1 $6d \rightarrow 5d$

Every $6d$ SCFT $\mathfrak{T}$ is believed to admit a tensor branch of vacua $\mathcal{T}$ on which the low energy effective theory is a non-abelian gauge theory interacting with some tensor multiplets. The scalars $\phi_{6d,a}$ in tensor multiplets parametrize $\mathcal{T}$. When we compactify $\mathfrak{T}$ on a circle of finite radius R , we can view it as a $5d$ KK theory. The KK theory admits a Coulomb branch of vacua $\mathcal{C}$ parametrized by scalars $\phi_{5d,i}$ which descend from $\phi_{6d,a}$ and from holonomies of the $6d$ gauge fields around the circle. Similarly, the mass parameters m_α for the KK theory descend from the holonomies of the continuous flavor symmetry groups of the $6d$ theory. Since our setup is on a circle, we have an additional mass parameter $m_B = R^{-1}$. The spectrum of BPS particles for a $5d$ KK theory arranges itself into towers such that the masses of two consecutive particles in a tower differ by m_B . This mass difference can be attributed to the central charge for the $U(1)_{KK}$ symmetry corresponding to translations along the compactification circle.

As we send R to zero, the KK towers disappear and we land on a $5d$ SCFT. There are multiple ways to send R to zero depending on how we tune $\langle \phi_{5d,i} \rangle$ and m_α in the process. Different ways of taking the $R \rightarrow 0$ limit give rise to different $5d$ SCFTs starting from the same $5d$ KK theory.

A $6d$ SCFT can be constructed by compactifying F-theory on a (generically singular) non-compact elliptically fibered Calabi-Yau threefold with a smooth base B that is required to satisfy some extra conditions. In particular, the compact holomorphic curves in B must be rational and their intersection pairing matrix must be negative definite. The latter condition follows from the fact that D3 branes wrapping curves in B give rise to BPS strings whose tensions are controlled by the volumes of the curves, and so the conformal point corresponding to tensionless strings exists only if all the curves in B can be shrunk to zero volume simultaneously; this physical requirement translates into a mathematical condition on the intersection pairing of the curves [17].

The Coulomb branch of KK theories corresponding to such $6d$ SCFTs on a circle of radius R can be described in terms of M-theory compactified on a smooth, resolved version of the same elliptically fibered Calabi-Yau threefold such that the size of the elliptic fiber is⁹ m_B [18–20]. The towers of KK particles descend from M2 branes wrapping a holomorphic curve C along with a multiple of the elliptic curve class.

Over a generic point on a compact rational curve C in the base B , the elliptic fiber degenerates into a collection of rational curves F_i intersecting with each other in some pattern. Each F_i in the collection is then fibered over C giving rise to a ruled surface¹⁰ $S_i \rightarrow C$ over a smooth base curve C . The ruled surfaces S_i then intersect according to the intersection pattern¹¹ of F_i forming the degenerate elliptic fiber, leading to a collection of

⁹Here the subscript B stands for the base of the elliptic fibration because m_B is the mass parameter associated to the base.

¹⁰We provide basic mathematical background about Hirzebruch surfaces and more general ruled surfaces in appendix A.2.

¹¹We actually find that the intersection pattern of F_i is only part of the full intersection structure of S_i . There can be other intersections of S_i which do not change the intersection pattern of F_i . For instance, see discussion between equations (4.14) and (4.15).

Figure 1. From left to right: a generic point in the base B carries an elliptic fiber. It degenerates to an I_3 fiber at the location of a holomorphic curve in B . The fiber degenerates further to an I_7 fiber at a point of intersection with another curve in B carrying an I_4 fiber. The split I_4 fiber is comprised of four rational curves F_i with index i taking values from 0 to 3. F_i moving over the base $\mathbb{P}^1$ gives rise to a Hirzebruch surface S_i with some blow-ups, three out of which come from collision with the I_3 fiber on the adjacent curve adjoining three extra rational curves to S_i .

surfaces $S_C = \cup_i S_i$ in the threefold that are associated to C . At the points of intersection of C with other curves, there is a collision of singular elliptic fibers leading to the presence of extra rational curves over the points of intersections. These extra rational curves show up as exceptional curves inside S_C . See figure 1. If two compact curves C and D in B intersect each other, then the collections S_C and S_D of surfaces over C and D are glued to each other such that components of elliptic fiber in S_C are glued to components of elliptic fiber in S_D .

3.2 Condensing the data of a Calabi-Yau threefold

We condense the data of a local Calabi-Yau threefold into a graph whose nodes represent compact surfaces S_i which are p_i -point blowups of ruled surfaces of degree n_i over a curve of genus g_i , denoted

$$S_i = \mathbb{F}_{n_i, g_i}^{p_i}. \quad (3.1)$$

An edge between S_i and S_j represents a transverse intersection between S_i and S_j . The locus of such an intersection corresponds to a curve C_i in S_i and some curve C_j in S_j . Thus, the intersection can also be viewed as an identification of a curve $C_i^{ij} \subset S_i$ with a curve $C_j^{ij} \subset S_j$. When a pair of curves in two different surfaces are identified with one another in this manner, we say that the surfaces S_i and S_j are *glued* to each other along the curve C_{ij} .

There are some conditions that two curves C_i and C_j have to satisfy for the gluing to be allowed in this setting. Clearly, the genus of C_i must be the same as the genus of C_j , i.e.

$$g(C_i^{ij}) = g(C_j^{ij}) = g. \quad (3.2)$$

Moreover, for such a gluing to be consistent with Calabi-Yau condition, it must be the case that

$$(C_i^{ij})_{S_i}^2 + (C_j^{ij})_{S_j}^2 = 2g - 2. \quad (3.3)$$

When $g = 0$, the nodes represent Hirzebruch surfaces $\mathbb{F}_{n_i}^{p_i}$. We graphically represent an intersection between two such surfaces as

$$(3.4)$$

which shows the label i of each surface S_i , the degree n_i of each surface, the number of blow-ups p_i in each surface, and the self-intersections of the gluing curves for each edge with their names C_i^{ij} and C_j^{ij} adjacent to the corresponding self-intersection numbers.

Sometimes, $2g_i$ blow-ups can be paired up for self-gluing of the surface S_i to itself. The resulting self-glued surface is then a degenerate limit of a ruled surface over a smooth curve of genus g . More precisely, if before the self gluing $S_i = \mathbb{F}_{n_i}^{p_i}$, then after the self-gluing it has transformed to $S'_i = \mathbb{F}_{n_i+2g_i, g_i}^{p_i-2g_i}$ which represents a ruled surface of genus g_i , degree $n_i + 2g_i$ with $p_i - 2g_i$ blowups. Keeping this in mind, we denote a general node as

$$(3.5)$$

Now, say S_i and S_j are glued along a number n_{ij} of curves which can be represented in S_i as $C_{i,\alpha}^{ij}$ and in S_j as $C_{j,\beta}^{ij}$ which are glued to each other if $\alpha = \beta$, but not otherwise. Also, say that $(C_{i,\alpha}^{ij})^2 = (C_i^{ij})^2$ and $(C_{j,\beta}^{ij})^2 = (C_j^{ij})^2$, i.e. the self-intersections of $C_{i,\alpha}^{ij}$ and $C_{j,\beta}^{ij}$ are independent of α and β . Then, we denote this configuration by

$$(3.6)$$

where we display the self-intersection of the sum of all the gluing curve between the two surfaces. This is the number that enters into the prepotential and not the individual self-intersections. In general, it can happen that the self-intersections of $C_{i,\alpha}^{ij}$ are not independent of α . But, this situation will not arise in this paper, so we do not make a notation for this more general case.

Finally, three distinct surfaces S_i , S_j and S_k can intersect in n_{ijk} number of points inside the threefold. We denote this by putting a number in the corresponding face as

shown below

The triple intersection $S_i \cdot S_j \cdot S_k = n_{ijk}$ is invariant under permutation of i, j, k , and can be computed inside any one of the surfaces by using the identity

$$S_i \cdot S_j \cdot S_k = \left[\left(\sum_{\alpha} C_{j,\alpha}^{ij} \right) \cdot \left(\sum_{\mu} C_{j,\mu}^{jk} \right) \right]_{S_j}. \quad (3.8)$$

We also have a special non-compact surface for the KK theories, namely the base B of the elliptic fibration. For each curve C in B , B joins S_C only along a single node, which we will refer to as the affine node and label it as S_0 . There is a curve C_0 in S_0 along which S_0 is glued to B . We display this curve separately in our figures but we do not associate an edge to this curve, as is evident in the following graph:

3.3 Singular elliptically fibered Calabi-Yau threefolds

We now discuss an explicit construction of a singular elliptically fibered Calabi-Yau threefold in detail, and the methods by which one can extract the data described in the previous subsection from such a construction. Let $X_0 \rightarrow B$ be an elliptically fibered Calabi-Yau threefold over a complex surface B . This elliptic fibration has an explicit realization as a hypersurface

$$W_0 = y^2 z + a_1 x y z + a_3 y z^2 - (x^3 + a_2 x^2 z + a_4 x z^2 + a_6 z^3) = 0 \quad (3.10)$$

of a rank 2 projective bundle $Y_0 \rightarrow B$ whose $\mathbb{P}^2$ fibers are parametrized by homogeneous coordinates $[x : y : z]$. Here we are using Tate form of the Weierstrass equation. The parameters a_n are sections of $\mathcal{K}_B^{-n}$, where $\mathcal{K}_B^{-1} \rightarrow B$ is the anti-canonical bundle of the base B . For a rank one 6d SCFT, B can be described as the total space of a local $\mathbb{P}^1$ with self intersection $-k$ in B ; we use the symbol C to denote this rational curve. Given an explicit realization of C as the zero locus $e_0 = 0$, the type of Kodaira singular fiber is

determined by specifying the order of vanishing q_n of the parameters a_n along C which can be read from table 2 of [21]; this procedure leads to an equation of the following form:

$$W_0 = y^2 z + a_{1,q_1} e_0^{q_1} x y z + a_{3,q_3} e_0^{q_3} y z^2 - (x^3 + a_{2,q_2} e_0^{q_2} x^2 z + a_{4,q_4} e_0^{q_4} x z^2 + a_{6,q_6} e_0^{q_6} z^3) = 0. \quad (3.11)$$

The singular locus of X_0 is

$$W_0 = \partial_i W_0 = 0, \quad (3.12)$$

where ∂_i denote partial derivatives with respect to the complex coordinates x, y, e_0 . In order to obtain a smooth elliptic fibration, we identify a sequence of blowups which do not change the canonical class of the threefold,

$$X_r \xrightarrow{f_r} X_{r-1} \xrightarrow{f_{r-1}} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0, \quad (3.13)$$

such that for some choice of positive $r < \infty$ the elliptic fibration $X_r \rightarrow B$ is smooth.

We now describe some general facts about blowups in more detail. Suppose the equation $W(y_i) = 0$ describes a singular projective variety $X \subset Y$ realized as a hypersurface of the ambient projective space Y with homogeneous coordinates y_i . We describe blowups in more detail. Let $W(y_i) = 0$ denote a singular projective variety $X \subset Y$ realized as a hypersurface of the ambient projective space Y with homogeneous coordinates y_i . A blowup can be described in terms of its center $(g_1, g_2, \dots)$, where $g_i(y_j)$ is a homogeneous polynomial in y_j , and a local parameter e whose zero locus $e = 0$ is the exceptional divisor E of the blowup. In practice, we use adopt the following succinct notation

$$(g_1, g_2, \dots | e), \quad (3.14)$$

which means we make the substitution

$$(g_1, g_2, \dots) \rightarrow (e g_1, e g_2, \dots) \quad (3.15)$$

and introduce a new ambient projective space $Y' \rightarrow Y$ in which the locus $g_1 = g_2 = \cdots = 0$ of the original projective bundle Y has been replaced by a projective space $[g_1 : g_2 : \cdots]$ located at $e = 0$ in Y' . This procedure defines a map of hypersurfaces $X' \rightarrow X$ where $X' \subset Y'$ is said to be the blowup of X along the center $g_1 = g_2 = \cdots = W = 0$ (note that center must intersect X). Checking smoothness of X' is equivalent to checking that the equations (3.12) have no solutions.

Note that the singular elliptic fibers F are the elliptic fibers located over the rational curve $C \subset B$ described by $e_0 = 0$. If one restricts to blowing up sub-loci of the hyperplane $e_0 = 0$ in Y_0 , the sequence of blowups $f : X_r \rightarrow X_0$ resolving X_0 can be viewed as blowups of singular points on F . The elliptic fibers F of the resolved threefold can therefore be viewed as a collection of smooth rational curves F_i with multiplicities m_i :

$$F = \sum_{i=0}^r m_i F_i. \quad (3.16)$$

The components F_i are the fibral divisors of the elliptic fiber F ; when the Kodaira singular fiber type is associated to a Lie algebra $\mathfrak{g}$, the F_i intersect in the pattern of the affine Dynkin diagram associated to $\mathfrak{g}$.

As the irreducible components F_i move over C , they sweep out complex surfaces in the threefold X_r . Thus the fibral divisors F_i define a natural basis of divisors S_i in X_r which have the structure of $\mathbb{P}^1$ bundles over C . Once we have explicitly computed a resolution $X_r \rightarrow X_0$ of the singular threefold X_0 , we can describe X_r as local neighborhood of a collection of transversely intersecting surfaces $\cup S_i$ by computing the triple intersection numbers $k_{ijk} = (S_i \cdot S_j \cdot S_k)_{X_r}$ and the degrees n_i of S_i . We explain how to perform these computations in the following subsections and illustrate them with detailed examples in appendix B.

Computing degrees. First, we explain how to compute the degrees n_i of the divisors S_i . For now, let us assume that the Kodaira fiber type is split. Using the fact that the class of the elliptic fiber in X_r can be decomposed into a sum over rational curves as in (3.16), we view each divisor S_i as a $\mathbb{P}^1$ bundle fibered over C , namely

$$S_i = \mathbb{P}_C[\mathcal{O} \oplus \mathcal{L}_i], \quad (3.17)$$

where $c_1(\mathcal{L}_i) = a_i K_B + b_i C$ for some integers a_i, b_i . The degree of the line bundle S_i can then be computed as

$$n_i = (C \cdot (a_i K_B + b_i C))_B = a_i(k-2) - b_i k. \quad (3.18)$$

In order to determine the number of blowups, we simply use the identity

$$k_{iii} = (S_i^3)_{X_r} = (K_{S_i}^2)_{S_i} = 8 - p_i \quad (3.19)$$

to read off p_i .

When the Kodaira fiber type is non-split, a given component F_i of the resolved elliptic fiber may be geometrically reducible, consisting of several components

$$F_i = \sum_j F_{i,j} \quad (3.20)$$

and moreover there can be non-trivial monodromies permuting the irreducible components $F_{i,j=1,\dots,s_i}$ of the resolved elliptic fiber. For notational clarity, assume in the forthcoming discussion that the elliptic fibration is defined with respect to a curve $C' \subset B$ (as opposed to the usual symbol C). In the case of non-split Kodaira fibers over C' , the relation between the (identical) line bundles $S'_i = \mathbb{P}[\mathcal{O} \oplus \mathcal{L}_i]_{j=1,\dots,s_i} \rightarrow C'$ and the divisor S_i may be such that $S_i \rightarrow S'_i$ is a ramified s_i -cover, ramification locus may consist of a collection of fibers.

Suppose the ruled surface $\mathbb{P}[\mathcal{O} \oplus \mathcal{L}_i]_j \rightarrow C'$ has degree n'_i . Then it is possible to view $S_i \rightarrow C'$ as a projective bundle over a different curve C , such that $C \rightarrow C'$ is a ramified s_i -cover (see [22] for a more precise discussion of this point in the context of the Stein factorization and its relevance to the non-split IV* model). The genus of the curve C in general depends on the genus of the curve C' along with the ramification locus of the

s_i -cover $S_i \rightarrow S'_i$. Suppose that the ramification locus of the s_i -cover $S_i \rightarrow S'_i$ consists of $2b$ fibers. Then, we have:

$$g_i(C) = \frac{s_i}{2} \left(2g_i(C') + b - 2 + \frac{2}{s_i} \right). \quad (3.21)$$

On the other hand, the degree n_i of the surface S_i depends on the degree $n'_i = -(C')^2_{S'_i}$ of the surface S'_i through the relation

$$(C^2)_{S_i} = s_i(C'^2)_{S'_i} = -s_i n'_i, \quad (3.22)$$

which is a consequence of the action of the pushforward map associated to the s_i cover $S_i \rightarrow S'_i$ on the intersection product $(C^2)_{S_i}$. See the non-split I_0^* model described in appendix B.1 for an example of this structure. Once we know $g_i(C)$, we can determine the number of blowups p_i using

$$k_{iii} = K_{S_i}^2 = 8 - 8g_i(C) - p_i. \quad (3.23)$$

Computing triple intersection numbers. We compute the triple intersection numbers of the divisors of $X_r \rightarrow B$ following the strategy outlined in [23]. Suppose that $W_r = 0$ describes the resolved threefold X_r as a hypersurface of the ambient projective bundle Y_r , and let $\pi \circ f : X_r \rightarrow B$ denote the projection of X_r to the base B . The triple intersection numbers of X_r can be considerably simplified by expressing them in terms of geometric data associated to B . This simplification can be accomplished by computing the pushforward of the intersection product $(S_i \cdot S_j \cdot S_k)_{X_r}$ to the base of the elliptic fibration:

$$(S_i \cdot S_j \cdot S_k)_{X_r} = \pi_* \circ f_*(S_i \cdot S_j \cdot S_k)_{X_r} = \pi_* \circ f_*(S_i \cdot S_j \cdot S_k \cdot [W_r])_{Y_r}. \quad (3.24)$$

To perform this computation, we need to expand the divisor classes $S_i, [W_r], [g_{i,1}], [g_{i,2}], \dots$ (where $[g_{i,j}]$ are the classes of the divisors $g_{i,j} = 0$, associated to the generators of the centers of the blowups $f_i : X_i \rightarrow X_{i-1}$) in the basis $f^*H, f^* \circ \pi^*K_B, E_{i=0,\dots,r}$, where $H = c_1(\mathcal{O}_{Y_0}(1))$ is the divisor class of a hyperplane in the ambient space $\mathbb{P}^2$ of the fibers, $E_{i=1,\dots,r}$ are the classes of the exceptional divisors of the blowups, and $E_0 = f^* \circ \pi^*C$ is the pullback of the class $C \subset B$ to X_r . Once we have expressed $S_i \cdot S_j \cdot S_k \cdot [W_r]$ in terms of this basis, we use the pushforward formula of [23] to express $\pi_* \circ f_*(S_i \cdot S_j \cdot S_k \cdot [W_r])_{Y_r}$ as an intersection product of the classes K_B, C in B .

4 KK theories for rank one $6d$ SCFTs

In this section, we collect the Calabi-Yau threefolds associated to rank one $6d$ SCFTs that we obtain via computations using the above mentioned tools. Some sample computations are illustrated in appendix B. Unless otherwise stated, we will only work with total transforms and avoid proper transforms. This is the reason why some curves can appear with negative sign in what follows. See appendix A.3 for a review of this terminology.

Before we start, let us provide a brief review of rank one $6d$ SCFTs constructed in the unfrozen phase of F-theory. The F-theory base for such theories involves a single compact

rational curve C in the base of a negative self-intersection $-k$ where k can take values $1 \leq k \leq 8$ or $k = 12$. The elliptic fiber degenerates over C to some fiber of Kodaira type which can potentially have a monodromy as one encircles loops in C . As is well-known, each Kodaira fiber type along with the specification of monodromy leads to a particular simple gauge algebra $\mathfrak{g}$ in the resulting $6d$ theory. The self-intersection $-k$ of the curve almost always uniquely fixes the associated matter content [24]. Sometimes we can tune the corresponding Weierstrass model to change the matter content. In the context of this paper, this is only possible in the situation when $k = 1$ and the Kodaira fiber type is I_6 without a monodromy. Generically this model gives rise to $SU(6)$ with a hyper in two-index antisymmetric plus 14 hypers in fundamental. A tuned version of the model gives rise to $SU(6)$ with a half-hyper in three-index antisymmetric plus 15 hypers in fundamental.

Below, each subsection is devoted to a particular class of rank one $6d$ SCFTs. In the starting of each subsection, we specify both the F-theory construction, and the gauge algebra and matter content on the tensor branch of the resulting rank one $6d$ SCFT. We then proceed to provide our choice of resolution and a graphical description of the resulting geometry.

4.1 $SU(n)$ with $2n$ fundamental hypers

The F-theory construction involves a split I_n fiber over a -2 curve in the base. For even $n = 2m > 0$, Weierstrass model is defined by the following orders of vanishing:

$$a_1 = a_1, \quad a_2 = a_{2,1}s, \quad a_3 = 0, \quad a_4 = a_{4,m}e_0^m, \quad a_6 = a_{6,2m}e_0^{2m}. \quad (4.1)$$

We consider the resolution [25] defined by the following sequence of blowup centers:

$$(x, y, e_0|e_1), \quad (y, e_1|e_2), \quad (x, e_2|e_3), \quad \dots, \quad (x, e_{2m-2}|e_{2m-1}) \quad (4.2)$$

and find that the associated Calabi-Yau threefold is

$$(4.3)$$

where $\sum x_i$ denotes the sum over (the total transforms of) all the exceptional curves created due to the $2n$ blowups. Since $h - \sum x_i$ is a single curve, all the blow-ups are restricted to happen on the proper transform of the h curve, which means that the blow-ups are not completely generic. We will adopt this notation in what follows unless otherwise stated.

A degenerate case in this class of models is $n = 0$, in which case the F-theory configuration is an I_0 fiber over a -2 curve. This configuration preserves 16 supercharges and

hence the surface describing the local threefold splits into a product $S = \mathbb{P}^1 \times T^2$. Since S is a product the self-intersections of $\mathbb{P}^1$ and T^2 inside S are both zero and their mutual intersection is one.

For odd $n = 2m + 1 > 1$, the Weierstrass model is defined by

$$a_1 = a_1, \quad a_2 = a_{2,1}e_0, \quad a_3 = a_{3,m}e_0^r, \quad a_4 = a_{4,m+1}e_0^{m+1}, \quad a_6 = a_{6,2m+1}e_0^{2m+1}. \quad (4.4)$$

We consider the resolution [25] defined by the following sequence of blowup centers:

$$(x, y, e_0|e_1), \quad (y, e_1|e_2), \quad (x, e_2|e_3), \quad \dots, \quad (y, e_{2m-1}|e_{2m}) \quad (4.5)$$

and we find

A degenerate case in this class of models is $n = 1$, in which case the F-theory configuration is an I_1 fiber over a -2 curve. Even though this configuration preserves 8 supercharges, it is known that the corresponding $6d$ theory is $A_1(2, 0)$ theory which has 16 supercharges. By taking a limit of geometries described above, the geometry for this case can be predicted to be

involving a self-gluing. Thus, we have found two geometries for the KK theory corresponding to $A_1(2, 0)$ theory. We will see an interesting difference between the two geometries in section 5.2.

4.2 $SO(n)$ with $n - 8$ hypers in fundamental

For even $n = 2r$, the F-theory setup involves a split I_{r-4}^* fiber, with $r > 4$, over a -4 curve in the base. The Weierstrass model engineering $SO(n = 2r)$ for r odd is defined by

$$a_1 = a_{1,1}e_0, \quad a_2 = a_{2,1}e_0, \quad a_3 = a_{3,\frac{r-1}{2}}e_0^{\frac{r-1}{2}}, \quad a_4 = a_{4,\frac{r+1}{2}}e_0^{\frac{r+1}{2}}, \quad a_6 = a_{6,r}e_0^r. \quad (4.8)$$

We consider the resolution

$$(x, y, e_0|e_1), (y, e_1|e_2), \dots, (x, e_{r-3}|e_{r-2}), (y, e_{r-2}|e_{r-1}), (e_{r-3}, e_{r-2}|e_r). \quad (4.9)$$

The Weierstrass model for r even and is defined by

$$a_1 = a_{1,1}e_0, \quad a_2 = a_{2,1}e_0, \quad a_3 = a_{3,\frac{r}{2}}e_0^{\frac{r}{2}}, \quad a_4 = a_{4,\frac{r}{2}}e_0^{\frac{r}{2}}, \quad a_6 = a_{6,r-1}e_0^{r-1}. \quad (4.10)$$

This model requires an additional split condition, namely

$$\left. \frac{a_4^2 - 4a_2a_6}{e_0^r} \right|_{e_0=0} \quad (4.11)$$

must be a perfect square. In practice we satisfy this condition by imposing

$$a_{6,r-1} = 0. \quad (4.12)$$

We consider the resolution

$$(x, y, e_0|e_1), (y, e_1|e_2), \dots, (y, e_{r-3}|e_{r-2}), (x, e_{r-2}|e_{r-1}), (e_{r-3}, e_{r-2}|e_r). \quad (4.13)$$

In combination, we find that the associated collection of surfaces is

$$(4.14)$$

where the exceptional curves in S_r have been divided into two sets denoted by x_i and y_i , $1 \leq i \leq 2r - 8$.

Notice that the surfaces intersect in the fashion of an affine D_r Dynkin diagram, but there are extra intersections between the two single valent nodes towards the right end. As we will show now, these extra intersections however do not change the intersection pattern for the components of the degenerate elliptic fiber for I_{r-4}^* . The intersection is computed in a non-compact surface N intersecting (blow-ups of) Hirzebruch surfaces S_i along the total transform of their fibers f_i . Then, $(f_{r-1} \cdot f_r)_N = f \cdot (f - x_j - y_j)_{S_r} = 0$ where we have picked a pair of blow-ups x_j, y_j . Similar comments apply to all the following cases where we have extra intersections between the surfaces not accounted for by the corresponding affine Dynkin graphs.

For odd $n = 2r + 1$, the F-theory setup involves a non-split I_{r-3}^* fiber over a -4 curve, where $r > 3$. The Weierstrass model for r even is defined by

$$a_1 = a_{1,1}e_0, \quad a_2 = a_{2,1}e_0, \quad a_3 = a_{3,\frac{r}{2}}e_0^{\frac{r}{2}}, \quad a_4 = a_{4,\frac{r}{2}+1}e_0^{\frac{r}{2}+1}, \quad a_6 = a_{6,r}e_0^r. \quad (4.15)$$

We consider the following resolution:

$$(x, y, e_0|e_1), \quad (y, e_1|e_2), \quad (x, e_2|e_3), \quad \cdots, \quad (y, e_{r-3}|e_{r-2}), \quad (x, e_{r-2}|e_{r-1}), \quad (e_{r-2}, e_{r-1}|e_r). \quad (4.16)$$

The Weierstrass model for r odd is defined by

$$a_1 = a_1, \quad a_2 = a_{2,1}e_0, \quad a_3 = a_{3,\frac{r+1}{2}}e_0^{\frac{r+1}{2}}, \quad a_4 = a_{4,\frac{r+1}{2}}e_0^{\frac{r+1}{2}}, \quad a_6 = a_{6,r}e_0^r. \quad (4.17)$$

We consider the following resolution:

$$(x, y, e_0|e_1), \quad (y, e_1|e_2), \quad (x, e_2|e_3), \quad \cdots, \quad (x, e_{r-3}|e_{r-2}), \quad (y, e_{r-2}|e_{r-1}), \quad (e_{r-2}, e_{r-1}|e_r). \quad (4.18)$$

In combination, they give rise to

$$(4.19)$$

4.3 $\text{Sp}(n)$ with $2n + 8$ fundamental hypers

An F-theory setup for this model involves a non-split I_{2n+1} fiber, with $n > 0$, over a -1 curve in the base. The Weierstrass model is defined by [23]

$$a_1 = 0, \quad a_2 = a_2, \quad a_3 = 0, \quad a_4 = a_{4,n+1}e_0^{n+1}, \quad a_6 = a_{6,2n+1}e_0^{2n+1}. \quad (4.20)$$

We study the resolution

$$(x, y, e_0|e_1), \quad (x, y, e_1|e_2), \quad \cdots, \quad (x, y, e_{n-1}|e_n) \quad (4.21)$$

and find that the corresponding KK theory is described by (note that the nodes in the following graph “grow” from right to left with increasing values of n)

$$(4.22)$$

We consider the resolution

$$(x, y, e_0|e_1), (y, e_1|e_2), (x, e_2|e_3), (y, e_3|e_4), (e_2, e_3|e_5), (e_2, e_4|e_6), (e_4, e_5|e_7) \quad (4.28)$$

and find the KK theory to be described by

We notice that our answer matches that of [12] for $k = 8$ but differs for $k = 7$. Noticeably, we have extra edges¹² between the nodes of the affine E_7 (which implies non-trivial triple intersection numbers between three distinct surfaces) which reference [12] do not have. Moreover, our answer is not flop equivalent to theirs because one cannot get rid of all of the extra edges by doing flops. According to [12], there is a consistency condition that the correct answer for $k = 7$ must satisfy. Namely, one should be able to do flop transitions (along with sending a curve to infinite size) to reach a point where we have two disjoint collection of surfaces. One of them should be $\mathbb{F}_5 - \mathbb{F}_3 - \mathbb{P}^2$ describing an orbifold CFT and the other one should describe a non-orbifold CFT. In our case, the corresponding flop transition is the flop of -1 curve inside $S_2 = \mathbb{F}_1$. Expanding the flopped curve to infinite size, we see that we also obtain $\mathbb{F}_5 - \mathbb{F}_3 - \mathbb{P}^2$ as one of the disjoint pieces. So, the disagreement in the proposal of [12] and our proposal can be phrased as a disagreement in the identification of the non-orbifold piece in the above mentioned limit.

Now, there are two reasons for us to trust that our result is the correct one. First, we found it by an honest computation using the tools described section 3.3 starting from the Weierstrass model defining this $6d$ SCFT. Second, we give a uniform answer for all k , including $k < 7$.

4.6 Pure E_8

The F-theory construction involves a II^* fiber over a -12 curve in the base. The Weierstrass model is defined by

$$a_1 = 0, \quad a_2 = 0, \quad a_3 = 0, \quad a_4 = a_{4,4}e_0^4, \quad a_6 = a_{6,5}e_0^5. \quad (4.30)$$

¹²Even though the surfaces do not intersect in the pattern of an affine E_7 , the components of the degenerate fiber do. This can be seen by a computation similar to the one in between equations (4.14) and (4.15).

We consider the resolution

$$(x, y, e_0|e_1), (y, e_1|e_2), (x, e_2|e_3), (y, e_3|e_4), (e_2, e_3|e_5), (e_4, e_5|e_6), (e_2, e_4, e_6|e_7), (e_4, e_7|e_8) \quad (4.31)$$

and determine the geometry to be

$$\quad (4.32)$$

We notice that our answer matches that of [12].

4.7 F_4 with $(5 - k)$ fundamental hypers

The F-theory construction involves a non-split IV^* fiber over a $-k$ curve in the base, where $1 \leq k \leq 5$. The Weierstrass model is defined by

$$a_1 = 0, \quad a_2 = 0, \quad a_3 = 0, \quad a_4 = a_{4,3}e_0^3, \quad a_6 = a_{6,4}e_0^4. \quad (4.33)$$

We consider the resolution

$$(x, y, e_0|e_1), (y, e_1|e_2), (x, e_2|e_3), (e_2, e_3|e_4). \quad (4.34)$$

In this case, we compute the geometry to be

$$\quad (4.35)$$

which matches that of [12] for $k = 5$, along with the results of [22] for general k .

4.8 G_2 with $(10 - 3k)$ fundamental hypers

The F-theory construction involves a non-split I_0^* fiber over a $-k$ curve in the base, where $1 \leq k \leq 3$. The Weierstrass model is defined by

$$a_1 = 0, \quad a_2 = 0, \quad a_3 = 0, \quad a_4 = a_{4,2}e_0^2, \quad a_6 = a_{6,3}e_0^3. \quad (4.36)$$

We consider the resolution

$$(x, y, e_0|e_1), (y, e_1|e_2), \quad (4.37)$$

and the geometry for the KK theory turns out to be

$$\quad (4.38)$$

4.9 $SU(n)$ with one antisymmetric hyper and $n + 8$ fundamental hypers

The F-theory construction involves a split I_n fiber over a -1 curve in the base. The Weierstrass models were already written down in section 4.1. For even $n = 2m$, we find the following Calabi-Yau

$$(4.39)$$

and for odd $n = 2m + 1$ we find

$$(4.40)$$

For the degenerate case of $SU(3)$ we have

$$(4.41)$$

4.10 $SU(6)$ with half-hyper in three-index antisymmetric and 15 fundamental hypers

The F-theory construction of this theory involves an alternate tuning of a split I_{2n} fiber over a -1 curve [26]. The Weierstrass model for general n is defined by

$$a_1 = a_1, \quad a_2 = e_0^2 a_{2,2}, \quad a_3 = a_{3,n-1} e_0^{n-1}, \quad a_4 = a_{4,n+1} e_0^{n+1}, \quad a_6 = a_{6,2n} e_0^{2n}. \quad (4.42)$$

For $n = 3$, which is the only case of interest for 6d SCFTs, we consider the resolution

$$(x, y, e_0|e_1), \quad (y, e_1|e_2), \quad (x, e_2|e_3), \quad (y, e_3|e_4), \quad (y, e_4|e_5). \quad (4.43)$$

and we identify the geometry to be

4.11 $SO(7)$ with $8 - 2k$ spinor hypers and $3 - k$ fundamental hypers

The F-theory construction involves a semi-split I_0^* fiber over a $-k$ curve in the base, where $1 \leq k \leq 3$. The Weierstrass model is defined by

$$a_1 = 0, \quad a_2 = a_{2,1}e_0, \quad a_3 = 0, \quad a_4 = a_{4,2}e_0^2, \quad a_6 = a_{6,4}e_0^4. \quad (4.45)$$

We consider the resolution

$$(x, y, e_0|e_1), \quad (y, e_1|e_2), \quad (x, e_2|e_3), \quad (4.46)$$

and the corresponding geometry is described by

4.12 $SO(8)$ with $4 - k$ fundamentals, $4 - k$ spinors and $4 - k$ conjugate spinors

The F-theory construction involves a split I_0^* fiber over a $-k$ curve in the base, where $1 \leq k \leq 4$. We use the Weierstrass model defined by the following orders of vanishing:

$$a_1 = a_{1,1}e_0, \quad a_2 = a_{2,1}e_0, \quad a_3 = a_{3,2}e_0^2, \quad a_4 = a_{4,2}e_0^2, \quad a_6 = a_{6,2}e_0^4, \quad (4.48)$$

where we also impose the split condition

$$4a_{4,2} - a_{2,1}^2 = \alpha^2. \quad (4.49)$$

We consider the resolution

$$(x, y, e_0|e_1), \quad (y, e_1|e_2), \quad (x, e_2|e_3), \quad (e_0\alpha_2^\pm z + 2e_3x, e_2|e_4) \quad (4.50)$$

where $\alpha_2^\pm = \alpha \pm a_2$. The geometry is

where we have split the $16 - 4k$ blow-ups on S_3 into four equal pieces denoted by x_i , y_i , z_i and w_i . We have also chosen to hide the affine node S_0 so that the diagram would be planar. S_0 has degree $2 - k$ and no blow-ups, and is attached to S_1 along the curve e in S_1 . This matches the answer of [12] for $k = 4$.

4.13 $\text{SO}(9)$ with $4 - k$ spinor hypers and $5 - k$ fundamental hypers

The F-theory construction involves a non-split I_1^* fiber over a $-k$ curve in the base, where $1 \leq k \leq 4$. The Weierstrass model appears in section 4.2. The geometry is

4.14 $\text{SO}(10)$ with $4 - k$ spinor hypers and $6 - k$ fundamental hypers

The F-theory construction involves a split I_1^* fiber over a $-k$ curve in the base, where $1 \leq k \leq 4$. The Weierstrass model appears in section 4.2. The threefold is

where we have split the blow-ups on S_4 into $(6 - k) + (6 - k) + (4 - k)$ blowups denoted respectively by x_i , y_i and z_i .

4.15 $\text{SO}(11)$ with $\frac{4-k}{2}$ spinor hypers and $7 - k$ fundamental hypers

The F-theory construction involves a non-split I_2^* fiber over a $-k$ curve in the base, where $1 \leq k \leq 4$. The Weierstrass model was written down in section 4.2. We find the corresponding threefold to be

(4.54)

4.16 $\text{SO}(12)$ with $\frac{4-k}{2}$ spinor hypers and fundamental $8 - k$

The F-theory construction involves a split I_2^* fiber over a $-k$ curve in the base, where $1 \leq k \leq 4$. The Weierstrass model was written down in section 4.2. In this case, our answer is

(4.55)

4.17 Pure SU(3)

This is constructed by taking a split IV fiber over a -3 curve in the base. The corresponding collection of surfaces is

which represents three $\mathbb{F}_1$ glued together along their e curve. There are actually four surfaces glued along a single locus because the base is also glued to the e curve of S_0 . Such a multi-valent gluing still satisfies the Calabi-Yau condition 3.3 pairwise as all the curves involved in the gluing are -1 curves. The data of triple intersection numbers can be depicted as

but it should be kept in mind that the latter figure does not represent the actual geometry and is only a bookkeeping device for triple intersection numbers.

5 RG flows

5.1 Geometric criteria

An RG flow is induced by taking the volumes of some compact curves to infinity. We cannot simply expand a compact curve C living inside a compact surface S to infinite volume, because that sends the volume of S to infinity as well, leading to a rank changing RG flow. This would seem to suggest that there are no RG flows possible. However, we should recall that there are two ways in which a mass parameter can be taken to infinity, i.e. either to $+\infty$ or to $-\infty$. This suggests that we should try to send the volumes of some curves to negative infinity. This is indeed possible if we have a rational curve E of

Figure 2. From left to right: two surfaces S and T are glued to each other along a -1 curve E . The dashed line represents a collection of surfaces joining S and T in some other direction. Flopping E leads to the second figure with S' being a blow-down of S and T' being a blow-down of T . The volume of E' can then be sent to infinity leading to a geometry (shown in the last figure) in which S' and T' are joined only via the collection of surfaces denoted by dashed line.

self-intersection -1 (henceforth referred to as a “ -1 curve”) inside a compact surface S . The volume of E can be sent to negative infinity, which is a formal way of saying that we first flop E to another -1 curve E' and then send the volume of E' to infinity. This RG flow is allowed whenever E' lives in a non-compact surface inside the threefold, as then sending the volume of E' to infinity doesn’t remove any BPS string from the spectrum.

In all, we can divide the analysis of rank preserving RG flows into three cases:

1. If E is part of the gluing locus between S and some other compact surface T , then after the flop E' will separate S' and T' (where S' and T' are the images of S and T after the flop). This can lead to a rank preserving RG flow only if S' and T' are linked by a chain of surfaces, which is equivalent to saying that S and T were part of a loop of surfaces before the flop. See figure 2. In other words, the RG flow is implemented by blowing down E inside S and T , and deleting the gluing corresponding to E between S and T .
2. If the E is neither a part of gluing locus between any two compact surfaces nor intersects the gluing locus between any two compact surfaces, then it can be flopped into an adjacent non-compact surface and its volume can subsequently be sent to infinity leading to an RG flow. See figure 3. In other words, the geometry after the RG flow is simply described by blowing down E inside S , and everything else remains the same.
3. If E is not a part of gluing locus between any two compact surfaces, but rather intersects the gluing locus between S and some other compact surface T at some point, then it can be flopped into T . Now we have a collection of surfaces in which S and T have been replaced by S' and T' , which might carry new -1 curves each of which can then be subsequently flopped. In other words, we have blow down E inside S and blown-up the point inside T corresponding to intersection with E . This

We first flop one of the exceptional curves, say x_1 , to obtain

(5.2)

Since the gluing curve itself is a -1 curve now, we can flop it and, since the surface was glued to itself in a loop before the flop, we obtain a rank preserving RG flow to the geometry described by a local $\mathbb{F}_0$

(5.3)

This geometry is indeed known to describe the Coulomb branch of $5d$ $SU(2)$ $\mathcal{N} = 1$ pure SYM.

6 Future direction: higher rank

It seems straightforward to extend our results to higher rank $6d$ SCFTs. Say C and D are two curves in the base which intersect each other transversely. The intersection corresponds to gluing S_C and S_D with each other along the components of the degenerate elliptic fibers on C and D . However, there is a subtlety that one might have to do some flops (corresponding to going to a different chamber of the Coulomb branch) on either S_C or S_D for a gluing to be permissible. We will discuss some examples of such gluings in this section.¹³

- **A_2 (2,0) SCFT:** the F-theory construction involves two -2 curves in the base intersecting each other at one point. The surface associated to each curve is $\mathbb{P}^1 \times T^2$. We simply glue the two T^2 together, which satisfies the Calabi-Yau condition 3.3 because the self-intersection of each T^2 is zero and the genus is one.
- **Bifundamental between $SU(m)$ and $SU(n)$:** we have two -2 curves C and D in the base intersecting each other at a point with I_m fiber on C and I_n fiber on D . We have computed S_C and S_D in section 4.1. To glue them, we first have to make the blow-ups slightly more non-generic. We require n out of the $2m$ blow-ups on h in $\mathbb{F}_m$ inside S_C to take place such that $x_i - x_{i+1}$ for $i = 1, \dots, n$ are effective curves after blow-ups. In this way, we obtain $n - 1$ number of -2 curves, namely $x_i - x_{i+1}$ for $i = 1, \dots, n$. We also get two -1 curves, namely x_n and $f - x_1$. See figure 5 for a description of the resulting surface. Similarly we make m out of $2n$ blowups on $\mathbb{F}_n$

¹³A general story for higher rank is currently a work in progress.

Figure 5. A non-generic blow-up of $\mathbb{F}_m$ as explained in the text. The edges represent curves inside the surface.

in S_D to be non-generic in exactly the same fashion leading to $m - 1$ number of -2 curves and two -1 curves. The gluing can now be described as follows. The $n - 1$ number of -2 curves in $\mathbb{F}_m^{2m}$ are glued to the f curves inside all the surfaces in S_D except for $\mathbb{F}_n^{2n}$. Notice that, since S_D contains n surfaces, the f curves participating in the gluing are $n - 1$ in number too. Similarly, the $m - 1$ number of -2 curves in $\mathbb{F}_n^{2n}$ are glued to the f curves inside all the surfaces in S_C except for $\mathbb{F}_m^{2m}$. The two -1 curves in $\mathbb{F}_m^{2m}$ are glued to the two -1 curves inside $\mathbb{F}_n^{2n}$.

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A Mathematical background

A.1 Some useful geometric notions

Let X be a smooth projective variety. The set of all curve classes in X with holomorphic representatives is called the *Mori cone*, where we view two representatives as being equivalent if they have identical intersection numbers with all divisors of X . The Mori cone can

be represented as the real span of a set of generators C_μ ; consequently, an arbitrary curve C with holomorphic representative can be expressed as a non-negative linear combination of these generators:

$$C = a_\mu C_\mu, \quad a_\mu \in \mathbb{Z}_{\geq 0}, \quad (\text{A.1})$$

Note that C_μ themselves are linear combinations of a basis C_i of all (possibly non-holomorphic) curve classes. The dual of the Mori cone (in the sense of convex geometry) is called the nef cone, and consists of all divisors D such that

$$(D \cdot C)_X \geq 0 \quad (\text{A.2})$$

for all C belonging to the Mori cone of X .

Let $X = S$ be a surface with canonical class K . Suppose $\pi : S' \rightarrow S$ is the blowup of S at a single point p on a curve C , and assume that C has multiplicity m at p . Then the proper transform C' of C in S' is denoted by

$$C' = \pi^*(C) - mE. \quad (\text{A.3})$$

Moreover, the canonical class K' of S' can be expressed in terms of K, S as

$$K' = \pi^*K + E. \quad (\text{A.4})$$

Finally, consider gluing a surface S to itself by identifying two curves $C_1, C_2 \in S$. This induces a birational map $\pi : S \rightarrow S'$. The canonical class K' of S' is then given by

$$\pi^*K' = K + C_1 + C_2. \quad (\text{A.5})$$

In practice we omit explicit pullback maps when the context is clear.

A.2 Ruled surfaces

A ruled surface $\mathbb{F}_{n,g}$ is a projective surface which can be viewed as a $\mathbb{P}^1$ bundle $\mathbb{F}_{n,g} \rightarrow e$ over a curve e with genus g and degree $n = -(e^2)_{\mathbb{F}_{n,g}}$. When $g = 0$, $\mathbb{F}_{n,0} \equiv \mathbb{F}_n = \mathbb{P}[\mathcal{O} \oplus \mathcal{O}(-n)] \rightarrow e$ is a Hirzebruch surface. The Mori cone of $\mathbb{F}_{n,g}$ is the positive real span of the curve classes e, f , namely all curves of the form

$$af + be, \quad (a, b) \in \mathbb{Z}_{\geq 0}^2, \quad (\text{A.6})$$

where

$$(e^2)_{\mathbb{F}_{n,g}} = -n, \quad (e \cdot f)_{\mathbb{F}_{n,g}} = 1, \quad (f^2)_{\mathbb{F}_{n,g}} = 0. \quad (\text{A.7})$$

It is also useful to define the curve class

$$h = e + nf, \quad (h^2)_{\mathbb{F}_{n,g}} = n, \quad (h \cdot f)_{\mathbb{F}_{n,g}} = 1, \quad (h \cdot e)_{\mathbb{F}_{n,g}} = 0, \quad (\text{A.8})$$

in terms of which the canonical class is

$$K = -2h + (2g - 2 + n)f. \quad (\text{A.9})$$

(Note that the canonical class can be derived by starting with the parametrization $K = af + be$ and then demanding that adjunction is satisfied for curves of known genus.) Given a curve $c = af + be$, the genus $g(c)$ can be expressed as a function of the genus and self-intersection of e using the adjunction formula:

$$2g(c) - 2 = ((K + c) \cdot c)_{\mathbb{F}_{n,g}} = 2a(b - 1) + b(-bn + 2g + n - 2). \quad (\text{A.10})$$

A.3 Total transforms versus proper transforms

In this paper, we often distinguish between the total transform and the proper transform of a curve in a surface. In this subsection, we explain the distinction between these two notions, and what this distinction implies about self-intersections of curve classes. Let $f : S' \rightarrow S$ be the blowup of a surface S at a point $p \in S$. The exceptional divisor of this blowup is a curve $E \cong \mathbb{P}^1$. Now, consider a smooth curve $C \subset S$ which passes through the point p . By construction,

$$f^*(C) = C' + E, \quad f_*(C') = C, \quad f_*(E) = 0. \quad (\text{A.11})$$

We call the curve C' the *proper transform* of C , while we call the curve $f^*(C)$ the *total transform* of the curve C . Informally, the proper transform can be thought of as the inverse image of all parts of the curve C away from the blowup locus. More formally, C' is the closure (in the Zariski topology) of the curve $f^{-1}(C - \{p\})$. The total transform, by contrast, is simply the inverse image of the entire curve C .

What is the self-intersection of the curve C' in S' ? Observe that the curve C' meets E transversally at one point and hence we have

$$(C' \cdot E)_{S'} = 1. \quad (\text{A.12})$$

Using the above, it follows from a useful result called the *projection formula* that

$$0 = f^*(f_*(E) \cdot C)_S = (E \cdot f^*(C))_{S'} = (E^2)_{S'} + (E \cdot C')_{S'} \quad (\text{A.13})$$

and hence

$$(E \cdot f^*(C))_{S'} = 0, \quad (E^2)_{S'} = -(E \cdot C')_{S'} = -1. \quad (\text{A.14})$$

Therefore, we may write

$$(C'^2)_S = (f^*(C) - E)_{S'}^2 = (f^*(C))_{S'}^2 + (E^2)_{S'} - 2(f^*(C) \cdot E)_{S'} = C_S^2 - 1. \quad (\text{A.15})$$

Thus we see that the self-intersection of the curve C' is reduced by 1. In practice, we suppress the pushforward and pullback notation and simply write

$$C' = C - E, \quad (\text{A.16})$$

keeping in mind that $(C \cdot E)_{S'} = 0$. This construction has a simple generalization to the case where several points $p_i \in S$ are blown up. Let C be a curve which passes through each point p_i with multiplicity m_i . Then the proper transform of C is

$$C' = C - \sum m_i E_i, \quad (\text{A.17})$$

where we have $(C \cdot E_i)_{S'} = 0, (E_i^2)_{S'} = -1$.

B Example computations

In this appendix, we provide explicit details in some examples where we compute the geometry of a resolved Calabi-Yau threefold starting from a singular elliptically fibered one. In the following examples, we use the notation established in section 3.3. We compute the degrees and some example triple intersection numbers associated to the fibral divisors S_i following the strategies explained there. For brevity, we suppress notation indicating pullbacks with respect to the blowups and projection of the elliptic fibration whenever the context is clear.

We have chosen these examples carefully to illustrate some key points. In appendix B.3, we discuss theories with $\text{SO}(10)$ gauge group along with $4-k$ hypers in spinor representation and $6-k$ hypers in fundamental representation, for $1 \leq k \leq 4$. For $k=4$, we only have fundamental hypers and we naively expect to see surfaces intersecting in the fashion of an affine D_5 Dynkin graph. However, our computations reveal that the surfaces have some extra intersections which do not modify the intersection pattern of the components of the degenerate D_5 elliptic fiber. In appendix B.1, we discuss rank one $6d$ SCFTs having G_2 gauge group with $n_f = 1, 4, 7$ hypers in fundamental representation. These examples illustrate that non-split fibers can sometimes lead to ruled surfaces over curves of non-zero genus. In appendix B.2, we discuss theories with $\text{SU}(4)$ gauge group along with $16-4k$ hypers in fundamental and $2-k$ hypers in antisymmetric for $k=1, 2$. This is the simplest example where a non-fundamental matter representation first appears.

B.1 G_2 theories

The non-split I_0^* model engineering G_2 is defined by the following Weierstrass equation:

$$W_0 = y^2 - (x^3 + e_0^2 a_{4,2} x + e_0^3 a_{6,3}) = 0. \quad (\text{B.1})$$

For convenience, we work in the open set $z=1$. In the following discussion we describe the resolution

$$(x, y, e_0 | e_1), \quad (y, e_1 | e_2). \quad (\text{B.2})$$

The $\mathbb{P}^2$ ambient spaces of the elliptic fibers are described by homogeneous coordinates $[x : y : z = 1]$. The singular locus $W_0 = \partial_i W_0 = 0$ is given by

$$x = y = e_0 = 0 \quad (\text{B.3})$$

and hence we blow up X_0 along the center (x, y, e_0) . The homogeneous coordinates of the new ambient space Y_1 are $[e_1 x : e_1 y : z = 1][x : y : e_0]$. Introducing new coordinates $(x, y, e_0) \rightarrow (e_1 x, e_1 y, e_1 e_0)$ and factoring out two copies of the exceptional divisor E_1 defined by the local equation $e_1 = 0$, we find that the total transform X_1 is given by

$$W_1 = y^2 - (e_1 x^3 + e_1 e_0^2 a_{4,2} x + e_1 e_0^3 a_{6,3}) = 0. \quad (\text{B.4})$$

The singular locus of X_1 is a subset of the locus

$$y = e_1 = 0. \quad (\text{B.5})$$

We thus take (y, e_1) to be the center of the second blowup and make the replacements $(y, e_1) \rightarrow (e_2y, e_2e_1)$. The homogeneous coordinates of the new ambient space Y_2 are $[e_2e_1x : e_2^2e_1y : z = 1][x : e_2y : e_0][y : e_1]$. Factoring out a single copy of the exceptional divisor E_2 with local equation $e_2 = 0$, the total transform X_2 is given by

$$W_2 = e_2y^2 - (e_1x^3 + e_1e_0^2a_{4,2}x + e_1e_0^3a_{6,3}) = 0. \quad (\text{B.6})$$

One can verify by explicit computation that X_2 is smooth, i.e. that $W_2 = \partial_i W_2 = 0$ has no solutions.

We now turn our attention to computing the geometry of these divisors, noting for reference the following divisor classes:

$$\begin{aligned} [x] &= -2K_B - E_1, & [y] &= -3K_B - E_1 - E_2, & [e_0] &= E_0 - E_1 \\ [e_1] &= E_1 - E_2, & [e_2] &= E_2. \end{aligned} \quad (\text{B.7})$$

We remark that by fixing $z = 1$ using the scaling freedom of the original $\mathbb{P}^2$ coordinates $[x : y : z] \cong [\lambda_0x : \lambda_0y : \lambda_0z]$ of the fibers of Y_0 , we have eliminated the dependence of $[x], [y]$ on the divisor class H . The irreducible components F_i of the resolved elliptic fiber $F = F_0 + 2F_1 + (\sum_{j=1}^3 F_{2,j})$ are parametrized as follows:

$$\begin{aligned} F_0 &: e_2y^2 - e_1x^3 = 0 \quad \subset \quad [\lambda_1x : \lambda_1e_2y][\lambda_2\lambda_1y : (\lambda_2/\lambda_1)e_1] \\ F_1 &: [\lambda_1x : \lambda_1e_0][\lambda_2\lambda_1y : 0] \\ F_2 &: x^3 + a_{4,2}e_0^2x + a_{6,3}e_0^3 = 0 \quad \subset \quad [\lambda_1x : \lambda_1e_0][\lambda_2\lambda_1y : (\lambda_2/\lambda_1)e_1] \end{aligned} \quad (\text{B.8})$$

where $\lambda_i \in \mathbb{C}^\times$ are scale factors associated to the scaling symmetries of the above homogeneous coordinates. We ignore the dependence of F_i on the coordinates of the original $\mathbb{P}^2$, $[e_2e_1x : e_2^2e_1y : 1]$, since the only curve with a non-trivial dependence on these coordinates is F_0 which has a rational parametrization in terms of a single complex coordinate x/ye_2 — see (B.9) below. We restrict F_i to open sets and fix the projective scaling freedom by setting λ_i equal to convenient choices of nonzero coordinates in order to determine the degrees of their associated projective bundles:

$$\begin{aligned} F_0 &: (x/ye_2)^3 (ye_1e_2^2) - 1 = 0 \quad \subset \quad [x/ye_2 : 1][ye_1e_2^2 : 1] \\ F_1 &: [x : e_0] \\ F_2 &: (x/e_0)^3 + a_{4,2}(x/e_0) + a_{6,3} = 0 \quad \subset \quad [x/e_0 : 1][y/e_0 : e_0e_1]. \end{aligned} \quad (\text{B.9})$$

In the case of F_2 , we see the defining equation above parametrizes three disjoint points on the complex line $\mathbb{C}$ with coordinate x/e_0 , and hence the fibral divisor S_2 is a triple cover of the ruled surface over C' with fiber coordinates $[y/e_0 : e_0e_1]$. The projective bundles $\mathbb{P}_C[\mathcal{O} \oplus \mathcal{L}_i]$ are defined by the following divisor classes L_i dual to the line bundles $\mathcal{L}_i$:

$$\begin{aligned} L_0 &= [x/ye_2] = -2K_B - E_1 - (H - 3K_B - E_1 - E_2) - E_2 = K_B \\ L_1 &= [x/e_0] = -2K_B - E_1 - (E_0 - E_1) = -2K_B - E_0 \\ L_2 &= [y/e_0^2e_1] = -3K_B - E_1 - E_2 - 2(E_0 - E_1) - (E_1 - E_2) = -3K_B - 2E_0, \end{aligned} \quad (\text{B.10})$$

in terms of which the degrees $n_i = (\pi_* \circ f_*(L_i) \cdot C')_B$ are

$$n_0 = k - 2, \quad n_1 = k - 4, \quad n_2 = k - 6. \quad (\text{B.11})$$

In the above equations we have used the fact that $(C')_B^2 = -k$ and $g(C') = 0$. Since the non-split I_0^* model contains non-split Kodaira fibers, the divisor S_2 is a triple cover of the projective bundle $\mathbb{P}_{C'}[\mathcal{O} \oplus \mathcal{L}_2]$, and may be viewed as a ruled surface of degree $3(k - 6)$ over a curve C which according to (3.21) has genus

$$g(C) = \frac{3}{2} \left(b - 2 + \frac{2}{3} \right) = \frac{3}{2}b - 2. \quad (\text{B.12})$$

In order to determine the genus of C we compute the number of fibers in the ramification locus. Referring once again to the defining equation for F_2 , we see that the ramification locus is the vanishing of the discriminant of the equation

$$(x/e_0)^3 + a_{4,2}(x/e_0) + a_{6,3} = 0, \quad (\text{B.13})$$

namely

$$4a_{4,2}^3 + 27a_{6,3}^2 = 0, \quad (\text{B.14})$$

where two of the solutions of the above cubic equation are conjugate. Since $[4a_{4,2}^3 + 27a_{6,3}^2]/3 = -4K_B - 2C$, we find that the discriminant vanishes at

$$b = 2(4 - k) \quad (\text{B.15})$$

points along C , which implies

$$g(C) = 3(4 - k) - 2 = 10 - 3k. \quad (\text{B.16})$$

Next, we compute the triple intersection numbers of this resolved non-split I_0^* model. The smooth elliptic threefold X_2 has a natural basis of divisors S_i given by

$$S_0 = E_0 - E_1, \quad S_1 = E_1 - E_2, \quad S_2 = E_2. \quad (\text{B.17})$$

We also note the class of $W_2 = 0$ is

$$[W_2] = 3H - 6K_B - 2E_1 - E_2. \quad (\text{B.18})$$

Following procedure outlined in [23], we compute triple intersection numbers in terms of the following pushforwards of intersection products. As a concrete example, we compute one triple intersection number explicitly, namely $(D_0^3)_{X_2}$. First we compute the pullback of this intersection product to the ambient projective bundle Y_2 and express it in a basis of exceptional divisors:

$$(D_0^3)_{X_2} = (D_0^3[W_2])_{Y_2} = ((S - E_1)^3(-E_2 - 2E_1 + 3H - 6K_B))_{Y_2}. \quad (\text{B.19})$$

Next, we compute a sequence of pushforwards:

$$\begin{aligned} f_{2*}((E_0 - E_1)^3(-E_2 - 2E_1 + 3H - 6K_B))_{Y_2} \\ = ((E_0 - E_1)^3(-2E_1 + 3H - 6K_B))_{Y_1} \end{aligned} \quad (\text{B.20})$$

$$\begin{aligned} f_{1*} \circ f_{2*}((E_0 - E_1)^3(-E_2 - 2E_1 + 3H - 6K_B))_{Y_2} \\ = (E_0(H - 2K_B)(H^2 + H(-7K_B - 4E_0) + 3(E_0 + 2K_B)^2))_{Y_0} \end{aligned} \quad (\text{B.21})$$

$$\begin{aligned} \pi_* \circ f_{1*} \circ f_{2*}((E_0 - E_1)^3(-E_2 - 2E_1 + 3H - 6K_B))_{Y_2} \\ = -4(C \cdot (K_B + C))_B \end{aligned} \quad (\text{B.22})$$

where in the last line above we have used $\pi_* E_0 = C$. Finally, we use the fact that $C^2 = -k$ to compute

$$-4(C \cdot (K_B + C))_B = 8. \quad (\text{B.23})$$

The complete set of triple intersection numbers are summarized in (4.38). Ultimately, we find

$$S_0 = \mathbb{F}_{k-2}, \quad S_1 = \mathbb{F}_{k-4}, \quad S_2 = \mathbb{F}_{3(k-6), 10-k}. \quad (\text{B.24})$$

B.2 SU(4) theories

The split I_4 model engineering SU(4) is described by the following Weierstrass equation:

$$W_0 = y^2 + a_1 xy - (x^3 + a_{2,1} e_0 x^2 + a_{4,1} e_0 x + a_{6,2} e_0^2) = 0. \quad (\text{B.25})$$

Again for convenience, we work in the open set $z = 1$. We study the resolution

$$(x, y, e_0 | e_1), \quad (y, e_1 | e_2), \quad (x, e_2 | e_3). \quad (\text{B.26})$$

The ambient spaces of the elliptic fibers are described by projective coordinates $[x : y : z = 1]$. The singular locus $W_0 = \partial_i W_0 = 0$ is given by

$$x = y = e_0 = 0. \quad (\text{B.27})$$

We first blow up X_0 along the center (x, y, e_0) by making the substitution $(x, y, e_0) \rightarrow (e_1 x, e_1 y, e_1 e_0)$, which introduces an exceptional divisor E_1 with local equation $e_1 = 0$. The homogeneous coordinates of the ambient space Y_1 are now $[e_0 x : e_0 y : z = 1][x : y : e_0]$. Factoring out two copies of E_1 , the total transform X_1 is given by

$$W_1 = y^2 + a_1 xy - (e_1 x^3 + e_1 e_0 a_{2,1} x^2 + e_1 e_0^2 a_{4,2} x + e_1^2 a_{6,4} e_0^4). \quad (\text{B.28})$$

The singular locus of X_1 is a subset of the locus

$$y = e_1 = 0, \quad (\text{B.29})$$

hence we select (y, e_1) as the center of the second blowup. Making the substitution $(y, e_1) \rightarrow (e_2 y, e_2 e_1)$ and introducing the exceptional divisor E_2 with local equation $e_2 = 0$, we find

that the ambient space Y_2 is described by the homogeneous coordinates $[e_1 e_0 x : e_1^2 e_0 : z = 1][x : e_2 y : e_0][y : e_1]$. Factoring out a single copy of E_2 , the proper transform X_2 is given by

$$W_2 = e_2 y^2 + a_1 x y - (e_1 x^3 + e_1 e_0 a_{2,1} x^2 + e_1 e_0^2 a_{4,2} x + e_2 e_1^2 e_0^4 a_{6,4}). \quad (\text{B.30})$$

The singular locus of X_2 is a subset of the locus

$$x = e_2 = 0. \quad (\text{B.31})$$

We perform one final blowup along the center (x, e_2) , making the substitution $(x, e_2) \rightarrow (e_3 x, e_3 e_2)$ and introducing the exceptional divisor E_3 with local equation $e_3 = 0$. The ambient space Y_3 is described by the homogeneous coordinates $[e_3^2 e_2 e_1 x : e_3^2 e_2^2 e_1 y : z = 1][e_3 x : e_3 e_2 y : e_0][y : e_1][x : e_2]$. Factoring out a single copy of E_3 , the proper transform X_3 is given by

$$W_3 = e_2 y^2 + a_1 x y - (e_3^2 e_1 x^3 + e_3 e_1 e_0 a_{2,1} x^2 + e_1 e_0^2 a_{4,2} x + e_2 e_1^2 e_0^4 a_{6,4}). \quad (\text{B.32})$$

One can verify by explicit computation that the $W_3 = \partial_i W_3 = 0$ has no solutions and hence X_3 is a smooth elliptic fibration.

Our next task is to determine the geometry of the divisors S_i . For reference, we note the following divisor classes

$$\begin{aligned} [x] &= -2K_B - E_1 - E_3, & [y] &= -3K_B - E_1 - E_2, & [e_0] &= E_0 - E_1 \\ [e_1] &= E_1 - E_2, & [e_2] &= E_2 - E_3, & [e_3] &= E_3. \end{aligned} \quad (\text{B.33})$$

Again, we have used the scaling symmetry of the $\mathbb{P}^2$ fiber coordinates of Y_0 to set $z = 1$, which removes the dependence of the classes $[x], [y]$ on the divisor class H in the above expressions.

We first study the irreducible components $F_{i=1,\dots,3}$ of the resolved elliptic fiber $F = F_0 + F_1 + F_2 + F_3$ (we ignore F_0 because as in the previous example F_0 has a rational parametrization.) The projective coordinates and scaling symmetries introduced by the blowups are

$$[\lambda_1 e_3 x : \lambda_1 e_3 e_2 y : \lambda_1 e_0][\lambda_2 \lambda_1 y : (\lambda_2/\lambda_1) e_1][\lambda_3 \lambda_1 x : e_2(\lambda_3/\lambda_2)], \quad (\text{B.34})$$

where $\lambda_i \in \mathbb{C}^\times$. After fixing the various projective scaling symmetries, we find the irreducible components admit the following parametrizations:

$$\begin{aligned} F_1 &: [x e_3 : e_0] \\ F_2 &: [x e_3 : e_0] \\ F_3 &: A(x/e_0) + B e_2 = 0 \quad \subset \quad [(y/e_0) : e_0 e_1][(x/e_0) : e_2] \end{aligned} \quad (\text{B.35})$$

where

$$A = a_1(y/e_0) - a_{4,2} e_0 e_1, \quad B = (y/e_0)^2 - a_{6,4}(e_0 e_1)^2. \quad (\text{B.36})$$

The above computation implies the projective bundles $\mathbb{P}_C[\mathcal{O} \oplus \mathcal{L}_i]$ are defined by the following divisor classes:

$$\begin{aligned} L_0 &= -K_B \\ L_1 &= [xe_3/e_0] = -2K_B - E_0 \\ L_2 &= [xe_3/e_0] = -2K_B - E_0 \\ L_3 &= [y/e_0^2 e_1] = -K_B. \end{aligned} \tag{B.37}$$

We thus find the following degrees $n_i = (\pi_* \circ f_*(L_i) \cdot C)_B$:

$$n_0 = k - 2, \quad n_1 = k - 4, \quad n_2 = k - 4, \quad n_3 = k - 6. \tag{B.38}$$

We next compute the triple intersection numbers. The threefold X_3 has a natural basis of divisors S_i given by

$$S_0 = E_0 - E_1, \quad S_1 = E_1 - E_2, \quad S_2 = E_2 - E_3, \quad S_3 = E_3. \tag{B.39}$$

The divisor class $[W_3]$ of $W_3 = 0$ is

$$[W_3] = 3H - 6K_B - 2E_1 - E_2 - E_3. \tag{B.40}$$

The triple intersection numbers and above degrees together imply

$$S_0 = \mathbb{F}_{k-2}, \quad S_1 = \mathbb{F}_{k-4}, \quad S_2 = \mathbb{F}_{k-4}^{2(2-k)}, \quad S_3 = \mathbb{F}_{k-6}^{4(4-k)}. \tag{B.41}$$

One very interesting property of the I_4^* model in the case $k = 1$ is the existence of unusual intersection patterns, i.e. intersections which do not fit into the structure of a Dynkin diagram, due to the presence of localized matter in the model. In particular, we find that S_1 and S_2 intersect non-trivially. Notice that the irreducible components of the elliptic fiber only intersect over specific points in B :

$$F_1 \cap F_2 : a_1 = 0. \tag{B.42}$$

Notice that a_1 vanishes at $(C \cdot [a_1])_B = -(K_B \cdot C)_B = 2 - k$ points along $C \subset B$.

B.3 SO(10) theories

The split I_1^* model engineering gauge group $\text{SO}(10)$ is defined by the following Weierstrass equation:

$$W_0 = y^2 + e_0^2 a_3 y + e_0 a_1 x y - (x^3 + e_0 a_2 x^2 + e_0^3 a_4 x + e_0^5 a_6) = 0. \tag{B.43}$$

We study the resolution

$$(x, y, e_0|e_1), \quad (y, e_1|e_2), \quad (x, e_2|e_3), \quad (y, e_3|e_4), \quad (e_2, e_3|e_5). \tag{B.44}$$

The ambient spaces of the elliptic fibers are described by the projective coordinates $[x : y : z = 1]$. The singular locus of $W_0 = \partial_i W_0 = 0$ as in previous cases is given by

$$x = y = e_0. \tag{B.45}$$

Since we have already explained the procedure for computing resolutions in previous examples, we skip the details of the resolution and simply record the final result. The smooth elliptic threefold X_5 is a hypersurface

$$W_5 = e_2 e_4 y^2 + a_1 e_1 e_2 e_3 e_4 e_5 e_0 x y + a_3 e_1 e_2 e_0^2 y - (e_1 e_3^2 e_4 e_5 x^3 + a_2 e_1 e_3 e_0 x^2 + a_4 e_1^2 e_2 e_3 e_5 e_0^3 x + a_6 e_1^3 e_2^2 e_3 e_5^2 e_0^5) = 0 \quad (\text{B.46})$$

in the ambient projective bundle Y_5 with homogeneous coordinates (including their scaling symmetries)

$$[e_1 e_2 e_3^2 e_4^2 e_5^3 x : e_1 e_2^2 e_3^2 e_4^3 e_5^4 y : z = 1][e_3 e_4 e_5 \lambda_1 x : e_2 e_3 e_4^2 e_5^2 \lambda_1 y : \lambda_1 s] \left[e_4 \lambda_1 \lambda_2 y : \frac{e_1 \lambda_2}{\lambda_1} \right] \left[\lambda_1 \lambda_3 x : \frac{e_2 e_5 \lambda_3}{\lambda_2} \right] \left[\lambda_1 \lambda_2 \lambda_4 y : \frac{e_3 e_5 \lambda_4}{\lambda_3} \right] \left[\frac{e_2 \lambda_3 \lambda_5}{\lambda_2} : \frac{e_3 \lambda_4 \lambda_5}{\lambda_3} \right]. \quad (\text{B.47})$$

One can verify by explicit computation that $W_5 = \partial_i W_5 = 0$ and hence X_5 is smooth.

To determine the geometry of the divisors, as in previous examples, we must compute the degrees of the projective bundles S_i . Our starting point is again the irreducible components F_i , whose explicit parametrizations we will use to retrieve the degrees of their projective bundles. We note the following divisor classes:

$$\begin{aligned} [x] &= 2L - E_1 - E_3, & [y] &= 3L - E_1 - E_2 - E_4, & [e_0] &= E_0 - E_1 \\ [e_1] &= E_1 - E_2, & [e_2] &= E_2 - E_3 - E_5, & [e_3] &= E_3 - E_4 - E_5 \\ [e_4] &= E_4, & [e_5] &= E_5. \end{aligned} \quad (\text{B.48})$$

After fixing the various projective scaling symmetries, we find the irreducible components of the elliptic fiber admit the following parametrizations in X_5 :

$$\begin{aligned} F_1 &: [e_5 : e_0 / (x e_3 e_4)] \\ F_2 &: [(y e_4 / e_0) : e_0 e_1] \\ F_3 &: [(x / e_0) : e_0 e_1 e_2 e_5] \\ F_4 &: [(x / e_0^2 e_1) : e_5] \\ F_5 &: [y e_4 : s^2 e_1]. \end{aligned} \quad (\text{B.49})$$

The projective bundles $\mathbb{P}_C[\mathcal{O} \oplus \mathcal{L}_i]$ are defined in terms of the following line bundles L_i :

$$\begin{aligned} L_0 &= K_B \\ L_1 &= [x e_3 e_4 e_5 / e_0] = E_0 + 2K_B \\ L_2 &= [y e_4 / e_0^2 e_1] = 2E_0 + 3K_B \\ L_3 &= [x / e_0^2 e_1 e_2 e_5] = 2K_B + 2E_0 \\ L_4 &= [x / e_0^2 e_1 e_5] = 2K_B + 2E_0 \\ L_5 &= [y e_4 / s^2 e_1] = 2E_0 + 3K_B. \end{aligned} \quad (\text{B.50})$$

It follows that the degrees $n_i = (\pi_* \circ f_*(L_i) \cdot C)_B$ are

$$n_0 = k - 2, \quad n_1 = k - 4, \quad n_2 = k - 6, \quad n_3 = 4, \quad n_4 = 4, \quad n_5 = k - 6. \quad (\text{B.51})$$

Next, we compute the triple intersection numbers. The threefold X_5 has a natural basis of fibral divisors given by

$$\begin{aligned} S_0 &= E_0 - E_1, & S_1 &= E_1 - E_2, & S_2 &= -E_1 + 2E_2 - E_3 - E_5 \\ S_3 &= E_3 - E_4 - E_5, & S_4 &= E_4, & S_5 &= E_5. \end{aligned} \quad (\text{B.52})$$

The divisor class of $W_5 = 0$ is given by

$$[W_5] = 3H - 6K_B - 2E_1 - E_2 - E_3 - E_4 - E_5. \quad (\text{B.53})$$

The triple intersection numbers are summarized in (4.53). The degrees and triple intersections together imply that the divisors have the following geometry:

$$S_0 = \mathbb{F}_{k-2}, \quad S_1 = \mathbb{F}_{k-4}, \quad S_2 = \mathbb{F}_{k-6}, \quad S_3 = \mathbb{F}_4, \quad S_4 = \mathbb{F}_4^{16-3k}, \quad S_5 = \mathbb{F}_{6-k}^{4-k}. \quad (\text{B.54})$$

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