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Mink₄ × S² solutions of 10 and 11 dimensional supergravity

Andrea Legramandi^a and Niall T. Macpherson^b

^a*Dipartimento di Fisica, Università di Milano-Bicocca and INFN — Sezione di Milano-Bicocca, Piazza della Scienza 3, I-20126 Milano, Italy*

^b*SISSA International School for Advanced Studies and INFN — Sezione di Trieste, Via Bonomea 265, 34136 Trieste*

E-mail: a.legramandi@campus.unimib.it, nmacpher@sissa.it

ABSTRACT: We complete the classification of Mink₄ solutions preserving $\mathcal{N} = 2$ supersymmetry and SU(2) R-symmetry parameterised by a round S² factor. We consider eleven-dimensional supergravity and relax the assumptions of earlier works in type II theories. We show that, using chains of dualities, all solutions of this type can be generated from one of two master classes: an SU(2)-structure in M-theory and a conformal Calabi-Yau in type IIB. Finally, using our results, we recover AdS₅ × S² solutions in M-theory and construct a compact Minkowski solution with Atiyah-Hitchin singularity.

KEYWORDS: Flux compactifications, M-Theory, Superstring Vacua, p-branes

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1 Introduction

Some of the most physically relevant solutions in supergravity are those exhibiting a warped Minkowski factor. From early on, the main reason for this is that Minkowski vacua of string and M-theory are required to make contact with known particle physics phenomena in four dimensions, where one should arrange for the co-dimensions to be compact. Another reason, which clearly gained considerable traction with the advent of the AdS/CFT correspondence, is that all AdS solutions admit a description in terms of a foliation of Minkowski over a non compact interval — namely the Poincaré patch.

The most simple way to realise a Minkowski vacua from ten or eleven dimensions is to assume that the compact internal space accommodates some holonomy group, specifically $SU(3)$ or G_2 for compactifications of string theory or M-theory down to four dimensions respectively. Such solutions preserve (at least) $\mathcal{N} = 2$ supersymmetry and have been well studied in the literature [1–5], with a resurgence of interest in the G_2 case in recent years [6–9] (see also [10–13] for G_2 arising in a heterotic context). However, such manifolds support neither fluxes nor a warping of the Minkowski directions, so it is reasonable to expect that they represent a rather small region of the space of possible solutions. The inclusion of fluxes requires one to generalise the notion of holonomy group to structure group (or G-structure) [14–17]. It is well known that no compact regular solutions of this type exist [18–20] — indeed a necessary element of such constructions are localised singularities, namely O-planes and their generalisations through string dualities, or lifts to M-theory (e.g. Atiyah-Hitchin singularities). Most progress constructing compact solutions with fluxes has happened within another restrictive ansatz, with the internal space assumed to be conformally a holonomy manifold [21–23], allowing one to broadly use the same mathematical tools as before. At this point all $Mink_d$ solutions for $d > 1$ (with the exception of $Mink_2$ in eleven dimensions) have been classified (see [24–28] for eleven dimensions, [29–36] for ten dimensions), however finding solutions beyond the ansatz of warped holonomy has proved challenging — see [38, 39] for success in this direction. The issue appears to be one of tractability, so it would be helpful to have some additional guiding principle.

AdS solutions necessitate the inclusion of fluxes, early examples were also constructed by studying warped holonomy manifolds (albeit now non compact) using the Freund-Rubin ansatz [40]. By now the state of affairs for AdS solutions beyond this class is rather more developed than for Minkowski. Many AdS solutions preserving 16 supercharges or more, when they exist, are either completely known (see the cases of AdS_7 [41], AdS_6 [42] in IIA/M-theory $AdS_5 \times S^2$ [43] in IIB¹), or known locally up to solving comparatively simple partial differential equations (see the cases of AdS_6 [44, 45] and AdS_4 [45] in IIB and $AdS_5 \times S^2$ in M-theory/IIA² [46, 48]). The exceptions are $AdS_{2/3}$ (see [50–52] for certain ansätze³), AdS_4 in IIA/M-theory, and AdS_5 with the requisite R-symmetry not realised by a round S^2 — but these will not be our focus here. Many cases with less supersymmetry

¹There is only the \mathbb{Z}_k orbifold of $AdS_5 \times S^5$.

²See [47] for some explicit examples in IIA.

³Recently there was also [37] which makes no assumption beyond the existence of a time-like Killing vector.

are also well studied, in particular, with the recent addition of [52], all AdS_d solutions with $d > 2$ and at least minimal supersymmetry have been classified (see [24, 28, 30, 52–58], and the $\text{AdS}_{6,7}$ classifications above⁴) — broadly speaking these classes have been more successfully used to find solutions beyond restrictive ansätze than their Minkowski counter-parts. The AdS/CFT obviously motivated these classifications, but in addition to the extra impetus this provided, these classifications benefit from additional symmetry with respect to the Minkowski cases. One issue however, is most classifications for AdS, while quite detailed, assume global AdS factors from the start, so are not particularly useful for studying certain non conformal behaviors such as RG flows. For this reason it would be useful to embed the AdS classes into a more general set up.⁵

Here and in the earlier works of [48, 49], the philosophy is to learn from the successes of the AdS classifications and perform a Mink_4 classification that assumes some additional symmetry so as to enable a more detailed description than previous efforts. A good starting point for this is to classify $\mathcal{N} = 2$ that preserve an $\text{SU}(2)$ R-symmetry in the form of a round S^2 factor (see also [36] where $\text{Mink}_3 \times S^3$ are classified) — one can then try to break some of this (super)symmetry and generate many more solutions in the spirit of [62]. An $\text{SU}(2)$ R-symmetry is a necessary part of the super-conformal algebra in $d = 5, 6$ and $d = 4$ with $\mathcal{N} = 2$. As such a corollary to this classification endeavor is that it provides an embedding of the known half-BPS AdS_d solutions (with $d > 4$) into a broader context still preserving $\text{SU}(2)_R$. In [48, 49] such classifications were performed in types IIA and IIB respectively, under a certain simplifying assumptions. The purpose of this work is to complete this program by classifying solutions in eleven-dimensional supergravity and relaxing the assumptions made in the earlier works.

The lay out of the paper is as follows: in section 2 we classify $\mathcal{N} = 2$ Mink_4 solutions in M-theory that realise an $\text{SU}(2)$ R-symmetry in terms of a round S^2 factor in there internal space, which leaves a five-manifold to be determined by geometric supersymmetry constraints. We find that there are two classes of solution, we refer to as A and B. Class A is governed by an $\text{SU}(2)$ -structure in five dimensions while Class B is the M5-branes with $\text{SO}(3)$ rotational invariance in it’s co-dimensions. The physical fields of case B also solve the supersymmetry conditions of case A, however the Killing spinor and G-structure vielbein are different, and this difference can become physical upon reduction to IIA. Additionally we make some simple ansätze for Case A, specifically a conformal CY2 ansatz and one where the internal space contains an additional squashed S^3 .

In section 3 we turn our attention to $\text{Mink}_4 \times S^2$ in ten dimensions. We drop the assumption of equal Majorana-Weyl spinor norm made in [48, 49] making the classification of type II completely general. We find that all solutions that lie outside the existing classifications always have a $\text{U}(1) \times \text{U}(1)$ flavour symmetry and can be generated from a “parent” system in M-theory (the case B M5-brane) via chains of dualities.

Section 4 elucidates the connection between the different classes of solution contained in [48, 49]. We are able to show that all such $\text{Mink}_4 \times S^2$ solutions in type II can be

⁴We don’t speak of $d > 7$ because such AdS solutions break SUSY, however see [59] for examples of AdSs.

⁵See [60] and [61] for some recent progress in this direction using exceptional field theory and consistent truncation respectively.

generated from two master systems using chains of dualities — a conformal Calabi-Yau system in IIB and the SU(2)-structure in in M-theory (Case A).

Finally in section 5 we close with some simple examples contained within M-theory case A. Specifically we establish how the $\mathcal{N} = 2$ AdS₅ class in M-theory is embedded in the SU(2)-structure of case A and we point the way towards some compact Mink₄ solutions with fluxes and Atiyah-Hitchin singularities.

2 Mink₄ × S² in M-theory

In this section we will classify $\mathcal{N} = 2$ supersymmetric warped Mink₄ solutions in eleven dimensions realising an SU(2) R-symmetry with a round S².

2.1 The spinor ansatz

We are interested in supersymmetric solutions to eleven-dimensional supergravity with a warped four-dimensional Minkowski factor. As such we decompose the metric as

$$ds^2 = e^{2\Delta} ds^2(\mathbb{R}_{1,3}) + ds^2(M_7), \tag{2.1}$$

where $e^{2\Delta}$ is a function with support on M_7 only and the four-form flux G is necessarily purely magnetic. As we seek $\mathcal{N} = 2$ solutions respecting the warped product $\mathbb{R}_{1,3} \times M_7$, our eleven-dimensional spinor will decompose as

$$\epsilon = \sum_{a=1}^2 \left(\zeta_+^a \otimes \chi^a + (\zeta_+^a)^c \otimes (\chi^a)^c \right) \tag{2.2}$$

where ζ_+^a is a doublet of positive chirality Mink₄ spinors, χ^a a doublet of seven-dimensional spinors such that

$$\|\chi^a\|^2 = e^\Delta, \quad \bar{\chi}^a \chi^a = 0, \tag{2.3}$$

and c denotes Majorana conjugation. It is argued in [27], that it is possible to be slightly more general than this, however we show in appendix A that (2.2) is sufficient for our considerations. We will also assume that our spinors χ^a are charged under an SU(2) R-symmetry, realised by a round S² factor in the internal geometry so that the metric and flux on M_7 further decompose as

$$ds^2(M_7) = e^{2C} ds^2(S^2) + ds^2(M_5), \quad G = F_4 + e^{2C} \text{Vol}(S^2) \wedge F_2, \tag{2.4}$$

i.e., as a foliation of S² over M₅, where e^{2C} is a function depending on the coordinates on the five-dimensional manifold. At the level of the spinors the SU(2) R-symmetry will be realised by decomposing our doublet of spinors as

$$\chi^a = \xi^a \otimes \eta^1 + \hat{\xi}^a \otimes \eta^2, \tag{2.5}$$

where ξ^a and $\hat{\xi}^a$ are two SU(2) doublets formed from the Killing spinors on S², ξ , which obeys

$$\nabla_\mu \xi = \frac{i}{2} \sigma_\mu \xi, \tag{2.6}$$

where we used the Pauli matrices to represent the Clifford algebra on S^2 . As established in [48], the doublets take the form

$$\xi^a = \begin{pmatrix} \xi \\ \xi^c \end{pmatrix}, \quad \hat{\xi}^a = \begin{pmatrix} \sigma_3 \xi \\ \sigma_3 \xi^c \end{pmatrix} \quad (2.7)$$

where σ_3 is the 2d chirality matrix and $\xi^c = \sigma_2 \xi^*$ is the Majorana conjugate. Plugging ϵ into the the eleven-dimensional Killing spinor equation we get two independent systems of equations for the seven dimensional spinors χ^1 and χ^2 respectively. However, given the doublet transformation property under the spinorial Lie derivative

$$\mathcal{L}_{K_i} \xi^a = \frac{i}{2} (\sigma^i)^a_b \xi^b, \quad \mathcal{L}_{K_i} \hat{\xi}^a = \frac{i}{2} (\sigma^i)^a_b \hat{\xi}^b, \quad (2.8)$$

where K_i are the Killing vectors of $SU(2)$, one finds that each system is implied by the other through this map, provided a solution respects the $SU(2)$ R-symmetry. As such it is sufficient to solve for just one $\mathcal{N} = 1$ sub-sector governed by for instance χ^1 , and impose that all physical fields of a solution depend on the invariant forms of $SU(2)$, then the second $\mathcal{N} = 1$ sub-sector governed by χ^2 is implied.

In the following section we shall establish a set of geometric conditions for the five-dimensional submanifold M_5 , that are necessary and sufficient for supersymmetry.

2.2 Supersymmetry conditions from seven to five dimensions

The $\mathcal{N} = 1$ supersymmetry conditions for a warped product of $\mathbb{R}_{1,3} \times M_7$ with spinors of the form

$$\epsilon = \zeta_+ \otimes \chi + \text{m.c.} \quad (2.9)$$

were studied in [25, 26], where necessary and sufficient geometric conditions were derived for the preservation of supersymmetry in the particular case $\bar{\chi}\chi = 0$. In the convention of [28] these are

$$d(e^{2\Delta} K) = 0, \quad (2.10a)$$

$$d(e^{4\Delta} J) = -e^{4\Delta} \star_7 G, \quad (2.10b)$$

$$d(e^{3\Delta} \Omega) = 0, \quad (2.10c)$$

$$d(e^{2\Delta} J \wedge J) = -2e^{2\Delta} G \wedge K \quad (2.10d)$$

where

$$e^\Delta = \chi^\dagger \chi, \quad e^\Delta K_a = \chi^\dagger \gamma_a \chi, \quad e^\Delta J_{ab} = -i \chi^\dagger \gamma_{ab} \chi, \quad e^\Delta \Omega_{abc} = -i \chi^{c\dagger} \gamma_{abc} \chi, \quad (2.11)$$

define an $SU(3)$ -structure on the 7 dimensional internal space. In fact these geometric conditions combined with Bianchi identity of G_4 are necessary and sufficient conditions for a solution with $\mathcal{N} = 1$ supersymmetry to exist [63].

Since we seek a solution respecting the $SU(2)$ isometry of S^2 , we can extract conditions on M_5 from (2.10a)–(2.10d) by imposing (2.4) and that our seven-dimensional spinor takes the form

$$\chi = e^{\frac{\Delta}{2}} (\xi \otimes \eta^1 + \sigma_3 \xi \otimes \eta^2), \quad (2.12)$$

where η^i and Δ are independent of the S^2 directions.

The first condition we encounter comes from the left hand expression in (2.3), which given (2.12) yields

$$\chi^\dagger \chi = e^\Delta (\eta_1^\dagger \eta_1 + \eta_2^\dagger \eta_2) + y_3 e^\Delta (\eta_1^\dagger \eta_2 + \eta_2^\dagger \eta_1). \quad (2.13)$$

This is only consistent with $e^\Delta = \chi^\dagger \chi$ being an SU(2) singlet if

$$\|\eta_1\|^2 + \|\eta_2\|^2 = 1, \quad \text{Re}(\eta_1^\dagger \eta_2) = 0. \quad (2.14)$$

For similar reasons the right hand expression of (2.3) imposes

$$\bar{\eta}_1 \eta_2 = 0. \quad (2.15)$$

Let's see how the seven dimensional bi-linears decompose into products of two- and five-dimensional ones. To do this we use the following gamma-matrix representation

$$\gamma_i^{(7)} = e^C \sigma_a \otimes \mathbb{I}_4, \quad \gamma_a^{(7)} = \sigma_3 \otimes \sigma_a, \quad B_7 = \sigma_2 \otimes \sigma_1 \otimes \sigma_2. \quad (2.16)$$

We also decompose the five-dimensional spinors in a common basis in terms of a unit norm spinor η as

$$\eta_1 = q_1 \eta, \quad \eta_2 = q_2 \left(i \cos \alpha \eta + \frac{1}{2} \sin \alpha \bar{w} \eta \right), \quad q_1^2 + q_2^2 = 1, \quad (2.17)$$

which is the most general parametrisation consistent with (2.14), (2.15). To calculate the forms in (2.11) we will make repeated use of the bilinear product identity

$$[\xi^1 \otimes \eta^1] \otimes [\xi^2 \otimes \eta^2]^\dagger = (\eta^1 \otimes \eta^{2\dagger})_+ \wedge (\xi^1 \otimes \xi^{2\dagger}) + (\eta^1 \otimes \eta^{2\dagger})_- \wedge (\sigma_3 \xi^1 \otimes \xi^{2\dagger}) \quad (2.18)$$

where \pm denotes the even/odd degree components of a form only, while the presence of σ_3 depends on our parametrisation of the gamma matrices (2.16). The bi-linears that follow from η are given in [58] and read:

$$\begin{aligned} \eta \otimes \eta^\dagger &= \frac{1}{4} (1 + v) \wedge e^{-ij2}, & \eta \otimes \eta^{c\dagger} &= \frac{1}{4} (1 + v) \wedge \omega_2, \\ \omega_2 &= w \wedge u, & j_2 &= \frac{i}{2} (w \wedge \bar{w} + u \wedge \bar{u}), \end{aligned} \quad (2.19)$$

where

$$v, w_1 = \text{Re}w, w_2 = \text{Im}w, u_1 = \text{Re}u, u_2 = \text{Im}u \quad (2.20)$$

defines a vielbein in five dimensions. And finally the bi-linears that follow from ξ are [48],

$$\begin{aligned} \xi \otimes \xi^\dagger &= \frac{1}{2} (1 + k_3 - iy_3 \text{Vol}(S^3)), & \xi \otimes \xi^{c\dagger} &= -\frac{1}{2} (k_1 + ik_2 - i(y_1 + iy_2) \text{Vol}(S^2)), \\ \sigma_3 \xi \otimes \xi^\dagger &= \frac{1}{2} (y_3 + idy_3 - i \text{Vol}(S^3)), & \sigma_3 \xi \otimes \xi^{c\dagger} &= -\frac{1}{2} (y_1 + iy_2 - id(y_1 + iy_2)), \end{aligned} \quad (2.21)$$

where y_i are coordinates embedding S^2 into \mathbb{R}^3 and k_i are one forms dual to the Killing vectors of SU(2) which may be parameterised as

$$k_i = \epsilon_{ijk} y_j dy_k. \quad (2.22)$$

The first thing we calculate is the one form

$$K = -q_1 q_2 e^C \cos \alpha dy_3 + q_1 q_2 \sin \alpha w_1 + q_2^2 y_3 \cos \alpha (\cos \alpha v - \sin \alpha w_2) + \frac{1}{2}(q_1^2 - q_2^2)(k_3 + y_3 u_2). \quad (2.23)$$

The appearance of k_3 here is opportune because $dk_i = 2y_i \text{Vol}(S^2)$ which means that the only way to make K consistent with (2.10a) is to set the coefficient of k_3 to zero — thus we can without loss of generality take

$$q_1 = q_2 = \frac{1}{\sqrt{2}}, \quad (2.24)$$

and then rotate the five-dimensional frame such that (2.23) becomes simply

$$K = -e^C \cos \alpha dy_3 + \sin \alpha w_1 + y_3 \cos \alpha v. \quad (2.25)$$

In this frame the other forms become

$$\begin{aligned} J &= e^C (\sin \alpha dy_3 \wedge w_2 - v \wedge k_3) + y_3 (\sin \alpha w_2 \wedge V + e^{2C} \text{Vol}(S^2)) \\ &\quad + \cos \alpha w_1 \wedge w_2 + u_1 \wedge u_2, \\ \Omega &= e^C u \wedge (i \sin \alpha v \wedge (k_1 + i k_2) - (\cos \alpha w_1 + i w_2) \wedge (y_1 + i dy_2)), \\ \frac{1}{2} J \wedge J &= e^{2C} (\sin \alpha w_2 \wedge v + y_3 (\cos \alpha w_1 \wedge w_2 + u_1 \wedge u_2) \wedge \text{Vol}(S^2)) \\ &\quad - e^C (w_2 \wedge u_1 \wedge u_2 \wedge dy_3 + (\cos \alpha w_1 \wedge w_2 + u_1 \wedge u_2) \wedge v \wedge k_3) \\ &\quad + \cos \alpha w_1 \wedge w_2 \wedge u_1 \wedge u_2 + y_3 \sin \alpha w_2 \wedge u_1 \wedge u_2 \wedge v \end{aligned} \quad (2.26)$$

where in simplifying the last of these we make use of the identity

$$k_{(i)} \wedge dy_{(i)} + y_{(i)}^2 \text{Vol}(S^2) = \text{Vol}(S^2). \quad (2.27)$$

We now plug (2.26) into (2.10a)–(2.10d) and factor out all dependence on S^2 . The result of this operation is the following set of five-dimensional form constraints

$$\sin \alpha d(e^{2\Delta} w) = \sin \alpha d(e^{-\Delta} u) = d(e^{2\Delta+C}) + e^{2\Delta} v = d\alpha = 0, \quad (2.28a)$$

$$d(e^\Delta (\cos \alpha w_1 + i w_2) \wedge u) = F_4 = 0, \quad (2.28b)$$

$$\sin \alpha (d(e^{2\Delta+2C} w_2 \wedge v) + e^{2\Delta+2C} w_1 \wedge F_2) = 0, \quad (2.28c)$$

$$d(e^{-2\Delta} (\cos \alpha w_1 \wedge w_2 + u_1 \wedge u_2) + e^{-2\Delta} \cos \alpha F_2 \wedge v) = 0, \quad (2.28d)$$

$$d(e^{4\Delta} (\cos \alpha w_1 \wedge w_2 + u_1 \wedge u_2)) = e^{4\Delta} \star_5 F_2. \quad (2.28e)$$

Clearly the behavior is quite different depending on whether or not $\alpha = 0$, so one should look at these cases separately, which we now proceed to do in the next section. Note

that solving these conditions and the Bianchi identity for the flux, implies the rest of the equations of motion [34].

Before we move on let briefly examine the form of the SU(3)-structure in the internal six dimensions orthogonal to K . Any SU(3)-structure can be expressed in canonical form in terms of a complex vielbein E^i as

$$J = \frac{i}{2}(E^1 \wedge \bar{E}^1 + E^2 \wedge \bar{E}^2 + E^3 \wedge \bar{E}^3),$$

$$\Omega = E^1 \wedge E^2 \wedge E^3. \tag{2.29}$$

With a little effort, one can reverse engineer a complex vielbein by manipulating (2.26), we find

$$E^1 = \cos \alpha w_1 + i w_2 + \sin \alpha (e^C dy_3 - y_3 v), \quad E^2 = u, \quad E^3 = (y_1 + i y_2) v - e^C d(y_1 + i y_2),$$

where it is easy to confirm that this does indeed lead to a factorised metric $M_7 = S^2 \times M_5$

$$K^2 + E^i \bar{E}^i = e^{2C} \text{Vol}(S^2) + v^2 + w_1^2 + w_2^2 + u_1^2 + u_2^2. \tag{2.30}$$

We shall now proceed to classify the solutions that follow from the five-dimensional supersymmetry conditions, (2.28a)–(2.28e) — there are two cases contained in sections 2.3 and 2.4.

2.3 Case A: $\alpha = 0$, SU(2)-structure

In this case η_1 and η_2 are proportional to one another, specifically $\eta_1 = i\eta_2$, and so they define a SU(2)-structure on M_5 ,

$$\omega_2 = u \wedge w, \quad j = \frac{i}{2}(u \wedge \bar{u} + w \wedge \bar{w}). \tag{2.31}$$

This means that we cannot find local expression for the vielbein u, w without any further assumptions. The purely geometric supersymmetry conditions are simply

$$d(e^{2\Delta+C}) + e^{2\Delta} v = 0, \tag{2.32}$$

$$d(e^\Delta \omega_2) = 0, \tag{2.33}$$

while those involving the flux read

$$d(e^{4\Delta} j_2) - e^{4\Delta} \star_5 F_2 = d(e^{-2\Delta} j_2) + e^{-2\Delta} F_2 \wedge v = 0. \tag{2.34}$$

We have a solution whenever we solve (2.32), (2.32), (2.34) and the Bianchi identity of G is satisfied, which requires

$$d(e^{2C} F_2) = 0, \tag{2.35}$$

away from localised sources.⁶ We can solve (2.33) without loss of generality by introducing a local coordinate ρ such that

$$\rho = e^{2\Delta+C}, \quad v = -e^{-2\Delta} d\rho. \tag{2.36}$$

⁶The inclusion of such sources would put delta-function sources on the left hand side of the equality.

Solutions in this class then take the form

$$\begin{aligned}
 ds^2 &= e^{2\Delta} ds^2(\text{Mink}_4) + e^{-4\Delta} (d\rho^2 + \rho^2 ds^2(S^2)) + ds^2(M_4), \\
 G &= e^{-8\Delta} \rho^2 \text{Vol}(S^2) \wedge \star_5 d(e^{4\Delta} j_2)
 \end{aligned}
 \tag{2.37}$$

where M_4 supports the $SU(2)$ -structure, that given (2.33) is complex.

In the next subsections we shall make some assumptions about the form of the $SU(2)$ -structure to obtain more detailed classes.

2.3.1 Simple ansatz: warped $SU(2)$ -holonomy

The easiest way to solve (2.33)–(2.34) is to impose that the part of the metric orthogonal to v has warped $SU(2)$ -holonomy, i.e.

$$e^\Delta \omega_2 = \tilde{\omega}_2 = e^\Delta j_2 = \tilde{j}_2, \tag{2.38}$$

with $\tilde{\omega}_2, \tilde{j}_2$ closed. The local form of the solution is then

$$\begin{aligned}
 ds^2 &= H^{-2/3} ds^2(\mathbb{R}_{1,3}) + H^{1/3} \left(ds^2(M_4) + H (d\rho^2 + \rho^2 ds^2(S^2)) \right), \\
 e^{2C} F_2 &= c \tilde{j}_2, \quad H = 1 + \frac{c}{\rho}.
 \end{aligned}
 \tag{2.39}$$

with M_4 any $SU(2)$ -manifold. These solutions are all non compact and the warp factor is not indicative of a simple brane set up. Following [72], this could be a KK6-M2 system or something more exotic like the lift of a O6-D2 system.

2.3.2 Squashed S^3 ansatz

Let's consider the simple case where the $SU(2)$ -structure is defined on a squashed S^3 trivially fibered over an interval y . Here we take the vielbein to be

$$v = -e^{-2\Delta} d\rho, \quad w = \frac{e^{C_1}}{2} (\omega_1 + i\omega_2), \quad u = \frac{e^{C_2}}{2} \omega_3 + i e^{K-2\Delta+h(y)} dy, \tag{2.40}$$

where C_1, C_2, K, Δ are warp factors which depend on $\rho = e^{2\Delta+C}$ and y only, h is an arbitrary function of y and ω_i are the right-invariant one-form defined on S^3 , which satisfies

$$d\omega_i = -\frac{1}{2} \epsilon_{ijk} \omega^j \wedge \omega^k. \tag{2.41}$$

Notice that this means we are enhancing the R-symmetry to $SU(2) \times U(1)$ as $(\omega_1 + i\omega_2)$ comes with a phase $e^{i\psi}$, where ∂_ψ is the $U(1)$ of the Hopf fibration. From (2.33) we get the following three equations:

$$\partial_\rho e^{C_1+C_2+\Delta} = \partial_\rho e^{C_1+K-\Delta} = 0, \quad \partial_y e^{C_1+C_2+\Delta} = 2e^{C_1+K-\Delta+h}. \tag{2.42}$$

These are solved by defining C_1 and C_2 in terms of two functions $f(y)$ and $g(y)$ such that

$$e^{C_1} = e^{\Delta-K-h} \frac{f}{g}, \quad e^{C_2} = e^{K-2\Delta+h} g, \quad g = \frac{2f}{f'}. \tag{2.43}$$

At this point, because it simplifies later expressions, we choose the arbitrary function h and use diffeomorphism invariance in y to fix f as

$$e^h = f', \quad f = 2\sqrt{y}, \tag{2.44}$$

without loss of generality. This fixes the metric as

$$ds^2 = e^{2\Delta} ds^2(\text{Mink}_4) + e^{-4\Delta} \left(d\rho^2 + \rho^2 ds^2(S^2) + \frac{e^{2K}}{y} dy^2 + 4ye^{2K} (d\psi + \eta) \right) + \frac{e^{2\Delta-2K}}{4} ds^2(\tilde{S}^2) \tag{2.45}$$

where η is a potential for the Kahler form on \tilde{S}^2 . The two conditions involving the flux in (2.34) impose that

$$e^{-6\Delta+2K} = \frac{1}{8} \partial_y (e^{-2K}), \tag{2.46}$$

which allows one to define Δ in terms of K . What remains of (2.34) just defines the flux F_2 , which can be used to express the M-theory 4-form as

$$G = \frac{\rho^2}{4} \left(\partial_\rho (e^{-2K}) \omega_1 \wedge \omega_2 + \partial_{\rho y}^2 (e^{-2K}) \omega_3 \wedge dy - \frac{y}{2} \partial_y^2 (e^{-4K}) \omega_3 \wedge d\rho \right) \wedge \text{Vol}(S^2).$$

All we need to do now is solve the Bianchi identity of the flux, which away from localised sources leads to just one partial differential equation (PDE) in e^K , namely

$$\frac{1}{\rho^2} \partial_\rho (\rho^2 \partial_\rho (e^{-2K})) + \frac{y}{2} \partial_y^2 (e^{-4K}) = 0. \tag{2.47}$$

This bares a striking similarity to the PDE governing the intersecting D8-D6-NS5 brane system of [71] — the only difference is the y factor. We also find deformations of this PDE in section C.2.2 of the appendix. Of course here this is only a formal similarity and the class of solutions currently under investigation is unlikely to have any relation to a Mink₆ class in massive IIA. At any rate, the form of the metric in (2.45), and a relatively simple PDE governing the physical fields, makes this class appear promising for finding compact solutions in M-theory⁷

2.4 Case B: $\alpha \neq 0$, the M5-brane

In this case α is non zero, however one can see that it actually factors out from all of the supersymmetry conditions and in the end nothing physical depends on its specific value, as long as $\sin \alpha \neq 0$ (in which case we fall into the class of the previous section). The 1-form supersymmetry constraints are

$$d(e^{2\Delta+C}) + e^{2\Delta} v = d(e^{2\Delta} w) = d(e^{-\Delta} u) = 0 \tag{2.48}$$

⁷One might also wonder if AdS solutions exist here. We checked that all of the SUSY constraint for the most general AdS₅ solution one can embed in this class can be solved — but the flux always has a leg in the AdS direction breaking the SO(2,4) symmetry; moreover, the Bianchi identity is not satisfied.

which we can be solved without loss of generality by introducing local coordinates x_1, x_2, x_3, x_4 and

$$\rho = e^{2\Delta+C}, \tag{2.49}$$

such that the five-dimensional vielbein becomes

$$v = -e^{-2\Delta}d\rho, \quad w = e^{-2\Delta}(dx_1 + idx_2), \quad u = e^\Delta(dx_3 + idx_4), \quad \rho = e^{2\Delta+C}, \tag{2.50}$$

which span an identity-structure — so we have completely local expressions for solutions in this class. The rest of the supersymmetry conditions, that are not implied by the one-forms, involve the flux, namely

$$d(e^{4\Delta}u_1 \wedge u_2) + e^{4\Delta} \star_5 F_2 = 0, \tag{2.51}$$

$$d(e^{2C+2\Delta}w_2 \wedge v) + e^{2C+2\Delta}w_1 \wedge F_2 = 0, \tag{2.52}$$

$$\cos \alpha (d(e^{-2\Delta}(w_1 \wedge w_2) + e^{-2\Delta}F_2 \wedge v) = 0. \tag{2.53}$$

The definition of F_2 can be read from (2.51) where we can make use of (2.50) to take the Hodge dual:

$$F_2 = -\rho^2 e^{-2C} (\partial_\rho e^{-6\Delta} dx_1 \wedge dx_2 + \partial_{x_2} e^{-6\Delta} d\rho \wedge dx_1 + \partial_{x_1} e^{-6\Delta} dx_2 \wedge d\rho). \tag{2.54}$$

We then plug this back into (2.52) which imposes

$$\partial_{x_3} e^{2\Delta} = \partial_{x_4} e^{2\Delta} = 0, \tag{2.55}$$

so that $\partial_{x_3}, \partial_{x_4}$ are necessarily isometries of the solution — this automatically solves (2.53) without imposing any further restriction on α , then nothing physical depends on it. Thus the local form of all solutions in this class is

$$ds^2 = e^{2\Delta} ds^2(\text{Mink}_6) + e^{-4\Delta} (dx_1^2 + dx_2^2 + d\rho^2 + \rho^2 ds^2(S^2)), \tag{2.56}$$

$$G = -\rho^2 (\partial_\rho e^{-6\Delta} dx_1 \wedge dx_2 + \partial_{x_2} e^{-6\Delta} d\rho \wedge dx_1 + \partial_{x_1} e^{-6\Delta} dx_2 \wedge d\rho) \wedge \text{Vol}(S^2), \tag{2.57}$$

where the Bianchi identity of G imposes that the warp factor obeys

$$\frac{1}{\rho^2} \partial_\rho (\rho^2 \partial_\rho e^{-6\Delta}) + \partial_{x_1}^2 e^{-6\Delta} + \partial_{x_2}^2 e^{-6\Delta} = 0 \tag{2.58}$$

away from localised sources. This is a five-dimensional Laplace equation expressed in coordinates that make the the $SO(3)$ symmetry of it's solutions manifest — thus this entire class of solutions is nothing more than M5-branes with some rotational symmetry in their co-dimensions. It is easy to check that the supersymmetry conditions (2.50)–(2.53) are actually compatible with the supersymmetry conditions (2.32)–(2.34) of case A, which means that the physical solution of Case B can actually be embedded in Case A by taking (2.50) to define the $SU(2)$ -structure as in (2.31).

We shall see in the next section that all solutions in type II which have internal Killing spinors with non equal norm, descend via dimensional reduction and T-duality, from this class.

3 Classes of solution in type II with non equal spinor norm

Supersymmetric type II solutions with $\text{Mink}_4 \times S^2$ factors were partially classified in [48, 49]. Similar to the M-theory classification of section 2, the starting point is a decomposition of the ten-dimensional Majorana-Weyl Killing spinors of the form

$$\epsilon_1 = \zeta_+ \otimes (\xi \otimes \tilde{\eta}^1 + \sigma_3 \xi \otimes \hat{\gamma} \tilde{\eta}^1) + \text{m.c.}, \quad \epsilon_2 = \zeta_+ \otimes (\xi \otimes \tilde{\eta}^2 \pm \sigma_3 \xi \otimes \hat{\gamma} \tilde{\eta}^2) + \text{m.c.}, \quad (3.1)$$

where \pm is taken in IIB/IIA. Once more ξ is a Killing spinor on S^2 obeying (2.6) and now η^i defines a non-chiral spinor in four dimensions. The metric decomposes as

$$ds^2 = e^{2A} ds^2(\text{Mink}_4) + e^{2C} ds^2(S^2) + ds^2(M_4), \quad (3.2)$$

where e^{2A} and e^{2C} depend on the coordinates on M_4 only. However, unlike the M-theory cases, the classifications in ten dimensions [48, 49] are not completely general, as we will now explain. A consequence of supersymmetry is that

$$d(e^{\mp A} (|\eta_1|^2 \pm |\eta_2|^2)) = 0, \quad (3.3)$$

which means we can define constants c_{\pm} such that

$$|\eta_1|^2 \pm |\eta_2|^2 = c_{\pm} e^{\pm A}, \quad (3.4)$$

c_+ can be tuned to any positive constant without loss of generality, but tuning c_- can have marked physical effects — namely the physics of solutions with generic c_- is quite different to the physics when $c_- = 0$, i.e. the case of equal spinor norm. In [48, 49] it is only the equal norm case that is classified, so in this section we shall study the possible solutions in type II for generic c_- , i.e. non equal spinor norm. As we shall see, it will turn out that such solutions always exhibit two uncharged $U(1)$ isometries, so all type IIB solutions are contained in the classification of type IIA up to a T-duality, and further, all type IIA solutions have the Romans mass set to zero — so descend from our M-theory classification.

In the next section we shall study the unequal norm cases in type IIA.

3.1 Non-equal norm in IIA

We begin our analysis in IIA, our goal here is not to give a detailed classification, rather we seek to show that all solutions in this class descend from M-theory. This statement turns out to actually be true for any warped Mink_4 solution — so we will not actually assume an S^2 factor here, merely a decomposition of the metric of the form

$$ds^2 = e^{2A} ds^2(\text{Mink}_4) + ds^2(M_6), \quad (3.5)$$

and likewise for the fluxes, with all physical fields supported by M_6 only.

In the conventions of [64], supersymmetry depends on the existence of a non-chiral IIA spinor on warped $\text{Mink}_4 \times M_6$ that takes the form

$$\epsilon = \zeta_+ \otimes \chi + \zeta_+^c \otimes \chi^c \quad (3.6)$$

where ζ_+ is a positive chirality spinor in four dimensions, χ a non-chiral six-dimensional spinor and the c superscript labels Majorana conjugation⁸ such that $\epsilon^c = \epsilon$. Supersymmetry is then implied by the following spinorial conditions on M_6

$$\left[\partial A + \frac{e^\Phi}{4} (F_0 + F_2 \hat{\gamma}^{(6)} + F_4 + i \star_6 F_6) \right] \chi = 0, \quad (3.7a)$$

$$\left[\partial \Phi + \frac{1}{2} H \hat{\gamma}^{(6)} + \frac{e^\Phi}{4} (5F_0 + 3F_2 \hat{\gamma}^{(6)} + F_4 - i \star_6 F_6) \right] \chi = 0, \quad (3.7b)$$

$$\left[\nabla_a + \frac{1}{4} H_a \hat{\gamma}^{(6)} + \frac{e^\Phi}{8} (F_0 - F_2 \hat{\gamma}^{(6)} + F_4 - i \star_6 F_6) \gamma_a^{(6)} \right] \chi = 0, \quad (3.7c)$$

where Φ is the dilaton, $\hat{\gamma}^{(6)}$ the chiral operator and where in (3.7c) we correct a typo appearing in [64]. Here F_n labels the RR flux magnetically coupled to a space-filling brane

$$F_{RR} = F + e^{4A} \text{Vol}_4 \wedge \star_6 \lambda(F) \quad (3.8)$$

however, as we shall see shortly, the 0, 2, 4 forms will turn out to be purely magnetic so we need not worry about this distinction for long. Using (3.7a) and (3.7c) it is possible to establish that $d(e^{-A} |\chi|^2) = d(e^A \chi^\dagger \hat{\gamma} \chi) = 0$, while (3.7b) implies that $d(e^{2A-\Phi} \bar{\chi} \chi) = 0$, which one can use to fix

$$|\chi|^2 = c_+ e^A, \quad \chi^\dagger \hat{\gamma}^{(6)} \chi = c_- e^{-A}, \quad \bar{\chi} \chi = c e^{-2A+\Phi}, \quad \bar{\chi} \hat{\gamma}^{(6)} \chi = 0, \quad (3.9)$$

where the final condition follows in general whenever $(\hat{\gamma}^{(6)})^T = -\hat{\gamma}^{(6)}$. c_\pm, c are constants and we set $c_+ = 2$ without loss of generality, $c_- = 0$ is the cases of equal internal spinor norms and c parameterises the G-structure of the internal 6 manifold; in particular, $c = 0$ corresponds to an orthogonal SU(2)-structure. Using the relations

$$\begin{aligned} (\chi^\dagger \gamma_{a_1, \dots, a_n}^{(6)} \chi)^* &= (-1)^{\frac{n}{2}(n-1)} \chi^\dagger \gamma_{a_1, \dots, a_n}^{(6)} \chi, \\ (\chi^\dagger \gamma_{a_1, \dots, a_n}^{(6)} \hat{\gamma}^{(6)} \chi)^* &= (-1)^{\frac{n}{2}(n+1)} \chi^\dagger \gamma_{a_1, \dots, a_n}^{(6)} \hat{\gamma}^{(6)} \chi, \end{aligned} \quad (3.10)$$

which actually hold for Hermitian gamma matrices in any dimension, we find that the rank 2,3,6 forms which come from the bi-linear $\chi \otimes \chi^\dagger$ and the rank 1,2,5,6 forms constructed from $\hat{\gamma}^{(6)} \chi \otimes \chi^\dagger$ are purely imaginary, while the rest are real. Using this fact, it is possible to show from the inner-products of (3.7a)–(3.7b) with χ and $\hat{\gamma}^{(6)} \chi$ that

$$\star_6 F_6 = F_0 \chi^\dagger \hat{\gamma}^{(6)} \chi = 0, \quad (3.11)$$

which come from the imaginary and real parts of the inner products respectively. Equal spinor norm is equivalent to $\chi^\dagger \hat{\gamma}^{(6)} \chi = 0$ in this language, so it is clear that a non vanishing Romans mass is only possible for equal spinor norm. As such all type IIA solutions with non equal spinor norm have $F_0 = 0$ so can be lifted to M-theory.

As we shall show later in section 3.3, the $\text{Mink}_4 \times S^2$ non equal norm solutions in IIA necessarily descend from section 2.4 by dimensional reduction. In the next section we turn our attention to type IIB solutions with non equal norm, as we shall see, this requires a more detailed analysis.

⁸One can without loss of generality take $B^{(10)} = \mathbb{I} \otimes B^{(6)}$ where $B^{(6)} B^{(6)*} = \mathbb{I}$ as the relevant intertwiner so that $\zeta_+^c = \zeta_+^*$ and $\chi^c = B^{(6)} \chi^*$.

3.2 Non equal norm in IIB

In this section we study the Killing spinor conditions in IIB. Our goal is to establish that when the spinor norms are not equal, there are always two flavour U(1) isometries — however, most of the results in this section apply irrespective of the particular values of the norms.

In IIB we shall define our ten-dimensional Killing spinors as in 3.1, but redefine the 4d spinors as

$$\tilde{\eta}^1 = \frac{1}{2}(\eta^1 + \eta^2), \quad \tilde{\eta}^2 = \frac{1}{2}(\eta^1 - \eta^2). \quad (3.12)$$

As we establish in appendix B, supersymmetry imposes the following set of four-dimensional spinorial constraints

$$\partial A \eta^1 + \frac{e^\Phi}{4} \left[- (f_1 \hat{\gamma} + i g_3) \eta^2 + (f_3 \hat{\gamma} + i g_1) \eta^1 \right] = 0, \quad (3.13a)$$

$$\partial A \eta^2 - \frac{e^\Phi}{4} \left[- (f_1 \hat{\gamma} + i g_3) \eta^1 + (f_3 \hat{\gamma} + i g_1) \eta^2 \right] = 0, \quad (3.13b)$$

$$\partial \Phi \eta_1 - \frac{1}{2} (H_3 + i H_1 \hat{\gamma}) \eta^2 + e^\Phi \left[- f_1 \hat{\gamma} \eta^2 + \frac{1}{2} (f_3 \hat{\gamma} + i g_1) \eta^1 \right] = 0, \quad (3.13c)$$

$$\partial \Phi \eta_2 - \frac{1}{2} (H_3 + i H_1 \hat{\gamma}) \eta^1 - e^\Phi \left[- f_1 \hat{\gamma} \eta^1 + \frac{1}{2} (f_3 \hat{\gamma} + i g_1) \eta^2 \right] = 0, \quad (3.13d)$$

$$\partial C \eta^1 - i e^{-C} \hat{\gamma} \eta^2 - \frac{i}{2} H_1 \hat{\gamma} \eta^2 + \frac{e^\Phi}{4} \left[- (f_1 \hat{\gamma} - i g_3) \eta^2 + (f_3 \hat{\gamma} - i g_1) \eta^1 \right] = 0, \quad (3.13e)$$

$$\partial C \eta^2 - i e^{-C} \hat{\gamma} \eta^1 - \frac{i}{2} H_1 \hat{\gamma} \eta^1 - \frac{e^\Phi}{4} \left[- (f_1 \hat{\gamma} - i g_3) \eta^1 + (f_3 \hat{\gamma} - i g_1) \eta^2 \right] = 0, \quad (3.13f)$$

$$\nabla_a \eta_1 - \frac{1}{4} \left((H_3)_a + i (H_1)_a \hat{\gamma} \right) \eta^2 + \frac{e^\Phi}{8} \left[- (f_1 \hat{\gamma} - i g_3) \gamma_a \eta_2 + (f_3 \hat{\gamma} - i g_1) \gamma_a \eta_1 \right] = 0, \quad (3.13g)$$

$$\nabla_a \eta_2 - \frac{1}{4} \left((H_3)_a + i (H_1)_a \hat{\gamma} \right) \eta^1 - \frac{e^\Phi}{8} \left[- (f_1 \hat{\gamma} - i g_3) \gamma_a \eta_1 + (f_3 \hat{\gamma} - i g_1) \gamma_a \eta_2 \right] = 0, \quad (3.13h)$$

where f, g, H_1, H_3 are forms related to the NS and RR fluxes defined as in appendix B while Φ, C, A are function on M_4 .

From the six-dimensional zero-form conditions (B), imposing that the physical quantities are SU(2) singlets leads to the following constraints for the four-dimensional spinors:

$$\begin{aligned} |\eta^1|^2 &= |\eta^2|^2 = e^A, & \eta^{1\dagger} \hat{\gamma} \eta^2 + \eta^{2\dagger} \hat{\gamma} \eta^1 &= 0, \\ \eta^{1\dagger} \eta^2 &= \eta^{2\dagger} \eta^1 = c_- e^{-A}, & \eta^{1\dagger} \hat{\gamma} \eta^1 + \eta^{2\dagger} \hat{\gamma} \eta^2 &= 0, \end{aligned} \quad (3.14)$$

Working a bit harder one can establish another scalar condition from the difference of $\overline{\eta^2} (2(3.13a) + 2(3.13e) - (3.13c))$ and $\overline{\eta^1} (2(3.13b) + 2(3.13f) - (3.13d))$,⁹ namely

$$\overline{\eta^2} \hat{\gamma} \eta^1 = 0. \quad (3.15)$$

⁹We define $\bar{\eta} = (\eta^c)^\dagger$ where $\eta^c = B \eta^*$ and the conventions for B are given in appendix B.

One can solve all of the scalar conditions with the following decomposition of η^i in terms of a single spinor η , and complex functions a, b

$$\begin{aligned} e^{-\frac{1}{2}A}\eta^1 &= \sin\left(\frac{\alpha}{2}\right)\eta + \cos\left(\frac{\alpha}{2}\right)(a\hat{\gamma}\eta + b\eta^c), \\ e^{-\frac{1}{2}A}\eta^2 &= \sin\left(\frac{\alpha}{2}\right)\eta - \cos\left(\frac{\alpha}{2}\right)(a\hat{\gamma}\eta + b\eta^c) \end{aligned} \quad (3.16)$$

where

$$|\eta|^2 = 1, \quad \eta^\dagger\hat{\gamma}\eta = 0, \quad |a|^2 + |b|^2 = 1, \quad e^{2A}\cos\alpha = -c_-, \quad (3.17)$$

so that α parametrises the difference in norms of the spinors, which are equal only when $\cos\alpha = 0$.

As shown in [36, 44], the spinor η is sufficient to define an identity-structure on M_4 , which is expressed in terms of the vielbein $\{v_1, v_2, w_1, w_2\}$

$$\eta^\dagger\gamma_a\eta = v_1, \quad \eta^\dagger\gamma_a\hat{\gamma}\eta = iv_2, \quad \bar{\eta}\gamma_a\hat{\gamma}\eta = w = w_1 + iw_2. \quad (3.18)$$

The definition of w is such that $\eta^c = \frac{1}{2}\bar{w}\hat{\gamma}\eta$, which we can use to fix the phase of b in (3.16) by rotating the vector w , so we choose b to be real and expand a as

$$b = |b|, \quad a = a_1 + ia_2. \quad (3.19)$$

Let us now consider the vector

$$k = \bar{\eta}^2\gamma^a\hat{\gamma}\eta^1\partial_a, \quad (3.20)$$

which can be rewritten as a one form in terms of the vielbein (3.18)

$$e^{-A}k = a(1 + \cos\alpha)(bv_1 + (a_1w_1 - a_2w_2)) - (\cos\alpha w_1 - iw_2). \quad (3.21)$$

We will spend the bulk of this section proving that k is a Killing vector parameterising a $U(1) \times U(1)$ flavour symmetry in the solutions. However let us first perform some consistency checks: we notice that for generic values of α both the real and imaginary components of k cannot vanish, so k necessarily defines 2 real and non zero vectors. The exception is when $\cos\alpha = 0$, which is the case of equal spinor norm studied in [49] — here the number of isometries we find depends on the values of a_1, a_2, b . In particular there is the following reduced isometry structure when the functions of the spinor ansatz are tuned as follows

$$0 \text{ isometries: } \quad \cos\alpha = a_1 = b = 0,$$

$$1 \text{ isometries: } \quad \cos\alpha = a_1 = 0, \quad b \neq 0,$$

which can happen because k is respectively zero or purely imaginary in these two cases only — so k is consistent with the isometry structure of the known classes with equal spinor norm, a strictly non vanishing for non equal norm, a promising sign we are on the correct track.

The first thing to establish is that k is an isometry with respect to scalar physical fields. To do so one can take the sum of the inner products of (3.13a), (3.13b) with $\hat{\gamma}\eta^2$,

$\hat{\gamma}\eta^1$ respectively and repeat this operation for (3.13c), (3.13d) and (3.13e), (3.13f). This leads to the conditions

$$k^a \partial_a A = k^a \partial_a \Phi = k^a \partial_a C = 0 \quad (3.22)$$

which means that A, C, Φ do not depend on the thk directions.

Now let's prove that k is Killing with respect to the metric, i.e., that $\nabla_{(a} k_{b)} = 0$. A straightforward calculation leads to:

$$\begin{aligned} \nabla_a k_b &= \frac{1}{4} H_{3abd} (\bar{\eta}^1 \gamma^d \hat{\gamma} \eta^1 + \bar{\eta}^2 \gamma^d \hat{\gamma} \eta^2) + \frac{e^\Phi}{8} (\star f_1 + i g_3)_{abd} (\bar{\eta}^1 \gamma^d \hat{\gamma} \eta^1 - \bar{\eta}^2 \gamma^d \hat{\gamma} \eta^2) \\ &\quad + \frac{e^\Phi}{4} (f_3 + i \star g_1)_{abd} \bar{\eta}^2 \gamma^d \eta^1, \end{aligned}$$

and by the manifest anti-symmetry of the right hand side, it is immediate to notice that k is a Killing vector. This expression can be further simplified, indeed, using the combinations of equations (2(3.13a) + (3.13e) - (3.13c)) and (2(3.13b) + (3.13f) - (3.13d)), which read

$$\partial(2A + C - \Phi)\eta^1 + \frac{1}{2} H_3 \eta^2 - i e^{-C} \hat{\gamma} \eta^2 + \frac{e^\Phi}{4} \left[-(-f_1 \hat{\gamma} + i g_3) \eta^2 + (f_3 \hat{\gamma} - i g_1) \eta^1 \right] = 0, \quad (3.23)$$

$$\partial(2A + C - \Phi)\eta^2 + \frac{1}{2} H_3 \eta^1 - i e^{-C} \hat{\gamma} \eta^1 - \frac{e^\Phi}{4} \left[-(-f_1 \hat{\gamma} + i g_3) \eta^1 + (f_3 \hat{\gamma} - i g_1) \eta^2 \right] = 0, \quad (3.24)$$

one may establish from the sum of $\bar{\eta}^2 \gamma_{ab} \hat{\gamma}$ (3.23) and $\bar{\eta}^1 \gamma_{ab} \hat{\gamma}$ (3.24), that

$$\begin{aligned} &\frac{1}{4} H_{3abd} (\bar{\eta}^1 \gamma^d \hat{\gamma} \eta^1 + \bar{\eta}^2 \gamma^d \hat{\gamma} \eta^2) + \frac{e^\Phi}{8} (\star f_1 + i g_3)_{abd} (\bar{\eta}^1 \gamma^d \hat{\gamma} \eta^1 - \bar{\eta}^2 \gamma^d \hat{\gamma} \eta^2) \\ &\quad + \frac{e^\Phi}{4} (f_3 + i \star g_1)_{abd} \bar{\eta}^2 \gamma^d \eta^1 \\ &= -2\partial_{[a} (2A + C - \Phi) k_{b]} + \frac{i}{2} e^{-C} (\bar{\eta}^1 \gamma_{ab} \eta^1 + \bar{\eta}^2 \gamma_{ab} \eta^2). \end{aligned} \quad (3.25)$$

We therefore arrive at

$$\nabla_a k_b = -2\partial_{[a} (2A + C - \Phi) k_{b]} + \frac{i}{2} e^{-C} (\bar{\eta}^1 \gamma_{ab} \eta^1 + \bar{\eta}^2 \gamma_{ab} \eta^2). \quad (3.26)$$

This expression is particularly useful because allows us to establish that the isometry group of k is $U(1) \times U(1)$. To prove this we need to show that the Lie bracket $[k, \bar{k}] = 0$ which is equivalent to

$$\bar{k}^a \nabla_a k_b \in \mathbb{R}. \quad (3.27)$$

Given our spinor decomposition and the definition of the four-dimensional vielbein in terms of a single spinor η , one can show that

$$(\bar{\eta}^1 \gamma_{ab} \eta^1 + \bar{\eta}^2 \gamma_{ab} \eta^2) = -4i e^A k_{[a} (v_2)_{b]}. \quad (3.28)$$

It is then a simple matter to calculate

$$\bar{k}^a \nabla_a k_b = -|k|^2 (\partial_b (2A - \Phi + C) - e^{A-C} (v_2)_b), \quad (3.29)$$

which is clearly real — so the isometry group of k is indeed $U(1) \times U(1)$.

The next task is to prove that $U(1) \times U(1)$ is a flavour symmetry — which means that the spinors $\eta^{1,2}$ are not charged under these isometries. This statement can be cast as the vanishing of the spinorial Lie derivative along k , i.e.

$$\mathcal{L}_k \eta^i = k^a \nabla_a \eta^i + \frac{1}{4} \nabla_a k_b \gamma^{ab} \eta^i = 0. \quad (3.30)$$

To prove this we will need the following algebraic relations, which can be demonstrated using standard Fierz identity techniques:

$$k\eta^1 = (\overline{\eta^2} \eta^1) \hat{\gamma} \eta^1, \quad k\eta^2 = -(\overline{\eta^2} \eta^1) \hat{\gamma} \eta^2, \quad ie^A v_2 \eta^2 = \hat{\gamma} \eta^1, \quad ie^A v_2 \eta^1 = \hat{\gamma} \eta^2, \quad (3.31)$$

where k and v_2 must now be understood as gamma matrices (via Clifford map). Let us fix $i = 1$ (the discussion for $i = 2$ is specular) and start by evaluating the first term in (3.30). Contracting (3.13g) with k^a and using the first two relations of (3.31) one finds:

$$k^a \left(\nabla_a \eta^1 - \frac{1}{4} ((H_3)_a + i(H_1)_a \hat{\gamma}) \eta^2 \right) + \frac{e^\phi}{8} (\overline{\eta^2} \eta^1) (-(-f_1 \hat{\gamma} + ig_3) \hat{\gamma} \eta^2 + (f_3 \hat{\gamma} - ig_1) \hat{\gamma} \eta^1) = 0. \quad (3.32)$$

Now we can substitute for the RR fluxes using (3.23) multiplied by $\hat{\gamma}$. With some manipulations one realises

$$k^a \left(\nabla_a \eta_1 - \frac{1}{4} (\star H_3 + iH_1)_a \hat{\gamma} \eta^2 \right) - \frac{1}{2} (\overline{\eta^2} \eta^1) (\partial(2A + C - \Phi) \hat{\gamma} \eta^1 + ie^{-C} \eta^2) = 0. \quad (3.33)$$

The term $k^a (\star H_3 + iH_1)_a$ can be evaluated from the inner product of (3.13c) with η^1 and (3.13d) with η^2 , indeed this leads to

$$k^a (\star H_3 + iH_1)_a = k^a (f_1)_a = 0, \quad (3.34)$$

note that this implies that when the norms are not equal, so that k defines two non vanishing Killing vectors in general, H_1 has no legs in the isometry directions, while H_3 is parallel to both. We are left with the expression

$$k^a \nabla_a \eta_1 = \frac{1}{2} (\overline{\eta^2} \eta^1) (\partial(2A + C - \Phi) \hat{\gamma} \eta^1 + ie^{-C} \eta^2) = \frac{1}{2} (\partial(2A + C - \Phi) k \eta^1 - ie^{-C} \hat{\gamma} k \eta^2). \quad (3.35)$$

Now we consider the second term in (3.30): using (3.26) one may write

$$(\nabla_a k_b) \gamma^{ab} \eta^1 = -2(\partial_a(2A + C - \phi) k_b - e^{A-C} k_a (v_2)_b) \gamma^{ab} \eta^1. \quad (3.36)$$

Since k_a is orthogonal to $\partial_a(2A + C - \phi)$ and $(v_2)_a$, we can take $\gamma^{ab} = \gamma^a \gamma^b$ in the previous expression and, using (3.31) we find

$$(\nabla_a k_b) \gamma^{ab} \eta^1 = -2\partial(2A + C - \phi) k \eta^1 + 2ie^{-C} \hat{\gamma} k \eta^2. \quad (3.37)$$

Now one simply combines the contributions from (3.35) and (3.37) to establish that

$$\mathcal{L}_k \eta^1 = 0. \quad (3.38)$$

It is easy to perform the same steps for $\mathcal{L}_k \eta^2$ and we therefore find that $U(1) \times U(1)$ is a flavour symmetry.

The last step is to prove that also the fluxes are not charged under the isometries. Extrapolating from the classification of [30], we know that the RR fluxes are defined in terms of bi-linears of η^i and the NSNS 3-form. Since the spinors are uncharged, the same is true of the bi-linears and so we need only establish that the NSNS 3-form is a $U(1) \times U(1)$ singlet. We begin by considering $\bar{\eta}^1$ (3.13a), $\bar{\eta}^1$ (3.13c) + $\bar{\eta}^2$ (3.13d), $\bar{\eta}^2 \hat{\gamma}$ (3.13a), $\bar{\eta}^2 \hat{\gamma}$ (3.13e); these, together with (3.34), allow us to find the following relations:

$$k^a (H_1)_a = k^a (\star H_3)_a = k^a (f_1)_a = k^a (\star f_3)_a = k^a (g_1)_a = k^a (\star g_3)_a = 0. \quad (3.39)$$

The first, with the Bianchi identity $d(e^{2C} H_1) = 0$ implies $\mathcal{L}_k(e^{2C} H_1) = 0$. Now let's consider the equation of motions for the B-field, this implies the following equation

$$d(e^{4A+2C-2\Phi} \star H_3) = e^{4A+2C} (f_1 \wedge \star f_3 + \star g_3 \wedge g_1). \quad (3.40)$$

Contracting this with k and using (3.39) we get that

$$\iota_k d(\star H_3) + d(\iota_k \star H_3) = \mathcal{L}_k(\star H_3) = 0, \quad (3.41)$$

which proves that H_1 and H_3 are singlets under $U(1) \times U(1)$.

We have now established that all $Mink_4 \times S^2$ solutions in type IIB with non equal spinor norm necessarily have a $U(1) \times U(1)$ flavour symmetry. In the next section we shall show that they all actually follow from IIA via T-duality and can then be lifted to class B in M-theory.

3.3 Relation to the M5-brane

We have proved that all supersymmetric $Mink_4 \times S^2$ solutions with non equal spinor norm have an uncharged $U(1) \times U(1)$ isometry when they are in type IIB and can be lifted to M-theory when they are in IIA. We shall now prove that they all descend from Case B in M-theory — which is the M5-brane. First let us establish that Case B implies the IIA solutions.

Whenever one has a $U(1)$ isometry in M-theory one can reduce down to type IIA along it. When one reduces from eleven dimensions to ten, it is convenient to choose a frame such that

$$e_{11}^a = e^{-\frac{1}{3}\Phi} e_{10}^a, \quad a = 0, 1, \dots, 9, \quad e_{11}^{10} = e^{\frac{2}{3}\Phi} (d\psi + C_1) \quad (3.42)$$

where e_d^a is the vielbein in d space-time dimensions, ∂_ψ is the isometry direction, Φ the dilaton and C_1 the potential of the RR 2-form in IIA. When one reduces, the gamma-matrix corresponding to e_{11}^{10} then becomes the chirality matrix in IIA. The non equal norm condition in IIA, in the conventions of section 3.1, is equivalent to $\chi^\dagger \hat{\gamma}^{(6)} \chi \neq 0$ and, up to an overall warping not relevant for our discussion, the six-dimensional non-chiral spinor in IIA and seven-dimensional spinor in M-theory are the same. The non equal norm condition in M-theory therefore reads $\chi^\dagger \gamma_m^{(7)} \chi (e_{11}^{10})^m = K_m (e_{11}^{10})^m \neq 0$. Therefore we can establish

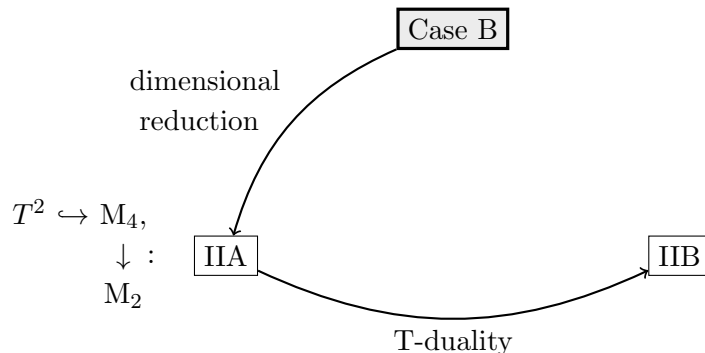


Figure 1. Depiction of the chains of dualities leading to $\text{Mink}_4 \times S^2$ classes in type II with non equal spinor norm. The relative complexity of solutions at each step as one moves from M-theory to IIA to IIB can increase considerably, with specific choices of $U(1)$ the duality is performed on. However all solutions are governed by the M5 Laplacian.

whether or not we reduce to a non-equal-norm class in IIA by studying the M-theory one-form K defined in (2.25). Specifically, we generate non equal norm in IIA whenever K has a leg in the isometry direction we reduce on.

There are two ways to get a round S^2 factor in IIA via dimensional reduction of a parent solution in M-theory. 1) The parent solution has a round S^2 and a $U(1)$ isometry orthogonal to this to reduce on. 2) The parent has a $U(1)$ fibered over S^2 , ie an S^3 that may be squashed along the Hopf fiber direction only. We can immediately rule out option 2) as in this case $K \sim d\psi + \eta + \dots$, with $d\eta = \text{Vol}(S^2)$, which can never solve the one-form condition $d(e^{2\Delta}K) = 0$ — as such IIA non equal norm must descend from Case A or B in M-theory. Here the only place¹⁰ the $U(1)$ can lie that will give rise to non equal norm in IIA is w_1 . As w_1 is only turned on when $\sin \alpha \neq 0$, we can realise all non equal norm solutions in type IIA through reduction from Case B in section 2.4, which is simply the M5-brane class with $SO(3)$ rotational symmetry in it’s co-dimensions, which is rather remarkable. The local form of this solution is given in (2.56), where giving w_1 a leg along the reduction isometry is equivalent to making the local coordinate ∂_{x_1} an isometry without loss of generality.¹¹ Together with the two a priori $U(1)$ ’s in section 2.4 (packaged as part of a Mink_6 factor there) this gives three flavour $U(1)$ isometries, which implies that two will always remain when we reduce to IIA, just as we found for IIB.

We have shown that all non equal norm solutions in IIA descend from Case B in M-theory and all such type II solutions come with a flavour $U(1) \times U(1)$ isometry. Naively,

¹⁰The 1-form K generically has legs along w_1 , dy_3 and v . But dy_3 contains no $U(1)$ and the supersymmetry equation for v in both M-theory classes is incompatible with it having a leg along a(n uncharged) $U(1)$ isometry direction.

¹¹This is not the only way to form a $U(1)$ to reduce on, in general one could take

$$x_1 = \psi + c_1x, \quad x_2 = c_2\psi + x, \quad dc_i = 0, \tag{3.43}$$

and impose that ∂_ψ is an isometry — however in IIA c_i can be turned off with a coordinate transformation and rescaling of g_s . One can also form a $U(1)$ of the metric and flux by expressing x_1, x_2 , in polar coordinates, however this is not an isometry of the G-structure, as (2.25) makes clear. Indeed such an isometry is charged, so reducing on it would brake supersymmetry.

one might then conclude that all the IIB solutions follow from IIA via T-duality — but there is another possibility to rule out. Much like option 2) for the M-theory reduction, round S^2 in IIB could follow by T-dualising on the Hopf fiber of a squashed S^3 . Such a solution in IIB would necessarily have

$$H_1 \sim d\psi \wedge \text{Vol}(S^2), \tag{3.44}$$

but fortunately for us, this was already ruled out for non equal norm in the previous section by the necessary condition $k^a(H_1)_a = 0$.

We have shown that the parent or master class of all non equal norm $\text{Mink}_4 \times S^2$ solutions in type II is an M5-brane with $SO(2) \times SO(3)$ rotational symmetry in it's co-dimensions. All solutions in IIA and IIB are then generated from this by a chain of dualities depicted in figure 1. However, we should stress this does not mean that the type II classes are completely trivial — there are several distinct ways to reduce to type IIA on a $U(1)$ subgroup of the available $U(1) \times U(1) \times U(1)$ uncharged isometry, and yet further ways to T-dualise inside the residual $U(1) \times U(1)$ to get to IIB — the result of this chain of dualities can potentially end up being quite complicated. Rather we would like to stress that if one is interested in constructing a non equal norm $\text{Mink}_4 \times S^2$ solution in type II, this is best done from the M5-brane perspective.

In the next section we shall provide master classes for the remaining $\text{Mink}_4 \times S^2$ solutions in type II, namely those with equal spinor norm.

4 Master classes for $\text{Mink}_4 \times S^2$

In [48, 49], $\text{Mink}_4 \times S^2$ solutions with equal spinor norm were classified. They fall into three distinct classes characterised by the minimum number of uncharged $U(1)$ isometries the internal four-manifold M_4 contain a priori.¹² Specifically one has

$$\begin{array}{lll} \text{Case I: } M_4 = M_2 \times \Sigma_2, & \text{Case II: } S^1 \hookrightarrow M_4, & \text{Case III: } T^2 \hookrightarrow M_4. \\ & \downarrow & \downarrow \\ & M_3 & M_2 \end{array}$$

Class I has no a priori isometries on M_4 , for Class II one generically has a $U(1)$ bundle over a three-manifold, while generic solutions in Class III have a T^2 bundle over a two-manifold — they can be found in sections 4.3, 4.4 and 4.5 of [48, 49] for IIA and IIB respectively. Cases II and III contain one and two constant parameters respectively — when they are set to zero one reduces to classes that have local $\text{Mink}_5 \times S^2$ and $\text{Mink}_6 \times S^2$ factors. More surprising is that all solutions in class III, all in IIB class II, and many in class II IIA, are governed by the same PDE's irrespective of whether or not these parameters are turned on — such generic solutions in these classes can be viewed as parametric deformations of un-fibered $\text{Mink}_4 \times S^1 \times S^2$ and $\text{Mink}_4 \times T^2 \times S^2$ solutions respectively, where the first two factors share a common warping in each instance.¹³

¹²This is in turn related to which inner products of the two independent non-chiral four-dimensional spinors are assumed to be non vanishing, see [48, 49] for details.

¹³When this is true, locally, there is no difference between $\text{Mink}_4 \times T^n$ and Mink_{4+n} . However, for the parametric deformations the warp factors are no longer the same and T^n becomes fibered over a base. As such it makes more sense physically for only the Mink_4 directions to be non compact in these cases.

The existence of parametric deformations raise an obvious question — is some sort of duality at play? Two obvious candidates are formal U-dualities of the type in [67] and T-s-T transformations [68] — which are both solution generating techniques involving chains of string dualities and coordinate transformations (specifically shifts between U(1) isometries) that do not commute with these dualities. As we shall see this is enough to explain case III and case II in IIB, but case II in IIA is more subtle.

In the previous section we established that all solutions in type II supergravity with non equal norm can be obtained from the M5-brane, by reduction and T-duality — so one can view the M5-brane as a master class for all such solutions. In this section, in addition to explaining the origin of the parametric deformations, we establish master classes for equal norm in type II. The first of these is M-theory Case A in section 2.3, the second is class I in IIB — note that after imposing a U(1) isometry in the M-theory class and reducing on it one end up in class I in IIA. We shall quote these solutions in the next sub-sections for convenience. All other cases can be generated from these via certain chains of dualities, as we shall show — the maps between solutions are also summarised in figure 2.

4.1 Master class in type IIB: case I in IIB

The master class in Type IIB has an NS sector that takes the local form

$$\begin{aligned}
 ds^2 &= e^{2A} ds^2(\text{Mink}_4) + e^{-2A} \left(\frac{1}{f} (dx_1^2 + dx_2^2 + x_2^2 ds^2(S^2)) \right) + ds^2(\Sigma_2), \\
 ds^2(\Sigma_2) &= e^{-2A} (dx_3^2 + dx_4^2), \quad B = g \left(\text{Vol}(S^2) + \frac{dx_1 \wedge dx_2}{x_2^2} \right), \quad e^{-\Phi} = f, \quad (4.1)
 \end{aligned}$$

where e^{2A} depends on all four-dimensional local coordinates x_i , and f, g have support on Σ_2 only. The RR sector is generically non trivial, with F_1, F_3, F_5 all turned on, and can be found in [49]. Solving the Bianchi identities of these fluxes (away from localised sources) requires the following PDE's to be solved

$$\square_2 f = 0, \quad \square_2 (fg) = 0, \quad \square_2 = \partial_{x_3}^2 + \partial_{x_4}^2, \quad (4.2a)$$

$$\partial_{x_1}^2 (e^{-4A}) + \frac{1}{x_2^2} \partial_{x_2} (x_2^2 \partial_{x_2} (e^{-4A})) + \square_2 (e^{-4A} f^{-1}) + \frac{1}{x_2^4} \square_2 (fg^2) = 0. \quad (4.2b)$$

The six dimensions orthogonal to Mink_4 in-fact support a conformal Calabi-Yau. Clearly by imposing that one of $\partial_{x_3}, \partial_{x_4}$ is an isometry of all fields and by T-dualising on it we end up with a local $\text{Mink}_5 \times S^2$ class in IIA — this exhausts this class. If we make both isometries (and g constant) then T-dualise on both we exhaust the $\text{Mink}_6 \times S^2$ solutions in IIB, there is just the flat space D5 brane with rotational symmetry in it's condennations. Of course a solution still has a U(1) isometries when g is a polynomial of order one in (x_3, x_4) — when this is the case we can gauge transform such that B respects the isometry, but has a leg in it. This generically leads to the U(1) becoming fibered over the S^2 in the T-dual — thus one can additionally generate classes of solution with Minkowski factors and a squashed 3-sphere in this fashion.

4.2 IIA reduction of master class in M-theory: case I in IIA

Case I in IIA can be generated from M-theory by imposing a $U(1)$ isometry within the $SU(2)$ -structure of the M-theory class in section 2.3 and reducing on it. This class is rather more complicated than its IIB counter-part, with the six-dimensional space orthogonal to $Mink_4$ supporting an orthogonal $SU(2)$ -structure. The local form of the NS sector in this class is

$$\begin{aligned}
 ds^2 &= e^{2A} ds^2(Mink_4) + e^{-4A+2\Phi} \left(dx_2^2 + x_2^2 ds^2(S^2) \right) \\
 &\quad + e^{4A-2\Phi} \left(dx_1 + B_0 e^{-4A+2\Phi} dx_2 \right)^2 + ds^2(\Sigma_2), \\
 ds^2(\Sigma_2) &= e^{-2A} (dx_3^2 + dx_4^2), \quad B = x_2^2 e^{-4A+2\Phi} B_0 \text{Vol}(S^2),
 \end{aligned} \tag{4.3}$$

where e^{2A} , e^Φ , B_0 each generically depends on all of the four-dimensional local coordinates x_i . Supersymmetry additionally requires the following PDE's to be solved

$$\begin{aligned}
 \partial_{x_2} (e^{2A-2\Phi}) &= \partial_{x_1} (e^{-2A} B_0), \\
 \partial_{x_2} (x_2^2 e^{-2A} B_0) &= \partial_{x_1} (x_2^2 e^{-6A+2\Phi} (1 + B_0^2)).
 \end{aligned} \tag{4.4}$$

The RR sector is rather involved, but can be found in [48], however there is no Romans mass. Ensuring that the fluxes obey the correct Bianchi identities imposes another set of PDE's

$$\begin{aligned}
 \partial_{x_3}^2 (e^{2A-2\Phi}) + \partial_{x_4}^2 (e^{2A-2\Phi}) + \partial_{x_1}^2 (e^{-4A}) &= 0, \\
 \partial_{x_3}^2 (e^{-2A} B_0) + \partial_{x_4}^2 (e^{-2A} B_0) + \partial_{x_1} \partial_{x_2} (e^{-4A}) &= 0, \\
 \partial_{x_3}^2 (x_2^2 e^{-6A+2\Phi} (1 + B_0^2)) + \partial_{x_4}^2 (x_2^2 e^{-6A+2\Phi} (1 + B_0^2)) + \partial_{x_2} (x_2^2 \partial_{x_2} (e^{-4A})) &= 0.
 \end{aligned} \tag{4.5}$$

One can generate the $Mink_5 \times S^2$ and $Mink_6 \times S^2$ cases in IIB and IIA respectively by imposing that Σ_2 contains isometry directions, that dB has no leg(s) in, and then T-dualising as before.

4.3 Generating Case II

In this section we shall establish how Case II in IIA and then IIB are realised from Case I in IIB and IIA respectively. The IIA case is a little subtle as generic solutions in this class are not in general related to the their $Mink_5$ sub-case by duality. In IIB things are more simple, indeed the generic solutions are essentially the U-dual of the $Mink_5$ sub-case.

4.3.1 Case II in IIA

It was already observed in [48] that when one T-dualises on the $U(1)$ isometry of Case II in IIA, one is mapped to a conformal Calabi-Yau class — this is contained in case I of IIB. To see this, let us explicitly generate case II in IIA from it. We start with (4.1) and perform the coordinate transformation

$$x_1 \rightarrow x_1 - c_0 x_4 \tag{4.6}$$

the NS sector of case I in IIB then takes the form

$$\begin{aligned}
 ds^2 &= e^{2A} ds^2(\text{Mink}_4) \\
 &+ e^{-2A} \left(\frac{1}{f} (b^2 dx_1^2 + dx_2^2 + x_2^2 ds^2(S^2)) + dx_3^2 + \frac{1}{b^2} \left(dx_4 + \frac{ba_2}{\sqrt{f}} dx_1 \right)^2 \right), \\
 B &= g\mathcal{C}_2 - \frac{a_2\sqrt{f}}{b} g \frac{dx_2 \wedge dx_4}{x_2^2}, \quad \mathcal{C}_2 = \text{Vol}(S^2) + \frac{dx_1 \wedge dx_2}{x_2^2},
 \end{aligned} \tag{4.7}$$

with the dilaton unchanged and where

$$b^2 + a_2^2 = 1, \quad \frac{a_2\sqrt{f}}{b} = c_0. \tag{4.8}$$

If we now impose that ∂_{x_4} is an isometry, we have

$$f(x_3, x_4) \rightarrow g(x_3), \quad g(x_3, x_4) \rightarrow g(x_3), \quad A(x_1, x_2, x_3, x_4) \rightarrow A(x_1, x_2, x_3) \tag{4.9}$$

and the PDE's governing the system become

$$\partial_{x_3}^2 f = 0, \quad \partial_{x_3}^2 (fg) = 0, \tag{4.10a}$$

$$\frac{1}{b^2} \partial_{x_1}^2 (e^{-4A}) + \frac{1}{x_2^2} \partial_{x_2} (x_2^2 \partial_{x_2} (e^{-4A})) + \partial_{x_3}^2 (f^{-1} e^{-4A}) + \frac{1}{x_2^4} \partial_{x_3}^2 (fg^2) = 0, \tag{4.10b}$$

i.e. exactly what one has for case II in IIA (cf. (5.59)–(4.60) of [48]). If we now T-dualise on ∂_{x_4} , the Mink₄ part of the metric is unchanged and the rest of the NS sector becomes

$$\begin{aligned}
 ds_6^2 &= e^{2A} b^2 \left(dx_4 - \frac{a_2 g \sqrt{f}}{b x_2^2} dx_2 \right)^2 + \frac{e^{-2A}}{f} \left(b^2 dx_1^2 + dx_2^2 + f dx_3^2 + x_2^2 ds^2(S^2) \right), \\
 B &= g\mathcal{C}_2 + \frac{a_2 b}{\sqrt{f}} dx_1 \wedge \left(dx_4 - \frac{a_2 g \sqrt{f}}{b x_2^2} dx_2 \right), \quad e^{-\Phi_{IIA}} = \frac{f}{b} e^{-A}.
 \end{aligned} \tag{4.11}$$

which reproduces the case II NS sector in IIA presented in [48] — we have checked that the same is true for the RR fluxes. After the T-duality the effect of (4.6) can no longer be turned off with a coordinate transformation so generic (4.11) is a deformation of its $c_0 = 0$ limit. However the PDEs with $c_0 = 0$ differ from the deformed case generically, due to the b^{-2} factor in (4.10b) that appears only in the later — which means this procedure is not generically a duality. An exception is if we assume that ∂_{x_3} is also an isometry, then (4.10b) is the same for all values of c_0 . This is because when ∂_{x_3} is also an isometry the diffeomorphism + T-duality procedure reproduces the effect of performing a T-s-T transformation on the $c_0 = 0$ limit of this case.

4.3.2 Case II in IIB

To generate case II in IIB one starts from case I in IIA. One first needs to make one direction in Σ_2 an isometry to T-dualise on, we will take ∂_{x_4} (∂_{x_3} is physically equivalent).

The NS sector (4.3) is then locally mapped to

$$\begin{aligned}
 ds^2 &= e^{2A} ds^2(\text{Mink}_4) + ds^2(\text{M}_6), \quad B = x_2^2 e^{-6A+2\Phi} B_0 \text{Vol}(S^2), \\
 ds^2(\text{M}_6) &= e^{2A} ds^2(S^1) + e^{-6A+2\Phi} \left(dx_2^2 + x_2^2 ds^2(S^2) \right) \\
 &\quad + e^{6A-2\Phi} \left(dx_1 + B_0 e^{-6A+2\Phi} dx_2 \right)^2 + e^{-2A} dx_3^2,
 \end{aligned} \tag{4.12}$$

where $ds^2(S^1) = dx_4^2$, Φ is now the IIB dilaton and all functions have support in (x_1, x_2, x_3) only. The supersymmetry PDE's then become

$$\begin{aligned}
 \partial_{x_2} (e^{4A-2\Phi}) &= \partial_{x_1} (e^{-2A} B_0), \\
 \partial_{x_2} (x_2^2 e^{-2A} B_0) &= \partial_{x_1} (x_2^2 e^{-8A+2\Phi} (1 + B_0^2)).
 \end{aligned} \tag{4.13}$$

The RR sector also gets mapped, we will not concern ourselves with the details (see (4.20) of [49] for explicit expressions), other than the fact that only (F_1, F_3) are non trivial. The Bianchi identities of these fluxes require

$$\begin{aligned}
 \partial_{x_3}^2 (e^{4A-2\Phi}) + \partial_{x_1}^2 (e^{-4A}) &= 0, \\
 \partial_{x_3}^2 (e^{-2A} B_0) + \partial_{x_1} \partial_{x_2} (e^{-4A}) &= 0, \\
 \partial_{x_3}^2 (x_2^2 e^{-8A+2\Phi} (1 + B_0^2)) + \partial_{x_2} (x_2^2 \partial_{x_2} (e^{-4A})) &= 0.
 \end{aligned} \tag{4.14}$$

This is locally the Mink_5 limit of case II with no parametric deformation. One may then generate generic solutions in this class via a formal U-duality that introduces a parameter c_0 as we will now demonstrate.

One begins by T-dualising on the three space-like $U(1)$ isometries contained in Mink_4 — which maps to a solution in IIA with only (F_4, H) non trivial. One lifts this solution to M-theory and performs a formal boost along the M-theory circle with isometry ∂_z

$$dt \rightarrow b_1 + c_0 dz, \quad dz \rightarrow b_2 dt + dz. \tag{4.15}$$

Here c_0, b_1, b_2 are all constants, c_0 is the deformation parameter, while b_1, b_2 should be non zero and satisfy $-b_1 + b_2 c_0 > 0$, but are otherwise arbitrary. After the boost, the M-theory circle becomes fibred over dt and additional components are turned on in the 4-form flux. In addition, the warp factors of the, now fibred, M-theory circle and time direction are modified. As such, when one reduces to IIA the dilaton and warp factors of the metric are modified, and one now has non trivial (F_2, F_4, H) fluxes. Finally one performs an additional three T-dualities within the \mathbb{R}^3 factor of the solution to get back to IIB. After rescaling $dt \rightarrow -(b_1 - b_2 c_0) dt$ all dependence on b_1, b_2 drops out of the solution and one finds the new metric and dilaton

$$d\hat{s}^2 = \frac{e^{2A}}{\kappa_\perp} ds^2(\text{Mink}_4) + \kappa_\perp ds^2(\text{M}_6), \quad e^{\hat{\Phi}} = e^\Phi, \tag{4.16}$$

where the deformed physical fields are given hats, and un-hatted fields are the original ones in IIB before the U-duality. We also introduce $(\kappa_\parallel, \kappa_\perp)$ such that

$$\kappa_\parallel = c_0 e^{4A-2\Phi}, \quad \kappa_\parallel^2 + \kappa_\perp^2 = 1, \tag{4.17}$$

to make contact with the notation of [49]. The deformed fluxes are given in terms of the original ones and $(\kappa_{\parallel}, \kappa_{\perp})$ as

$$\begin{aligned}\hat{F}_1 &= F_1, \\ \hat{F}_3 &= F_3 - \kappa_{\parallel} e^{-\Phi} \star_6 H_3, \\ \hat{H} &= dB + \kappa_{\parallel} e^{\Phi} \star_6 F_3, \\ \hat{F}_5 &= (1 + \hat{\star}) \text{Vol}(\text{Mink}_4) \wedge \left(\frac{e^{4A - \Phi} \kappa_{\parallel}}{\kappa_{\perp}^2} \right),\end{aligned}\tag{4.18}$$

where \star_6 is taken on the unwarped M_6 and $\hat{\star}$ on the full ten-dimensional space. It is then not hard to check that if we define $e^{2\hat{A}}, \hat{B}_0$ such that

$$e^{2\hat{A}} = \kappa_{\perp} e^{2A}, \quad B_0 = \kappa_{\perp} \hat{B}_0\tag{4.19}$$

and substitute un-hatted functions for hatted ones in (4.13), (4.14), (4.16) and (4.18) one precisely reproduces the metric, dilaton, fluxes and PDE's of section 4.5 in of [49]. Thus we have shown that case II in IIB follows from case I in IIA by first T-dualising to IIB, then performing a U-duality.

4.4 Generating Case III

In this section we show how Case III in type IIA and IIB can be generated from one of the master systems. As this requires performing both a T-s-T transformation and U-duality, a detailed description becomes rather protracted, so we will only sketch the procedure.

4.4.1 Case III in IIB

Case III in IIB can be generated from case I in IIB by first imposing that $\Sigma_2 = T^2$ so that one has two isometries to work with and setting

$$g = 0.\tag{4.20}$$

Notice that this makes the dilaton constant and reduces (4.2b) to a Laplacian in four dimensions; indeed, if we now T-dualise on both directions in $\Sigma_2 = T^2$ we are mapped to D5-brane in flat space. We can generate a double parametric deformation of the D5-brane system as follows

1. Formal U-duality on spatial Mink_4 directions.¹⁴
2. T-dualise on ∂_{x_4} and add an exact to the NS 2-form.
3. Shift $x_3 \rightarrow x_3 + \gamma x_4$.
4. T-dualise on ∂_{x_4} .

One needs to supplement this by rescaling the dilaton, Minkowski and local coordinates, but after doing this carefully one is mapped to Case III in IIB.

¹⁴A T-duality on each of the 3 spacial directions, lift to M-theory, boost along the M-theory circle, reduce back to IIA, redo the spatial T-dualities to return to IIB.

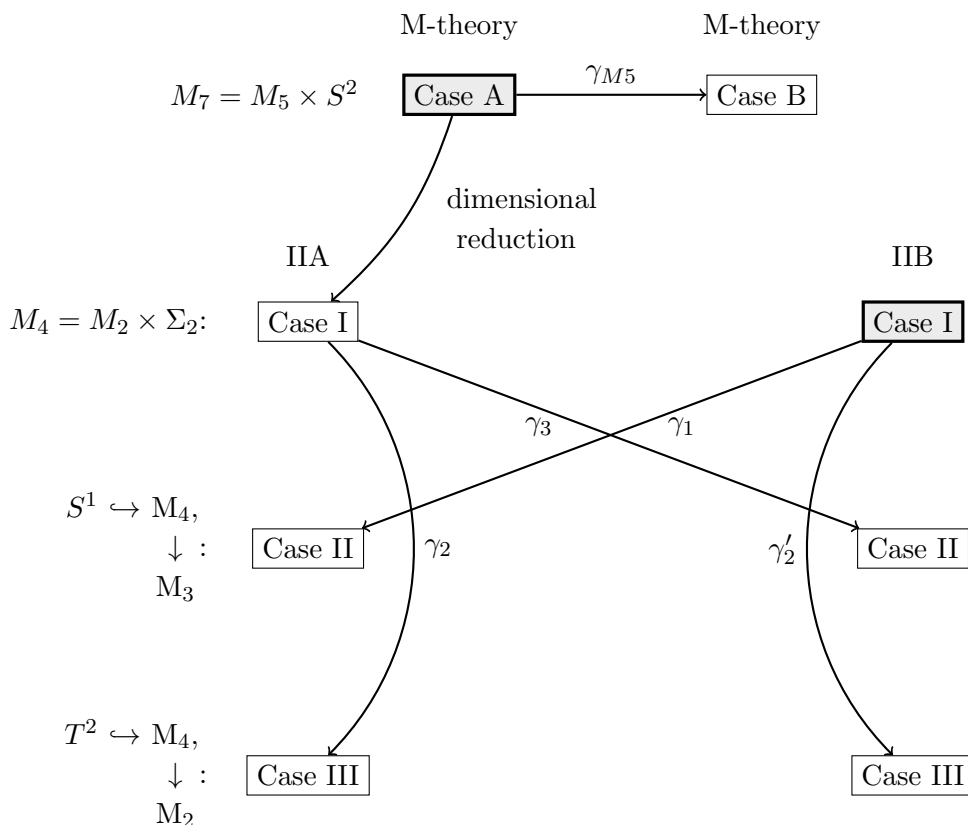


Figure 2. Depiction of the chains of dualities leading to the various $\text{Mink}_4 \times S^2$ class in type II with equal spinor norm. Cases A in M-theory and I in IIB (shaded grey) are the truly fundamental master cases from which all else can be generated. Specifically γ_1 represent a transformations where one performs a diffeomorphism mixing the M_2 and Σ_2 factors in IIB case I, and introducing a parameter c_0 . One then imposes an isometry and T-dualising on it. γ_2, γ'_2 represent the following: impose $\Sigma_2 = T^2$ in case I and T-dualise on both directions, lets call this case III_0 , — one then performs a formal U-duality followed by a T-s-T transformation with T^2 — which generates a two-parameter deformation of case III_0 governed by the same PDE's, i.e. case III. γ_3 represents imposing an isometry in the Σ_2 factor of IIA case I, then T-dualiing to get to IIB. One then performs a formal U-duality on the spacial directions of Mink_4 . Finally, as explained in the end of section 2.4, the physical fields (but not the spinor) of Case B can be obtained from Case A by specialising the vielbein to a M5-brane solution with some rotational symmetry in the co-dimensions; this operation in the figure is named γ_{M5} .

4.4.2 Case III in IIA

Case III in IIA can be generated in a similar fashion. We again fix $\Sigma_2 = T^2$ and T-dualise on both $U(1)$'s therein. We are mapped to a Mink_6 system studied in [71] (see also section 4.1 of [48]), with non trivial NS and RR 0- and 2-form fluxes. We have checked that the following chain of dualities, boosts and shifts maps this to section Case III in IIA

1. Four T-dualities performed on the spatial Mink_4 directions and one direction in T^2 , say x_4 .
2. Lift to M-theory, boost along the M-theory isometry, reduce back to IIA.

3. Shift $x_3 \rightarrow x_3 + \gamma x_4$.
4. Four more T-dualities once more on the spatial Mink_4 directions and x_4 .

Along the way some additional rescaling of coordinates is required, but this gives the general idea — i.e. Case III in IIA is generated by a combination of U-duality and T-s-T transformations.

5 Examples

In this section we give some simple examples that follow from case A in M-theory, we exclude case B because it is merely the M5 brane — of course one AdS solution can be embedded here, $\text{AdS}_7 \times S^4/\mathbb{Z}_k$, but as this just requires solving a simple Laplace equation, we omit the details. Another result that is easy to establish is that there exist no $\text{Mink}_5 \times S^2$ solution in M-theory that are not locally $\text{Mink}_6 \times S^2$. From which it follows that there also exist no AdS_6 solutions in M-theory.

Here we shall focus on the following: in section 5.1 we show how $\mathcal{N} = 2$ AdS_5 solutions are embedded in M-theory while in section 5.2 we provide some evidence that the squashed S^3 ansatz of section 2.3.2 likely contains compact Mink_4 solutions in M-theory with non trivial fluxes.

5.1 $\mathcal{N} = 2$ $\text{AdS}_5 \times S^2$ in M-theory

AdS_5 solutions preserving $\mathcal{N} = 2$ in M-theory were classified in [46] (which was shown to be an exhaustive class in [69]), in this section we will show how they are embedded within M-theory class A.

The metric of these solutions is of the form

$$ds^2 = e^{2\lambda} \left(4ds^2(\text{AdS}_5) + y^2 e^{-6\lambda} ds^2(S^2) \right) + \frac{4}{(1 - y\partial_y D)} e^{2\lambda} (d\psi + V)^2, \\ + \frac{-\partial_y D}{y} e^{2\lambda} \left(dy^2 + e^D (d\hat{x}_1^2 + d\hat{x}_2^2) \right), \tag{5.1}$$

$$e^{-6\lambda} = \frac{-\partial_y D}{y(1 - y\partial_y D)}, \quad V = \frac{1}{2} (\partial_{x_2} d\hat{x}_1 - \partial_{\hat{x}_1} d\hat{x}_2), \tag{5.2}$$

where D is a function of y, x_1, x_2 and ∂_ψ is an isometry; this one together with the S^2 factor realises the $\text{SU}(2) \times \text{U}(1)$ R-symmetry of the $\mathcal{N} = 2$ super-conformal algebra in four dimensions. Comparing this with (2.37) we see that if one realise AdS as

$$ds^2(\text{AdS}_5) = e^{2r} ds^2(\text{Mink}_4) + dr^2, \tag{5.3}$$

then one needs to fix

$$\rho = e^{2r} y, \quad e^{2\Delta} = e^{2\lambda+2r}, \quad v = -e^{-2\lambda} (2dr + dy). \tag{5.4}$$

In order to get a metric that actually respects the isometries of AdS, we need the radial component to point along one of the vielbein that make up the $\text{SU}(2)$ in such a way that

the metric has no dr cross terms, and has a common warp factor for all the putative AdS directions — the $SU(2)$ structure should also be charged under ∂_ψ . After some work, one is able to establish that the following four-dimensional vielbein satisfies all the $SU(2)$ -structure conditions

$$u = \sqrt{\frac{-\partial_y D}{y}} e^{\lambda + \frac{1}{2}D} (dx_1 + i dx_2), \quad w = 2e^{i\psi} e^{-2\lambda} \sqrt{\frac{y}{-\partial_y D}} \left(dr + \frac{1}{2} \partial_y D dy + i(d\psi + V) \right), \quad (5.5)$$

and yields a metric diagonal in r provided that D satisfies the Toda equation

$$(\partial_{x_1}^2 + \partial_{x_2}^2)D + \partial_y^2 e^D = 0, \quad (5.6)$$

away from localised sources. The flux F_2 that follows from this is then

$$F_2 = e^{4\lambda} y^{-2} \left(2(d\psi + V) \wedge d(y^3 e^{-6\lambda}) + 2y(1 - y^2 e^{-6\lambda})dV - \partial_y e^D dx_1 \wedge dx_2 \right), \quad (5.7)$$

which reproduces the correct M-theory 4-form through $G = e^{2C} \text{Vol}(S^2) \wedge F_2$ and obeys the Bianchi identity.

5.2 Towards compact Mink_4 in M-theory with fluxes

The ansatz of section 2.3.2, contains a squashed S^3 in addition to the $\text{Mink}_4 \times S^2$ factor generic to all solutions we considered. As such it is a prime candidate for solutions with a Taub-NUT or Atiyah-Hitchin factor. Metrics with Atiyah-Hitchin singularities (which are the lift of $O6$ planes) are a possible way to achieve compact Minkowski solutions in M-theory with non trivial fluxes. Here we show that it is indeed possible to find metrics with such singularities inside section 2.3.2 with a simple example.

The solutions in section 2.3.2 are defined in terms of a single undetermined function e^{-2K} that is governed by a relatively simple PDE in two variables (ρ, y) , namely (2.47). The simplest way to make progress with this PDE is by a separation of variables ansatz

$$e^{-2K} = p(\rho)q(y). \quad (5.8)$$

The PDE then reduces to two ODE's of the form

$$\frac{1}{\rho^2} \partial_\rho (\rho^2 \partial_\rho p) + c_1 p^2 = 0, \quad \partial_y^2 (q^2) = \frac{c_1 q}{c + y}, \quad (5.9)$$

where c, c_1 are constants. These ODE's are still difficult to solve, so in principle one can proceed numerically, however our aim is just to show the plausibility of finding compact solutions with non trivial fluxes in the squashed S^3 ansatz, so let us just make the most brutal non trivial assumption we can, namely

$$c_1 = 0, \quad p = 1. \quad (5.10)$$

As a consequence of this

$$q = L\sqrt{y - b} \quad (5.11)$$

for L, b constants. We necessarily have $G = 0$ while the part of the metric spanned by (ρ, S^2) becomes simply \mathbb{R}^3 or equivalently T^3 . The metric for this solution is given (for $L = 1$) by

$$ds^2 = ds^2(\text{Mink}_4) + ds^2(T^3) + \frac{dy^2}{(y+c)\sqrt{y+b}} + \frac{\sqrt{y+b}}{4} ds^2(\tilde{S}^2) + \frac{y+c}{4\sqrt{y+b}} (d\psi + \eta)^2, \quad (5.12)$$

i.e. a squashed S^3 foliated over an interval — which is in fact bounded. One can check that close to $y = -b$ the metric exhibits an Atiyah-Hitchin singularity while as y approaches $-c$ the warp factor of \tilde{S}^2 becomes constant and the remaining directions vanish regularly as \mathbb{R}^2 in polar coordinates — thus if we assume $-c < -b$ the interval is bounded between $-c < y < -b$ and the manifold is compact.

This example of course has no fluxes turned on, but it seems likely that, even within the separation of variables ansatz, it will be possible to find similar solutions, with $G \neq 0$, and where $T^3 \rightarrow (\rho, \text{Vol}(S^2))$ with the metric warped in terms of ρ . It would be interesting to pursue this — but such a detailed study is outside of the scope of this work.

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A Killing spinors in M-theory

In this appendix we derive some of the conditions that will be useful in section 2 and we prove that the M-theory case with a G_2 structure is actually a subcase of M-theory classification.

The eleven-dimensional supersymmetry condition can be written as

$$D_M \epsilon + \frac{1}{24} (3G\gamma_M - \gamma_M G) \epsilon = 0. \quad (A.1)$$

where G is defined in (2.1). The first step is to perform a decomposition from 11 to $4 + 7$ dimensions

$$\epsilon = \zeta_+ \otimes \chi + \zeta_- \otimes \chi^c \quad (A.2)$$

where ζ_+ is a four-dimensional spinor with positive chirality, $\zeta_+^c = \zeta_-$, and χ is a seven-dimensional one.

Evaluating (A.1) on Mink_4 directions we get the following algebraic condition:

$$\left(e^{-\Delta} \partial_a e^{\Delta} \gamma^a - \frac{1}{3} i f + \frac{1}{6} F \right) \chi = 0 \quad (A.3)$$

while taking the internal index

$$\left(D_a + \frac{1}{24}(3F\gamma_a - \gamma_a F) - \frac{i}{12}f\gamma_a\right)\chi = 0. \quad (\text{A.4})$$

where we have chosen $\gamma_5 = i\text{Vol}_4$. Multiplying (A.3) by $\gamma_a/4$ we can further simplify (A.4):

$$\left(D_a + \frac{1}{8}F\gamma_a - \frac{i}{6}f\gamma_a + \frac{1}{4}e^{-\Delta}\partial^b e^\Delta \gamma_a \gamma_b\right)\chi = 0. \quad (\text{A.5})$$

From these conditions we can easily prove that $f = 0$, indeed contracting (A.3) by χ^\dagger and make the difference with the conjugate of the same expression we get

$$\begin{aligned} \chi^\dagger \left(e^{-\Delta}\partial_a e^\Delta \gamma^a - \frac{1}{3}if + \frac{1}{6}F\right)\chi - \bar{\chi} \left(-e^{-\Delta}\partial_a e^\Delta \gamma^a + \frac{1}{3}if + \frac{1}{6}F\right)\chi^c \\ = 2i \left(e^{-\Delta}\partial_a e^\Delta \text{Im}(\chi^\dagger \gamma^a \chi) - \frac{1}{6}\text{Im}(\chi^\dagger F\chi) - \frac{1}{3}f\right) \\ = -\frac{2i}{3}f\chi^\dagger \chi = 0 \end{aligned} \quad (\text{A.6})$$

since the bilinears of $\chi\chi^\dagger$ are real if they have degree 0,1,4,5 and purely imaginary otherwise.

When $\chi^c = \chi$, we can only define a G_2 structure on M_7 . From the sum and the difference of (A.3) with its conjugate we have that $F\chi = 0$ and $\partial_a \Delta \gamma^a \chi = 0$. From the second condition we get:

$$(\partial_a \Delta \gamma^a)^2 \chi = (\partial_a \Delta \partial^a \Delta)\chi = 0 \quad (\text{A.7})$$

which implies $\partial^a \Delta = 0$ since a sum of squares vanishes if and only if each individual term in the sum does. Moreover from (A.4)

$$D_a \chi + D_a \chi^c = 2D_a \chi = 0 \quad (\text{A.8})$$

and then we can set $F = 0$. So the internal seven dimensional-manifold actually has G_2 holonomy and no warping factor on Mink_4 . Although such solutions may exist with $M_7 = S^2 \times M_5$ we need not consider them explicitly; indeed there are no Killing spinors on S^2 such that $\xi = \xi^c$, and there is no invariant form on S^2 that maps the Killing spinor to its Majorana conjugate — as such imposing $\chi^c = \chi$ makes χ contain two separable five-dimensional systems, one coupling to ξ and one to ξ^c — this just imposes additional constraints on the five-dimensional system that follows from $\chi^c \neq \chi$, so cases with G_2 holonomy are special cases of the systems we consider in the main text. We thus restrict ourselves to $SU(3)$ structure case $\chi \neq \chi^c$.

Now let's analyse the zero-form constraints on M_7 :

$$\begin{aligned} \partial_a(\chi^\dagger \chi) &= D_a \chi^\dagger \chi + \chi^\dagger D_a \chi = \frac{1}{12} \left(\chi^\dagger \gamma_a F \chi + \chi^\dagger F \gamma_a \chi\right) \\ &= \frac{1}{2} \partial_b \Delta \left(\chi^\dagger \gamma_a \gamma_b \chi + \chi^\dagger \gamma_b \gamma_a \chi\right) = \partial_a \Delta \chi^\dagger \chi, \end{aligned} \quad (\text{A.9})$$

from which we have

$$d(e^{-\Delta} \chi^\dagger \chi) = 0 \quad (\text{A.10})$$

and so we can set, without loss of generality, $||\chi||^2 = e^\Delta$. Moreover, we can also calculate

$$\partial_a(\bar{\chi}\chi) = -\frac{1}{6}(\bar{\chi}\gamma_a F\chi - \bar{\chi}F\gamma_a\chi) = -2\partial_a\Delta\chi^\dagger\chi \tag{A.11}$$

which implies

$$d(e^{2\Delta}\bar{\chi}\chi) = 0. \tag{A.12}$$

This equation tells us that the phase of $\bar{\chi}\chi$ is constant and then we can set it to zero choosing the following parametrisation:

$$\bar{\chi}\chi = ce^{-2\Delta} \tag{A.13}$$

where c is a real constant.

Let's analyse the two zero-form equations using our ansatz (2.12). The first condition in terms of the five- and two-dimensional spinors reads:

$$\frac{1}{2}(\eta_1^\dagger\eta_1 + \eta_2^\dagger\eta_2) + \frac{y_3}{2}(\eta_1^\dagger\eta_2 + \eta_2^\dagger\eta_1) = e^\Delta, \tag{A.14}$$

to preserve the SU(2) R-symmetry the warp factor cannot depend on the coordinate on the sphere, so we have

$$||\eta_1||^2 + ||\eta_2||^2 = 2e^\Delta, \quad \text{Re}(\eta_1^\dagger\eta_2) = 0. \tag{A.15}$$

On the other hand (A.12) reads

$$-(y_1 - iy_2)\bar{\eta}_1\eta_2 = ce^{-2\Delta} \tag{A.16}$$

and then we must have

$$\bar{\eta}_1\eta_2 = 0, \quad c = 0. \tag{A.17}$$

These considerably simplify the supersymmetry conditions, and in particular we can restrict to (2.10), as we have done in the main text.

B Mink₄ × S² Killing spinor conditions in type IIB

In this appendix we derive the four-dimensional Killing spinor conditions a supersymmetric solutions of type IIB on Mink₄ × S² × M₄ must obey. In the conventions of [66] the type IIB supersymmetry conditions are

$$\partial\Phi\epsilon_1 - \frac{1}{2}H\epsilon_1 + e^\Phi\left(F_1 + \frac{1}{2}F_3\right)\epsilon_2 = 0, \tag{B.1a}$$

$$\partial\Phi\epsilon_2 + \frac{1}{2}H\epsilon_2 + e^\Phi\left(-F_1 + \frac{1}{2}F_3\right)\epsilon_1 = 0, \tag{B.1b}$$

$$\left(\nabla_M - \frac{1}{4}H_M\right)\epsilon_1 - \frac{e^\Phi}{8}\left(F_1 + F_3 + \frac{1}{2}F_5\right)\Gamma_M\epsilon_2 = 0, \tag{B.1c}$$

$$\left(\nabla_M + \frac{1}{4}H_M\right)\epsilon_1 - \frac{e^\Phi}{8}\left(-F_1 + F_3 - \frac{1}{2}F_5\right)\Gamma_M\epsilon_1 = 0, \tag{B.1d}$$

where ϵ_i are two positive chirality Majorana spinors. If we seek a solution that contains a Mink_4 factor foliated over the internal space by a warp factor e^{2A} , we can decompose the ten-dimensional Clifford algebra as

$$\Gamma_\mu = e^A \gamma_\mu^{(4)} \otimes \mathbb{I}, \quad \Gamma_a = \gamma_5^{(4)} \otimes \gamma_a^{(6)}, \quad \mu = 0, \dots, 3, \quad a = 4, \dots, 9 \quad (\text{B.2})$$

and the ten-dimensional spinors as

$$\epsilon_i = \zeta_+ \otimes \chi_+^i + \text{m.c.} \quad (\text{B.3})$$

where ζ_+, χ_+^i are positive chirality spinors in four- and six- dimensions respectively, and (B.1a)–(B.1d) become

$$\partial A \chi_+^1 + \frac{e^\Phi}{4} (F_1 + F_3 + \tilde{F}_5) \chi_+^2 = 0, \quad (\text{B.4a})$$

$$\partial A \chi_+^2 + \frac{e^\Phi}{4} (-F_1 + F_3 - \tilde{F}_5) \chi_+^1 = 0, \quad (\text{B.4b})$$

$$\partial \Phi \chi_+^1 - \frac{1}{2} H \chi_+^1 + e^\Phi \left(F_1 + \frac{1}{2} F_3 \right) \chi_+^2 = 0, \quad (\text{B.4c})$$

$$\partial \Phi \chi_+^2 + \frac{1}{2} H \chi_+^2 + e^\Phi \left(-F_1 + \frac{1}{2} F_3 \right) \chi_+^1 = 0, \quad (\text{B.4d})$$

$$\left(\nabla_a - \frac{1}{4} H_a \right) \chi_+^1 - \frac{e^\Phi}{8} (F_1 + F_3 + \tilde{F}_5) \gamma_a^{(6)} \chi_+^2 = 0, \quad (\text{B.4e})$$

$$\left(\nabla_a + \frac{1}{4} H_a \right) \chi_+^2 - \frac{e^\Phi}{8} (-F_1 + F_3 - \tilde{F}_5) \gamma_a^{(6)} \chi_+^1 = 0, \quad (\text{B.4f})$$

where F_1, F_3, H_3 are purely magnetic and \tilde{F}_5 is the magnetic component of F_5 .

Using (B.4e) and (B.4f) together with (B.4a)–(B.4d) multiplied by γ_a , we get the following conditions on the zero degree bilinears:

$$\begin{aligned} |\chi_+^1|^2 + |\chi_+^2|^2 &= 2e^A, & |\chi_+^1|^2 - |\chi_+^2|^2 &= 2c_- e^{-A}, \\ \partial_a (e^{2A-\Phi} \chi_+^{2\dagger} \chi_+^1) - e^{2A-\Phi} \partial_a (\chi_+^{1\dagger} \chi_+^2) &= i e^{3A} (\star \tilde{F}_5)_a - c_- e^A (F_1)_a. \end{aligned} \quad (\text{B.5})$$

If we now assume we have an $\text{SU}(2)$ R-symmetry parameterised by a round S^2 factor in our solution we have again to factorise the Clifford algebra, and we choose the following conventions:

$$\begin{aligned} \gamma_i^{(6)} &= \sigma_i \otimes \mathbb{I}, \quad i = 1, 2 & \gamma_{a+2}^{(6)} &= \sigma_3 \otimes \gamma_a, \quad a = 1, \dots, 4 \\ B^{(6)} &= \sigma_2 \otimes B, & B &= B^{-1} = -B^* = -B^T, & \gamma_a^* &= B^{-1} \gamma_a B, \end{aligned} \quad (\text{B.6})$$

where e^C is the warping function for S^2 . Moreover, we have to decompose the spinors

$$\begin{aligned} \chi_+^1 &= \frac{1}{2} \left[\xi \otimes (\eta^1 + \eta^2) + \sigma_3 \xi \otimes \hat{\gamma}^{(6)} (\eta^1 + \eta^2) \right], \\ \chi_+^2 &= \frac{1}{2} \left[\xi \otimes \hat{\gamma}^{(6)} (\eta^1 - \eta^2) + \sigma_3 \xi \otimes (\eta^1 - \eta^2) \right], \end{aligned} \quad (\text{B.7})$$

and the fluxes

$$F = f + e^{2C} \text{Vol}(S^2) \wedge g, \quad H = H_3 + e^{2C} \text{Vol}(S^2) \wedge H_1, \quad (\text{B.8})$$

one can then show that (B.4a)–(B.4f) is equivalent to (3.13a)–(3.13h) in the main text.

C Towards solutions for the general IIA case

In this appendix we will find a large ansatz that allows us to simplify Case I in IIA in such a way to have a diagonal metric. In this case supersymmetry is implied by the following equations:

$$d(e^A w) = 0, \quad d(e^{2A+C-\Phi}) + e^{2A-\Phi} v_2 = 0, \quad d(e^{-2A+\Phi}(v_1 + B_0 v_2)) = 0 \quad (\text{C.1a})$$

$$d(e^{-\Phi} v_1) \wedge w \wedge \bar{w} = 0, \quad d(e^{2C-\Phi}(B_0 v_1 - v_2)) \wedge w \wedge \bar{w} = 0, \quad B_2 = 0. \quad (\text{C.1b})$$

In the present case equations (C.1) automatically give us a definition of a diagonal vielbein except for the component v_1 , which we diagonalise by hand with the introduction of an arbitrary function μ :

$$v_1 = e^{\Phi+\mu} dx_1, \quad v_2 = -e^{-2A+\Phi} dx_2, \quad w_1 = e^{-A} dx_3, \quad w_2 = e^{-A} dx_4. \quad (\text{C.2})$$

where $x_2 = e^{2A+C-\Phi}$. The internal metric thus reads

$$ds_6^2 = e^{-4A+2\Phi}(dx_2^2 + x_2^2 ds^2(S^2)) + e^{2\Phi+2\mu} dx_1^2 + e^{-2A}(dx_3^2 + dx_4^2). \quad (\text{C.3})$$

It turns out to be useful define the following functions:

$$f = e^{-2A+2\Phi+\mu}, \quad g = e^{2C} B_0. \quad (\text{C.4})$$

The other equations of (C.1) imply that $f = f(x_1, x_2)$, $g = g(x_1, x_2)$ and $\mu = \mu(x_1, x_3, x_4)$. Moreover we have the following PDEs:

$$\partial_{x_1} g = -x_2^2 \partial_{x_2} f, \quad (\text{C.5a})$$

$$\partial_{x_2} g = x_2^2 e^{-\mu} \partial_{x_1} (f e^{-4A-\mu}). \quad (\text{C.5b})$$

The fluxes are

$$B_2 = g \text{Vol}(S^2), \quad (\text{C.6a})$$

$$F_2 = (\partial_{x_4} e^\mu dx_3 - \partial_{x_3} e^\mu dx_4) \wedge dx_1 - f^{-1} \partial_{x_1} e^{-4A} dx_3 \wedge dx_4, \quad (\text{C.6b})$$

$$F_4 = B_2 \wedge F_2 + \text{Vol}(S^2) \wedge (x_2^2 f (\partial_{x_3} e^{-4A-\mu} dx_4 - \partial_{x_4} e^{-4A-\mu} dx_3) \wedge dx_2 \\ - g (\partial_{x_4} e^\mu dx_3 - \partial_{x_3} e^\mu dx_4) \wedge dx_1 + (x_2^2 \partial_{x_2} e^{-4A} + g f^{-1} \partial_{x_1} e^{-4A}) dx_3 \wedge dx_4), \quad (\text{C.6c})$$

and the Bianchi identities for these fluxes impose the following PDEs:

$$\partial_{x_2} (f^{-1} \partial_{x_1} e^{-4A}) = 0 \quad (\text{C.7a})$$

$$(\partial_{x_3}^2 + \partial_{x_4}^2) e^\mu + \partial_{x_1} (f^{-1} \partial_{x_1} e^{-4A}) = 0 \quad (\text{C.7b})$$

$$f (\partial_{x_3}^2 + \partial_{x_4}^2) e^{-4A-\mu} + \frac{1}{x_2^2} \partial_{x_2} (x_2^2 \partial_{x_2} e^{-4A}) + \frac{1}{x_2^2} \partial_{x_2} g f^{-1} \partial_{x_1} e^{-4A} = 0. \quad (\text{C.7c})$$

We can notice that (C.7a) is just a restriction on the coordinate dependence of the physical fields, while (C.5) can be used to define H . If this is the case, then we can define f from this consistency equation

$$\frac{1}{x_2^2} \partial_{x_2} (x_2^2 \partial_{x_2} f) + \partial_{x_1} (e^{-\mu} \partial_{x_1} (f e^{-4A-\mu})) = 0 \tag{C.8}$$

which is redundant whenever g gets defined rather than its derivatives.

At this point, solving these PDEs in full generality is hard; however, this class of solutions is large enough to allow us to impose some further ansatz and still obtaining interesting intersecting-brane systems.

The first thing that needs to be addressed is that the l.h.s. of (C.5b) is independent of (x_3, x_4) , which means the r.h.s. should also be, which is not true a priori. We can deal with this by making an ansatz for e^{-4A} , we find that

$$\text{Case I : } e^{-4A} = f^{-1} e^{\mu} T(x_2, x_3, x_4), \tag{C.9a}$$

$$\text{Case II : } e^{-4A} = f^{-1} H(x_1, x_2) S(x_3, x_4)^2, \tag{C.9b}$$

$$e^{\mu} = S(x_3, x_4) \tag{C.9c}$$

achieve the stated aims.¹⁵ We will look at these cases separately in the following subsections.

C.1 Case I

Here (C.5a)–(C.5b) imply

$$g'(x_1) = f_1(x_1), \quad f = f_0(x_1) + \frac{f_1(x_1)}{x_2}. \tag{C.10}$$

The first thing we need to do is solve (C.7a), which becomes

$$\partial_{x_2} \left[\frac{1}{(x_2 f_0 + f_1)} \partial_{x_1} \left(\frac{e^{\mu} x_2^2 T}{x_2 f_0 + f_1} \right) \right] = 0. \tag{C.11}$$

The obvious way to make progress with this is to do one of the following

$$f_0 = 0, \quad f_1 = 0, \quad \text{or} \quad f = c_0 + \frac{c_1}{x_2} \text{ for } c_i \text{ constant.} \tag{C.12}$$

The first two of these turn out to be subcases of the last up to a redefinition of x_1 so we without loss of generality set

$$T = \frac{\tilde{T}(x_2, x_3, x_4)}{x_2^2 (c_1 + c_2 x_2)}, \quad e^{\mu} = \tilde{S}(x_1, x_3, x_4), \quad f(x_2) = c_0 + \frac{c_1}{x_2} \tag{C.13}$$

and then (C.7a) PDE's becomes a Youm-like condition

$$\partial_{x_1} \tilde{S} \partial_{x_2} \tilde{U} = 0. \tag{C.14}$$

¹⁵In principle one can add a generic function $e^{\mu_0(x_1)}$ in front of the expressions for e^{-4A} and e^{μ} in the second case, but it is easy to check that this can be reabsorbed into f by performing a change of coordinates. Another non-trivial generalisation of the second case is to define $e^{-4A} = f^{-1} [H(x_1, x_2) S(x_3, x_4)^2 + T(x_2, x_3, x_4)]$. This case however seems to require further assumptions which so far didn't lead to something interesting.

C.1.1 $\partial_{x_1}\tilde{S} = 0$: localised D4 and smeared D6-NS5 system

For this case \tilde{T} is not particularly useful, instead we define

$$T = fU(x_2, x_3, x_4), \quad \tilde{S} = S(x_3, x_4). \tag{C.15}$$

We are then left with the PDE's

$$\square_2 S = 0, \quad f\square_2 U + \frac{S}{x_2^2} \partial_{x_2}(x_2^2 \partial_{x_2} U) = 0 \tag{C.16}$$

which define the system. The physical fields are defined by the solutions to these PDEs as

$$e^{2A} = \frac{1}{\sqrt{US}}, \quad e^{2\Phi+2\mu} = f\sqrt{\frac{U}{S}}, \quad e^{-4A+2\Phi} = f\sqrt{\frac{S}{U}}, \quad e^{2C} B_0 = c_1 x_1. \tag{C.17}$$

Finally we note that the only dependence on x_1 appears in the NS 2-form as $B = f_1 x_1 \text{Vol}(S^2)$, which gives rise to

$$H = c_1 dx_1 \wedge \text{Vol}(S^2), \tag{C.18}$$

so ∂_{x_1} is an isometry of any solution within this class. Comparing (C.17) and (C.3) with ([70], sections 4.2 and 4.5) we see that we have an ansatz for localised D4 stretched between D6-branes and ending on NS5 branes that are both smeared along x_1 .

C.1.2 $\partial_{x_2}\tilde{T} = 0$: smeared D4-D6-NS5 system

Here we define

$$\tilde{T} = U(x_3, x_4), \quad \tilde{S} = S(x_1, x_3, x_4) \tag{C.19}$$

and then the remaining PDE's become

$$\square_2 U = 0, \quad \square_2 S + U \partial_{x_1}^2 S = 0, \tag{C.20}$$

The physical fields are defined as:

$$e^{2A} = \frac{1}{\sqrt{fSU}}, \quad e^{2\Phi+2\mu} = \sqrt{\frac{fS}{U}}, \quad e^{-4A+2\Phi} = f^{-3/2} \sqrt{\frac{S}{U}}, \quad e^{2C} B_0 = c_1 x_1 \tag{C.21}$$

We thus have a system of D4-D6-NS5 branes where the NS5 is smeared along x_1 while the D4 along x_2 .

C.2 Case II

Here (C.5b) becomes

$$\partial_{x_2} g = x_2^2 e^{-\mu_0} \partial_{x_1} H \tag{C.22}$$

The discussion turns out to be different if we consider the warping A to depend or not from x_1 .

C.2.1 $\partial_{x_1} e^{-4A} = 0$: localised D4-NS5 smeared D6 system

To make e^{-4A} independent of x_1 we need to set

$$H = f\lambda(x_2). \tag{C.23}$$

We can solve the (C.5a) by introducing an auxiliary function $h(x_1, x_2)$ such that

$$f = \partial_{x_1} h, \quad g = -x_2^2 \partial_{x_2} h. \tag{C.24}$$

We are then left with the PDE's

$$\partial_{x_2}(x_2^2 \partial_{x_2} \lambda) = 0, \quad \square_2 S = 0, \quad \frac{1}{x_2^2} \partial_{x_2}(x_2^2 \partial_{x_2} h) + \lambda \partial_{x_1}^2 h = 0, \tag{C.25}$$

to solve, with physical fields given by

$$e^{2A} = \frac{1}{S\sqrt{\lambda}}, \quad e^{2\Phi+2\mu} = \frac{\partial_{x_1} h}{\sqrt{\lambda}}, \quad e^{-4A+2\Phi} = \sqrt{\lambda} \partial_{x_1} h, \quad e^{2C} B_0 = -x_2^2 \partial_{x_2} h. \tag{C.26}$$

When $S = 1$ this is simply the massless system of [71] describing localised D6-NS5 brane intersection up to two T-dualities along dx_3 and dx_4 . On the other hand when $S \neq 0$ but $\partial_{x_1} h = 1$ we have a D4-D6 system where the D6 brane is smeared along x_1 , indeed its harmonic function is $S = S(x_3, x_4)$, while the harmonic function of the D4 factorise in λS . therefore we can interpret our solution as a localised D4-NS5 smeared D6 system.

C.2.2 $\partial_{x_1} e^{-4A} \neq 0$: “massive” D4-D6-NS5 system

We now allow e^{-4A} to depend on x_1 . Let's absorb f^{-1} into H . From (C.7a) we can define f from H up to an arbitrary function $G(x_1) \neq 0$ (otherwise we will come back to the previous case)

$$f = \frac{\partial_{x_1} H}{G}. \tag{C.27}$$

Moreover condition (C.7b) imposes that this function has to be at least linear in x_1 , $G = c_1 x_1 + c_2$. We then have that supersymmetry follows from the following two PDEs:

$$\square_2 S = -c_1 S^2, \quad \frac{1}{x_2^2} \partial_{x_2}(x_2^2 \partial_{x_2} H) + \frac{1}{2} \partial_{x_1}^2 H^2 = c_1 \frac{\partial_{x_1} H^2}{c_1 x_1 + c_2}. \tag{C.28}$$

The fields are defined as:

$$e^{2A} = \frac{1}{S\sqrt{H}}, \quad e^{2\Phi+2\mu} = \frac{1}{\sqrt{HG}} \partial_{x_1} H, \quad e^{-4A+2\Phi} = \frac{\sqrt{H}}{G} \partial_{x_1} H$$

$$\partial_{x_1}(e^{2C} B_0) = \frac{x_2^2}{G} \partial_{x_1} \partial_{x_2} H, \quad \partial_{x_2}(e^{2C} B_0) = \frac{x_2^2}{2} \partial_{x_1} \left(\frac{\partial_{x_1} H^2}{G} \right). \tag{C.29}$$

We can see that for $c_1 = 0$ this is equivalent to the previous case but with the T-dual of the massive system of localised D6-NS5 of [71]; therefore for $c_1 = 0$ we have a localised “massive” D4-NS5 smeared D6 system. When $c_1 \neq 0$ we have something more exotic which physical interpretation is less clear, but the fluxes suggest a system of (possibly localised) D4-D6-NS5 branes.

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