Toric quiver asymptotics and Mahler measure: $\mathcal{N} = 2$ BPS states

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ABSTRACT: The BPS sector in $\mathcal{N} = 2$, four-dimensional toric quiver gauge theories has previously been studied using the crystal melting model and dimer model. We introduce the Mahler measure associated with the statistical dimer model to study the large N limit of these quiver gauge theories. In this limit, the generating function of BPS states in a general toric quiver theory is studied and entropy, growth rate of BPS states and free energy of the quiver are obtained in terms of the Mahler measure. Moreover, a possible third-order phase transition in toric quivers is discussed. Explicit computations of the profile function, entropy density, BPS growth rate and phase structure of quivers are presented in concrete examples of the clover $\mathbb{C}^3$, and resolved conifold $\mathcal{C}$ quivers. Finally, BPS growth rates of Hirzebruch $\mathbb{F}_0$, and $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold quivers are obtained and a possible interpretation of the results for certain BPS black holes is discussed.

KEYWORDS: D-branes, Lattice Integrable Models, Nonperturbative Effects, Supersymmetric Gauge Theory

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1 Introduction

In the context of the AdS/CFT correspondence, quiver gauge theories are engineered by the low energy limit of D-branes on singularities of Calabi-Yau manifolds [1]. In this way, half-BPS states in $\mathcal{N} = 2$ four-dimensional quiver gauge theories can be seen as D6-D2-D0 branes bound states in type IIA string theory. Recently, the BPS sector, its rich structure and interesting features such as wall-crossing and relations to topological strings have been studied extensively, [2–4]. Beside AdS/CFT, quiver representation theory and combinatorial lattice models constructed from that, such as the dimer model and the crystal model play an important role in the study of non-perturbative aspects of quiver gauge theories [5].

Despite intense studies of the BPS counting and interesting behavior of the BPS index, the asymptotic analysis of the index is less considered, while, some interesting features of the theory such as phase transitions appear in the asymptotic limit. The main goal of this paper is to study the half BPS sector of $\mathcal{N} = 2$, $4d$ toric quiver gauge theories on the universal cover of quiver diagram, and to obtain large N asymptotic of the BPS index. Similar problems in the asymptotic limit of different gauge theories have been studied using other plausible methods. Recently, asymptotic counting of chiral operators in a free quiver gauge theory have been studied via multivariate asymptotic analysis of generating

functions [6]. In another direction, the phase structure of a topological gauge theory, Chern-Simons matter theory has been studied by using random matrix theory [7].

Quiver lattice models and related mathematical techniques provide efficient methods for counting BPS states and its asymptotic limit. Crystal melting models capture the BPS indices of toric quiver gauge theories. The crystal melting model is also interesting from the asymptotic point of view. In fact, the thermodynamic limit of the crystal model is studied via toroidal dimer model [8]. Therefore, statistical mechanics of the dimer model on a toroidal bipartite graph and related techniques from tropical geometry and Mahler measure theory provide a powerful framework to study the asymptotics of BPS sector of toric quiver gauge theories. Furthermore, we use generalized Mahler measures (on arbitrary tori) of certain Newton polynomials related to particular toric Calabi-Yau threefolds, $\mathbb{C}^3$ and resolved conifold, to explicitly compute some analytic results in the quiver theory.

The generalized Mahler measure on the tori of the toroidal dimer model, with a specific parameterization explained in section 3.1, coincides with the Ronkin function that is the main studying tool in dimer model [8]. However, the Mahler measure theory and its techniques in number theory are developed independent of the Ronkin function and its relations to tropical geometry and dimer model. The practical difference between the two approaches, the Mahler measure vs. Ronkin function, lies in the explicit computational techniques that Mahler measure theory brings into our study, which should be considered as a complement to the Ronkin function technique in the dimer model.

Using analytic techniques from [8], the thermodynamic limit of the generating function of BPS states is studied via the dimer model and the free energy of the quiver theory are obtained in [9]. Following this approach and adopting methods from the statistical toroidal dimer model and associated Mahler measure theory in the quiver gauge theory, we obtain parameters and observables of the quiver theory, such as the profile of ranks of gauge groups of the quiver, free energy, entropy, BPS growth rate and phase structure of the theory, in the asymptotic limit.

In explicit computations in concrete examples of quivers, we use the dilogarithm function and its associated special functions in Mahler measure theory. Using Lobachevsky and Bloch-Wigner functions, we develop the statistical mechanics of the clover and resolved conifold quivers in the thermodynamic limit. We explicitly compute the observables of these quiver gauge theories and obtain new analytic expressions for them.

One of our first observations in this work is that the BPS growth rate of any toric quiver can be obtained, up to a scaling factor, from the surface tension of the profile function and in terms of the Mahler measure of the Calabi-Yau threefold. The actual computations of the BPS growth rate is performed explicitly in the most important and fundamental examples, such as $\mathbb{C}^3$, the resolved conifold $\mathcal{C}$, local $\mathbb{P}^1 \times \mathbb{P}^1$, and orbifold quivers.

The free energy of quiver is shown to be equal to the partition function of genus zero topological strings [9]. Following this result, we compute the finite part of the topological string partition function, in certain approximations, in terms of Mahler measure and toric data.

To study the phase structure of the toric quiver gauge theory, we use the tropical geometric methods in toroidal dimer model [8]. Similar to the dimer model, the phase

structure of the quiver is obtained from the different fluctuating behavior of the profile function in different regions of the Amoeba, i.e. the image of the zero set of the Newton polynomial under the logarithmic map. Using the entropy density of the quiver, we find evidence for a possible third order phase transition at the boundaries of Amoeba.

The crystal melting model for $\mathbb{C}^3/\mathbb{Z}_p \times \mathbb{Z}_p$ orbifold quiver in the large p limit, explains certain BPS black holes [10]. We propose physical applications and possible interpretations of our results for BPS black holes in this example of large orbifold quivers.

This paper is organized as follow. In section 2, we review the crystal melting model for BPS states of toric quiver theories. In section 3, we first briefly review some background in statistical dimer models and their connection to Mahler measure theory and special functions, then the thermodynamics of the quiver theories are explained via the Mahler measure. In section 4, explicit computations of some concrete examples are presented. In section 5, we conclude our paper and discuss some directions for future research.

2 BPS quivers and crystal model

In this section, we briefly review the BPS bound states of D6-D2-D0 branes on a toric Calabi-Yau threefold singularity and its relation to the crystal melting model [5]. We consider a single non-compact D6-brane wrapping the whole Calabi-Yau threefold and an arbitrary number of point-like D0 branes, and D2-branes wrapping compact two-cycles in the Calabi-Yau.

The D-brane bound states are $\frac{1}{2}$ -BPS states in $\mathcal{N} = 2, 4d$ gauge theories. They carry a charge $\gamma = (n, l)$ where n and l are the number of D0 and D2 branes, respectively. The number of BPS states with a given charge γ is denoted by $\Omega(\gamma)$ and is called the BPS index. Defining the fugacity factors q and Q_I for D0 and D2 charges, the generating function of the BPS index is given by

$$\mathcal{Z}_{\text{BPS}} = \sum_{n,l} \Omega(n,l) Q^l q^n, \quad (2.1)$$

where $Q = \{Q_I\}$ and $Q^l = \prod_I Q_I^{l_I}$ where I runs over two-cycles in the Calabi-Yau. Writing $q = e^{-g_s}$ and $Q_I = e^{t_I}$, we can identify g_s with the coupling constant of the topological string and t_I as the Kähler moduli of the Calabi-Yau threefold.

In order to compute the index, we consider the low energy effective theory on the D-branes, which is a $d = 1, \mathcal{N} = 4$ toric quiver quantum mechanics. In this way, the BPS index reduces to the Witten index. Moreover, the field content and the superpotential term of quantum mechanics is encoded in the quiver diagram. The quiver diagram $Q = (Q_0, Q_1)$ consists of a set of nodes Q_0 with a gauge group $U(N_i)$ associated with each node i , and a set of arrows Q_1 connecting the nodes with a bifundamental field associated with each arrow. The set of faces of the quiver is denoted by Q_2 . The dual diagram of the quiver is denoted by Q' and is called brane tiling. The quiver diagram is related to the brane tiling by the following dictionary: $Q'_0 = Q_2$, $Q'_1 = Q_1$, and $Q'_2 = Q_0$. The superpotential W consists of terms which are represented as a loop around a face in the quiver diagram. For further details of how D0, D2 and D6 branes appear in this picture, see [5]. Assigning a

black or white color to each vertex of Q'_0 depending on the orientation of the corresponding face, the brane tiling defines a toroidal bipartite dimer model [8]. The statistical mechanics of this dimer model play a crucial role in this study.

To compute the Witten index, we need to identify the moduli space of vacua and this can be done in terms of the quiver representation. There is a path algebra $\mathbb{C}Q$ associated with a quiver diagram, with paths between nodes of the quiver as elements and concatenation of paths as product. Imposing F-term relations on the path algebra, we obtain a factor algebra as a quotient, $A = \mathbb{C}Q/F$, where F is the ideal of path algebra generated by $\partial W/\partial X_a$ for $X_a \in Q_1$. This is often called the A-module. Furthermore, imposing D-term relations, we get a so-called stable A-module which is in one-to-one correspondence with supersymmetric vacua of the quiver quantum mechanics.

Consider the universal cover of the quiver diagram $\tilde{Q}$. Any path starting at a reference node $i_0 \in \tilde{Q}_0$ and ending at $j \in \tilde{Q}_0$ in the A-module can be given in the form $v_{i_0j} \omega^n$, for some $n \in \mathbb{N}$, where v_{i_0j} is the shortest path between vertices i_0 and j and ω is a superpotential term, i.e. a loop around some face of $\tilde{Q}$. The crystal is defined by putting an atom on ending point of each path on the vertices of the quiver. The paths of the form v_{i_0j} correspond to atoms on the face of the crystal, while paths $v_{i_0j} \omega^n$ with $n > 0$ are atoms on the node j at the depth n inside the crystal. Putting atoms for all elements of the factor algebra, we construct a three-dimensional crystal on the universal cover of periodic quiver.

Now, the Witten index can be computed in terms of the stable A-module, as a sum over the fixed-points of the torus action, more precisely $U(1)^3 \times U(1)_R$, on the moduli space. The fixed-points of the moduli space are equivalent to $U(1)^3$ invariant stable A-modules, which are in one-to-one correspondence with the ideals of factor algebra. The ideals of factor algebra are in fact the molten configurations of crystal denoted by π_i for $i \in \mathbb{Z}$. The set of all possible molten configurations of the crystal is denoted by π . Thus, the fixed points are in bijection with the molten crystal configurations, and generating function of the Witten index, up to a sign, can be written as a sum over molten configurations $\pi_i \in \pi$,

$$\mathcal{Z}_{\text{BPS}} = \mathcal{Z}_{\text{CM}} = \sum_{\pi_i \in \pi} \prod_{i=0}^{L-1} q_i^{|\pi_i|}, \quad (2.2)$$

where q_i is a periodic fugacity factor ($q_i = q_{i+L}$ for fixed integer L) associated with $|\pi_i|$, i.e. the number of atoms in the configuration π_i which is associated with the i -th quiver node. By comparing generating functions (2.1) and (2.2), and the following relations between the parameters of the two (modulo a sign) [11],

$$q = q_0 q_1 \dots q_{L-1}, \quad Q_I = q_i \quad (I = i = 1, \dots, L-1), \quad (2.3)$$

we see that the Witten index is equal to the number of molten crystal configurations where n is the total number of atoms removed from the crystal, and the relative numbers of different types of atoms removed from the crystal are specified by l_I . For further details and explicit examples such as the conifold quiver consult [12]. The main goal of this paper is the asymptotic counting of BPS states, $\Omega(n, l)$ in the large n limit and obtaining the BPS growth rate, i.e. logarithm of BPS index in the asymptotic limit.

3 Asymptotic analysis in BPS sector

In this section we explain a method of asymptotic analysis in the BPS sector of toric quivers. The BPS sector of quiver gauge theories is a quantum mechanical system and the statistical mechanics of the crystal model explain this sector in the asymptotic regime. To study the asymptotic regime, we adopt the statistical mechanics of the toroidal dimer model and Mahler measure theory in the context of quiver gauge theories. Before that, we review some background material.

3.1 Mahler measure in toroidal dimer model

In this section, we review essential background material and methods for statistical mechanics of bipartite dimer models [8], and their associated Mahler measures.

Dimer models on bipartite planar graphs are the statistical mechanics of sets of dimer coverings of the graph, i.e. a pairing of adjacent vertices such that each vertex is covered only by one dimer. Consider a toroidal bipartite graph $G = (V, E)$, as a set of vertices $v \in V$ and edges $e \in E$, with oriented closed curves γ_z and γ_w on $\mathbb{T}^2$, transverse to G , representing a basis of $H_1(\mathbb{T}^2)$. Each of the edges of the graph is oriented from a black vertex to a white vertex. The weight $\nu(e)$ of edge e is $\nu(e) = z^{\gamma_z \cdot e} w^{\gamma_w \cdot e}$, with $\cdot$ denotes the signed intersection number of edge and closed curve. A Kasteleyn matrix $K(z, w)$ is an oriented adjacency matrix of G , indexed by vertices, and the matrix elements of K are $\pm \nu(e)$ with the sign given by the orientation of that edge. Orient the edges of G such that each face of graph has an odd number of clockwise oriented edges. The characteristic polynomial of the dimer model, called the Newton polynomial, is defined by

$$P(z, w) = \det K(z, w).$$

A fundamental result is the number of the dimer coverings of G , obtained in [8],

$$Z(G) = \frac{1}{2} | -P(1, 1) + P(-1, 1) + P(1, -1) + P(-1, -1) |.$$

Now, let G be a periodic bipartite planar graph. Then, for an integer $M > 0$, define a finite graph on a torus, $G_M = G/M\mathbb{Z}^2$. Then, the logarithm of the partition function of the toroidal dimer model on the universal cover of the planar graph, which is the growth rate of the toroidal dimer model, is obtained in [8],

$$\log Z(G) := \lim_{M \rightarrow \infty} \frac{1}{M^2} \log Z(G_M) = \frac{1}{(2\pi i)^2} \int_{S^1 \times S^1} \log |P(z, w)| \frac{dz}{z} \frac{dw}{w}, \quad (3.1)$$

where $S^1 \times S^1$ is a unit torus defined by $|z| = 1, |w| = 1$. The above integral in eq. (3.1) is called the Mahler measure of Newton polynomial, and is denoted by $m(P)$. A general logarithmic Mahler measure is defined as an average of the logarithm of a given non-zero polynomial $P(z_1, z_2, \dots, z_n) \in \mathbb{C}[z_1^\pm, z_2^\pm, \dots, z_n^\pm]$ over the real n -torus $\mathbb{T}^n \in \mathbb{C}^n$,

$$m(P) = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(z_1, z_2, \dots, z_n)| \prod_{i=1}^n \frac{dz_i}{z_i}. \quad (3.2)$$

A generalization of the two-variable Mahler measure on arbitrary tori,

$$m_{a,b}(P) = \frac{1}{(2\pi i)^2} \int_{|z|=a} \int_{|w|=b} \log |P(z, w)| \frac{dz}{z} \frac{dw}{w}, \quad (3.3)$$

has been considered recently [13]. For our purposes, the generalized Mahler measure, with specific choice of parameters $a = e^x$ and $b = e^y$, coincides with the Ronkin function introduced in eq. (3.11). In this study, we consider two explicit examples of this generalization of the Mahler measure and apply them to study the asymptotic limit of quiver gauge theories. Explicit computations of the Mahler measure often involve the dilogarithm function and its extensions such as the Lobachevsky and Bloch-Wigner functions. In the last part of this section, we briefly review these functions and their properties.

Lobachevsky and Bloch-Wigner functions. The Lobachevsky function $L(\theta)$ is defined by the following integral

$$L(\theta) = - \int_0^\theta \log |2 \sin(t)| dt. \quad (3.4)$$

This function is a well defined, continuous, odd and π -periodic function for all θ , [14]. For any positive integer n , $L(\theta)$ satisfies

$$L(n\theta) = n \sum_{j=0}^{n-1} L\left(\theta + \frac{j\pi}{n}\right). \quad (3.5)$$

The Bloch-Wigner function, for $z \in \mathbb{C} \setminus \{0, 1\}$ is defined as

$$D(z) = \Im(Li_2(z)) + \log |z| \arg(1 - z), \quad (3.6)$$

where $\Im$ denotes the imaginary part and $Li_2(z)$ is the dilogarithm function. For $z \in \mathbb{R} \cup \{\infty\}$, $D(z) = 0$, and for imaginary $z = e^{i\theta}$, $\theta \in \mathbb{R}$, we have

$$D(e^{i\theta}) = \Im[Li_2(e^{i\theta})] = 2L\left(\frac{\theta}{2}\right). \quad (3.7)$$

The Bloch-Wigner function is a real-valued analytic function on $\mathbb{C}$ except at 0 and 1 where it is continuous but not differentiable. It satisfies the following 6-fold symmetry [15],

$$D(z) = D\left(1 - \frac{1}{z}\right) = D\left(\frac{1}{1-z}\right) = -D\left(\frac{1}{z}\right) = -D(1-z) = -D\left(\frac{-z}{1-z}\right). \quad (3.8)$$

In addition we have

$$D(\bar{z}) = -D(z), \quad 2D(z) + 2D(-z) = D(z^2). \quad (3.9)$$

The Bloch-Wigner function is related to the Lobachevsky function in the following way [16]. For $\zeta \in \mathbb{C} \setminus \mathbb{R}$, $\theta_1 = \arg(\zeta)$, $\theta_2 = \arg(1 - \bar{\zeta})$, $\theta_3 = \arg(\frac{1}{1-\bar{\zeta}})$, we have

$$D(\zeta) = L(\theta_1) + L(\theta_2) + L(\theta_3) = L(\theta_1) + L(\theta_2) - L(\theta_1 + \theta_2). \quad (3.10)$$

Having reviewed essential background, we apply these methods to quiver gauge theories.

3.2 Toric quivers in asymptotic limit

In this section, we will use asymptotics of the dimer model and method of Mahler measure theory to study thermodynamics of quiver gauge theory.

The construction of the quiver crystal, discussed in section 2, implies that the profile function of ranks of the gauge groups of the quiver is the height function of the crystal model. In the large N limit, with appropriate scaling of the profile function and locations of quiver nodes, the profile becomes a smooth convex function, called limit shape of the crystal model, and it is given by a generalized Mahler measure (Ronkin function) of the Newton Polynomial associated with the quiver [8],

$$\begin{aligned} N(x, y) &= \frac{1}{(2\pi i)^2} \int_{S^1 \times S^1} \log |P(e^x z, e^y w)| \frac{dz}{z} \frac{dw}{w} \\ &= m\left(P(e^x z, e^y w)\right) = m_{e^x, e^y}(P). \end{aligned} \quad (3.11)$$

The slope of the profile is defined by

$$\frac{\partial N}{\partial x} = n_x, \quad \frac{\partial N}{\partial y} = n_y. \quad (3.12)$$

The results in [8] imply that surface tension of the profile is the Legendre transform $\mathcal{L}[\cdot]$ of the profile of the quiver,

$$-\sigma(n_x, n_y) = \mathcal{L}[N] = N(x, y) - n_x x - n_y y, \quad (3.13)$$

where n_x and n_y are given by eq. (3.12). The Legendre duality also implies $\frac{\partial \sigma}{\partial n_x} = x$, $\frac{\partial \sigma}{\partial n_y} = y$.

The Newton Polygon $\text{NP}(P)$ of a polynomial $P(z, w) \in \mathbb{C}[z^\pm, w^\pm]$ is a convex hull of the exponents of the monomials with nonzero coefficients. Then, it is shown that the slope (n_x, n_y) of the toroidal dimer model lies inside the Newton polygon, $(n_x, n_y) \in \text{NP}(P)$ [8]. As we will see in the following, the surface tension of the quiver is the entropy density of the BPS states and determines the BPS growth rate.

We consider the toric quiver crystals with crystal lattice spacing g_s , on the $M \times M$ cover of the torus on the (x, y) plane. One can also think of g_s as inverse temperature. To study quiver crystals in the continuum and thermodynamic limits, the distance between atoms in crystal are taken to zero; $g_s \rightarrow 0$ and we put the quiver on the universal cover; $M \rightarrow \infty$ such that $g_s M$ is a large but fixed. In this way, the number of atoms removed from crystal is taken to infinity while the volume of the crystal $g_s^3 n$ is large but fixed. We call these limits the asymptotic limit.

In the asymptotic limit, the generating function of the BPS states can be approximated by a dominant configuration or a typical state, coming from the profile of the quiver, in a continuous integral form in which M and g_s dependence are both manifest [9],

$$\mathcal{Z}_{\text{BPS}} \sim \exp \left(M^2 \left(\int_0^1 -\sigma(n_x, n_y) dx dy - g_s M \int_0^1 N(x, y) dx dy \right) \right). \quad (3.14)$$

With the choice of scaling $M^3 \int_0^1 N(x, y) dx dy = n$, the volume factor in generating function (2.1), $q^n = e^{-g_s n}$, is given by the second integral in the right hand side of generating

function (3.14). One-to-one correspondence between the dimer model and the crystal model implies that the number of dimer covering is (up to a multiplicative factor) equal to the number of molten configurations in the crystal model, and the number of BPS states in toric quivers. In the asymptotic limit, the first integral in eq. (3.14) gives the asymptotic number of dimer coverings. In the asymptotic regime, the correspondence holds up to a scaling power. The BPS growth rate is obtained by the number of (fluctuating) stepped surfaces lying close to the limit shape. Since the first integral in generating function (3.14), is a volume term, thus its third root gives the correct dimension and with the choice of scaling $n \sim M^3$, the BPS growth rate becomes

$$\lim_{n \rightarrow \infty} \log \Omega(n) \sim n^{\frac{2}{3}} \left(\int_0^1 -\sigma(n_x, n_y) dx dy \right)^{\frac{1}{3}}. \quad (3.15)$$

Notice that in the asymptotic limit, the Q^l term in the generating function (2.1) is finite and thus asymptotically irrelevant.

3.3 Entropy density and BPS growth rate

First, by using eqs. (3.11) and (3.13), we find the relations between the profile function, entropy density and Mahler measure as

$$-\sigma(n_x^*, n_y^*) = N(0, 0) = m(P), \quad (3.16)$$

where $\sigma(n_x^*, n_y^*)$ is the extremum of entropy density obtained by solving

$$\frac{\partial \sigma}{\partial n_x} = x = 0, \quad \frac{\partial \sigma}{\partial n_y} = y = 0. \quad (3.17)$$

For a better approximation, we maximize the entropy density in eq. (3.15), and by using eq. (3.16), we obtain the BPS growth rate

$$\log \Omega(n) \sim n^{\frac{2}{3}} \left(\int_0^1 -\sigma(n_x^*, n_y^*) dx dy \right)^{\frac{1}{3}} = n^{\frac{2}{3}} \left(\int_0^1 N(0, 0) dx dy \right)^{\frac{1}{3}}. \quad (3.18)$$

We thus obtain the numerical coefficient of the BPS growth rate from the asymptotic rank of the quiver at origin, which is the Mahler measure,

$$\log \Omega(n) \sim N(0, 0)^{1/3} n^{2/3} = m_P^{1/3} n^{2/3}. \quad (3.19)$$

3.4 Free energy of quivers and topological strings

In the infinite volume limit $g_s M \rightarrow \infty$, the free energy of the quiver which is defined as the logarithm of the generating function (2.1), can be obtained, by using the Legendre transformation (3.13) and putting the integral of the total derivative to zero [9],

$$\mathcal{F} = \log \mathcal{Z}_{\text{BPS}} \sim \frac{1}{g_s^2} \int_{\mathbb{R}^2} N(x, y) dx dy. \quad (3.20)$$

There is only g_s dependence remained in above formula and as it suggests, this free energy matches with the genus zero topological string partition function on toric Calabi-Yau

threefolds, as it is shown in [9]. Moreover, at the origin $x = y = 0$, it reproduces eq. (3.1) and thus counts the number of dimer coverings. The above integral is divergent because of infinite contribution from facets of the profile function in the unbounded complement components of the Amoeba. Amoeba $\mathcal{A}(P)$ of a Newton polynomial is the image of the zero set $\{(z, w) \in \mathbb{C}^2 | P(z, w) = 0\}$, under the logarithmic map $(z, w) \rightarrow (\log |z|, \log |w|)$, [17].

For g_s small but fixed, the finite part of the free energy is obtained by removing the infinite contributions and thus restricting the domain of integration to the Amoeba,

$$\mathcal{F}_f \sim \frac{1}{g_s^2} \int_{\mathcal{A}} N(x, y) \, dx \, dy. \quad (3.21)$$

Using the following results about the area of amoeba, a plausible approximation to the integral can be found. The area of the Amoeba is bounded from above by the area of the corresponding Newton polygon Δ , [18],

$$\text{Area}(\mathcal{A}) \leq \pi^2 \text{Area}(\Delta). \quad (3.22)$$

The Newton polynomials of the toroidal dimer models are real curves of special type, called Harnack curves [8]. Inequality (3.22) saturates for the Amoeba of the Harnack curves and thus for the Amoeba of the dimer model. We approximate the finite part of the free energy (3.21), at origin by using eq. (3.16), and above theorems,

$$\mathcal{F}_f^* \sim \frac{1}{g_s^2} \int_{\mathcal{A}} N(0, 0) \, dx \, dy = \frac{m_P}{g_s^2} \text{Area}(\mathcal{A}) = \frac{\pi^2 m_P}{g_s^2} \text{Area}(\Delta). \quad (3.23)$$

This is in fact proportional to the number of dimer coverings inside the Amoeba, which is the liquid phase of the dimer model. The relation between the topological vertex formalism and crystal melting model was discovered in [19]. In the asymptotic limit, the Ronkin function method provides a novel approach to obtain the genus-zero Gromov-Witten invariants, [9, 19]. For further comparison of the results obtained from this method and topological vertex technique consult [12]. Moreover, as one of the advantages of our work, based on the established link between the crystal melting model, its Ronkin function and topological vertex method, and by extending the use of the Mahler measure and tropical geometry techniques associated with dimer model, we are able to explicitly compute the genus-zero Gromov-Witten invariants which are a priori comparable and consistent with the results by other plausible methods. In section 4, we obtain analytic results for profile function and the Mahler measure which can be used to explicitly compute the partition function of genus-zero topological string. However, in this paper we do not focus in this line of research. We hope to come back to this interesting problem in the future studies.

3.5 Phase structure of quivers and PDEs

As quivers are dual to brane tilings, thus by construction, the phase structure of the toric quiver theory is the same as the phase structure of the toroidal dimer model. This phase structure is determined by the Amoeba [8]. There are solid, liquid, and gas phases, according to different behavior of fluctuations of the profile function in different regions of the Amoeba [8]. In unbounded complement components of the Amoeba which correspond

to facets of the crystal, the profile function has no fluctuation and these regions are called the solid phase. The liquid phase is the interior of the Amoeba, corresponding to melting part of the crystal in which profile function has bounded fluctuations. The gas phase is the bounded component of the complement of the Amoeba and inside this region, the profile function has unbounded fluctuations. Gas phases are associated with the points inside the Newton polygon. The phase borders are the boundaries of the Amoeba between different phases.

From the entropy density point of view, phase transitions and their orders are obtained from the discontinuity of the entropy and its derivatives, across the boundaries of the Amoeba. In our case, entropy is defined as a double integral of the entropy density, i.e the first integral in generating function (3.14). Although we are not computing contributions of the fluctuations of the profile in the entropy, but we can study whether the entropy density or its derivatives jumps across the boundaries of the amoeba, since there are different fluctuation contributions to the entropy density in different regions of the amoeba. Notice that boundaries of the Amoeba correspond to borders of the facets of crystal in which the profile function is constant. On one facet of quiver and its boundary, slope (n_x, n_y) is zero. Thus, phase transition on one of the boundaries of the Amoeba can be studied by considering the zero slope limit, $n_x, n_y \rightarrow 0$, of the entropy density and its derivatives for $n_x, n_y \in \text{NP}(P)$. Heuristically, the discontinuity in $\lim_{n_x, n_y \rightarrow 0} \frac{\partial^k \sigma(n_x, n_y)}{\partial n_x^k}$ and/or $\lim_{n_x, n_y \rightarrow 0} \frac{\partial^k \sigma(n_x, n_y)}{\partial n_y^k}$ across the boundaries of Amoeba suggests a possible phase transition of the order $k + 2$, since the entropy is a double integral of entropy density.

Using other plausible methods from statistical mechanics of the dimer model, it is conjectured that there is a third order phase transition in a dimer model [20]. In this article, we obtain some evidence toward a possible third order phase transition in concrete examples of clover and resolved conifold quivers, in section 4.

PDEs for Profile and Entropy Density

As we explained, we expect that some derivatives of entropy density are divergent and generally not continuous near the boundaries of facets of the quiver. However, the second order derivatives of the profile function and entropy density satisfy the Monge-Ampère differential equation inside the Amoeba and near its boundaries. We define the Hessian matrix of the Ronkin function, $\mathcal{H}_{ij}(N) := \partial_{ij} N$ and the Monge-Ampère operator $\mathcal{M}(N) := \det \mathcal{H}(N)$. Using results for the Ronkin function in [8, 18], we observe that the profile function satisfy a Monge-Ampère equation,

$$\mathcal{M}(N) = \partial_{xx} N \partial_{yy} N - \partial_{xy} N \partial_{yx} N = \pi^2. \quad (3.24)$$

Legendre duality between the profile function and the entropy density implies [21],

$$\mathcal{M}(N)\mathcal{M}(\sigma) = 1, \quad \partial_{n_j n_i} \sigma = \frac{\mathcal{H}(N)_{(ij)}}{\mathcal{M}(N)}, \quad \partial_{j i} N = \frac{\mathcal{H}(\sigma)_{(ij)}}{\mathcal{M}(\sigma)}, \quad (3.25)$$

where $\mathcal{H}_{(ij)}$ is the co-factor of i -th row and j -th column of the Hessian matrix. Then, eqs. (3.24) and (3.25) imply the following Monge-Ampère equation for the entropy density,

$$\mathcal{M}(\sigma) = \partial_{n_x n_x} \sigma \partial_{n_y n_y} \sigma - \partial_{n_x n_y} \sigma \partial_{n_y n_x} \sigma = 1/\pi^2. \quad (3.26)$$

Figure 1. Clover quiver diagram.

Figure 2. Triangle for the clover quiver.

Having explained our general results for arbitrary toric quiver gauge theory, in the next section we explicitly compute the observables in some concrete examples.

4 Some examples

In this section, we consider basic concrete examples of quivers and obtain analytic formulas for the profile function, the entropy density, phase structure, and BPS growth rate.

4.1 Clover quiver $\mathbb{C}^3$

The clover quiver, figure 1 is characterized by the Newton polynomial of $\mathbb{C}^3$ Calabi-Yau, $P(z, w) = 1 + z + w$. The Amoeba of this quiver is shown in figure 10. The Following result for the Mahler measure of $P(z, w) = c + az + bw$, is essential in understanding the thermodynamics of the clover quiver.

Theorem (Maillot [22]). *For $a, b, c \in \mathbb{C}^3$, and $(\log |a|, \log |b|, \log |c|) \in \mathcal{A}_P$*

$$\pi m(c + az + bw) = D\left(\frac{|c|}{|a|} e^{i\theta_2}\right) + \theta_1 \log |a| + \theta_2 \log |b| + \theta_3 \log |c|, \quad (4.1)$$

where D is the Bloch-Wigner function, reviewed in section 3.1.

Outside the Amoeba, we have

$$m = \max(\log |a|, \log |b|, \log |c|). \quad (4.2)$$

$|a|, |b|, |c|$ can be seen as sides of a triangle, figure 2, and the Mahler measure in this case is given by eq. (4.1). The profile of the clover quiver is the Ronkin function of $\mathbb{C}^3$ and by using eq. (3.11), it can be obtained from the Mahler measure in "Maillot Theorem" by putting $a = e^x$, $b = e^y$ and $c = 1$. For $x, y \in \mathcal{A}_{\mathbb{C}^3}$ ($|a|, |b|, |c| \in \Delta$), we obtain

$$N_{\mathbb{C}^3}(x, y) = m(1 + e^x z + e^y w) = \frac{1}{\pi} D(e^{-x+i\theta_2}) + \frac{\theta_1}{\pi} x + \frac{\theta_2}{\pi} y, \quad (4.3)$$

Figure 3. Profile of clover quiver.

and in unbounded complement of the Amoeba we have,

$$N_{\mathbb{C}^3}(x, y) = \max(x, y, 0). \quad (4.4)$$

Comparing with the Legendre transformation in eq. (3.13), we observe that

$$\sigma_{\mathbb{C}^3} = -\frac{1}{\pi} D(e^{-x+i\theta_2}), \quad n_x = \frac{\partial N_{\mathbb{C}^3}}{\partial x} = \frac{\theta_1}{\pi}, \quad n_y = \frac{\partial N_{\mathbb{C}^3}}{\partial y} = \frac{\theta_2}{\pi}. \quad (4.5)$$

Using the trigonometry of the triangle in figure 2, the slopes of $\mathbb{C}^3$ quiver is explicitly obtained as

$$n_x = \begin{cases} 0 & x < \log |e^y - 1| \\ \frac{1}{\pi} \cos^{-1} \left(\frac{e^{2y} - e^{2x} + 1}{2e^y} \right) & \log |e^y - 1| \leq x \leq \log |e^y + 1|, \\ 1 & x > \log |e^y + 1| \end{cases} \quad (4.6)$$

and

$$n_y = \begin{cases} 0 & y < \log |e^x - 1| \\ \frac{1}{\pi} \cos^{-1} \left(\frac{e^{2x} - e^{2y} + 1}{2e^x} \right) & \log |e^x - 1| \leq y \leq \log |e^x + 1|, \\ 1 & y > \log |e^x + 1| \end{cases} \quad (4.7)$$

The slope can also be obtained directly by residue computation of the derivatives of the Ronkin function [9]. Using eqs. (4.3), (4.6) and (4.7), profile of the quiver, can be explicitly obtained in terms of (x, y) coordinates,

$$N_{\mathbb{C}^3}(x, y) = x n_x(x, y) + y n_y(x, y) + \frac{1}{\pi} D(e^{-x+i\pi n_y(x, y)}), \quad (4.8)$$

Figure 4. Minus entropy density of clover quiver.

as it is plotted in figure 3. To compute the entropy density $\sigma_{\mathbb{C}^3}(n_x, n_y)$ as a function of the slope, we need to solve the system of equations (4.6) and (4.7) for x , so we obtain

$$e^{-x} = \csc(\pi n_x) \sin\left(\pi(n_x - n_y)\right), \quad (4.9)$$

and then replacing the result in (4.5), we have

$$\sigma_{\mathbb{C}^3}(n_x, n_y) = -\frac{1}{\pi} D\left(e^{i\pi n_y} \csc(\pi n_x) \sin\left(\pi(n_x - n_y)\right)\right). \quad (4.10)$$

The slope of the dimer model is restricted to Newton polygon, $(n_x, n_y) \in \text{NP}(P)$, see section 3.1, and the entropy density in this region is plotted in figure 4.

In order to explore the phase structure, we use a helpful expression for the entropy density as a function of slopes. Eq. (3.10) together with properties of the Bloch-Wigner function in section 3.1, applied in eq. (4.5), imply

$$\sigma_{\mathbb{C}^3}(n_x, n_y) = -\frac{1}{\pi} \left(L(\pi n_x) + L(\pi n_y) + L(\pi(1 - n_x - n_y)) \right). \quad (4.11)$$

This result, for the surface tension of the honeycomb dimer model is also obtained by other plausible methods in [23].

To compute the BPS growth rate we need the Mahler measure $m_{\mathbb{C}^3} = m(1 + z + w)$. This can be obtained from eq. (4.3), by choosing $a = 1$, $b = 1$ and $c = 1$ in eq. (4.1), or equivalently by the Ronkin function at origin

$$m_{\mathbb{C}^3} = N_{\mathbb{C}^3}(0, 0) = -\sigma_{\mathbb{C}^3}(2/3, 1/3) = \frac{1}{\pi} D(e^{i\pi/3}) = \frac{3}{\pi} L(\pi/3) \approx 0.32, \quad (4.12)$$

where the extremum point of the entropy density is obtained as $(n_x^*, n_y^*) = (2/3, 1/3)$. This is consistent with the solution of $x = y = 0$ in eqs. (4.6) and (4.7), since for two opposite

angles of triangle, θ_1 and θ_2 , angle θ_1 is $\pi - \theta_1$ in the unit circle. Thus, the BPS growth rate in this quiver can be computed explicitly,

$$\log \Omega_{\mathbb{C}^3}(n) \sim m_{\mathbb{C}^3}^{1/3} n^{2/3} \approx 0.68 n^{2/3}. \quad (4.13)$$

This is in agreement with the asymptotics $(\zeta(3)/4)^{1/3} n^{2/3}$ of the crystal model of $\mathbb{C}^3$ quiver, the plane partition.

4.2 Resolved conifold quiver $\mathcal{C}$

The resolved conifold quiver, figure 5, is characterized by the Newton polynomial $P(z, w) = e^{-t}zw + z + w - 1$. The Amoeba of the resolved conifold is plotted in figure 10. The statistical mechanics of the resolved conifold quiver is encoded in following result for the Mahler measure of $\mathcal{C}$ -type Calabi-Yau.

Theorem (Vandervelde [24]). *Consider Newton polynomial $P(z, w) = azw + bz + cw + d$ for $a, b, c, d \in \mathbb{R}$. Suppose there is a non-degenerate convex cyclic quadrilateral $ABCD$ with edge lengths $|a|, |b|, |d|, |c|$ in cyclic order, inscribed in a circle and angles $\alpha, \beta, \delta, \gamma$, of arcs cut off by the edges of the quadrilateral, see figure 6. Then, the Mahler measure, inside the Amoeba, is given by*

$$\pi m = D\left(\frac{au}{b}\right) - D\left(\frac{cu}{d}\right) + \alpha \log |a| + \beta \log |b| + \gamma \log |c| + \delta \log |d|, \quad (4.14)$$

and in the non-quadrilateral case, outside Amoeba, we have

$$m = \max(\log |a|, \log |b|, \log |c|, \log |d|), \quad (4.15)$$

where u is the solution for w of the following equations,

$$P(z, w) = 0, \quad |z| = 1, \quad |w| = 1. \quad (4.16)$$

To obtain the profile of the resolved conifold quiver from eq. (3.11), we need to make the following choice $a = e^{-t+x+y}, b = e^x, c = e^y, d = -1$ in the above Mahler measure in “Vandervelde Theorem”, and thus for x, y inside the Ameoba we have

$$\pi N_{\mathcal{C}_t}(x, y; t) = D(e^{-t+y}u_t) - D(-e^y u_t) + (\alpha + \beta)x + (\alpha + \gamma)y - \alpha t, \quad (4.17)$$

otherwise,

$$N_{\mathcal{C}_t}(x, y; t) = \max(-t + x + y, x, y, 0), \quad (4.18)$$

where we obtain u from eq. (4.16),

$$u_t(x, y) = \frac{e^{-y}}{2(1 + e^{2x-t})} \left(r + \sqrt{r^2 - 4(e^y + e^{y+2x-t})^2} \right), \quad (4.19)$$

Figure 5. Conifold quiver diagram.

Figure 6. Cyclic quadrilateral for resolved conifold.

with $r = 1 + e^{2y} - e^{2x} - e^{2x+2y-2t}$. The angles $\alpha, \beta, \gamma, \delta$ of the cyclic quadrilateral can be found in terms of x, y, t , using our choice of the parameters a, b, c, d and the trigonometry of the cyclic quadrilateral,

$$\begin{aligned} \cos(\alpha + \beta) &= -\frac{|a|^2 + |b|^2 - |c|^2 - |d|^2}{2(|a||b| + |c||d|)}, & \cos(\alpha + \gamma) &= \frac{|b|^2 + |d|^2 - |a|^2 - |c|^2}{2(|b||d| + |a||c|)}, \\ \cos(\beta + \delta) &= \frac{|a|^2 + |c|^2 - |b|^2 - |d|^2}{2(|a||c| + |b||d|)}, & \alpha + \beta + \gamma + \delta &= \pi. \end{aligned} \quad (4.20)$$

Comparing eqs. (4.17) and (3.13), we obtain the entropy density and slope of the resolved conifold quiver,

$$\sigma_{\mathcal{C}_t} = -\frac{1}{\pi} \left(D(e^{-t+y}u_t) - D(-e^y u_t) - \alpha t \right), \quad (4.21)$$

$$n_x = \frac{\partial N_{\mathcal{C}_t}}{\partial x} = \frac{\alpha + \beta}{\pi}, \quad n_y = \frac{\partial N_{\mathcal{C}_t}}{\partial y} = \frac{\alpha + \gamma}{\pi}. \quad (4.22)$$

By using eqs. (4.19), (4.20) and (4.22), one can make the following observation

$$u_t(x, y) = e^{i\pi n_x}. \quad (4.23)$$

The entropy density can be written in terms of slope,

$$\sigma_{\mathcal{C}_t}(n_x, n_y) = -\frac{1}{\pi} \left(D(f(n_x, n_y)) - D(g(n_x, n_y)) - h(n_x, n_y)t \right), \quad (4.24)$$

where $f(n_x, n_y)$, $g(n_x, n_y)$, and $h(n_x, n_y)$ are the inverse functions of $e^{-t+y}u_t$, $-e^y u_t$, and α , respectively. They can be obtained explicitly using eq. (4.20) with the choice of parameters for resolved conifolds and equations for slopes in eq. (4.21).

Next we consider an important limit $t \rightarrow 0$ in the resolved conifold, to obtain the conifold quivers with $P(z, w) = -1 + z + w + zw$. To obtain the profile function and entropy density of the conifold, we take the limit $t \rightarrow 0$ of eq. (4.17),

$$\begin{aligned}\pi N_{\mathcal{C}}(x, y) &= D(e^y u_0) - D(-e^y u_0) + (\alpha' + \beta')x + (\alpha' + \gamma')y, \\ \sigma_{\mathcal{C}} &= -\frac{1}{\pi} \left(D(e^y u_0) - D(-e^y u_0) \right).\end{aligned}\quad (4.25)$$

where α', β', γ' are $t \rightarrow 0$ limit of α, β, γ , and $u_0(x, y) = \lim_{t \rightarrow 0} u_t(x, y)$.

Solving eqs. (4.20) for y , with the choice of conifold parameters $a = e^{x+y}, b = e^x, c = e^y, d = -1$, we obtain,

$$e^y = \pm \sqrt{\csc^2(\pi n_y) (1 - \cos(2\pi n_x) \cos^2(\pi n_y) + 2I)}, \quad (4.26)$$

where $I = \sqrt{\cos^2(\pi n_y) (1 - \cos^2(\pi n_x) \cos^2(\pi n_y)) \sin^2(\pi n_x)}$. Using eqs. (4.25) and (4.26), we can write the entropy density in terms of the slopes of the conifold,

$$\sigma_{\mathcal{C}} = -\frac{1}{\pi} \left(D(e^y e^{i\pi n_x}) - D(e^y e^{i\pi(n_x+1)}) \right), \quad (4.27)$$

where $\pi n_x = \alpha' + \beta'$ and $\pi n_y = \alpha' + \gamma'$. The entropy density is plotted in figure 8. Furthermore, the profile function can be written in terms of x, y coordinates by using eqs. (4.19) and (4.20) in the first line of eq. (4.25). The profile function of the conifold quiver is plotted in figure 7.

We can find the entropy density of the conifold in terms of the Lobachevsky function. First, define $\xi = e^y u_0$, then by using properties of the Bloch-Wigner function and Lobachevsky function, from eq. (4.25) we obtain

$$\begin{aligned}\sigma_{\mathcal{C}} &= -\frac{1}{\pi} \left(D(\xi) + D(1 + \xi) \right) \\ &= -\frac{1}{\pi} \left(L(\arg(\xi)) + L\left(\arg\left(1 - \frac{1}{\xi}\right)\right) + L\left(\arg\left(\frac{1}{1 - \xi}\right)\right) \right. \\ &\quad \left. + L\left(\arg\left(-\frac{1}{\xi}\right)\right) + L\left(\arg\left(\frac{\xi}{1 + \xi}\right)\right) + L(\arg(1 + \xi)) \right).\end{aligned}\quad (4.28)$$

Remember that L is an odd and π -periodic function, thus we have $L(\arg(\xi)) + L(\arg(-\frac{1}{\xi})) = 0$ and

$$\begin{aligned}\sigma_{\mathcal{C}} &= -\frac{1}{\pi} \left(L\left(\arg\left(1 - \frac{1}{\xi}\right)\right) + L\left(\arg\left(\frac{1}{1 - \xi}\right)\right) + L\left(\arg\left(\frac{\xi}{1 + \xi}\right)\right) + L(\arg(1 + \xi)) \right) \\ &= -\frac{1}{\pi} \left(L(\alpha') + L(\beta') + L(\gamma') + L(\delta') \right) \\ &= -\frac{1}{\pi} \left(L(\alpha') + L(\beta') + L(\gamma') - L(\alpha' + \beta' + \gamma') \right).\end{aligned}\quad (4.29)$$

This result exactly matches with the following result in domino tiling [25],

$$\sigma_{\mathcal{C}} = -\frac{1}{\pi} \sum_{i=1}^4 L(p_i), \quad (4.30)$$

where p_i is the arc length on the circle cut off by i -th edge of the quadrilateral.

Figure 7. Profile of the conifold quiver.

Limit $t \rightarrow \infty$ is another interesting limit, in which we have $a \rightarrow 0$ and $\alpha \rightarrow 0$. Moreover, using the relation between the angle and length of the chord on the unit circle, $\alpha = \arcsin(\frac{e^{-t+x+y}}{2})$, we obtain $\lim_{t \rightarrow \infty} \alpha t = \lim_{t \rightarrow \infty} \arcsin(\frac{e^{-t+x+y}}{2})t \rightarrow 0$. Thus, in this limit, using eq. (4.21), we obtain

$$\lim_{t \rightarrow \infty} \sigma_{C_t} = \frac{1}{\pi} D(-\zeta), \quad (4.31)$$

where we defined $\zeta = \lim_{t \rightarrow \infty} e^y u_t(x, y)$, and from eq. (4.19)

$$\lim_{t \rightarrow \infty} u_t(x, y) = \frac{1}{2} e^{-y} \left(1 + e^{2y} - e^{2x} + \sqrt{(-1 - e^{2y} + e^{2x})^2 - 4e^{2y}} \right). \quad (4.32)$$

In the resolved conifold, the parameter t , for large values of t , can be seen as the length of the internal leg in the Amoeba of the resolved conifold, figure 10. The amoeba of the resolved conifold consists of two $\mathbb{C}^3$ vertices with a phase difference of π and separated by an internal leg of length t . As $t \rightarrow \infty$ we have two isolated vertices. In this limit, the length of the internal leg of the Amoeba is infinitely large and, as we expect, we get the π -rotated $\mathbb{C}^3$ vertex with entropy density $\sigma_{\pi(\mathbb{C}^3)} = \frac{1}{\pi} D(-\zeta)$ from eq. (3.10). This is consistent with the entropy of the clover quiver in section 4.1, since we observe $\gamma = \theta_2, \delta = \theta_1$ in the limit $\alpha \rightarrow 0$.

To study the BPS growth rate of the resolved conifold, we need to find the Mahler measure with the following choice of parameters $a = e^{-t}, b = c = 1, d = -1$. First, we find $u_{C_t}(t)$ by solving eq. (4.16),

$$u_{C_t}(t) = \frac{1 - e^{-2t} + \sqrt{(-3 + e^{-t})(1 + e^{-t})^3}}{2 + 2e^{-t}}. \quad (4.33)$$

Figure 8. Minus entropy density of the conifold quiver.

The dilogarithmic part of the Mahler measure of the resolved conifold can be obtained from eq. (4.17) by putting $x = 0, y = 0$ and removing linear term $-\alpha t$,

$$m_{\mathcal{C}_t}(t) = \frac{1}{\pi} \left(D(e^{-t} u_{\mathcal{C}_t}(t)) - D(-u_{\mathcal{C}_t}(t)) \right). \quad (4.34)$$

Then, the associated BPS growth rate to this Mahler measure is given by

$$\log \Omega_{\mathcal{C}_t}(n) \sim m_{\mathcal{C}_t}^{1/3} n^{2/3}, \quad (4.35)$$

and the numerical coefficient $m_{\mathcal{C}_t}^{1/3}$ is plotted in figure 9.

In the limit $t \rightarrow 0$, we can obtain the Mahler measure of the conifold,

$$\lim_{t \rightarrow 0} \left(m_{\mathcal{C}_t}(t) - \frac{\alpha}{\pi} t \right) = m_{\mathcal{C}} = N_{\mathcal{C}}(0, 0) = -\sigma_{\mathcal{C}}(1/2, 1/2), \quad (4.36)$$

$$m_{\mathcal{C}} = \frac{1}{\pi} \left(D(i) - D(-i) \right) = \frac{2}{\pi} D(e^{i\pi/2}) = \frac{4}{\pi} L(\pi/2) = \frac{2}{\pi} \Im[Li_2(e^{i\pi/2})]. \quad (4.37)$$

We observe the Catalan constant G appears in this case,

$$m_{\mathcal{C}} = \frac{2G}{\pi} \approx 0.58. \quad (4.38)$$

This result matches with the asymptotic number of domino tilings [26, 27]. Thus we obtain the BPS growth rate of conifold

$$\log \Omega_{\mathcal{C}}(n) \sim m_{\mathcal{C}}^{1/3} n^{2/3} = \left(\frac{2G}{\pi} \right)^{1/3} n^{2/3} \approx 0.83 n^{2/3}. \quad (4.39)$$

The Mahler measure in the limit $t \rightarrow \infty$, becomes

$$\lim_{t \rightarrow \infty} m_{\mathcal{C}_t} = -\frac{1}{\pi} D(e^{-2i\pi/3}) = -\frac{1}{\pi} \Im[Li_2(e^{-2i\pi/3})] \approx 0.21. \quad (4.40)$$

Figure 9. BPS growth rate of resolved conifold.

Compare to eq. (4.12), we observe a rotation by π in the clover quiver result,

$$\lim_{t \rightarrow \infty} m_{C_t} = -m_{\pi(\mathbb{C}^3)} = -\frac{1}{\pi} D(e^{i(\pi+\pi/3)}). \quad (4.41)$$

This is consistent from the geometrical point of view, as we discussed before. The BPS growth rate in the limit becomes

$$\lim_{t \rightarrow \infty} \log \Omega_{C_t}(n) \sim \lim_{t \rightarrow \infty} (m_{C_t})^{1/3} n^{2/3} \approx 0.59 n^{2/3}. \quad (4.42)$$

However, in the limit $t \rightarrow \infty$, the term $e^{-t}zw$ in the Newton polynomial vanishes and we expect to reproduce the clover quiver result. This result can be recovered by a π -transformation ($-2\pi/3 \rightarrow \pi - (-2\pi/3) = -\pi/3$) in our result. This is because, for a π -rotated $\mathbb{C}^3$ vertex, the appropriate angle to consider in the unit circle is obtained as $\pi - \theta$ instead of θ . Finally, twisted BPS growth rate in this case is

$$\lim_{t \rightarrow \infty} \log \Omega_{C_t}^*(n) \sim \left(-\frac{1}{\pi} D(e^{-i\pi/3}) \right)^{1/3} n^{2/3} = m_{\mathbb{C}^3}^{1/3} n^{2/3} \approx 0.68 n^{2/3}. \quad (4.43)$$

Phase structure and PDEs for clover and conifold quivers. The phase structures of clover and conifold quivers are the Amoebas of associated Newton polynomials, plotted in 10. As we explained in section 3.5, the zero slope limit, $n_x, n_y \rightarrow 0$ gives the boundary of the phase transition. We consider a facet of quiver, in which $(s, t) = (0, 0)$ and study the continuity of the entropy density and its derivative across the boundary of this facet. Using eq. (4.11), and the following approximation of the Lobachevsky function near $\theta = 0$,

$$L(\theta) \approx -\theta \log \theta + O(\theta^2), \quad (4.44)$$

the entropy density of clover quiver near the zero slope becomes

$$\sigma_{\mathbb{C}^3}(n_x, n_y) \approx n_x \log \left(\frac{n_x}{n_x + n_y} \right) + n_y \log \left(\frac{n_y}{n_x + n_y} \right) + O(n_x^2, n_y^2). \quad (4.45)$$

Figure 10. Amoebas of the clover (left), conifold (middle), and resolved conifold (right) quivers.

Then, we observe that the entropy density vanishes in the zero slope limit since $\lim_{n_x \rightarrow 0} \sigma(n_x, n_y) = \lim_{n_y \rightarrow 0} \sigma(n_x, n_y) = 0$ and this confirms its continuity across the boundary of facet. However, we observe that the first derivative of entropy density in the limits $n_x \rightarrow 0$ and $n_y \rightarrow 0$ does not commute and in fact it diverges as $\lim_{n_x \rightarrow 0} \frac{\partial \sigma(n_x, n_y)}{\partial n_x} \rightarrow \infty$. Thus, the first derivative of the entropy density is not continuous on the boundary of the facet and we observe a phase transition. According to the discussion in section 3.5, this result suggests a possible third order phase transition.

In the case of the conifold quiver, the slope is given by eq. (4.22), and the zero slope limit implies $\alpha, \beta, \gamma \rightarrow 0$. Thus, using eqs. (4.29) and (4.30), close to zero slope, in the leading order, we obtain

$$\sigma_C \approx \frac{1}{\pi} (\alpha' \log \alpha' + \beta' \log \beta' + \gamma' \log \gamma' - (\alpha' + \beta' + \gamma') \log(\alpha' + \beta' + \gamma')). \quad (4.46)$$

Similar to the clover quiver, the entropy density is continuous across the boundary of the zero slope facet but since the first derivative of $\theta \log \theta$ term diverges at zero, we observe a discontinuity in the entropy density of the conifold quiver, and thus a possible third-order phase transition. Considering the entropy density expressed as a Bloch-Wigner function, the result of this section is consistent with the continuity but non-differentiability of the Bloch-Wigner function at the two points 0 and 1, mentioned in [15]. A proper study of phase structure of the quivers considering all the facets of quiver requires further work and remains for future investigation.

From the Monge-Ampère equation (3.26) in the $\mathbb{C}^3$ quiver with entropy density in eqs. (4.5) and (4.11), we find an explicit solution for the Monge-Ampère equation,

$$\mathcal{M}\left(D(e^{-x+i\pi n_y})\right) = \mathcal{M}\left(L(\theta_1) + L(\theta_2) - L(\theta_1 + \theta_2)\right) = 1/\pi. \quad (4.47)$$

Furthermore, by using eqs. (4.27) and (4.29) in the conifold quiver we have

$$\begin{aligned} \mathcal{M}\left(D(e^{y+i\pi n_x}) - D(-e^{y+i\pi n_x})\right) &= \mathcal{M}\left(L(\alpha') + L(\beta') + L(\gamma') - L(\alpha' + \beta' + \gamma')\right) \\ &= 1/\pi \end{aligned} \quad (4.48)$$

We make an interesting observation that the Lobachevsky and Bloch-Wigner functions are solutions of Monge-Ampère equation. We are not aware of any possible application or implications of this result for PDEs.

Figure 11. Amoebas of $\mathbb{F}_0$ quiver at $k = 4$ (left) and $k = 5$ (right).

4.3 Hirzebruch quiver $\mathbb{F}_0$

The next example is $\mathbb{F}_0$ quiver characterized with Newton polynomial $P(z, w) = z + w + \frac{1}{z} + \frac{1}{w} - k$. This quiver is dual to the square-octagon dimer model [28]. The Amoeba of this quiver is plotted in figure 11 for two different values of parameter k . The bounded component of complement of the Amoeba is the gas phase of the quiver and its area is a function of $k - 4$. For $k > 4$, this is the simplest quiver with a gaseous phase.

The current techniques in Mahler measure theory do not allow us to explicitly compute the Mahler measure of $P(e^x z, e^y w)$. Thus, we cannot develop statistical mechanics of this quiver to obtain the Ronkin function and entropy density of this quiver. However, we can obtain the BPS growth rate from the Mahler measure of $P(z, w)$ for $k > 4$, computed in [29],

$$m_{\mathbb{F}_0}(k) = \frac{1}{2} \left(\log k^{-2} - 4k^{-2} {}_4F_3 \left(\begin{matrix} 3/2 & 3/2 & 1 & 1 \\ 2 & 2 & 2 \end{matrix} ; 16k^{-2} \right) \right), \quad (4.49)$$

where F is a hypergeometric function,

$${}_pF_q \left(\begin{matrix} a_1 & a_2 & \dots & a_p \\ b_1 & b_2 & \dots & b_q \end{matrix} ; x \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_p)_n}{(b_1)_n (b_2)_n \dots (b_q)_n} \frac{x^n}{n!}, \quad (4.50)$$

and $(c)_n = \frac{\Gamma(c+n)}{\Gamma(c)}$. Using the Mahler measure, we can obtain the BPS growth rate in this quiver,

$$\log \Omega_{\mathbb{F}_0}(k) \sim m_{\mathbb{F}_0}(k)^{1/3} n^{2/3}. \quad (4.51)$$

The dependence on k for the numerical coefficient of BPS growth rate, for $k \geq 4$, is plotted in figure 12. From figure 12, we observe that BPS growth rate is increasing with the size of the gaseous phase, k . At $k = 4$, one can show that

$$m_{\mathbb{F}_0}(4) = \frac{4G}{\pi} = 4\pi^{-1} D(e^{i\pi/2}) = 4\pi^{-1} \Im[Li_2(i\pi/2)] \approx 1.16, \quad (4.52)$$

and we observe $m_{\mathbb{F}_0}(4) = 2m_{\mathcal{C}}$. Finally, the minimum of the BPS growth rate is

$$\log \Omega_{\mathbb{F}_0}(4) \sim m_{\mathbb{F}_0}(4)^{1/3} n^{2/3} = \left(\frac{4G}{\pi} \right)^{1/3} n^{2/3} \approx 1.05 n^{2/3}. \quad (4.53)$$

Figure 12. BPS growth rate of $\mathbb{F}_0$ quiver.

4.4 $\mathbb{C}^3/\mathbb{Z}_p \times \mathbb{Z}_p$ orbifold quivers and BPS black holes

As a natural generalization of clover and conifold examples, we consider orbifold quivers [30]. The Mahler measure of orbifold $\mathbb{C}^3/\mathbb{Z}_p \times \mathbb{Z}_p$ for $p \geq 2$, denoted by $\mathcal{O}_p$, can be written as

$$m_{\mathcal{O}_p} = \int_{|z|=1} \int_{|w|=1} \log \left| 1 + \sum_{i=1}^p (z^i + w^i) + \sum_{i=1}^{p-1} z^i w^{p-i} \right| \frac{dz}{z} \frac{dw}{w}. \quad (4.54)$$

Current techniques in Mahler measure theory, do not allow for an explicit computation of the above integral for a general p , however, the simplest case $p = 2$, $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold quiver with the Newton polynomial $P(z, w) = 1 + z + z^2 + w + w^2 + zw$, has been analytically computed [31],

$$m_{\mathcal{O}_2} = \frac{1}{\pi} \left(2D(i) - \frac{3}{4} D(e^{\frac{2\pi i}{3}}) \right) = \frac{1}{\pi} \left(2G - \frac{3}{4} \Im [Li_2(e^{\frac{2\pi i}{3}})] \right) \approx 0.42. \quad (4.55)$$

Thus, we can obtain the BPS growth rate for this orbifold,

$$\log \Omega_{\mathcal{O}_2} \sim m_{\mathcal{O}_2}^{1/3} n^{2/3} \approx 0.74 n^{2/3}. \quad (4.56)$$

In the following we propose a physical application of our results for certain type of BPS black holes associated with the orbifolds of $\mathbb{C}^3$, constructed in [10]. We will interpret the BPS growth rate as the entropy of such black holes. A bijection between crystal melting configurations and BPS black holes given by $D0$ -, $D2$ - and $D4$ -branes wrapped on collapsed cycles on the large p orbifolds $\mathbb{C}^3/\mathbb{Z}_p \times \mathbb{Z}_p$ is established in [10]. The spectrum of BPS black hole configurations are generated by geometric transitions of brane configurations and parametrized by crystal melting configurations. In the limit of large charges, a stable configuration in the orbifold will produce a local model for a BPS black hole in four dimensions. At low energies, these charge configurations are explained by quiver quantum

mechanics. The profile of the partially melted crystal is inscribed in the ranks of the gauge groups in the quiver. These ranks of the gauge groups determine the D-brane charges.

The untwisted RR D0-brane charge of a black hole configuration is realized by branes wrapped on cycles in the orbifolds. The crystal melting model for counting $D0$ brane bound states in a $D6$ brane filling $\mathbb{C}^3$, gives the RR $D0$ -brane charges of the orbifold BPS black holes. Up to a factor $1/p^2$ this charge is given by the total number of atoms in the corresponding crystal melting model and thus the charge is equal to sum over the ranks of gauge groups in the quiver quantum mechanics, $Q(\text{BH}) = \frac{1}{|\Gamma|} \sum_i N_i$ with $|\Gamma| = p^2$ for $\mathbb{C}^3/\mathbb{Z}_p \times \mathbb{Z}_p$. The $D0$ charge characterizes the energy of the ensemble of the black holes, $E_{\text{BH}} = |\Gamma|Q(\text{BH})$. Therefore, the partition function of the black hole is given by the crystal melting model partition function

$$\mathcal{Z}_{\text{BH}} = \sum_{\text{BH}} q^{|\Gamma|Q(\text{BH})} = \sum_{\text{Quiver}} q^{\sum_i N_i} = \prod_{k \geq 1} \frac{1}{(1 - q^k)^k}. \quad (4.57)$$

The charge of the BPS black hole in the large p limit can be obtained from the profile function of the quiver in the thermodynamic limit. Thus, the Ronkin function can be interpreted as the charge density function of the heavy state of the black hole and the integral of charge density gives the charge of the black hole $Q^* = \frac{1}{p^2} \int N(x, y) dx dy$. In the thermodynamic limit, consider the statistical average of ranks of the gauge groups in the quiver. Then, the average charge of the black hole can be calculated as

$$\begin{aligned} \langle Q \rangle &= \sum_i \frac{1}{p^2} \langle N_i \rangle = \frac{1}{p^2} q \frac{\partial}{\partial q} \log \mathcal{Z}_{\text{BH}} = \frac{1}{p^2} \sum_{k=1}^{\infty} \frac{k^2 q^k}{1 - q^k} \\ &\sim \frac{1}{p^2 g_s^3} \int_0^{\infty} du \frac{u^2 e^{-u}}{1 - e^{-u}} = \frac{2\zeta(3)}{p^2 g_s^3}. \end{aligned} \quad (4.58)$$

Since $\text{var}(Q) \sim O(g_s^{-4}) \rightarrow 0$, as $g_s \rightarrow 0$ we have $\langle Q \rangle = Q^*$. Therefore, the black hole states in the thermodynamic limit tend to the typical state corresponding to the limit shape and this typical state is in fact the state with the average charge. Similarly, the entropy density of the clover quiver can be interpreted as the entropy density of the black hole with a certain charge and the Mahler measure as the BH growth rate, up to a scaling power. Interpreting the BH growth rate as the entropy of the black hole, using eqs. (3.19), (4.13) and (4.58) we obtain

$$S_{\text{BH}} = \log \Omega_{\text{BH}} = \lim_{p \rightarrow \infty} (p^2 Q^*)^{\frac{2}{3}} m_{\mathbb{C}^3}^{\frac{1}{3}} = \frac{\zeta(3)}{g_s^2}. \quad (4.59)$$

5 Conclusion and discussion

In this work, we used the toroidal dimer model and Mahler measure theory to obtain the asymptotic observables of the toric quiver gauge theories in terms of the Mahler measure. Our focus in this work was on explicit computations of the observables in concrete examples. As a result of these examples we developed a new statistical mechanics of the resolved conifold quiver.

The initial goal of this work was to explicitly obtain the numerical values of BPS growth rates in different examples of quivers. We found that the BPS growth rate is proportional to the Mahler measure up to a certain scaling power, however, the factor of proportionality is not yet explicitly obtained. we conjecture that this factor is $1/c$ where c is the number of different types of dimer in a specific dimer model. For example, in the honey-comb dimer model we have three different types of dimers and with the factor of one-third inserted in eq. (4.13), it exactly matches with the asymptotic number of plane partitions. Following this conjecture and using obtained results for Mahler measures, we can compare the asymptotic BPS index of different quivers,

$$\Omega_{\mathbb{F}_0} > \Omega_{\mathcal{C}} > \Omega_{\mathcal{O}_2} > \Omega_{\mathbb{C}^3}. \quad (5.1)$$

Explicit computations of the profile function, entropy density and BPS growth rate in other examples of the toric quivers such as orbifolds of $\mathbb{C}^3$ and $\mathcal{C}$ are very interesting. A related direction for future studies is the full asymptotic analysis of $\mathbb{F}_0$ quiver in which the Amoeba has a gas phase and in principle we can count D6-D4-D2-D0 bound states. Analytic computations of the profile function and entropy density are not known in this example.

The class of isoradial dimer model is of special interest because of their relations to $\mathcal{N} = 1$ superconformal field theories and brane tilings [32]. The surface tension and Mahler measure of isoradial dimer model and their relation to hyperbolic volumes of certain polyhedrons have been partially studied recently [33, 34]. In fact, the entropy density of clover and conifold quivers, obtained in this paper, are the Hyperbolic volumes of some ideal tetrahedron and pyramid. Further understanding of the connection between entropy density in this work or topological indices of $\mathcal{N} = 1$ quiver theory with hyperbolic volumes using techniques of dimer model is a possible continuation of this work.

In this work, we mainly considered the leading order asymptotics to the BPS counting problem. The fluctuations around the Ronkin function contribute to higher order corrections. Presumably, these corrections capture the partition function of higher genus topological strings. Another interesting direction for future studies is to develop the asymptotic analysis for the D4-D2-D0 BPS bound states and corresponding the two-dimensional crystal model [35]. A plausible approach would be to consider the zero slope limit of the results of the three-dimensional crystal model.

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