Two-loop $\mathcal{N} = 2$ SQCD amplitudes with external matter from iterated cuts

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ABSTRACT: We develop an iterative method for constructing four-dimensional generalized unitarity cuts in $\mathcal{N} = 2$ supersymmetric Yang-Mills (SYM) theory coupled to fundamental matter hypermultiplets ($\mathcal{N} = 2$ SQCD). For iterated two-particle cuts, specifically those involving only four-point amplitudes, this implies simple diagrammatic rules for assembling the cuts to any loop order, reminiscent of the rung rule in $\mathcal{N} = 4$ SYM. By identifying physical poles, the construction simplifies the task of extracting complete integrands. In combination with the duality between color and kinematics we construct all four-point massless MHV-sector scattering amplitudes up to two loops in $\mathcal{N} = 2$ SQCD, including those with matter on external legs. Our results reveal chiral infrared-finite integrands closely related to those found using loop-level BCFW recursion. The integrands are valid in $D \leq 6$ dimensions with external states in a four-dimensional subspace; the upper bound is dictated by our use of six-dimensional chiral $\mathcal{N} = (1, 0)$ SYM as a means of dimensionally regulating loop integrals.

KEYWORDS: Duality in Gauge Field Theories, Scattering Amplitudes, Supersymmetric Gauge Theory

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1 Introduction

Supersymmetric Yang-Mills (SYM) theories are well known to have simpler scattering amplitudes than the most physically interesting gauge theory — quantum chromodynamics (QCD). In planar $\mathcal{N} = 4$ SYM theory, for instance, modern methods have enabled five-loop six-point amplitude computations [1], with the four- and five- point amplitudes known to all loop orders for more than a decade [2, 3]. Even more is known about amplitude integrands in $\mathcal{N} = 4$ SYM, with all-loop n -point results in the maximally helicity-violating (MHV) sector [4–6], their two-loop extensions beyond MHV [7], and higher-loop but lower-point results beyond the leading-color (planar) limit [8–13].

Such studies of SYM theories, together with impressive developments beyond next-to-leading order [14–37], have helped mature modern on-shell methods [38–45]. Together with other techniques, these methods are widely used in current state-of-the-art QCD calculations — nowadays involving two-loop five-parton amplitudes [46–53]. Moreover, amplitudes in supersymmetric gauge theories can often be viewed as specific contributions to QCD amplitudes, at least at tree [54–56] and one-loop level [38, 57, 58]. In these ways SYM calculations have paved the way to new results in QCD.

Recent all-loop BCFW constructions of four-dimensional amplitude integrands in $\mathcal{N} = 4$ SYM [4–6] were preceded by a more pedestrian way of constructing integrands, often referred to as the “rung rule” [59, 60]. It is based on an analysis of two- and three-particle unitarity cuts and their iterative structure. The idea is to directly obtain $(L + 1)$ - from L -loop integrands by attaching rungs to the individual diagrams:

$$\begin{array}{c} \xrightarrow{\ell_1} \cdots \\ \xrightarrow{\ell_2} \cdots \end{array} \rightarrow -i(\ell_1 + \ell_2)^2 \times \begin{array}{c} \xrightarrow{\ell_1} \cdots \\ \text{---} \text{---} \text{---} \\ \xrightarrow{\ell_2} \cdots \end{array} \quad (1.1)$$

where each rung comes with a kinematic factor. Despite its known shortcomings (it does not give unique representations of the integrand, and if one desires integrands obeying color-kinematics duality [61–63] then the results are not always compatible) the rung rule has been instrumental to initial progress in $\mathcal{N} = 4$ SYM.

In this paper we develop an iterative approach for computing generalized unitarity cuts in $\mathcal{N} = 2$ supersymmetric QCD (SQCD),¹ which is reminiscent of the rung rule and helps us construct amplitude integrands. This theory is equivalent to $\mathcal{N} = 2$ SYM coupled to

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¹Our methods are in particular inspired by supersum technologies developed in ref. [9].

N_f copies of massless $\mathcal{N} = 2$ matter multiplets in the (anti-)fundamental representation of the (arbitrary) gauge group G . It is therefore more similar to ordinary QCD than $\mathcal{N} = 4$ SYM, while retaining considerable simplifications with respect to the former. This makes it an ideal theory from which to study the effect of reducing supersymmetry on the analytic structure of gauge theories — an open question that is crucial should one wish to extend the impressive progress in $\mathcal{N} = 4$ SYM to QCD.

Our concrete results, obtained using the rung-rule-like iterative structure of the unitarity cuts, are the complete set of massless four-point MHV amplitude integrands up to two loops, including those with external matter states. The one-loop amplitude with four external matter states has already been computed using an orbifold construction [64], the one- and two-loop amplitudes with four external gluons were determined in refs. [65, 66], and the rest were previously unknown. All of these full-color amplitudes are obtained in a form that respects color-kinematics duality [61–63]. They can therefore be used to produce amplitudes in $\mathcal{N} \geq 2$ pure or matter-coupled supergravities via an array of related double-copy constructions [61, 62, 64, 65, 67–75]. Other four-point one- and two-loop results in $\mathcal{N} = 2$ SQCD include refs. [76–79].

An intriguing new aspect of our approach is the appearance of Dirac traces in the kinematic numerators, which make infrared (IR) properties manifest. Their structure echoes the BCFW-derived expressions in $\mathcal{N} = 4$ SYM, which are known for having well-behaved IR structure; they are also similar to traces appearing in the planar two-loop all-plus amplitudes in non-supersymmetric Yang-Mills theory [80], which are well known thanks to their one-loop-like simplicity [46–48, 81–84].² A careful exposition of the IR properties of the two-loop $\mathcal{N} = 2$ SQCD integrands will be reported elsewhere, while in this paper we limit ourselves to explanatory comments during the derivation of our results.

The paper is organized as follows. In section 2 we review the relevant aspects of $\mathcal{N} = 2$ SQCD and its scattering amplitudes, and introduce the necessary tools to deal with color-kinematics duality in this theory. In section 3 we compare the structure of iterated two-particle cuts in this theory with that in $\mathcal{N} = 4$ SYM, and formulate diagrammatic generalized rung rules for the former. We use these rules in sections 4 and 5 to motivate — and, in some cases, fully derive — the kinematic numerators of all four-point amplitudes in $\mathcal{N} = 2$ SQCD, first at one loop and then at two loops. In section 6 we show the limits of applicability of our rung rules by studying more general unitarity cuts. We conclude in section 7 by discussing the interesting features of our results and their derivations, and outline our next steps in the analysis of the integrand structure of (S)QCD.

2 Review: $\mathcal{N} = 2$ SQCD

In this section we explain our approach to scattering amplitudes in $\mathcal{N} = 2$ SQCD, to a considerable degree following refs. [65, 66] but updating the notation as necessary to prepare for later sections. In particular, we introduce a new notation to compactly write four-point

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²In fact, local-integrand representations of all-plus amplitudes were inspired by those of $\mathcal{N} = 4$ SYM amplitudes based on a dimension-shifting relationship between their one-loop integrands [85] which persists at two loops.

tree-level amplitudes involving fundamental hypermultiplets on external legs. We also summarize the off-shell constraints to be placed on kinematic numerators, in addition to those required by color-kinematics duality [61, 62].

2.1 On-shell particle content

The on-shell content of four-dimensional $\mathcal{N} = 2$ SQCD is most easily described by comparison with that of $\mathcal{N} = 4$ SYM. The latter contains $2^4 = 16$ states, and forms a vector supermultiplet [86]:

$$\mathcal{V}_{\mathcal{N}=4}(\eta^I) = A^+ + \eta^I \psi_I^+ + \frac{1}{2} \eta^I \eta^J \varphi_{IJ} + \frac{1}{3!} \epsilon_{IJKL} \eta^I \eta^J \eta^K \psi_-^L + \eta^1 \eta^2 \eta^3 \eta^4 A_- . \quad (2.1)$$

The four-dimensional chiral superspace coordinates η^I carry SU(4) R-symmetry indices $\{I, J, \dots\}$. For later use, let us remark that this multiplet is CPT self conjugate and can equally well be written in terms of anti-chiral superspace coordinates $\bar{\eta}_I$ using

$$\mathcal{V}_{\mathcal{N}=4}(\bar{\eta}_I) = \int d^4 \eta e^{\eta^I \bar{\eta}_I} \mathcal{V}_{\mathcal{N}=4}(\eta^I) = A_- + \bar{\eta}_I \psi_-^I + \dots + \bar{\eta}_1 \bar{\eta}_2 \bar{\eta}_3 \bar{\eta}_4 A^+ , \quad (2.2)$$

where the measure is $d^4 \eta = d\eta^1 d\eta^2 d\eta^3 d\eta^4$.

The $\mathcal{N} = 4$ multiplet naturally decomposes on η^3 and η^4 into $\mathcal{N} = 2$ multiplets:

$$\mathcal{V}_{\mathcal{N}=4} = V_{\mathcal{N}=2}^+ + \eta^3 \Phi_{\mathcal{N}=2} + \eta^4 \bar{\Phi}_{\mathcal{N}=2} + \eta^3 \eta^4 V_{\mathcal{N}=2}^- . \quad (2.3)$$

Here the $\mathcal{N} = 2$ vector multiplets are

$$V_{\mathcal{N}=2}^+(\eta^I) = A^+ + \eta^I \psi_I^+ + \eta^1 \eta^2 \varphi_{12} , \quad V_{\mathcal{N}=2}^-(\eta^I) = \varphi_{34} + \epsilon_{I34J} \eta^I \psi_-^J + \eta^1 \eta^2 A_- , \quad (2.4)$$

where the SU(2) indices $I, J = 1, 2$ are inherited from SU(4); the hypermultiplets (hypers) are

$$\Phi_{\mathcal{N}=2}(\eta^I) = \psi_3^+ - \eta^I \varphi_{I3} + \eta^1 \eta^2 \psi_-^4 , \quad \bar{\Phi}_{\mathcal{N}=2}(\eta^I) = \psi_4^+ - \eta^I \varphi_{I4} - \eta^1 \eta^2 \psi_-^3 . \quad (2.5)$$

All four have $\mathcal{N} = 2$ supersymmetries represented by the remaining Grassmann variables η^1 and η^2 . Moreover, $V_{\mathcal{N}=2}^+$ is related to $V_{\mathcal{N}=2}^-$ by CPT conjugation, and likewise for $\Phi_{\mathcal{N}=2}$ and $\bar{\Phi}_{\mathcal{N}=2}$.

The on-shell content of $\mathcal{N} = 2$ SQCD is obtained by switching the representation of the hypers $\Phi_{\mathcal{N}=2}$ and $\bar{\Phi}_{\mathcal{N}=2}$ from the adjoint to the fundamental and anti-fundamental representations, respectively. This explicitly breaks supersymmetry on η^3 and η^4 , but not on η^1 and η^2 . Furthermore, the hypers can be generalized to an arbitrary number $N_f = \delta_\alpha^\alpha$ of flavors by attaching flavor indices $\{\alpha, \beta, \dots\}$ to them, as in $(\Phi_{\mathcal{N}=2})^\alpha$ and $(\bar{\Phi}_{\mathcal{N}=2})_\alpha$. Alternatively, the flavor indices can be conflated with the color indices, implying reducible gauge-group representations for the matter multiplets. In this paper will use the tree amplitudes for $N_f = 1$ to construct unitarity cuts; the diagrammatic form of the resulting loop integrands will allow for an arbitrary N_f .

By analogy to QCD, $V_{\mathcal{N}=2}^+$ and $V_{\mathcal{N}=2}^-$ act as positive- and negative-helicity gluons, A^+ and A_- , and can be regarded as their respective on-shell supersymmetrizations. The

hel. \ th.	$\mathcal{N}=4$ SYM	$\mathcal{N}=2$ SQCD	QCD
+1	1	1 $\overbrace{\quad}^{N_f}$	1 $\overbrace{\quad}^{N_f}$
+1/2	4	2 $\begin{smallmatrix} 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 & 1 \end{smallmatrix}$
0	6	1 1 2 2	
-1/2	4	2 1 1	1 1
-1	1	1	1
rep.	adj.	adj. $\frac{\text{fund.}}{\text{fund.}}$	adj. $\frac{\text{fund.}}{\text{fund.}}$

Table 1. Helicity content of $\mathcal{N} = 2$ supersymmetric QCD in comparison to $\mathcal{N} = 4$ super-Yang-Mills theory and conventional QCD. For these theories, the helicities and the representations of the particles are listed in the left column and the lower row, respectively.

hypermultiplets $(\Phi_{\mathcal{N}=2})^\alpha$ and $(\bar{\Phi}_{\mathcal{N}=2})_\alpha$ play the roles of massless quarks and anti-quarks. In this way, $\mathcal{N} = 2$ SQCD can be viewed as the middle ground between $\mathcal{N} = 4$ SYM and the actual QCD, as illustrated in table 1. Although it is less well studied than the other two, its one-loop MHV and NMHV amplitudes are known for any number of external gluons [39, 87, 88] via their relation to those in $\mathcal{N} = 1, 2, 4$ SYM [38].

2.2 Tree-level amplitudes

Tree-level $\mathcal{N} = 2$ SQCD amplitudes are simply related to those of $\mathcal{N} = 4$ SYM. In the maximally helicity-violating (MHV) sector, to which we specialize in this paper, planar tree-level $\mathcal{N} = 4$ SYM amplitudes are given by the famous Parke-Taylor formula [86, 89]:³

$$A_n^{(0),\text{MHV}}(\mathcal{V}_{\mathcal{N}=4}, \mathcal{V}_{\mathcal{N}=4}, \dots, \mathcal{V}_{\mathcal{N}=4}) = \frac{i\delta^8(Q)}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}. \quad (2.6)$$

Here the Grassmann delta function imposes conservation of supercharges; for $\mathcal{N}$ supersymmetries it is

$$\delta^{2\mathcal{N}}(Q) = \delta^{2\mathcal{N}}\left(\sum_{i=1}^n |i\rangle \eta_i\right) = \prod_{I=1}^{\mathcal{N}} \sum_{i < j}^n \eta_i^I \langle i j \rangle \eta_j^I. \quad (2.7)$$

Tree-level $\mathcal{N} = 2$ SQCD amplitudes with $N_f = 1$ massless hypermultiplet flavors are obtained by projecting out the relevant multiplets using the decomposition given in eq. (2.3). The two extra superspace coordinates η^3 and η^4 , associated with the two broken supersymmetries, serve to identify the multiplets: $\Phi_{\mathcal{N}=2}$ carries η^3 , $\bar{\Phi}_{\mathcal{N}=2}$ carries η^4 , $V_{\mathcal{N}=2}^+$ carries neither, and $V_{\mathcal{N}=2}^-$ carries both. Several n -point examples are given in ref. [66].

In this paper we will mostly use four-point amplitudes. In order to track the $\mathcal{N} = 2$ multiplets on their external legs, we introduce the superspace combination

$$\kappa_{(ab)(cd)}(1, 2, 3, 4) \equiv \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle} \delta^4(Q) \eta_a^3 \langle a b \rangle \eta_b^3 \eta_c^4 \langle c d \rangle \eta_d^4, \quad (2.8)$$

³We adopt the usual spinor-helicity notation — see for example refs. [54, 90].

where $\{a, b, c, d\} \in \{1, 2, 3, 4\}$. Here the spinor-helicity prefactor is permutation invariant and is familiar from the commonly used $\mathcal{N} = 4$ amplitude prefactor

$$\kappa(1, 2, 3, 4) \equiv \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle} \delta^8(Q). \quad (2.9)$$

In refs. [65, 66] a similar notation was used to label the two anti-chiral $V_{\mathcal{N}=2}^-$ vector multiplets in the MHV sector: $\kappa_{ab} \equiv \kappa_{(ab)(ab)}$. Our updated notation is more flexible, as it allows us to also track the hypermultiplets $\Phi_{\mathcal{N}=2}$ and $\bar{\Phi}_{\mathcal{N}=2}$ on external legs. For instance, we can now compactly write

$$A_4^{(0),\text{MHV}}(V_{\mathcal{N}=2}^-, V_{\mathcal{N}=2}^+, V_{\mathcal{N}=2}^-, V_{\mathcal{N}=2}^+) = -\frac{i}{st} \kappa_{(13)(13)} = -\frac{i}{st} \kappa_{13}, \quad (2.10a)$$

$$A_4^{(0),\text{MHV}}(V_{\mathcal{N}=2}^-, \Phi_{\mathcal{N}=2}, \bar{\Phi}_{\mathcal{N}=2}, V_{\mathcal{N}=2}^+) = -\frac{i}{st} \kappa_{(12)(13)}, \quad (2.10b)$$

$$A_4^{(0),\text{MHV}}(\Phi_{\mathcal{N}=2}, \bar{\Phi}_{\mathcal{N}=2}, \Phi_{\mathcal{N}=2}, \bar{\Phi}_{\mathcal{N}=2}) = -\frac{i}{st} \kappa_{(13)(24)}, \quad (2.10c)$$

$$A_4^{(0),\text{MHV}}(\Phi_{\mathcal{N}=2}, \Phi_{\mathcal{N}=2}, \bar{\Phi}_{\mathcal{N}=2}, \bar{\Phi}_{\mathcal{N}=2}) = -\frac{i}{st} \kappa_{(12)(34)}, \quad (2.10d)$$

where $s = (p_1 + p_2)^2$ and $t = (p_2 + p_3)^2$ are the usual Mandelstam variables.

Using CPT invariance of the theory we can equally well study $\overline{\text{MHV}}$ amplitudes. These are related to the MHV by exchanging $|i\rangle \leftrightarrow |i]$ and $\eta_i^I \leftrightarrow \bar{\eta}_{i,I}$. For instance, the n -point tree-level $\overline{\text{MHV}}$ amplitude is also given by a Parke-Taylor formula:

$$A_n^{(0),\overline{\text{MHV}}}(\mathcal{V}_{\mathcal{N}=4}, \mathcal{V}_{\mathcal{N}=4}, \dots, \mathcal{V}_{\mathcal{N}=4}) = \frac{i\delta^8(\bar{Q})}{[12][23] \cdots [n1]}, \quad (2.11)$$

where the anti-chiral supermomentum-conserving delta function is defined as

$$\delta^{2\mathcal{N}}(\bar{Q}) = \prod_{I=1}^{\mathcal{N}} \sum_{i < j}^n \bar{\eta}_i^I [i j] \bar{\eta}_j^I. \quad (2.12)$$

To compare amplitudes formulated in different superspaces we switch between chiral and anti-chiral superspace coordinates using Fourier transforms:

$$A_n(\eta_i^I) = \int d^4 \bar{\eta}_1 \cdots d^4 \bar{\eta}_n e^{\bar{\eta}_{1,I} \eta_1^I} \cdots e^{\bar{\eta}_{n,I} \eta_n^I} A_n(\bar{\eta}_{i,I}). \quad (2.13)$$

Note that even with $\mathcal{N} = 2$ supersymmetries we continue to use the four-dimensional Grassmann integration $d^4 \eta_i$ — the superspace variables of the broken supersymmetry are retained in order to discern between the $\mathcal{N} = 2$ multiplets.

At four points the MHV and $\overline{\text{MHV}}$ amplitudes are equivalent, so related by the Fourier transform (2.13). We can therefore also track the external state configuration using

$$\bar{\kappa}_{(ab)(cd)}(1, 2, 3, 4) \equiv \frac{\langle 12 \rangle \langle 34 \rangle}{[12][34]} \delta^4(\bar{Q}) \bar{\eta}_{a,3} [a b] \bar{\eta}_{b,3} \bar{\eta}_{c,4} [c d] \bar{\eta}_{d,4}. \quad (2.14)$$

Under the Fourier transform (2.13) this maps into $\kappa_{(\overline{ab})(\overline{cd})}(1, 2, 3, 4)$, where a barred pair of indices $\{\overline{a}, \overline{b}\} \equiv \{1, 2, 3, 4\} \setminus \{a, b\}$ denotes the complement with respect to the set of external labels.

2.3 Loop-level amplitudes

Proceeding now to consider loop-level amplitudes, we adopt diagrammatic representations from the outset. In a general Yang-Mills theory, the trivalent nature of the gauge group generators allows us to write any L -loop amplitude as a sum of cubic graphs:

$$\mathcal{A}_n^{(L)} = i^{L-1} g^{n+2L-2} \sum_{\text{cubic graphs } \Gamma_i} \int \frac{d^D \ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i c_i}{D_i}. \quad (2.15)$$

Here g is the coupling, S_i are the symmetry factors, D_i are the usual products of massless propagators, and n_i are the kinematic numerators associated with each graph, depending on both external and loop momenta. To regulate potentially divergent integrals, we use dimensional regularization in $D = 4 - 2\epsilon$ dimensions. Finally, we assume $N_f = 1$ hypermultiplet flavors; one can generalize to $N_f \neq 1$ by assigning flavor-conserving delta functions to each diagram, with $N_f = \delta_\alpha^\alpha$ for closed matter loops (see refs. [55, 56, 63] for more details).

One of the main advantages of such a cubic representation is that the color factors c_i are unambiguously assigned to the graphs. There are two kinds of trivalent vertices: pure-adjoint, and those with a particle in each of the adjoint, fundamental and anti-fundamental representations. These are associated with the structure constant $\tilde{f}^{abc} = \text{tr}([T^a, T^b]T^c)$ and generator T_{ij}^a , respectively.⁴

$$\tilde{f}^{abc} = c \left(\begin{array}{c} a \quad b \\ \quad c \end{array} \right), \quad T_{ij}^a = c \left(\begin{array}{c} a \\ i \quad j \end{array} \right). \quad (2.16)$$

Both are antisymmetric: $\tilde{f}^{abc} = -\tilde{f}^{acb}$, $T_{ij}^a \equiv -T_{ji}^a$ (the latter relationship defines T_{ij}^a). Fundamental structure constants are normalized such that $\text{tr}(T^a T^b) = \delta^{ab}$.

We seek loop-level amplitude representations obeying color-kinematics duality [61, 62]. In this case, the same linear identities satisfied by the color factors c_i should also be satisfied by the kinematic numerators n_i , which we refer to as color-dual. Such relationships include commutation relations

$$\begin{aligned} \tilde{f}^{ba_3 a_4} T_{i_1 \bar{i}_2}^b &= T_{i_1 \bar{j}}^{a_3} T_{j \bar{i}_2}^{a_4} - T_{i_1 \bar{j}}^{a_4} T_{j \bar{i}_2}^{a_3} = [T^{a_3}, T^{a_4}]_{i_1 \bar{i}_2}, \\ c \left(\begin{array}{c} 4 \quad 1 \\ 3 \quad 2 \end{array} \right) &= c \left(\begin{array}{c} 4 \quad 1 \\ 3 \quad 2 \end{array} \right) - c \left(\begin{array}{c} 3 \quad 1 \\ 4 \quad 2 \end{array} \right), \end{aligned} \quad (2.17)$$

and their adjoint-representation counterparts — the Jacobi identities

$$\begin{aligned} \tilde{f}^{a_1 a_2 b} \tilde{f}^{ba_3 a_4} &= \tilde{f}^{a_4 a_1 b} \tilde{f}^{ba_2 a_3} - \tilde{f}^{a_2 a_4 b} \tilde{f}^{ba_3 a_1}, \\ c \left(\begin{array}{c} 4 \quad 1 \\ 3 \quad 2 \end{array} \right) &= c \left(\begin{array}{c} 4 \quad 1 \\ 3 \quad 2 \end{array} \right) - c \left(\begin{array}{c} 3 \quad 1 \\ 4 \quad 2 \end{array} \right). \end{aligned} \quad (2.18)$$

Color-kinematics duality requires that

$$c_i = c_j - c_k \quad \Leftrightarrow \quad n_i = n_j - n_k, \quad (2.19)$$

⁴In this paper, we label cubic diagrams by their graphical representations. Their explicit layout encodes a sign due to the antisymmetry of the vertices.

The usual motivation for finding so-called color-dual representations is to enable use of the double copy [61, 62], which allows supergravity amplitudes to be obtained by replacing the color factors c_i with a second copy of the kinematic numerators n_i in the amplitude (2.15). Fundamental-representation hypers play an important role when dealing with $\mathcal{N} < 4$ supergravities, for example as they allow unwanted additional vector multiplets to be subtracted from the resulting supergravity multiplet [65]. For instance, refs. [65, 66] described how pure $\mathcal{N} = 4$ supergravity amplitudes could be obtained from a double copy of $\mathcal{N} = 2$ SYM with itself, the hypermultiplets being used internally to remove unwanted $\mathcal{N} = 4$ SYM multiplets from the supergravity theory.

There are, however, considerable advantages to finding color-dual representations even if the goal is merely efficient computation of gauge-theory amplitudes. First, such representations are cubic, so the assignment of color factors to diagrams is trivial (for alternative non-cubic constructions of full-color integrands from unitarity cuts see e.g. refs. [47, 91, 92]). Moreover, the kinematic numerators being interlinked by commutation and Jacobi relations implies that only a limited subset of the numerators need to be calculated directly. The corresponding graphs, which are referred to as masters, are chosen to ensure that the numerators of all other graphs can be obtained using commutation and Jacobi identities. For instance, using the commutation relation (2.17) implies

$$n \left(\begin{array}{c} 4 \\ \text{triangle} \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) = n \left(\begin{array}{c} 4 \\ \text{box} \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) - n \left(\begin{array}{c} 4 \\ \text{box} \\ 3 \end{array} \begin{array}{c} 2 \\ 1 \end{array} \right). \quad (2.20)$$

In this case, the triangle is uniquely determined by the two boxes; these, in turn, are related by a symmetry through the horizontal axis (after relabeling $p_3 \leftrightarrow p_4$). The box is in this case a master: it is on these masters that we will focus our attention in later sections.

In general, the existence of a consistent set of color-dual numerators is not trivial. At tree level, it is proven [93, 94] for gauge theories, in which color-ordered amplitudes satisfy the BCJ relations [61, 95–97]. For (super-)Yang-Mills theories with arbitrary fundamental matter, which is less studied, the corresponding BCJ relations [63, 65] have been proven in the case of QCD [98].

The task of finding color-dual representations is further complicated by the fact that, for a given amplitude, such representations are generally not unique. For this reason, a large part of ref. [66] was devoted to finding additional constraints to be imposed on the numerators, with the intention of shrinking the space of allowed solutions while manifesting certain desirable properties. These constraints also reduce the number of masters to be computed. In the remainder of this section we discuss the constraints that we have found helpful. Note that not all the integrands presented here have all the properties presented below; appendix B summarizes the various representations and their properties.

2.3.1 Two-term identities

If imposed, the two-term identities require that, for indistinguishable matter multiplets,

$$c \left(\begin{array}{c} 4 \\ \text{triangle} \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) \stackrel{?}{=} c \left(\begin{array}{c} 4 \\ \text{box} \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right). \quad (2.21)$$

Although this is not true in general, it holds if the gauge group is chosen as $G = \text{U}(1)$, or for specific tensor representations of $\text{U}(N_c)$ [65]. Like the commutation and Jacobi relations, we impose these identities on the numerators whose graphs contain internal subgraphs of the above form. For instance, these one-loop box and triangle numerators are equated:

$$n \left(\begin{array}{c} 4 \quad 1 \\ \text{box} \\ 3 \quad 2 \end{array} \right) = n \left(\begin{array}{c} 4 \quad 1 \\ \text{triangle} \\ 3 \quad 2 \end{array} \right). \quad (2.22)$$

One can also regard the two-term identities as their own kind of commutation relations, the difference being that the u -channel graphs are excluded as their routing of fundamental matter lines is not sensible.

As we shall see in section 3.2.1, the two-term identities have their origin in the structure of the $\mathcal{N} = 2$ cuts, and the diagrammatic rules will help to clarify this. In ref. [66], these identities allowed all numerators with two matter loops to be reduced to those with one matter loop. The two-term identities are equally useful in restricting the set of masters when matter is taken on external legs.

2.3.2 CPT conjugation

Amplitudes respect the CPT invariance of the theory, and we can extend this to a manifest off-shell symmetry acting on individual numerators. CPT conjugation acts by transforming $|i\rangle \leftrightarrow |\bar{i}]$ and $\eta_i^I \leftrightarrow \bar{\eta}_{i,I}$; graphically, this corresponds to flipping the helicity of external vectors and reversing arrow directions on hypermultiplets. The transformation should correspond to a replacement of $\kappa_{(ab)(cd)}$ by its complement and an additional sign flip of parity-odd terms:

$$n_i(1, 2, 3, 4; \ell_1, \ell_2) = \bar{n}_i(1, 2, 3, 4; \ell_1, \ell_2) \Big|_{\kappa_{(ab)(cd)} \rightarrow \kappa_{(\overline{ab})(\overline{cd})}, |i\rangle \leftrightarrow |\bar{i}]}, \quad (2.23)$$

where $\bar{n}_i$ stands for the numerator of the graph with flipped arrows on matter lines; $\{\overline{a}, \overline{b}\} \equiv \{1, 2, 3, 4\} \setminus \{a, b\}$. For instance, we could equate

$$n \left(\begin{array}{c} 4 \quad 1 \\ \text{box} \\ 3 \quad 2 \end{array} \right) = n \left(\begin{array}{c} 4 \quad 1 \\ \text{box} \\ 3 \quad 2 \end{array} \right) \Big|_{\kappa_{(ab)(cd)} \rightarrow \kappa_{(\overline{ab})(\overline{cd})}, |i\rangle \leftrightarrow |\bar{i}]}. \quad (2.24)$$

Note that a change of direction of external hyper lines together with the conjugation of the indices of κ lands us back on the same external state configuration as we started from.

2.3.3 Matter-reversal symmetry

The matter multiplets $\Phi_{\mathcal{N}=2}$ and $\bar{\Phi}_{\mathcal{N}=2}$ are identical up to R-symmetry indices and the gauge-group representation, which leads to another potential off-shell symmetry of the numerators. In ref. [66] only vector multiplets were allowed on external legs, so the symmetry was invariance under arrow reversal for all numerators containing matter loops. The symmetry held for each matter loop individually.

With hypermultiplets on external legs the situation is more subtle. One cannot simply equate numerators with reversed hypermultiplets, as they carry different external states.

But this is easily remedied: by inspection of the $\mathcal{N} = 4$ state decomposition (2.3) the symmetry clearly exchanges $\eta^3 \leftrightarrow \eta^4$ (with no effect on $V_{\mathcal{N}=2}^+$ or $V_{\mathcal{N}=2}^-$). With this additional exchange imposed, we can implement the same identity, for instance

$$n \left(\begin{array}{c} 4 \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \\ \swarrow \quad \searrow \\ 3 \end{array} \begin{array}{c} 1 \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \\ \swarrow \quad \searrow \\ 2 \end{array} \right) = - n \left(\begin{array}{c} 4 \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \\ \swarrow \quad \searrow \\ 3 \end{array} \begin{array}{c} 1 \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \\ \swarrow \quad \searrow \\ 2 \end{array} \right) \Bigg|_{\eta_2^4 \rightarrow \eta_2^3, \eta_3^3 \rightarrow \eta_3^4} . \quad (2.25)$$

This symmetry is required if one considers matter multiplets in a pseudo-real representation [71, 99].

2.3.4 Matching with $\mathcal{N} = 4$ SYM

As we have already seen in eq. (2.3), the vector multiplet $\mathcal{V}_{\mathcal{N}=4}$'s $2^4 = 16$ states can be distributed between $V_{\mathcal{N}=2}^+$, $V_{\mathcal{N}=2}^-$, $\Phi_{\mathcal{N}=2}$ and $\bar{\Phi}_{\mathcal{N}=2}$. This offers another constraint on the color-dual $\mathcal{N} = 2$ SQCD numerators: that summing them over the internal multiplets corresponding to the on-shell content of $\mathcal{N} = 4$ SYM should reproduce those same numerators. For instance, we can demand that

$$\begin{aligned} n^{[\mathcal{N}=4]} \left(\begin{array}{c} 4 \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \\ \swarrow \quad \searrow \\ 3 \end{array} \begin{array}{c} 1 \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \\ \swarrow \quad \searrow \\ e_2 \end{array} \right) &= n \left(\begin{array}{c} 4 \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \\ \swarrow \quad \searrow \\ 3 \end{array} \begin{array}{c} 1 \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \\ \swarrow \quad \searrow \\ e_2 \end{array} \right) + n \left(\begin{array}{c} 4 \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \\ \swarrow \quad \searrow \\ 3 \end{array} \begin{array}{c} 1 \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \\ \swarrow \quad \searrow \\ e_2 \end{array} \right) + n \left(\begin{array}{c} 4 \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \\ \swarrow \quad \searrow \\ 3 \end{array} \begin{array}{c} 1 \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \\ \swarrow \quad \searrow \\ e_2 \end{array} \right) + n \left(\begin{array}{c} 4 \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \\ \swarrow \quad \searrow \\ 3 \end{array} \begin{array}{c} 1 \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \\ \swarrow \quad \searrow \\ e_2 \end{array} \right) \\ &+ n \left(\begin{array}{c} 4 \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \\ \swarrow \quad \searrow \\ 3 \end{array} \begin{array}{c} 1 \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \\ \swarrow \quad \searrow \\ e_2 \end{array} \right) + n \left(\begin{array}{c} 4 \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \\ \swarrow \quad \searrow \\ 3 \end{array} \begin{array}{c} 1 \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \\ \swarrow \quad \searrow \\ e_2 \end{array} \right) + n \left(\begin{array}{c} 4 \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \\ \swarrow \quad \searrow \\ 3 \end{array} \begin{array}{c} 1 \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \\ \swarrow \quad \searrow \\ e_2 \end{array} \right) , \end{aligned} \quad (2.26)$$

where in this case a suitable expression for the $\mathcal{N} = 4$ double-box numerator is simply

$$n^{[\mathcal{N}=4]} \left(\begin{array}{c} 4 \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \\ \swarrow \quad \searrow \\ 3 \end{array} \begin{array}{c} 1 \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \\ \swarrow \quad \searrow \\ 2 \end{array} \right) = s(\kappa_{12} + \kappa_{13} + \kappa_{14} + \kappa_{23} + \kappa_{24} + \kappa_{34}) . \quad (2.27)$$

Here we have projected the $\mathcal{N} = 2$ vector multiplets out of the supersymmetric delta function $\delta^8(Q)$. This statement must be true for kinematic configurations where the propagators are taken on-shell, i.e. on the maximal cut, because summing over these diagrams then simply amounts to taking η^3 and η^4 integrals on the right-hand side. This logic also holds when hypers are taken on external legs. The non-trivial observation is that we can demand it also be true for off-shell loop momenta.

3 Iterated two-particle cuts

In this section we describe the iterative two-particle cut construction, as it applies to both $\mathcal{N} = 4$ and $\mathcal{N} = 2$ SYM in strictly four dimensions (in section 4 we will explain how to find higher-dimensional corrections for the purpose of dimensional regularization). The construction is underpinned by that fact that when two four-point amplitudes are glued together to form a cut, the result is proportional to another four-point tree amplitude. This allows the gluing procedure to be iterated, leading to diagrammatic rules for cut assembly without the need to perform intermediate supersums. As we shall see, in $\mathcal{N} = 4$ SYM this construction leads to the so-called “rung rule” for assembling Mondrian-type diagrams [59, 60, 100].

3.1 $\mathcal{N} = 4$ SYM

To understand the iterated nature of two-particle cuts in $\mathcal{N} = 4$ SYM, we begin with the four-point one-loop s -channel cut

$$\begin{aligned}
 \text{Diagram: } & \text{A four-point one-loop } s\text{-channel cut diagram with external legs } 1, 2, 3, 4 \text{ and internal loop momenta } l_1, l_2. \\
 &= \int d^4\eta_{l_1} d^4\eta_{l_2} A_4^{(0),\text{MHV}}(1, 2, l_1, l_2) A_4^{(0),\text{MHV}}(3, 4, -l_2, -l_1) \\
 &= -i \frac{st}{(l_1 + p_2)^2 (l_1 - p_3)^2} A_4^{(0),\text{MHV}}(1, 2, 3, 4).
 \end{aligned} \tag{3.1}$$

The fact that this cut is proportional to the tree amplitude is a well-known result, following from Green, Schwarz and Brink’s original computation of the one-loop amplitude [101]. If we extract the physical poles from the tree amplitudes using the MHV formula

$$A_4^{(0),\text{MHV}}(1, 2, 3, 4) = -\frac{i}{st} \kappa(1, 2, 3, 4), \tag{3.2}$$

then using $s = s_{l_1 l_2}$ (in this case) — where $s_{ij} = (p_i + p_j)^2$, which we extend to include loop momenta — the cut identity can be more compactly written as

$$\int d^4\eta_{l_1} d^4\eta_{l_2} \kappa(1, 2, l_1, l_2) \kappa(3, 4, -l_2, -l_1) = s_{l_1 l_2}^2 \kappa(1, 2, 3, 4). \tag{3.3}$$

A similar construction was presented in ref. [9]; for the sake of completeness we give a proof of this relation in appendix A.

3.1.1 $\mathcal{N} = 4$ diagrammatic rules

The two-particle cut being proportional to the tree-level amplitude allows for an iterated construction. We can attach more four-point tree-level amplitudes to the cut and glue two pairs of legs at a time by using eq. (3.3). So any iterated two-particle cut is a product of terms coming from its four-point tree-level amplitudes and two-particle supersums. The following diagrammatic rules summarize the construction:

$$\begin{aligned}
 & \text{Diagram: } \text{A four-point tree-level amplitude with external legs } a, b, c, d. \\
 & \rightarrow -\frac{i}{s_{ab}s_{ac}}, \quad \text{Diagram: } \text{A two-particle cut diagram with internal momenta } l_1, l_2. \\
 & \rightarrow s_{l_1 l_2}^2, \quad \text{Diagram: } \text{A four-point tree-level amplitude with external legs } q, r, s, t. \\
 & \rightarrow \kappa(q, r, s, t).
 \end{aligned} \tag{3.4}$$

The first rule comes from eq. (3.2); for each tree-amplitude constituent, we must insert its physical poles. The second rule is the result of eq. (3.3); it tells us that a factor $s_{l_1 l_2}^2$ is obtained whenever two tree-level amplitudes are glued together. Finally, the last “external” rule tells us that, once all partons have been assembled, we should multiply by an overall κ factor to encode the configuration of external states.

The s -channel cut from before is now easily assembled:

$$\text{Diagram: } \text{A four-point one-loop } s\text{-channel cut diagram} = \frac{-i}{s_{12}s_{1l_2}} \times s_{l_1 l_2}^2 \times \frac{-i}{s_{34}s_{4(-l_2)}} \times \kappa = -\frac{s_{l_1 l_2}^2 \kappa}{s^2 s_{1l_2} s_{4(-l_2)}}. \tag{3.5}$$

This pattern agrees with the three-loop amplitude [4, 5, 59].

However, we should recognize the circumstances under which the rung rule is too naïve. While cuts are unique, off-shell numerators are not: we are free to shift terms between numerators by adding terms that vanish on support of the on-shell conditions. So the numerators may require modification, for instance, if

- the same diagram contributes to cuts that suggest different on-shell forms;
- additional off-shell constraints — like color duality — are demanded.

A classic example where the rung rule fails to provide color-dual numerators is the 4-point, 3-loop MHV amplitude [62, 102] — the numerators given above are not color-dual. Therefore, as we now proceed to consider $\mathcal{N} = 2$ SQCD, we will use the iteration only to construct cuts, and remember that off-shell numerators may require modification.

3.2 $\mathcal{N} = 2$ SQCD

The iterative cut construction works in $\mathcal{N} = 2$ SYM for the same reason as in $\mathcal{N} = 4$ SYM: because the result of gluing together a pair of tree-level amplitudes is proportional to another tree-level amplitude. The generalization of the supersum (3.3), written in terms of $\kappa_{(ab)(cd)}$ as introduced in section 2.2, is

$$\int d^4\eta_{l_1} d^4\eta_{l_2} \kappa_{(ab)(cd)}^{(L)} \kappa_{(ef)(gh)}^{(R)} = s_{l_1 l_2} \langle ab \rangle [\overline{cd}] \langle ef \rangle [\overline{gh}] [qr] \langle \overline{st} \rangle \frac{\kappa_{(qr)(st)}}{s_{qr} s_{st}}, \quad (3.12)$$

which we will prove below; we denote

$$\begin{aligned} \kappa_{(ab)(cd)}^{(L)} &= \kappa_{(ab)(cd)}(1, 2, l_1, l_2), & \kappa_{(ef)(gh)}^{(R)} &= \kappa_{(ef)(gh)}(3, 4, -l_2, -l_1), \\ \kappa_{(qr)(st)} &= \kappa_{(qr)(st)}(1, 2, 3, 4), \end{aligned} \quad (3.13)$$

where $(qr)(st)$ denotes the overall state configuration. Bars denote the complement with respect to external legs on a tree amplitude; for instance, $\{\overline{c}, \overline{d}\} = \{1, 2, l_1, l_2\} \setminus \{c, d\}$. We omit the overall sign since it depends on the ordering of the complement, which affects the spinor-helicity brackets.

As we shall discuss in section 6, similar relations work for higher-point tree amplitudes. In the conclusions, we will also discuss possible generalizations to $\mathcal{N} = 0$ and $\mathcal{N} = 1$ SYM amplitudes.

3.2.1 $\mathcal{N} = 2$ diagrammatic rules

The relationship (3.12) leads to simple rules for assembling any iterated two-particle cut. For each four-point constituent, regardless of the configuration of external or intermediate states, we include the same physical poles as we did for $\mathcal{N} = 4$ SYM:

$$\begin{array}{c} d \quad a \\ \diagdown \quad \diagup \\ \text{---} \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ c \quad b \end{array} \rightarrow -\frac{i}{s_{ab} s_{bc}}. \quad (3.14)$$

The solid lines are used to indicate that it does not matter whether the states are vectors or hypermultiplets. For each amplitude there is now an additional factor: $\langle ab \rangle [\overline{cd}]$ for the

left-hand side of the cut, and $\langle ef \rangle [\overline{gh}]$ for the right-hand side. This gives a numerator contribution depending on the particle content:

$$d^+ a^- \rightarrow \langle ab \rangle [cd], \quad (3.15a)$$

$$d \begin{array}{c} \swarrow \\ \text{\tiny{---}}\circlearrowleft \\ \nwarrow \end{array} \begin{array}{c} \nearrow \\ \text{\tiny{---}}\circlearrowright \\ \searrow \end{array} a^- \rightarrow \langle a|c|b\rangle, \quad (3.15b)$$

$$\begin{array}{c} d \quad a \\ \diagdown \quad \diagup \\ \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ c \quad b \end{array} \rightarrow s_{ac} = s_{bd}, \quad (3.15c)$$

where the ordering of legs is irrelevant. When gluing two tree-level amplitudes, we multiply by

$$\begin{array}{c} l_1 \\ \rightarrow \\ \text{---} \\ \leftarrow \\ l_2 \end{array} \rightarrow s_{l_1 l_2} , \quad (3.16)$$

where again the type of particles is irrelevant. Finally, the leftover $[qr]\langle\overline{st}\rangle$, as well as the poles in s_{qr} and s_{st} , yield an overall factor depending on the configuration of the four external legs:

$$\begin{array}{c} t^+ \\ \vdots \\ s^+ \end{array} \begin{array}{c} \cdots \\ \cdots \\ \cdots \end{array} \begin{array}{c} q^- \\ \vdots \\ r^- \end{array} \rightarrow [qr] \langle st \rangle \hat{\kappa}_{(qr)(qr)} , \quad (3.17a)$$

$$\begin{array}{c} t \\ \swarrow \\ s \end{array} \begin{array}{c} \cdots \\ \cdots \\ \cdots \end{array} \begin{array}{c} q^- \\ \searrow \\ r^+ \end{array} \rightarrow [q|s|r\rangle \hat{\kappa}_{(qs)(qt)}, \quad (3.17b)$$

$$\begin{array}{c} t \\ \nearrow \\ \circ \\ \nwarrow \\ q \\ \nearrow \\ \circ \\ \nwarrow \\ r \end{array} \rightarrow s_{rt} \hat{\mathcal{K}}_{(qs)(rt)} = s_{qs} \hat{\mathcal{K}}_{(qs)(rt)}, \quad (3.17c)$$

where we have introduced

$$\hat{\kappa}_{(qr)(st)} \equiv \frac{\kappa_{(qr)(st)}}{s_{qr}s_{st}}. \quad (3.18)$$

This completes the set of rules required to assemble any cut obtained by gluing four-point amplitudes.

3.2.2 Derivation of $\mathcal{N} = 2$ diagrammatic rules

A convenient way to derive the recursion formula given in eq. (3.12) is by treating chiral and anti-chiral superspace coordinates democratically. We begin with

$$\int d^4\eta_{l_1} d^4\eta_{l_2} \kappa_{(ab)(cd)}^{(L)} \kappa_{(ef)(gh)}^{(R)} = [l_1 l_2]^2 \langle ab \rangle \langle cd \rangle \langle ef \rangle \langle gh \rangle [qr] [st] \frac{\kappa_{(qr)(st)}}{s_{qr} s_{st}}, \quad (3.19a)$$

$$\int d^4\bar{\eta}_{l_1} d^4\bar{\eta}_{l_2} \bar{\kappa}_{(\overline{ab})(\overline{cd})}^{(\text{L})} \bar{\kappa}_{(\overline{ef})(\overline{gh})}^{(\text{R})} = \langle l_1 l_2 \rangle^2 [\overline{ab}][\overline{cd}][\overline{ef}][\overline{gh}] \langle \overline{qr} \rangle \langle \overline{st} \rangle \frac{\bar{\kappa}_{(\overline{qr})(\overline{st})}}{s_{\overline{qr}} s_{\overline{st}}}, \quad (3.19b)$$

which are CPT conjugates of each other; the former is proved in appendix A. The Fourier transform (2.13) that brings the second expression back into the chiral superspace amounts to replacing

$$\bar{\kappa}_{(\overline{qr})(\overline{st})} \rightarrow \kappa_{(qr)(st)} \, , \quad (3.20)$$

which reflects the equivalence of four-point MHV and $\overline{\text{MHV}}$ amplitudes. As $s_{qr} = s_{\overline{q}\overline{r}}$, $s_{st} = s_{\overline{s}\overline{t}}$ by momentum conservation, this implies that the strings of spinor-helicity brackets are equal across the two formulas.

We can therefore treat chiral and anti-chiral spinor variables democratically. This is naturally accomplished using a square root:

$$\begin{aligned} & \int d^4\eta_{l_1} d^4\eta_{l_2} \kappa_{(ab)(cd)}^{(L)} \kappa_{(ef)(gh)}^{(R)} \\ &= s_{l_1 l_2} \left(\langle ab \rangle [\overline{ab}] \langle cd \rangle [\overline{cd}] \langle ef \rangle [\overline{ef}] \langle gh \rangle [\overline{gh}] [qr] \langle \overline{qr} \rangle [st] \langle \overline{st} \rangle \right)^{\frac{1}{2}} \frac{\kappa_{(qr)(st)}}{s_{qr} s_{st}}, \end{aligned} \quad (3.21)$$

where the sign is left ambiguous. The last step is to show that, for the left-hand tree amplitude, $\langle ab \rangle [\overline{cd}] = \pm [\overline{ab}] \langle cd \rangle$; this follows by examination for the three possible external helicity configurations listed in the diagrammatic rules (3.15). A similar identity holds for the right-hand side, $\langle ef \rangle [\overline{gh}] = \pm [\overline{ef}] \langle gh \rangle$, and for the external configuration, $\langle \overline{qr} \rangle [st] = \pm [qr] \langle \overline{st} \rangle$. Using these identities we can eliminate the square root, yielding the iterative formula (3.12).

3.2.3 Locality

An important property of the recursive formula (3.12) is that unphysical poles cancel between successive iterations. Suppose we glued a third tree amplitude, with external states encoded by $\kappa_{(ij)(kl)}^{(X)}$. The additional rung carries two new loop momenta l_3 and l_4 , where the orientation is irrelevant. Using $\kappa_{(qr)(st)}$ to denote the new external state configuration, we can write

$$\begin{aligned} & \int d^4\eta_{l_1} d^4\eta_{l_2} d^4\eta_{l_3} d^4\eta_{l_4} \kappa_{(ab)(cd)}^{(L)} \kappa_{(ef)(gh)}^{(R)} \kappa_{(ij)(kl)}^{(X)} \\ &= s_{l_1 l_2} \frac{\langle ab \rangle [\overline{cd}] \langle ef \rangle [\overline{gh}] [wx] \langle \overline{yz} \rangle}{s_{wx} s_{yz}} \int d^4\eta_{l_3} d^4\eta_{l_4} \kappa_{(wx)(yz)}^{(L+R)} \kappa_{(ij)(kl)}^{(X)} \\ &= s_{l_1 l_2} \frac{\langle ab \rangle [\overline{cd}] \langle ef \rangle [\overline{gh}] [wx] \langle \overline{yz} \rangle}{s_{wx} s_{yz}} s_{l_3 l_4} \langle wx \rangle [\overline{yz}] \langle ij \rangle [\overline{kl}] [qr] \langle \overline{st} \rangle \frac{\kappa_{(qr)(st)}}{s_{qr} s_{st}} \\ &= s_{l_1 l_2} s_{l_3 l_4} \langle ab \rangle [\overline{cd}] \langle ef \rangle [\overline{gh}] \langle ij \rangle [\overline{kl}] [qr] \langle \overline{st} \rangle \frac{\kappa_{(qr)(st)}}{s_{qr} s_{st}}, \end{aligned} \quad (3.22)$$

having used $s_{wx} = s_{\overline{y}\overline{z}}$. We are left only with the poles of the external state configuration, s_{qr} and s_{st} . The above expression is consistent with the diagrammatic rules given in section 3.2.1.

The upshot is that all poles are handled transparently. Physical poles coming from the trees are directly accommodated for by the first rule (3.14); the only other poles are those associated with the external rule (3.17), and which we absorb into $\hat{\kappa}_{(qr)(st)} \equiv \kappa_{(qr)(st)} / (s_{qr} s_{st})$. They are relics of the spinor-helicity notation, arising from the representation of four-dimensional gluon polarization vectors in terms of spinor-helicity brackets (see e.g. refs. [54, 90]):

$$\varepsilon_+^\mu(p; q) = \frac{[p | \sigma^\mu | q \rangle}{\sqrt{2} \langle qp \rangle}, \quad \varepsilon_-^\mu(p; q) = \frac{[q | \sigma^\mu | p \rangle}{\sqrt{2} [pq]}. \quad (3.23)$$

The resulting numerators will evidently vanish whenever $\ell \rightarrow 0$, $\ell \rightarrow -p_a$, or $\ell \rightarrow p_b$, which correspond to the soft regions of loop integration. The numerators will also vanish in the collinear regions where ℓ becomes collinear to p_a or p_b . Such behavior makes a lot of sense, as only vectors should give rise to soft and collinear divergences in the loop integrals, not hypermultiplets (which in our case are also massless) — see e.g. refs. [103–105]. As we shall see (here and in future work currently in progress), this vanishing of the numerators in specific regions serves to block potentially singular regions arising for the hypers.

4 One-loop examples

To illustrate the iterative method of calculating cuts in $\mathcal{N} = 2$ SYM, we begin by considering all three four-point one-loop amplitudes in the MHV sector: with zero, one, and two external hypermultiplet pairs. In each case, the formation of the four-dimensional cuts tells us what structure we should expect in the off-shell numerators, and we find it unnecessary to use ansätze. A complete listing of the non-zero numerators for all three solutions is provided in appendix B.

Given our desire to regulate integrals in $D = 4 - 2\epsilon$ dimensions, it is also necessary for us to obtain unitarity cuts from higher-dimensional trees. As explained in ref. [66], a convenient method is to calculate cuts arising from the tree amplitudes of six-dimensional $\mathcal{N} = (1, 0)$ SYM — the dimensional uplift of four-dimensional $\mathcal{N} = 2$ — using the six-dimensional spinor-helicity formalism [106–112]. One restricts the six-dimensional external momenta to a four-dimensional subspace, and re-interprets the extra two loop momentum components as complex masses: $\mu^2 = \bar{\ell}^2 - \ell^2$, where $\bar{\ell}$ is the four-dimensional part of ℓ .

From these six-dimensional cuts, terms proportional to μ^2 are found by subtracting the previously obtained four-dimensional cuts. These terms are sufficiently simple that they can be lifted off shell without interference to the color-kinematic structure of the four-dimensional numerators. We use a D -dimensional Clifford algebra to write Dirac traces involving ℓ [113, 114]; $\text{tr}_\pm$ are defined in terms of $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, which anticommutes with elements of the four-dimensional subalgebra but commutes with the rest.⁶ Computing six-dimensional cuts also provides a check on all of the four-dimensional cuts computed in this paper.

4.1 External vectors

The one-loop amplitudes with four external vector multiplets, previous versions of which have been obtained in refs. [64, 65, 68, 116–118], are particularly simple to determine. An $\mathcal{N} = 4$ matching identity (see section 2.3.4) relates the pure-adjoint box numerator to the fundamental:

$$n^{[\mathcal{N}=4]} \left(\text{Box}(4, 1, 3, 2) \right) = n \left(\text{Box}(4, 1, 3, 2) \right) + n \left(\text{Box}(4, 1, 3, 2) \right) + n \left(\text{Box}(4, 1, 3, 2) \right), \quad (4.1)$$

⁶A recent review of dimensional-regularization schemes was given in ref. [115].

where the $\mathcal{N} = 4$ box numerator is given by κ and can be rewritten as

$$n^{[\mathcal{N}=4]} \left(\begin{array}{c} 4 \\ \text{box} \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) = s^2(\hat{\kappa}_{12} + \hat{\kappa}_{34}) + t^2(\hat{\kappa}_{23} + \hat{\kappa}_{14}) + u^2(\hat{\kappa}_{13} + \hat{\kappa}_{24}). \quad (4.2)$$

All possible combinations of external $\mathcal{N} = 2$ vector multiplets are projected out of κ as coefficients of $\hat{\kappa}_{ij} = \kappa_{ij}/s_{ij}^2$. The fundamental box is the only master — from it we can uniquely fix all other numerators.

We isolate this box numerator from the following family of four-dimensional cuts, determined using the diagrammatic rules given in section 3.2.1. Negative helicities are placed on different external legs to yield coefficients of different $\hat{\kappa}_{ij}$:

$$\begin{array}{c} 4^+ \\ l_2 \uparrow \\ 3^+ \end{array} \begin{array}{c} \text{box} \\ \text{cut} \end{array} \begin{array}{c} 1^- \\ l_1 \downarrow \\ 2^- \end{array} = 0, \quad (4.3a)$$

$$\begin{array}{c} 4^+ \\ l_2 \uparrow \\ 3^- \end{array} \begin{array}{c} \text{box} \\ \text{cut} \end{array} \begin{array}{c} 1^- \\ l_1 \downarrow \\ 2^+ \end{array} = -\frac{\langle 3|l_2|4 \rangle}{s l_2^2} \times s \times \frac{\langle 1|l_1|2 \rangle}{s l_1^2} \times [13]\langle 24 \rangle \hat{\kappa}_{13} = \frac{\text{tr}_-(1l_1 24 l_2 3)}{s l_1^2 l_2^2} \hat{\kappa}_{13}, \quad (4.3b)$$

$$\begin{array}{c} 4^- \\ l_2 \uparrow \\ 3^+ \end{array} \begin{array}{c} \text{box} \\ \text{cut} \end{array} \begin{array}{c} 1^- \\ l_1 \downarrow \\ 2^+ \end{array} = -\frac{[3|l_2|4 \rangle}{s l_2^2} \times s \times \frac{\langle 1|l_1|2 \rangle}{s l_1^2} \times [14]\langle 23 \rangle \hat{\kappa}_{14} = \frac{\text{tr}_-(1l_1 23 l_2 4)}{s l_1^2 l_2^2} \hat{\kappa}_{14}. \quad (4.3c)$$

The box is separated from the triangles and bubbles which also contribute by further cutting into the l_1 and l_2 propagators; in that case $\text{tr}_\pm(1l_1 24 l_2 3) = -s \text{tr}_\pm(1l_1 l_2 3)$ and $\text{tr}_\pm(1l_1 23 l_2 4) = 0$. The $\hat{\kappa}_{23}$, $\hat{\kappa}_{24}$, and $\hat{\kappa}_{34}$ coefficients are related by CPT conjugation; the descendants are determined using commutation relations:

$$n \left(\begin{array}{c} 4 \\ \text{box} \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) = \hat{\kappa}_{13} \text{tr}_-(1(\ell - p_1)(\ell + p_4)3) + \hat{\kappa}_{24} \text{tr}_+(1(\ell - p_1)(\ell + p_4)3) + \mu^2 (s(\hat{\kappa}_{12} + \hat{\kappa}_{34}) + t(\hat{\kappa}_{23} + \hat{\kappa}_{14}) + u(\hat{\kappa}_{13} + \hat{\kappa}_{24})), \quad (4.4a)$$

$$n \left(\begin{array}{c} 1 \\ \ell \uparrow \\ 4 \end{array} \begin{array}{c} \text{box} \\ \text{cut} \end{array} \begin{array}{c} 2 \\ 3 \end{array} \right) = (\hat{\kappa}_{13} + \hat{\kappa}_{34}) \text{tr}_-(1(\ell - p_1)(\ell + p_4)3) + (\hat{\kappa}_{12} + \hat{\kappa}_{24}) \text{tr}_+(1(\ell - p_1)(\ell + p_4)3) + (\hat{\kappa}_{12} + \hat{\kappa}_{34}) t \ell^2, \quad (4.4b)$$

$$n \left(\begin{array}{c} 4 \\ \text{box} \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) = 2\ell \cdot (p_{12} - \ell) [t(\hat{\kappa}_{23} + \hat{\kappa}_{14}) - u(\hat{\kappa}_{13} + \hat{\kappa}_{24})], \quad (4.4c)$$

By resubstituting back into the cuts given in eq. (4.3), we confirm the color duality of these kinematic numerators.⁷

Box numerators involving four-term traces of this kind have previously appeared in the context of D -dimensional local integrands [80, 119]. A box integral with numerator $\text{tr}_\pm[1(\ell - p_1)(\ell + p_4)3]$ was there found to be free of both UV and IR divergences — in fact, it is proportional to the $(D + 2)$ -dimensional scalar box integral, which does not diverge. The mechanism blocking these divergences was discussed at the end of section 3.2.4: when ℓ

⁷There are other descendants with bubbles on external legs or tadpoles, listed in appendix B. However, as explained in ref. [66], these contributions vanish upon integration as they lack a proper mass scale.

Figure 1. The three one-loop masters with mixed external particle content.

enters any potentially soft or collinear region, vanishing of the trace blocks the divergence. The descendants have similar properties, so these are our first examples of numerators with an IR structure that manifests vanishing of soft and collinear limits of matter-line momenta.

4.2 External vectors + matter

There are three masters, drawn in figure 1. We examine the first on its s -channel cut:

$$\begin{array}{c} 4^+ \\ \swarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ 3 \end{array} \begin{array}{c} \xrightarrow{l_1} \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ 2 \end{array} \begin{array}{c} 1^- \\ \swarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ 2 \end{array} = \frac{\text{tr}_+(4l_1l_2l_112)}{s s_{4l_1} s_{2l_2}} \hat{\kappa}_{(12)(13)} = -\frac{\text{tr}_+(4l_112)}{s_{4l_1} s_{2l_2}} \hat{\kappa}_{(12)(13)}, \quad (4.5)$$

where the Dirac algebra is used to cancel the s -channel pole. This implies that, similar to the rung rule in $\mathcal{N} = 4$ SYM, there are no triangles; only the first box drawn in figure 1 contributes. The second and third masters are isolated on near-identical cuts: the former by exchanging $p_3 \leftrightarrow p_4$, and the latter by also exchanging $p_1 \leftrightarrow p_2$. As the diagrammatic rules for the numerators do not depend on the ordering of the tree amplitudes (except for relabeling) the same result is obtained. The three masters are

$$n \left(\begin{array}{c} 4 \\ \swarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ 3 \end{array} \begin{array}{c} \xrightarrow{\ell} \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ 2 \end{array} \begin{array}{c} 1 \\ \swarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ 2 \end{array} \right) = \text{tr}_+(4\ell12) \hat{\kappa}_{(12)(13)} + \text{tr}_-(4\ell12) \hat{\kappa}_{(24)(34)}, \quad (4.6a)$$

$$n \left(\begin{array}{c} 4 \\ \swarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ 3 \end{array} \begin{array}{c} \xrightarrow{\ell} \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ 2 \end{array} \begin{array}{c} 1 \\ \swarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ 2 \end{array} \right) = \text{tr}_+(3\ell12) \hat{\kappa}_{(12)(14)} + \text{tr}_-(3\ell12) \hat{\kappa}_{(23)(34)}, \quad (4.6b)$$

$$n \left(\begin{array}{c} 4 \\ \swarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ 3 \end{array} \begin{array}{c} \xrightarrow{\ell} \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ 2 \end{array} \begin{array}{c} 1 \\ \swarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ 2 \end{array} \right) = \text{tr}_+(3\ell21) \hat{\kappa}_{(12)(24)} + \text{tr}_-(3\ell21) \hat{\kappa}_{(13)(34)}, \quad (4.6c)$$

where in all three cases the coefficient of tr_- is related by CPT conjugation. It can be checked using six-dimensional cuts that continuation to $D = 4 - 2\epsilon$ does not introduce terms proportional to μ^2 ; this can also be argued from the absence of quadratic ℓ terms in the numerators.

As further confirmation of these expressions we can also examine the t -channel cuts. The first master contributes to

$$\begin{array}{c} 4^+ \\ \swarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ 3 \end{array} \begin{array}{c} \xrightarrow{l_1} \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ 2 \end{array} \begin{array}{c} 1^- \\ \swarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ \text{wavy} \\ \downarrow \\ 2 \end{array} = -\left(\frac{1}{t} + \frac{1}{l_2^2}\right) \frac{\text{tr}_+(4l_112)}{l_1^2} \hat{\kappa}_{(12)(13)}. \quad (4.7)$$

In this case there are two contributors; as explained in section 3.2.4, the four-hyper amplitude naturally separates this cut into two contributions with equal numerators:

$$n \left(\begin{array}{c} 1 \\ \ell \uparrow \\ 4 \end{array} \begin{array}{c} \text{---} \end{array} \begin{array}{c} 2 \\ \text{---} \\ 3 \end{array} \right) = n \left(\begin{array}{c} 1 \\ \ell \uparrow \\ 4 \end{array} \begin{array}{c} \text{---} \end{array} \begin{array}{c} 2 \\ \text{---} \\ 3 \end{array} \right), \quad (4.8)$$

which is a two-term identity (see section 2.3.1). The t -channel cut for the second master is redundant as it related by symmetry to the s -channel one. For the third master, it gives

$$\begin{array}{c} l_1 \\ 4 \rightarrow \text{---} 1 \\ \text{---} \text{---} \text{---} \\ 3 \leftarrow \text{---} 2 \\ l_2 \end{array} = \begin{array}{c} 4 \rightarrow \text{---} 1 \\ \text{---} \text{---} \text{---} \\ 3 \leftarrow \text{---} 2 \end{array} = \begin{array}{c} 4 \rightarrow \text{---} 1 \\ \text{---} \text{---} \text{---} \\ 3 \leftarrow \text{---} 2 \end{array} = - \left(\frac{\text{tr}_+(12l_13)}{l_1^2 l_2^2} - \frac{2 \text{tr}_+(12l_23)}{t l_2^2} \right) \hat{\kappa}_{(12)(24)}. \quad (4.9)$$

We are required to sum over the two possible helicity configurations, and the Dirac algebra is used to expose the two contributors: the box and another triangle. They are automatically related by a commutation relation:

$$\begin{aligned} n \left(\begin{array}{c} 1 \\ \ell \uparrow \\ 4 \end{array} \begin{array}{c} \text{---} \end{array} \begin{array}{c} 2 \\ \text{---} \\ 3 \end{array} \right) &= n \left(\begin{array}{c} 1 \\ \ell \uparrow \\ 4 \end{array} \begin{array}{c} \text{---} \end{array} \begin{array}{c} 2 \\ \text{---} \\ 3 \end{array} \right) - n \left(\begin{array}{c} 1 \\ \ell \uparrow \\ 4 \end{array} \begin{array}{c} \text{---} \end{array} \begin{array}{c} 3 \\ \text{---} \\ 2 \end{array} \right) \\ &= -2 \text{tr}_+(4\ell 12) \hat{\kappa}_{(12)(13)} - 2 \text{tr}_-(4\ell 12) \hat{\kappa}_{(24)(34)}. \end{aligned} \quad (4.10)$$

This completes the set of non-zero numerators, which are also listed in appendix B.2; all of the extra off-shell identities described in section 2.3 are satisfied by these numerators including $\mathcal{N} = 4$ matching identities:

$$n \left(\begin{array}{c} 4 \rightarrow \text{---} 1 \\ \text{---} \text{---} \text{---} \\ 3 \leftarrow \text{---} 2 \end{array} \right) + n \left(\begin{array}{c} 4 \rightarrow \text{---} 1 \\ \text{---} \text{---} \text{---} \\ 3 \leftarrow \text{---} 2 \end{array} \right) = su(\hat{\kappa}_{(12)(13)} + \hat{\kappa}_{(24)(34)}), \quad (4.11a)$$

$$n \left(\begin{array}{c} 4 \rightarrow \text{---} 1 \\ \text{---} \text{---} \text{---} \\ 3 \leftarrow \text{---} 2 \end{array} \right) + n \left(\begin{array}{c} 4 \rightarrow \text{---} 1 \\ \text{---} \text{---} \text{---} \\ 3 \leftarrow \text{---} 2 \end{array} \right) = st(\hat{\kappa}_{(12)(14)} + \hat{\kappa}_{(23)(34)}), \quad (4.11b)$$

$$n \left(\begin{array}{c} 1 \\ \ell \uparrow \\ 4 \end{array} \begin{array}{c} \text{---} \end{array} \begin{array}{c} 2 \\ \text{---} \\ 3 \end{array} \right) + 2n \left(\begin{array}{c} 1 \\ \ell \uparrow \\ 4 \end{array} \begin{array}{c} \text{---} \end{array} \begin{array}{c} 2 \\ \text{---} \\ 3 \end{array} \right) = 0. \quad (4.11c)$$

These numerators have good IR behavior — taking the loop momentum associated with an edge carrying hypermultiplets in any one of them to zero, the numerator does indeed vanish; this does not happen for internal gluon lines. This indicates that soft divergences can indeed develop, but only as a result of soft vectors being exchanged, not soft hypers. Collinear divergences can also develop, but only at vertices connecting an internal gluon line.

4.3 External matter

A color-dual representation for four external matter multiplets has previously been obtained in ref. [64] via an orbifold construction.⁸ A single master is sufficient; we choose the box

⁸We are especially thankful to Marco Chiodaroli for sharing unpublished material containing the explicit orbifold construction for hypermultiplets.

Figure 2. Three two-loop masters with four external vector multiplets.

The resulting set of numerators satisfies the cuts conditions, color-kinematics duality and the two-term identities but violates the $\mathcal{N} = 4$ matching conditions. Moreover, an unfortunate consequence of the above numerator rearrangement is that the descendants become both numerous and more complicated (for instance, there are now non-vanishing bubbles and tadpoles). This suggests that, should one wish to expose the IR behavior in this way, maintaining manifest color-kinematics duality is not always the best approach. Nevertheless, when scattering four hypermultiplets at two loops we will find that such rearrangements become necessary in order to obtain valid color-dual numerators.

5 Two-loop examples

We now proceed to the main result of this paper: the complete set of two-loop four-point MHV amplitudes in $\mathcal{N} = 2$ SQCD. A color-dual representation of the amplitude with four external vector multiplets was already found in ref. [66] — in fact, two solutions were found which emphasized different physical properties. We begin by re-deriving one of these two solutions from iterative two-particle cuts; when expressed in terms of Dirac traces, the resulting solution is written far more compactly than as originally presented. We then proceed to calculate the two-loop amplitudes with one and two external hypermultiplet pairs.

As we explained at one loop, using cuts in strictly four dimensions misses extra-dimensional terms $\mu_{ij} = \bar{\ell}_i \cdot \bar{\ell}_j - \ell_i \cdot \ell_j$ needed for dimensional regularization ($\bar{\ell}_i$ is the four-dimensional part of ℓ_i). By evaluating cuts in six-dimensional $\mathcal{N} = (1, 0)$ SYM we recover the missing terms, which are simple enough not to interfere with the color duality of the four-dimensional numerators (again, see ref. [66] for details). A new feature is the antisymmetric object $\epsilon(\mu_1, \mu_2)$, which is an extra-dimensional echo of the six-dimensional Levi-Civita tensor:

$$\epsilon(\mu_1, \mu_2) = \frac{\epsilon^{(6)}(v_1, v_2, v_3, v_4, \ell_1, \ell_2)}{\epsilon(v_1, v_2, v_3, v_4)}, \quad (5.1)$$

where v_i are four-dimensional vectors. Its appearance is due to the unavoidably chiral nature of certain six-dimensional internal states; although it vanishes upon integration, we keep it here as it gives rise to non-chiral contributions after the double copy: $\epsilon(\mu_1, \mu_2)^2 = \mu_{11}\mu_{22} - \mu_{12}^2$.

5.1 External vectors

First we summarize the main result of ref. [66], updating the notation as necessary to make use of Dirac traces. The solution we are interested in was chosen to satisfy two-term and $\mathcal{N} = 4$ identities, as well as matter-reversal symmetry and CPT conjugation. A suitable

choice of three masters is displayed in figure 2. The pentagon triangle vanishes on its maximal cut; by setting it to zero, while demanding locality of all numerators, it was found that the other two masters are uniquely fixed. This left a total of 19 non-zero numerators.

The $\mathcal{N} = 4$ identities are particularly useful, as they ensure that all numerators with pure-adjoint content (no hyper loops) can be uniquely written in terms of those with internal hyper loops using $\mathcal{N} = 4$ identities. There are only two non-zero two-loop four-point $\mathcal{N} = 4$ SYM numerators in the MHV sector:

$$n^{[\mathcal{N}=4]} \left(\text{Diagram 1} \right) = n^{[\mathcal{N}=4]} \left(\text{Diagram 2} \right) = s \sum_{i < j} \kappa_{ij}, \quad (5.2)$$

where the $\mathcal{N} = 2$ content has been projected out. Using the two-term identities, any numerator with two hypermultiplet loops can be uniquely specified in terms of one with a single loop. So only numerators with a single hypermultiplet loop need to be specified — there are 10 of these.

Up to relabeling of loop momenta and overall constants, four are equal:⁹

$$\begin{aligned} n \left(\text{Diagram 3} \right) &= n \left(\text{Diagram 4} \right) = n \left(\text{Diagram 5} \right) = -\frac{1}{2} n \left(\text{Diagram 6} \right) \\ &= \hat{\kappa}_{13} \text{tr}_-(1\ell_1 24\ell_2 3) + \hat{\kappa}_{14} \text{tr}_-(1\ell_1 23\ell_2 4) + \hat{\kappa}_{23} \text{tr}_+(1\ell_1 23\ell_2 4) + \hat{\kappa}_{24} \text{tr}_+(1\ell_1 24\ell_2 3) \\ &\quad - s\mu_{12} (s(\hat{\kappa}_{12} + \hat{\kappa}_{34}) + t(\hat{\kappa}_{23} + \hat{\kappa}_{14}) + u(\hat{\kappa}_{13} + \hat{\kappa}_{24})) \\ &\quad + i\epsilon(\mu_1, \mu_2) s^2 (\hat{\kappa}_{12} - \hat{\kappa}_{34}). \end{aligned} \quad (5.3)$$

There is another non-planar double box with a similar structure, given by

$$\begin{aligned} n \left(\text{Diagram 7} \right) &= s(\hat{\kappa}_{12} \text{tr}_+(3\ell_{12}\ell_2 4) + \hat{\kappa}_{34} \text{tr}_-(3\ell_{12}\ell_2 4)) \\ &\quad + \hat{\kappa}_{13} \text{tr}_-(1\ell_2 42\ell_{12} 3) + \hat{\kappa}_{23} \text{tr}_+(1\ell_{12} 32\ell_2 4) \\ &\quad + \hat{\kappa}_{14} \text{tr}_-(1\ell_{12} 32\ell_2 4) + \hat{\kappa}_{24} \text{tr}_+(1\ell_2 42\ell_{12} 3) \\ &\quad + s(\mu_{12} + \mu_{22}) [s(\hat{\kappa}_{12} + \hat{\kappa}_{34}) + t(\hat{\kappa}_{23} + \hat{\kappa}_{14}) + u(\hat{\kappa}_{13} + \hat{\kappa}_{24})] \\ &\quad + i\epsilon(\mu_1, \mu_2) [t^2(\hat{\kappa}_{23} - \hat{\kappa}_{14}) + u^2(\hat{\kappa}_{13} - \hat{\kappa}_{24})], \end{aligned} \quad (5.4)$$

where $\ell_{12} = \ell_1 + \ell_2$. The only non-zero pentagon triangle with internal matter is

$$\begin{aligned} n \left(\text{Diagram 8} \right) &= -s(\hat{\kappa}_{12} \text{tr}_+(3\ell_{12}\ell_2 4) + \hat{\kappa}_{34} \text{tr}_-(3\ell_{12}\ell_2 4)) \\ &\quad - t(\hat{\kappa}_{23} \text{tr}_+(1\ell_{12}\ell_2 4) + \hat{\kappa}_{14} \text{tr}_-(1\ell_{12}\ell_2 4)) \\ &\quad - u(\hat{\kappa}_{13} \text{tr}_+(2\ell_{12}\ell_2 4) + \hat{\kappa}_{24} \text{tr}_-(2\ell_{12}\ell_2 4)) \\ &\quad - i\epsilon(\mu_1, \mu_2) [s^2(\hat{\kappa}_{12} - \hat{\kappa}_{34}) + t^2(\hat{\kappa}_{23} - \hat{\kappa}_{14}) + u^2(\hat{\kappa}_{13} - \hat{\kappa}_{24})], \end{aligned} \quad (5.5)$$

⁹In this section we use both κ_{ab} and $\hat{\kappa}_{ab} \equiv \kappa_{(ab)(ab)} = \kappa_{(ab)}/s_{ab}^2$ to our convenience.

Finally, the other four non-zero numerators are

$$\begin{aligned}
 n \left(\text{diagram 1} \right) &= -2\ell_1 \cdot \ell_2 \sum_{i < j} \kappa_{ij}, & n \left(\text{diagram 2} \right) &= -4\ell_1 \cdot \ell_2 \sum_{i < j} \kappa_{ij}, \\
 n \left(\text{diagram 3} \right) &= 2\ell_2 \cdot (p_4 - \ell_2) \sum_{i < j} \kappa_{ij}, & n \left(\text{diagram 4} \right) &= 4\ell_2 \cdot p_4 \sum_{i < j} \kappa_{ij}.
 \end{aligned} \tag{5.6}$$

As explained in ref. [66], the three diagrams with bubbles on external legs or tadpoles (5.6) are of no concern as they vanish upon integration.

This solution has good soft behavior: setting the loop momentum of any internal edge carrying hypermultiplets to zero, the corresponding numerator vanishes. In fact, double-box integrals involving the six-term traces appearing in eq. (5.3) have already been calculated by Caron-Huot and Larsen [120]; they were suggested as forming part of a basis of IR-finite integrals. From their results, we conclude that the integral of the first double box with hypermultiplets circulating the outside edge is both UV and IR finite to all orders in ϵ .

To see how the solution arises, we strategically choose cuts to yield information about the non-vanishing masters. Beginning with the first double box, we consider

$$\text{diagram 5.7a} = 0, \tag{5.7a}$$

$$\text{diagram 5.7b} = \left(\frac{1}{s} + \frac{1}{l_3^2} \right) \frac{\text{tr}_-(1l_1 24l_2 3)}{l_1^2 l_2^2} \hat{\kappa}_{13}, \tag{5.7b}$$

$$\text{diagram 5.7c} = \left(\frac{1}{s} + \frac{1}{l_3^2} \right) \frac{\text{tr}_-(1l_1 23l_2 4)}{l_1^2 l_2^2} \hat{\kappa}_{14}. \tag{5.7c}$$

These are almost identical to the one-loop cuts with four external vectors given in eq. (4.3); the new feature is the central tree amplitude insertion which, as explained in section 3.2.4, naturally implies two numerators equated by a two-term identity:

$$n \left(\text{diagram 5.8 left} \right) = n \left(\text{diagram 5.8 right} \right). \tag{5.8}$$

The double box can also be isolated from the double triangle on

$$\begin{array}{c} 4^+ \\ \downarrow l_2 \\ 3^+ \end{array} \begin{array}{c} \text{diagram} \\ \uparrow l_3 \end{array} \begin{array}{c} 1^- \\ \downarrow l_1 \\ 2^- \end{array} = 0, \quad (5.9a)$$

$$\begin{array}{c} 4^+ \\ \downarrow l_2 \\ 3^- \end{array} \begin{array}{c} \text{diagram} \\ \uparrow l_3 \end{array} \begin{array}{c} 1^- \\ \downarrow l_1 \\ 2^+ \end{array} = -\frac{\langle 3|l_2|4 \rangle}{(l_2 + p_4)^2(l_2 - p_3)^2} \times s \times \frac{\langle 1|l_1|2 \rangle}{s l_1^2} \times [13] \langle 24 \rangle \hat{\kappa}_{13} \\ = \frac{\text{tr}_-(1l_1 24 l_2 3)}{l_1^2(l_2 + p_4)^2(l_2 - p_3)^2} \hat{\kappa}_{13}, \quad (5.9b)$$

$$\begin{array}{c} 4^- \\ \downarrow l_2 \\ 3^+ \end{array} \begin{array}{c} \text{diagram} \\ \uparrow l_3 \end{array} \begin{array}{c} 1^- \\ \downarrow l_1 \\ 2^+ \end{array} = -\frac{\langle 4|l_2|3 \rangle}{(l_2 + p_4)^2(l_2 - p_3)^2} \times s \times \frac{\langle 1|l_1|2 \rangle}{s l_1^2} \times [14] \langle 23 \rangle \hat{\kappa}_{14} \\ = \frac{\text{tr}_-(1l_1 23 l_2 4)}{l_1^2(l_2 + p_4)^2(l_2 - p_3)^2} \hat{\kappa}_{14}. \quad (5.9c)$$

In the second two cases, we have re-used the one-loop cut given in eq. (4.5) for the two amplitudes on the left-hand side — we simply stripped away the part of the expression given by the external rules. So only the double box contributes, and we reproduce the same expression. We are also reminded that the pentagon triangle can safely be set to zero.

Finally, the other double box can be determined from

$$\begin{array}{c} 4^+ \\ \downarrow l_2 \\ 3^+ \end{array} \begin{array}{c} \text{diagram} \\ \uparrow l_3 \end{array} \begin{array}{c} 1^- \\ \downarrow l_1 \\ 2^- \end{array} = 0, \quad (5.10a)$$

$$\begin{array}{c} 4^+ \\ \downarrow l_2 \\ 3^- \end{array} \begin{array}{c} \text{diagram} \\ \uparrow l_3 \end{array} \begin{array}{c} 1^- \\ \downarrow l_1 \\ 2^+ \end{array} = -\frac{\text{tr}_-(1l_1 24 l_3 3)}{l_1^2 l_2^2 l_3^2} \hat{\kappa}_{13} - \frac{2 \text{tr}_-(1l_1 24 l_2 3)}{s l_1^2 l_2^2} \hat{\kappa}_{13}, \quad (5.10b)$$

$$\begin{array}{c} 4^- \\ \downarrow l_2 \\ 3^+ \end{array} \begin{array}{c} \text{diagram} \\ \uparrow l_3 \end{array} \begin{array}{c} 1^- \\ \downarrow l_1 \\ 2^+ \end{array} = -\frac{\text{tr}_-(1l_1 23 l_3 4)}{l_1^2 l_2^2 l_3^2} \hat{\kappa}_{14} - \frac{2 \text{tr}_-(1l_1 23 l_2 4)}{s l_1^2 l_2^2} \hat{\kappa}_{14}, \quad (5.10c)$$

This time we re-used the one-loop cut given in eq. (4.9), saving us the need to sum over helicity configurations of the vector multiplets. The two contributing numerators, a double box and double triangle, are related by a commutation relation:

$$n \left(\begin{array}{c} 4 \\ \downarrow \ell_2 \\ 3 \end{array} \begin{array}{c} \text{diagram} \\ \uparrow \ell_3 \end{array} \begin{array}{c} 1 \\ \downarrow \ell_1 \\ 2 \end{array} \right) = n \left(\begin{array}{c} 4 \\ \downarrow \ell_2 \\ 3 \end{array} \begin{array}{c} \text{diagram} \\ \uparrow \ell_3 \end{array} \begin{array}{c} 1 \\ \downarrow \ell_1 \\ 2 \end{array} \right) - n \left(\begin{array}{c} 3 \\ \downarrow \ell_2 \\ 4 \end{array} \begin{array}{c} \text{diagram} \\ \uparrow \ell_3 \end{array} \begin{array}{c} 1 \\ \downarrow \ell_1 \\ 2 \end{array} \right). \quad (5.11)$$

This works by precise analogy to the one-loop relation given in eq. (4.10).

5.2 External matter

In this case, assuming that the matter-reversal symmetry discussed in section 2.3.3 holds for non-tadpole diagrams, there are two masters displayed in figure 3. The first is most naturally isolated from

$$\begin{array}{c} 4 \\ \downarrow l_2 \\ 3 \end{array} \begin{array}{c} \text{diagram} \\ \uparrow l_3 \end{array} \begin{array}{c} 1 \\ \downarrow l_1 \\ 2 \end{array} = \frac{s^3}{l_1^2 l_2^2 l_3^2} \hat{\kappa}_{(12)(34)}. \quad (5.12)$$

Figure 3. The two two-loop masters with external hypermultiplets.

We encountered a similar pattern when dealing with four external hypermultiplets at one loop (4.12), and there it was simple to read off a color-dual box numerator. As explained in section 3.2.4, the new tree amplitude insertion implies that the double box should be given by a rung rule from the first box numerator in eq. (4.13), so in this case simply $s^3 \hat{\kappa}_{(12)(34)}$.

Unfortunately, we have confirmed by direct calculation that such a choice is incompatible with a color-dual representation of the complete amplitude assuming two-term identities. From the one-loop discussion in section 4.3 we learned that there is a second solution if we change the off-shell continuation to carry loop momenta. Converting one factor of s into $2(l_1 + p_1) \cdot (p_2 - l_1)$ or $2(l_2 + p_4) \cdot (p_3 - l_2)$ inside eq. (5.12), we make the following simple ansatz for the numerator:

$$n \left(\begin{array}{c} 4 \swarrow \quad \searrow 1 \\ \ell_2 \downarrow \quad \downarrow \ell_1 \\ 3 \swarrow \quad \searrow 2 \end{array} \right) = s^2 [c_1 s + c_2 (\ell_1 + p_1) \cdot (\ell_1 - p_2) + c_3 (\ell_2 + p_4) \cdot (\ell_2 - p_3)] \hat{\kappa}_{(12)(34)}, \quad (5.13)$$

where c_i are numerical coefficients to be determined. The same terms are also suggested by cuts of the form

$$\begin{array}{c} 4 \swarrow \quad \searrow 1 \\ \text{cut} \\ 3 \swarrow \quad \searrow 2 \end{array}, \quad \begin{array}{c} 4 \swarrow \quad \searrow 1 \\ \text{cut} \\ 3 \swarrow \quad \searrow 2 \end{array}, \quad (5.14)$$

which can again be obtained by recycling the one-loop results.

The situation is even more difficult for the second master in figure 3. The three cuts above naturally propose the same off-shell continuation, which does not conform to a color-dual representation (assuming two-term identities), as we explicitly checked. We therefore use an ansatz construction for this master along the lines of ref. [66]. It is, however, simplified by the diagrammatic rule (3.17c), which suggests an overall factor of s_{ab} , where a and b denote the two external hyper legs. Combining with the ansatz (5.13) and applying the constraints described in section 2.3, we arrive at a solution with a single free parameter. In our final numerators

$$n \left(\begin{array}{c} 4 \swarrow \quad \searrow 1 \\ \ell_2 \downarrow \quad \downarrow \ell_1 \\ 3 \swarrow \quad \searrow 2 \end{array} \right) = -s^2 [(\ell_1 + p_1) \cdot (\ell_1 - p_2) + (\ell_2 + p_4) \cdot (\ell_2 - p_3)] \hat{\kappa}_{(12)(34)}, \quad (5.15a)$$

$$n \left(\begin{array}{c} 4 \swarrow \quad \searrow 1 \\ \ell_2 \downarrow \quad \downarrow \ell_1 \\ 3 \swarrow \quad \searrow 2 \end{array} \right) = \frac{1}{2} s t [(\ell_1 - p_1 - p_3)^2 - 2(\ell_2 \cdot \ell_3)] \hat{\kappa}_{(14)(23)}, \quad (5.15b)$$

we fixed it to have the shortest possible expression for the second master.

5.3 External vectors + matter

The last — and most difficult — two-loop external MHV state configuration is with a single hypermultiplet pair and two vectors of opposite chirality. The number of masters

Figure 4. Two-loop masters with two external vector and matter multiplets.

given in figure 4 is significantly larger than for all the other cases. Similar to what we have seen with four external matter states, a color-dual representation does not always agree with the off-shell continuation suggested by the cuts. Nevertheless, we are able to find a valid color-dual representation for two of the master numerators in figure 4 directly from the cuts. Re-using computations from the corresponding one-loop example, we bring the cuts for these two masters into the form

$$\begin{aligned}
 \text{Diagram 1} &= -\frac{\text{tr}_-(4l_231)(s+l_3^2)(s+l_1^2)}{sl_1^2l_2^2l_3^2}\hat{\kappa}_{(41)(42)} \\
 &\quad -\frac{\text{tr}_+(4l_231)(s+l_3^2)(s+l_1^2)}{sl_1^2l_2^2l_3^2}\hat{\kappa}_{(31)(32)}, \quad (5.16a)
 \end{aligned}$$

$$\begin{aligned}
 \text{Diagram 2} &= -\frac{\text{tr}_-(4l_231)(s+l_1^2)}{l_1^2l_2^2l_3^2}\hat{\kappa}_{(41)(42)} - \frac{\text{tr}_+(4l_231)(s+l_1^2)}{l_1^2l_2^2l_3^2}\hat{\kappa}_{(31)(32)}, \quad (5.16b)
 \end{aligned}$$

$$\begin{aligned}
 \text{Diagram 3} &= \frac{[s \text{tr}_+(3l_341) + 2l_3^2 \text{tr}_+(3l_241)](s+l_1^2)}{sl_1^2l_2^2l_3^2}\hat{\kappa}_{(41)(42)} \\
 &\quad + \frac{[s \text{tr}_-(3l_341) + 2l_3^2 \text{tr}_-(3l_241)](s+l_1^2)}{sl_1^2l_2^2l_3^2}\hat{\kappa}_{(31)(32)}, \quad (5.16c)
 \end{aligned}$$

$$\begin{aligned}
 \text{Diagram 4} &= \frac{\text{tr}_+(3l_341)(s+l_1^2)}{l_1^2l_2^2l_3^2}\hat{\kappa}_{(41)(42)} + \frac{\text{tr}_-(3l_341)(s+l_1^2)}{l_1^2l_2^2l_3^2}\hat{\kappa}_{(31)(32)}. \quad (5.16d)
 \end{aligned}$$

This leads us to simple expressions for the corresponding double-box numerators:

$$n\left(\text{Diagram 1}\right) = -s \text{tr}_-(4l_231)\hat{\kappa}_{(41)(42)} - s \text{tr}_+(4l_231)\hat{\kappa}_{(31)(32)}, \quad (5.17a)$$

$$n\left(\text{Diagram 2}\right) = s \text{tr}_+(3l_341)\hat{\kappa}_{(41)(42)} + s \text{tr}_-(3l_341)\hat{\kappa}_{(31)(32)}, \quad (5.17b)$$

and these do form a valid representation.

The other seven masters fall out of the pattern and had to be computed through an ansatz construction. After implementing as many constraints as possible from section 2.3 the solution contains one free parameter, that we fix by hand to obtain the shortest possible representation. The expressions for all numerators are given in the [supplementary material](#) attached to this publication and described in appendix B.

6 Multi-particle cuts

We generalize the recursion to multi-particle cuts, deriving general formulas for cuts of MHV or $\overline{\text{MHV}}$ amplitudes built from MHV and $\overline{\text{MHV}}$ trees. The structure of supersums for less than maximal supersymmetries has previously been studied in ref. [121], using similar on-shell superspace techniques as we will employ here. As there is no natural generalization of κ at higher points, our generalizations of the $\mathcal{N} = 4$ and $\mathcal{N} = 2$ supersymmetric recursion formulas, given in eqs. (3.3) and (3.12), are built out of full tree-level amplitudes containing physical poles in their Parke-Taylor factors; the resulting cut formulas are therefore less compact. Nevertheless, if the final result is a four-point cut then there is a simple mechanism to reintroduce κ and cancel all unphysical poles. These expressions can be used for an iteration to any loop order.

In order to determine higher-loop or higher-point amplitudes one generally also requires cuts containing non-MHV (and non- $\overline{\text{MHV}}$) amplitudes. However, to obtain two-loop MHV amplitudes this is not necessary; the techniques described here have been used to check the two-loop representations detailed in the previous section.

6.1 $\mathcal{N} = 4$ SYM

Consider a cut of the form

$$\begin{aligned} \text{Diagram} &= \int d^4\eta_{l_1} \cdots d^4\eta_{l_r} A_{k+m}^{(0)}(1, \dots, k, l_1, \dots, l_r) \\ &\quad \times A_{n-k+m}^{(0)}(k+1, \dots, n, -l_m, \dots, -l_1). \end{aligned} \quad (6.1)$$

We are interested in cuts for which the individual trees and the full external state configuration live in either the MHV or $\overline{\text{MHV}}$ sector. There are two ways this can happen: (i) one of the trees is MHV and the other is $\overline{\text{MHV}}$, in which case we require $k = 2$ or $k = n - 2$; (ii) both trees are MHV or $\overline{\text{MHV}}$, in which case we require $m = 2$ (a two-particle cut). When $k = m = 2$ both cases should reduce to the existing iterated two-particle cuts; for more than a three-particle cut ($m > 3$) this will give only specific contributions to the full cut. We consider the two possibilities in turn.

6.1.1 $\overline{\text{MHV}} \times \text{MHV}$

In this configuration the cut (6.1) is given by the superspace integration

$$\text{Diagram} = - \int d^4\eta_{l_1} \cdots d^4\eta_{l_m} \frac{\delta^8(\bar{Q}_L)}{[12][2l_1] \cdots [l_m 1]} \frac{\delta^8(Q_R)}{\langle 34 \rangle \cdots \langle n l_m \rangle \cdots \langle l_1 3 \rangle}, \quad (6.2)$$

where we have inserted the Parke-Taylor formulas (2.6) and (2.11) for right-hand MHV and left-hand $\overline{\text{MHV}}$ tree amplitudes respectively; using the symmetry between chiral and anti-chiral superspace we can specialize to $k = 2$ without loss of generality. We also implicitly assume a Fourier transform (2.13) of the first delta function to bring it into the chiral superspace.

To see the iterated structure some manipulation is required. The product of the two Parke-Taylor denominators is brought into the form

$$\begin{aligned} & \frac{1}{[12][2l_1] \cdots [l_m 1]} \frac{1}{\langle 34 \rangle \cdots \langle n l_m \rangle \cdots \langle l_1 3 \rangle} \\ &= \frac{\langle 12 \rangle}{[12]} \underbrace{\frac{\langle 2|3|l_1|2 \rangle \langle 1|l_m|n|1 \rangle}{s_{2l_1} s_{l_m 1} s_{n(-l_m)} s_{(-l_1)3} s_{l_1 l_2} \cdots s_{l_{m-1}, l_m}}}_{\text{phys. poles}} \underbrace{\frac{1}{\langle 12 \rangle \cdots \langle n1 \rangle}}_{\text{Parke-Taylor}}, \end{aligned} \quad (6.3)$$

which exposes another Parke-Taylor factor. The overall supersum is given by

$$\int d^{\mathcal{N}} \eta_{l_1} \cdots d^{\mathcal{N}} \eta_{l_m} \delta^{2\mathcal{N}}(\bar{Q}_L) \delta^{2\mathcal{N}}(Q_R) = [12]^{\mathcal{N}} \delta^{2\mathcal{N}}(Q), \quad (6.4)$$

where Q is the overall supermomentum. Putting the pieces together,

$$\begin{array}{c} 2 \\ \circlearrowleft \\ 1 \end{array} \begin{array}{c} l_1 \\ \rightarrow \\ l_m \end{array} \begin{array}{c} 3 \\ \circlearrowright \\ n \end{array} = \frac{i s_{12} \text{tr}_-(23l_1 21l_m n1)}{s_{2l_1} s_{(-l_1)3} s_{1l_m} s_{(-l_m)n} s_{l_1 l_2} \cdots s_{l_{m-1} l_m}} A_n^{(0), \text{MHV}}(1, \dots, n). \quad (6.5)$$

Via CPT conjugation the cut with $\text{MHV} \leftrightarrow \overline{\text{MHV}}$ is given by replacing $|i\rangle \leftrightarrow |i]$, which exchanges $\text{tr}_+ \leftrightarrow \text{tr}_-$.

If, at the end of several iteration steps, we are left with the cut of a four-point amplitude the result is simplified by reinstating κ :

$$\begin{array}{c} 2 \\ \circlearrowleft \\ 1 \end{array} \begin{array}{c} l_1 \\ \rightarrow \\ l_m \end{array} \begin{array}{c} 3 \\ \circlearrowright \\ 4 \end{array} = - \frac{\text{tr}_-(43l_1 21l_m)}{s_{2l_1} s_{(-l_1)3} s_{1l_m} s_{(-l_m)4} s_{l_1 l_2} \cdots s_{l_{m-1} l_m}} \kappa. \quad (6.6)$$

We recover the two-particle cut (3.5) for $m = 2$ using $\text{tr}_-(43l_1 21l_2) = -s_{1l_2} s_{4(-l_2)} s_{l_1 l_2}$. For $m = 3$ this construction determines the full cut, given by

$$\begin{array}{c} 2 \\ \circlearrowleft \\ 1 \end{array} \begin{array}{c} l_1 \\ \rightarrow \\ l_3 \end{array} \begin{array}{c} 3 \\ \circlearrowright \\ 4 \end{array} + \begin{array}{c} 2 \\ \circlearrowright \\ 1 \end{array} \begin{array}{c} l_1 \\ \rightarrow \\ l_3 \end{array} \begin{array}{c} 3 \\ \circlearrowleft \\ 4 \end{array} = - \frac{\text{tr}(43l_1 21l_3)}{s_{2l_1} s_{3(-l_1)} s_{1l_3} s_{4(-l_3)} s_{l_1 l_2} s_{l_2 l_3}} \kappa. \quad (6.7)$$

The trace arises as $\text{tr} = \text{tr}_+ + \text{tr}_-$ from the two contributions.

6.1.2 MHV $\times$ MHV

The computation of two-particle cuts involving two MHV trees is analogous:

$$\begin{array}{c} k \\ \circlearrowleft \\ 1 \end{array} \begin{array}{c} l_1 \\ \rightarrow \\ l_2 \end{array} \begin{array}{c} k+1 \\ \circlearrowright \\ n \end{array} = i \frac{\text{tr}_+(l_1 k(k+1) l_1 l_2 n1 l_2)}{s_{1l_2} s_{kl_1} s_{(k+1)(-l_1)} s_{n(-l_2)}} A_n^{(0), \text{MHV}}. \quad (6.8)$$

When $k = 2$ and $n = 4$ we recover the two-particle cut (3.5) using $\text{tr}_+(l_1 23l_1 l_2 41l_2) = -s t s_{3(-l_1)} s_{2l_1}$. The two MHV Parke-Taylor factors from the trees have been manipulated using

$$\begin{aligned} & \frac{1}{\langle l_2 1 \rangle \cdots \langle k l_1 \rangle \langle l_1 l_2 \rangle} \times \frac{1}{\langle l_1(k+1) \rangle \cdots \langle n l_2 \rangle \langle l_2 l_1 \rangle} \\ &= - \underbrace{\frac{[l_1|k|k+1|l_1][l_2|n|1|l_2][l_1 l_2]^2}{s_{1l_2} s_{kl_1} s_{(k+1)(-l_1)} s_{n(-l_2)} s_{l_1 l_2}^2}}_{\text{phys. poles}} \underbrace{\frac{1}{\langle 12 \rangle \cdots \langle n1 \rangle}}_{\text{Parke-Taylor}}, \end{aligned} \quad (6.9)$$

Larsen [120], which are IR finite when integrated to all orders in $\epsilon = (4 - D)/2$. Their integrands are closely related to the chiral, BCFW-derived “local integrands” developed in planar $\mathcal{N} = 4$ SYM, which also have good IR properties [4, 5, 7]. We also wonder whether such simplicity persists at three loops.

However, in other cases we encountered obstacles to finding simple color-dual integrands. We first noticed this in the one-loop solution with external matter multiplets — while a simple color-dual representation exists, it does not exhibit the IR properties we have come to expect. This persisted at two loops: in both cases with external matter, while we always found compact expressions for the cuts, extending them to off-shell color-dual numerators required more work. Nevertheless, knowing the terms appearing in different cuts allowed us to restrict our ansätze to certain terms, thus simplifying the computation.

Such obstacles are not uncommon in the pursuit of color-dual loop integrands. At three loops in $\mathcal{N} = 4$ SYM, the rung rule does not give four-point color-dual numerators [4, 5, 59, 62, 102]; instead, it gives a representation matching the one in ref. [4]. Efforts to find a color-dual representation of the four-point, five-loop $\mathcal{N} = 4$ integrand have faced similar impediments [11] — however, the need for such a representation to perform the double copy has now been circumvented to compute UV divergences in $\mathcal{N} = 8$ supergravity [123–125]. Another conspicuous example is the five-gluon two-loop all-plus integrand: while non-local color-dual numerators were found requiring twelve powers of loop momentum [126], a planar local-integrand-based presentation is far more compact, with only four non-zero numerators [80, 119].

This suggests that our requirement of color-kinematics duality is, in some cases, creating tension between the off-shell numerators, so if our objective is not to use the double copy, we should consider relaxing it. Doing so would allow us to directly lift expressions for the numerators from their cuts; however, each numerator would then need to be computed separately. It would also be necessary to ensure that expressions for numerators coming from different cuts overlap with each other. The reward may be more chiral integrand structures for the two-loop solutions with external hypermultiplets.

A study of non-planar structures would also be required — we believe these can be isolated from iterated two-particle cuts. Local-integrand-like non-planar integrands have already been developed for $\mathcal{N} = 4$ SYM [127]; the relevant integrals have logarithmic singularities manifested by expressing them in $d\log$ forms [12, 128]. In this paper, we have found examples of non-planar chiral integrands in the two-loop four-vector amplitude, which warrant further study.

We intend to explore these concepts further in an upcoming work, in which our main objective will be a better understanding of the interplay between local integrand representations and the IR structure of two-loop $\mathcal{N} = 2$ SQCD integrands.

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A Superspace calculus

In this section we detail the derivation of eqs. (6.10), (3.3) and (3.12). Grassmann variables are widely used in the literature to trivialize supersymmetric state sums, see notably refs. [121, 129]. Here we also restore some relative signs that we chose to omit in the main text.

We start by pointing out that already in eq. (2.7) there is a sign ambiguity due to an unspecified order of Grassmann multiplication inside $\delta^{2\mathcal{N}}(Q)$. We fix that sign by taking the right-hand side of eq. (2.7) as the definition of the Grassmann delta function. Then we can use the Schouten identity to derive the following identity valid for any p and q such that $\langle pq \rangle \neq 0$:

$$\delta^{2\mathcal{N}}\left(\sum_{i=1}^n |i\rangle \eta_i\right) \equiv \prod_{I=1}^{\mathcal{N}} \sum_{i < j}^n \langle i j \rangle \eta_i^I \eta_j^I = \frac{1}{\langle pq \rangle^{\mathcal{N}}} \delta^{\mathcal{N}}\left(\sum_{i=1}^n \langle pi \rangle \eta_i\right) \delta^{\mathcal{N}}\left(\sum_{i=1}^n \langle qi \rangle \eta_i\right). \quad (\text{A.1})$$

Using this identity twice, we compute the supersum (6.10) relevant for a general two-particle MHV $\times$ MHV:

$$\begin{aligned} & \int d^{\mathcal{N}} \eta_{l_1} d^{\mathcal{N}} \eta_{l_2} \delta^{2\mathcal{N}}(Q_L) \delta^{2\mathcal{N}}(Q_R) \\ &= \int d^{\mathcal{N}} \eta_{l_1} d^{\mathcal{N}} \eta_{l_2} \delta^{2\mathcal{N}}\left(|l_1\rangle \eta_{l_1} + |l_2\rangle \eta_{l_2} + \sum_{i=1}^k |i\rangle \eta_i\right) \delta^{2\mathcal{N}}\left(-|l_1\rangle \eta_{l_1} - |l_2\rangle \eta_{l_2} + \sum_{i=k+1}^n |i\rangle \eta_i\right) \\ &= \frac{1}{\langle l_1 l_2 \rangle^{2\mathcal{N}}} \int d^{\mathcal{N}} \eta_{l_1} d^{\mathcal{N}} \eta_{l_2} \delta^{\mathcal{N}}\left(\langle l_1 l_2 \rangle \eta_{l_2} + \sum_{i=1}^k \langle l_1 i \rangle \eta_i\right) \delta^{\mathcal{N}}\left(\langle l_2 l_1 \rangle \eta_{l_1} + \sum_{i=1}^k \langle l_2 i \rangle \eta_i\right) \\ &\quad \times \delta^{\mathcal{N}}\left(\langle l_1 l_2 \rangle \eta_{l_2} - \sum_{i=k+1}^n \langle l_1 i \rangle \eta_i\right) \delta^{\mathcal{N}}\left(\langle l_2 l_1 \rangle \eta_{l_1} - \sum_{i=k+1}^n \langle l_2 i \rangle \eta_i\right) \\ &= \delta^{\mathcal{N}}\left(\sum_{i=1}^n \langle l_1 i \rangle \eta_i\right) \delta^{\mathcal{N}}\left(\sum_{i=1}^n \langle l_2 i \rangle \eta_i\right) = \langle l_1 l_2 \rangle^{\mathcal{N}} \delta^{2\mathcal{N}}(Q). \end{aligned} \quad (\text{A.2})$$

It is now effortless to verify the $\mathcal{N} = 4$ supersum in eq. (3.3):

$$\begin{aligned} & \int d^4 \eta_{l_1} d^4 \eta_{l_2} \kappa(1, 2, l_1, l_2) \kappa(3, 4, -l_2, -l_1) = \frac{[12][34][l_1 l_2]^2}{\langle 12 \rangle \langle 34 \rangle \langle l_1 l_2 \rangle^2} \int d^4 \eta_{l_1} d^4 \eta_{l_2} \delta^8(Q_L) \delta^8(Q_R) \\ &= \frac{[12][34]\langle l_1 l_2 \rangle^2 [l_1 l_2]^2}{\langle 12 \rangle \langle 34 \rangle} \delta^8\left(\sum_{i=1}^4 |i\rangle \eta_i\right) = s_{l_1 l_2}^2 \kappa(1, 2, 3, 4). \end{aligned} \quad (\text{A.3})$$

The $\mathcal{N} = 2$ supersum in eq. (3.12) is handled similarly

$$\begin{aligned}
& \int d^4\eta_{l_1} d^4\eta_{l_2} \kappa_{(ab)(cd)}(1, 2, l_1, l_2) \kappa_{(ef)(gh)}(3, 4, -l_2, -l_1) \\
&= \frac{[12][34][l_1 l_2]^2}{\langle 12 \rangle \langle 34 \rangle \langle l_1 l_2 \rangle^2} \int d^4\eta_{l_1} d^4\eta_{l_2} \delta^4(Q_L) \langle ab \rangle \langle cd \rangle \eta_a^3 \eta_b^3 \eta_c^4 \eta_d^4 \times \delta^4(Q_R) \langle ef \rangle \langle gh \rangle \eta_e^3 \eta_f^3 \eta_g^4 \eta_h^4 \\
&= \frac{[12][34][l_1 l_2]^2}{\langle 12 \rangle \langle 34 \rangle \langle l_1 l_2 \rangle^2} \int d^2\eta_{l_1} d^2\eta_{l_2} \delta^4(Q_L) \delta^4(Q_R) \\
&\quad \times \int d\eta_{l_1}^4 d\eta_{l_1}^3 d\eta_{l_2}^4 d\eta_{l_2}^3 (\eta_a^3 \eta_b^3 \eta_c^4 \eta_d^4) (\eta_e^3 \eta_f^3 \eta_g^4 \eta_h^4) \langle ab \rangle \langle cd \rangle \langle ef \rangle \langle gh \rangle \\
&= \frac{[12][34][l_1 l_2]^2}{\langle 12 \rangle \langle 34 \rangle} \delta^4(Q) \int d\eta_{l_1}^4 d\eta_{l_1}^3 d\eta_{l_2}^4 d\eta_{l_2}^3 (\eta_a^3 \eta_b^3 \eta_c^4 \eta_d^4) (\eta_e^3 \eta_f^3 \eta_g^4 \eta_h^4) \langle ab \rangle \langle cd \rangle \langle ef \rangle \langle gh \rangle \\
&= \text{sgn}(abcd) \text{sgn}(efgh) [l_1 l_2]^2 \langle ab \rangle \langle cd \rangle \langle ef \rangle \langle gh \rangle [qr][st] \frac{\kappa_{(qr)(st)}}{s_{qr} s_{st}},
\end{aligned} \tag{A.4}$$

up to the last step in the derivation, where we have used the fact that the broken-superspace variables must factorize onto the external ones comprising $\kappa_{(qr)(st)}$ and the internal ones annihilated by the remaining Grassmann integration,

$$(\eta_a^3 \eta_b^3 \eta_c^4 \eta_d^4) (\eta_e^3 \eta_f^3 \eta_g^4 \eta_h^4) = (\eta_{l_1}^3 \eta_{l_1}^4 \eta_{l_2}^3 \eta_{l_2}^4) (\eta_q^3 \eta_r^3 \eta_s^4 \eta_t^4). \tag{A.5}$$

The signs $\text{sgn}(abcd)$ and $\text{sgn}(efgh)$ are determined by the permutation signatures with respect to $\{q, r, l_1, l_2\}$ and $\{s, t, l_1, l_2\}$, respectively.

B All integrands summarized

In this section we summarize the full color-dual representation of all one- and two-loop integrands for $\mathcal{N} = 2$ SQCD. The one-loop results are short enough to be explicitly written out here. The two-loop integrands are quite lengthy, but they, as well as their one-loop counterpart, can be downloaded as [supplementary material](#). Table 2 summarizes which representations fulfill the properties discussed in section 2.3.

All representations are attached in a machine-readable format in the [supplementary material](#) of this paper. The ancillary files for each solution are named:

- One-loop external vectors: `ancillaryLeq1Vectors.m`
- One-loop mixed: `ancillaryLeq1Mixed.m`
- One-loop external matter: `ancillaryLeq1Matter.m`
- Two-loop external vectors: `ancillaryLeq2Vectors.m`
- Two-loop mixed: `ancillaryLeq2Mixed.m`
- Two-loop external matter: `ancillaryLeq2Matter.m`

The files are optimized for usage with Mathematica, but the format is general enough to allow for an import into any other computer algebra system. Each file contains a short overview of its contents in the start.

	Two-term id.	Manifest CPT	Matter reversal	$\mathcal{N} = 4$
1-loop vectors	✓	✓	✓	✓
1-loop mixed	✓	✓	✓	✓
1-loop matter	✓	✓	✓	✓
1-loop matter alt.	✓	✓	$\times^*$	$\times$
2-loop vectors	✓	✓	✓	✓
2-loop mixed	✓	✓	✓	✓
2-loop matter	✓	✓	$\times^*$	$\times$

Table 2. Properties of the various solutions summarized: two-term identities (see section 2.3.1), manifest CPT invariance (see section 2.3.2), matter-reversal symmetry (see section 2.3.3), and adding up to $\mathcal{N} = 4$ (see section 2.3.4). *Matter-reversal symmetry works for all numerators except for some of those with matter tadpoles. The symmetry can still be used to reduce the set of masters for all other topologies.

B.1 One-loop external vectors

$$n \left(\begin{array}{c} \text{4} \quad \ell \quad \text{1} \\ \text{3} \quad \text{2} \end{array} \right) = \hat{\kappa}_{13} \text{tr}_-(1(\ell - p_1)(\ell + p_4)3) + \hat{\kappa}_{24} \text{tr}_+(1(\ell - p_1)(\ell + p_4)3) + \mu^2 (s(\hat{\kappa}_{12} + \hat{\kappa}_{34}) + t(\hat{\kappa}_{23} + \hat{\kappa}_{14}) + u(\hat{\kappa}_{13} + \hat{\kappa}_{24})), \quad (\text{B.1a})$$

$$n \left(\begin{array}{c} \text{1} \quad \text{2} \\ \ell \uparrow \quad \text{3} \\ \text{4} \end{array} \right) = (\hat{\kappa}_{13} + \hat{\kappa}_{34}) \text{tr}_-(1(\ell - p_1)(\ell + p_4)3) + (\hat{\kappa}_{12} + \hat{\kappa}_{24}) \text{tr}_+(1(\ell - p_1)(\ell + p_4)3) + (\hat{\kappa}_{12} + \hat{\kappa}_{34}) t \ell^2, \quad (\text{B.1b})$$

$$n \left(\begin{array}{c} \text{4} \quad \ell \quad \text{1} \\ \text{3} \quad \text{2} \end{array} \right) = 2\ell \cdot (p_{12} - \ell) [t(\hat{\kappa}_{23} + \hat{\kappa}_{14}) - u(\hat{\kappa}_{13} + \hat{\kappa}_{24})], \quad (\text{B.1c})$$

$$n \left(\begin{array}{c} \text{4} \quad \ell \quad \text{1} \\ \text{3} \quad \text{2} \end{array} \right) = 2\ell \cdot (p_4 + \ell) [u(\hat{\kappa}_{13} + \hat{\kappa}_{24}) - t(\hat{\kappa}_{14} + \hat{\kappa}_{23})], \quad (\text{B.1d})$$

$$n \left(\begin{array}{c} \text{4} \quad \ell \downarrow \quad \text{1} \\ \text{3} \quad \text{2} \end{array} \right) = 4(\ell \cdot p_4) [u(\hat{\kappa}_{13} + \hat{\kappa}_{24}) - t(\hat{\kappa}_{14} + \hat{\kappa}_{23})], \quad (\text{B.1e})$$

$$n \left(\begin{array}{c} \text{4} \quad \ell \downarrow \quad \text{1} \\ \text{3} \quad \text{2} \end{array} \right) = 4(\ell \cdot p_{34}) [u(\hat{\kappa}_{13} + \hat{\kappa}_{24}) - t(\hat{\kappa}_{14} + \hat{\kappa}_{23})], \quad (\text{B.1f})$$

$$n \left(\begin{array}{c} \text{4} \quad \text{1} \\ \text{3} \quad \text{2} \end{array} \right) = n^{[\mathcal{N}=4]} \left(\begin{array}{c} \text{4} \quad \text{1} \\ \text{3} \quad \text{2} \end{array} \right) - 2n \left(\begin{array}{c} \text{4} \quad \text{1} \\ \text{3} \quad \text{2} \end{array} \right), \quad (\text{B.1g})$$

$$n \left(\begin{array}{c} \text{1} \quad \text{2} \\ \ell \uparrow \quad \text{3} \\ \text{4} \end{array} \right) = -2n \left(\begin{array}{c} \text{1} \quad \text{2} \\ \ell \uparrow \quad \text{3} \\ \text{4} \end{array} \right), \quad (\text{B.1h})$$

$$n \left(\begin{array}{c} \ell \\ \text{diagram} \end{array} \right) = -2n \left(\begin{array}{c} \ell \\ \text{diagram} \end{array} \right), \quad n \left(\begin{array}{c} \ell \\ \text{diagram} \end{array} \right) = -2n \left(\begin{array}{c} \ell \\ \text{diagram} \end{array} \right), \quad (\text{B.1i})$$

$$n \left(\begin{array}{c} \ell \\ \text{diagram} \end{array} \right) = -2n \left(\begin{array}{c} \ell \\ \text{diagram} \end{array} \right), \quad n \left(\begin{array}{c} \ell \\ \text{diagram} \end{array} \right) = -2n \left(\begin{array}{c} \ell \\ \text{diagram} \end{array} \right). \quad (\text{B.1j})$$

B.2 One-loop external vectors + matter

$$n \left(\begin{array}{c} \ell \\ \text{diagram} \end{array} \right) = \hat{\kappa}_{(12)(13)} \text{tr}_+(4\ell 12) + \hat{\kappa}_{(24)(34)} \text{tr}_-(4\ell 12), \quad (\text{B.2a})$$

$$n \left(\begin{array}{c} \ell \\ \text{diagram} \end{array} \right) = \hat{\kappa}_{(12)(14)} \text{tr}_+(3\ell 12) + \hat{\kappa}_{(23)(34)} \text{tr}_-(3\ell 12), \quad (\text{B.2b})$$

$$n \left(\begin{array}{c} \ell \\ \text{diagram} \end{array} \right) = \hat{\kappa}_{(12)(24)} \text{tr}_+(3\ell 21) + \hat{\kappa}_{(13)(34)} \text{tr}_-(3\ell 21), \quad (\text{B.2c})$$

$$n \left(\begin{array}{c} \ell \\ \text{diagram} \end{array} \right) = -\frac{1}{2} n \left(\begin{array}{c} \ell \\ \text{diagram} \end{array} \right) = \hat{\kappa}_{(12)(13)} \text{tr}_+(4\ell 12) + \hat{\kappa}_{(24)(34)} \text{tr}_-(4\ell 12). \quad (\text{B.2d})$$

B.3 One-loop external matter

$$n \left(\begin{array}{c} \text{diagram} \end{array} \right) = -su \hat{\kappa}_{(13)(24)}, \quad n \left(\begin{array}{c} \text{diagram} \end{array} \right) = s^2 \hat{\kappa}_{(12)(34)}, \quad (\text{B.3a})$$

$$n \left(\begin{array}{c} \text{diagram} \end{array} \right) = -n \left(\begin{array}{c} \text{diagram} \end{array} \right) = -su \hat{\kappa}_{(13)(24)}, \quad (\text{B.3b})$$

$$n \left(\begin{array}{c} \text{diagram} \end{array} \right) = -\frac{1}{2} n \left(\begin{array}{c} \text{diagram} \end{array} \right) = -su \hat{\kappa}_{(13)(24)}. \quad (\text{B.3c})$$

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