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Out of time ordered quantum dissipation

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Abstract: We consider a quantum Brownian particle interacting with two harmonic baths, which is then perturbed by a cubic coupling linking the particle and the baths. This cubic coupling induces non-linear dissipation and noise terms in the influence functional/master equation of the particle. Its effect on the Out-of-Time-Ordered Correlators (OTOCs) of the particle cannot be captured by the conventional Feynman-Vernon formalism.We derive the generalised influence functional which correctly encodes the physics of OTO fluctuations, response, dissipation and decoherence. We examine an example where Markovian approximation is valid for the OTO dynamics.

If the original cubic coupling has a definite time-reversal parity, the leading order OTO influence functional is completely determined by the couplings in the usual master equation via OTO generalisation of Onsager-Casimir relations. New OTO fluctuation-dissipation relations connect the non-Gaussianity of the thermal noise to the thermal jitter in the damping constant of the Brownian particle.

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Contents

1 Motivation

The dynamics of a Brownian particle interacting with a thermal bath is a topic that has been studied for over a hundred years. A systematic understanding of a quantum Brownian particle emerged in the 1960s with the works of Schwinger [1] and Feynman-Vernon [2]. In these works, an effective theory was derived for a quantum Brownian particle by tracing out the thermal bath's degrees of freedom.

These analyses were later extended by Caldeira and Leggett [3] to a concrete model of a particle linearly coupled to a harmonic bath. The bath degrees of freedom can be exactly integrated out to get a non-local, non-unitary theory describing the evolution of the particle. In this simple model, Caldeira and Leggett managed to show the following interesting result: first, if the distribution of the bath oscillator frequencies is chosen appropriately then the bath correlators decay with time. Consequently, at sufficiently long time-scales, one obtains a local effective theory for the particle which in classical limit reduces to the standard Langevin dynamics. This model of Caldeira-Leggett and its generalisations [4, 5] (see [6–8] for textbook level discussion) have been crucial in understanding dissipation and decoherence in quantum systems.

In this work, we seek to generalise Caldeira-Leggett like models. We extend such models in two directions: one extension is to go beyond the linear Langevin description to non-linear generalisations of Langevin equation. While the non-linear versions have indeed been considered in different contexts before (see references above), the authors of this work are not aware of a systematic classification of all possible terms at the leading level of non-linearity for a Brownian particle coupled to a generic bath. Such a systematic study along with the concomitant generalisation of the famous fluctutation dissipation theorems and Onsager's reciprocal relations is one of our objectives.

The power of Caldeira-Leggett model lies in its ability to relate the effective couplings of a dissipative/open quantum system description to the underlying microscopic physics. At a superficial level, the idea is the one familiar in usual effective theories: one computes correlators in the effective description and in the microscopic description and then matches the two. But, in open quantum systems, the integrated out degrees of freedom are not quite the heavy degrees of freedom. Thus, the integrated degrees of freedom can go on-shell resulting in the non-unitarity of the effective description. For this reason, the matching is not merely one of comparing low frequency behaviour.

At high temperatures, a whole band of frequency domain contributes right upto the thermal scale. Further, this contribution is modulated by how effectively the particle couples to the various states of the bath. We will take these aspects into account by writing the effective couplings as frequency integrals over complex contours with appropriate $i\epsilon$ prescription that picks up the correct causal response. The integrands would be the *spectral functions* of the bath which encode the effective number of bath states accessible to the particle in a given process. From a mathematical point of view, we will write down expressions that relate the effective couplings of the Langevin description to generalised discontinuities of the appropriate (in general, out of time ordered) bath correlators.

A second related objective is the generalisation of Caldeira-Leggett like models to take into account out of time order correlations(OTOCs) and how they get transmitted from

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the bath to the Brownian particle. Two of us [9] recently considered the general formalism to tackle the question of how the OTOCs of a probe records information about the OTOCs of a system. This work is motivated by trying to find a concrete realisation of the ideas considered in [9].

The perspective provided by extending the ideas about open systems to include out of time order correlations is crucial to our work. As we shall show explicitly, the fluctuation-dissipation theorems and Onsager's reciprocal relations relate parameters governing time-ordered correlations to those controlling out of time ordered correlations. The straightforward way to understand the relations between time-ordered couplings is via the relations between out of time-ordered correlations and time-ordered correlations. This is the macroscopic counterpart to the observation that including out of time ordered correlations simplify the structure of thermal correlators in a general quantum field theory [10–12].

We illustrate our results about non-linear corrections to Langevin theory by considering a simple extension of Caldeira-Leggett model. Our model consists of a Brownian particle coupled to two sets of oscillators at the same temperature. Apart from the usual linear coupling, we will also turn on weak three-body interactions involving the particle and two other bath oscillators, one drawn from each set. We show that under appropriate distribution of couplings, the bath continues to be Markovian. We derive the non-linear corrections to Langevin equation that result from such an interaction in the Markovian limit and check that these couplings indeed satisfy the correct generalisations of fluctuation-dissipation/Onsager reciprocal relations.

In the following sections, we will elaborate on these ideas and summarise our results. We will begin by reviewing the model by Caldeira-Leggett in section 2. This is followed by a detailed description of a non-linear generalisation of Langevin theory in section 3, where we also summarise our main results on such non-linear corrections for a general bath. This is followed by a description of the particular model we work with. A complete specification of the qXY model is given in section 4. The derivation of the non-linear Langevin theory from qXY model is the subject of section 5. We give a general analysis of both the generalised Onsager and fluctuation-dissipation relations in the non-linear Langevin theory in the section 6, before our concluding remarks in the section 7.

Appendix A enumerates the dimension of various physical quantities that appear in this work. In appendix B, we give a brief review of the rules that constrain the form of effective theories that can arise by integrating out the bath. This is followed by appendix C, where we have summarised various contour integrals that are useful in computation of the effective Langevin couplings from our microscopic model.

2 Review of Caldeira-Leggett model

We will now begin with a brief review of the salient features of the quantum Brownian particle and later, how these get generalised in the context of our work. We will sketch the basic structural features postponing detailed derivations for later. The reader familiar with this material is encouraged to quickly skim over this description, making note of our notation.

2.1 Review of Langevin theory and fluctuation-dissipation theorem

Consider, for definiteness, a Brownian particle whose evolution is described by Langevin equation:

$$\frac{d^2q}{dt^2} + \gamma \frac{dq}{dt} = \langle f^2 \rangle \, \mathcal{N}(t) \, . \tag{2.1}$$

Here γ is the damping constant of the particle. The term $\langle f^2 \rangle \mathcal{N}(t)$ is the fluctuating force ('the noise term') which is commonly approximated to be Gaussian and delta-correlated, viz., its only non-zero cumulant is taken to be of the form

$$\langle \mathcal{N}(t)\mathcal{N}(t') \rangle_{\text{noise}} \equiv \frac{1}{\langle f^2 \rangle} \, \delta(t-t'). \tag{2.2}$$

Here $\mathcal{N}(t)$ is normalised to have the inverse dimension of velocity for later convenience, so that $\langle f^2 \rangle$ measures the statistical variance of the fluctuating force per unit mass (and has the dimension of acceleration$^2\times$time). The cumulant statement above can also be recast into a statement about the probability distribution governing the noise ensemble:

$$P[\mathcal{N}] \propto \exp\left\{ -\frac{\langle f^2 \rangle}{2} \int dt \, \mathcal{N}^2 \right\} . \tag{2.3}$$

The significance of Caldeira-Leggett model is that it shows how such a system of equations can arise from an underlying quantum mechanical model, under appropriate approximations, via integrating out the effects of a harmonic bath.

Another triumph of Caldeira-Leggett model is that it can reproduce the well-known fluctuation-dissipation relation [13–16] between the parameter $\langle f^2 \rangle$ characterising thermal fluctuations and the parameter γ characterising dissipation. Let us remind the reader the classical argument why such a relation should be expected: the thermal fluctuations in the bath which source the noise also cause the dissipative effects of the bath. On one hand, the rate of damping γ is roughly determined by the effective number of degrees of freedom of the bath with which it interacts. On the other hand, the thermal noise is produced by the energy fluctuations in the bath which is roughly $k_B T$ times the number of degrees of freedom (here, T denotes the temperature of the bath and k_B is the Boltzmann constant). This suggests a relationship of the form $\langle f^2 \rangle \sim \gamma \frac{k_B T}{m_p}$ between the fluctuation and the dissipation parameters. Here, we have introduced the mass of the particle m_p to match the dimensions.

A more precise argument is as follows: at the level of classical stochastic dynamics, one can integrate the Langevin equation to obtain

$$\frac{dq}{dt} = e^{-\gamma(t-t_0)} \left(\frac{dq}{dt} \right)_{t_0} + \langle f^2 \rangle \int_{t_0}^{t} dt' e^{-\gamma(t-t')} \mathcal{N}(t') \, . \tag{2.4}$$

We square this equation and average over the noise under the assumption that the initial velocity distribution and the fluctuating force are uncorrelated. At large times, this yields for the long-time average kinetic energy for a particle of mass m_p:

$$\lim_{t\to\infty} \left\langle \frac{1}{2} m_p \left(\frac{dq}{dt} \right)^2 \right\rangle_{\text{noise}} = \frac{m_p \langle f^2 \rangle}{4\gamma} \, . \tag{2.5}$$

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Demanding that this average kinetic energy approach the thermal equipartition value $\frac{1}{2}k_BT$, we obtain the fluctuation-dissipation relation

$$\langle f^2\rangle = 2\gamma \frac{k_BT}{m_p} \equiv 2\gamma v_{th}^2 \,, \tag{2.6}$$

where we have introduced the rms thermal velocity v_{th} of the Brownian particle defined via

$$v_{th}^2 \equiv \frac{k_BT}{m_p} \,. \tag{2.7}$$

The physics behind such a classical stochastic argument is clear: the kicks of the fluctuating force should, on average, replenish the energy lost due to dissipation so that the eventual balance is achieved at the Maxwell-Boltzmann value for the average kinetic energy.

As pointed out by Kubo, Martin and Schwinger [17, 18], the origins of this relation can be traced back, in the underlying quantum description, to the mathematical structure of thermal correlators. In the linear response theory, both the fluctuating force and damping rate felt by the Brownian particle can be computed from a characteristic *spectral function* of the bath denoted by $\rho(\omega)$, related to the Fourier domain commutator of bath operators that couple to the particle (see below). We then have the frequency integrals (often termed the *sum rules*):

$$\begin{aligned} m_p^2\langle f^2\rangle &= \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{d\omega}{2\pi i}\frac{\rho(\omega)}{\omega^2}\ \left(1+2\mathfrak{f}_\omega\right)\hbar\omega \ \ , \\ m_p\gamma &= \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{d\omega}{2\pi i}\frac{\rho(\omega)}{\omega^2}\ \ , \end{aligned} \tag{2.8}$$

where $\beta \equiv \frac{\hbar}{k_BT}$ is the periodicity of the thermal circle and $\mathfrak{f}_\omega \equiv \frac{1}{e^{\beta\omega}-1}$ is the Bose-Einstein distribution. These relations make precise the afore-mentioned intuition that the same effective degrees of freedom (encoded in the single function $\rho(\omega)$ of the bath) determine both $\langle f^2\rangle$ and γ.

Note the factor $(1 + 2\mathfrak{f})$ whose three terms describe spontaneous emission, stimulated emission and absorption of the fluctuation of frequency ω, all of which add incoherently into the variance of noise. At high temperature limit (i.e., small β) one then recovers eq. (2.6). The $i\epsilon$ prescription for the frequency integral is the frequency domain analogue of step functions that appear in time domain retarded Green functions. For example, the formula above for γ can also be written in the form

$$m_p\gamma = \int d\tau\ (\tau-t)\Theta(t-\tau)\int \frac{d\omega}{2\pi i}\rho(\omega)e^{-i\omega(t-\tau)} \ \ , \tag{2.9}$$

which can be thought of as coming in turn from the approximation of a non-local retarded Green function:

$$m_p\gamma\dot{q}(t) \in \int d\tau\ \left\{\Theta(t-\tau)\int \frac{d\omega}{2\pi i}\rho(\omega)e^{-i\omega(t-\tau)}\right\}\ q(\tau) \ \ . \tag{2.10}$$

While such time domain expressions clearly exhibit the causality properties, we will find it convenient to work in frequency domain where the structure of thermal correlators can be examined clearly. All our frequency domain contour integrals can, if needed, be readily converted into time domain integrals with appropriate step functions.

Let us examine in some more detail how this works: consider a probe particle coupled to a bath in a general time-independent state and assume that at long enough time scales, a local autonomous description is still possible for the particle (i.e., Markovian approximation can be justified and any effect of the memory of the bath can be ignored). Under these assumptions, the effect of the bath can be integrated out following the method of Schwinger [1] and Feynman-Vernon [2] to get a local Schwinger-Keldysh effective action (or equivalently a local Feynman-Vernon influence functional). We can match this local effective action against the generating functional for the Langevin correlators following the method of Martin-Siggia-Rose [19]-De Dominicis-Peliti [20]-Janssen [21] (see [22] for a recent review). This results in a general expression for the Langevin couplings:

$$\begin{aligned} m_p^2\langle f^2\rangle &= \int_{-\infty-i\epsilon}^{\infty-i\epsilon}\frac{d\omega_1}{2\pi i}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}\frac{d\omega_2}{2\pi}\ \frac{\rho[12_+]}{\omega_1}\ , \\ m_p\gamma &= \int_{-\infty-i\epsilon}^{\infty-i\epsilon}\frac{d\omega_1}{2\pi i}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}\frac{d\omega_2}{2\pi}\ \frac{\rho[12]}{\omega_1^2}\ . \end{aligned} \tag{2.11}$$

Here the integrands are the Fourier transformed expectation values of the commutator/anti-commutator

$$\begin{aligned} \rho[12] &\equiv \frac{1}{\hbar}\int dt_1\int dt_2\ e^{i(\omega_1 t_1+\omega_2 t_2)}\ \langle[\mathcal{O}(t_1),\mathcal{O}(t_2)]\rangle_B, \\ \rho[12_+] &\equiv \int dt_1\int dt_2\ e^{i(\omega_1 t_1+\omega_2 t_2)}\langle\{\mathcal{O}(t_1),\mathcal{O}(t_2)\}\rangle_B\,. \end{aligned} \tag{2.12}$$

In the above, $\mathcal{O}$ is the bath operator that couples linearly to the Brownian particle position q and the expectation values are evaluated in an appropriate time-independent state of the bath.[1]

The reader should also note the specific energy-conserving $i\epsilon$ prescription needed to write the sum-rules above:

$$\int_{\mathcal{C}_2}\equiv\int_{-\infty-i\epsilon}^{\infty-i\epsilon}\frac{d\omega_1}{2\pi}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}\frac{d\omega_2}{2\pi}\ . \tag{2.13}$$

The above expressions relating the microscopic dynamics of the bath degrees of freedom to the effective couplings can be obtained by standard procedures of effective theory: for instance by matching the correlators predicted by the effective theory against the microscopic computations. The $i\epsilon$ prescription then naturally appears when comparing appropriately retarded correlators.

[1]In the following, we stick to the convention that every commutator is divided by $\hbar$ in the definition of spectral functions. This has the advantage that the classical limit of the spectral function $\rho[12]$ is just the Fourier transform of the Poisson bracket. For example, eq. (2.11) is also valid in classical statistical mechanics, provided $\rho[12]$ is computed via Poisson brackets.

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Another important property of the complex contour $\mathcal{C}_2$ is the following: it remains invariant under a simultaneous complex conjugation and the reversal of frequencies. It follows then that, if the integrands are similarly invariant (as the above integrands indeed are), the resultant answers are real.

The expressions above imply that, from the point of view of thermal correlators, whereas the damping constant γ is related to the *commutator* of bath operators, the fluctuation $\langle f^2 \rangle$ is related to the *anti-commutator*. If the bath state is thermal, the Kubo-Martin-Schwinger(KMS) conditions relate these frequency domain functions as

$$\rho[12_+] = \hbar(1 + 2\mathfrak{f}_1)\rho[12] = -\hbar(1 + 2\mathfrak{f}_2)\rho[12]\,. \tag{2.14}$$

Using time translation invariance to write $\rho[12] = \rho(\omega_1)\ 2\pi\delta(\omega_1 + \omega_2)$, we then recover eq. (2.8). The eq. (2.14) can be motivated by the following relation between thermal averages in a harmonic oscillator of frequency ω:

$$\langle \{a, a^\dagger\} \rangle_\beta = \hbar\,(1 + 2\mathfrak{f}_\omega)\,\frac{1}{\hbar} \langle [a, a^\dagger] \rangle_\beta\,. \tag{2.15}$$

KMS showed that such a relation continues to hold true for a general quantum system, thus giving rise to eqs. (2.8) and (2.6) under very general assumptions.

To summarise, the microscopic justification of fluctuation-dissipation theorem lies in the following steps:

- In the first step, one identifies the relevant effective couplings of the system (here $\langle f^2 \rangle$ and γ of the Brownian particle) and connects it with the appropriate bath correlators (via sum rules like the ones in eq. (2.11)). This step already assumes the emergence of a Markovian description which can be checked explicitly in a simple model like that of Caldeira-Leggett.

- In the next step, one uses thermality to derive KMS relations akin to eq. (2.14). Note that this step, by itself, does not immediately result in a simple relation between the effective couplings of the local Markovian description.

  KMS condition merely says that two couplings are related to two *different moments* of the same spectral function (see eq. (2.8)). These two moments are a priori two independent numbers provided by the theory behind the bath. Thus, the KMS condition alone is of limited utility to an experimentalist probing the local dynamics of the Brownian particle.[2]

- In the last step, we take a high temperature limit to get a fluctuation-dissipation relation of the form eq. (2.6). It is in this step that one obtains the fluctuation-dissipation relation between the effective couplings.

[2]One could however imagine a fine-grained experiment sensitive to non-Markovian/memory effects of the bath (such experiments are within the realm of possibility [23–26]) and think of the integrands in eq. (2.8) as part of the memory functions of the bath. For this reason, many authors (see, for example, Stratonovich [27, 28]) refer to equations like eq. (2.14) as *non-Markovian* fluctuation-dissipation theorems.

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Apart from the fluctuation-dissipation relations, there are other useful relations that can be derived when one has more than one Brownian degree of freedom and when one can assume additional symmetries. An example of such a symmetry is the microscopic time-reversal invariance whose consequences were explored by Onsager [29, 30] and Casimir [31] (we refer the reader to the monograph by Stratonovich [27] for extensions and a detailed exposition). Say we had many degrees of freedom denoted by the coordinate q_A, which undergo coupled Langevin dynamics governed by a matrix of Langevin couplings γ_{AB} and $\langle f^2_{AB}\rangle$. In that case, the sum rules of the discussion above generalise to

$$\begin{aligned} m_p^2\langle f^2_{AB}\rangle &= \int_{-\infty-i\epsilon}^{\infty-i\epsilon}\frac{d\omega_1}{2\pi i}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}\frac{d\omega_2}{2\pi}\,\frac{\rho[1_A 2_{B+}]}{\omega_1}\ , \\ m_p\gamma_{AB} &= \int_{-\infty-i\epsilon}^{\infty-i\epsilon}\frac{d\omega_1}{2\pi i}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}\frac{d\omega_2}{2\pi}\,\frac{\rho[1_A 2_B]}{\omega_1^2}\ . \end{aligned} \tag{2.16}$$

Here $\mathcal{O}_A$ is the bath operator that couples to the coordinate q_A and we have used a matrix of spectral functions

$$\begin{aligned} \rho[1_A 2_B] &\equiv \frac{1}{\hbar}\int dt_1\int dt_2\ e^{i(\omega_1 t_1+\omega_2 t_2)}\ \langle[\mathcal{O}_A(t_1),\mathcal{O}_B(t_2)]\rangle, \\ \rho[1_A 2_{B+}] &\equiv \int dt_1\int dt_2\ e^{i(\omega_1 t_1+\omega_2 t_2)}\langle\{\mathcal{O}_A(t_1),\mathcal{O}_B(t_2)\}\rangle\,. \end{aligned} \tag{2.17}$$

Say the dynamics and the initial state of the bath are microscopic time-reversal invariant and we will assume that the Hermitian operators $\{\mathcal{O}_A\}$ have definite time-reversal parities $\{\eta_A\}$. The microscopic time-reversal then acts by the simultaneous exchange of ω_1 and ω_2 along with complex conjugation.[3] One then obtains the famous *Onsager reciprocal relations* [29, 30]:

$$\langle f^2_{AB}\rangle = \eta_A\eta_B\ \langle f^2_{BA}\rangle\,,\quad \gamma_{AB} = \eta_A\eta_B\ \gamma_{BA}\,. \tag{2.18}$$

In this work, we will mainly confine ourselves to studying the dynamics of a single degree of freedom, which can in principle couple to many bath operators $\{\mathcal{O}_A\}$ with different couplings g_A. Then, the Onsager reciprocal relations can be interpreted as statements relating the contributions proportional to $g_A g_B$ to the damping constant and the noise variance. We will later meet an example of similar relations, when we study non-linear corrections to the dynamics of a single degree of freedom (see the discussion in section 6.2).

2.2 Model of the harmonic bath in Caldeira-Leggett like models

We will now move from the general description to particular models of the bath. The equation (2.11) for the damping constant γ naively suggests the following: if we can engineer a bath of harmonic oscillators with a linear spectral function of the form[4]

$$\rho[12] \equiv \rho(\omega_1)2\pi\delta(\omega_1+\omega_2)\ \sim m_p\gamma\omega_1\ 2\pi\delta(\omega_1+\omega_2)\,, \tag{2.19}$$

[3] We note that the contour $\mathcal{C}_2$ is left invariant under this operation. Hence, the symmetry property of integrands under this operation is inherited by the effective couplings.

[4] We note that, for Markovian approximation to hold, the spectral function should be sufficiently analytic near real axis of the frequency domain [4, 5, 32].

a naive residue integral seems to pickup the required contribution. This is however misleading. In fact, the integral for γ with the linear spectral function is UV divergent and needs to be regulated appropriately. A simple and commonly used regulator is to assume a Lorentz-Drude form for the spectral function, with the linear growth at low frequencies:

$$\rho(\omega) = 2\ \mathrm{Im}\left\{ m_p\gamma \frac{i\Omega^2}{\omega + i\Omega} \right\} = 2m_p\gamma\omega \frac{\Omega^2}{\omega^2 + \Omega^2}\,, \tag{2.20}$$

which does give back γ after the contour integral is performed.

How can such a distribution be obtained from the underlying microscopic dynamics of the harmonic bath? Say we are interested in the thermal harmonic bath where $q(t)$ of the Brownian particle is coupled linearly to the bath oscillators via

$$\mathcal{O}(t) = \sum_i g_{x,i}\ X_i(t)\,. \tag{2.21}$$

We want to now examine what set of couplings $g_{x,i}$ and masses $m_{x,i}$ for the bath oscillators will result in a Lorentz-Drude spectral function. Computing the commutator of the bath operator in the thermal state and Fourier transforming yields

$$\begin{aligned} \rho[12] &= \frac{1}{2\omega_1}\ 2\pi\delta(\omega_1+\omega_2) \sum_i \frac{g_{x,i}^2}{m_{x,i}} 2\pi\delta(|\omega_1| - \mu_i) \\ &\equiv 2\pi\delta(\omega_1+\omega_2) \times \int_0^\infty \frac{d\mu_x}{2\pi}\ (2\pi)\ \mathrm{sgn}(\omega_1)\delta(\omega_1^2 - \mu_x^2) \left\langle\!\left\langle \frac{g_x^2}{m_x} \right\rangle\!\right\rangle, \end{aligned} \tag{2.22}$$

where we have defined a function of μ_x:

$$\left\langle\!\left\langle \frac{g_x^2}{m_x} \right\rangle\!\right\rangle \equiv \sum_i \frac{g_{x,i}^2}{m_{x,i}} 2\pi\delta(\mu_x - \mu_i)\,. \tag{2.23}$$

A spectral function $\rho(\omega)$ as in eq. (2.20) is then obtained, if we take a continuum of oscillators whose couplings add up to give

$$\left\langle\!\left\langle \frac{g_x^2}{m_x} \right\rangle\!\right\rangle = m_p\gamma \frac{4\mu_x^2\Omega^2}{\mu_x^2 + \Omega^2}\,. \tag{2.24}$$

Thus, the continuum approximation with an infinite set of oscillators gives the required smooth form for the spectral function. Only in this limit can the set of bath oscillators be idealised as a perfect thermal bath into which the Brownian particle can irreversibly dissipate into. It is also only in this limit that the information about the particle is quickly forgotten by the thermal bath, thus allowing us to ignore any memory effect. This last point can be explicitly checked by computing the bath correlators and confirming that they indeed decay at time scales set by Ω^{-1}. Thus, we expect a local description to be good when there is a hierarchy of frequency scales:

$$\gamma \ll \Omega \ll \frac{k_B T}{\hbar}\,.$$

The continuum approximation and the resultant irreversibility are good approximations at time scales much smaller than the inverse of the typical gap in the bath spectrum.

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This ends our brief review of the standard Langevin theory. One main goal of this work is to see many of these ideas and expressions generalise to higher point functions, out of time order correlations and to non-linear Langevin dynamics.

3 Introduction to non-linear Langevin equation

It is a natural question to ask how these results generalise once we go beyond the linear Langevin description. A particular non-linear generalisation is of relevance to this work, which we shall now describe. Consider a non-linear generalisation of Langevin theory described by the following stochastic equation:

$$\mathcal{E}[q] \equiv \frac{d^2q}{dt^2} + (\gamma + \zeta_\gamma \mathcal{N})\frac{dq}{dt} + (\bar{\mu}^2 + \zeta_\mu \mathcal{N})\, q + \left(\overline{\lambda}_3 - \overline{\lambda}_{3\gamma}\, \frac{d}{dt}\right)\frac{q^2}{2!} - F = \langle f^2 \rangle \mathcal{N} \ . \tag{3.1}$$

Here, we will take $\mathcal{N}$ to be a random noise drawn from the non-Gaussian probability distribution

$$P[\mathcal{N}] \propto \exp\left\{ -\frac{1}{2\langle f^2\rangle}\int dt\, \left(\langle f^2\rangle \mathcal{N} - \zeta_N \mathcal{N}^2\right)^2 - \frac{1}{2} Z_I \int dt\, \dot{\mathcal{N}}^2 \right\} . \tag{3.2}$$

We will assume that the corrections to the Langevin equation are small: this is equivalent to assuming that the parameters $\{\zeta_\gamma, \zeta_\mu, \overline{\lambda}_3, \overline{\lambda}_{3\gamma}, \zeta_N, Z_I\}$ are small.

The physical meaning of these non-linear parameters should be evident: ζ_γ, ζ_μ characterise the thermal jitter in the damping constant γ and the (renormalised) natural frequency $\bar{\mu}^2$ of the Brownian oscillator. The parameters $\overline{\lambda}_3, \overline{\lambda}_{3\gamma}$ control the anharmonicity in the model whereas ζ_N characterises the non-Gaussianity of the thermal noise. The above equation includes all terms upto quadratic in amplitudes of q and $\mathcal{N}$ and upto one time derivative acting on q (except for the inertial term $\frac{d^2q}{dt^2}$ which was already present before the bath came into picture). In this sense, this is indeed the most generic leading non-linear correction to the linear Langevin theory.

Another equivalent way to define these couplings would be to state how they occur in the long-time three point cumulants of the Brownian particle which starts off from the harmonic oscillator vacuum at an initial time $t = t_0$. While specification in terms of cumulants does not have the immediate intuitive appeal of the description above, it is a very useful characterisation for computing these couplings from a microscopic model. Consider long enough time scales such that the memory effects of the bath can be ignored and the Markovian approximation is valid. We will however be interested in the time scales much smaller than γ^{-1}, the time scale at which damping effects become substantial. In this time window, one can write down universal expressions for the vacuum cumulants of the Brownian particle in terms of the effective Langevin couplings. In the semi-classical limit (i.e., ignoring $O(\hbar)$ terms), they take the following form [9]: for $t_1 > t_2 > t_3$, we get

$$
\begin{aligned}
\frac{1}{\hbar^2}\langle[q(t_1)q(t_2)q(t_3)]\rangle_c &= \Big(\overline{\lambda}_3 - \overline{\lambda}_{3\gamma}\frac{\partial}{\partial t_1}\Big)Q_{123} + O(\hbar)\,,\\
\frac{1}{\hbar^2}\langle[q(t_3)q(t_2)q(t_1)]\rangle_c &= \Big(\overline{\kappa}_3 + \overline{\kappa}_{3\gamma}\frac{\partial}{\partial t_3}\Big)Q_{321} + O(\hbar)\,,\\
\frac{i}{\hbar}\langle[q(t_1)q(t_2)q(t_3)_+]\rangle_c &= O(\hbar),\\
\frac{i}{\hbar}\langle[q(t_3)q(t_2)q(t_1)_+]\rangle_c &= 2m_p\Big[\zeta_\mu + 2\zeta_\gamma\frac{\partial}{\partial t_1} - \frac{2}{3}\widehat{\kappa}_{3\gamma}\Big(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2}\Big)\Big]Q_{321} + O(\hbar)\,,\\
\frac{i}{\hbar}\langle[q(t_1)q(t_3)q(t_2)_+]\rangle_c &= -2m_p\Big[\zeta_\mu + 2\zeta_\gamma\frac{\partial}{\partial t_2} + \frac{2}{3}\widehat{\kappa}_{3\gamma}\Big(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2}\Big)\Big]Q_{321} + O(\hbar)\,,\\
\langle[q(t_1)q(t_2)_+q(t_3)_+]\rangle_c &= \frac{2}{\overline{\mu}^4}\zeta_N\Big[2\cos(\overline{\mu}(t_{13}+t_{23})) + 6\cos(\overline{\mu}t_{12}) + \cos(\overline{\mu}(t_{10}+t_{20}+t_{30}))\\
&\quad - 3\cos(\overline{\mu}(t_{12}-t_{30})) - 3\cos(\overline{\mu}(t_{23}-t_{10})) - 3\cos(\overline{\mu}(t_{31}-t_{20}))\Big]\\
&\quad + O(\hbar)\,.
\end{aligned} \tag{3.3}
$$

In the above, we have defined the function

$$
Q_{ijk} \equiv \frac{1}{6(m_p\overline{\mu}^2)^2}\Big\{\cos[\overline{\mu}(t_{ij}+t_{ik})] - \cos[\overline{\mu}(t_{ji}+t_{jk})] - 3\cos[\overline{\mu}t_{ik}] + 3\cos[\overline{\mu}t_{jk}]\Big\}\,, \tag{3.4}
$$

with $t_{ij} \equiv t_i - t_j$ and we have used the square bracket notation to indicate nested commutators (with a + subscript indicating anticommutator). For example,

$$
\begin{aligned}
\langle[q(t_1)q(t_2)q(t_3)]\rangle_c &\equiv \langle[[q(t_1),q(t_2)],q(t_3)]\rangle_c\,,\\
\langle[q(t_1)q(t_2)q(t_3)_+]\rangle_c &\equiv \langle\{[q(t_1),q(t_2)],q(t_3)\}\rangle_c\,,\\
\langle[q(t_1)q(t_2)_+q(t_3)_+]\rangle_c &\equiv \langle\{\{q(t_1),q(t_2)\},q(t_3)\}\rangle_c\,.
\end{aligned} \tag{3.5}
$$

The six correlators given above cover all possible time orderings with three operators. In the above, we have divided by a factor of $\hbar$ for every commutator in l.h.s., so that the commutators smoothly go to Poisson brackets in the classical limit.

The reader should also note that apart from the non-linear Langevin/1-OTO couplings,[5] three new 'out of time ordered' 2-OTO couplings $\overline{\kappa}_3, \overline{\kappa}_{3\gamma}$ and $\widehat{\kappa}_{3\gamma}$ are needed to fit the long-time behaviour of arbitrarily ordered cumulants. On general grounds, we expect correlators with four out of the six time orderings to be computable via standard Schwinger-Keldysh/Feynman-Vernon influence functionals: these are the 1-OTO correlators in the classification of [10]. A basis of such correlators is provided by

$$
\begin{aligned}
&\langle[q(t_1)q(t_2)q(t_3)]\rangle_c\,,\quad \langle[q(t_1)q(t_2)q(t_3)_+]\rangle_c\,,\quad \langle[q(t_1)q(t_2)_+q(t_3)_+]\rangle_c\,,\\
&\langle[q(t_1)q(t_2)_+q(t_3)]\rangle_c = -\langle[q(t_3)q(t_2)q(t_1)_+]\rangle_c + \langle[q(t_1)q(t_3)q(t_2)_+]\rangle_c\,,
\end{aligned} \tag{3.6}
$$

all of which can be written down in terms of the standard non-linear Langevin/1-OTO couplings, as is evident from the expressions above. The two other remaining time-orderings

[5]These labels characterise how much out of time ordered a particular correlator/coupling actually is. The numbers here represent the number of minimum number of time-folds required to define a particular correlator/coupling [10].

are however genuinely out of time ordered which can neither be captured by the standard Schwinger-Keldysh/Feynman-Vernon influence functionals nor can they be written solely in terms of the standard non-linear Langevin couplings.

A historical aside may be in order: many authors have studied non-linear generalisation of Langevin equations in a variety of contexts (See, for example [33–36]). However, as far as the authors are aware, there is no systematic discussion in the literature including all leading nonlinear corrections allowed on general grounds, nor a microscopic model within which Markovian approximation is justified and all couplings derived. Similarly, despite a long and rich literature on non-linear fluctuation-dissipation/Onsager relations [16, 27, 28, 37–44], the relations we derive here for non-linear Langevin theory are new, as far as we know.

In the rest of this section, we will give a detailed summary of our results listing the integrals that relate the couplings that appear in the above equation to the bath correlators in the microscopic theory. The reader desirous of a briefer summary is encouraged to consult the subsection 3.6 at the end of this section.

3.1 Linear Langevin couplings

We will now summarise our results regarding how the above coefficients could be computed starting from a non-linear generalisation of Caldeira-Leggett like setup. Assume the Langevin degree of freedom $q(t)$ is still linearly coupled to some bath operator $\mathcal{O}(t)$.[6] As in the discussion above eq. (2.11), take the bath to be in a general time-independent state and assume a Markovian description at long enough time scales.

Under these assumptions, a general expression can be written for the Langevin couplings in leading order causal perturbation theory in particle-bath coupling (or equivalently by matching the local influence functional to MSRDPJ action of the above stochastic equation to leading order in perturbation theory):

$$\begin{aligned} m_p\gamma &= \int_{\mathcal{C}_2} \frac{\rho[12]}{i\omega_1^2}\,, \quad m_p\Delta\bar{\mu}^2 = -\int_{\mathcal{C}_2} \frac{\rho[12]}{\omega_1}\,, \quad \Delta m_p = \int_{\mathcal{C}_2} \frac{\rho[12]}{\omega_1^3}\,, \\ m_p^2\langle f^2\rangle &= \int_{\mathcal{C}_2} \frac{\rho[12_+]}{i\omega_1} = \int_{\mathcal{C}_2} \frac{\hbar}{i\omega_1}(1+2\mathfrak{f}_1)\rho[12]\,, \\ m_p^2 Z_I &= -\int_{\mathcal{C}_2} \frac{\rho[12_+]}{i\omega_1^3} = -\int_{\mathcal{C}_2} \frac{\hbar}{i\omega_1^3}(1+2\mathfrak{f}_1)\rho[12]\,, \end{aligned} \tag{3.7}$$

where as before, we have an integral over the causal contour

$$\int_{\mathcal{C}_2} \equiv \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{d\omega_1}{2\pi} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{d\omega_2}{2\pi}\ . \tag{3.8}$$

Here Δm_p and $\Delta\bar{\mu}^2$ denote the renormalisation of mass and frequency of the Brownian oscillators.

The spectral functions appearing in the integrand are defined via Fourier integrals

$$\begin{aligned} \rho[12_+] &\equiv \int dt_1 \int dt_2\ e^{i(\omega_1 t_1+\omega_2 t_2)}\ \langle\{\mathcal{O}(t_1),\mathcal{O}(t_2)\}\rangle_B\,, \\ \rho[12] &\equiv \frac{1}{\hbar}\int dt_1 \int dt_2\ e^{i(\omega_1 t_1+\omega_2 t_2)}\ \langle[\mathcal{O}(t_1),\mathcal{O}(t_2)]\rangle_B\,. \end{aligned} \tag{3.9}$$

[6] We assume that the thermal 1-point function of this bath operator is zero.

Further, in writing the second equality for the couplings above, we have assumed that the bath state is thermal and consequently, we have used the Kubo-Martin-Schwinger (KMS) relation to convert the anti-commutator to commutator.

The complex contour $\mathcal{C}_2$ has the following *reality* property: it remains invariant under a simultaneous complex conjugation and the reversal of frequencies;

$$\mathcal{C}_2^*[\omega_i^c \ \rightarrow \ -\omega_i] = \mathcal{C}_2[\omega_i], \tag{3.10}$$

where ω_i^c denotes the complex conjugate of the frequency ω_i. Further, the hermiticity of bath operator $\mathcal{O}(t)$ yields the result

$$(\rho[12])^*_{\omega_i^c \ \rightarrow \ -\omega_i} = -\rho[12]\,, \qquad (\rho[12_+])^*_{\omega_i^c \ \rightarrow \ -\omega_i} = \rho[12_+]\,. \tag{3.11}$$

Thus, integrating the conjugated, frequency-reversed spectral function over the conjugated, frequency-reversed contour gives the same answer as the original spectral function integrated over the original contour. Since the relabelling of the integration variables $\omega_i^c \ \rightarrow \ -\omega_i$ should not change the value of the integral, the above assertion is equivalent to the statement of reality of the Langevin couplings.

The complex contour $\mathcal{C}_2$ also has the following *time-reversal property*:[7] it remains invariant under a simultaneous complex conjugation and the exchange of frequencies ω_1 and ω_2. Say the spectral function has a specific time-reversal parity η_{12} inherited from the time parity of operators that define it. We then have

$$(\rho[12])^*_{\omega_1^c\rightarrow\omega_2,\ \omega_2^c\rightarrow\omega_1} = -\eta_{12}\rho[12]\,, \qquad (\rho[12_+])^*_{\omega_1^c\rightarrow\omega_2,\ \omega_2^c\rightarrow\omega_1} = \eta_{12}\rho[12_+]\,, \tag{3.12}$$

thus guaranteeing that the couplings derived above obey Onsager reciprocal relations using a similar argument to the above argument for reality.

A comment on the force term. The force per unit mass F appearing in the particle's equation of motion (3.1) is determined at leading order in the particle-bath coupling by the 1-point function of the operator $\mathcal{O}(t)$. For the qXY model introduced in section 4, this thermal 1-point function is zero. This, in turn, leads to the vanishing of the leading order term in F. At sub-leading order, it can receive contributions from the 3-point spectral functions of the bath. We will ignore such sub-leading corrections to the couplings in this paper.

3.2 Anharmonicity parameters: time ordered and out of time ordered

A similar exercise can be carried out for 3-point functions: for the anhormonic couplings we have

$$m_p\overline{\lambda}_3 = 2\int_{\mathcal{C}_3} \frac{\rho[123]}{\omega_1\omega_3}\,, \quad m_p\overline{\lambda}_{3\gamma} = \int_{\mathcal{C}_3} \frac{1}{i\omega_1\omega_3}\left(\frac{2}{\omega_1} - \frac{1}{\omega_3}\right)\rho[123] = \int_{\mathcal{C}_3} \frac{(2\omega_3 - \omega_1)}{i(\omega_1\omega_3)^2}\rho[123]\,. \tag{3.13}$$

[7]We refer the reader to section 6 for a detailed discussion of time-reversal invariance and its action on causal contours in the frequency domain.

As before, we have defined here the spectral function

$$\rho[123] \equiv \frac{1}{\hbar^2} \int dt_1 \int dt_2 \ \int dt_3 \ e^{i(\omega_1 t_1 + \omega_2 t_2 + \omega_3 t_3)} \ \langle [[\mathcal{O}(t_1), \mathcal{O}(t_2)], \mathcal{O}(t_3)] \rangle_B \,. \tag{3.14}$$

and the frequency domain contour which picks up the causal part of three point function:

$$\int_{\mathcal{C}_3} \equiv \int_{-\infty-i\epsilon_1}^{\infty-i\epsilon_1} \frac{d\omega_1}{2\pi} \int_{-\infty+i\epsilon_1-i\epsilon_3}^{\infty+i\epsilon_1-i\epsilon_3} \frac{d\omega_2}{2\pi} \int_{-\infty+i\epsilon_3}^{\infty+i\epsilon_3} \frac{d\omega_3}{2\pi} \tag{3.15}$$

where $\epsilon_1, \epsilon_3 > 0$. We will find it convenient to not fix a particular ordering between ϵ_1 and ϵ_3: the ordering will not matter, provided we take care that our integrands do not have poles/branch cuts in ω_2 near the real axis. For example, the above expression for the effective couplings is valid provided the spectral function has no discontinuities or branch cuts in the real axis:

$$\text{Disc}_{\omega_2} \rho[123] = 0\,. \tag{3.16}$$

In cases where there are discontinuities, they are known to lead to long time memory effects which, in turn, lead to a breakdown of Markovian approximation [32]. Markovian approximation also needs the following constraint on the spectral function:

$$\lim_{\omega_3 \to 0} \rho[123] = 0\,. \tag{3.17}$$

The complex contour $\mathcal{C}_3$ has the following *reality property*: it remains invariant under a simultaneous complex conjugation and the reversal of frequencies (similar to our discussion above for the two-point causal contour $\mathcal{C}_2$). It follows then that, if the spectral function $\rho[123]$ is invariant under this operation, then the resultant answers for the anharmonic couplings are real.

The line of argument which establishes this invariance is identical to the argument outlined for two point functions: the hermiticity of bath operator $\mathcal{O}(t)$ guarantees the reality of the double commutator $\langle [[\mathcal{O}(t_1), \mathcal{O}(t_2)], \mathcal{O}(t_3)] \rangle_B$ and in turn, the invariance of $\rho[123]$ under simultaneous complex conjugation and the reversal of frequencies, viz.,

$$(\rho[123])^*_{\omega_i^c \,\to\, -\omega_i} = \rho[123]\,. \tag{3.18}$$

This along with the invariance of the contour guarantees reality of the couplings.

The complex contour $\mathcal{C}_3$ also has the following *time-reversal property*: it remains invariant under a simultaneous complex conjugation and the exchange of frequencies ω_1 and ω_3 (with the concomitant exchange of their imaginary parts ϵ_1 and ϵ_3). If there are no poles/branch cuts in ω_2 near the real axis, we can deform the contour back to $\mathcal{C}_3$: it is in this sense that the frequency contour $\mathcal{C}_3$ is time-reversal invariant. This is similar to the time-reversal invariance of the two-point causal contour $\mathcal{C}_2$. However, unlike the two point function case, the action on the *integrand* cannot be simply described: the spectral function $\rho[123]$ gets mapped to a new function $\rho[321]$

$$(\rho[123])^*_{\omega_1^c \to \omega_3,\ \omega_2^c \to \omega_2,\ \omega_3^c \to \omega_1} = \eta_{123}\, \rho[321]\,, \tag{3.19}$$

where η_{123} is the time parity of the spectral function inherited from the time parities of the underlying operators. In the above, the function $\rho[321]$ can be thought of as the time-reversed/out of time order (OTO) spectral function:

$$\rho[321] \equiv \frac{1}{\hbar^2}\int dt_1 \int dt_2 \int dt_3\; e^{i(\omega_1 t_1+\omega_2 t_2+\omega_3 t_3)}\; \langle[[\mathcal{O}(t_3),\mathcal{O}(t_2)],\mathcal{O}(t_1)]\rangle_B\,. \tag{3.20}$$

While the expression for $\rho[321]$ looks formally similar to that of $\rho[123]$, note that these are two different functions on the $\mathcal{C}_3$ contour due to the different $i\epsilon$ prescriptions. For a bath with no microscopic time-reversal invariance, these two spectral functions are a priori unrelated. On the other hand, in the presence of time-reversal invariance, the analogue of Onsager relations for cubic couplings relate the Langevin anharmonic couplings to anharmonic couplings in the time-reversed stochastic dynamics.

The above statement can be made precise by introducing the out of time order (or time-reversed) anharmonic couplings

$$m_p\overline{\kappa}_3 \equiv 2\int_{\mathcal{C}_3}\frac{\rho[321]}{\omega_1\omega_3}\,,\quad m_p\overline{\kappa}_{3\gamma} \equiv \int_{\mathcal{C}_3}\frac{1}{i\omega_1\omega_3}\left(\frac{1}{\omega_1}-\frac{2}{\omega_3}\right)\rho[321] = \int_{\mathcal{C}_3}\frac{(\omega_3-2\omega_1)}{i(\omega_1\omega_3)^2}\rho[321]\,. \tag{3.21}$$

Say the bath operators $O_k(t)$ with a definite time-reversal parity η_k couple to the single degree of freedom of the Brownian particle. For simplicity, assume that the relevant spectral function gets contribution only from the product of three operators $O_{k=1,2,3}$. We can then write down the non-linear, OTO analogue of the Onsager reciprocal relations for the resultant couplings as

$$\overline{\lambda}_3 = \eta_{_O}\;\overline{\kappa}_3\,,\qquad \overline{\lambda}_{3\gamma} = \eta_{_O}\;\overline{\kappa}_{3\gamma}\,, \tag{3.22}$$

where $\eta_{_O} \equiv \eta_1\eta_2\eta_3$ is the total time-reversal parity of the given spectral function. Thus, rather than constraining the cubic couplings in the non-linear Langevin dynamics, the time-reversal invariance ends up relating the standard anharmonic couplings to the OTO anharmonic couplings.

3.3 Frequency noise parameter ζ_μ

We now move on to quote the results for the other non-linear couplings that appear in the non-linear Langevin equation. We begin with the parameter ζ_μ that governs the thermal jitter in the frequency of the Brownian oscillator:

$$\begin{aligned} m_p^2\zeta_\mu &= \int_{\mathcal{C}_3}\frac{1}{2i\omega_1\omega_3}\Big(\rho[12_+3]+\rho[123_+]\Big)\\ &= \int_{\mathcal{C}_3}\frac{1}{2i\omega_1\omega_3}\Big(\rho[231_+]+\rho[132_+]+\rho[123_+]\Big)\\ &= \int_{\mathcal{C}_3}\frac{\hbar}{i\omega_1\omega_3}\Big\{(1+\mathfrak{f}_1+\mathfrak{f}_2)\rho[321]-(1+\mathfrak{f}_2+\mathfrak{f}_3)\rho[123]\Big\}\,. \end{aligned} \tag{3.23}$$

Here, the spectral functions are defined as before by Fourier transforming the appropriate nested commutators/anti-commutators.[8] For example, we have

$$\begin{aligned}\rho[12_+3] &\equiv \frac{1}{\hbar}\int dt_1 \int dt_2 \int dt_3 \; e^{i(\omega_1 t_1+\omega_2 t_2+\omega_3 t_3)}\langle [\{\mathcal{O}(t_1),\mathcal{O}(t_2)\},\mathcal{O}(t_3)]\rangle_B \,,\\ \rho[123_+] &\equiv \frac{1}{\hbar}\int dt_1 \int dt_2 \int dt_3 \; e^{i(\omega_1 t_1+\omega_2 t_2+\omega_3 t_3)}\langle \{[\mathcal{O}(t_1),\mathcal{O}(t_2)],\mathcal{O}(t_3)\}\rangle_B \,,\end{aligned} \tag{3.24}$$

where the subscript + indicates the anti-commutator. The integral is over the same causal contour $\mathcal{C}_3$ as before. The first line gives the appropriate causal Green function from which the effective coupling ζ_μ can be extracted.

The expression in the second line above arise from generalised Jacobi identities of the form

$$\rho[12_+3] = \rho[231_+] + \rho[132_+]\,, \qquad \rho[123] + \rho[231] + \rho[312] = 0 \tag{3.25}$$

and permutations thereof, and the last line which follows from Kubo-Martin-Schwinger (KMS) relation for three point functions:

$$\rho[123_+] = -\hbar(1+2\mathfrak{f}_3)\rho[123] \tag{3.26}$$

and permutations thereof. Here $\mathfrak{f}_i \equiv \mathfrak{f}(\omega_i)$ are the Bose-Einstein functions of the respective frequencies. Thus, one can express the coupling as a causal contour integral over the spectral function $\rho[123]$ and the out of time order spectral function $\rho[321]$ multiplied by the appropriate Bose-Einstein factors.

The discussion on the reality properties is similar to before: the causal contour $\mathcal{C}_3$ and the commutator spectral functions are invariant under simultaneous conjugation and frequency reversal (assuming no ω_2 discontinuities on the real axis). The anti-commutator spectral functions are odd under simultaneous conjugation and frequency reversal:

$$(\rho[123_+])^*_{\omega_i\,\to\,-\omega_i} = -\rho[123_+]\,, \quad (\rho[12_+3])^*_{\omega_i\,\to\,-\omega_i} = -\rho[12_+3]\,. \tag{3.27}$$

This, then ensures the reality of ζ_μ. The reality can also be shown using the expression in terms of Bose-Einstein distributions, however the argument involved is slightly more subtle: the following property of Bose-Einstein functions

$$\mathfrak{f}(-\omega) = -\{1+\mathfrak{f}(\omega)\}\,, \tag{3.28}$$

should be used along with the assumption that the potential discontinuity due to $\mathfrak{f}_2$ near $\omega_2 \to 0$ is cancelled among the two terms.[9]

As discussed in the last subsection, time-reversal acts on the causal contour $\mathcal{C}_3$ via a simultaneous complex conjugation and the exchange of frequencies ω_1 and ω_3. If the bath

[8]Including a factor of inverse $\hbar$ for every commutator to guarantee smooth classical limit.
[9]One can explicitly prove that this indeed happens in the Markovian models of the bath we use.

operator $\mathcal{O}(t)$ has a definite time-reversal parity $\eta_{_O}$, we can then write down the non-linear Onsager reciprocal relation for ζ_μ as

$$\zeta_\mu = \eta_{_O}\ \zeta_\mu, \tag{3.29}$$

i.e., one obtains a trivial relation for time-reversal even bath operators and for time-reversal odd bath operators, $\zeta_\mu = 0$.

At large temperatures (or small β), one can approximate

$$\hbar\mathfrak{f}(\omega) \approx \frac{\hbar}{\beta\omega} = \frac{m_p v_{th}^2}{\omega}\,. \tag{3.30}$$

We then get a high temperature formula for ζ_μ of the form

$$\begin{aligned} m_p\zeta_\mu &= v_{th}^2 \int_{\mathcal{C}_3} \frac{1}{i\omega_1\omega_3}\left\{\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)\rho[321] - \left(\frac{1}{\omega_2}+\frac{1}{\omega_3}\right)\rho[123]\right\} \\ &= v_{th}^2 \int_{\mathcal{C}_3} \frac{1}{i\omega_2}\left\{\frac{\rho[123]}{\omega_3^2} - \frac{\rho[321]}{\omega_1^2}\right\}, \end{aligned} \tag{3.31}$$

where, in the last line, we have used the fact that the spectral functions are proportional to $\delta(\omega_1+\omega_2+\omega_3)$ for time-independent state of the bath.

We now turn to possible fluctuation-dissipation type relation involving ζ_μ. While the integrand which appears in the first line above has structural similarities with the integrands in the sum rules of the couplings $\overline{\lambda}_{3\gamma}$ and $\overline{\kappa}_{3\gamma}$, we have not succeeded in establishing any general fluctuation-dissipation type relation between these couplings. Nevertheless, in many explicit models with time-reversal invariance,[10] we find the following relation:

$$\begin{aligned} &\int_{\mathcal{C}_3} \frac{1}{i\omega_1\omega_3}\left\{\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)\rho[321] - \left(\frac{1}{\omega_2}+\frac{1}{\omega_3}\right)\rho[123]\right\} \\ &\quad = \int_{\mathcal{C}_3} \frac{1}{i\omega_1\omega_3}\left(\frac{2}{\omega_1}-\frac{1}{\omega_3}\right)\rho[123] = \int_{\mathcal{C}_3} \frac{1}{i\omega_1\omega_3}\left(\frac{1}{\omega_1}-\frac{2}{\omega_3}\right)\rho[321]\,, \end{aligned} \tag{3.32}$$

where the last equality is the relation $\overline{\lambda}_{3\gamma} = \overline{\kappa}_{3\gamma}$ which holds for even time-reversal bath operators. More generally, we find that the following integral vanishes in many explicit examples:

$$\begin{aligned} &v_{th}^2\left(\overline{\lambda}_{3\gamma}+\overline{\kappa}_{3\gamma}\right) - 2\zeta_\mu \\ &= \frac{v_{th}^2}{m_p}\int_{\mathcal{C}_3} \frac{1}{i\omega_1\omega_3}\left\{\left(\frac{2}{\omega_1}+\frac{2}{\omega_2}+\frac{1}{\omega_3}\right)\rho[123] - \left(\frac{1}{\omega_1}+\frac{2}{\omega_2}+\frac{2}{\omega_3}\right)\rho[321]\right\} = 0\,, \end{aligned} \tag{3.33}$$

thus resulting in a relation of the form

$$\zeta_\mu = \frac{1}{2} v_{th}^2\left(\overline{\lambda}_{3\gamma}+\overline{\kappa}_{3\gamma}\right)\,. \tag{3.34}$$

[10] We will describe one such model in detail in the following section.

3.4 Dissipative noise parameter ζ_γ and its OTO counterpart

We will now move on to the parameter ζ_γ which describes the jitter in the dissipative constant γ of the Brownian particle. The sum rule(s) for computing ζ_γ is given by

$$\begin{aligned} m_p^2\zeta_\gamma &= \int_{\mathcal{C}_3} \frac{1}{(2\omega_1\omega_3)^2}\Big((\omega_3-2\omega_1)\ \rho[12_+3]-\omega_2\ \rho[123_+]\Big) \\ &= \int_{\mathcal{C}_3} \frac{1}{(2\omega_1\omega_3)^2}\Big((2\omega_1-\omega_3)\ \rho[321_+]+(\omega_3-2\omega_1)\ \rho[132_+]+(\omega_1+\omega_3)\ \rho[123_+]\Big) \\ &= \int_{\mathcal{C}_3} \frac{\hbar}{(2\omega_1\omega_3)^2} \\ &\quad\times\Big\{2(\omega_3-2\omega_1)\Big(1+\mathfrak{f}_1+\mathfrak{f}_2\Big)\rho[321]-(\omega_3-2\omega_1)(1+2\mathfrak{f}_2)\rho[123]+\omega_2(1+2\mathfrak{f}_3)\rho[123]\Big\}. \end{aligned} \tag{3.35}$$

In the first line, we have the representation in terms of the causal correlator. The second line comes from generalised Jacobi identities and the last line from Kubo-Martin-Schwinger (KMS) relations. Following arguments very similar to the ones sketched in the previous subsection, one can argue that ζ_γ produced by the above integral is real.

Next we turn to examine the behaviour of the integrand under the exchange of frequencies ω_1 and ω_3 (which would be relevant for the action of time-reversal on this coupling). We have the following identity:

$$\begin{aligned} &(2\omega_1-\omega_3)\ \rho[321_+]+(\omega_3-2\omega_1)\ \rho[132_+]+(\omega_1+\omega_3)\ \rho[123_+] \\ &\quad+(2\omega_3-\omega_1)\ \rho[123_+]+(\omega_1-2\omega_3)\ \rho[312_+]+(\omega_3+\omega_1)\ \rho[321_+] \\ &\quad= 3\Big(\omega_1\ \rho[321_+]+(\omega_3-\omega_1)\ \rho[132_+]+\omega_3\ \rho[123_+]\Big)\ . \end{aligned} \tag{3.36}$$

This implies that if the bath operator $\mathcal{O}(t)$ has a definite time-reversal parity $\eta_{_O}$, then the generalised Onsager reciprocal relation for ζ_γ becomes

$$\zeta_\gamma = \eta_{_O}\ (\widehat{\kappa}_{3\gamma}-\zeta_\gamma)\,, \quad \widehat{\kappa}_{3\gamma} = \eta_{_O}\ \widehat{\kappa}_{3\gamma}\,, \tag{3.37}$$

where we have defined a new OTO coupling $\widehat{\kappa}_{3\gamma}$ via

$$\begin{aligned} m_p^2\ \widehat{\kappa}_{3\gamma} &\equiv 3\int_{\mathcal{C}_3} \frac{1}{(2\omega_1\omega_3)^2}\Big(\omega_1\ \rho[321_+]+(\omega_3-\omega_1)\ \rho[132_+]+\omega_3\ \rho[123_+]\Big) \\ &= -3\int_{\mathcal{C}_3} \frac{1}{(2\omega_1\omega_3)^2}\Big(\omega_1\ \rho[12_+3]+\omega_3\ \rho[32_+1]\Big) \\ &= -3\int_{\mathcal{C}_3} \frac{\hbar}{(2\omega_1\omega_3)^2}\bigg(\Big\{2\omega_1\Big(1+\mathfrak{f}_1+\mathfrak{f}_2\Big)-\omega_3(1+2\mathfrak{f}_2)\Big\}\rho[321] \\ &\qquad+\Big\{2\omega_3\Big(1+\mathfrak{f}_2+\mathfrak{f}_3\Big)-\omega_1(1+2\mathfrak{f}_2)\Big\}\rho[123]\bigg)\,. \end{aligned} \tag{3.38}$$

We conclude that if $\eta_{_O}=1$, one has the constraint $\widehat{\kappa}_{3\gamma}=2\zeta_\gamma$ whereas for $\eta_{_O}=-1$, one has the constraint $\widehat{\kappa}_{3\gamma}=0$. Thus, $\widehat{\kappa}_{3\gamma}$ can be thought of as the part of ζ_γ even under time-reversal, whereas the part odd under time-reversal is given by

$$\begin{aligned} m_p^2(2\zeta_\gamma-\widehat{\kappa}_{3\gamma}) &\equiv \int_{\mathcal{C}_3} \frac{1}{(2\omega_1\omega_3)^2}\Big((\omega_1-2\omega_3)\ \rho[321_+]+3(\omega_3-\omega_1)\ \rho[132_+]+(2\omega_1-\omega_3)\ \rho[123_+]\Big) \\ &= \int_{\mathcal{C}_3} \frac{\hbar}{(2\omega_1\omega_3)^2}\Big(\Big\{2\omega_2\Big(1+\mathfrak{f}_1+\mathfrak{f}_2\Big)+3\omega_3(1+2\mathfrak{f}_1)\Big\}\rho[321] \\ &\qquad -\Big\{2\omega_2\Big(1+\mathfrak{f}_2+\mathfrak{f}_3\Big)+3\omega_1(1+2\mathfrak{f}_3)\Big\}\rho[123]\Big)\,. \end{aligned} \tag{3.39}$$

At high temperature, these expressions become

$$\begin{aligned} m_p\zeta_\gamma &= v_{th}^2\int_{\mathcal{C}_3}\frac{1}{2\omega_1\omega_3} \\ &\quad\times\left\{\left(\frac{1}{\omega_1}-\frac{2}{\omega_3}\right)\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)\rho[321]+\left(\frac{2}{\omega_2\omega_3}-\frac{1}{\omega_2\omega_1}+\frac{\omega_2}{\omega_1\omega_3^2}\right)\rho[123]\right\} \\ &= v_{th}^2\int_{\mathcal{C}_3}\frac{1}{\omega_1\omega_3}\left\{\left(\frac{1}{\omega_1^2}+\frac{3}{\omega_1\omega_2}\right)\rho[321]+\left(\frac{3}{\omega_3\omega_2}-\frac{1}{\omega_3^2}\right)\rho[123]\right\}, \end{aligned} \tag{3.40}$$

or equivalently

$$m_p\ \ (2\zeta_\gamma-\widehat{\kappa}_{3\gamma}) = v_{th}^2\int_{\mathcal{C}_3}\frac{1}{\omega_1\omega_3}\left\{\frac{1}{\omega_1^2}\rho[321]-\frac{1}{\omega_3^2}\rho[123]\right\}\ , \tag{3.41}$$

and

$$\begin{aligned} m_p\ \widehat{\kappa}_{3\gamma} &= -3v_{th}^2\int_{\mathcal{C}_3}\frac{1}{2\omega_1\omega_3} \\ &\quad\times\left\{\Big(\frac{1}{\omega_1\omega_3}+\frac{1}{\omega_3\omega_2}-\frac{1}{\omega_1\omega_2}\Big)\rho[321]+\Big(\frac{1}{\omega_1\omega_3}+\frac{1}{\omega_1\omega_2}-\frac{1}{\omega_3\omega_2}\Big)\rho[123]\right\} \\ &= 3v_{th}^2\int_{\mathcal{C}_3}\frac{1}{\omega_1\omega_2\omega_3}\left\{\frac{1}{\omega_1}\rho[321]+\frac{1}{\omega_3}\rho[123]\right\}\ . \end{aligned} \tag{3.42}$$

In simplifying these expressions, we have used the identity

$$\frac{1}{\omega_1\omega_3}+\frac{1}{\omega_1\omega_2}+\frac{1}{\omega_3\omega_2}=0\,, \tag{3.43}$$

which holds because of the $\delta(\omega_1+\omega_2+\omega_3)$ inside the spectral functions.

3.5 Non-Gaussianity ζ_N and its fluctuation-dissipation relation

We finally turn our attention to the last parameter of the non-linear Langevin model which is the non-Gaussianity parameter ζ_N. We find the sum rule(s) for computing ζ_N as

$$m_p^3\zeta_N=\int_{\mathcal{C}_3}\frac{\rho[12_+3_+]}{4\omega_1\omega_3}=\int_{\mathcal{C}_3}\frac{\hbar^2}{4\omega_1\omega_3}(1+2\mathfrak{f}_3)\Big\{(1+2\mathfrak{f}_2)\rho[123]-2(1+\mathfrak{f}_1+\mathfrak{f}_2)\rho[321]\Big\}, \tag{3.44}$$

where we have used the condition for the nested anti-commutator to write the second equality. The relevant spectral function is

$$\rho[12_+3_+]\equiv\int dt_1\int dt_2\int dt_3\ e^{i(\omega_1t_1+\omega_2t_2+\omega_3t_3)}\langle\{\{\mathcal{O}(t_1),\mathcal{O}(t_2)\},\mathcal{O}(t_3)\}\rangle_B\,. \tag{3.45}$$

It can be checked that the integral above does give a real coupling as an answer using methods described before. In order to study time-reversal, consider the following time-reversed/OTO counterpart of the spectral function appearing above:

$$\rho[32_+1_+] = \rho[12_+3_+] + \hbar^2 \left(\rho[123] - \rho[321]\right) , \tag{3.46}$$

a relation which can be checked by a simple expansion of the nested commutators/anti-commutators. It follows that the combination of spectral functions with good time-reversal properties is

$$\rho[12_+3_+] + \rho[32_+1_+] = 2\ \rho[12_+3_+] + \hbar^2 \left(\rho[123] - \rho[321]\right) . \tag{3.47}$$

With this in mind, we examine the integral

$$\begin{aligned} \int_{\mathcal{C}_3} \frac{\rho[12_+3_+] + \rho[32_+1_+]}{4\omega_1\omega_3} &= \int_{\mathcal{C}_3} \frac{2\ \rho[12_+3_+] + \hbar^2 \left(\rho[123] - \rho[321]\right)}{4\omega_1\omega_3} \\ &= 2m_p^3\zeta_N + \frac{\hbar^2}{8} m_p(\overline{\lambda}_3 - \overline{\kappa}_3) . \end{aligned} \tag{3.48}$$

Here, we have used the sum rules quoted before to write a combination of couplings with good time-reversal properties. Microscopic time-reversal invariance thus implies

$$2m_p^3\zeta_N + \frac{\hbar^2}{8} m_p(\overline{\lambda}_3 - \overline{\kappa}_3) = \eta_{_O} \left\{ 2m_p^3\zeta_N + \frac{\hbar^2}{8} m_p(\overline{\lambda}_3 - \overline{\kappa}_3) \right\} , \tag{3.49}$$

which is trivially satisfied for $\eta_{_O} = +1$. For $\eta_{_O} = -1$, we get a generalised Onsager condition of the form

$$\zeta_N = -\frac{\hbar^2}{16m_p^2} (\overline{\lambda}_3 - \overline{\kappa}_3) . \tag{3.50}$$

Thus, the non-Gaussianity in thermal noise is quantum suppressed for time-reversal odd coupling to the bath.

At high temperature limit, we get

$$\begin{aligned} m_p \left\{ \zeta_N + \frac{\hbar^2}{16m_p^2} (\overline{\lambda}_3 - \overline{\kappa}_3) \right\} &= v_{th}^4 \int_{\mathcal{C}_3} \frac{1}{\omega_1\omega_3} \left\{ \frac{1}{\omega_3\omega_2} \rho[123] - \left(\frac{1}{\omega_1\omega_3} + \frac{1}{\omega_3\omega_2} \right) \rho[321] \right\} \\ &= v_{th}^4 \int_{\mathcal{C}_3} \frac{1}{\omega_1\omega_3} \left\{ \frac{1}{\omega_1\omega_2} \rho[321] + \frac{1}{\omega_3\omega_2} \rho[123] \right\} . \end{aligned} \tag{3.51}$$

where we have used eq. (3.43). Comparing this against the high temperature limit of $\widehat{\kappa}_{3\gamma}$, we obtain the fluctuation-dissipation relation:

$$\zeta_N + \frac{\hbar^2}{16m_p^2} (\overline{\lambda}_3 - \overline{\kappa}_3) = \frac{1}{3} \widehat{\kappa}_{3\gamma} v_{th}^2 . \tag{3.52}$$

We note that, in this way of writing both sides of these equations have the same time-reversal property.

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3.6 Summary of relations in non-linear Langevin theory

We will now summarise all the Onsager type relations between the couplings of the non-linear Langevin theory:

$$\begin{aligned}&\overline{\lambda}_3 = \eta_O\ \overline{\kappa}_3\,,\quad \overline{\lambda}_{3\gamma} = \eta_O\ \overline{\kappa}_{3\gamma}\,,\quad \zeta_\mu = \eta_O\ \zeta_\mu\,,\quad \zeta_\gamma = \eta_O\ (\widehat{\kappa}_{3\gamma} - \zeta_\gamma)\,,\quad \widehat{\kappa}_{3\gamma} = \eta_O\ \widehat{\kappa}_{3\gamma}\,,\\ &2m_p^3\zeta_N + \frac{\hbar^2}{8}m_p(\overline{\lambda}_3 - \overline{\kappa}_3) = \eta_O\left\{2m_p^3\zeta_N + \frac{\hbar^2}{8}m_p(\overline{\lambda}_3 - \overline{\kappa}_3)\right\}.\end{aligned} \tag{3.53}$$

In addition, we have the consequence of KMS relations:

$$\zeta_\mu = \frac{1}{2}v_{th}^2\left(\overline{\lambda}_{3\gamma} + \overline{\kappa}_{3\gamma}\right)\,,\qquad \zeta_N + \frac{\hbar^2}{16m_p^2}(\overline{\lambda}_3 - \overline{\kappa}_3) = \frac{1}{3}\widehat{\kappa}_{3\gamma}v_{th}^2\,. \tag{3.54}$$

Note that, in general, each of these conditions relate the time-ordered Langevin couplings to OTO Langevin couplings. These relations can be combined to solve for 5 of the non-linear Langevin couplings in terms of the 3 remaining couplings $\{\overline{\lambda}_3, \overline{\lambda}_{3\gamma}, \zeta_\gamma\}$:

$$\begin{aligned}&\overline{\kappa}_3 = \eta_O\ \overline{\lambda}_3\,,\quad \overline{\kappa}_{3\gamma} = \eta_O\ \overline{\lambda}_{3\gamma}\,,\quad \widehat{\kappa}_{3\gamma} = (1+\eta_O)\,\zeta_\gamma\,,\\ &\zeta_\mu = \frac{1}{2}(1+\eta_O)\,\overline{\lambda}_{3\gamma}v_{th}^2\,,\qquad \zeta_N = \frac{1}{3}(1+\eta_O)\,\zeta_\gamma v_{th}^2 - \frac{\hbar^2}{16m_p^2}(1-\eta_O)\,\overline{\lambda}_3\,.\end{aligned} \tag{3.55}$$

These relations are one of the main results of this work. We collect in table 4, the Onsager pairs related to each other by time-reversal.

The following tables summarise the integrands $\mathcal{I}[g]$ associated with each coupling g. The tables 1, 2 and 3 summarise the integrals for quadratic couplings that are induced, in the leading order in perturbation theory, from a bath in a general time-independent state, by a general thermal bath and by a bath at high temperature respectively. The couplings are given by expressions of the form

$$g = \int_{\mathcal{C}_2} \mathcal{I}[g]. \tag{3.56}$$

Similarly, the tables 4, 5 and 6 summarise the integrals for cubic couplings that are induced, in the leading order in perturbation theory, from a bath in a general time-independent state, by a general thermal bath and by a bath at high temperature respectively. In this case, the couplings are given by

$$g = \int_{\mathcal{C}_3} \mathcal{I}[g]. \tag{3.57}$$

4 Introduction to the qXY model

In this section, we will begin by describing a microscopic model of an oscillator coupled to bath oscillator degrees of freedom in a way that results in an effective non-linear Langevin equation for the original oscillator. Our motivation here is to construct a physical microscopic description in which one can check the Markovian assumption and the relation between the effective couplings that emerge therein.

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g	$\mathcal{I}[g]$
Δm_p	$\frac{1}{\omega_1^3}\rho[12]$
Z_I	$-\frac{1}{im_p^2\omega_1^3}\rho[12_+]$
$\Delta\bar{\mu}^2$	$-\frac{1}{m_p\omega_1}\rho[12]$
$\langle f^2\rangle$	$\frac{1}{im_p^2\omega_1}\rho[12_+]$
γ	$\frac{1}{im_p\omega_1^2}\rho[12]$

Table 1. Quadratic Couplings (general environment).

g	$\mathcal{I}[g]$
Δm_p	$\frac{1}{\omega_1^3}\rho[12]$
Z_I	$-\frac{\hbar}{im_p^2\omega_1^3}(1+2\mathfrak{f}_1)\rho[12]$
$\Delta\bar{\mu}^2$	$-\frac{1}{m_p\omega_1}\rho[12]$
$\langle f^2\rangle$	$\frac{\hbar}{im_p^2\omega_1}(1+2\mathfrak{f}_1)\rho[12]$
γ	$\frac{1}{im_p\omega_1^2}\rho[12]$

Table 2. Sum rules for Quadratic Couplings (Thermal environment).

g	$\mathcal{I}[g]$
Δm_p	$\frac{1}{\omega_1^3}\rho[12]$
Z_I	$-\frac{2v_{th}^2}{im_p}\frac{1}{\omega_1^4}\rho[12]$
$\Delta\bar{\mu}^2$	$-\frac{1}{m_p\omega_1}\rho[12]$
$\langle f^2\rangle$	$\frac{2v_{th}^2}{im_p}\frac{1}{\omega_1^2}\rho[12]$
γ	$\frac{1}{im_p\omega_1^2}\rho[12]$

Table 3. Sum rules for quadratic Couplings (High Temperature limit).

4.1 Model of the bath

We will now begin with the Caldeira-Leggett model and then modify it to suit our requirements. As described before, the model is that of a single system oscillator (denoted by a degree of freedom q) coupled to a bath of oscillators (denoted by degrees of freedom X). One starts with a distribution of couplings and masses of bath oscillators specified by a characteristic distribution function, defined by

$$\left\langle\left\langle \frac{g_x^2}{m_x}\right\rangle\right\rangle \equiv \sum_i \frac{g_{x,i}^2}{m_{x,i}} 2\pi\delta(\mu_x-\mu_i)\,. \tag{4.1}$$

g	$\mathcal{I}[g]$
$\overline{\lambda}_3$	$\frac{2}{m_p\omega_1\omega_3}\rho[123]$
$\overline{\kappa}_3$	$\frac{2}{m_p\omega_1\omega_3}\rho[321]$
$\overline{\lambda}_{3\gamma}$	$\frac{1}{im_p\omega_1\omega_3}\left(\frac{2}{\omega_1}-\frac{1}{\omega_3}\right)\rho[123]$
$\overline{\kappa}_{3\gamma}$	$\frac{1}{im_p\omega_1\omega_3}\left(\frac{1}{\omega_1}-\frac{2}{\omega_3}\right)\rho[321]$
ζ_γ	$\frac{1}{(2m_p\omega_1\omega_3)^2}\Big((2\omega_1-\omega_3)\ \rho[321_+]+(\omega_3-2\omega_1)\ \rho[132_+]+(\omega_1+\omega_3)\ \rho[123_+]\Big)$
$\widehat{\kappa}_{3\gamma}-\zeta_\gamma$	$\frac{1}{(2m_p\omega_1\omega_3)^2}\Big((\omega_1+\omega_3)\ \rho[321_+]+(2\omega_3-\omega_1)\ \rho[132_+]+(2\omega_3-\omega_1)\ \rho[123_+]\Big)$
$\widehat{\kappa}_{3\gamma}$	$\frac{3}{(2m_p\omega_1\omega_3)^2}\Big(\omega_1\ \rho[321_+]+(\omega_3-\omega_1)\ \rho[132_+]+\omega_3\ \rho[123_+]\Big)$
ζ_μ	$\frac{1}{2im_p^2\omega_1\omega_3}\Big(\rho[123_+]-\rho[321_+]+\rho[132_+]\Big)$
$2\zeta_N+\frac{\hbar^2}{8m_p^2}\left(\overline{\lambda}_3-\overline{\kappa}_3\right)$	$\frac{1}{8m_p^3\omega_1\omega_3}\Big(\rho[12_+3_+]+\rho[32_+1_+]\Big)$

Table 4. Cubic couplings: doublets/singlets under microscopic time-reversal (general environment).

g	$\mathcal{I}[g]$
$\overline{\lambda}_3$	$\frac{2}{m_p\omega_1\omega_3}\rho[123]$
$\overline{\kappa}_3$	$\frac{2}{m_p\omega_1\omega_3}\rho[321]$
$\overline{\lambda}_{3\gamma}$	$\frac{1}{im_p\omega_1\omega_3}\left(\frac{2}{\omega_1}-\frac{1}{\omega_3}\right)\rho[123]$
$\overline{\kappa}_{3\gamma}$	$\frac{1}{im_p\omega_1\omega_3}\left(\frac{1}{\omega_1}-\frac{2}{\omega_3}\right)\rho[321]$
ζ_γ	$\frac{\hbar}{(2m_p\omega_1\omega_3)^2}\Big\{2(\omega_3-2\omega_1)\Big(1+\mathfrak{f}_1+\mathfrak{f}_2\Big)\rho[321]-(\omega_3-2\omega_1)(1+2\mathfrak{f}_2)\rho[123]$ $+\,\omega_2(1+2\mathfrak{f}_3)\rho[123]\Big\}$
$\widehat{\kappa}_{3\gamma}$	$-3\frac{\hbar}{(2m_p\omega_1\omega_3)^2}\Big(\Big\{2\omega_1\Big(1+\mathfrak{f}_1+\mathfrak{f}_2\Big)-\omega_3(1+2\mathfrak{f}_2)\Big\}\rho[321]$ $+\Big\{2\omega_3\Big(1+\mathfrak{f}_2+\mathfrak{f}_3\Big)-\omega_1(1+2\mathfrak{f}_2)\Big\}\rho[123]\Big)$
ζ_μ	$\frac{\hbar}{im_p^2\omega_1\omega_3}\Big\{(1+\mathfrak{f}_1+\mathfrak{f}_2)\rho[321]-(1+\mathfrak{f}_2+\mathfrak{f}_3)\rho[123]\Big\}$
ζ_N	$\frac{\hbar^2}{4m_p^3\omega_1\omega_3}(1+2\mathfrak{f}_3)\Big\{(1+2\mathfrak{f}_2)\rho[123]-2(1+\mathfrak{f}_1+\mathfrak{f}_2)\rho[321]\Big\}$

Table 5. Sum rules for Coupling (Thermal environment).

This distribution function multiplied by the spectral contribution of each bath oscillator can be summed to give the Caldeira-Leggett spectral function:

$$\rho[12]_{CL}\equiv\int_0^\infty\frac{d\mu_x}{2\pi}\ 2\pi\delta(\omega_1+\omega_2)\times(2\pi)\ \mathrm{sgn}(\omega_1)\delta(\omega_1^2-\mu_x^2)\ \left\langle\!\left\langle\frac{g_x^2}{m_x}\right\rangle\!\right\rangle. \tag{4.2}$$

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g	$\mathcal{I}[g]$
$\overline{\lambda}_3$	$\frac{2}{m_p\omega_1\omega_3}\rho[123]$
$\overline{\kappa}_3$	$\frac{2}{m_p\omega_1\omega_3}\rho[321]$
$\overline{\lambda}_{3\gamma}$	$\frac{1}{im_p\omega_1\omega_3}\left(\frac{2}{\omega_1}-\frac{1}{\omega_3}\right)\rho[123]$
$\overline{\kappa}_{3\gamma}$	$\frac{1}{im_p\omega_1\omega_3}\left(\frac{1}{\omega_1}-\frac{2}{\omega_3}\right)\rho[321]$
ζ_γ	$\frac{v_{th}^2}{m_p}\frac{1}{\omega_1\omega_3}\left\{\left(\frac{1}{\omega_1^2}+\frac{3}{\omega_1\omega_2}\right)\rho[321]+\left(\frac{3}{\omega_3\omega_2}-\frac{1}{\omega_3^2}\right)\rho[123]\right\}$
$\widehat{\kappa}_{3\gamma}$	$3\frac{v_{th}^2}{m_p}\frac{1}{\omega_1\omega_2\omega_3}\left\{\frac{1}{\omega_1}\rho[321]+\frac{1}{\omega_3}\rho[123]\right\}$
ζ_μ	$\frac{v_{th}^2}{m_p}\frac{1}{i\omega_2}\left\{\frac{1}{\omega_3^2}\rho[123]-\frac{1}{\omega_1^2}\rho[321]\right\}$
ζ_N	$\frac{v_{th}^4}{m_p}\frac{1}{\omega_1\omega_2\omega_3}\left\{\frac{1}{\omega_1}\rho[321]+\frac{1}{\omega_3}\rho[123]\right\}$

Table 6. Sum rules for Coupling (High Temperature limit).

To obtain a Lorentz-Drude spectral function, we consider a continuum of oscillators whose couplings add up to give a distribution of the form

$$\left\langle\left\langle\frac{g_x^2}{m_x}\right\rangle\right\rangle = m_p\gamma_x\frac{4\mu_x^2\Omega^2}{\mu_x^2+\Omega^2}\,, \tag{4.3}$$

where γ_x denotes the contribution of X oscillators to the damping constant γ of the particle. The contribution to the noise is determined by fluctutation-dissipation relation

$$\langle f^2\rangle = 2\gamma v_{th}^2.$$

To get a simple nonlinear generalisation, we will double the number of oscillators into two kinds of bath both at same temperature. We imagine them to be two different sets of bath oscillators distinguished by the letters X and Y. For simplicity, we will assume that the Y type oscillators also have similar coupling distribution as X type oscillators

$$\left\langle\left\langle\frac{g_y^2}{m_y}\right\rangle\right\rangle = m_p\gamma_y\frac{4\mu_y^2\Omega^2}{\mu_y^2+\Omega^2}\,, \tag{4.4}$$

and they add to the damping due to X type oscillators. Thus, $\gamma = \gamma_x + \gamma_y$ and the noise contributions from the two sets of oscillators also add up. Till now, this is merely a relabelling of the original model, and the model is hence exactly solvable and yields linear Langevin theory.

We will now introduce non-linearity into this theory by introducing a very small 3-body interaction term of the form qXY. More precisely, one considers a system of oscillators with the Lagrangian

$$L[q,X,Y] = L_B[X,Y] + \frac{1}{2}m_{p0}(\dot{q}^2 - \overline{\mu}_0^2q^2) + q\left(\sum_i g_{x,i}X_i + \sum_j g_{y,j}Y_j + \sum_{i,j} g_{xy,ij}X_iY_j\right) \tag{4.5}$$

where we denote the oscillator's position by q and $L_B[X,Y]$ is the free Lagrangian of the harmonic bath. The operator that acts on the Hilbert space of the bath and couples to q is

$$\mathcal{O} \equiv \sum_i g_{x,i} X_i + \sum_j g_{y,j} Y_j + \sum_{i,j} g_{xy,ij} X_i Y_j . \tag{4.6}$$

When we integrate out the effects of the bath degrees of freedom, the effective couplings are now induced for the oscillator and there are corrections to the linear Langevin theory. These Langevin effective couplings have their microscopic origin in the thermal correlators/spectral functions of the operator above.

One can think of the above as a toy model for say an atom coupled to photons whereby, apart from the standard, dominant linear dipole coupling responsible for single photon emission/absorption processes, one also has two photon processes involving two photons of two different frequencies. The physics here is familiar one say from Brillouin scattering of photons against phonons or the inelastic Raman scattering of photons against molecules. At the level of linear Langevin couplings, both the noise and the damping constant receive contributions from the inelastic 3-body scattering: first, there is an effect due to two 'photon' emission and absorption into/from the thermal bath which gives a contribution proportional to

$$\hbar(\mu_x + \mu_y) \ \left[(1+\mathfrak{f}_x)(1+\mathfrak{f}_y) - \mathfrak{f}_x \mathfrak{f}_y\right] \tag{4.7}$$

in the spectral function. The second effect is due to inelastic scattering whose contribution is proportional to

$$\hbar(\mu_x - \mu_y)\left[\mathfrak{f}_x(1+\mathfrak{f}_y) - \mathfrak{f}_y(1+\mathfrak{f}_x)\right] . \tag{4.8}$$

Putting these effects together, we get a spectral function[11]

$$\begin{aligned}
\rho[12] = &(2\pi)\delta(\omega_1+\omega_2)\int_0^\infty \frac{d\mu_x}{2\pi}\left\langle\!\left\langle \frac{g_x^2}{m_x}\right\rangle\!\right\rangle (2\pi)\text{sgn}(\omega_1)\delta(\omega_1^2-\mu_x^2)\\
&+(2\pi)\delta(\omega_1+\omega_2)\int_0^\infty \frac{d\mu_y}{2\pi}\left\langle\!\left\langle \frac{g_y^2}{m_y}\right\rangle\!\right\rangle (2\pi)\text{sgn}(\omega_1)\delta(\omega_1^2-\mu_y^2)\\
&+(2\pi)\delta(\omega_1+\omega_2)\int_0^\infty \frac{d\mu_x}{2\pi}\int_0^\infty \frac{d\mu_y}{2\pi}\left\langle\!\left\langle \frac{g_{xy}^2}{m_x m_y}\right\rangle\!\right\rangle\\
&\frac{1}{(2\mu_x)(2\mu_y)}\Big\{2\hbar(\mu_x+\mu_y)\left[(1+\mathfrak{f}_x)(1+\mathfrak{f}_y)-\mathfrak{f}_x\mathfrak{f}_y\right](2\pi)\text{sgn}(\omega_1)\delta\left(\omega_1^2-(\mu_x+\mu_y)^2\right)\\
&+2\hbar(\mu_x-\mu_y)\left[\mathfrak{f}_x(1+\mathfrak{f}_y)-\mathfrak{f}_y(1+\mathfrak{f}_x)\right](2\pi)\text{sgn}(\omega_2)\delta\left(\omega_2^2-(\mu_x-\mu_y)^2\right)\Big\} .
\end{aligned} \tag{4.10}$$

[11] The correction to the spectral function appearing in the last two lines comes from computing the Fourier transform of the thermal commutator

$$\langle [X(t_1)Y(t_1), X(t_2)Y(t_2)]\rangle . \tag{4.9}$$

Here we have introduced the cubic coupling distribution

$$\left\langle\left\langle\frac{g_{xy}^2}{m_x m_y}\right\rangle\right\rangle \equiv \sum_{ij}\frac{g_{xy,ij}^2}{m_{x,i}m_{y,j}}2\pi\delta(\mu_x-\mu_i)2\pi\delta(\mu_y-\mu_j)\,. \tag{4.11}$$

This cubic coupling distribution should be judiciously chosen so that its dynamics does not destroy the Markovian approximation. We find it convenient to choose a distribution of couplings such that

$$\left\langle\left\langle\frac{g_{xy}^2}{m_x m_y}\right\rangle\right\rangle \sim \left\langle\left\langle\frac{g_x^2}{m_x}\right\rangle\right\rangle\left\langle\left\langle\frac{g_y^2}{m_y}\right\rangle\right\rangle. \tag{4.12}$$

More precisely, we take

$$\left\langle\left\langle\frac{g_{xy}^2}{m_x m_y}\right\rangle\right\rangle = \Gamma_{xy}\,\frac{4\mu_x^2\Omega^2}{\mu_x^2+\Omega^2}\,\frac{4\mu_y^2\Omega^2}{\mu_y^2+\Omega^2}\,. \tag{4.13}$$

As we will show later, this distribution function is sufficient to give a fast decay of correlator at timescales larger than Ω^{-1}. In the large temperature limit i.e. $\beta\to 0$, this distribution function gives the following form of $\rho[12]$ upto $O(\beta^0)$:

$$\rho[12] = 2m_p\left[\frac{1}{2}\Gamma_{xy}\Omega\, v_{th}^2\frac{4\Omega^2}{\omega_1^2+4\Omega^2}+(\gamma_x+\gamma_y)\frac{\Omega^2}{\omega_1^2+\Omega^2}\right]\omega_1(2\pi)\delta(\omega_1+\omega_2)\,. \tag{4.14}$$

Thus, the damping constant acquires a correction of the order $\Gamma_{xy}\Omega\, v_{th}^2$. This is a small correction provided we have

$$\Gamma_{xy} \ll \frac{\gamma}{\Omega v_{th}^2} = \frac{m_p\gamma}{\Omega k_B T},$$

a condition we will assume from now on.

We can now turn to the three point spectral functions obtained by Fourier transforming the nested commutators of the bath operator $\mathcal{O}$ defined in eq. (4.6). This yields

$$\begin{aligned}\rho[123] = (2\pi)\delta(\omega_1+\omega_2+\omega_3)&\int_0^\infty\frac{d\mu_x}{2\pi}\int_0^\infty\frac{d\mu_y}{2\pi}\left\langle\left\langle\frac{g_x g_y g_{xy}}{m_x m_y}\right\rangle\right\rangle\\ &\Big\{\text{sgn}(\omega_3)(2\pi)\delta(\omega_3^2-\mu_x^2)\,\Big[\text{sgn}(\omega_2)(2\pi)\delta(\omega_2^2-\mu_y^2)-\text{sgn}(\omega_1)(2\pi)\delta(\omega_1^2-\mu_y^2)\Big]\\ &+\text{sgn}(\omega_3)(2\pi)\delta(\omega_3^2-\mu_y^2)\Big[\text{sgn}(\omega_2)(2\pi)\delta(\omega_2^2-\mu_x^2)-\text{sgn}(\omega_1)(2\pi)\delta(\omega_1^2-\mu_x^2)\Big]\Big\}.\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}\rho[321] = (2\pi)\delta(\omega_1+\omega_2+\omega_3)&\int_0^\infty\frac{d\mu_x}{2\pi}\int_0^\infty\frac{d\mu_y}{2\pi}\left\langle\left\langle\frac{g_x g_y g_{xy}}{m_x m_y}\right\rangle\right\rangle\\ &\Big\{\text{sgn}(\omega_1)(2\pi)\delta(\omega_1^2-\mu_x^2)\,\Big[\text{sgn}(\omega_2)(2\pi)\delta(\omega_2^2-\mu_y^2)-\text{sgn}(\omega_3)(2\pi)\delta(\omega_3^2-\mu_y^2)\Big]\\ &+\text{sgn}(\omega_1)(2\pi)\delta(\omega_1^2-\mu_y^2)\Big[\text{sgn}(\omega_2)(2\pi)\delta(\omega_2^2-\mu_x^2)-\text{sgn}(\omega_3)(2\pi)\delta(\omega_3^2-\mu_x^2)\Big]\Big\},\end{aligned} \tag{4.16}$$

where we have defined the distribution function

$$\left\langle\left\langle \frac{g_x g_y g_{xy}}{m_x m_y} \right\rangle\right\rangle \equiv \sum_{ij} \frac{g_{x,i} g_{y,j} g_{xy,ij}}{m_{x,i} m_{y,j}} 2\pi\delta(\mu_x - \mu_i) 2\pi\delta(\mu_y - \mu_j) \,. \tag{4.17}$$

We will find it convenient to assume

$$|\left\langle\left\langle \frac{g_x g_y g_{xy}}{m_x m_y} \right\rangle\right\rangle|^2 \sim \left\langle\left\langle \frac{g_{xy}^2}{m_x m_y} \right\rangle\right\rangle \left\langle\left\langle \frac{g_x^2}{m_x} \right\rangle\right\rangle \left\langle\left\langle \frac{g_y^2}{m_y} \right\rangle\right\rangle . \tag{4.18}$$

More precisely, we take

$$\left\langle\left\langle \frac{g_x g_y g_{xy}}{m_x m_y} \right\rangle\right\rangle = \frac{m_p}{4} \Gamma_3 \frac{4\mu_x^2 \Omega^2}{\mu_x^2 + \Omega^2} \frac{4\mu_y^2 \Omega^2}{\mu_y^2 + \Omega^2}. \tag{4.19}$$

Here, the parameter Γ_3 can roughly be thought of as an inverse penetration depth for the three body scattering. This induces a small correction to the usual Langevin dynamics if

$$\Gamma_3 \ll \frac{\gamma}{v_{th}} = \frac{m_p \gamma}{k_B T},$$

a condition we will assume from now on. In the next subsection, we will examine the correlation functions of the bath and to what extent they justify a Markovian appproximation.

4.2 KMS relations and decay of bath correlations

Consider the model described in the above section of a single oscillator coupled to two harmonic baths. In this subsection, we will be interested in the evolution starting from an initial time t_0 and whether and how the bath correlators decay with time.

We will assume that the bath and the oscillator are unentangled at an initial time t_0. Therefore, the initial density matrix of the oscillator and the bath is given by

$$\rho(t_0) = \frac{e^{-\frac{H_B}{k_B T}}}{Z_B} \otimes \rho_{\rm p} \,, \tag{4.20}$$

where H_B is the Hamiltonian of the bath, Z_B is its partition function, and $\rho_{\rm p}$ is the initial density matrix of the oscillator at time t_0.

The effective theory of the Brownian particle is obtained after integrating out the degrees of freedom of the thermal bath. In the process, the bath correlation functions (in general out-of-time-order) imprint themselves on the effective couplings of the Brownian particle. In this section we are interested in studying the out-of-time-order bath correlators of the operator $\mathcal{O}$. We will use these correlators later to determine the effective couplings of the particle.

Since the bath is in a thermal state, not all the Wightman correlators of $\mathcal{O}$ are independent. The bath correlators that are related to each other by cyclic permutations of insertions, satisfy the KMS (Kubo-Martin-Schwinger) conditions [11, 17, 18, 45–52]. For a thermal n-point function of $\mathcal{O}(t)$, the KMS condition in time domain gives the following condition on the connected parts (cumulants) of the bath correlators:

$$\langle \mathcal{O}(t_1)\mathcal{O}(t_2)\ldots\mathcal{O}(t_n)\rangle_c = \langle \mathcal{O}(t_n - i\beta)\mathcal{O}(t_1)\mathcal{O}(t_2)\ldots\mathcal{O}(t_{n-1})\rangle_c. \tag{4.21}$$

In frequency space the KMS condition simplifies to

$$\langle \mathcal{O}(\omega_1)\mathcal{O}(\omega_2)\dots\mathcal{O}(\omega_n)\rangle_c = e^{-\beta\omega_n}\langle \mathcal{O}(\omega_n)\mathcal{O}(\omega_1)\dots\mathcal{O}(\omega_{n-1})\rangle_c. \tag{4.22}$$

This follows straightforwardly from the fact that the frequency domain analogue of $\mathcal{O}(t_n - i\beta)$ is $e^{-\beta\omega_n}\mathcal{O}(\omega_n)$.

At the level of the two point function, the statement implies that there is only one independent two-point correlator of $\mathcal{O}$. We choose that to be $\rho[12]$. Then, using the KMS relations, the expectation value of the anticommutator is given by

$$\rho[12_+] = -\hbar\,(1+2\mathfrak{f}_2)\,\rho[12]\ , \tag{4.23}$$

where $\mathfrak{f}_1 = \frac{1}{e^{\beta\omega_1}-1}$ and $\mathfrak{f}_2 = \frac{1}{e^{\beta\omega_2}-1}$ are the Bose-Einstein distribution functions.

Similarly all the three point functions of $\mathcal{O}$ are determined by $\rho[123]$ and $\rho[321]$. The other 3-point correlators are related to the two by the following KMS relations [11]

$$\begin{aligned} \rho[123_+] &= -\hbar(1+2\mathfrak{f}_3)\rho[123]\,,\\ \rho[321_+] &= -\hbar(1+2\mathfrak{f}_1)\rho[321]\,,\\ \rho[12_+3] &= -\hbar(1+2\mathfrak{f}_2)\rho[123] + 2(1+\mathfrak{f}_1+\mathfrak{f}_2)\rho[321]\rangle\,,\\ \rho[12_+3_+] &= \hbar^2(1+2\mathfrak{f}_3)\Big[(1+2\mathfrak{f}_2)\rho[123] - 2(1+\mathfrak{f}_1+\mathfrak{f}_2)\rho[321]\Big]. \end{aligned} \tag{4.24}$$

Hence the spectral functions $\rho[12]$, $\rho[123]$ and $\rho[321]$ are sufficient to determine all two-point and three-point bath correlators.

For the two-point function the correlator of the anticommutator in equation (4.23) provides a measure of the thermal noise arising from the thermal fluctuations in the bath whereas the correlation function of the commutator in that equation gives a measure of the dissipation/damping in the bath due to the motion of the Brownian particle [2]. We denote the connected part of nested commutators of operators in time domain by a tilde in the following:

$$\widetilde{\langle[123]\rangle} \equiv \langle[[O(t_1), O(t_2)], O(t_3)]\rangle_c\,. \tag{4.25}$$

For the cumulants of the nested anticommutators we use a similar notation with an extra '+' sign inside the square bracket indicating the position of the anticommutator as follows

$$\begin{aligned} \langle\widetilde{[12_+3]}\rangle &\equiv \langle[\{O(t_1), O(t_2)\}, O(t_3)]\rangle_c\,,\\ \langle\widetilde{[321_+]}\rangle &\equiv \langle\{[O(t_3), O(t_2)], O(t_1)\}\rangle_c\,. \end{aligned} \tag{4.26}$$

We can use the forms of the spectral functions to get the bath correlators in time domain. The bath correlators decay with increase in separation between any two insertions. In the following, we provide the two-point and three-point cumulants for the slowest decaying modes with frequency Ω.

For our model, the two point cumulants decay as

$$
\begin{aligned}
\frac{i}{\hbar}\langle\widetilde{[12]}\rangle &= \Omega^2\left[m_p(\gamma_x+\gamma_y)\exp(-\Omega t_{12})+\hbar\Gamma_{xy}\Omega^2\exp(-2\Omega t_{12})\cot\left(\frac{\beta\Omega}{2}\right)\right], \\
\langle\widetilde{[12_+]}\rangle &= \frac{\Omega^2}{2}\left[(\gamma_x+\gamma_y)\hbar\csc\left(\frac{\beta\Omega}{2}\right)\exp(-\Omega t_{12})+\hbar^2\Gamma_{xy}\Omega^2\left(\cot^2\left(\frac{\beta\Omega}{2}\right)-1\right)\exp(-2\Omega t_{12})\right].
\end{aligned}
\tag{4.27}
$$

In the high temperature limit, this yields

$$
\begin{aligned}
\frac{i}{\hbar}\langle\widetilde{[12]}\rangle &= m_p\Omega^2\Big[(\gamma_x+\gamma_y)\exp(-\Omega t_{12})+2\Gamma_{xy}\Omega v_{th}^2\exp(-2\Omega t_{12})\Big], \\
\langle\widetilde{[12_+]}\rangle &= m_p^2 v_{th}^2\Omega\Big[(\gamma_x+\gamma_y)\exp(-\Omega t_{12})+2\Gamma_{xy}\Omega v_{th}^2\exp(-2\Omega t_{12})\Big].
\end{aligned}
\tag{4.28}
$$

The three point nested cumulants are given by

$$
\begin{aligned}
\frac{i^2}{\hbar^2}\langle\widetilde{[321]}\rangle &= m_p\frac{\Gamma_3\Omega^4}{2}\exp(-\Omega t_{13})\Big(1+\exp(-\Omega t_{23})\Big), \\
\frac{i^2}{\hbar^2}\langle\widetilde{[123]}\rangle &= m_p\frac{\Gamma_3\Omega^4}{2}\exp(-\Omega t_{13})\Big(1+\exp(-\Omega t_{12})\Big).
\end{aligned}
\tag{4.29}
$$

The other cumulants can also be computed to yield

$$
\begin{aligned}
\frac{i}{\hbar}\langle\widetilde{[321_+]}\rangle &= -m_p\Gamma_3\Omega^4\frac{\hbar}{2}\cot\left(\frac{\beta\Omega}{2}\right)\exp(-\Omega t_{13})\Big(1+\exp(-\Omega t_{23})\Big), \\
\frac{i}{\hbar}\langle\widetilde{[123_+]}\rangle &= m_p\Gamma_3\Omega^4\frac{\hbar}{2}\cot\left(\frac{\beta\Omega}{2}\right)\exp(-\Omega t_{13})\Big(1+\exp(-\Omega t_{13})\Big), \\
\frac{i}{\hbar}\langle\widetilde{[12_+3]}\rangle &= m_p\Gamma_3\Omega^4\frac{\hbar}{2}\cot\left(\frac{\beta\Omega}{2}\right)\exp(-\Omega t_{13})\Big(1+2\exp(-\Omega t_{23})+\exp(-\Omega t_{12})\Big)
\end{aligned}
\tag{4.30}
$$

and

$$
\langle\widetilde{[12_+3_+]}\rangle=\hbar^2 m_p\frac{\Gamma_3\Omega^4}{2}\exp(-\Omega t_{13})\left[\cot^2\left(\frac{\beta\Omega}{2}\right)\{1+\exp(-\Omega t_{12})+\exp(-\Omega t_{23})\}-\exp(-\Omega t_{23})\right].
\tag{4.31}
$$

In the high temperature limit, this yields

$$
\begin{aligned}
\frac{i}{\hbar}\langle\widetilde{[321_+]}\rangle &= -\Gamma_3\ m_p^2 v_{th}^2\Omega^3\exp(-\Omega t_{13})\Big(1+\exp(-\Omega t_{23})\Big), \\
\frac{i}{\hbar}\langle\widetilde{[123_+]}\rangle &= \Gamma_3\ m_p^2 v_{th}^2\Omega^3\exp(-\Omega t_{13})\Big(1+\exp(-\Omega t_{13})\Big), \\
\frac{i}{\hbar}\langle\widetilde{[12_+3]}\rangle &= \Gamma_3\ m_p^2 v_{th}^2\Omega^3\exp(-\Omega t_{13})\Big(1+2\exp(-\Omega t_{23})+\exp(-\Omega t_{12})\Big)
\end{aligned}
\tag{4.32}
$$

and

$$
\langle\widetilde{[12_+3_+]}\rangle = 2\Gamma_3\ m_p^3 v_{th}^4\Omega^4\exp(-\Omega t_{13})\left(1+\exp(-\Omega t_{12})+\exp(-\Omega t_{23})\right).
\tag{4.33}
$$

Thus, given the decay of the memory in the bath at time-scales much larger than $(\frac{1}{\Omega})$ we expect to obtain a local effective theory for the particle at long time-scales. In the next section, we will describe how such an effective theory be obtained starting from the microscopic description.

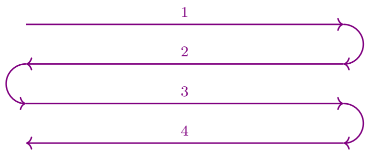

Figure 1. A contour with 2 time-folds.

5 Effective theory of the Brownian particle

5.1 OTO influence phase

We will begin by describing the procedure we employ to derive the effective theory for the particle at long time-scales, closely following [9]. Our aim is to obtain the couplings in the effective action of the Brownian particle (after systematically integrating out the degrees of freedom of the bath) in a way that keeps track of out-of-time order correlations. Thus, we are interested in an effective theory which has sufficient number of effective couplings that can compute along with the time-ordered correlators, also the out-of-time order correlators (OTOCs) of the particle.

We use the generalised Schwinger-Keldysh path integral formalism to arrive at an effective action/generalised influence phase for the Brownian particle. The path integral representation of 2-OTO correlators (correlators with two insertions whose immediate neighbours lie to their pasts)[12] is then defined on a contour with two time-folds as shown in figure 1. There are four copies of the degrees of freedom of the particle and bath $\{q_1, X_{i1}, Y_{j1}\}, \{q_2, X_{i2}, Y_{j2}\}, \{q_3, X_{i3}, Y_{j3}\}$ and $\{q_4, X_{i4}, Y_{j4}\}$ living on the four legs of the contour. The action that enters in the path integral is

$$S_{\text{2-fold}} = \int_{t_0}^{T} dt \Big\{ L[q_1, X_{i1}, Y_{j1}] - L[-q_2, X_{i2}, Y_{j2}] + L[q_3, X_{i3}, Y_{j3}] - L[-q_4, X_{i4}, Y_{j4}] \Big\}. \tag{5.1}$$

The degrees of freedom of the particle are identified at the turning points of the contour while performing the path integral. After integrating out the degrees of freedom of the bath, one obtains an out-of-time-order generalised influence phase W for the particle which can be expanded in powers of the particle-bath coupling:

$$W = W_1 + W_2 + W_3 + \dots \quad . \tag{5.2}$$

The n-th term in this perturbative expansion is given by

$$W_n = \frac{i^{n-1}}{n!\hbar^{n-1}} \int_{t_0}^{T} dt_1 \cdots \int_{t_0}^{T} dt_n \sum_{i_1,\cdots,i_n=1}^{4} \langle \mathcal{T}_C \mathcal{O}_{i_1}(t_1) \cdots \mathcal{O}_{i_n}(t_n) \rangle_c \; q_{i_1}(t_1) \cdots q_{i_n}(t_n) \tag{5.3}$$

Here the subscripts denote the contour legs and the expectation values are contour ordered cumulants computed in the initial state of the bath.

[12] E.g. for $t_1 > t_2 > t_3 > t_0$, $\langle \mathcal{O}(t_1)\mathcal{O}(t_3)\mathcal{O}(t_2)\rangle = Tr\,(\rho(t_0)\mathcal{O}(t_1)\mathcal{O}(t_3)\mathcal{O}(t_2))$ has $\mathcal{O}(t_1)$ and $\mathcal{O}(t_2)$ whose immediate neighbours lie to their pasts. Hence this is an example of a 2-OTO correlator.

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5.2 Markovian approximation and effective action

The correlators of the Brownian particle calculated from the generalised influence phase can be obtained from a 1-PI effective action. The 1-PI effective action is generally non-unitary and non-local. However since in our model the cumulants of $\mathcal{O}(t)$ decay sufficiently fast, with an increase in separation between the insertions compared to the natural time scale of the particle, we can work in a Markovian limit [6]. In this limit we get a local, non-unitary 1-PI effective action. We assume that in the action the terms with two or more time derivatives on q are negligible. Such a local 1-PI effective Lagrangian for the Brownian particle has the following form [9, 53]

$$L_{\text{1PI}} = L^{(1)}_{\text{1PI}} + L^{(2)}_{\text{1PI}} + L^{(3)}_{\text{1PI}} + \cdots \tag{5.4}$$

where the $L^{(1)}_{\text{1PI}}$, $L^{(2)}_{\text{1PI}}$ and $L^{(3)}_{\text{1PI}}$ correspond to terms that are linear, quadratic and cubic in q's respectively. The linear term is given by

$$L^{(1)}_{\text{1PI}} = \widehat{F}(q_1 + q_2 + q_3 + q_4). \tag{5.5}$$

The quadratic term is given by

$$\begin{aligned} L^{(2)}_{\text{1PI}} = &\frac{1}{2}Z(\dot{q}_1^2 + \dot{q}_3^2) - \frac{1}{2}Z^*(\dot{q}_2^2 + \dot{q}_4^2) + i\ Z_\Delta \sum_{i<j} \dot{q}_i\dot{q}_j \\ &- \frac{m^2}{2}(q_1^2 + q_3^2) + \frac{(m^2)^*}{2}(q_2^2 + q_4^2) \\ &- im_\Delta^2 \sum_{i<j} q_iq_j + \frac{\widehat{\gamma}}{2}\sum_{i<j}(q_i\dot{q}_j - \dot{q}_iq_j), \end{aligned} \tag{5.6}$$

The cubic term $L^{(3)}_{\text{1PI}}$ can be split into 2 parts: one part, which reduces to the cubic terms in the Schwinger-Keldysh effective theory under identification of the degrees of freedom on any two successive legs, is given by

$$\begin{aligned} L^{(3)}_{\text{1PI,SK}} = &-\frac{\lambda_3}{3!}(q_1^3 + q_3^3) - \frac{\lambda_3^*}{3!}(q_2^3 + q_4^3) \\ &+ \frac{\sigma_3}{2!}\left[q_1^2(q_2 + q_3 + q_4) - q_2^2(q_3 + q_4) + q_3^2q_4\right] \\ &+ \frac{\sigma_3^*}{2!}\left[q_1(q_2^2 - q_3^2 + q_4^2) - q_2(q_3^2 - q_4^2) + q_3q_4^2\right] \\ &+ \frac{\sigma_{3\gamma}}{2!}\left[q_1^2(\dot{q}_2 + \dot{q}_3 + \dot{q}_4) - q_2^2(\dot{q}_3 + \dot{q}_4) + q_3^2\dot{q}_4 - (q_2^2\dot{q}_2 + q_4^2\dot{q}_4)\right] \\ &+ \frac{\sigma_{3\gamma}^*}{2!}\left[\dot{q}_1(q_2^2 - q_3^2 + q_4^2) - \dot{q}_2(q_3^2 - q_4^2) + \dot{q}_3q_4^2 - (q_1^2\dot{q}_1 + q_3^2\dot{q}_3)\right], \end{aligned} \tag{5.7}$$

The other part, which vanishes under such identifications, is given by

$$\begin{aligned} L^{(3)}_{\text{1PI,2-OTO}} = &-\left(\kappa_3 + \frac{1}{2}\text{Re}[\lambda_3 - \sigma_3]\right)(q_1 + q_2)(q_2 + q_3)(q_3 + q_4) \\ &- (q_2 + q_3)\Big[\left(\kappa_{3\gamma} - \text{Re}[\sigma_{3\gamma}]\right)(\dot{q}_1 + \dot{q}_2)(q_3 + q_4) \\ &\qquad + \left(\kappa_{3\gamma}^* - \text{Re}[\sigma_{3\gamma}]\right)(q_1 + q_2)(\dot{q}_3 + \dot{q}_4)\Big]. \end{aligned} \tag{5.8}$$

The effective action constructed from this Lagrangian has to satisfy certain conditions as elaborated in the appendix B. The conditions arise due to the fact that the Brownian particle and the bath together comprise a closed unitary quantum system.

5.3 Comparison with the nonlinear Langevin system

We will now relate the description in terms of generalised influence phase and generalised Schwinger Keldysh effective action to the classical stochastic description in terms of non-linear Langevin equation. This relation, familiar in the stochastic quantisation literature, can be studied at various levels: we can for example, study the correlators predicted by the two descriptions and match them against each other. We will instead derive the dictionary between the quantum and stochastic descriptions by deriving a path integral which generates the non-linear Langevin correlators and then matching its terms against the influence phase obtained by ignoring out of time ordered contributions.

We are interested in the non-linear Langevin theory described by the stochastic equation:

$$\mathcal{E}[q] \equiv \frac{d^2q}{dt^2} + (\gamma + \zeta_\gamma \mathcal{N})\frac{dq}{dt} + (\bar{\mu}^2 + \zeta_\mu \mathcal{N})\, q + \left(\overline{\lambda}_3 - \overline{\lambda}_{3\gamma}\, \frac{d}{dt}\right)\frac{q^2}{2!} - F = \langle f^2 \rangle \mathcal{N} \ . \tag{5.9}$$

Here, we will take $\mathcal{N}$ to be a random noise drawn from the non-Gaussian probability distribution

$$P[\mathcal{N}] \propto \exp\left\{ -\frac{1}{2\langle f^2 \rangle} \int dt\, \left(\langle f^2 \rangle \mathcal{N} - \zeta_N \mathcal{N}^2\right)^2 - \frac{1}{2} Z_I \int dt\, \dot{\mathcal{N}}^2 \right\} . \tag{5.10}$$

We will assume that the corrections to the Langevin equation are small: this is equivalent to assuming the parameters $\{\zeta_\gamma, \zeta_\mu, \overline{\lambda}_3, \overline{\lambda}_{3\gamma}, \zeta_N, Z_I\}$ are small.

The equation above is a non-linear stochastic ODE with *multiplicative noise*, i.e., the noise variable $\mathcal{N}$ appears in the equation multiplied by the functions of the fundamental stochastic variable q of the differential equation. In the theory of stochastic ODEs, such ODEs need a definite prescription for equal time stochastic products to be well-defined. In this work, we will adopt a time-symmetric (or Stratonovich) prescription for equal time stochastic products. But we will only need leading order corrections due to the multiplicative noise terms, where the subtleties regarding various prescriptions for stochastic products will not matter.

To study this non-linear Langevin theory in the context of path integrals, one can employ the following method (often attributed to Martin-Siggia-Rose [19]-De Dominicis-Peliti [20]-Janssen [21]): we start by thinking about the functional integral[13] over noise realisations along with the imposition of the non-linear Langevin equation on a variable $q_a(t)$:

$$\begin{aligned} &\int [dq_a][d\mathcal{N}]\ P[\mathcal{N}]\ \delta\Big[\langle f^2 \rangle \mathcal{N} - \mathcal{E}[q_a]\Big] \\ &= \int [dq_a][dq_d][d\mathcal{N}]\ P[\mathcal{N}]\ \exp\left\{ i\frac{m_p}{\hbar} \int dt\ q_d \Big[\langle f^2 \rangle \mathcal{N} - \mathcal{E}[q_a]\Big] \right\} , \end{aligned} \tag{5.11}$$

[13]In this integral, we ignore the Jacobian det $\left[\frac{\delta \mathcal{E}[q(t)]}{\delta q(t')}\right]$ as it does not correct the coefficients of the terms obtained in (5.12) up to leading order in the particle-bath coupling.

where we have given the standard functional integral representation of the delta function. We can now discretise the noise integral, add appropriate counterterms and perform the path integral perturbatively in the small parameters $\{\zeta_\gamma, \zeta_\mu, \overline{\lambda}_3, \overline{\lambda}_{3\gamma}, \zeta_N, Z_I\}$. This exercise yields

$$\begin{aligned}
&\lim_{\delta t\to 0}\int [d\mathcal{N}] e^{-i\frac{m_p}{\hbar}\frac{3\zeta_N}{\langle f^2\rangle \delta t}\int dt\ q_d}\ P[\mathcal{N}]\exp\left\{i\frac{m_p}{\hbar}\int dt\ q_d\Big[\langle f^2\rangle\mathcal{N}-\mathcal{E}[q_a]\Big]\right\}\\
&\approx \exp\left\{\frac{i}{\hbar}\int dt\left[\frac{i}{2}\frac{m_p^2}{\hbar}\langle f^2\rangle q_d^2-\frac{i}{2}\frac{m_p^2}{\hbar}Z_I\dot{q}_d^2-\frac{m_p^3}{\hbar^2}\zeta_N q_d^3-m_p q_d\mathcal{E}[q_a]_{\mathcal{N}=0}-i\frac{m_p^2}{\hbar}q_d^2\frac{\partial\mathcal{E}[q_a]}{\partial\mathcal{N}}\right]\right\}.
\end{aligned} \tag{5.12}$$

This MSRDPJ effective action can be connected to Schwinger Keldysh effective action by identifying $q_d = q_1 + q_2$, $q_a = \frac{1}{2}(q_1 - q_2)$. We will refer the reader to [54] for a textbook level discussion of why this is the correct identification that maps the Schwinger Keldysh boundary conditions on the quantum side to the causal boundary conditions of the stochastic path integral. Using this map, we can write the above effective action in the form:

$$\begin{aligned}
L^{(1)}_{\text{1PI,SK}} &= \widehat{F}(q_1+q_2),\\
L^{(2)}_{\text{1PI,SK}} &= \frac{1}{2}Z\dot{q}_1^2-\frac{1}{2}Z^*\dot{q}_2^2+i\ Z_\Delta\dot{q}_1\dot{q}_2-\frac{m^2}{2}q_1^2+\frac{(m^2)^*}{2}q_2^2-im_\Delta^2 q_1q_2+\frac{\widehat{\gamma}}{2}(q_1\dot{q}_2-\dot{q}_1q_2),\\
L^{(3)}_{\text{1PI,SK}} &= -\frac{\lambda_3}{3!}q_1^3-\frac{\lambda_3^*}{3!}q_2^3+\frac{\sigma_3}{2!}q_1^2q_2+\frac{\sigma_3^*}{2!}q_1q_2^2+\frac{\sigma_{3\gamma}}{2!}q_1^2\dot{q}_2+\frac{\sigma_{3\gamma}^*}{2!}\dot{q}_1q_2^2\,,
\end{aligned} \tag{5.13}$$

where the Schwinger Keldysh effective couplings are given in terms of Langevin couplings via

$$\begin{aligned}
\widehat{F} &\equiv m_pF\,, & \widehat{\gamma} &\equiv m_p\gamma\,,\\
Z &\equiv m_p - i\frac{m_p^2}{\hbar}Z_I\,, & Z_\Delta &\equiv -\frac{m_p^2}{\hbar}Z_I\,,\\
m^2 &\equiv m_p\bar{\mu}^2 - i\frac{m_p^2}{\hbar}\langle f^2\rangle\,, & m_\Delta^2 &\equiv -\frac{m_p^2}{\hbar}\langle f^2\rangle\,,\\
\lambda_3 &\equiv \frac{3}{4}m_p\overline{\lambda}_3+6\frac{m_p^3}{\hbar^2}\zeta_N+3i\frac{m_p^2}{\hbar}\zeta_\mu\,,\\
\sigma_3 &\equiv \frac{1}{4}m_p\overline{\lambda}_3-6\frac{m_p^3}{\hbar^2}\zeta_N-i\frac{m_p^2}{\hbar}\zeta_\mu\,,\\
\sigma_{3\gamma} &\equiv -\frac{1}{2}m_p\overline{\lambda}_{3\gamma}+2i\frac{m_p^2}{\hbar}\zeta_\gamma\,.
\end{aligned} \tag{5.14}$$

We recognise the above form as the most general Schwinger Keldysh effective action obtained by collapsing two successive time contours as mentioned in the previous subsection. As will be described elsewhere [55], one can extend the nonlinear Langevin theory to an ‘out of time order’ stochastic theory which can capture all the couplings of the generalised Schwinger Keldysh effective action. For now, we will content ourselves with matching the out of time-ordered couplings by looking at the OTOCs of the system. This yields a map

$$\begin{aligned}
\kappa_3 &\equiv -\frac{1}{2}m_p\overline{\kappa}_3\,,\\
\kappa_{3\gamma} &\equiv -\frac{1}{2}m_p\overline{\kappa}_{3\gamma}+\frac{2}{3}i\frac{m_p^2}{\hbar}\widehat{\kappa}_{3\gamma}\,.
\end{aligned} \tag{5.15}$$

The couplings in the 2-OTO effective theory, and hence the non-linear Langevin couplings are then determined by the bath correlators which enter into the generalised influence phase eq. (5.2). Such relations between the couplings and the correlators can be obtained by computing the particle's correlators with generalised influence phase given in eq. (5.2) and then comparing them with the results obtained from the effective theory. The expressions of the influence phase couplings in terms of correlators in the time domain were given in [9] and in frequency domain, they take the forms summarised earlier in section 3. In the following subsection, we will use the expressions quoted in section 3 to compute explicitly the effective couplings for the qXY model.

5.4 Influence couplings in the qXY model

We will now describe the computation of non-linear Langevin couplings starting from the spectral functions of the qXY model. In our model, quadratic spectral function is given at high temperatures by the expression

$$\rho[12] = 2m_p \left[(\gamma_x + \gamma_y)\frac{\Omega^2}{\omega_1^2 + \Omega^2} + \frac{1}{2}\Gamma_{xy}\Omega\, v_{th}^2 \frac{4\Omega^2}{\omega_1^2 + 4\Omega^2}\right] \omega_1 (2\pi)\delta(\omega_1 + \omega_2)\,. \tag{5.16}$$

This two point spectral function obeys the following integral identities

$$\begin{aligned}
\int_{\mathcal{C}_2} \frac{\rho[12]}{\omega_1} &= m_p\Omega(\gamma_x + \gamma_y + \Gamma_{xy}\Omega v_{\rm th}^2)\,, \\
\int_{\mathcal{C}_2} \frac{\rho[12]}{i\omega_1^2} &= m_p(\gamma_x + \gamma_y + \Gamma_{xy}\Omega v_{\rm th}^2)\,, \\
\int_{\mathcal{C}_2} \frac{\rho[12]}{\omega_1^3} &= -\frac{m_p}{\Omega}(\gamma_x + \gamma_y + \frac{1}{4}\Gamma_{xy}\Omega v_{\rm th}^2)\,, \\
-\int_{\mathcal{C}_2} \frac{\rho[12]}{i\omega_1^4} &= \frac{m_p}{\Omega^2}(\gamma_x + \gamma_y + \frac{1}{8}\Gamma_{xy}\Omega v_{\rm th}^2)\,,
\end{aligned} \tag{5.17}$$

which yield the following quadratic couplings at high temperature:

$$\begin{aligned}
\Delta m_p \equiv m_p - m_{p0} &= -\frac{m_p}{\Omega}\left(\gamma_x + \gamma_y + \frac{1}{4}\Gamma_{xy}\Omega\, v_{th}^2\right)\,, \\
Z_I &= \frac{v_{th}^2}{\Omega^2}\left(2\gamma_x + 2\gamma_y + \frac{1}{4}\Gamma_{xy}\Omega\, v_{th}^2\right)\,, \\
\Delta\overline{\mu}^2 \equiv \overline{\mu}^2 - \overline{\mu}_0^2 &= -\Omega\left(\gamma_x + \gamma_y + \Gamma_{xy}\Omega\, v_{th}^2\right)\,, \\
\langle f^2\rangle &= 2v_{th}^2\left(\gamma_x + \gamma_y + \frac{1}{2}\Gamma_{xy}\Omega\, v_{th}^2\right)\,, \\
\gamma &= \gamma_x + \gamma_y + \frac{1}{2}\Gamma_{xy}\Omega\, v_{th}^2\,.
\end{aligned} \tag{5.18}$$

Similarly, the cubic spectral functions are given by

$$\begin{aligned}
\rho[123] &= 2\pi\delta(\omega_1+\omega_2+\omega_3) \times 2m_p\Gamma_3(\omega_1^2 - \omega_2^2)\left(1 - \frac{\omega_1\omega_2}{\Omega^2}\right) \times \prod_{k=1}^{3}\frac{\Omega^2}{\omega_k^2 + \Omega^2}\,, \\
\rho[321] &= 2\pi\delta(\omega_1+\omega_2+\omega_3) \times 2m_p\Gamma_3(\omega_3^2 - \omega_2^2)\left(1 - \frac{\omega_3\omega_2}{\Omega^2}\right) \times \prod_{k=1}^{3}\frac{\Omega^2}{\omega_k^2 + \Omega^2}\,.
\end{aligned} \tag{5.19}$$

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These three point spectral functions obeys the following integral identities at high temperature:

$$\int_{\mathcal{C}_3} \frac{\rho[123]}{\omega_1\omega_3} = \int_{\mathcal{C}_3} \frac{\rho[321]}{\omega_1\omega_3} = -\frac{3}{4} m_p \Gamma_3 \Omega^2 \,, \tag{5.20}$$

$$\begin{aligned} \frac{2}{3}\int_{\mathcal{C}_3} \frac{\rho[123]}{i\omega_1\omega_3^2} &= \frac{4}{5}\int_{\mathcal{C}_3} \frac{\rho[321]}{i\omega_1\omega_3^2} = -\frac{4}{5}\int_{\mathcal{C}_3} \frac{\rho[123]}{i\omega_1^2\omega_3} = -\frac{2}{3}\int_{\mathcal{C}_3} \frac{\rho[321]}{i\omega_1^2\omega_3} = \int_{\mathcal{C}_3} \frac{\rho[132]}{i\omega_1\omega_2\omega_3} \\ &= -\frac{1}{4}\int_{\mathcal{C}_3} \frac{1}{i\omega_2}\left\{\frac{\rho[123]}{\omega_3^2} - \frac{\rho[321]}{\omega_1^2}\right\} = \frac{1}{2} m_p \Gamma_3 \Omega \,, \end{aligned} \tag{5.21}$$

$$\frac{4}{3}\int_{\mathcal{C}_3} \frac{1}{\omega_1^3\omega_3}\rho[321] = \frac{4}{3}\int_{\mathcal{C}_3} \frac{1}{\omega_1\omega_3^3}\rho[123] = \int_{\mathcal{C}_3} \frac{1}{\omega_1\omega_2\omega_3}\left\{\frac{1}{\omega_1}\rho[321] + \frac{1}{\omega_3}\rho[123]\right\} = m_p \Gamma_3 \,, \tag{5.22}$$

which yields the following values for cubic couplings

$$\begin{aligned} \overline{\lambda}_3 &= \overline{\kappa}_3 = -\frac{3}{2}\Gamma_3\Omega^2 \,, \\ \overline{\lambda}_{3\gamma} &= \overline{\kappa}_{3\gamma} = -2\Gamma_3\Omega \,, \qquad \zeta_\mu = -2\Gamma_3\Omega \; v_{th}^2 \,, \\ \zeta_\gamma &= \frac{1}{2}\widehat{\kappa}_{3\gamma} = \frac{3}{2}\Gamma_3 v_{th}^2 \,, \qquad \zeta_N = \Gamma_3 v_{th}^4 \;. \end{aligned} \tag{5.23}$$

We provide more details about these integrals and the poles which contribute via their residues in the appendix C of this work.

It is evident from the expressions above that many of the couplings in the effective theory are related to each other by a series of relations. As we will elaborate in the next section, quite a few of these relations can be explained on general grounds using the fact that the bath correlators exhibit microscopic time-reversal invariance and obey KMS conditions.

6 Relations between effective couplings

In this section we discuss the origin of the relations between the cubic effective couplings given in eq. (5.23). As we show in the following two subsections, most of these relations are based on the following two general features of our model:

1. Microscopic time-reversal invariance in the bath,

2. KMS relations between bath correlators.

While discussing the consequences of these features, we will first give a general proof of the relations between the couplings, and then describe why our particular model satisfies the conditions that go into the proof. The arguments in the following two subsections will show that most of the relations between the effective couplings in our model are not just particular features of our model alone. Rather, they are generally valid for a broad class of systems, whenever the two conditions mentioned above are satisfied.

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6.1 Consequences of time-reversal invariance

First, let us discuss the consequence of microscopic time-reversal invariance in the bath. As we mentioned in the introduction, the implications of such microscopic time-reversal invariance for systems with multiple degrees of freedom were analysed by Onsager in [29, 30] where he showed that the quadratic effective couplings such as γ_{AB} and $\langle f^2_{AB}\rangle$ are symmetric under the exchange of the indices. The derivation of such reciprocal relations relied on the operators $\{\mathcal{O}_A\}$ being invariant under time-reversal. These relations were later generalised by Casimir [31] to the case where the operators $\{\mathcal{O}_A\}$ have the parities $\{\eta_{_A}\}$ under time-reversal. The corresponding generalisation of the Onsager relations is as follows:

$$\begin{aligned} \gamma_{AB} &= \eta_{_A}\eta_{_B}\ \gamma_{BA}, \\ \langle f^2_{AB}\rangle &= \eta_{_A}\eta_{_B}\ \langle f^2_{BA}\rangle. \end{aligned} \tag{6.1}$$

Here, we generalise the Onsager-Casimir reciprocal relations to cubic couplings in the OTO effective theory. We find that microscopic time-reversal invariance in the bath leads to certain relations between the 2-OTO couplings and the 1-OTO couplings which are derived below. We would like to point out that, unlike the scenario considered by Onsager and Casimir, our system (the Brownian particle) has a single degree of freedom. Nevertheless, the relations that we obtain between the couplings are based on principles similar to those for the reciprocal relations.

To keep the discussion precise, let us note that the operator $\mathcal{O}(t)$ is defined with respect to some particular reference point in time when it coincides with $\mathcal{O}$ (the Schrodinger-picture operator). While calculating the contribution of the correlators of this operator to the particle's dynamics, this reference point must be the instant t_0 at which the particle starts interacting with the bath. However, if the bath's initial state (described by the density matrix ρ_B) is time-translation invariant i.e

$$[\rho_B, H_B] = 0, \tag{6.2}$$

then such correlators are independent of the choice of the reference point and depend only on the intervals between the insertions. This is true, for instance, in our model where the bath is assumed to be in a thermal state.

For such an initial state of the bath, one can shift the reference point to $t = 0$ which can be chosen well into the domain of validity of the particle's effective theory. With respect to this new reference point, the bath's correlators with insertions at both positive and negative values of time are relevant for the particle's dynamics. In the following discussion we are going to assume time-translation invariance of the initial state of the bath and choose the reference point for the operators to be at the origin of time $t = 0$.

Let us now assume that the bath's dynamics has time-reversal invariance and the initial state of the bath respects this symmetry. Then, there exists an anti-linear and anti-unitary time-reversal operator $\mathbf{T}$ such that[14]

$$[\mathbf{T}, H_B] = 0, \quad \mathbf{T}\rho_B\mathbf{T}^\dagger = \rho_B\,. \tag{6.3}$$

[14] See [56] for a proof of the existence of such an operator.

At the level of correlators, this symmetry implies

$$\text{Tr}\Big[\mathbf{T}\rho_B\mathbf{T}^\dagger\mathbf{T}\mathcal{O}(t_1)\mathbf{T}^\dagger\cdots\mathbf{T}\mathcal{O}(t_n)\mathbf{T}^\dagger\Big]=\text{Tr}\Big[\rho_B\mathcal{O}(t_1)\cdots\mathcal{O}(t_n)\Big]^*. \tag{6.4}$$

Now, say the operator $\mathcal{O}$ has a definite time parity i.e.,

$$\mathbf{T}\mathcal{O}\mathbf{T}^\dagger=\eta_{\mathcal{O}}\mathcal{O}\,, \tag{6.5}$$

where $\eta_{\mathcal{O}}=\pm1$. The fact that $\mathbf{T}$ is an anti-linear operator which commutes with H_B, implies

$$\mathbf{T}\mathcal{O}(t)\mathbf{T}^\dagger=\eta_{\mathcal{O}}\mathcal{O}(-t)\,. \tag{6.6}$$

Inserting this transformation of $\mathcal{O}(t)$ into the equation (6.4) and imposing the time-reversal invariance of the initial state, we get

$$\langle\mathcal{O}(-t_1)\cdots\mathcal{O}(-t_n)\rangle=\eta_{\mathcal{O}}^n\langle\mathcal{O}(t_1)\cdots\mathcal{O}(t_n)\rangle^*. \tag{6.7}$$

As the operator $\mathcal{O}$ is Hermitian, the complex conjugation in the r.h.s. of the above equation implies reversing the order of the insertions. So we have

$$\langle\mathcal{O}(-t_1)\cdots\mathcal{O}(-t_n)\rangle=\eta_{\mathcal{O}}^n\langle\mathcal{O}(t_n)\cdots\mathcal{O}(t_1)\rangle. \tag{6.8}$$

Such relations, in general, imply that correlators with different OTO numbers (i.e., the number of minimum time-folds required to compute the correlators [10]) get related to each other. In case of 3-point functions, as was mentioned earlier, we have at most 2-OTO correlators. For three time instants $t_1>t_2>t_3$, the 2-OTO correlators are:

$$\langle\mathcal{O}(t_1)\mathcal{O}(t_3)\mathcal{O}(t_2)\rangle \text{ and } \langle\mathcal{O}(t_2)\mathcal{O}(t_3)\mathcal{O}(t_1)\rangle. \tag{6.9}$$

In both these correlators, we have 2 future turning point insertions: $\mathcal{O}(t_1)$ and $\mathcal{O}(t_2)$, and hence they are 2-OTO correlators. From (6.8) we can see that these correlators are related to their time-reversed counterparts as follows:

$$\begin{aligned}\langle\mathcal{O}(t_1)\mathcal{O}(t_3)\mathcal{O}(t_2)\rangle&=\eta_{\mathcal{O}}\langle\mathcal{O}(-t_2)\mathcal{O}(-t_3)\mathcal{O}(-t_1)\rangle,\\ \langle\mathcal{O}(t_2)\mathcal{O}(t_3)\mathcal{O}(t_1)\rangle&=\eta_{\mathcal{O}}\langle\mathcal{O}(-t_1)\mathcal{O}(-t_3)\mathcal{O}(-t_2)\rangle.\end{aligned} \tag{6.10}$$

The correlators in the r.h.s. of the above equations are 1-OTO correlators. So we see that all 2-OTO 3-point correlators of the bath get related to 1-OTO correlators.

It is natural to ask whether such relations between the bath's correlators lead to any relation between the 2-OTO couplings and the 1-OTO couplings in the effective theory of the particle. To answer this question, first notice that the relation (6.7) implies that, when all the frequencies are real, correlators of $\mathcal{O}$ in frequency space are either purely real or purely imaginary i.e.

$$\langle\mathcal{O}(\omega_1)\cdots\mathcal{O}(\omega_n)\rangle=\eta_{\mathcal{O}}^n\langle\mathcal{O}(\omega_1)\cdots\mathcal{O}(\omega_n)\rangle^*. \tag{6.11}$$

This reality property gets carried over to the connected parts i.e. the cumulants of these correlators. Equivalently, the spectral functions satisfy relations of the form

$$\rho[1\dots n]=\eta_{\mathcal{O}}^n\rho[1\dots n]^*. \tag{6.12}$$

g	$\mathcal{I}[g]$
$\overline{\lambda}_3$	$\frac{2}{m_p\omega_1\omega_3}\rho[123]$
$\overline{\kappa}_3$	$\frac{2}{m_p\omega_1\omega_3}\rho[321]$
$\overline{\lambda}_{3\gamma}$	$\frac{1}{im_p\omega_1\omega_3}\left(\frac{2}{\omega_1}-\frac{1}{\omega_3}\right)\rho[123]$
$\overline{\kappa}_{3\gamma}$	$\frac{1}{im_p\omega_1\omega_3}\left(\frac{1}{\omega_1}-\frac{2}{\omega_3}\right)\rho[321]$
ζ_γ	$\frac{1}{(2m_p\omega_1\omega_3)^2}\Big((2\omega_1-\omega_3)\ \rho[321_+]+(\omega_3-2\omega_1)\ \rho[132_+]+(\omega_1+\omega_3)\ \rho[123_+]\Big)$
$\widehat{\kappa}_{3\gamma}-\zeta_\gamma$	$\frac{1}{(2m_p\omega_1\omega_3)^2}\Big((\omega_1+\omega_3)\ \rho[321_+]+(2\omega_3-\omega_1)\ \rho[132_+]+(2\omega_3-\omega_1)\ \rho[123_+]\Big)$
$\widehat{\kappa}_{3\gamma}$	$\frac{3}{(2m_p\omega_1\omega_3)^2}\Big(\omega_1\ \rho[321_+]+(\omega_3-\omega_1)\ \rho[132_+]+\omega_3\ \rho[123_+]\Big)$
ζ_μ	$\frac{1}{2im_p^2\omega_1\omega_3}\Big(\rho[123_+]-\rho[321_+]+\rho[132_+]\Big)$
$2\zeta_N+\frac{\hbar^2}{8m_p^2}\left(\overline{\lambda}_3-\overline{\kappa}_3\right)$	$\frac{1}{8m_p^3\omega_1\omega_3}\Big(\rho[12_+3_+]+\rho[32_+1_+]\Big)$

Table 7. Coupling doublets/singlets under microscopic time-reversal (general environment).

We use this property of the cumulants in frequency space to derive relations between the 2-OTO and 1-OTO couplings.

From the expressions of the cubic couplings, we see that the couplings can be divided into doublets and singlets as shown in the table 7. We will show that the pair of couplings in each doublet are related to each other due to time-reversal invariance. On the other hand, time-reversal maps the singlets to themselves upto a factor of $\eta_{\mathcal{O}}$. So when $\eta_{\mathcal{O}}=1$, these relations are trivial. But when $\eta_{\mathcal{O}}=-1$, these relations imply that these singlets vanish.

As the couplings mentioned in table 7 are all real, their complex conjugates are equal to them. This can be used to obtain alternative expressions for these couplings by complex conjugating the integrals. Such a complex conjugation in the frequency space maps the contour of integration from $\mathcal{C}_3$:

$$\int_{\mathcal{C}_3}\equiv\int_{-\infty-i\epsilon_1}^{\infty-i\epsilon_1}\frac{d\omega_1}{2\pi}\int_{-\infty+i\epsilon_1-i\epsilon_3}^{\infty+i\epsilon_1-i\epsilon_3}\frac{d\omega_2}{2\pi}\int_{-\infty+i\epsilon_3}^{\infty+i\epsilon_3}\frac{d\omega_3}{2\pi} \tag{6.13}$$

to $\mathcal{C}_3^*$ where the frequencies run over the following values:

$$\int_{\mathcal{C}_3^*}\equiv\int_{-\infty+i\epsilon_1}^{\infty+i\epsilon_1}\frac{d\omega_1^c}{2\pi}\int_{-\infty-i\epsilon_1+i\epsilon_3}^{\infty-i\epsilon_1+i\epsilon_3}\frac{d\omega_2^c}{2\pi}\int_{-\infty-i\epsilon_3}^{\infty-i\epsilon_3}\frac{d\omega_3^c}{2\pi}\,. \tag{6.14}$$

Note that the integration over ω_1^c in the $\mathcal{C}_3^*$ contour runs just above the real axis, exactly like the integration over ω_3 in the $\mathcal{C}_3$ contour. Similarly, integration over ω_3^c in the $\mathcal{C}_3^*$ contour runs just below the real axis exactly like the integration over ω_1 in the $\mathcal{C}_3$ contour. Therefore, under the following redefinitions:

$$\omega_1\equiv\omega_3^c,\quad \omega_2\equiv\omega_2^c,\quad \omega_3\equiv\omega_1^c, \tag{6.15}$$

the contour of integration gets mapped back to $\mathcal{C}_3$ with ϵ_1 and ϵ_3 exchanged (and the imaginary part of ω_2 reversed). This exchange of ϵ_1 and ϵ_3 and the concomitant reversal of the imaginary part of ω_2 can be undone by a contour deformation, provided our integrands have no ω_2 discontinuities near real axis (as required for the validity of Markovian approximation). To conclude, assuming appropriate analyticity in ω_2, the complex conjugation and the above redefinition leave $\mathcal{C}_3$ contour invariant.

Now, let us turn to how the integrands are modified under the above operation. Notice that each term in the integrand has the following form: it is a product of a rational function of the frequencies and the bath cumulants. The modification of the rational functions is simple: the rational functions are modified by complex conjugating them and then performing the above frequency redefinition. This has the effect of replacing any explicit i by $(-i)$ and exchanging ω_1 and ω_3.

Turning to the cumulants, time-reversal invariance implies that, when the frequencies are real, these cumulants are either purely real or purely imaginary depending on their time-reversal parity $\eta_{\mathcal{O}}$. Thus, the cumulants can be complex-conjugated by conjugating the frequencies in the argument of the cumulants followed by a multiplication by $\eta_{\mathcal{O}}$. The frequency redefinition above then results in the exchange of ω_1 and ω_3 in the arguments. To summarise, the modified integrands are obtained from the original ones by the following rules:

- Replace i by $-i$ in the coefficients,
- Exchange ω_1 and ω_3 in the rational functions and the cumulants,
- Multiply by $\eta_{\mathcal{O}}$.

After re-expressing the couplings in terms of these modified integrals over the same contour $\mathcal{C}_3$, one can compare them with the expression given in table 7 and find the following relations:

$$\begin{aligned}&\overline{\kappa}_3=\eta_{\mathcal{O}}\overline{\lambda}_3\,,\quad \overline{\kappa}_{3\gamma}=\eta_{\mathcal{O}}\overline{\lambda}_{3\gamma}\,,\quad \widehat{\kappa}_{3\gamma}-\zeta_\gamma=\eta_{\mathcal{O}}\zeta_\gamma\,,\quad \zeta_\mu=\eta_{\mathcal{O}}\zeta_\mu\,,\\ &2\zeta_N+\frac{\hbar^2}{8m_p}(\overline{\lambda}_3-\overline{\kappa}_3)=\eta_{\mathcal{O}}\left(2\zeta_N+\frac{\hbar^2}{8m_p}(\overline{\lambda}_3-\overline{\kappa}_3)\right).\end{aligned} \tag{6.16}$$

This, in turn, implies

$$\begin{aligned}&\overline{\kappa}_3=\eta_{\mathcal{O}}\overline{\lambda}_3\,,\quad \overline{\kappa}_{3\gamma}=\eta_{\mathcal{O}}\overline{\lambda}_{3\gamma}\,,\quad \widehat{\kappa}_{3\gamma}=(1+\eta_{\mathcal{O}})\zeta_\gamma\,,\quad \widehat{\kappa}_{3\gamma}=\eta_{\mathcal{O}}\widehat{\kappa}_{3\gamma}\,,\\ &\zeta_\mu=\eta_{\mathcal{O}}\zeta_\mu\,,\quad 2(1-\eta_{\mathcal{O}})\zeta_N=(\eta_{\mathcal{O}}-1)\frac{\hbar^2}{8m_p}(\overline{\lambda}_3-\overline{\kappa}_3)\,.\end{aligned} \tag{6.17}$$

When the operator $\mathcal{O}$ is even under time-reversal i.e. when $\eta_{\mathcal{O}}=1$, the last two relations are trivial. The other relations reduce to

$$\overline{\kappa}_3=\overline{\lambda}_3\,,\quad \overline{\kappa}_{3\gamma}=\overline{\lambda}_{3\gamma}\,,\quad \widehat{\kappa}_{3\gamma}=2\zeta_\gamma\,. \tag{6.18}$$

On the other hand, when the operator $\mathcal{O}$ is odd under time-reversal i.e. when $\eta_{\mathcal{O}}=-1$, then the relations in (6.16) reduce to

$$\overline{\kappa}_3=-\overline{\lambda}_3\,,\quad \overline{\kappa}_{3\gamma}=-\overline{\lambda}_{3\gamma}\,,\quad \widehat{\kappa}_{3\gamma}=0\,,\quad \zeta_\mu=0\,,\quad \zeta_N=-\frac{\hbar^2}{16m_p}(\overline{\lambda}_3-\overline{\kappa}_3)\,. \tag{6.19}$$

6.2 Time-reversal invariance of the bath in qXY model

We found in the preceding discussion that the bath needs to satisfy the following conditions for the relations given in equation (6.18) to hold true:

1. Time-translation invariance of the initial state,
2. Time-reversal invariance in the dynamics,
3. Time-reversal invariance of the initial state,
4. Time-reversal invariance of the operator that couples to the particle.

Let us check whether these conditions are satisfied in our model one by one.

As we have already mentioned, the initial state of the bath is a thermal state and hence it is invariant under time-translations.

To see the time-reversal invariance in the bath's dynamics, first note that the bath consists of two sets of harmonic oscillators. We can denote the lowering and raising operators of these oscillators by a_i and $a_i^\dagger$ for the X-type oscillators and by b_j and $b_j^\dagger$ for the Y-type oscillators. Therefore, the Hamiltonian of the bath is given by

$$H_B = \sum_i \hbar\mu_{x,i}\left(a_i^\dagger a_i + \frac{1}{2}\right) + \sum_j \hbar\mu_{y,j}\left(b_j^\dagger b_j + \frac{1}{2}\right). \tag{6.20}$$

Now, the action of the time-reversal operator on the raising and lowering operators is as follows:

$$\begin{aligned} \mathbf{T}a_i\mathbf{T}^\dagger &= a_i, \quad & \mathbf{T}b_j\mathbf{T}^\dagger &= b_j, \\ \mathbf{T}a_i^\dagger\mathbf{T}^\dagger &= a_i^\dagger, \quad & \mathbf{T}b_j^\dagger\mathbf{T}^\dagger &= b_j^\dagger. \end{aligned} \tag{6.21}$$

Using these transformations of the raising and lowering operators under time-reversal and the form of H_B given in (6.20) we see that

$$\mathbf{T}H_B\mathbf{T}^\dagger = H_B \implies [\mathbf{T}, H_B] = 0, \tag{6.22}$$

which means that time-reversal is a symmetry of the dynamics.

Now, the initial state is a thermal state i.e.

$$\rho_B = \frac{1}{Z_B} e^{-\frac{H_B}{k_B T}}. \tag{6.23}$$

Therefore, the commutation of the time-reversal operator with the Hamiltonian also implies

$$\mathbf{T}\rho_B\mathbf{T}^\dagger = \rho_B \tag{6.24}$$

i.e. the initial state is invariant under time-reversal.

Finally, the operator that couples to the particle is

$$\mathcal{O} = \sum_i g_{x,i}X_i + \sum_j g_{y,j}Y_j + \sum_{i,j} g_{xy,ij}X_iY_j. \tag{6.25}$$

Here the positions of the oscillators are give by

$$
\begin{aligned}
X_i &= \sqrt{\frac{\hbar}{2m_{x,i}\mu_{x,i}}}(a_i + a_i^\dagger), \\
Y_j &= \sqrt{\frac{\hbar}{2m_{y,j}\mu_{y,j}}}(b_j + b_j^\dagger).
\end{aligned}
\tag{6.26}
$$

Therefore, using the transformations in (6.21) we have

$$
\begin{aligned}
\mathbf{T}X_i\mathbf{T}^\dagger &= X_i, \\
\mathbf{T}Y_j\mathbf{T}^\dagger &= Y_j.
\end{aligned}
\tag{6.27}
$$

This implies that the operator $\mathcal{O}$ is invariant under time-reversal i.e.

$$
\mathbf{T}\mathcal{O}\mathbf{T}^\dagger = \mathcal{O}\,. \tag{6.28}
$$

So, all the conditions necessary for the relations (6.18) between the effective couplings are satisfied in our model.

Example of a time-reversal odd operator coupling to the particle. We can slightly modify the qXY model to introduce a piece in the operator $\mathcal{O}$ that is odd under time-reversal. For this, consider the following particle-bath interaction:

$$
L_{\text{int}} = \left(\sum_i g_{x,i}X_i + \sum_i g_{y,j}Y_j\right)q - \sum_i \widetilde{g}_{xy,ij}X_iY_j\dot{q}\,. \tag{6.29}
$$

Integrating by parts, we see that the operator that couples to the particle's position is

$$
\mathcal{O} = \sum_i g_{x,i}X_i + \sum_i g_{y,j}Y_j + \sum_i \widetilde{g}_{xy,ij}\dot{X}_iY_j + \sum_i \widetilde{g}_{xy,ij}X_i\dot{Y}_j\,. \tag{6.30}
$$

Here, the operators $\dot{X}_i$ and $\dot{Y}_j$ are odd under time-reversal i.e.

$$
\mathbf{T}\dot{X}_i\mathbf{T}^\dagger = -\dot{X}_i\,, \quad \mathbf{T}\dot{Y}_j\mathbf{T}^\dagger = -\dot{Y}_j\,. \tag{6.31}
$$

The thermal correlators of 3 point functions $\mathcal{O}$ receive contributions only from terms of the form:

$$
\begin{aligned}
\langle\mathcal{O}(t_1)\mathcal{O}(t_2)\mathcal{O}(t_3)\rangle &= \sum_{i,j} g_{x,i}g_{y,j}\widetilde{g}_{xy,ij}\langle X_i(t_1)Y_j(t_2)\dot{X}_i(t_3)Y_j(t_3)\rangle \\
&\quad + \sum_{i,j} g_{x,i}g_{y,j}\widetilde{g}_{xy,ij}\langle X_i(t_1)Y_j(t_2)X_i(t_3)\dot{Y}_j(t_3)\rangle + \cdots
\end{aligned}
\tag{6.32}
$$

All such terms are correlators with 3 position operators (which are time-reversal even) and one velocity operator (which is time-reversal odd). Therefore, the overall correlator satisfies the following relation:

$$
\langle\mathcal{O}(t_1)\mathcal{O}(t_2)\mathcal{O}(t_3)\rangle = -\langle\mathcal{O}(-t_3)\mathcal{O}(-t_2)\mathcal{O}(-t_1)\rangle\,. \tag{6.33}
$$

This, in turn, means that the relations in equation (6.19) are satisfied in this model.

A brief comment on couplings of quartic and higher degree terms. The fact that all the 2-OTO couplings get related to 1-OTO couplings is not generally true for higher degree terms in the particle's effective action. To see this consider the quartic couplings which receive contributions from 4-point cumulants of the operator $\mathcal{O}$. Such 4-point functions can again be at most 2-OTO. But not all of these 2-OTO correlators are related to 1-OTO correlators by time-reversal. For instance, consider the following correlator:

$$\langle \mathcal{O}(t_1)\mathcal{O}(t_3)\mathcal{O}(t_2)\mathcal{O}(t_4)\rangle \tag{6.34}$$

where $t_1 > t_2 > t_3 > t_4$. This is a 2-OTO correlator as there are 2 future-turning point insertions in it, viz., $\mathcal{O}(t_1)$ and $\mathcal{O}(t_2)$.

Under time-reversal, this gets related to the another correlator as follows:

$$\langle \mathcal{O}(t_1)\mathcal{O}(t_3)\mathcal{O}(t_2)\mathcal{O}(t_4)\rangle = \langle \mathcal{O}(-t_4)\mathcal{O}(-t_2)\mathcal{O}(-t_3)\mathcal{O}(-t_1)\rangle\,. \tag{6.35}$$

Now the correlator on r.h.s. of the above equation also has two future-turning point insertions, viz., $\mathcal{O}(-t_4)$ and $\mathcal{O}(-t_3)$. Thus, this is a genuine 2-OTO correlator which is not related to a 1-OTO correlator under time-reversal. Such 2-OTO correlators may lead to genuinely new 2-OTO quartic couplings in the effective dynamics of the particle, unrelated to the 1-OTO coupling [57].

6.3 Consequence of KMS relations: generalised fluctuation-dissipation relations

We have already seen that, at the high temperature limit, the KMS relations between thermal 2-point functions of the operator $\mathcal{O}$ leads to a relation between the damping coefficient γ of the particle and the strength of the additive noise $\langle f^2\rangle$ that it experiences:

$$\langle f^2\rangle = 2\gamma v_{th}^2\,. \tag{6.36}$$

This is the fluctuation-dissipation relation that was originally discovered through the studies of Brownian motion by Einstein, Smoluchowski and Sutherland. An analogous relation in electrical circuits was discovered by Johnson [13] and was theoretically derived by Nyquist [14]. A general proof of such relations was worked out by Callen and Welton in [15] which was further generalised by Stratonovich [27].

All these fluctuation-dissipation relations are relations between what we now understand to be 1-OTO couplings and are rooted in the KMS relations between the 1-OTO correlators of the bath. However, in [11] it was pointed out that the KMS relations can also relate 2-OTO correlators to 1-OTO correlators. For example, consider the following 2-OTO correlator:

$$\langle \mathcal{O}(t_1)\mathcal{O}(t_3)\mathcal{O}(t_2)\rangle \tag{6.37}$$

where $t_1 > t_2 > t_3$. By KMS relations, this 2-OTO correlator is related via analytic continuation to a 1-OTO correlator as follows:

$$\langle \mathcal{O}(t_1)\mathcal{O}(t_3)\mathcal{O}(t_2)\rangle = \langle \mathcal{O}(t_2 - i\beta)\mathcal{O}(t_1)\mathcal{O}(t_3)\rangle. \tag{6.38}$$

It is natural to wonder what imprint do such relations have on the effective dynamics of the particle. Do they lead to generalisations of the relation in (6.36)? If so, then do such relations connect 2-OTO couplings with the 1-OTO ones?

In section 3, we indeed saw such a relation between the 2-OTO coupling $\widehat{\kappa}_{3\gamma}$ and the 1-OTO coupling ζ_N at high temperature:

$$\zeta_N + \frac{\hbar^2}{16m_p^2}(\overline{\lambda}_3 - \overline{\kappa}_3) = \frac{1}{3}\widehat{\kappa}_{3\gamma}v_{th}^2. \tag{6.39}$$

As we mentioned there, this followed from using the KMS relations between the 3-point thermal correlators of the bath to express the couplings in terms of the spectral functions $\rho[123]$ and $\rho[321]$, and keeping only the leading order term in β-expansion. Thus, this relation can be considered to be a generalisation of the fluctuation-dissipation relation between the quadratic couplings in (6.36).

Microscopic time-reversal invariance with $\eta_{\mathcal{O}} = 1$ implies $\overline{\lambda}_3 = \overline{\kappa}_3$ and $\widehat{\kappa}_{3\gamma} = 2\zeta_\gamma$. In this case, the above relation reduces to

$$\zeta_N = \frac{1}{3}\widehat{\kappa}_{3\gamma}v_{th}^2 = \frac{2}{3}\zeta_\gamma v_{th}^2. \tag{6.40}$$

This is hence a relation between the coefficient ζ_N of the cubic non-gaussian term in the probability distribution of the noise and the jitter ζ_γ in the damping coefficient of the particle.

On the other hand, if $\eta_{\mathcal{O}} = -1$, then $\widehat{\kappa}_{3\gamma} = 0$ along with

$$\zeta_N = -\frac{\hbar^2}{16m_p^2}(\overline{\lambda}_3 - \overline{\kappa}_3)\,. \tag{6.41}$$

This implies that the relation in (6.39) is trivially satisfied.

Apart from these relations, we also see another analogous fluctuation-dissipation relation in our model which is given by

$$\zeta_\mu = v_{\rm th}^2\overline{\kappa}_{3\gamma} = v_{\rm th}^2\overline{\lambda}_{3\gamma}\,. \tag{6.42}$$

As we discussed in the last subsection, the relation between $\overline{\kappa}_{3\gamma}$ and $\overline{\lambda}_{3\gamma}$ in the second half of the above equation is a consequence of time-reversal invariance in the bath. This equation relates the cubic 1-derivative anharmonicity in the particle's motion to the jitter in its frequency.

As yet, we do not know how generic this relation is. However, we suspect that, with some assumptions about the properties of the spectral functions, a general relation of the following form can be proven:

$$\zeta_\mu = \frac{v_{\rm th}^2}{2}\,(\overline{\kappa}_{3\gamma} + \overline{\lambda}_{3\gamma})\,. \tag{6.43}$$

Such a relation is consistent with time-reversal invariance, since for the operator $\mathcal{O}$ having a definite parity $\eta_{\mathcal{O}}$, the two sides of the above equation transform similarly under time-reversal (see the relations in (6.16)). When $\eta_{\mathcal{O}} = -1$ both sides of the equation are equal

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to zero and hence the relation (6.43) is trivially true. Moreover, we find it to be satisfied in the qXY model and a variety of related models. Hence, we expect it to be true for a broad class of models than the one studied in this work. It will be interesting to check this expectation and determine the exact conditions which are required for this relation to hold.

7 Conclusion and discussion

In this paper, we have constructed an effective theory of a Brownian particle which goes beyond the standard Langevin dynamics. We remind the reader that the standard Langevin theory describes a Brownian particle subject to linear damping and a Gaussian thermal noise. The effective theory described in this paper includes in addition anharmonic couplings $\overline{\lambda}_3$ and $\overline{\lambda}_{3\gamma}$ along with a thermal jitter ζ_μ in the frequency and a jitter ζ_γ in the damping constant. Apart from these parameters and the usual Langevin couplings, this theory also contains a parameter ζ_N which is the strength of the non-Gaussianity in the thermal noise experienced by the particle.

When out of time ordered correlations (or more specifically, 2-OTO correlators) transmitted from the bath are kept track of, one has to add three more OTO couplings $\overline{\kappa}_3, \overline{\kappa}_{3\gamma}$ and $\widehat{\kappa}_{3\gamma}$ which are related by time-reversal to the standard (1-OTO) couplings $\overline{\lambda}_3, \overline{\lambda}_{3\gamma}$ and ζ_γ respectively. To demonstrate how these couplings affect the dynamics of the particle, we have expressed the classical limits of the particle's correlators in terms of these couplings. We find that the OTO couplings show up in the out of time ordered nested Poisson brackets of the particle.

We explore the constraints imposed by microscopic time-reversal invariance of the (particle+bath) combined system on the effective theory of the particle. Such an invariance of the overall dynamics of the combined system under time-reversal holds when the interaction term between the particle and the bath is even under this transformation. This fixes the OTO cubic effective couplings in terms of the parameters in the nonlinear Langevin theory via the relations

$$\overline{\kappa}_3 = \overline{\lambda}_3 \,, \quad \overline{\kappa}_{3\gamma} = \overline{\lambda}_{3\gamma} \,, \quad \widehat{\kappa}_{3\gamma} = 2\zeta_\gamma \,.$$

These relations between the cubic couplings are the generalisations of the well known Onsager-Casimir reciprocal relations that originate from the microscopic time-reversal invariance of the combined system.

Since the bath is in a thermal state, the bath correlators satisfy the KMS relations. In the high temperature limit, these further give rise to a generalised fluctuation-dissipation relation between the 2-OTO cubic derivative coupling $\widehat{\kappa}_{3\gamma}$ and the 1-OTO cubic non-derivative coupling ζ_N. Combining time-reversal invariance and the generalised fluctuation-dissipation relation, the coefficient of the thermal jitter ζ_γ in the damping term of the non-linear Langevin equation gets related to the coefficient of non-Gaussianity ζ_N of thermal noise.

To provide a concrete model where these general results are justified, we have constructed an OTO-effective theory of a Brownian particle interacting with a dissipative thermal bath composed of two sets of harmonic oscillators. To this, we add a small 3-body interaction coupling the particle to two other oscillators, one from each set. For this model, we show that all the bath correlators decay exponentially at late times, leading to a local non-unitary effective theory for the particle.

Working out the effective couplings of the particle in this model, we find that the above mentioned relations between these couplings are indeed satisfied. Furthermore, in this model, we find another fluctuation-dissipation type relation between the strength (ζ_μ) of the thermal jitter in the frequency and the coefficient ($\overline{\lambda}_{3\gamma}$) of the anharmonic term with a single time derivative.

An immediate future direction would be to check whether the relations between the effective couplings (arising from time-reversal invariance and thermality of the bath) are valid at higher orders in the system-bath coupling. Moreover, it will be interesting to explore quartic and higher order terms in the effective action of the particle. At the quartic order, there will be genuinely new 2-OTO effective couplings that are not fixed in terms of the Schwinger-Keldysh 1-OTO effective couplings by time-reversal invariance.

It will be useful to understand the generalisation of fluctuation-dissipation relations and the consequence of time reversal invariance in the quartic case. Furthermore, such an extension to quartic terms in the effective theory may lead to observing chaotic motion of the particle.[15] If that is indeed the case, then it may be insightful to connect the already existing studies on quantum chaos based on OTOCs with this example.

Our techniques can potentially be extended to the context of quantum optics [58, 59]. As a toy model one can consider an atom that is interacting electromagnetically with a gas of photons [60–63]. One can write down an effective theory for the atom and calculate its OTOCs. Such predictions for the behaviour of OTOCs may be verified with development in experimental techniques to measure them [64–69]. Moreover, our prediction of a generalised fluctuation-dissipation relation between the thermal jitter in the damping and the non-gaussianity in the noise may be testable in such a setup.

Another possible extension is to 1-D spin chains in a thermal environment [70–72]. It will be interesting to explore the possibility of writing a similar effective theory of the chain (or a part of it) which would aid in studying the thermalisation of its OTOCs and comparing them with those of time-ordered correlators. Such comparisons may be useful in classifying systems according to the behaviour of their OTOCs.

In this work, we saw that the OTO couplings are not just a feature of the quantum mechanical theory of the particle, but they show up in out of time ordered Poisson brackets in the classical limit as well (Similar classical limits of OTOCs have been discussed in [73–76]). A stochastic interpretation of this classical OTO behaviour would be useful in understanding the significance of such OTOCs in the quantum mechanical framework (where OTO dynamics has been mostly studied up till now) as well as devising experiments to measure them. In this context, it will also be interesting to extend the idea of decoherence of quantum systems to their out of time ordered dynamics.

[15]Such chaotic motion is seen, for example, in periodically driven anharmonic oscillators such as the Van der Pol oscillator and the Duffing oscillator.

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One potentially useful way to study the OTO dynamics of open systems connected to holographic baths would be to use the AdS/CFT correspondence for mapping this to a problem in gravity. It would be nice to systematically derive the spectral functions from gravity following [77–79] and calculate OTOCs in AdS gravity theories.

Another interesting direction to pursue would be to systematically develop diagrammatic methods for OTO perturbation theory [55]. This would provide a simpler way of computing OTOCs of the system. It would also be nice to derive cutting rules for OTOCs along the lines of [51, 80, 81].

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A Dimensional analysis

In this appendix, we list the dimensions of various couplings etc. that appear in this work. Though we do not do so in this work, many of the computations described in this work are simplified by setting $\hbar = m_p = 1$, which is equivalent to counting dimensions by setting $M = T^0$ and $L^2 = T$.

- Dimensions of position and noise variables:

$$[q] = L\,, \quad [\mathcal{N}] = L^{-1}T\,. \tag{A.1}$$

- Dimensions of couplings:

$$\begin{aligned} &[m_p] = M\,, \quad [F] = LT^{-2}\,, \qquad [\gamma] = T^{-1}\,, \quad [\bar{\mu}^2] = T^{-2}\,, \\ &[\zeta_\gamma] = [\widehat{\kappa}_{3\gamma}] = LT^{-2}\,, \qquad [\zeta_\mu] = LT^{-3}\,, \\ &[\overline{\lambda}_3] = [\overline{\kappa}_3] = L^{-1}T^{-2}\,, \qquad [\overline{\lambda}_{3\gamma}] = [\overline{\kappa}_{3\gamma}] = L^{-1}T^{-1}\,. \end{aligned} \tag{A.2}$$

- Dimensions of Noise parameters:

$$[\langle f^2\rangle] = L^2T^{-3}\,, \quad [\zeta_N] = L^3T^{-4}\,, \quad [Z_I] = L^2T^{-1}. \tag{A.3}$$

- Dimensions of Spectral functions:

$$[\rho[12]] = MT^{-1}\,, \qquad [\rho[12_+]] = M^2L^2T^{-2}\,, \tag{A.4}$$

$$[\rho[123]] = ML^{-1}T^{-1}\,, \quad [\rho[123_+]] = [\rho[12_+3]] = M^2LT^{-2}\,, \quad [\rho[12_+3_+]] = M^3L^3T^{-3}\,. \tag{A.5}$$

- Dimensions of System Bath couplings

$$[g_x] = [g_y] = MT^{-2}\,, \quad [g_{xy}] = ML^{-1}T^{-2}\,. \tag{A.6}$$

- Dimensions of System Bath coupling distribution functions

$$\begin{aligned} &\left[\left\langle\!\left\langle \frac{g_x^2}{m_x} \right\rangle\!\right\rangle\right] = \left[\left\langle\!\left\langle \frac{g_y^2}{m_y} \right\rangle\!\right\rangle\right] = MT^{-3}\,, \quad \left[\left\langle\!\left\langle \frac{g_{xy}^2}{m_x m_y} \right\rangle\!\right\rangle\right] = L^{-2}T^{-2}\,, \\ &\left[\left\langle\!\left\langle \frac{g_x g_y g_{xy}}{m_x m_y} \right\rangle\!\right\rangle\right] = ML^{-1}T^{-4}. \end{aligned} \tag{A.7}$$

- Dimensions of Spectral function parameters

$$[\gamma_x] = [\gamma_y] = [\Omega] = T^{-1}\,, \quad [\Gamma_{xy}] = L^{-2}T^2\,, \quad [\Gamma_3] = L^{-1}\,. \tag{A.8}$$

- Dimensions of Thermal parameters

$$[\beta] = T\,, \quad [v_{th}^2] = L^2T^{-2}\,. \tag{A.9}$$

B Structure of 1PI effective action

The dynamics of the combined system of particle and bath obeys unitary time evolution. Consequently the 1-PI effective action of the Brownian particle has to satisfy the following conditions [9, 10, 12, 53, 82].

1. *Collapse conditions.*

   The contour-ordered correlator of the particle picks up a sign when one slides an operator insertion across a turning point of the time contour. This is true, provided as we move from one leg to another, we return to the same real time instant and there is no operator insertion in between. The sign change is due to our choice of putting an extra minus sign in the q's on even legs over the usual convention followed in discussions on the Schwinger-Keldysh formalism [10, 45, 82, 83].

   At the level of the 1-PI effective action this implies that under identification of the degrees of freedom on any two consecutive legs, the effective action becomes independent of the common degree of freedom of these two legs i.e. the effective action becomes independent of $\tilde{q}$ under any one of the following identifications [9, 10, 12, 82]:

   (a) **(1,2) collapse:** $q_1 = -q_2 = \tilde{q}$

   (b) **(2,3) collapse:** $q_2 = -q_3 = \tilde{q}$

   (c) **(3,4) collapse:** $q_3 = -q_4 = \tilde{q}$.

   Under any of these collapses, the 1-PI effective action reduces to the Schwinger-Keldysh 1-PI effective action for the degrees of freedom on the remaining two legs. These rules further impose the following relations between the couplings [9, 53]:

$$Z_\Delta = \mathrm{Im}[Z], \; m_\Delta^2 = \mathrm{Im}[m^2], \; \mathrm{Im}[\lambda_3 + 3\sigma_3] = 0\,. \tag{B.1}$$

2. *Reality condition.*

   A correlation function of hermitian operators gets complex conjugated when the operators are inserted in the reverse order within the correlation function. This is assured when the 1PI effective action picks up a negative sign under simultaneous complex conjugation of all the effective couplings and exchange of the degrees of freedom in the following way:

$$q_1 \leftrightarrow -q_4\,, \; q_2 \leftrightarrow -q_3\,.$$

   This constraint is satisfied when the effective couplings $\widehat{F}$, $\widehat{\gamma}$ and κ_3 are real.

C Contour integrals and poles for the effective couplings

In this appendix, we will present some more details about the contour integrals that have to be evaluated to obtain the effective couplings in our model.

For each coupling, we write down the explicit integrals over which double contour integration[16] needs to be performed. In table 8, we tabulate the poles in each integrand whose residues add to give the final coupling as we perform the contour integral first over ω_1 and then over ω_3. We will write our integrands performing an integral over the frequency delta function:

$$\int_{\mathcal{C}_3'} \equiv \int_{\mathcal{C}_3} 2\pi\delta(\omega_1+\omega_2+\omega_3)\,. \tag{C.1}$$

The explicit integrals for the cubic couplings are given by the following expressions:

$$\begin{aligned}
\overline{\lambda}_3 &= -4\Gamma_3\int_{\mathcal{C}_3'}\frac{(\omega_1-\omega_2)}{\omega_1}\left(1-\frac{\omega_1\omega_2}{\Omega^2}\right)\times\prod_{k=1}^{3}\frac{\Omega^2}{\omega_k^2+\Omega^2} = -\frac{3\Gamma_3\Omega^2}{2}\,,\\
\overline{\kappa}_3 &= -4\Gamma_3\int_{\mathcal{C}_3'}\frac{(\omega_3-\omega_2)}{\omega_3}\left(1-\frac{\omega_2\omega_3}{\Omega^2}\right)\times\prod_{k=1}^{3}\frac{\Omega^2}{\omega_k^2+\Omega^2} = -\frac{3\Gamma_3\Omega^2}{2}\,,\\
\overline{\lambda}_{3\gamma} &= 2i\Gamma_3\int_{\mathcal{C}_3'}\frac{(2\omega_3-\omega_1)(\omega_1-\omega_2)}{\omega_1^2\omega_3}\left(1-\frac{\omega_1\omega_2}{\Omega^2}\right)\times\prod_{k=1}^{3}\frac{\Omega^2}{\omega_k^2+\Omega^2} = -2\Gamma_3\Omega\,,\\
\overline{\kappa}_{3\gamma} &= 2i\Gamma_3\int_{\mathcal{C}_3'}\frac{(\omega_3-2\omega_1)(\omega_3-\omega_2)}{\omega_1\omega_3^2}\left(1-\frac{\omega_2\omega_3}{\Omega^2}\right)\times\prod_{k=1}^{3}\frac{\Omega^2}{\omega_k^2+\Omega^2} = -2\Gamma_3\Omega\,,
\end{aligned}\tag{C.2}$$

$$\begin{aligned}
\zeta_\gamma &= -\Gamma_3 v_{th}^2\int_{\mathcal{C}_3'}\left[\left(\frac{2}{\omega_1^2}+\frac{3}{\omega_1\omega_3}\right)\left(1+\frac{\omega_3^2}{\Omega^2}\right)-\left(\frac{2}{\omega_3^2}+\frac{3}{\omega_1\omega_3}\right)\left(1+\frac{\omega_1^2}{\Omega^2}\right)\right.\\
&\qquad\left.-\frac{3}{\omega_1\omega_3}\left(3+2\frac{\omega_3^2+\omega_1\omega_3+\omega_1^2}{\Omega^2}\right)\right]\times\prod_{k=1}^{3}\frac{\Omega^2}{\omega_k^2+\Omega^2} = \frac{3}{2}\Gamma_3 v_{th}^2\,,
\end{aligned}\tag{C.3}$$

$$\begin{aligned}
\widehat{\kappa}_{3\gamma} &= 6\Gamma_3 v_{th}^2\int_{\mathcal{C}_3'}\frac{1}{\omega_1\omega_3}\left(3+2\frac{\omega_3^2+\omega_1\omega_3+\omega_1^2}{\Omega^2}\right)\times\prod_{k=1}^{3}\frac{\Omega^2}{\omega_k^2+\Omega^2} = 3\Gamma_3 v_{th}^2\,,\\
\zeta_N &= 2\Gamma_3 v_{th}^4\int_{\mathcal{C}_3'}\frac{1}{\omega_1\omega_3}\left(3+2\frac{\omega_3^2+\omega_1\omega_3+\omega_1^2}{\Omega^2}\right)\times\prod_{k=1}^{3}\frac{\Omega^2}{\omega_k^2+\Omega^2} = \Gamma_3 v_{th}^4\,,\\
\zeta_\mu &= \frac{2\Gamma_3 v_{th}^2}{im_p\Omega^2}\int_{\mathcal{C}_3'}\frac{(\omega_1-\omega_3)}{\omega_1\omega_3}\left(2\Omega^2+\omega_1\omega_3+2(\omega_1^2+\omega_1\omega_3+\omega_3^2)\right)\times\prod_{k=1}^{3}\frac{\Omega^2}{\omega_k^2+\Omega^2}\\
&= -2\Gamma_3\Omega v_{th}^2\,.
\end{aligned}\tag{C.4}$$

[16] The integral over ω_2 is trivial due to the presence of a delta function coming from energy conservation. So, one has to integrate over the frequencies ω_1 and ω_3.

	Poles in ω_1	Poles in ω_3		Poles in ω_1	Poles in ω_3
$\bar{\kappa}_3$	$i\Omega + i\epsilon_1$	$i\Omega - i\epsilon_2$	$\overline{\lambda}_3$	$i\Omega + i\epsilon_1$	$i\Omega - i\epsilon_2$
	$-\omega_3 + i\Omega + i\epsilon_1 - i\epsilon_2$	$2i\Omega - i\epsilon_2$		$-\omega_3 + i\Omega + i\epsilon_1 - i\epsilon_2$	$i\Omega - i\epsilon_2$
$\overline{\kappa}_{3\gamma}$	$i\epsilon_1$	$i\Omega - i\epsilon_2$	$\widehat{\kappa}_{3\gamma}$	$i\epsilon_1$	$i\Omega - i\epsilon_2$
	$i\Omega + i\epsilon_1$	$i\Omega - i\epsilon_2$		$i\Omega + i\epsilon_1$	$i\Omega - i\epsilon_2$
	$-\omega_3 + i\Omega + i\epsilon_1 - i\epsilon_2$	$i\Omega - i\epsilon_2$		$-\omega_3 + i\Omega + i\epsilon_1 - i\epsilon_2$	$i\Omega - i\epsilon_2$
	$-\omega_3 + i\Omega + i\epsilon_1 - i\epsilon_2$	$2i\Omega - i\epsilon_2$		$-\omega_3 + i\Omega + i\epsilon_1 - i\epsilon_2$	$2i\Omega - i\epsilon_2$
$\overline{\lambda}_{3\gamma}$	$i\epsilon_1$	$i\Omega - i\epsilon_2$	ζ_μ	$i\epsilon_1$	$i\Omega - i\epsilon_2$
	$i\Omega + i\epsilon_1$	$i\Omega - i\epsilon_2$		$-\omega_3 + i\Omega + i\epsilon_1 - i\epsilon_2$	$i\Omega - i\epsilon_2$
	$-\omega_3 + i\Omega + i\epsilon_1 - i\epsilon_2$	$i\Omega - i\epsilon_2$		$-\omega_3 + i\Omega + i\epsilon_1 - i\epsilon_2$	$2i\Omega - i\epsilon_2$
ζ_N	$i\epsilon_1$	$i\Omega - i\epsilon_2$	ζ_γ	$i\epsilon_1$	$i\Omega - i\epsilon_2$
	$i\Omega + i\epsilon_1$	$i\Omega - i\epsilon_2$		$i\Omega + i\epsilon_1$	$i\Omega - i\epsilon_2$
	$-\omega_3 + i\Omega + i\epsilon_1 - i\epsilon_2$	$i\Omega - i\epsilon_2$		$-\omega_3 + i\Omega + i\epsilon_1 - i\epsilon_2$	$i\Omega - i\epsilon_2$
	$-\omega_3 + i\Omega + i\epsilon_1 - i\epsilon_2$	$2i\Omega - i\epsilon_2$		$-\omega_3 + i\Omega + i\epsilon_1 - i\epsilon_2$	$2i\Omega - i\epsilon_2$

Table 8. Poles for determining cubic couplings.

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