Half-BPS vertex operators of the $AdS_5 \times S^5$ superstring

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ABSTRACT: Using the pure spinor formalism for the superstring in an $AdS_5 \times S^5$ background, a simple expression is found for half-BPS vertex operators. At large radius, these vertex operators reduce to the usual supergravity vertex operators in a flat background. And at small radius, there is a natural conjecture for generalizing these vertex operators to non-BPS states.

KEYWORDS: String Field Theory, AdS-CFT Correspondence, BRST Quantization

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1 Introduction

Although the computation of superstring scattering amplitudes in an $AdS_5 \times S^5$ background is complicated by the nonlinear form of the worldsheet action, the presence of maximal supersymmetry and the duality with $d = 4$ $N = 4$ super-Yang-Mills gives reasons to be optimistic that progress will be made. Since the RNS formalism can only be used to describe infinitesimal Ramond-Ramond backgrounds [1, 2], one needs to use either the Green-Schwarz or pure spinor formalisms to fully describe $AdS_5 \times S^5$. The Green-Schwarz light-cone formalism is convenient for computing the physical spectrum of “long” strings [3], but amplitude computations using this formalism are complicated even in a flat background.

The pure spinor formalism in an $AdS_5 \times S^5$ background has the advantage over the Green-Schwarz formalism of allowing manifestly $PSU(2, 2|4)$ -covariant quantization [4]. Although less studied, this formalism was used to derive the quantum structure of the infinite set of nonlocal conserved currents in [5] and to compute the physical spectrum of “long” strings in [6]. And in a flat background, the pure spinor formalism has been used for computing multiloop superstring amplitudes [7] that have not yet been computed using either the RNS or Green-Schwarz formalisms.

To generalize these amplitude computations to an $AdS_5 \times S^5$ background, the first step is to explicitly construct the superstring vertex operators for half-BPS states. Although the behavior of half-BPS vertex operators near the $AdS_5 \times S^5$ boundary was computed in [8], the complete BRST-invariant vertex operator was only previously known for some special states [9] such as the moduli for the AdS radius [10] and for the β -deformation [11].

In this paper, simple expressions will be constructed for general half-BPS vertex operators in an $AdS_5 \times S^5$ background using the pure spinor formalism. These simple expressions

were very recently proposed in [12] and will be shown here to be manifestly BRST-invariant and to closely resemble the vertex operators for Type IIB supergravity states in a flat background. Hopefully, these simple expressions for vertex operators will soon be used for computing superstring scattering amplitudes in an $AdS_5 \times S^5$ background.

In section 2, the BRST-invariant vertex operator for Type IIB supergravity states in a flat background will be constructed in terms of the chiral supergravity superfield whose lowest components are the dilaton and axion. In section 3, this vertex operator will be expressed in a simple form using picture-changing operators. And in section 4, this simple expression for the Type IIB supergravity vertex operator in a flat background will be generalized to half-BPS vertex operators in an $AdS_5 \times S^5$ background. Finally, section 5 will discuss the recent conjecture of [12] for generalizing this construction to non-BPS states in an $AdS_5 \times S^5$ background at small radius.

2 Supergravity vertex operators

In any Type IIB supergravity background, the massless closed superstring vertex operator in unintegrated form in the pure spinor formalism is [13]

$$V = \lambda_L^\alpha \lambda_R^\beta A_{\alpha\beta}(x, \theta_L, \theta_R) \quad (2.1)$$

where $A_{\alpha\beta}$ are bispinor superfields depending on the $N = 2B$ $d = 10$ superspace variables $(x^m, \theta_L^\alpha, \theta_R^\alpha)$, $\alpha = 1$ to 16 are Majorana-Weyl spinor indices, and λ_L^α and λ_R^α are left and right-moving pure spinor variables satisfying $\lambda_L \gamma^m \lambda_L = \lambda_R \gamma^m \lambda_R = 0$ for $m = 0$ to 9. The onshell equations of motion and gauge invariances are implied by $QV = 0$ and $\delta V = Q\Omega$ where

$$Q = \lambda_L^\alpha \nabla_{L\alpha} + \lambda_R^\alpha \nabla_{R\alpha} \quad (2.2)$$

and $\nabla_{L\alpha}$ and $\nabla_{R\alpha}$ are the 32 fermionic covariant derivatives in the supergravity background. These equations of motion and gauge invariances imply that $A_{\alpha\beta}$ satisfies

$$\gamma_{abcde}^{\alpha\gamma} \nabla_{L\alpha} A_{\gamma\beta} = \gamma_{abcde}^{\alpha\beta} \nabla_{R\alpha} A_{\gamma\beta} = 0, \quad \delta A_{\alpha\beta} = \nabla_{L\alpha} \Omega_{R\beta} + \nabla_{R\beta} \Omega_{L\alpha}, \quad (2.3)$$

where $\Omega_{L\alpha}$ and $\Omega_{R\alpha}$ satisfy $\gamma_{abcde}^{\alpha\beta} \nabla_{L\alpha} \Omega_{L\beta} = \gamma_{abcde}^{\alpha\beta} \nabla_{R\alpha} \Omega_{R\beta} = 0$.

2.1 Flat background

To construct solutions to (2.3) in a flat background, it is convenient to choose a reference frame where the momentum is only in the $k_+ = k_0 + k_9$ direction so that the covariant fermionic derivatives reduce to

$$\begin{aligned} \nabla_{La} &\equiv (\gamma^- \nabla_L)_a = \frac{\partial}{\partial \theta_L^a} + \theta_{La} \partial_+, & \nabla_{L\dot{a}} &\equiv (\gamma^+ \nabla_L)_{\dot{a}} = \frac{\partial}{\partial \bar{\theta}_L^{\dot{a}}}, \\ \nabla_{Ra} &\equiv (\gamma^- \nabla_R)_a = \frac{\partial}{\partial \theta_R^a} + \theta_{Ra} \partial_+, & \nabla_{R\dot{a}} &\equiv (\gamma^+ \nabla_R)_{\dot{a}} = \frac{\partial}{\partial \bar{\theta}_R^{\dot{a}}}, \end{aligned} \quad (2.4)$$

where $a, \dot{a}$ are SO(8) chiral and antichiral spinor indices and

$$\theta_{La} = (\gamma^+ \theta_L)_a, \quad \bar{\theta}_{L\dot{a}} = (\gamma^- \theta_L)_{\dot{a}}, \quad \theta_{Ra} = (\gamma^+ \theta_R)_a, \quad \bar{\theta}_{R\dot{a}} = (\gamma^- \theta_R)_{\dot{a}}. \quad (2.5)$$

Since k_+ is nonzero, (2.3) implies one can gauge-fix $A_{a\dot{b}} = A_{ab} = A_{\dot{a}b} = 0$, so that

$$V = \bar{\lambda}_L^{\dot{a}} \bar{\lambda}_R^{\dot{b}} A_{\dot{a}\dot{b}}(x, \theta_L, \theta_R) \quad (2.6)$$

where $\lambda_L^a = (\gamma^+ \lambda_L)^a$, $\bar{\lambda}_L^{\dot{a}} = (\gamma^- \lambda_L)^{\dot{a}}$, $\lambda_R^a = (\gamma^+ \lambda_R)^a$, $\bar{\lambda}_R^{\dot{a}} = (\gamma^- \lambda_R)^{\dot{a}}$. In the gauge of (2.6), $QV = 0$ together with $\bar{\lambda}_L^{\dot{a}} \lambda_L^a \sigma_{a\dot{a}}^j = \bar{\lambda}_R^{\dot{a}} \lambda_R^a \sigma_{a\dot{a}}^j = 0$ implies that

$$\frac{\partial}{\partial \bar{\theta}_L^{\dot{a}}} A_{b\dot{c}} = \frac{\partial}{\partial \bar{\theta}_R^{\dot{a}}} A_{b\dot{c}} = 0, \quad \nabla_{La} A_{b\dot{c}} = \frac{1}{8} \sigma_{ab}^j \sigma_j^{cd} \nabla_{Lc} A_{d\dot{c}}, \quad \nabla_{Ra} A_{b\dot{c}} = \frac{1}{8} \sigma_{ac}^j \sigma_j^{cd} \nabla_{Rc} A_{b\dot{d}} \quad (2.7)$$

where $\sigma_{a\dot{a}}^j$ are the SO(8) Pauli matrices.

One method of solving (2.7) is to take the left-right product of the open superstring solutions of [14], but it will be useful to describe another method which can be easily generalized to the $AdS_5 \times S^5$ background. This method is based on the SO(8) chiral superfield Φ satisfying $\nabla_-^a \Phi = 0$ where $\nabla_{\pm}^a \equiv \nabla_L^a \pm i \nabla_R^a$ is a linear combination of the left and right-moving fermionic derivatives. In terms of $(x^+, \theta_L^a, \theta_R^a)$,

$$\Phi(x^+, \theta_L^a, \theta_R^a) = e^{ik_+(x^+ + i\theta_L^a \theta_R^a)} f(\theta_-) \quad (2.8)$$

where $\theta_-^a = \theta_L^a - i\theta_R^a$. The superfield Φ will be defined to satisfy the reality condition $(\nabla_+)^4_{abcd} \Phi = \frac{1}{24} \epsilon_{abcdefgh} (\nabla_-)^4_{efgh} \bar{\Phi}$, and the 2^8 components of Φ describe the Type IIB supergravity multiplet where, at zeroth order in θ_- , the real part of Φ is the Type IIB dilaton and the imaginary part of Φ is the Type IIB axion.

To construct the vertex operator of (2.6) for this multiplet, first consider the vertex operator

$$V_0 = \bar{\lambda}_L^{\dot{a}} \bar{\lambda}_R^{\dot{a}} \Phi. \quad (2.9)$$

Using the relation $\lambda_L^a \bar{\lambda}_L^{\dot{a}} = -\frac{1}{4} (\sigma^{jk} \lambda_L)^a (\sigma_{jk} \bar{\lambda}_L)^{\dot{a}}$ and $\lambda_R^a \bar{\lambda}_R^{\dot{a}} = -\frac{1}{4} (\sigma^{jk} \lambda_R)^a (\sigma_{jk} \bar{\lambda}_R)^{\dot{a}}$, one finds that

$$QV_0 = (\lambda_- \nabla_+ + \lambda_+ \nabla_-) (\bar{\lambda}_L \bar{\lambda}_R) \Phi = (\lambda_- \nabla_+) (\bar{\lambda}_L \bar{\lambda}_R) \Phi = -\frac{1}{4} (\lambda_+ \sigma^{jk} \nabla_+) (\bar{\lambda}_L \sigma_{jk} \bar{\lambda}_R) \Phi \quad (2.10)$$

where $\lambda_{\pm}^a = \lambda_L^a \pm i\lambda_R^a$. Now consider the vertex operator

$$V_1 = \frac{1}{32ik_+} (\bar{\lambda}_L \sigma_{jk} \bar{\lambda}_R) (\nabla_+ \sigma^{jk} \nabla_+) \Phi. \quad (2.11)$$

Since $\{\nabla_-, \nabla_+\} = 4\partial_+$, (2.10) implies that $QV_0 = -(\lambda_+ \nabla_-) V_1$. Furthermore, a similar argument implies that $(\lambda_- \nabla_+) V_1 = -(\lambda_+ \nabla_-) V_2$ where

$$V_2 = -\frac{1}{2048k_+^2} (\bar{\lambda}_L \sigma_{jklm} \bar{\lambda}_R) (\nabla_+ \sigma^{jk} \nabla_+) (\nabla_+ \sigma^{lm} \nabla_+) \Phi. \quad (2.12)$$

Continuing this argument, one finds that $QV = 0$ where $V = V_0 + V_1 + V_2 + V_3 + V_4$ and

$$V_n = \frac{1}{n!(32ik_+)^n} (\bar{\lambda}_L \sigma_{j_1 k_1 \dots j_n k_n} \bar{\lambda}_R) (\nabla_+ \sigma^{j_1 k_1} \nabla_+) \dots (\nabla_+ \sigma^{j_n k_n} \nabla_+) \Phi. \quad (2.13)$$

Note that $(\lambda_- \nabla_+) V_4 = 0$ since $(\nabla_+)^9 \Phi = 0$.

So the BRST-invariant vertex operator with momentum k_+ in this gauge is

$$V = \bar{\lambda}_L^{\dot{a}} \bar{\lambda}_R^{\dot{b}} e^{ik_+ x^+} A_{\dot{a}\dot{b}}(\theta_L, \theta_R) = V_0 + V_1 + V_2 + V_3 + V_4, \quad (2.14)$$

and one can easily verify that at $\theta_L^a = \theta_R^a = 0$, $A_{\dot{a}\dot{b}}$ is the bispinor Ramond-Ramond field in light-cone gauge

$$A_{\dot{a}\dot{b}} = \delta_{\dot{a}\dot{b}} a + \sigma_{\dot{a}\dot{b}}^{jk} a_{jk} + \sigma_{\dot{a}\dot{b}}^{jklm} a_{jklm}. \quad (2.15)$$

It will be useful to note that one would end up with the same expression of (2.14) for V if one had instead started with the superfield Φ_{1234} which is annihilated by $\nabla_-^a \equiv \nabla_L^a - i(\sigma_{1234} \nabla_R)^a$. In this case, $V_0 = (\bar{\lambda}_L \sigma_{1234} \bar{\lambda}_R) \Phi_{1234}$ and

$$V_n = \frac{1}{n!(32ik_+)^n} (\bar{\lambda}_L \sigma_{j_1 k_1 \dots j_n k_n} \sigma_{1234} \bar{\lambda}_R) (\nabla_+ \sigma^{j_1 k_1} \nabla_+) \dots (\nabla_+ \sigma^{j_n k_n} \nabla_+) \Phi_{1234} \quad (2.16)$$

where $\nabla_+^a \equiv \nabla_L^a + i(\sigma_{1234} \nabla_R)^a$.

In the following section, $\nabla_{\pm}^a$ will always denote $\nabla_{\pm}^a \equiv \nabla_L^a \pm i(\sigma_{1234} \nabla_R)^a$, $\theta_{\pm}^a$ will always denote $\theta_{\pm}^a = \theta_L^a \pm i(\sigma_{1234} \theta_R)^a$ and $\lambda_{\pm}^a$ will always denote $\lambda_{\pm}^a = \lambda_L^a \pm i(\sigma_{1234} \lambda_R)^a$.

3 Picture-changing

To generalize this construction to an $AdS_5 \times S^5$ background, it will be useful to first consider the vertex operator V for the lowest component of Φ_{1234} in (2.16), i.e. $\Phi_{1234} = \exp(ik_+ \hat{x}^+)$ where $\hat{x}^+ \equiv x^+ + i\theta_L \sigma_{1234} \theta_R$. Although this vertex operator of (2.16) has various terms $V_0 \dots V_4$ with different powers of $\theta_+^a = \theta_L^a + i(\sigma_{1234} \theta_R)^a$, it can be reduced to just one term by writing it in a different “picture” as

$$V_{-1} = PV = (\bar{\lambda}_L \sigma_{1234} \bar{\lambda}_R) e^{ik_+ \hat{x}^+} \prod_{a=1}^8 \theta_+^a \delta(\lambda_+^a) \quad (3.1)$$

where P is the “picture-lowering” operator

$$P = \prod_{a=1}^8 \theta_+^a \delta(\lambda_+^a) \quad (3.2)$$

and $\lambda_+^a = \lambda_L^a + i(\sigma_{1234} \lambda_R)^a$. Note that the 8 λ_+^a ’s in P are all independent so that $\prod_{a=1}^8 \delta(\lambda_+^a)$ is well-defined. Also note that P is BRST-invariant and is super-Poincaré invariant up to a BRST-trivial quantity. For example, under the supersymmetry transformation generated by q_1 ,

$$q_1 P = \delta(\lambda_+^1) \prod_{a=2}^8 \theta_+^a \delta(\lambda_+^a) = Q \left[-\theta_+^1 \delta'(\lambda_+^1) \prod_{a=2}^8 \theta_+^a \delta(\lambda_+^a) \right]. \quad (3.3)$$

The original vertex operator V of (2.14) is related to V_{-1} of (3.1) by picture-raising as $V = CV_{-1}$ where

$$C = \prod_{a=1}^8 Q(\xi_a) \quad (3.4)$$

is the picture-raising operator and $Q(\xi_a)$ is a formal expression whose action on V_{-1} is defined through the following procedure: using the notation of Friedan-Martinec-Shenker for picture-changing operators, $\delta(\gamma) = e^{-\phi}$ and $\xi\delta(\gamma) = \xi e^{-\phi} = \frac{1}{\gamma}$ where (γ, β) are chiral bosons which have been fermionized as $\gamma = \eta e^\phi$ and $\beta = \partial\xi e^{-\phi}$. Although λ_+^a and its conjugate w_a^+ are not chiral bosons, one can formally define

$$\lambda_+^a = \eta^a e^{\phi_a}, \quad w_a^+ = \partial\xi_a e^{-\phi_a} \quad (3.5)$$

so that

$$\xi_a \delta(\lambda_+^a) = \xi_a e^{-\phi_a} = \frac{1}{\lambda_+^a}. \quad (3.6)$$

Using this definition, CV_{-1} can be computed by using (3.6) to convert the factors of $\delta(\lambda_+^a)$ in V_{-1} into factors of $\frac{1}{\lambda_+^a}$. Furthermore, the BRST invariance of V_{-1} guarantees that CV_{-1} has no poles when $\lambda_+^a = 0$ and can be expressed in the form of (2.1) as $V = \lambda_L^\alpha \lambda_R^\beta A_{\alpha\beta}(x, \theta_L, \theta_R)$. To see why, note that $Q(F\delta(\lambda_+^a)) = 0$ implies that $Q(F)$ is proportional to λ_+^a . So $Q(\frac{F}{\lambda_+^a})$ has no poles when $\lambda_+^a = 0$. Also note that if F has (left,right)-moving ghost number equal to (g_L, g_R) , then $Q(\frac{F}{\lambda_+^a})$ also has (left,right) ghost number (g_L, g_R) . This is easy to see since terms in QF must either carry ghost number $(g_L + 1, g_R)$ or $(g_L, g_R + 1)$. So $QF = E\lambda_+^a$ for some E implies that E must carry ghost number (g_L, g_R) .

One can explicitly compute CV_{-1} for the vertex operator of (3.1) as

$$\begin{aligned} CV_{-1} &= \prod_{b=2}^8 Q(\xi_b) Q(\xi_1) V_{-1} = \prod_{b=2}^8 Q(\xi_b) Q(\xi_1 V_{-1}) \\ &= - \prod_{b=2}^8 Q(\xi_b) Q \left(\frac{\theta_+^1}{\lambda_+^1} \prod_{a=2}^8 \theta_+^a \delta(\lambda_+^a) (\bar{\lambda}_L \sigma_{1234} \bar{\lambda}_R) e^{ik_+ \hat{x}^+} \right) \\ &= - \prod_{b=2}^8 Q(\xi_b) \prod_{a=2}^8 \theta_+^a \delta(\lambda_+^a) (\bar{\lambda}_L \sigma_{1234} \bar{\lambda}_R) e^{ik_+ \hat{x}^+} \\ &= - \prod_{b=3}^8 Q(\xi_b) Q \left(\xi_2 \prod_{a=2}^8 \theta_+^a \delta(\lambda_+^a) (\bar{\lambda}_L \sigma_{1234} \bar{\lambda}_R) e^{ik_+ \hat{x}^+} \right) \\ &= \prod_{b=3}^8 Q(\xi_b) Q \left(\frac{\theta_+^2}{\lambda_+^2} \prod_{a=3}^8 \theta_+^a \delta(\lambda_+^a) (\bar{\lambda}_L \sigma_{1234} \bar{\lambda}_R) e^{ik_+ \hat{x}^+} \right) \\ &= \prod_{b=3}^8 Q(\xi_b) \left(1 + 2ik_+ (\lambda_-^1 \theta_+^1) \frac{\theta_+^2}{\lambda_+^2} \right) \prod_{a=3}^8 \theta_+^a \delta(\lambda_+^a) (\bar{\lambda}_L \sigma_{1234} \bar{\lambda}_R) e^{ik_+ \hat{x}^+} \\ &= \prod_{b=3}^8 Q(\xi_b) \left((\bar{\lambda}_L \sigma_{1234} \bar{\lambda}_R) + \frac{i}{2} k_+ \theta_+^1 \theta_+^2 \sigma_{12}^{jk} (\bar{\lambda}_L \sigma^{jk} \sigma_{1234} \bar{\lambda}_R) \right) \prod_{a=3}^8 \theta_+^a \delta(\lambda_+^a) e^{ik_+ \hat{x}^+} \end{aligned} \quad (3.7)$$

where we have used that $\lambda_-^1 (\bar{\lambda}_L \sigma_{1234} \bar{\lambda}_R) = \frac{1}{4} (\sigma^{jk} \lambda_+)^1 (\bar{\lambda}_L \sigma^{jk} \sigma_{1234} \bar{\lambda}_R)$. Note that the term proportional to $k_+ \theta_+^1 \theta_+^2$ in (3.7) is related to V_1 in (2.16) where each of the two ∇_+ 's

contributes a factor of $k_+\theta_+$. Continuing with this procedure of converting $\xi_a\delta(\lambda_+^a)$ into $(\lambda_+^a)^{-1}$ to compute the product with $Q(\xi_a)$, it is expected that CV_{-1} will reproduce V of (2.14).

4 $AdS_5 \times S^5$ vertex operators

4.1 Parameterization of $AdS_5 \times S^5$

To generalize this construction for half-BPS states in an $AdS_5 \times S^5$ background, parameterize $AdS_5 \times S^5$ using the supercoset $g \in \frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}$ as

$$g(\theta, X, Y) = F(\theta)G(X)H(Y) \quad (4.1)$$

where $F(\theta) = \exp(\theta_R^J q_J^R + \theta_J^R q_R^J)$ is a fermionic $\frac{PSU(2,2|4)}{SO(4,2) \times SO(6)}$ coset, (q_J^R, q_R^J) are the 32 fermionic generators of $PSU(2,2|4)$, $R = 1$ to 4 are $SO(4,2)$ spinor indices, $J = 1$ to 4 are $SO(6)$ spinor indices, $G(X)$ is an $\frac{SO(4,2)}{SO(4,1)}$ coset for AdS_5 and $H(Y)$ is an $\frac{SO(6)}{SO(5)}$ coset for S^5 . Under global $PSU(2,2|4)$ transformations, $\delta g = \Sigma g$ where $\Sigma \in PSU(2,2|4)$, and under BRST transformations,

$$\delta g = g[(\lambda_L + i\lambda_R)\tilde{J}_{\tilde{R}} q_J^R + (\lambda_L - i\lambda_R)\tilde{R}_{\tilde{J}} q_R^J] \quad (4.2)$$

where $\tilde{R} = 1$ to 4 is an $SO(4,1)$ spinor index, $\tilde{J} = 1$ to 4 is an $SO(5)$ spinor index, and $(\lambda_L)_{\tilde{J}}^{\tilde{R}}$ and $(\lambda_R)_{\tilde{J}}^{\tilde{R}}$ are the left and right-moving pure spinors. Note that $SO(4,1)$ and $SO(5)$ spinor indices can be raised and lowered using the matrices $\sigma_6^{\tilde{R}\tilde{S}}$ and $\sigma_6^{\tilde{J}\tilde{K}}$ which commute with $SO(4,1)$ and $SO(5)$ rotations.

The cosets $G(X)$ and $H(Y)$ are defined up to local $SO(4,1) \times SO(5)$ gauge transformations parameterized by $\Omega \in SO(4,1)$ and $\hat{\Omega} \in SO(5)$ as

$$G(X) \sim G(X)\Omega, \quad H(Y) \sim H(Y)\hat{\Omega} \quad (4.3)$$

where the left and right-moving pure spinors λ_L and λ_R transform as $SO(4,1) \times SO(5)$ spinors. More explicitly, $G_{\tilde{R}}^R$ and H_J^J are 4×4 matrices which transform under the gauge transformations as

$$\begin{aligned} G_{\tilde{R}}^R &\rightarrow G_{\tilde{S}}^R \Omega_{\tilde{R}}^{\tilde{S}}, & H_J^J &\rightarrow H_{\tilde{K}}^J \hat{\Omega}_{\tilde{J}}^{\tilde{K}}, \\ (\lambda_L)_{\tilde{J}}^{\tilde{R}} &\rightarrow (\lambda_L)_{\tilde{K}}^{\tilde{S}} \Omega_{\tilde{S}}^{\tilde{R}} \hat{\Omega}_{\tilde{J}}^{\tilde{K}}, & (\lambda_R)_{\tilde{J}}^{\tilde{R}} &\rightarrow (\lambda_R)_{\tilde{K}}^{\tilde{S}} \Omega_{\tilde{S}}^{\tilde{R}} \hat{\Omega}_{\tilde{J}}^{\tilde{K}}, \end{aligned} \quad (4.4)$$

and the AdS_5 coordinate $X^{RS} = -X^{SR}$ and S^5 coordinate $Y^{JK} = -Y^{KJ}$ are defined in terms of $G_{\tilde{R}}^R$ and H_J^J by

$$X^{RS} = G_{\tilde{R}}^R \sigma_6^{\tilde{R}\tilde{S}} G_{\tilde{S}}^S, \quad Y^{JK} = H_J^J \sigma_6^{\tilde{J}\tilde{K}} H_{\tilde{K}}^K. \quad (4.5)$$

Defining $X_{RS} = \frac{1}{2}\epsilon_{RSTU}X^{TU}$ and $Y_{JK} = \frac{1}{2}\epsilon_{JKLM}Y^{LM}$, (4.5) implies $X^{RS}X_{RS} = 4$ and $Y^{JK}Y_{JK} = 4$.

4.2 Half-BPS vertex operator

To construct the vertex operator for a half-BPS state in an $AdS_5 \times S^5$ background, consider the state dual to the super-Yang-Mills gauge-invariant operator

$$\text{Tr}[(y_0^{JK} \Phi_{JK}(x))^n] \quad (4.6)$$

where $\Phi_{JK}(x)$ are the six scalars located at the position x^m on the AdS_5 boundary and y_0^{JK} is a fixed null six-vector satisfying $\epsilon_{JKLM} y_0^{JK} y_0^{LM} = 0$. It will be convenient to define the null six-vector

$$x_0^{RS} = (\epsilon^{AB}, x^m \sigma_m^{AA}, (x^m x_m) \epsilon^{\dot{A}\dot{B}}) \quad (4.7)$$

where $R = (A, \dot{A})$ with $A, \dot{A} = 1$ to 2 . x_0^{RS} transforms covariantly under $SO(4, 2)$ conformal transformations of the AdS_5 boundary and satisfies $\epsilon_{RSTU} x_0^{RS} x_0^{TU} = 0$.

The choice of y_0^{JK} breaks $SO(6)$ R -symmetry to $U(1) \times SO(4)$, and J will be defined to be the charge with respect to this $U(1)$. Similarly, the choice of x_0^{RS} breaks $SO(4, 2)$ conformal symmetry to $SO(1, 1) \times SO(3, 1)$, and Δ will be defined to be the charge with respect to the $SO(1, 1)$. The half-BPS state of (4.6) carries $J = n$ and $\Delta = n$ and is preserved by the 24 spacetime supersymmetries which carry $J - \Delta \geq 0$.

In analogy with the construction of the vertex operator of V_{-1} in a flat background, it will now be argued that the BRST-invariant vertex operator for the state (4.6) is [12]

$$V_{-1} = (\lambda_L)^{\tilde{J}}_{\tilde{R}} (\lambda_R)^{\tilde{R}}_{\tilde{J}} P \left(\frac{Y \cdot y_0}{X \cdot x_0} \right)^n \quad (4.8)$$

where the picture-lowering operator P is defined as

$$P = \prod_{a=1}^8 \theta_+^a \delta(Q(\theta_+^a)) \quad (4.9)$$

and θ_+^a are the 8 θ 's which carry charge $J - \Delta = 1$. In terms of x_0^{RS} and y_0^{JK} ,

$$\theta_+^a = [(x_0)^{RS} (y_0)_{JK} \theta_S^K, (x_0)_{RS} (y_0)^{JK} \theta_K^S] \quad (4.10)$$

where only 8 of the 32 components of $(x_0)^{RS} (y_0)_{JK} \theta_S^K$ and $(x_0)_{RS} (y_0)^{JK} \theta_K^S$ are independent since $(x_0)^{RS} (x_0)_{ST} = (y_0)^{JK} (y_0)_{KL} = 0$.

To show that V_{-1} of (4.8) carries the same charges and is invariant under the same 24 supersymmetries as (4.6), note that $Y \cdot y_0$ carries $J = 1$ and $X \cdot x_0$ carries $\Delta = -1$ so that V_{-1} carries $J = \Delta = n$. Furthermore, both $Y \cdot y_0$ and $X \cdot x_0$ are invariant under the 8 supersymmetries with $J - \Delta = 1$. And under the 16 supersymmetries with $J - \Delta = 0$, $\frac{Y \cdot y_0}{X \cdot x_0}$ transforms into terms which contain at least one θ with $J - \Delta = 1$. However, all 8 θ 's with $J - \Delta = 1$ are contained in the picture-lowering operator P of (4.9). So V_{-1} is invariant under all 24 supersymmetries which carry $J - \Delta \geq 0$.

Similarly, under the BRST transformation of (4.2), $\frac{Y \cdot y_0}{X \cdot x_0}$ transforms into terms containing products of $Q(\theta)$ with θ 's where either $Q(\theta)$ carries $J - \Delta = 1$ or at least one of the θ 's carries $J - \Delta = 1$. In both cases, the BRST transformation is killed by $P = \prod_{a=1}^8 \theta_+^a \delta(Q(\theta_+^a))$ of (4.9). And since P and $(\lambda_L)^{\tilde{J}}_{\tilde{R}} (\lambda_R)^{\tilde{R}}_{\tilde{J}}$ are also BRST-invariant, it has been shown that V_{-1} of (4.8) is BRST-invariant.

4.3 Explicit example

For example, consider the state corresponding to $\text{Tr}[(\Phi_{12}(0))^n]$ which carries $\Delta = J = n$ where Δ is the dilatation charge and J is the $U(1)$ charge. To simplify the vertex operator, parameterize the supercoset $g \in \frac{\text{PSU}(2,2|4)}{\text{SO}(4,1) \times \text{SO}(5)}$ as

$$g = \exp(\theta_- q_+ + \theta_0 q_0 + xK + yR) \exp(\theta_+ q_-) \exp(z\Delta + wJ) \quad (4.11)$$

where (q_+, q_0, q_-) are the $(8, 16, 8)$ fermionic isometries with $(+1, 0, -1)$ charge with respect to $J - \Delta$, and K and R are the four conformal boosts and four R -symmetries with charge $J - \Delta = 1$. Since the vertex operator V is annihilated by $(q_+, q_0, K, R, \Delta - J)$, the parameterization of (4.11) implies that V is independent of $(\theta_-, \theta_0, x, y, w + z)$ and only depends on $(\theta_+, z - w)$ and the pure spinor ghosts.

Using the picture-lowering operator $P = \prod_{a=1}^8 \theta_+^a \delta(Q(\theta_+^a))$, the vertex operator of (4.8) is

$$V_{-1} = (\bar{\lambda}_L \sigma_{1234} \bar{\lambda}_R) e^{n(w-z)} \prod_{a=1}^8 \theta_+^a \delta(\lambda_+^a) \quad (4.12)$$

where θ_+^a are the 8 θ 's which carry charge $J - \Delta = 1$ and $(\lambda_+^a, \lambda_-^a, \bar{\lambda}_L^{\dot{a}}, \bar{\lambda}_R^{\dot{a}})$ for $a, \dot{a} = 1$ to 8 are defined by

$$\begin{aligned} \lambda_+^a &\equiv \left[e^{\frac{1}{2}(z-w)} (\lambda_L + i\lambda_R)_{\tilde{R}}^{\tilde{J}} \text{ for } \tilde{J} = 1, 2, \tilde{R} = 1, 2; \quad e^{\frac{1}{2}(z-w)} (\lambda_L - i\lambda_R)_{\tilde{R}}^{\tilde{J}} \text{ for } \tilde{J} = 3, 4, \tilde{R} = 3, 4 \right] \\ \lambda_-^a &\equiv \left[e^{\frac{1}{2}(w-z)} (\lambda_L + i\lambda_R)_{\tilde{R}}^{\tilde{J}} \text{ for } \tilde{J} = 3, 4, \tilde{R} = 3, 4; \quad e^{\frac{1}{2}(w-z)} (\lambda_L - i\lambda_R)_{\tilde{R}}^{\tilde{J}} \text{ for } \tilde{J} = 1, 2, \tilde{R} = 1, 2 \right] \\ \bar{\lambda}_L^{\dot{a}} &\equiv [(\lambda_L)_{\tilde{R}}^{\tilde{J}} \text{ for } \tilde{J} = 3, 4, \tilde{R} = 1, 2; \quad (\lambda_L)_{\tilde{R}}^{\tilde{J}} \text{ for } \tilde{J} = 1, 2, \tilde{R} = 3, 4] \\ \bar{\lambda}_R^{\dot{a}} &\equiv [(\lambda_R)_{\tilde{R}}^{\tilde{J}} \text{ for } \tilde{J} = 3, 4, \tilde{R} = 1, 2; \quad (\lambda_R)_{\tilde{R}}^{\tilde{J}} \text{ for } \tilde{J} = 1, 2, \tilde{R} = 3, 4] \end{aligned} \quad (4.13)$$

and we have used that $(\lambda_L)_{\tilde{R}}^{\tilde{J}} (\lambda_R)_{\tilde{J}}^{\tilde{R}} = \bar{\lambda}_L^{\dot{a}} (\sigma_{1234})_{\dot{a}\dot{b}} \bar{\lambda}_R^{\dot{b}}$ when $\lambda_+^a = 0$.

In the large radius limit where the $AdS_5 \times S^5$ background approaches flat space, one can easily verify that V_{-1} of (4.12) approaches the flat space vertex operator V_{-1} of (3.1) where $k_+ = n$ and ix^+ is identified with $w - z$. And the vertex operator for all other half-BPS states in an $AdS_5 \times S^5$ background are obtained from (4.12) by acting with the appropriate $\text{PSU}(2, 2|4)$ transformations, and reduce in the flat space limit to the vertex operators of other supergravity states in the multiplet of (3.1).

Finally, one can relate V_{-1} of (4.12) to the supergravity vertex operator $V = \lambda_L^\alpha \lambda_R^\beta \cdot A_{\alpha\beta}(x, \theta)$ of (2.1) by defining

$$V = CV^{-1} \quad (4.14)$$

where $C = \prod_{a=1}^8 Q(\xi_a)$ and the 8 λ_+^a 's of (4.13) have been fermionized as in (3.5). Using the same procedure as in (3.7), this construction will produce an $AdS_5 \times S^5$ vertex operator of the form $V = \lambda_L^\alpha \lambda_R^\beta A_{\alpha\beta}(\theta_+, z - w)$ where, as in a flat background, the potential poles coming from $\xi^a \delta(\lambda_+^a) = \frac{1}{\lambda_+^a}$ are absent because of the BRST invariance of V_{-1} .

5 Summary

In this paper, a simple BRST-invariant vertex operator was constructed for half-BPS states in an $AdS_5 \times S^5$ background. One possible application of this paper is to use these vertex operators to compute scattering amplitudes. Much is known about scattering amplitudes of half-BPS states in $AdS_5 \times S^5$, and it would be very interesting to show how to compute these amplitudes using superstring vertex operators even for the simplest 3-point amplitude.

Another possible application of this paper is to construct $AdS_5 \times S^5$ vertex operators for non-BPS states. As discussed in [12], the half-BPS vertex operator can be expressed as

$$V = (\lambda_L)_{\tilde{J}}^{\tilde{R}} (\lambda_R)_{\tilde{R}}^{\tilde{J}} \left(C P \frac{Y \cdot y_0}{X \cdot x_0} \right)^n \quad (5.1)$$

if one adds $(n-1)$ picture-raising operators C and $(n-1)$ picture-lowering operators P to $V = CV_{-1}$ of (4.14). Since all states at zero 't Hooft coupling can be described as “spin chains” constructed from n super-Yang-Mills fields, it is natural to express the half-BPS vertex operator of (5.1) as

$$V = (\lambda_L)_{\tilde{J}}^{\tilde{R}} (\lambda_R)_{\tilde{R}}^{\tilde{J}} C E C E \dots C E \quad (5.2)$$

where $E \equiv P \frac{Y \cdot y_0}{X \cdot x_0}$ corresponds to the Yang-Mills field $y_0^{JK} \phi_{JK}(x_0)$ on the spin chain. Therefore, a natural conjecture for general non-BPS vertex operators is

$$V = (\lambda_L)_{\tilde{J}}^{\tilde{R}} (\lambda_R)_{\tilde{R}}^{\tilde{J}} : C E_1 C E_2 \dots C E_n : \quad (5.3)$$

where $E_1 \dots E_n$ describe n different super-Yang-Mills fields on the spin chain and are obtained from $P \frac{Y \cdot y_0}{X \cdot x_0}$ by performing the appropriate $PSU(2, 2|4)$ transformation. Since E and C are independently BRST-invariant, the vertex operator of (5.3) is BRST-invariant where $: \ :$ denotes a normal-ordering prescription which is defined to be invariant under cyclic permutations of the E 's. It would be very interesting to find evidence for this conjecture by using the topological description of [12] to study the $AdS_5 \times S^5$ superstring at small radius.

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