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Black holes and general Freudenthal transformations

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ABSTRACT: We study General Freudenthal Transformations (GFT) on black hole solutions in Einstein-Maxwell-Scalar (super)gravity theories with global symmetry of type E_7 . GFT can be considered as a 2-parameter, $a, b \in \mathbb{R}$, generalisation of Freudenthal duality: $x \mapsto x_F = ax + b\tilde{x}$, where x is the vector of the electromagnetic charges, an element of a Freudenthal triple system (FTS), carried by a large black hole and $\tilde{x}$ is its Freudenthal dual. These transformations leave the Bekenstein-Hawking entropy invariant up to a scalar factor given by $a^2 \pm b^2$. For any x there exists a one parameter subset of GFT that leave the entropy invariant, $a^2 \pm b^2 = 1$, defining the subgroup of Freudenthal rotations. The Freudenthal plane defined by $\text{span}_{\mathbb{R}}\{x, \tilde{x}\}$ is closed under GFT and is foliated by the orbits of the Freudenthal rotations. Having introduced the basic definitions and presented their properties in detail, we consider the relation of GFT to the global symmetries or U-dualities in the context of supergravity. We consider explicit examples in pure supergravity, axion-dilaton theories and $N = 2, D = 4$ supergravities obtained from $D = 5$ by dimensional reductions associated to (non-degenerate) reduced FTS's descending from cubic Jordan Algebras.

KEYWORDS: Supergravity Models, Black Holes, Supersymmetry and Duality

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1 Introduction

Recent observations, consistent with expectations for the shadow of a Kerr black hole (BH) as predicted by general relativity, have been, for the first time, presented [1, 2]. This result demonstrates yet again the effectiveness of general relativity, but also serves to emphasise the need to address the long-standing puzzles presented by its BH solutions. A classical stationary BH solution is characterised by its mass M , angular momentum J and charge Q alone. In particular, its horizon area is a simple function of these three quantities. Identifying the horizon area as an entropy (determined up to a numerical constant of proportionality), the classical mechanics of BHs obeys a set of laws are directly analogous to those of thermodynamics [3–5]. Hawking’s prediction [6, 7] that BHs quantum mechanically emit thermal radiation at the semiclassical level fixes the Bekenstein-Hawking area/entropy relation to be precisely (where the usual constants $c = \hbar = G = 1$),

$$S_{\text{BH}} = \frac{A_{\text{horizon}}}{4} \quad (1.1)$$

and suggests that the thermodynamic interpretation of BH mechanics is more than a mere analogy. However, it also presents an immediate question. A large BH carries a huge entropy, yet is classically characterised entirely by M, J and Q . Where, then, are the microscopic degrees of freedom underpinning the entropy?

Any complete theory of quantum gravity should address this challenge in some way or at least advance in this direction. String/M-theory provides an answer for a very special class of extremal dyonic BHs, where the calculations are made tractable by the presence of some preserved supersymmetries [8]. This result and its generalisations depend on a range of mathematical and theoretical insights. In particular, symmetries, duality transformations and the mathematical structures upon which they are realised, constitute important tools in the study of black hole solutions in general relativity and its supersymmetric extension, supergravity, which provides the low-energy effective field theory limit of string/M-theory. For instance, the non-compact global symmetries of supergravity theories [9], or U-dualities in the context of M-theory [10, 11], have played a particularly crucial role, starting with the work of [12]. For a large class of $\mathcal{N} \geq 2$ Poincaré supergravity theories with symmetric scalar manifolds¹ the U-duality groups are of “type E_7 ”, a class of groups sharing the same algebraic structure as the second largest exceptional Lie group E_7 [14]. Groups of type E_7 are axiomatically characterised by Freudenthal triple systems

¹For a survey of symmetric spaces in supergravity see [13].

(FTS) [14–16]. An FTS is a vector space $\mathfrak{F}$ with, in particular, a symmetric four-linear form $\Delta(x, y, z, w)$ (see section 2 for full details). The automorphism group $\text{Aut}(\mathfrak{F})$ of the FTS is the U-duality group G_4 of the associated 4d supergravity. The electromagnetic charges carried by the static extremal black hole solutions in such theories correspond to elements $x \in \mathfrak{F}$ and fall into linear representations of the associated U-duality groups. For such theories the leading-order Bekenstein-Hawking black hole entropy is given by

$$S_{\text{BH}} = \pi \sqrt{|\Delta(x)|}, \quad (1.2)$$

where $\Delta(x) := \Delta(x, x, x, x)$ is the unique U-duality invariant quartic polynomial of the BH charges.

In [17] it was shown that when the U-duality group is of type E_7 [14, 18], these black hole solutions enjoy a nonlinear symmetry, named *Freudenthal duality*, acting on their associated charge vectors x . This holds for instance in all $\mathcal{N} > 2$ -extended, $D = 4$ supergravities, as well as in all $\mathcal{N} = 2$ supergravities coupled to vector multiplets with symmetric scalar manifolds. However, supersymmetry is not a necessary ingredient (e.g. in the case of $\mathfrak{F}(J_3^{\text{C}_s})$ and $\mathfrak{F}(J_3^{\text{H}_s})$; cfr. table 1).

In [19] Freudenthal duality was then generalised to a symmetry not only of the Bekenstein-Hawking black hole entropy S_{BH} , but also of the critical points of the black hole effective potential V_{BH} : regardless of supersymmetry, such a formulation of Freudenthal duality actually holds for *any* Maxwell-Einstein system coupled to a non-linear sigma model of scalar fields, in four dimensions.

The role of Freudenthal duality in the structure of extremal black hole solutions was investigated in [20], in the framework of ungauged $\mathcal{N} = 8$, $D = 4$ maximal supergravity. In particular, the most general solution to the supersymmetric stabilisation equations were shown to be given by the F-dual of a suitably defined real 56-dimensional vector, whose components are real harmonic functions in $\mathbb{R}^3$ transverse space. Then, in [21] Freudenthal duality was also shown to be an *on-shell* symmetry of the effective, one-dimensional action describing the dynamics of scalar fields in the background of a static, spherically symmetric and asymptotically flat black hole in $\mathcal{N} = 2$, $D = 4$ supergravity. In [22] it was shown that the generalised, scalar-dependent Freudenthal duality introduced in [19] actually is a symmetry of the equations of motion of the full theory, and is not restricted to the extremal black hole solutions or their effective action. Remarkably, in [22] Freudenthal duality was also applied to world-sheet actions, such as the Nambu-Goto world-sheet action in any (t, s) -signature spacetime, then allowing for an F-dual formulation of Gaillard-Zumino duality [23–25] on the world-sheet.

It is also here worth remarking that, in recent years, groups of type E_7 , Freudenthal triple systems, and Freudenthal duality have also appeared in several indirectly related contexts, such as the relation to minimal coupling of vectors and scalars in cosmology and supergravity [26, 27], Freudenthal gauge theory (in which the scalar fields are $\mathfrak{F}$ -valued) [28], multi-centered BPS black holes [29], conformal isometries [30], Hitchin functionals and entanglement in quantum information theory [31–35].²

²Freudenthal duality in the context of entanglement and Hitchin functionals can actually be related back to its application to black holes via the *black-hole/qubit correspondence* [34, 36–39].

Our focus here is on the notion of general Freudenthal transformations (GFT), introduced in [29]. In this work it was shown

that F-duality can be generalised to an Abelian group of transformations

$$x \mapsto x_F = ax + b\tilde{x}. \quad (1.3)$$

The GFT leave the quartic form invariant up to a scalar factor $\lambda^2 = a^2 \pm b^2$,

$$\Delta(x_F) = \lambda^4 \Delta(x) \quad (1.4)$$

The entropy, ADM mass and, for multicenter solutions in some specific models, the inter-centre distances scale as

$$S_{BH} \rightarrow \lambda^2 S_{BH}, \quad M_{ADM} \rightarrow \lambda M_{ADM}, \quad r_{ab} \rightarrow \lambda r_{ab}, \quad (1.5)$$

while the scalars on the horizon and at infinity are left invariant.

The properties of GFT, in particular the properties of the quartic FTS invariant (or the Bekenstein-Hawking entropy in physical terms), can be traced back to the existence and properties of Freudenthal planes in $\mathfrak{F}$. This notion first appears in the mathematical literature in [14]. Given an F-dual pair x and $\tilde{x}$ we define the Freudenthal plane $\mathfrak{F}_x \subset \mathfrak{F}$ as the set of all elements

$$y_x = ax + b\tilde{x}, \quad a, b \in \mathbb{R}. \quad (1.6)$$

The $\mathfrak{F}_x$ -plane is closed under GFT. From (1.4) we see that the quartic form and, thus, the Bekenstein-Hawking entropy, is invariant under the special set of GFT with $\lambda = \pm 1$. In particular, for any $x, \Delta(x) \neq 0$ there exists a one-parameter subgroup of $\text{Aut}(\mathfrak{F})$ that preserves the $\mathfrak{F}_x$ -plane and the Bekenstein-Hawking entropy. These will be referred to as Freudenthal rotations. Although GFT are non-linear, there always exists a linearly acting “gauged” U-duality³ transformation that sends x to x_F .

In the present work we introduce in detail these constructions and develop their applications to black holes in supergravity, as summarised here. An extended treatment of GFT is presented in sections 2–5. In the following sections these mathematical tools will be applied to the physics of black holes solutions in supergravity. First, in section 6 we will study the entropy properties of $\mathcal{N} = 2, D = 4$ pure supergravity from the point of view of the FTS formalism. This provides an example of a degenerate FTS, where the quartic invariant is a positive definite perfect square. As a consequence the Freudenthal plane in this case coincides with the entire FTS and the GFT are transitive on the space of charges. The Freudenthal rotations correspond precisely to familiar electromagnetic duality. To go beyond electromagnetic duality we consider in section 7 the axion-dilaton model, an $\mathcal{N} = 2, d = 4$ supergravity minimally coupled to one vector multiplet, which can be considered a consistent truncation of $\mathcal{N} = 4$ supergravity. Again, this model is degenerate and cannot be uplifted to $D = 5$. This is reflected in the non-reduced character of the FTS; it is not built from an underlying cubic Jordan algebra. In section 8 we proceed to the analysis of $\mathcal{N} = 2, D = 4$ supergravities admitting a $D = 5$ origin. The

³Note, here we are considering continuous charges; when they are quantised this no longer holds.

mathematical structure of these models is that of a reduced FTS, which may be derived from a cubic Jordan Algebra, J_3 , so that $\mathfrak{F} \cong \mathfrak{F}(J_3)$. In first place we study the T^3 model, or in Freudenthal terminology $\mathfrak{F}(\mathbb{R})$.

In subsection 8.4 we study the question of orbit stratification of the $\Delta > 0$ locus of $\mathfrak{F}(\mathbb{R})$ and its preservation by GFT. In section 9 we show, in different examples, how the action of GFT, and, in particular, Freudenthal duality can be realised by U-duality transformations that are “gauged” in the sense that they depend on the element of $\mathfrak{F}$ to which they are applied. Finally in section 10 we present some further physical discussion, summary and conclusions. We study the properties of asymptotically small interacting black holes. In the different appendices we present a summary of formulae used throughout the work and further technical details.

2 Freudenthal triple systems: definitions and properties

In 1954 Freudenthal [15, 16] constructed the exceptional Lie group E_7 (of dimension 133) as the automorphism group of a structure based on the smallest, non-trivial E_7 irrepr. **56**, in turn related to the exceptional Jordan algebra J_3^O of 3×3 Hermitian octonionic matrices (also referred to as the Albert algebra) [40]. Freudenthal’s aforementioned construction is often referred to as a *Freudenthal triple system* (FTS) for reasons that shall become clear shortly.

At the end of 60’s, Meyberg [41] and Brown [14] elaborated the axioms on which the, completely symmetric, ternary structure underlying an FTS is based; in fact, the E_7 irrepr. **56** is just an example of a class of modules, characterising certain Lie groups as of groups “of type E_7 ”. The role of the FTS’s in $D = 4$ Maxwell-Einstein (super)gravity theories was discovered later [42–44] to be related to the representation of the electric-magnetic (dyonic) charges of black hole solutions.

A FTS is defined [14] as a finite dimensional vector space $\mathfrak{F}$ over a field $\mathbb{F}$ ⁴ (not of characteristic 2 or 3), such that:

1. $\mathfrak{F}$ possesses a non-degenerate antisymmetric bilinear form $\{x, y\}$.
2. $\mathfrak{F}$ possesses a completely symmetric four-linear form $\Delta(x, y, z, w)$ which is not identically zero. This quartic linear form induces a ternary product $T(x, y, z)$ defined on $\mathfrak{F}$ by

$$\{T(x, y, z), w\} = 2\Delta(x, y, z, w).$$

3. For the ternary product $T(x, y, z)$ it is required that

$$3\{T(x, x, y), T(y, y, y)\} = 2\{x, y\}\Delta(x, y, y, y). \quad (2.1)$$

In our case of interest, the semi-classical supergravity limit, the physical vector of charges x is to be regarded as continuous and the associated FTS is taken to be over $\mathbb{R}$ or $\mathbb{C}$.

⁴In the following treatment, we will consider $\mathbb{F} = \mathbb{R}$ (classical/(super)gravity level). The (quantum/Dirac-Schwinger-Zwanzinger-quantized) case (and further extensions thereof) will be investigated elsewhere. The complex case $\mathbb{F} = \mathbb{C}$ is relevant for quantum qubit entanglement applications.

The *automorphism* group of an FTS is defined as the set of invertible $\mathbb{F}$ -linear transformations preserving the quartic and quadratic forms:

$$\text{Aut}(\mathfrak{F}) \equiv \{\sigma \in \text{Iso}_{\mathbb{F}}(\mathfrak{F}) | \{\sigma x, \sigma y\} = \{x, y\}, \Delta(\sigma x) = \Delta(x)\}. \quad (2.2)$$

An important operation in what follows is the $\mathfrak{T}$ -dual,⁵ $' : \mathfrak{F} \rightarrow \mathfrak{F}$, defined by

$$x \mapsto x' := T(x, x, x) \equiv T(x). \quad (2.3)$$

Note that, the conditions $\{\sigma x, \sigma y\} = \{x, y\}$ and $\Delta(\sigma x) = \Delta(x)$ immediately imply the homogeneity of the $\mathfrak{T}$ -map

$$T(\sigma x) = \sigma T(x). \quad (2.4)$$

Hence $\text{Aut}(\mathfrak{F})$ is the set of automorphisms in the conventional sense.

The Lie algebra $\mathfrak{aut}(\mathfrak{F})$ of $\text{Aut}(\mathfrak{F})$ is given by

$$\mathfrak{aut}(\mathfrak{F}) = \{\phi \in \text{Hom}_{\mathbb{F}}(\mathfrak{F}) | \Delta(\phi x, x, x, x) = 0, \{\phi x, y\} + \{x, \phi y\} = 0, \forall x, y \in \mathfrak{F}\}, \quad (2.5)$$

as is easily verified [45]. The first of the conditions can be restated as $\{\phi x, x'\} = 0$.

The $\mathbb{F}$ -linear map $\Upsilon_x : \mathfrak{F} \rightarrow \mathfrak{F}$ defined by

$$\Upsilon_x(y) = 3T(x, x, y) + \{x, y\}x \quad (2.6)$$

is in $\mathfrak{aut}(\mathfrak{F})$. This is a direct consequence of axiom III eq. (2.6). In fact note that eq. (2.1) can be reexpressed as

$$\{\Upsilon_x(y), y'\} = 0.$$

Note that, in particular,

$$\Upsilon_x(x) = 3x',$$

but $'$ is not in $\mathfrak{aut}(\mathfrak{F})$.

The linear map Υ_x was introduced in this $\text{Aut}(\mathfrak{F})$ -covariant form in [46]. For $\Delta(x) \neq 0$ we may also define the normalized map

$$\overline{\Upsilon}_x \equiv \frac{1}{3\sqrt{|\Delta(x)|}} \Upsilon_x. \quad (2.7)$$

Linearizing eq. (2.6) with respect to x implies that $\Upsilon_{x,y} : \mathfrak{F} \rightarrow \mathfrak{F}$ defined by

$$\Upsilon_{x,y}(z) = 6T(x, y, z) + \{x, z\}y + \{y, z\}x \quad (2.8)$$

is also in $\mathfrak{aut}(\mathfrak{F})$ [47] (see also eq.11.b in [14]). In particular the following relation holds $\{\Upsilon_{x,y}(z), z'\} = 0$. We have also $\Upsilon_{x,x}(z) = 2\Upsilon_x(z)$ and, (see [14]), for any y, z , $\Upsilon_{y,y'}(z) = 0$

Following [46] an FTS element may be assigned a manifestly $\text{Aut}(\mathfrak{F})$ invariant *rank*, an integer function between 1 and 4 defined by the relations:

$$\begin{aligned} \text{Rank}(x) = 1 &\Leftrightarrow \Upsilon_x(y) = 0 \forall y, x \neq 0; \\ \text{Rank}(x) = 2 &\Leftrightarrow T(x) = 0, \exists y \text{ s.t. } \Upsilon_x(y) \neq 0; \\ \text{Rank}(x) = 3 &\Leftrightarrow \Delta(x) = 0, T(x) \neq 0; \\ \text{Rank}(x) = 4 &\Leftrightarrow \Delta(x) \neq 0. \end{aligned} \quad (2.9)$$

⁵Not to be confused with T-duality in string theory.

The Rank 1 conditions appeared before in [15]. We define the sets of elements of a given rank $\mathfrak{F}_{(k)} \equiv \{x \in \mathfrak{F} \mid \text{Rank}(x) = k\}$. The rank of a element can be related to the degree of supersymmetry preserved by the solution (see [48] and references therein).

Supergravity and the classification of FTS: an outline. An FTS is said to be reduced if it contains a strictly regular element: $\exists u \in \mathfrak{F}$ such that $T(u) = 0$ and $u \in \text{Range } L_{u,u}$ where $L_{x,y} : \mathfrak{F} \rightarrow \mathfrak{F}; \quad L_{x,y}(z) \equiv T(x, y, z)$. It can be proved [14, 49] that every simple *reduced* FTS $\mathfrak{F}$ is isomorphic to an FTS $\mathfrak{F}(J_3)$, where

$$\mathfrak{F}(J_3) \equiv \mathbb{F} \oplus \mathbb{F} \oplus J_3 \oplus \bar{J}_3, \quad (2.10)$$

with J_3 denoting a rank-3 Jordan algebra. All algebraic structures in $\mathfrak{F}(J_3)$ can be defined in terms of the basic Jordan algebra operations [14, 49] (also cfr. [50] and refs. therein). In a Maxwell-Einstein physical framework, the presence of an underlying Jordan algebra J_3 corresponds to the fact that the $D = 4$ Maxwell-Einstein (super)gravity theories can be obtained by dimensional reduction of a $D = 5$ theory, whose electric-magnetic (U-)duality⁶ is nothing but the reduced structure group of J_3 itself.

For $\mathfrak{F}(J_3^{\mathbb{A}})$, the automorphism group has a two element centre, and its quotient yields the simple groups listed *e.g.* in table 1 of [22], whereas for $\mathfrak{F}(\mathbb{R} \oplus \Gamma_{m,n})$ one obtains the semi-simple groups $\text{SL}(2, \mathbb{R}) \times \text{SO}(m+1, n+1)$ [14, 46, 54]. In all cases, $\mathfrak{F}$ fits into a symplectic representation of $\text{Aut}(\mathfrak{F})$, with dimensions listed *e.g.* in the rightmost column of table 1 of [22].

By confining ourselves to reduced FTS's $\mathfrak{F}(J_3)$ related to *simple* or *semi-simple* rank-3 Jordan algebras J_3 , one can exploit the Jordan-Von Neumann-Wigner classification [40], and enumerate the possible FTS's, depending on their dimension $\dim \mathfrak{F} = 2N$.⁷ A summary of this classification is presented in table 1.

Various $D = 4$ supergravities are listed in table 1: the semi-simple cases $\mathfrak{F}(\mathbb{R} \oplus \Gamma_{1,n-1})$ and $\mathfrak{F}(\mathbb{R} \oplus \Gamma_{5,n-1})$ correspond to $\mathcal{N} = 2$ resp. 4 Maxwell-Einstein supergravity, while $\mathfrak{F}(J_3^{\mathbb{A}}) \equiv \mathfrak{F}^{\mathbb{A}}$ correspond to the so-called $\mathcal{N} = 2$ “magic” Maxwell-Einstein supergravities⁸ [42]. Moreover, $\mathfrak{F}^{\mathbb{O}_s} \equiv \mathfrak{F}(J_3^{\mathbb{O}_s})$ pertains to maximal $\mathcal{N} = 8$ supergravity, and the simplest reduced FTS is $\mathfrak{F}(\mathbb{R})$, related to the so-called T^3 model of $\mathcal{N} = 2$, $D = 4$ supergravity (treated in subsection 8.1).

As evident from table 1, there are two (for $\mathbb{A} = \mathbb{R}$) or three (for $\mathbb{A} = \mathbb{C}, \mathbb{H}, \mathbb{O}$) possible FTS structures for $N = 3q + 4$, where $q = \dim(\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}) = 1, 2, 4, 8$, corresponding to $\mathfrak{F}^{\mathbb{A}}$ [42, 43] and $\mathfrak{F}^{\mathbb{A}_s}$ [51, 52, 57].

Finally, an FTS is said to be degenerate if its quartic form is identically proportional to the square of a quadratic polynomial. Note that FTS on “degenerate” groups of type E_7 (as

⁶Here U-duality is referred to as the “continuous” symmetries of [51, 52]. Their discrete versions are the U-duality non-perturbative string theory symmetries introduced by Hull and Townsend [53].

⁷Reduced FTS's have *at least* dimension $2N = 4$, namely they contain *at least* $N = 2$ Abelian vectors in $D = 4$. Within the $\mathcal{N} = 2$ interpretation, they are the $5D \rightarrow 4D$ Kaluza-Klein (KK) vector (*aka* the $D = 4$ graviphoton) and the $D = 5$ graviphoton (which becomes a matter photon in $D = 4$).

⁸The theories based on Lorentzian cubic Jordan algebras $J_{2,1}^{\mathbb{A}}$ and $J_{2,1}^{\mathbb{A}_s}$ correspond to certain classes of $\mathcal{N} = 2$ supergravities with non-homogeneous vector multiplets' scalar manifolds (cfr. *e.g.* [55, 56]).

N	J_3	$D = 4$ Maxwell-Einstein theory
2	$\mathbb{R}$	$\mathcal{N} = 2 \ T^3$
3	$\mathbb{R} \oplus \mathbb{R}$	$\mathcal{N} = 2, \ ST^2$
4	$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$	$\mathcal{N} = 2, \ STU$
5–6	$\mathbb{R} \oplus \Gamma_{n,3-n}, \ \mathbb{R} \oplus \Gamma_{n,4-n}$	$n=1 : \mathcal{N}=2, \ n_V=4, 5$
7	$\begin{cases} \mathbb{R} \oplus \Gamma_{n,5-n} \\ J_3^{\mathbb{R}} \end{cases}$	$\begin{cases} n=1 : \mathcal{N}=2, n_V=5, \ n=5 : \mathcal{N}=4, n_V=1 \\ \mathcal{N}=2 \text{ magic } \mathbb{R} \end{cases}$
8–9	$\mathbb{R} \oplus \Gamma_{n,6-n}, \ \mathbb{R} \oplus \Gamma_{n,7-n}$	$n=1 : \mathcal{N}=2, n_V=7, 8, \ n=5 : \mathcal{N}=4, n_V=2, 3,$
10	$\begin{cases} \mathbb{R} \oplus \Gamma_{n,8-n} \\ J_3^{\mathbb{C}}, \ J_3^{\mathbb{C}_s} \end{cases}$	$\begin{cases} n=1 : \mathcal{N}=2, n_V=9, \ n=5 : \mathcal{N}=4, n_V=3 \\ \mathcal{N}=2 \text{ magic } \mathbb{C}, \ \mathcal{N}=0 \text{ magic } \mathbb{C}_s \end{cases}$
11–15	$\mathbb{R} \oplus \Gamma_{n,9-n}, \dots, \ \mathbb{R} \oplus \Gamma_{n,13-n}$	$n=1 : \mathcal{N}=2, n_V=10-14, \ n=5 : \mathcal{N}=4, n_V=5-9,$
16	$\begin{cases} \mathbb{R} \oplus \Gamma_{n,14-n} \\ J_3^{\mathbb{H}}, \ J_3^{\mathbb{H}_s} \end{cases}$	$\begin{cases} n=1 : \mathcal{N}=2, n_V=15, \ n=5 : \mathcal{N}=4, n_V=10 \\ \mathcal{N}=2 \text{ magic } \mathbb{H}, \ \mathcal{N}=0 \text{ magic } \mathbb{H}_s \end{cases}$
17–27	$\mathbb{R} \oplus \Gamma_{n,15-n}, \dots, \ \mathbb{R} \oplus \Gamma_{n,25-n}$	$n=1 : \mathcal{N}=2, n_V=16-26, \ n=5 : \mathcal{N}=4, n_V=11-21$
28	$\begin{cases} \mathbb{R} \oplus \Gamma_{n,26-n} \\ J_3^{\mathbb{O}}, \ J_3^{\mathbb{O}_s} \end{cases}$	$\begin{cases} n=1 : \mathcal{N}=2, n_V=27, \ n=5 : \mathcal{N}=4, n_V=22 \\ \mathcal{N}=2 \text{ magic } \mathbb{O}, \ \mathcal{N}=8 \end{cases}$
> 28	$\mathbb{R} \oplus \Gamma_{n,N-2-n}$	$n=1 : \mathcal{N}=2, n_V > 27, \ n=5 : \mathcal{N}=4, n_V > 22$

Table 1. Classification of Freudenthal triple systems (see text for explanation).

defined in [27], and refs. therein) are not reduced and hence cannot be written as $\mathfrak{F}(J_3)$; they correspond to theories which cannot be uplifted to $D = 5$ dimensions consistently reflecting the lack of an underlying rank-3 Jordan algebra J_3 .

3 Freudenthal dualities and planes

F-duality. We have defined already the transformation $x' = T(x)$, valid for a vector of any rank (see eq. (2.3)). For rank-4 charge vectors $x \in \mathfrak{F}_{(4)}$, the black hole charge Freudenthal duality is defined by ($\epsilon \equiv \epsilon(x) \equiv \text{sgn } \Delta(x)$)

$$\tilde{\cdot} : \mathfrak{F}_{(4)} \rightarrow \mathfrak{F}_{(4)}, \quad x \mapsto \tilde{x} \equiv \epsilon(x) \frac{T(x)}{\sqrt{|\Delta(x)|}}. \quad (3.1)$$

The Freudenthal duality has the following elementary properties [48]:

- It preserves the quartic norm $\Delta(\tilde{x}) = \Delta(x)$; obviously $\Delta(\tilde{x}, x, x, x) = \{\tilde{x}, x'\} = 0$.
- It is an anti-involution: $\tilde{\tilde{x}} = -x$;
- It is not a U-duality, since it is non-linear and generically $\{\tilde{x}, \tilde{y}\} \neq \{x, y\}$. Also, in general, $\{\tilde{x}, y\} + \{x, \tilde{y}\} \neq 0$.

Note that, although the map $x \rightarrow \tilde{x}$ is not a U-duality, the map Υ_x (or $\bar{\Upsilon}$), for x fixed, is indeed:

- a) linear,
- b) $\Upsilon_x \in \mathbf{aut}(\mathfrak{F})$ and, finally
- c) $\bar{\Upsilon}_x(x) = \epsilon(x)\tilde{x}$.

It follows from (c), that for $\Delta(x) > 0$ the Freudenthal map $x \mapsto \tilde{x}$ can be considered as an “ x -dependent” U-duality.

The $\mathfrak{T}$ - and $\mathfrak{F}$ -planes: definition and general properties. For a general element in $\mathfrak{F}$, respectively an element $x \in \mathfrak{F}_{(4)}$, we define the associated $\mathfrak{T}$ - and $\mathfrak{F}$ -planes, respectively denoted $\mathfrak{T}_x, \mathfrak{F}_x$, as the $\mathbb{R}$ -linear spans of x, x' or $x, \tilde{x}$. In each case:

$$\mathfrak{T}_x \equiv \{y \in \mathfrak{F} | y = ax + bx', \ a, b \in \mathbb{R}\}, \quad (3.2)$$

$$\mathfrak{F}_x \equiv \{y \in \mathfrak{F}_{(4)} | y = ax + b\tilde{x}, \ a, b \in \mathbb{R}\}. \quad (3.3)$$

Naturally the $\mathfrak{F}$ -plane $\mathfrak{F}_x$ is only defined as long as $\Delta(x) \neq 0$ (maximal rank elements), while $\mathfrak{T}_x$ is defined for any x , although it degenerates to a $\mathfrak{T}$ -line for rank $x < 3$ elements. If they both exist, $\mathfrak{T}_x$ and $\mathfrak{F}_x$ are the same space. It is advantageous to study the properties of the $\mathfrak{T}$ -planes, and when needed, to specialise to $\mathfrak{F}$ -planes. We will follow this strategy in what follows.

Linearity of $\mathfrak{T}$ -transformations on the $\mathfrak{T}$ -plane. We first show the linearity of the $\mathfrak{T}$ -dual on the $\mathfrak{T}$ -plane: $\mathfrak{T}$ -planes are closed under $\mathfrak{T}$ -transformations. For any linear combination, one has, because of the multi-linearity of T , (a, b constants, $\Delta_x = \Delta(x)$),

$$\begin{aligned} T(ax + bx') &= a^3T(x) + b^3T(x') + 3a^2bT(x, x, x') + 3ab^2T(x, x', x') \\ &= a^3x' + ab^2\Delta_x x' - b^3\Delta_x^2 x - a^2b\Delta_x x \\ &= (a^2 + b^2\Delta_x)(-b\Delta_x x + ax'). \end{aligned} \quad (3.4)$$

Where we have used the properties ([14], lemma 11.(abcf)):

$$T(x, x, x') = -1/3\Delta_x x, \quad (3.5)$$

$$T(x, x', x') = 1/3\Delta_x x', \quad (3.6)$$

$$T(x', x', x') = -\Delta_x^2 x. \quad (3.7)$$

We can see that eq. (3.4) is equivalent to, or simply summarizes, the relations eqs. (3.5)–(3.7).

Using eq. (3.4) we can compute the map Δ for any element on the $\mathfrak{T}$ -plane. After a short explicit computation we have (using $2\Delta(x) = \{x', x\}$)

$$\Delta(ax + bx') = \frac{1}{2}\{T(ax + bx'), ax + bx'\} \quad (3.8)$$

$$= (a^2 + b^2\Delta(x))^2 \Delta(x). \quad (3.9)$$

The sign of Δ on the $\mathfrak{T}$ -plane is constant. Hence, in any $\mathfrak{F}$ -plane, there is an element $y \in \mathfrak{F}_x$ such $\Delta(y) = 0$ if and only if $\Delta(x)$ is negative.

Similarly, we have the following expressions describing the behaviour of the map Υ on the $\mathfrak{T}$ -plane:

$$\Upsilon_x(x') = 3T(x, x, x') + \{x, x'\}x = -\Delta_x x - 2\Delta_x x = -3\Delta_x x, \quad (3.10)$$

$$\Upsilon_x(ax + bx') = -3b\Delta(x)x + 3ax' = 3T(ax + bx')/(a^2 + b^2\Delta_x). \quad (3.11)$$

Further mathematical properties of the $\mathfrak{T}$ -planes are shown in section A and section B.

$\mathfrak{T}$ - and $\mathfrak{F}$ -transformations on the planes. For $\tilde{x}$ well-defined, we rewrite eq. (3.4), eq. (3.9), by a redefinition of the parameters a, b in terms of the F-dual eq. (3.1) as ($\epsilon = \text{sgn } \Delta(x)$)

$$T(ax + b\tilde{x}) = \epsilon (a^2 + \epsilon b^2) \sqrt{|\Delta(x)|} (-\epsilon bx + a\tilde{x}), \quad (3.12)$$

$$\Delta(ax + b\tilde{x}) = (a^2 + \epsilon b^2)^2 \Delta(x). \quad (3.13)$$

It is clear from this last expression that $a^2 + \epsilon b^2 = \pm 1$ defines a subset of elements in $\mathfrak{F}_x$ with fixed entropy: an $\text{SO}(2)$ or $\text{SO}(1, 1)$ symmetry. Moreover, $\text{sgn } \Delta(x) = \text{sgn } \Delta(ax + b\tilde{x}) = \epsilon$, unless $a^2 + \epsilon b^2 = 0$. In particular

$$\Delta(x \pm \tilde{x}) = (1 + \epsilon)^2 \Delta(x). \quad (3.14)$$

As consequence of eq. (3.12) and eq. (3.13), the F-dual of a linear combination is given by ($\Delta(x) \neq 0, \Delta(ax + b\tilde{x}) \neq 0, \text{sgn } \Delta(x) = \text{sgn } \Delta(ax + b\tilde{x}) = \epsilon$)

$$\widetilde{ax + b\tilde{x}} = \frac{\epsilon T(ax + b\tilde{x})}{\sqrt{|\Delta(ax + b\tilde{x})|}} \quad (3.15)$$

and, finally,

$$\widetilde{ax + b\tilde{x}} = \eta (-\epsilon bx + a\tilde{x}) = \begin{cases} -bx + a\tilde{x}, & (\Delta(x) > 0), \\ \eta (bx + a\tilde{x}), & (\Delta(x) < 0), \end{cases} \quad (3.16)$$

where $\epsilon = \text{sgn } \Delta(x), \eta = \text{sgn}(a^2 + b^2\epsilon)$. Clearly, if $\epsilon = 1$ then $\eta = 1$

As discussed in appendix subsection B.4 the $\mathfrak{F}$ -plane is a, quadratic, two dimensional, sub-FTS system with suitably restricted operations $\Delta_{\mathfrak{F}}, T_{\mathfrak{F}}, \{, \}_{\mathfrak{F}}$ and I_2 .

On the full FTS, for maximal rank elements, one can define an $(\text{Aut}(\mathfrak{F})$ -invariant “metric” by the (non quadratic) expression

$$(u, v) \equiv \frac{1}{4} [\{\tilde{u}, v\} + \{\tilde{v}, u\}]. \quad (3.17)$$

and a “pseudo-norm” by

$$\|u\| = (u, u) = \frac{1}{2} \{\tilde{u}, u\} = \epsilon \sqrt{|\Delta(u)|}. \quad (3.18)$$

If we fix x , and restrict ourselves to the $\mathfrak{F}_x$ plane we can use the expressions in subsection B.4 (see also further properties in ref. [58]) and connect $\|u\|$ with $I_2(u)$:

$$(u, v) \equiv \frac{1}{4} [\{\tilde{u}, v\} + \{\tilde{v}, u\}] \quad (3.19)$$

$$= \frac{1}{4} [\eta(u)\{\hat{S}u, v\} + \eta(v)\{\hat{S}v, u\}] \quad (3.20)$$

$$= \frac{1}{4} (\eta(u) + \eta(v))\{\hat{S}v, u\} \quad (3.21)$$

$$= \frac{1}{4} \epsilon (\eta(u) + \eta(v)) I_2(v, u) \quad (3.22)$$

where $\epsilon = \text{sgn } \Delta(x)$, $\eta(x) = \text{sgn } I_2(x)$ and $\hat{S}$ is a linear map given in section B. In particular

$$\|u\| = \frac{1}{2} \epsilon \eta(u) I_2(u) = \frac{1}{2} \epsilon |I_2(u)| \quad (3.23)$$

We arrive at the same conclusions as for the $\mathfrak{T}$ -plane. For $\epsilon = 1$, the pseudo-norm $(\cdot, \cdot)$ (or $I_2(u)$) is positive definite and the norm-preserving group is $\text{SO}(2)$; thus, the $\mathfrak{F}$ -plane $\mathfrak{F}_x$ undergoes a “*spherical foliation*”. On the other hand, for $\epsilon = -1$, the norm is positive semi-definite and the norm-preserving group is $\text{SO}(1, 1)$; thus, the $\mathfrak{F}$ -plane $\mathfrak{F}_x$ undergoes an “*hyperboloid-like foliation*”. While the norm is timelike or null, the vector u can be timelike, spacelike and null according to $\eta(x)$, the sign of $I_2(x)$.

As for the $\mathfrak{T}$ -operation, the F-duality $x \rightarrow \tilde{x}$, change the character of the vector. The vectors $x, \tilde{x}$ are “ I_2 -orthogonal”, by eq. (B.29), $(x, \tilde{x}) = I_2(x, \tilde{x}) = 0$. $\tilde{x}$ is timelike (resp. spacelike) if x is spacelike (resp. timelike):

$$x : \text{lightlike} \longleftrightarrow \tilde{x} : \text{lightlike}, \quad \tilde{x} = \pm x, \quad (3.24)$$

$$x : \text{timelike (spacelike)} \longleftrightarrow \tilde{x} : \text{spacelike (timelike)}. \quad (3.25)$$

It is noted that, although the metric $I_2(x, y)$ is defined only inside a concrete $\mathfrak{F}$ -plane, the character null, time or spacelike of a vector is an intrinsic property, as any given element belongs to one and only one $\mathfrak{F}$ -plane, “its” plane, from the disjointness of the $\mathfrak{F}$ -planes (see subsection B.2).

4 The orthogonal space $\mathfrak{F}_x^\perp$ and the orthogonal plane $\mathfrak{F}_{y \perp x}$

In general $\{x, \tilde{x}\} = -2\sqrt{|\Delta(x)|} \neq 0$. The bilinear form $\{\cdot, \cdot\}$ is non-degenerate on $\mathfrak{F}_x$ by construction, since x is necessarily of maximal rank ($\Delta(x) \neq 0$). Consequently, for a given $x \in \mathfrak{F}_{(4)}$, the FTS $\mathfrak{F}$ may be decomposed as

$$\mathfrak{F} = \mathfrak{F}_x \oplus \mathfrak{F}_x^\perp, \quad (4.1)$$

where $\mathfrak{F}_x$ is the 2-dimensional $\mathfrak{F}$ -plane and $\mathfrak{F}_x^\perp$ is its $(\dim_{\mathbb{F}} \mathfrak{F} - 2)$ -dimensional orthogonal complement w.r.t. the bilinear form $\{\cdot, \cdot\}$:

$$\mathfrak{F}_x^\perp = \{y \in \mathfrak{F} : \{x, y\} = \{x', y\} = 0\}. \quad (4.2)$$

Hence, for a given $x \in \mathfrak{F}$, any element $y \in \mathfrak{F}$ enjoys the decomposition

$$y = y_{\parallel x} + y_{\perp x}, \quad (4.3)$$

where $y_{\parallel x} \in \mathfrak{F}_x$ and $y_{\perp x} \in \mathfrak{F}_x^\perp$. Note that $y_{\parallel x}, y_{\perp x}$ (also denoted $y_{\parallel}, y_{\perp}$ if there is no risk of confusion) are uniquely determined by x . The coordinates of $y_{\parallel x} = ax + b\tilde{x}$ are uniquely determined by the expressions,

$$\{x, y_{\perp}\} = 0 \Leftrightarrow b = \frac{-1}{2\sqrt{|\Delta(x)|}} \{x, y\}, \quad (4.4)$$

$$\{\tilde{x}, y_{\perp}\} = 0 \Leftrightarrow a = \frac{1}{2\sqrt{|\Delta(x)|}} \{\tilde{x}, y\}. \quad (4.5)$$

Or, in compact notation (with respect a fixed element x),

$$y_{\parallel} = \frac{1}{2\sqrt{|\Delta(x)|}} \begin{vmatrix} x & \tilde{x} \\ \{x, y\} & \{\tilde{x}, y\} \end{vmatrix}, \quad (4.6)$$

$$= \frac{1}{\{x', x\}} \begin{vmatrix} x & x' \\ \{x, y\} & \{x', y\} \end{vmatrix}, \quad (4.7)$$

$$y_{\perp} = y - y_{\parallel}. \quad (4.8)$$

The following properties hold (for the parallel component with respect a fixed x):

$$\Delta(y_{\parallel}) = \frac{1}{16\Delta(x)} (\epsilon\{\tilde{x}, y\}^2 + \{x, y\}^2)^2 \quad (4.9)$$

$$(ay + b\tilde{y})_{\parallel} = ay_{\parallel} + b(\tilde{y})_{\parallel} \quad (4.10)$$

$$\widetilde{(y_{\parallel})} = \frac{\eta\epsilon}{2\sqrt{|\Delta(x)|}} (\epsilon\{\tilde{x}, y\}\tilde{x} + \{x, y\}x). \quad (4.11)$$

with $\eta = \text{sgn}(\epsilon\{\tilde{x}, y\}^2 + \{x, y\}^2)$. In particular, note the distributivity of $\parallel$ (second expression).

Note, however that

$$\widetilde{\tilde{y}}_{\parallel} \neq (\tilde{y})_{\parallel} = \frac{(\{\tilde{x}, \tilde{y}\}x - \{x, \tilde{y}\}\tilde{x})}{2\sqrt{|\Delta_x|}}. \quad (4.12)$$

Obviously, a similar construction can be performed for the $\mathfrak{T}$ -plane (see [14], from pg. 89 on, where such a space is used to build a Jordan Algebra for reduced FTSs).

This decomposition into “parallel” and “orthogonal” spaces and the further decomposition of the orthogonal space in orthogonal planes (to be defined in the next section) will be important in what follows.

The $\mathfrak{F}_{y\perp x}$ plane. Consider an arbitrary reference vector x (of maximal rank for simplicity) and a perpendicular vector $y \in \mathfrak{F}_x^\perp$. We define the space, $\mathfrak{F}_{y\perp x} := \text{span}\{y, \Upsilon_x(y)\}$. That is

$$\mathfrak{F}_{y\perp x} := \{ay + b\Upsilon_x(y) \mid y \in \mathfrak{F}_x^\perp \quad a, b \in \mathbb{R}\}. \quad (4.13)$$

We will show that the “planes” $\mathfrak{F}_x$ and $\mathfrak{F}_{y \perp x}$ are $\{, \}$ -orthogonal:

$$\mathfrak{F}_{y \perp x} \subseteq \mathfrak{F}_x^\perp.$$

For any $y \in \mathfrak{F}_x^\perp$, we can show that also $\Upsilon_x(y) \in \mathfrak{F}_x^\perp$. We have indeed (using Equation (11c) in [14] in the first line and axiom 3 in the second line)

$$\{x', \Upsilon_x(y)\} = 3\{T(x, x, x), T(x, x, y)\} = -\{y, x'\}\Delta(x) = 0, \quad (4.14)$$

$$\{x, \Upsilon_x(y)\} = \{x, 3T(x, x, y)\} = 3\{y, T(x, x, x)\} = 3\{y, x'\} = 0. \quad (4.15)$$

which implies $\Upsilon_x(y) \in \mathfrak{F}_x^\perp$. Obviously, the equality $\mathfrak{F}_{y \perp x} = \mathfrak{F}_x^\perp$ is only possible for $\dim(\mathfrak{F}) = 4$, as $\dim(\mathfrak{F}_x^\perp) = \dim(\mathfrak{F}) - 2$.

We show next that, in the same case, successive powers of Υ_x acting on y belong to the orthogonal plane. In fact, $\Upsilon_x^n(y)$ is proportional to y or $\Upsilon_x(y)$. We have for example (as for any $y \in \mathfrak{F}_x^\perp$)

$$\begin{aligned} \Upsilon_x^2(y) &\equiv \Upsilon_x(\Upsilon_x(y)) = 9T(x, x, T(x, x, y)) \\ &= -2\Delta(x, x, x, y)x - \Delta(x)y - \{y, x\}x' = -\{x', y\} - \Delta(x)y \end{aligned} \quad (4.16)$$

$$= -\Delta(x)y. \quad (4.17)$$

In the second line we have used the Lemma 1(11e) in [14]. In general for any n , we have, by using induction (for any $y \in \mathfrak{F}_x^\perp$), for $n \geq 1$,

$$\Upsilon_x^{2n}(y) = (-1)^n \Delta(x)^n y, \quad (4.18)$$

$$\Upsilon_x^{2n+1}(y) = (-1)^n \Delta(x)^n \Upsilon_x(y). \quad (4.19)$$

Let us remark that $\mathfrak{F}_{y \perp x}$ is not closed in general under $\mathfrak{T}$ -transformations, it is not a sub-FTS with the operations inherited from the parent FTS. The plane $\mathfrak{F}_{y \perp x}$ is however closed under the Υ_x map. For any element belonging to it ($u \in \mathfrak{F}_{y \perp x}$, $u = ay + b\Upsilon_x(y)$),

$$\Upsilon_x(u) = 3T(x, x, ay + b\Upsilon_x(y)) + \{x, ay + b\Upsilon_x(y)\}x \quad (4.20)$$

$$= 3aT(x, x, y) + 3bT(x, x, \Upsilon_x(y)) \quad (4.21)$$

$$= -b\Delta(x)y + a\Upsilon_x(y), \quad (4.22)$$

where in the second line we have used the equality expressed by eq. (4.16) (see also [14]). According to this $\Upsilon_x(u) \in \mathfrak{F}_x^\perp$.

4.1 Behaviour of Δ on the $\mathfrak{F}_{y \perp x}$ plane

We are interested in the behaviour of Δ on the $\mathfrak{F}_{y \perp x}$ plane. For any $u \in \mathfrak{F}_{y \perp x}$, $u = ay + b\Upsilon_x(y)$, we have, by combining eq. (B.10) with eq. (B.11)

$$\Delta(u) = (a^2 + b^2\Delta(x))^2 \Delta(y), \quad (4.23)$$

$$\Delta(\Upsilon_x(y)) = \Delta(x)^2 \Delta(y). \quad (4.24)$$

Or, in normalized terms

$$\Delta(\tilde{\Upsilon}_x(y)) \equiv \Delta\left(\frac{\Upsilon_x(y)}{\sqrt{|\Delta(x)|}}\right) = \Delta(y). \quad (4.25)$$

The $\tilde{\Upsilon}_x$ map thus preserves both the bilinear and quartic invariants in each of the $\mathfrak{F}_{y\perp x}$ planes. Applying twice eq. (4.25) we arrive to

$$\Delta(\tilde{\Upsilon}_x^2(y)) = \Delta(y) \quad (4.26)$$

and in general

$$\Delta(\tilde{\Upsilon}_x^n(y)) = \Delta(y). \quad (4.27)$$

Combining eqs. (4.23), (4.24) with (3.9) we arrive to an expression

$$\Delta(ay + b\Upsilon_x(y))\Delta(x) = \Delta(a + bx')\Delta(y). \quad (4.28)$$

which relates the behaviour of the quartic invariant Δ on the $\mathfrak{T}_x$ and $\mathfrak{F}_{y\perp x}$ planes.

The behaviour of Δ on the $\mathfrak{F}_x$ or $\mathfrak{F}_{y\perp x}$ planes is similar but with some important differences. In the $\mathfrak{F}_{y\perp x}$ case it depends on the signs of both $\Delta(x)$ and $\Delta(y)$. The overall sign of all the elements of the $\mathfrak{F}_{y\perp x}$ plane is the same as $\Delta(y)$ excluding the null elements such that

$$a^2 + b^2\Delta(x) = 0. \quad (4.29)$$

For example, any element of the form

$$z_{\pm} \propto \sqrt{|\Delta(x)|} |y \pm \Upsilon_x(y)| \quad (4.30)$$

is null, $\Delta(z_{\pm}) = 0$ (for $\Delta(x) < 0, y \in \mathfrak{F}_x^{\perp}$). We observe that the null elements of $\mathfrak{F}_x$ and any $\mathfrak{F}_{y\perp x}$ are aligned, they are given by the same eq. (4.29) which it is independent of y .

5 Freudenthal rotations: the Υ map and its exponential

The closure of $\mathfrak{F}_x$ under F-duality implies the existence of a one-parameter family of U-duality transformations stabilising $\mathfrak{F}_x$ as it will be shown in this section.

Recall, for any fixed x , Υ_x is in $\text{aut}(\mathfrak{F})$. In particular, the normalised version, $\bar{\Upsilon}_x$, given in eq. (2.7), acting on x itself maps x into its F-dual,

$$\bar{\Upsilon}_x(x) = \epsilon \tilde{x}. \quad (5.1)$$

Note, we also have the relation ($\epsilon = \text{sgn } \Delta(x)$)

$$\Upsilon_{\tilde{x}}(x) = 3T(\tilde{x}, \tilde{x}, x) + \{\tilde{x}, x\}\tilde{x} \quad (5.2)$$

$$= \frac{1}{|\Delta(x)|} (3T(x', x', x) + \{x', x\}x'), \quad (5.3)$$

$$= \frac{1}{|\Delta(x)|} (\Delta(x)x' + 2\Delta(x)x') = 3\epsilon x', \quad (5.4)$$

then

$$\bar{\Upsilon}_{\tilde{x}}(x) = \frac{\epsilon}{\sqrt{|\Delta(\tilde{x})|}} x' = \tilde{x}. \quad (5.5)$$

It is obvious that Υ_x (as well as $\Upsilon_{ax+b\tilde{x}}$) is a $\mathfrak{F}_x \rightarrow \mathfrak{F}_x$ map. Furthermore, the set of maps $\{\Upsilon_{ax+b\tilde{x}}\}_{a,b \in \mathbb{R}}$ for a fixed x forms an two-parametric automorphism subalgebra.

We are interested here in the action of the Υ_x map and the computation of its exponential. For this purpose, it is convenient to distinguish the action of any Υ_x on its particular associated $\mathfrak{F}_x$ plane and on the respective orthogonal complement $\mathfrak{F}_x^\perp$.⁹

5.1 The exponential map on the $\mathfrak{F}_x$ -plane

The action of the exponential of the (normalized) map $\bar{\Upsilon}_x$ reads as follows. For any rank-4, fixed, $x \in \mathfrak{F}$, ($\theta \in \mathbb{F}$, $\epsilon(x) = \text{sgn } \Delta(x)$)

$$\exp(\theta \bar{\Upsilon}_x)(x) = \cos(\sqrt{\epsilon}\theta) x + \sqrt{\epsilon} \sin(\sqrt{\epsilon}\theta) \tilde{x}, \quad (5.6)$$

$$\exp(\theta \bar{\Upsilon}_x)(\tilde{x}) = -\epsilon \sqrt{\epsilon} \sin(\sqrt{\epsilon}\theta) x + \cos(\sqrt{\epsilon}\theta) \tilde{x}, \quad (5.7)$$

$$\exp(\theta \bar{\Upsilon}_{\tilde{x}})(x) = \cos(\sqrt{\epsilon}\theta) x + \epsilon \sqrt{\epsilon} \sin(\sqrt{\epsilon}\theta) \tilde{x}, \quad (5.8)$$

$$\exp(\theta \bar{\Upsilon}_{\tilde{x}})(\tilde{x}) = -\sqrt{\epsilon} \sin(\sqrt{\epsilon}\theta) x + \cos(\sqrt{\epsilon}\theta) \tilde{x}, \quad (5.9)$$

where $x \in \mathfrak{F}$ and $\exp$ is defined by the usual infinite series.

The proof of eqs. (5.6)–(5.9) is based in the following properties:

$$\Upsilon_x^{2n}(x) = (-1)^n (3)^{2n} \Delta(x)^n x, \quad (5.10)$$

$$\Upsilon_x^{2n+1}(x) = (-1)^n (3)^{2n+1} \Delta(x)^n x', \quad (5.11)$$

$$\Upsilon_{\tilde{x}}^{2n}(x) = \Upsilon_x^{2n}(x), \quad (5.12)$$

$$\Upsilon_{\tilde{x}}^{2n+1}(x) = \epsilon \Upsilon_x^{2n+1}(x), \quad (5.13)$$

which are obtained by induction starting on with $\Upsilon_x(x) = 3x'$, $\Upsilon_x^2(x) = 9T(x, x, x') = -3\Delta(x)x$. By linearity $\exp \theta \Upsilon_x$ can be extended to the full $\mathfrak{F}_x$ plane.

Explicitly, for $\Delta_x = \Delta_{\tilde{x}} > 0 \Leftrightarrow \epsilon = 1$, it holds that

$$\exp(\theta \bar{\Upsilon}_x)(x) = \cos(\theta) x + \sin(\theta) \tilde{x}, \quad (5.14)$$

$$\exp(\theta \bar{\Upsilon}_x)(\tilde{x}) = -\sin(\theta) x + \cos(\theta) \tilde{x}, \quad (5.15)$$

$$\exp(\theta \bar{\Upsilon}_{\tilde{x}})(x) = \cos(\theta) x + \sin(\theta) \tilde{x} = \exp(\theta \bar{\Upsilon}_x)(x), \quad (5.16)$$

$$\exp(\theta \bar{\Upsilon}_{\tilde{x}})(\tilde{x}) = -\sin(\theta) x + \cos(\theta) \tilde{x} = \exp(\theta \bar{\Upsilon}_x)(\tilde{x}), \quad (5.17)$$

whereas for $\Delta_x = \Delta_{\tilde{x}} < 0 \Leftrightarrow \epsilon = -1$, it holds that

$$\exp(\theta \bar{\Upsilon}_x)(x) = \cosh(\theta) x - \sinh(\theta) \tilde{x}, \quad (5.18)$$

$$\exp(\theta \bar{\Upsilon}_x)(\tilde{x}) = -\sinh(\theta) x + \cosh(\theta) \tilde{x}, \quad (5.19)$$

$$\exp(\theta \bar{\Upsilon}_{\tilde{x}})(x) = \cosh(\theta) x + \sinh(\theta) \tilde{x}, \quad (5.20)$$

$$\exp(\theta \bar{\Upsilon}_{\tilde{x}})(\tilde{x}) = \sinh(\theta) x + \cosh(\theta) \tilde{x}. \quad (5.21)$$

⁹For convenience, we work indistinctly on the $\mathfrak{F}_x, \mathfrak{F}_x^\perp$ or on $\mathfrak{T}_x, \mathfrak{T}_x^\perp$. They are equivalent as long $\Delta(x) \neq 0$.

Hence, the set of transformations $\exp \theta \Upsilon_x$ form an automorphism subgroup $\text{Aut}(\mathfrak{F}_x) \subseteq \text{Aut}(\mathfrak{F})$ preserving the $\mathfrak{F}_x$ plane.

To summarise, as a consequence of eqs. (5.6) and (5.7), for any rank-4 $x \in \mathfrak{F}$, there exists a monparametric subgroup $\sigma_x(\theta) \in \text{Aut}(\mathfrak{F}_x)$ which is made of “rotations” in $\mathfrak{F}_x$ and whose generator is Υ_x :

$$\sigma_x(\theta) \equiv e^{\theta \Upsilon_x}.$$

Let us study the details of the automorphism subgroup $\sigma_x(\theta)$ depending on the sign of $\Delta(x)$. For $\Delta(x) > 0$, ($\epsilon = 1$) the subgroup $\sigma_x(\theta)$ is $\text{SO}(2)$. The Freudenthal rotation with $\theta = \pi/2$ is the U-duality transformation relating x to its F-dual. For $\mathcal{N} = 8$ black holes with $G_4 = E_{7(7)}$ the existence of a U-duality connecting x and $\tilde{x}$ was guaranteed since all x with the same $\Delta(x) > 0$ belong to the same $E_{7(7)}$ orbit. For $\mathcal{N} < 8$ not all x with the same $\Delta(x) > 0$ necessarily lie in the same U-duality orbit; the orbits are split by further U-duality invariant conditions. See [45, 59] and the references therein. Nonetheless, for $\epsilon = 1$ the Freudenthal rotation given by eq. (5.6) with $\theta = \pi/2$ implies that x and $\tilde{x}$ are in the same U-duality orbits for all FTS.

On the other hand, for $\Delta(x) < 0$, ($\epsilon = -1$) the subgroup $\sigma_x(\theta)$ is $\text{SO}_0(1, 1)$ which has three different kinds of orbits: the origin (a group fixed point), the four rays $\{(\pm t, \pm t), t > 0\}$, and the hyperbolae $a^2 - b^2 = \pm r^2$. The Freudenthal rotation cannot relate x to its F-dual (by inspection of eq. (5.7), the orbits of the exponential of the Υ are hyperbolic). Therefore $x, \tilde{x}$ lie in different branches. However, for any FTS, all x with the same $\Delta(x) < 0$ lie in the same U-duality orbit [45, 59]. Hence, there exists a U-duality transformation, which is determined by x , connecting x and $\tilde{x}$ for $\Delta(x) < 0$. But, this U-duality transformation is not represented by any member of $\sigma_x(\theta)$. In fact, as we shall see there is a one-parameter family of U-dualities which connects x and $\tilde{x}$ but does not preserve the $\mathfrak{F}_x$ -plane. We will return to this question in the next sections.

In summary, putting together the previous comments, we arrive to the conclusion that

- a) For all supergravities with E_7 -type duality group of any $\mathcal{N}$, large BH have charges x and $\tilde{x}$ in the same U-duality orbit, irrespective of the sign of $\Delta(x)$.
- b) For $\Delta(x) > 0$ the orbit of $\sigma_x(\theta)$, which relates the F-dual BHs, is contained in the $\mathfrak{F}_x$ -plane. For $\Delta(x) < 0$, the orbit of the one-parameter subgroup, introduced later, connecting x and $\tilde{x}$ does not preserve the $\mathfrak{F}_x$ -plane. It would perhaps be “natural” to conjecture that this orbit only intersects the $\mathfrak{F}_x$ -plane only at x and $\tilde{x}$. We will come back to this point later on.

Note, a similar treatment can be performed for the case of small BHs, $\Delta(x) = 0$. In this case the group generated by Υ_x has orbits corresponding to null rays.

6 Pure $\mathcal{N} = 2, D = 4$ supergravity and degenerate FTS

The simplest example of a FTS (which is, being two-dimensional, a Freudenthal plane with $\Delta(x) > 0$) in supergravity is provided by the one associated to “pure” $\mathcal{N} = 2, D = 4$ supergravity, whose purely bosonic sector is the simplest (scalarless) instance of Maxwell-Einstein gravity. In such a theory, the asymptotically flat, spherically symmetric, dyonic extremal Reissner-Nordström (RN) black hole (BH) solution has Bekenstein-Hawking entropy

$$\frac{S_{RN}}{\pi} = \frac{1}{2} (p^2 + q^2), \quad (6.1)$$

where p and q are the magnetic resp. electric fluxes associated to the unique Abelian vector field (which, in the $\mathcal{N} = 2$ supersymmetric interpretation, is the so-called graviphoton).

In this case, the associated FTS $\mathfrak{F}_{\mathcal{N}=2}$ “pure” has $\dim = 2$ (i.e., it has $N = 1$, within the previous treatment); it is immediate to realize that this *cannot* be a reduced FTS, because¹⁰ defining

$$x = (p, q)^T \quad (6.2)$$

then the associated quartic invariant $\Delta(x)$ is defined by

$$\Delta(x) = \frac{1}{4} (p^2 + q^2)^2 = \frac{S_{RN}^2}{\pi^2} \geq 0 \quad (6.3)$$

for any choice of p and q .

This system can be considered a BPS ($\Delta(x) > 0$) prototype. Let us start by doing some, simple, explicit computations. For this purpose let us choose (without any loss of generality) a vector given by

$$x = (p, 0)^T, \quad (6.4)$$

which corresponds to a purely magnetic extremal RN BH. For this configuration,

$$\Delta(x) = \frac{1}{4} p^4, \quad (6.5)$$

$$\frac{S_{RN}(x)}{\pi} = \frac{1}{2} p^2. \quad (6.6)$$

Introducing a basis¹¹ $\{e_M\}_{M=1}^{\dim \mathfrak{F}}$, the Freudenthal dual $\tilde{x}$ of x can be computed [17, 19] by using

$$\tilde{x}^M = \Omega^{MN} \frac{1}{\pi} \frac{\partial S(x)}{\partial x^N} = \Omega^{MN} \frac{\partial \sqrt{|\Delta(x)|}}{\partial x^N} = \frac{\epsilon}{2\sqrt{\epsilon\Delta(x)}} \Omega^{MN} \frac{\partial \Delta(x)}{\partial x^N}, \quad (6.7)$$

where we recall that $\epsilon \equiv \text{sgn } \Delta(x)$. Note, we have introduced here the $\dim \mathfrak{F} \times \dim \mathfrak{F} = 2N \times 2N$ symplectic matrix Ω , defined by

$$\{x, y\} \equiv x^T \Omega y \quad (6.8)$$

¹⁰The case $\Delta = 0$ corresponds to the uncharged limit $p = 0 = q$.

¹¹For $\dim \mathfrak{F} = 2N$, Latin capital indices are symplectic, and take values over $0, 1, \dots, N-1$ contravariant and $0, 1, \dots, N-1$ covariant indices.

For a generic FTS, we can choose a basis such that Ω is realized as follows:

$$\Omega = (\Omega^{MN}) = \begin{pmatrix} \mathbf{0} & -\mathbb{1} \\ \mathbb{1} & \mathbf{0} \end{pmatrix}, \quad (6.9)$$

where $\mathbf{0}$ and $\mathbb{1}$ denote the $N \times N$ zero and identity matrices, respectively.

In the present case

$$x^T \Omega y = qp' - pq', \quad x = (p, q)^T, y = (p', q')^T \quad (6.10)$$

and from eq. (6.3) we find

$$\tilde{x} \equiv (\tilde{p}, \tilde{q})^T = (0, p)^T. \quad (6.11)$$

A purely electric extremal RN BH is nothing else as the Freudenthal dual of purely magnetic extremal RN BH. The whole FTS $\mathfrak{F}_{\mathcal{N}=2}$ “pure” *coincides* with the Freudenthal plane $\mathfrak{F}_x$ associated to x :

$$\mathfrak{F}_{\mathcal{N}=2} \text{ “pure”} = \mathfrak{F}_x = \{ax + b\tilde{x}, a, b \in \mathbb{R}\} \quad (6.12)$$

and transverse space is obviously empty $\mathfrak{F}_x^\perp = \emptyset$. A general Freudenthal transformation (GFT) depending on the real parameters a, b is given by

$$x \rightarrow x_F = ax + b\tilde{x} \quad (6.13)$$

or, in this case

$$x_F = (q_F, p_F)^T = (ap, bp)^T, \quad (6.14)$$

with

$$\Delta(x_F) \equiv \Delta(ax + b\tilde{x}) = \frac{1}{4} (a^2 p^2 + b^2 q^2)^2 \quad (6.15)$$

$$= \frac{1}{4} (a^2 + b^2) p^4 \geq 0. \quad (6.16)$$

The corresponding extremal RN BH is supersymmetric and $\frac{1}{2}$ -BPS (in absence of scalar fields, supersymmetry implies extremality). For $a^2 + b^2 = 1$, the general Freudenthal transformation leaves invariant the entropy of the black hole. In this context a GFT is nothing else as an instance of EM duality. Automorphism algebra and group element as Υ and $\exp(\theta\Upsilon)$ can be explicitly and easily computed. $\Upsilon_x(ax + b\tilde{x}) \propto ax + b\tilde{x}$.

$\mathfrak{F}_{\mathcal{N}=2}$ “pure” provides the simplest case of *degenerate* FTS, in which 2Δ is the square of a quadratic polynomial I_2 :

$$\Delta = I_2^2, \quad (6.17)$$

and thus it is always positive. In fact, “pure” $\mathcal{N} = 2$, $D = 4$ supergravity is the $n_V = 0$ limit of the sequence of $\mathcal{N} = 2$, $D = 4$ supergravity “*minimally coupled*” to n_V vector multiplets¹² [61] (see also [62, 63]), in which the related FTS is degenerate $\forall n_V \in \mathbb{N} \cup \{0\}$; the corresponding scalar manifold is $\overline{\mathbb{CP}}^{n_V}$.

¹²Actually, such a sequence is the unique, *at least* among theories with homogeneous scalar manifolds, to admit the “pure” theory as the limit of $n_V = 0$ vector multiplets.

In the formalism discussed in section 3, in $\mathfrak{F}_{\mathcal{N}=2}$ “pure” = $\mathfrak{F}_x$ it holds that ($I, J = 1, 2$)

$$I_2 = \frac{1}{2} \|\mathbf{x}\|_\delta^2 = \frac{1}{2} \delta_{IJ} x^I x^J, \quad (6.18)$$

with $x^1 = x$, $x^2 = \tilde{x}$. The Euclidean nature of the metric structure defined on $\mathfrak{F}_{\mathcal{N}=2}$ “pure” = $\mathfrak{F}_x$ corresponds to a *spherical foliation* of $\mathfrak{F}_x$ for $\Delta > 0$.

Degenerate FTS’s *never* satisfy the reducibility condition [14], namely they are globally non-reduced; they have been treated e.g. in [64], and their application in supergravity has been discussed in [27] (see also and [63]). Other (infinite) examples of degenerate FTS’s are provided by the ones related to the n -parameterised sequence of $\mathcal{N} = 3$, $D = 4$ supergravity coupled to n matter (vector) multiplets [27, 62, 65]. On the other hand, $\mathcal{N} = 4$ [66] and $\mathcal{N} = 5$,¹³ $D = 4$ “pure” supergravities have FTS’s which do not satisfy the degeneracy condition eq. (6.17) in all symplectic frames, but rather eq. (6.17) is satisfied *at least* in the so-called “scalar-dressed” symplectic frame [62].

This FTS cannot be associated to any Jordan Algebra. Consistently, “pure” $\mathcal{N} = 2$, $D = 4$ supergravity does *not* admit an uplift to $D = 5$, or conversely it cannot be obtained by dimensionally reducing any $D = 5$ theory down to $D = 4$. In general, degenerate FTS’s are *not* built starting from rank-3 Jordan algebras, and therefore the corresponding Maxwell-Einstein (super)gravity models do *not* admit an uplift to $D = 5$; rather, degenerate FTS’s are based on Hermitian (Jordan) triple systems (*cfr. e.g.* [43, 64], and refs. therein).

As discussed in section 10 of [27], *at least* for the degenerate FTS’s relevant to $D = 4$ supergravities with symmetric scalar manifold (i.e., $\mathcal{N} = 2$ “minimally coupled” and $\mathcal{N} = 3$ theories¹⁴), Freudenthal duality is nothing but an anti-involutive U-duality mapping. This can be realized immediately in the aforementioned case of $\mathcal{N} = 2$, $D = 4$ “pure” supergravity; let us consider ($a = b = 1$)

$$\mathfrak{F}_{\mathcal{N}=2} \text{ “pure”} \ni y = (p, q)^T \Rightarrow \Delta(x + \tilde{x}) = \frac{1}{2} (p^2 + q^2)^2. \quad (6.19)$$

The Freudenthal dual $\tilde{y}$ of y can be computed (by recalling eq. (6.7) and using eq. (6.3)) to read

$$\tilde{y} = (-q, p)^T = \Omega_0 y, \quad (6.20)$$

where Ω_0 is nothing but the canonical symplectic 2×2 metric $\Omega_{2 \times 2}$:

$$\Omega_0 \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \equiv \Omega_{2 \times 2}. \quad (6.21)$$

Thus, Freudenthal duality in $\mathfrak{F}_{\mathcal{N}=2}$ “pure” is given by the application of the symplectic metric $\Omega \equiv \Omega_0$, and it is thus an *anti-involutive U-duality transformation*. The relation

¹³A particularly interesting case is provided by $\mathcal{N} = 5$, $D = 4$ supergravity [67], which is seemingly related to a non-reduced FTS which is non-degenerate, but also to a triple system denoted by $M_{2,1}(\mathcal{O}) \sim M_{1,2}(\mathcal{O})$ [42, 68, 69] which deserves a particular study.

¹⁴These cases pertain to simple, degenerate FTS’s [27]. No examples of semi-simple or non-semi-simple degenerate FTS’s relevant to (super)gravity ($D = 4$) models are known to us.

eq. (6.20) defines a $\mathbb{Z}_4$ symmetry in the 2-dim. FTS $\mathfrak{F}_{\mathcal{N}=2}$ “pure” = $\mathfrak{F}_x$, spanned by x eq. (6.6) and its Freudenthal dual $\tilde{x}$ eq. (6.11): in fact, the iteration of Freudenthal duality yields

$$(p, q)^T \xrightarrow{\sim} (-q, p)^T \xrightarrow{\sim} -(p, q)^T \xrightarrow{\sim} (q, -p)^T \xrightarrow{\sim} (p, q)^T. \quad (6.22)$$

This provides the realisation of the $\mathbb{Z}_4$ in the FTS $\mathfrak{F}_{\mathcal{N}=2}$ “pure” = $\mathfrak{F}_x$, as a consequence of the anti-involutivity of Freudenthal duality itself. The same symmetry will be also explicitly observed, for example, for the Freudenthal plane defined by the $D0 - D6$ brane charge configuration in reduced FTS’s, to be studied in latter sections.

7 The axion-dilaton $\mathcal{N} = 2$, $D = 4$ supergravity

Let us consider now $\mathcal{N} = 2$, $D = 4$ supergravity “minimally coupled” to one vector multiplet, in the so-called axion-dilaton (denoted by the subscript “ ad ”) symplectic frame. Ultimately, this is nothing but the $n_V = 1$ element of the sequence of $\mathbb{CP}^{n_V}$ “minimally coupled” models [61], but in a particular symplectic frame, which can be obtained as a consistent truncation of “pure” $\mathcal{N} = 4$ supergravity, in which only two of the six graviphoton survive (in this frame, the holomorphic prepotential reads $F(X) = -iX^0X^1$; cfr. e.g. the discussion in [70], and refs. therein).

The purely bosonic sector of such an $\mathcal{N} = 2$ theory may be regarded as the simplest instance of Maxwell-Einstein gravity coupled to one complex scalar field. In the axion-dilaton symplectic frame, in the particular charge configuration obtained by setting to zero two charges out of four and thus having only two non-vanishing charges,¹⁵ namely one magnetic and one electric charge p resp. q , the asymptotically flat, spherically symmetric, dyonic extremal BH solution has Bekenstein-Hawking entropy

$$\frac{S_{ad}}{\pi} = |pq|, \quad (7.1)$$

and it is non-supersymmetric¹⁶ (non-BPS). The expression eq. (7.1) is very reminiscent of the Bekenstein-Hawking entropy of a BH in a reduced FTS in the $D0 - D6$ charge configuration (to be treated later on, we refer to eqs. (8.3)–(8.4), $I_4 \equiv \Delta(x)$):

$$\frac{S_{BH}}{\pi} = \sqrt{|\Delta|} = |p^0 q_0|. \quad (7.2)$$

However, the $\mathcal{N} = 2$ axion-dilaton supergravity model, as the “pure” $\mathcal{N} = 4$, $D = 4$ supergravity from which it derives, *cannot* be uplifted to $D = 5$ (as instead all models related to reduced FTS’s can), consistently with its “minimally coupled” nature: in fact, the charges P and Q do *not* have the interpretation of the magnetic resp. electric charge of the KK vector in the $D = 5 \rightarrow 4$ dimensional reduction.

¹⁵In this case, the effective FTS $\mathfrak{F}_{\mathcal{N}=2 \text{ } ad}$ given by the truncation has dimension 2.

¹⁶Indeed, in presence of scalar fields (in this context stabilized at the event horizon of the BH by virtue of the attractor mechanism), extremality does not imply BPS nature, and extremal non-BPS solutions may exist.

This truncated system can be described by a two dimensional FTS characterized by a quartic form ($x \equiv (p, q)^T$)

$$\Delta(x) = -\frac{1}{2}(pq)^2. \quad (7.3)$$

One can recast this expression by defining

$$p \equiv \frac{1}{\sqrt{2}}(P + Q), \quad q \equiv \frac{1}{\sqrt{2}}(P - Q), \quad (7.4)$$

in the following form

$$\frac{S_{ad}}{\pi} = \frac{1}{2}|P^2 - Q^2|, \quad (7.5)$$

Let us start by choosing, without any loss of generality, a charge configuration given by

$$x = (P, 0)^T. \quad (7.6)$$

The corresponding entropy is given by

$$\frac{S_{ad}(x)}{\pi} = \frac{1}{2}P^2. \quad (7.7)$$

By virtue of eq. (6.7), one can compute the Freudenthal dual $\tilde{x}$ of x to read ($\epsilon \equiv \text{sgn}(P^2 - Q^2)$)

$$\tilde{x} = (\tilde{P}, \tilde{Q}) = (0, \epsilon P)^T. \quad (7.8)$$

Thus, one can define a GFT transformations and the 2-dim. Freudenthal plane $\mathfrak{F}_x$ associated to x inside the whole 4-dim. FTS $\mathfrak{F}_{\mathcal{N}=2 \text{ ad}}$:

$$\mathfrak{F}_{\mathcal{N}=2 \text{ ad}} \supset \mathfrak{F}_x = \{x_F \equiv ax + b\tilde{x}, \quad a, b \in \mathbb{R}\}, \quad (7.9)$$

with

$$\frac{S_{ad}(x_F)}{\pi} = \frac{1}{2}|a^2P^2 - b^2Q^2|. \quad (7.10)$$

The corresponding extremal BH is non-supersymmetric (non-BPS). In particular for the eq. (7.6) configuration

$$\frac{S_{ad}(x_F)}{\pi} = \frac{1}{2}|a^2 - Q^2| \frac{S_{ad}(x)}{\pi}. \quad (7.11)$$

The entropy is invariant for $a^2 - b^2 = \pm 1$.

Within the formalism discussed in section 3, in $\mathfrak{F}_x \subset \mathfrak{F}_{\mathcal{N}=2 \text{ ad}}$, it holds that

$$I_2 = \frac{1}{2} \|\mathbf{x}\|_\eta^2 = \frac{1}{2} \eta_{IJ} x^I x^J, \quad (7.12)$$

with $x^1 = x$, $x^2 = \tilde{x}$, and $\eta_{IJ} = \text{diag}(1, -1)$. The Kleinian nature of the metric structure defined on $\mathfrak{F}_x \subset \mathfrak{F}_{\mathcal{N}=2 \text{ ad}}$ corresponds to an *hyperbolic* (i.e., *hyperboloid-like*) *foliation* of $\mathfrak{F}_x$ for $\Delta < 0$. Therefore, notwithstanding the fact that $\mathcal{N} = 2$, $D = 4$ axion-dilaton supergravity is nothing but the $\overline{\mathbb{CP}}^1$ “minimally coupled” model in a particular (non-Fubini-Study) symplectic frame and thus with eq. (6.17) holding true, in the *peculiar*

(P, Q) charge configuration eq. (7.4), the corresponding $\mathfrak{F}_x \subset \mathfrak{F}_{\mathcal{N}=2 \text{ ad}}$ can be considered as a “degenerate” limit of the $\Delta < 0$ prototype of Freudenthal plane for reduced FTS’s.

It is instructive to consider the explicit action of the Freudenthal duality in the Freudenthal plane $\mathfrak{F}_x \subset \mathfrak{F}_{\mathcal{N}=2 \text{ ad}}$. Let us start and consider ($a = b = 1$; we disregard the coordinates in $\mathfrak{F}_{\mathcal{N}=2 \text{ ad}}$ pertaining to $\mathfrak{F}_x^\perp = \mathfrak{F}_{\mathcal{N}=2 \text{ ad}}/\mathfrak{F}_x$)

$$\mathfrak{F}_{\mathcal{N}=2 \text{ ad}} \supset \mathfrak{F}_x \ni y = (P, Q)^T \Rightarrow \frac{S_{ad}(y)}{\pi} = \frac{1}{2} |P^2 - Q^2|. \quad (7.13)$$

The Freudenthal dual $\tilde{y}$ of y can be computed (by recalling eq. (6.7)) to read¹⁷

$$\tilde{y} = \epsilon(Q, P)^T = \epsilon \hat{\mathbf{O}} y, \quad (7.14)$$

with

$$\hat{\mathbf{O}} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (7.15)$$

Note that $\hat{\mathbf{O}}$ eq. (7.15) is involutive:

$$\hat{\mathbf{O}}^2 = Id, \quad (7.16)$$

but since the Freudenthal duality on $\mathfrak{F}_x$ exchanges P and Q and thus flips $\epsilon (= \text{sgn}(P^2 - Q^2))$, it follows that the correct iteration of the Freudenthal duality on $\mathfrak{F}_x \subset \mathfrak{F}_{\mathcal{N}=2 \text{ ad}}$ is provided by the application of $\epsilon \hat{\mathbf{O}}$ and then necessarily of $-\epsilon \hat{\mathbf{O}}$, thus corresponding to $-\hat{\mathbf{O}}^2 = -Id$ acting on x , and thus correctly yielding

$$\tilde{\tilde{x}} = -x.$$

As at the end of previous subsection for “pure” $\mathcal{N} = 2$, $D = 4$ supergravity, in this case due to the relations eqs. (7.14)–(7.15), we can define a $\mathbb{Z}_4$ symmetry in the 2-dim. Freudenthal plane $\mathfrak{F}_x \subset \mathfrak{F}_{\mathcal{N}=2 \text{ ad}}$, spanned by x eq. (7.7) and its Freudenthal dual $\tilde{x}$ eq. (7.8): e.g., starting from $\epsilon = 1$, the iteration of Freudenthal duality yields

$$(P, Q)^T \xrightarrow{\sim} (Q, P)^T \xrightarrow{\sim} -(P, Q)^T \xrightarrow{\sim} -(Q, P)^T \xrightarrow{\sim} (P, Q)^T. \quad (7.17)$$

This provides the realisation of the $\mathbb{Z}_4$ in the Freudenthal plane $\mathfrak{F}_x \subset \mathfrak{F}_{\mathcal{N}=2 \text{ ad}}$, as a consequence of the anti-involutivity of Freudenthal duality itself.

8 $\mathcal{N} = 2, D = 4$ supergravities from $D = 5$: the reduced $\mathfrak{F}$ case

We will now proceed to present an analysis of the (non-degenerate) *reduced* FTS’s, of the properties of Freudenthal duality defined in them, and of the corresponding Freudenthal planes.

¹⁷By virtue of the discussion made at the end of the previous subsection (also cfr. section 10 of [27]), $\mathbf{O}$ eq. (7.15) can be completed to a 4×4 (consistently anti-involutive; cfr. discussion further below) transformation of the U-duality group $U(1, 1)$.

Unless otherwise noted, we will essentially confine ourselves *at least* to (non-degenerate) *reduced* FTS's $\mathfrak{F} = \mathfrak{F}(J_3)$, for which a $4D/5D$ special coordinates' symplectic frame can be defined.

A generic element x of the reduced FTS $\mathfrak{F}$ splits as

$$x = (x^0, x^i, x_0, x_i)^T \equiv (p^0, p^i, q_0, q_i)^T, \quad (8.1)$$

where the second renaming pertains to the identification of x with a dyonic charge configuration in $D = 4$ (super)gravity, where p 's and q 's are magnetic and electric charges, respectively; within the standard convention in supergravity, p^0, p^i, q_0 and q_i will usually be called $D6, D4, D2, D0$ (brane) charges, respectively.

In the canonical basis the symplectic product of two generic elements x and y in $\mathfrak{F}$ reads

$$\{x, y\} \equiv x^T \Omega y = -x^0 y_0 - x^i y_i + x_0 y^0 + x_i y^i \quad (8.2)$$

where Ω is a symplectic matrix.

At least within (non-degenerate) reduced FTS's, the quartic polynomial invariant $I_4 = 2\Delta(x)$ of $\text{Aut}(\mathfrak{F}(J_3)) \approx \text{Conf}(J_3)$ can be written¹⁸ as follows¹⁹ (cfr. e.g. [44, 72, 73]; ($i = 1, \dots, N-1, \dim \mathfrak{F} = 2N$)):

$$\Delta(x) \equiv I_4(x) = -(p^0 q_0 + p^i q_i)^2 + 4q_0 I_3(p) - 4p^0 I_3(q) + 4\{I_3(p), I_3(q)\} \quad (8.3)$$

$$= -(p^0 q_0 + p^i q_i)^2 + \frac{2}{3} q_0 d_{ijk} p^i p^j p^k - \frac{2}{3} p^0 d^{ijk} q_i q_j q_k + d_{ijk} d^{ilm} p^j p^k q_l q_m, \quad (8.4)$$

where

$$I_3(p) \equiv \frac{1}{3!} d_{ijk} p^i p^j p^k, \quad (8.5)$$

$$I_3(q) \equiv \frac{1}{3!} d^{ijk} q_i q_j q_k, \quad (8.6)$$

$$\{I_3(p), I_3(q)\} \equiv \frac{\partial I_3(p)}{\partial p^i} \frac{\partial I_3(q)}{\partial q_i}. \quad (8.7)$$

The symmetric quantities d^{ijk}, d_{ijk} follow the so-called *adjoint identity* of the Jordan algebra J_3 underlying the reduced FTS $\mathfrak{F}$ (cfr. e.g. [46, 73, 74] and refs. therein), which reads

$$d_{(ij|k} d_{l|mn)} d^{klp} = \frac{4}{3} \delta_{(i}^p d_{jmn)}. \quad (8.8)$$

The triple product map $T(x, y, z)$ reads (*up to contributions* $\in \mathfrak{F}_w^\perp = \mathfrak{F}/\mathfrak{F}_w$)

$$T(x, y, z)_M = \frac{\partial I_4(x, y, z, w)}{\partial w^M} = K_{MNPQ} x^N y^P z^Q, \quad (8.9)$$

where the capital Latin indices span the entire FTS $\mathfrak{F}$, and $K_{MNPQ} = K_{(MNPQ)}$ is the rank-4 completely symmetric tensor characterizing $\mathfrak{F}$ [14, 27, 68]. Note that, from its very definition eq. (8.9), $T(x, y, z)$ is completely symmetric in all its arguments [14].

¹⁸Recall that $d_{ijk} = d_{(ijk)}$ and $d^{ijk} = d^{(ijk)}$ throughout.

¹⁹*At least* in all reduced FTS's, $\text{Aut}(\mathfrak{F}(J_3))$ is “of type E_7 ” [14], and the ring of invariant polynomials is one-dimensional, and finitely generated (i.e., with no syzygies) by I_4 [71].

Then, by using Ω to raise the symplectic indices, one can compute

$$T(x, y, z)^M = \Omega^{MN} T(x, y, z)_N. \quad (8.10)$$

By direct computation, one gets

$$\frac{\partial I_4(x)}{\partial p^0} = -2(p^0 q_0 + p^i q_i) q_0 - \frac{2}{3} d^{ijk} q_i q_j q_k; \quad (8.11)$$

$$\frac{\partial I_4(x)}{\partial p^i} = -2(p^0 q_0 + p^j q_j) q_i + 2q_0 d_{ijk} p^j p^k + 2d_{ijk} d^{klm} p^j q_l q_m; \quad (8.12)$$

$$\frac{\partial I_4(x)}{\partial q_0} = -2(p^0 q_0 + p^i q_i) p^0 + \frac{2}{3} d_{ijk} p^i p^j p^k; \quad (8.13)$$

$$\frac{\partial I_4(x)}{\partial q_i} = -2(p^0 q_0 + p^j q_j) p^i - 2p^0 d^{ijk} q_j q_k + d_{jkl} d^{lin} p^j p^k q_n. \quad (8.14)$$

In appendix D we present some explicit expressions for the triple product and other maps.

8.1 The T^3 supergravity model and $\mathfrak{F}(\mathbb{R})$

The so-called T^3 model of $\mathcal{N} = 2$, $D = 4$ supergravity is the smallest model in which the plane $\mathfrak{F}_{y \perp x}$ can be defined; such a model is comprised within all models based on (non-degenerate) reduced FTS's (cfr. e.g. (table 1)). In this model, it holds that ($i = 1$, and $p^1 \equiv T$)

$$\frac{1}{3!} d_{ijk} p^i p^j p^k = T^3 \Leftrightarrow d_{111} = 6. \quad (8.15)$$

In the usual normalization of d -tensors used in supergravity literature,²⁰ it holds that (cfr. e.g. [73])

$$d^{111} = \frac{2}{9}. \quad (8.16)$$

In this case we have ($N = 2$, $i = 1$, $\dim \mathfrak{F} = 4$):

$$x = (p^0, p^1, q_0, q_1)^T, \quad (8.17)$$

$$I_4(x) = \Delta(x) = -(p^0 q_0 + p^1 q_1)^2 + 4q_0 (p^1)^3 - \frac{4}{27} p^0 (q_1)^3 + \frac{4}{3} (p^1 q_1)^2. \quad (8.18)$$

By direct computation, one gets

$$\frac{\partial I_4(x)}{\partial p^0} = -2(p^0 q_0 + p^1 q_1) q_0 - \frac{4}{27} q_1^3; \quad (8.19)$$

$$\frac{\partial I_4(x)}{\partial p^1} = -2(p^0 q_0 + p^1 q_1) q_1 + \frac{4}{9} q_0 (p^1)^2 + \frac{24}{9} p^1 q_1^2; \quad (8.20)$$

$$\frac{\partial I_4(x)}{\partial q_0} = -2(p^0 q_0 + p^1 q_1) p^0 + 4(p^1)^3; \quad (8.21)$$

$$\frac{\partial I_4(x)}{\partial q_1} = -2(p^0 q_0 + p^1 q_1) p^1 - \frac{4}{9} p^0 q_1^2 + \frac{12}{9} (p^1)^2 q_1. \quad (8.22)$$

which allows to compute the dual components by eq. (6.7).

²⁰Which, however, is not the one used e.g. in [68].

Since the T^3 model pertains to the unique reduced FTS for which $N = 2$ (cfr. table 1), for this model $\dim \mathfrak{F} = 2N = 4$ and the plane $\mathfrak{F}_{y \perp x}$ coincides with the whole space $\{, \}$ -orthogonal to the Freudenthal plane $\mathfrak{F}_x$:

$$N = 2 \Rightarrow \mathfrak{F}_{y \perp x} = \mathfrak{F}_x^\perp. \quad (8.23)$$

Thus, the FTS $\mathfrak{F}_{T^3} \equiv \mathfrak{F}(\mathbb{R})$, sitting in the spin- $\frac{3}{2}$ irrepr. **4** of $\text{Aut}(\mathfrak{F}_{T^3}) \approx \text{Conf}(J_3 = \mathbb{R}) = \text{SL}(2, \mathbb{R})$, gets decomposed as follows:

$$\mathfrak{F}_{T^3} \equiv \mathfrak{F}(\mathbb{R}) \equiv \begin{matrix} \mathbf{4} = \mathbf{1}_{-3} + \mathbf{1}_{-1} + \mathbf{1}_1 + \mathbf{1}_3 \\ \text{Conf}(J_3 = \mathbb{R}) = \text{SL}(2, \mathbb{R}) \longrightarrow \text{SO}(1, 1)_{KK} \end{matrix}, \quad (8.24)$$

where $\text{SO}(1, 1)_{KK}$ is related to the radius of the S^1 in the dimensional reduction from minimal ($\mathcal{N} = 2$) $D = 5$ “pure” supergravity down to $D = 4$ (giving rise to the T^3 model).

Let us start first with a particular configuration with $\Delta(x) < 0$. Specifying (8.52) and (6.7)–(8.59) for the T^3 model, one has ($\epsilon = \text{sgn}(x^0 x_0)$)

$$x = (x^0, 0, x_0, 0)^T; \quad (8.25)$$

$$\tilde{x} = \epsilon (-x^0, 0, x_0, 0)^T, \quad (8.26)$$

with

$$\Delta_0 \equiv \Delta(x) = \Delta(\tilde{x}) = -(x^0 x_0)^2 < 0. \quad (8.27)$$

Then, for a generic GFT transformation on x

$$x_F = ax + b\tilde{x} \in \mathfrak{F}_x,$$

it holds that

$$\Delta(x_F) = -(a^2 - b^2)^2 (x^0 x_0)^2 = (a^2 - b^2)^2 \Delta(x) < 0, \quad (8.28)$$

and therefore $\mathfrak{F}_x$ lies completely in the rank-4 $\Delta < 0$ orbit of $\text{Aut}(\mathfrak{F}(J_3 = \mathbb{R}))$.

Analogously, specifying eq. (8.63) and eq. (8.64) for the T^3 model, one obtains

$$y = (0, y^1, 0, y_1)^T \quad (8.29)$$

and, according to eq. (8.18),

$$\Delta(y) = \frac{1}{3} (y^1 y_1)^2 > 0, \quad (8.30)$$

where the strict inequality holds, because we assume y to be of maximal(= 4) rank in $\mathfrak{F}_{T^3}$. Note that, while x and $\tilde{x}$ lie in the $\Delta < 0$ orbit of $\text{Aut}(\mathfrak{F}(\mathbb{R}))$, y belongs to the other rank-4 orbit.²¹

Starting from the decomposition eq. (8.24), the Freudenthal plane $\mathfrak{F}_x$ related to x eq. (8.25) and the $\{, \}$ -orthogonal plane $\mathfrak{F}_{y \perp x} = \mathfrak{F}_x^\perp$ can respectively be identified as follows:

$$\mathfrak{F}_x = D6 \oplus_{x^0} D0 = \mathbf{1}_{-3} + \mathbf{1}_3; \quad (8.31)$$

$$\mathfrak{F}_{y \perp x} = \mathfrak{F}_x^\perp = D4 \oplus_{y^1} D2 = \mathbf{1}_{-1} + \mathbf{1}_1. \quad (8.32)$$

²¹As pointed out above, there is a unique $\Delta > 0$ orbit in the T^3 model.

Nicely, within the interpretation of $SO(1, 1)_{KK}$ as a non-compact analogue of $D = 4$ helicity of a would-be spin- $\frac{3}{2}$ (Rarita-Schwinger) particle, the Freudenthal plane $\mathfrak{F}_x$ pertains to the two massless helicity modes.

Let us recall that, while $\mathfrak{F}_x$ eq. (8.31) is a quadratic sub-FTS of $\mathfrak{F}_{T^3}$ (as discussed in section 8.1), $\mathfrak{F}_{y \perp x} = \mathfrak{F}_x^\perp$ eq. (8.32) is *not* a sub-FTS of $\mathfrak{F}$ (as discussed in section 5). This can be explicitly checked by relying on the treatment of subsection 8.2; in fact, for the T^3 model, $\mathfrak{F}_{y \perp x} = \mathfrak{F}_x^\perp$ (8.32) is not closed under T . Out of the four cases 1–4 listed at the end of subsection 8.2, only the last one (4) is to be considered: in this case, the condition of closure of $\mathfrak{F}_{y \perp x} = \mathfrak{F}_x^\perp$ eq. (8.32) under T is that y^1 is rank < 3 in $J_3 = \mathbb{R}$ and y_1 is rank < 3 in $\bar{J}_3 = \mathbb{R}$, namely

$$y_1 = 0 = y^1 \Leftrightarrow y = 0 \in \mathfrak{F}. \quad (8.33)$$

Thus, the condition of closure of $\mathfrak{F}_{y \perp x} = \mathfrak{F}_x^\perp$ eq. (8.32) under T implies, in the case of the T^3 model, an *absurdum*, namely that the rank-4 element $y \in \mathfrak{F}_x^\perp$ be the *null element* of the FTS $\mathfrak{F}_{T^3} \equiv \mathfrak{F}(\mathbb{R})$. Therefore, $\mathfrak{F}_{y \perp x} = \mathfrak{F}_x^\perp$ eq. (8.32) is *not* closed under T .

In other words, as also pointed out above, in order for $y = (0, y^1, 0, y_1)^T \in \mathfrak{F}_x^\perp$ to be rank-4 (as assumed throughout), it must have both components non-vanishing; from eq. (8.30) one can observe that in the T^3 model y belongs to the rank-4, $\Delta > 0$, $\text{Aut}(\mathfrak{F}_{T^3}) = \text{SL}(2, \mathbb{R})$ orbit, unless $y^1 = 0$ and/or $y_1 = 0$, in which case it has rank < 4. Therefore an element y of the form

$$y = (0, y^1, 0, y_1)^T \in \mathfrak{F}_x^\perp$$

is rank-4 (and necessarily in the unique $\Delta > 0$ orbit) *iff* $y^1 \neq 0$ and $y_1 \neq 0$. $\square$

Furthermore, we are interested in the behaviour of the quartic invariant Δ on the $D4 \oplus D2$ $\mathfrak{F}_{y \perp x}$ plane. General results are presented in section B, in particular in eq. (4.23) and eq. (4.24) which can be used here. According to these results²²

$$\Delta(\Upsilon_x(y)) = \Delta(x)^2 \Delta(y) \quad (8.35)$$

The sign of $\Delta(\Upsilon_x(y))$ depends only on the sign of $\Delta(y)$ implying that $\Upsilon_x(y)$ belongs to the same rank-4 ($\Delta > 0$) $\text{Aut}(\mathfrak{F}(\mathbb{R}))$ -orbit as y . Explicitly in this case

$$\Delta(\Upsilon_x(y)) = \Delta(x)^2 \Delta(y) = \frac{1}{3} (x^0 x_0)^4 (y^1 y_1)^2 > 0 \quad (8.36)$$

For a generic element r

$$r = ay + b\Upsilon_x(y) \in \mathfrak{F}_{y \perp x} = \mathfrak{F}_x^\perp \quad (a, b \in \mathbb{R}),$$

one gets eq. (4.23) and eq. (4.24)

$$\Delta(r) = (a^2 - |\Delta(x)| b^2)^2 \Delta(y) \geq 0. \quad (8.37)$$

²²We can explicitly write the Υ map (see eq. (2.6))

$$(\Upsilon_x(y))^0 = 0; (\Upsilon_x(y))^1 = \frac{1}{2} x^0 x_0 y^1; (\Upsilon_x(y))_0 = 0; (\Upsilon_x(y))_1 = -\frac{1}{2} x^0 x_0 y_1. \quad (8.34)$$

Thus, following the general behaviour explained in appendix B, $r \in \mathfrak{F}_{y \perp x}$ is not of the same (maximal = 4) rank orbit as y (and $\Upsilon_x(y)$) only when

$$\Delta(r) = 0 \Leftrightarrow a^2 - (x^0 x_0)^2 b^2 = 0 \Leftrightarrow a^2 = |\Delta(x)| b^2. \quad (8.38)$$

The conditions for $r \in \mathfrak{F}_{y \perp x} \subset \mathfrak{F}_x^\perp$ to lie in the rank-3, rank-2 or even rank-1 orbits might be easily studied using expressions eq. (4.23) and eq. (4.24)).

Let us study now the family of configurations with $D4 - D0$ charges. This family includes configurations with both $\Delta(x) > 0$ and $\Delta(x) < 0$ possibilities. Let us take,²³

$$x = (0, x^1, x_0, 0)^T, \quad (8.39)$$

then one obtains

$$\Delta(x) = \Delta(\tilde{x}) = 4x_0 (x^1)^3. \quad (8.40)$$

Thus the sign of $\Delta(x)$ equals the sign of $x_0 x^1$:

$$\text{sgn } \Delta(x) = \text{sgn}(x_0 x^1). \quad (8.41)$$

For a positive sign, $\text{sgn}(x_0 x^1) > 0$, the dual is a $D6 - D2$ configuration, it reads

$$\tilde{x} = \frac{1}{\sqrt{\Delta_0}} \left(- (x^1)^3, 0, 0, 3x_0 (x^1)^2 \right), \quad (8.42)$$

x and $\tilde{x}$ belong to the same (rank-4, $\Delta > 0$) orbit of $\text{Aut}((\mathfrak{F} = J_3))$, which is unique in this model (cfr. [75], and refs. therein). For a generic element $x_F = ax + b\tilde{x} \in \mathfrak{F}_x$, it holds that

$$\Delta(x_F) = (a^2 + b^2)^2 \Delta(x) > 0, \quad (8.43)$$

implying that $\mathfrak{F}_x$ lies completely in the unique rank-4 $\Delta > 0$ orbit of $\text{Aut}(\mathfrak{F}(\mathbb{R}))$.

Then, let us pick a rank-4 element $y \in \mathfrak{F}_x^\perp$, that means which is $\{, \}$ -orthogonal to x and $\tilde{x}$, one can show that the most general element of this kind is given by the charge configuration:

$$y = \left(\frac{x^1}{x_0} y_1, y^1, -3 \frac{x_0}{x^1} y^1, y_1 \right), \quad (8.44)$$

whose quartic invariant is given by

$$\Delta(y) = -\frac{8}{3} (y^1)^2 y_1^2 - 12 \frac{x_0}{x^1} (y^1)^4 - \frac{4}{27} \frac{x^1}{x_0} y_1^4, \quad (8.45)$$

$$= -\frac{4}{27(x_0 x^1)} (9x_0 (y^1)^2 + x^1 (y_1)^2)^2 \quad (8.46)$$

thus the signs of $\Delta(x)$ and $\Delta(y)$ are opposite

$$\text{sgn } \Delta(y) = -\text{sgn}(x_0 x^1) = -\text{sgn } \Delta(x). \quad (8.47)$$

²³This can be seen as an special case of eq. (8.81) and eqs. (8.83)–(8.85) for the T^3 model.

In the case of $(x_0 x^1) = 0$ then $\mathfrak{F}_x^\perp \sim D6 \oplus D4$ (for $x_0 = 0$) and $\mathfrak{F}_x^\perp \sim D2 \oplus D4$ (for $x^1 = 0$)
Moreover, according to eq. (4.23) and eq. (4.24)

$$\Delta(\Upsilon_x(y)) = \Delta(x)^2 \Delta(y). \quad (8.48)$$

The sign of $\Delta(\Upsilon_x(y))$ depends only on the sign of $\Delta(y)$. Both of them are negative in our current case. For a generic element r

$$r = ay + b\Upsilon_x(y) \in \mathfrak{F}_{y \perp x} = \mathfrak{F}_x^\perp$$

($a, b \in \mathbb{R}$), one gets eq. (4.23) and eq. (4.24)

$$\Delta(r) = (a^2 + \Delta(x)b^2)^2 \Delta(y). \quad (8.49)$$

implying that $\Upsilon_x(y)$ and for the case any $\Upsilon_x(r)$ lies in the same (maximal rank) $\text{Aut}(\mathfrak{F}(\mathbb{R}))$ -orbit as y .²⁴

8.2 General $D0 - D6/D2 - D4$ sectors

Let us consider a particular configuration with only $D0 - D6$ charges with an arbitrary number of them. We start by identifying x with the rank-4, strictly regular element of the FTS $\mathfrak{F}$ given by the $D0 - D6$ brane charge configuration

$$x_{D0D6} \equiv (p^0, 0^i, q_0, 0_i)^T \in \mathfrak{F}, \quad (8.52)$$

for any element of this configuration we have²⁵

$$\Delta(x_{D0D6}) = -(p^0 q_0)^2 < 0. \quad (8.53)$$

One can compute the Freudenthal dual $\tilde{x}$. Using the expressions

$$\frac{\partial I_4(x)}{\partial p^0} = -2(p^0 q_0) q_0, \quad (8.54)$$

$$\frac{\partial I_4(x)}{\partial p^1} = 0, \quad (8.55)$$

$$\frac{\partial I_4(x)}{\partial q_0} = -2(p^0 q_0) p^0, \quad (8.56)$$

$$\frac{\partial I_4(x)}{\partial q_1} = 0, \quad (8.57)$$

²⁴Explicitly, from eq. (8.98), one obtains

$$(\Upsilon_x(y))^0 = -6(x^1)^2 y^1; (\Upsilon_x(y))^1 = \frac{2}{3}(x^1)^2 y_1; (\Upsilon_x(y))_0 = -2x_0 x^1 y_1; (\Upsilon_x(y))_1 = -6x_0 x^1 y^1, \quad (8.50)$$

yielding

$$\Delta(\Upsilon_x(y)) = -\frac{4}{3}x_0^2 (x^1)^6 (y^1)^2 y_1^2 - 6x_0^3 (x^1)^5 (y^1)^4 - \frac{2}{27}x_0 (x^1)^7 y_1^4. \quad (8.51)$$

²⁵This characterizes $\mathfrak{F}$ as a reduced [14] FTS.

which allows to compute the dual components using eq. (6.7). We arrive to ($\epsilon = \text{sgn}(p^0 q_0)$)

$$\widetilde{x_{D0D6}} \equiv (\widetilde{p^0}, 0^i, \widetilde{q_0}, 0_i)^T \in \mathfrak{F}, \quad (8.58)$$

$$= \epsilon (-p^0, 0^i, q_0, 0_i). \quad (8.59)$$

Thus, depending on the sign²⁶ of $p^0 q_0$. Note that

$$\Delta(x) = \Delta(\widetilde{x}) < 0, \quad (8.60)$$

and thus $\widetilde{x}$ belongs to the same (unique) rank-4 $\Delta < 0$ orbit of $\text{Aut}(\mathfrak{F}(J_3))$ as x . Namely, when $p^0 q_0 > 0$, the action of Freudenthal duality on x_{D0D6} amounts to flipping p^0 only, whereas when $p^0 q_0 < 0$, the action of Freudenthal duality on x_{D0D6} amounts to flipping q_0 only.

Associated to a GFT transformation on x , one defines the Freudenthal plane $\mathfrak{F}_x \subset \mathfrak{F}$ ($\dim \mathfrak{F}_x = 2$), spanned by x and $\widetilde{x}$, whose generic element is

$$x_F = ax + b\widetilde{x} \in \mathfrak{F}_x \quad (a, b \in \mathbb{R}).$$

Within the choice above, $\mathfrak{F}_x$ is coordinatized by the charges of $D0$ and $D6$ branes, respectively being the electric and magnetic charges x_0 and x^0 of the KK Abelian vector in the reduction $D = 5 \rightarrow 4$. In other words, within the position (which does not imply any loss of generality for reduced FTS's), the Freudenthal plane $\mathfrak{F}_x$ is spanned (in a canonical Darboux symplectic frame - see below -) by the electric and magnetic charges x_0 and x^0 of the $D = 5 \rightarrow 4$ Kaluza-Klein Abelian vector (which is the $D = 4$ graviphoton in the $\mathcal{N} = 2$ supersymmetric interpretation).

Note that $\mathfrak{F}_x$ lies completely in the (unique) rank-4 $\Delta < 0$ orbit of $\text{Aut}(\mathfrak{F}(J_3))$, because eq. (3.9)

$$\Delta(X_F) = - (ax^0 + b\widetilde{x}^0)^2 (ax_0 + b\widetilde{x}_0)^2 \quad (8.61)$$

$$= - (a^2 - b^2)^2 (x^0 x_0)^2 = (a^2 - b^2)^2 \Delta(x) < 0. \quad (8.62)$$

This implies that s belongs to the same maximal ($= 4$) rank, $\Delta < 0$ $\text{Aut}(\mathfrak{F}(J_3))$ -orbit as x and $\widetilde{x}$, *unless* $a^2 = b^2$. This observation actually yields interesting consequences for multi-centered black hole physics, as briefly discussed in section 10.

The $D2 - D4$ orthogonal space $\mathfrak{F}_x^\perp$, and the plane $\mathfrak{F}_{y \perp x} \subset \mathfrak{F}_x^\perp$. One can choose a rank-4 element $y \in \mathfrak{F}$ which is $\{, \}$ -orthogonal to the generic $D0 - D6$ element x defined

²⁶The result eq. (8.59) defines a $\mathbb{Z}_4$ symmetry in the 2-dim. Freudenthal plane $\mathfrak{F}_x \subset \mathfrak{F}$, spanned by x_{D0D6} eq. (8.52) and its Freudenthal dual $\widetilde{x_{D0D6}}$ (8.59) (or, equivalently, in the Darboux canonical basis, by the magnetic and electric charges p^0 and q_0 of the $5D \rightarrow 4D$ KK Abelian vector — see below —): e.g., starting from p^0 and q_0 both positive (denoted by “(+, +)”), the iteration of Freudenthal duality yields

$$(+, +) \rightsquigarrow (-, +) \rightsquigarrow (-, -) \rightsquigarrow (+, -) \rightsquigarrow (+, +).$$

This provides a simple realisation of the $\mathbb{Z}_4$ symmetry characterizing every Freudenthal plane, as a consequence of the anti-involutivity of Freudenthal duality itself.

before and its dual and $\tilde{x}$. A possible, particular, choice is provided by a $D2 - D4$ brane charge configuration:

$$y \equiv (0, y^i, 0, y_i)^T \in \mathfrak{F}, \quad (8.63)$$

$$\Delta(y) = -(y^i y_i)^2 + d_{ijk} d^{ilm} y^j y^k y_l y_m \geq 0. \quad (8.64)$$

In eq. (8.64), the case of vanishing Δ has been excluded because y is chosen to be of maximal(= 4) rank in $\mathfrak{F}$. By recalling (8.2), one can immediately check that $(\dim \mathfrak{F}_x^\perp = 2N - 2)$

$$\{x, y\} = 0 = \{\tilde{x}, y\} \Leftrightarrow y \in \mathfrak{F}_x^\perp \equiv \mathfrak{F}/\mathfrak{F}_x. \quad (8.65)$$

One can compute the components of $\Upsilon_x(y)^M$ eq. (2.6) as given by

$$(\Upsilon_x(y))^0 = 0; \quad (8.66)$$

$$(\Upsilon_x(y))^i = x^0 x_0 y^i; \quad (8.67)$$

$$(\Upsilon_x(y))_0 = 0; \quad (8.68)$$

$$(\Upsilon_x(y))_i = -x^0 x_0 y_i; \quad (8.69)$$

thus, $\Upsilon_x(y)$ is still given by a rank-4 $D2 - D4$ brane charge configuration, and it holds that

$$\{x, \Upsilon_x(y)\} = 0 = \{\tilde{x}, \Upsilon_x(y)\} \Leftrightarrow \Upsilon_x(y) \in \mathfrak{F}_x^\perp \equiv \mathfrak{F}/\mathfrak{F}_x. \quad (8.70)$$

Consequently, one can define the 2-dim. plane $\mathfrak{F}_{y \perp x} \subset \mathfrak{F}_x^\perp$, spanned by y and $\Upsilon_x(y)$, whose generic element is $r = ay + b\Upsilon_x(y) \in \mathfrak{F}_{y \perp x}$ ($a, b \in \mathbb{R}$, in our classical/supergravity treatment. In particular, note that $\Upsilon_x(y)$ belongs to the same $\text{Aut}(\mathfrak{F}(J_3))$ -orbit as y , because (consistent with the general eq. (4.23) and eq. (4.24)), it holds that

$$\Delta(\Upsilon_x(y)) = (x^0 x_0)^4 \Delta(y) = (\Delta(x))^2 \Delta(y) \geq 0. \quad (8.71)$$

whose sign depends only on the sign of $\Delta(y)$.

It is worth remarking that $\Upsilon_x(y)$ automatically satisfies eq. (8.70) for every pair y^i and y_i , with $i = 1, \dots, N - 1$. In fact, regardless of d_{ijk} and d^{ijk} , when only a pair y^i and y_i for a *fixed* i is non-vanishing (among all y^i 's and y_i 's), then y is non-trivially of rank-4 in $\mathfrak{F}$, because generally $\Delta(y) \neq 0$, since at least the term $-(y^i y_i)^2$ is present (cfr. eqs. (8.3)–(8.4)). Therefore, one can define $N - 1$ *distinct* planes $(\mathfrak{F}_{y \perp x}^\perp)_i$'s, orthogonal to the Freudenthal plane $\mathfrak{F}_x$, cfr. subsection 8.3.

Moreover, note that $\mathfrak{F}_{y \perp x}$ intersects at least three orbits of $\text{Aut}(\mathfrak{F})$. Indeed, it holds that, using eq. (4.23) and eq. (4.24), ($c \equiv x^0 x_0 b$)

$$\Delta(r) = (a + c)^2 (a - c)^2 \Delta(y) = (a^2 - c^2)^2 \Delta(y) \quad (8.72)$$

$$= (a^2 - |\Delta(x)| b^2)^2 \Delta(y) \geq 0, \quad (8.73)$$

which implies $r \in \mathfrak{F}_{y \perp x} \subset \mathfrak{F}_x^\perp$ to be not of the same (maximal = 4) rank as y (and $\Upsilon_x(y)$) only when (recall eq. (8.64))

$$\Delta(r) = 0 \Leftrightarrow a^2 - c^2 = 0 \rightarrow a^2 - (x^0 x_0)^2 b^2 = 0. \quad (8.74)$$

The conditions for $r \in \mathfrak{F}_{y \perp x} \subset \mathfrak{F}_x^\perp$ to lie in the rank-3, rank-2 or even rank-1 orbits may be easily inferred.

Closure of the $D2-D4 \mathfrak{F}_{y \perp x}$ under T . The plane $\mathfrak{F}_{y \perp x}$ is not generally closed under the triple map T (or, equivalently, under Freudenthal duality $\sim$), see section 3.

Within the framework under consideration, namely within the $4D/5D$ special coordinates' symplectic frame of reduced FTS's and within the choice given by eq. (8.63) of the rank-4 element $x \in \mathfrak{F}$ (with $\Delta(x) < 0$) and of the rank-4 element $y \in \mathfrak{F}_x^\perp = \mathfrak{F}/\mathfrak{F}_x$ (with $\Delta(y) \geq 0$), we study now more in detail the condition of closure of the plane $\mathfrak{F}_{y \perp x}$ under T .

In order to determine the condition of closure of $\mathfrak{F}_{y \perp x}$ under T , we have to explicitly compute $T(r) \equiv T(r, r, r)$ for a generic element $r = ay + b\Upsilon_x(y) \in \mathfrak{F}_{y \perp x}$, and for any $D2-D4$ configuration y . This is given by, (see appendix D) ($c \equiv x^0 x_0 b$, $T(y) = T(y, y, y)$).

$$T(r)_M : \begin{cases} T(r)_0 = (a - c)^3 T(y)_0; \\ T(r)_i = (a^2 - c^2)(a - c)T(y)_i; \\ T(r)^0 = (a + c)^3 T(y)^0; \\ T(r)^i = (a^2 - c^2)(a + c)T(y)^i. \end{cases} \quad (8.75)$$

Then $T(r)^M = \Omega^{MN} T(r)_N$. Thus, the plane $\mathfrak{F}_{y \perp x}$ is closed under T iff

$$T(r)^0 = 0 = T(r)_0 \Leftrightarrow \begin{cases} (a - c)^3 d^{ijk} y_i y_j y_k = 0; \\ (a + c)^3 d_{ijk} y^i y^j y^k = 0. \end{cases} \quad (8.76)$$

There are various cases, as follows:

1. y^i is rank-3 in J_3 and y_i is rank-3 in $\bar{J}_3$, namely

$$d^{ijk} y_i y_j y_k \neq 0, \quad d_{ijk} y^i y^j y^k \neq 0. \quad (8.77)$$

In this case, no solutions exist to the system eq. (8.76), and $\mathfrak{F}_{y \perp x}$ is *not* closed under T .

2. y^i is rank < 3 in J_3 and y_i is rank-3 in $\bar{J}_3$, namely

$$d^{ijk} y_i y_j y_k \neq 0, \quad d_{ijk} y^i y^j y^k = 0. \quad (8.78)$$

In this case, $T(r)_0 = 0$ is automatically satisfied, while $T(r)^0 = 0$ has solution $a = x^0 x_0 b$. However, for a fixed x , this solution is a line in $\mathbb{F}^2 = \mathbb{R}^2$ spanned by (a, b) , and thus is codimension-1 in $\mathfrak{F}_{y \perp x}$. Therefore, only the x -dependent 1-dimensional locus $a = x^0 x_0 b$ in $\mathfrak{F}_{y \perp x} \subset \mathfrak{F}_x^\perp$, and *not* $\mathfrak{F}_{y \perp x}$ itself, is closed under T .

3. y^i is rank-3 in J_3 and y_i is rank < 3 in $\bar{J}_3$, namely

$$d^{ijk} y_i y_j y_k = 0, \quad d_{ijk} y^i y^j y^k \neq 0. \quad (8.79)$$

In this case, $T(r)^0 = 0$ is automatically satisfied, while $T(r)_0 = 0$ has solution $a = -x^0 x_0 b$. However, for a fixed x , this solution is a line in $\mathbb{F}^2 = \mathbb{R}^2$ spanned by (a, b) , and thus is codimension-1 in $F_{y \perp x}$. Therefore, only the x -dependent 1-dimensional locus $a = -x^0 x_0 b$ in $F_{y \perp x} \subset F_x^\perp$, and *not* $F_{y \perp x}$ itself, is closed under T .

4. y^i is rank < 3 in J_3 and y_i is rank < 3 in $\bar{J}_3$, namely

$$d^{ijk}y_iy_jy_k = 0 = d_{ijk}y^iy^jy^k. \quad (8.80)$$

In this case, the system eq. (8.76) is automatically satisfied $\forall a, b \in \mathbb{R}$, and $\mathfrak{F}_{y \perp x}$ is therefore closed²⁷ under T . Note that the condition eq. (8.80) is not inconsistent with the assumption of y (8.63) to be a rank-4 element of $\mathfrak{F}$. In fact, if both y^i and y_i are rank-2 elements in J_3 resp. $\bar{J}_3$, then $\Delta(y)$ eq. (8.64) is still generally non-vanishing, with the second term vanishing iff $d_{ijk}d^{ilm}y^jy^ky^ly^m = 0$ (in this latter case, when non-vanishing, $\Delta(y) < 0$, and y - and $\Upsilon_x(y)$ as well - would lie in the same $\Delta < 0$ $\text{Aut}(\mathfrak{F}(J_3))$ -orbit as x and $\tilde{x}$). On the other hand, if y^i and/or y_i are rank-1 elements in J_3 resp. $\bar{J}_3$, still y can be a rank-4 element of $\mathfrak{F}(J_3)$, because $\Delta(y) = -(y^iy_i)^2 \leq 0$ in this case, and thus (when the inequality strictly holds), y - and $\Upsilon_x(y)$ as well - would lie, as above, in the same $\Delta < 0$ $\text{Aut}(\mathfrak{F}(J_3))$ -orbit as x and $\tilde{x}$.

8.3 The general $D0 - D4$ sector

The Freudenthal plane $\mathfrak{F}_x$. We start by identifying x with the rank-4 element of the FTS $\mathfrak{F}$ given by the²⁸ $D0 - D4$ brane charge configuration:

$$x_{D0D4} \equiv (0, x^i, x_0, 0_i)^T \in \mathfrak{F}, \quad (8.81)$$

and we further impose that x_{D0D4} belongs to (one of) the $\Delta > 0$ $\text{Aut}(\mathfrak{F})$ -orbit(s) (see section 8.4).

$$\Delta(x_{D0D4}) = \frac{2}{3}x_0d_{ijk}x^ix^jx^k > 0. \quad (8.82)$$

From the definition eq. (6.7) (note that $\epsilon = 1$ in this case), one can compute that the Freudenthal-dual $\widetilde{x_{D0D4}}$ of the $D0 - D4$ configuration eq. (8.81) is a $D2 - D6$ configuration, namely²⁹ The dual is

$$\tilde{x} \equiv (\tilde{x}^0, 0^i, 0, \tilde{x}_i)^T \in \mathfrak{F}, \quad (8.83)$$

with eq. (6.7)

$$\tilde{x}^0 = -\frac{1}{3\sqrt{\Delta(x)}}d_{jkl}x^jx^kx^l; \quad (8.84)$$

$$\tilde{x}_i = \frac{1}{\sqrt{\Delta(x)}}x_0d_{ijk}x^jx^k. \quad (8.85)$$

By exploiting the *adjoint identity* (see [46, 73, 74] and refs. therein) of the Jordan algebra J_3 underlying the reduced FTS $\mathfrak{F}$ eq. (8.8) one can also check that Δ is invariant under Freudenthal duality:

$$\Delta(x_{D0D4}) = \Delta(\widetilde{x_{D0D4}}) > 0, \quad (8.86)$$

and thus that $\widetilde{x_{D0D4}}$ would lie in the same $\text{Aut}(\mathfrak{F}(J_3))$ -orbit as x_{D0D4} .

²⁷It could have also characterized as a 2-dimensional sub-FTS of $\mathfrak{F}$.

²⁸We might have as well started with a $D2 - D6$ configuration, and perform an equivalent treatment (obtaining, as evident from the treatment given below and from the anti-involutivity of Freudenthal duality, a $D0 - D4$ configuration as Freudenthal-dual of the starting $D2 - D6$ configuration).

²⁹This has been computed for the STU model in Example 1 of section V.A of [17] (cfr. (198)-(200) therein); see also the treatment of section 8.4.

Thus, one can define the Freudenthal plane $\mathfrak{F}_x$ ($\dim \mathfrak{F}_x = 2$), spanned by x eq. (8.81) and $\tilde{x}$ (8.83), whose generic element is $x_F = ax + b\tilde{x} \in \mathfrak{F}_x$, $a, b \in \mathbb{R}$. By using eq. (8.8) again, one can also compute that

$$\Delta(x_F) = (a^2 + b^2)^2 \Delta(x) > 0, \quad (8.87)$$

implying that $\mathfrak{F}_x$ lies completely in the rank-4 $\Delta > 0$ orbit³⁰ of $\text{Aut}(\mathfrak{F}(J_3))$.

The orthogonal space $\mathfrak{F}_x^\perp$, and the plane $\mathfrak{F}_{y \perp x} \subset \mathfrak{F}_x^\perp$. Then, one can pick another rank-4 element $y \in \mathfrak{F}$ which is $\{, \}$ -orthogonal to x eq. (8.81) and $\tilde{x}$ (8.83); the most general element of this kind is given by the charge configuration:

$$y \equiv (y^0, y^i, y_0, y_i)^T \in \mathfrak{F}, \quad (8.88)$$

constrained

$$\{x, y\} = 0 = \{\tilde{x}, y\} \Leftrightarrow \begin{cases} y^0 = \frac{1}{x_0} x^i y_i; \\ y_0 = -3x_0 \frac{d_{ijk} y^i x^j x^k}{d_{lmn} x^l x^m x^n}. \end{cases} \quad (8.89)$$

One can also compute that

$$\begin{aligned} \Delta(y) = & - \left(-3 \frac{d_{klm} y^k x^l x^m}{d_{qrs} x^q x^r x^s} x^j y_j + y^j y_j \right)^2 - 2x_0 \frac{d_{klm} y^k x^l x^m}{d_{qrs} x^q x^r x^s} d_{ijt} y^j y^j y^t \\ & - \frac{2}{3} \frac{1}{x_0} x^j y_j d^{klm} y_k y_l y_m + d_{ijk} d^{ilm} y^j y^k y_l y_m \gtrless 0. \end{aligned} \quad (8.90)$$

In eq. (8.90), the case of vanishing Δ has been excluded because y is chosen to be of rank-4 in $\mathfrak{F}$.

One can compute the components of $\Upsilon_x(y)^M$ eq. (2.6) as given by

$$(\Upsilon_x(y))^0 = -d_{ijk} x^i x^j y^k; \quad (8.91)$$

$$(\Upsilon_x(y))^i = -d_{klm} d^{mij} x^k x^l y_j + 2x^j y_j x^i; \quad (8.92)$$

$$(\Upsilon_x(y))_0 = -2x^j y_j x_0; \quad (8.93)$$

$$(\Upsilon_x(y))_i = 2x_0 d_{ijk} x^j y^k - 3x_0 \frac{d_{klm} y^k x^l x^m}{d_{qrs} x^q x^r x^s} d_{ijt} x^j x^t, \quad (8.94)$$

which for this case can be written as

$$(\Upsilon_x(y))^0 = -d_{ijk} x^i x^j y^k; \quad (8.95)$$

$$(\Upsilon_x(y))^i = -d_{klm} d^{mij} x^k x^l y_j + y^0 x_0 x^i; \quad (8.96)$$

$$(\Upsilon_x(y))_0 = -2y^0 x_0^2; \quad (8.97)$$

$$(\Upsilon_x(y))_i = 2x_0 d_{ijk} x^j y^k + y_0 d_{ijt} x^j x^t. \quad (8.98)$$

By exploiting (8.8), one can then check that $\Upsilon_x(y)$ eq. (8.98) automatically satisfies

$$\{x, \Upsilon_x(y)\} = 0 = \{\tilde{x}, \Upsilon_x(y)\} \Leftrightarrow \Upsilon_x(y) \in \mathfrak{F}_x^\perp \equiv \mathfrak{F}/\mathfrak{F}_x, \quad (8.99)$$

³⁰Again, as pointed out above, some subtleties may arise, related to the splitting of the $\Delta > 0$ locus of $\mathfrak{F}$ (cfr. section 8.4, and *e.g.* [50, 75, 76], and refs. therein).

for every pair y^i and y_i , with $i = 1, \dots, N-1$. In fact, regardless of d_{ijk} and d^{ijk} , when only a pair y^i and y_i for a *fixed* i is non-vanishing (among all y^i 's and y_i 's), then y is non-trivially of rank-4 in $\mathfrak{F}$, because generally $\Delta(y) \neq 0$, since at least the term $-(y^i y_i)^2$ is present (cfr. eqs. (8.3)–(8.4)). Therefore, one can define $N-1$ *distinct* planes $(\mathfrak{F}_{y \perp x}^\perp)_i$'s, orthogonal to the Freudenthal plane $\mathfrak{F}_x$, cfr. subsection 8.3.

Moreover, (consistently with eq. (4.23) and eq. (4.24)) one can compute that

$$\begin{aligned} \Delta(\Upsilon_x(y)) &= -\frac{8}{9}x_0^2 \left(d_{abc}x^a x^b x^c \right)^2 \left[\frac{1}{2} \left(-3 \frac{d_{klm}y^k x^l x^m}{d_{qrs}x^q x^r x^s} x^j y_j + y^j y_j \right)^2 + x_0 \frac{d_{klm}y^k x^l x^m}{d_{qrs}x^q x^r x^s} d_{ijt} y^j y^j y^t \right] \\ &\quad + \frac{1}{3} \frac{1}{x_0} x^j y_j d^{klm} y_k y_l y_m + d_{ijk} d^{ilm} y^j y^k y_l y_m \geq 0. \end{aligned} \quad (8.100)$$

implying that $\Upsilon_x(y)$ lies in the same $\text{Aut}(\mathfrak{F}(J_3))$ -orbit as y .

The same holds for a generic element $r = ay + b\Upsilon_x(y) \in \mathfrak{F}_{y \perp x}$ ($a, b \in \mathbb{R}$), which belongs to the same $\text{Aut}(\mathfrak{F})$ -orbit as y : indeed it can be checked that (consistently with eq. (4.23) and eq. (4.24))

$$\begin{aligned} \Delta(r) &= -2 \left(a^2 + \frac{2b^2}{3} x_0 d_{abc} x^a x^b x^c \right)^2 \left[\frac{1}{2} \left(-3 \frac{d_{klm}y^k x^l x^m}{d_{qrs}x^q x^r x^s} x^j y_j + y^j y_j \right)^2 + x_0 \frac{d_{klm}y^k x^l x^m}{d_{qrs}x^q x^r x^s} d_{ijt} y^j y^j y^t \right] \\ &\quad + \frac{1}{3} \frac{1}{x_0} x^j y_j d^{klm} y_k y_l y_m + d_{ijk} d^{ilm} y^j y^k y_l y_m \geq 0. \end{aligned} \quad (8.101)$$

The canonical Darboux symplectic frame. We recall that in the $4D/5D$ special coordinates' symplectic frame a generic element Q of the reduced FTS $\mathfrak{F}(J_3)$ splits as given by eq. (8.1), while the $2N \times 2N$ symplectic metric is given by (6.9). By a simple re-ordering of rows and columns (amounting to a relabelling of indices, one can switch to a *canonical Darboux symplectic frame*³¹ (in which the $4D/5D$ covariance is still manifest), in which x eq. (8.1) splits as follows:

$$x = - \left(p^0, q_0, p^1, q_1, \dots, p^{N-1}, q_{N-1} \right)^T, \quad (8.102)$$

and in which the symplectic metric eq. (6.9) acquires the following form

$$\Omega = -\mathbf{1} \otimes \varepsilon, \quad (8.103)$$

where ε is the 2×2 symplectic metric of the defining irrepr. **2** of $\text{Sp}(2) \approx \text{SL}(2)$ defined by eq. (6.21).

At a glance, in a physical (Maxwell-Einstein) framework eq. (8.102) suggests that the choice of the (canonical) Darboux symplectic frame defined by (8.102) (or, equivalently, by eq. (8.103 and eq. (6.21)), amounts to making manifest the splitting of the electric-magnetic fluxes of the Abelian 2-form field strengths, grouped, within the symplectic vector Q eq. (8.102), into the KK vector's fluxes (magnetic p^0 and electric q_0), and into the fluxes (magnetic p^i and electric q_i , $i = 1, \dots, N-1$) of each of the $N-1$ Abelian vectors with a

³¹For some applications of the canonical Darboux frame to supergravity, see e.g. [77–79], and refs. therein.

$D = 5$ origin. When specifying such a generic (supersymmetry-independent) interpretation for minimal $D = 5$ supergravity dimensionally reduced down to $\mathcal{N} = 2$, $D = 4$ supergravity, p^0 and q_0 are the magnetic resp. electric charges of the $D = 4$ graviphoton (the Abelian vector in the $\mathcal{N} = 2$ gravity multiplet), whereas each of the $N - 1$ pairs (p^i, q_i) denote the magnetic resp. electric charges of the Abelian vector belonging to each of the $N - 1$ vector supermultiplets coupled to the gravity one (these all have a $D = 5$ origin, thereby comprising the $D = 5$ graviphoton, as well).

Thus, the $(2N - 2)$ -dim. space $\mathfrak{F}_x^\perp$, $\{, \}$ -orthogonal to the 2-dim. Freudenthal plane $\mathfrak{F}_x$, gets decomposed into $N - 1$ 2-dim. subspaces, all mutually orthogonal with respect to the symplectic product $\{, \}$ defined by (8.103): each of them corresponds to the electric-magnetic flux degrees of freedom of a vector supermultiplet in the corresponding $\mathcal{N} = 2$, $D = 4$ supergravity, or, more generally, to the electric-magnetic fluxes of a $D = 4$ Abelian vector fields with a five-dimensional origin.

8.4 F-duality preserves the $\Delta > 0$ $\text{Aut}(\mathfrak{F})$ -orbits: the STU model

At least for (non-degenerate) reduced FTS's, $\text{Aut}(\mathfrak{F})$ has a transitive action on elements with a given $\Delta < 0$. Thus, the $\Delta < 0$ locus corresponds to a one-parameter family of $\text{Aut}(\mathfrak{F})$ -orbits,³² consequently, Freudenthal duality trivially preserves the orbit structure for $\Delta < 0$.

The story is more complicated for the $\Delta > 0$ locus, which, again *at least* for (non-degenerate) reduced FTS's, generically (with the unique exception of the T^3 model) has two or more $\text{Aut}(\mathfrak{F})$ -orbits. However, the existence of the Freudenthal rotations presented in section 5.1 ensures that x and $\tilde{x}$ always lie in the same orbit. Here we explicitly present the non-trivial orbit structure of the $\Delta > 0$ locus and its properties under Freudenthal duality for the STU model using only the (discrete) U-duality invariants characterising the orbits.

The STU model, introduced independently in [81, 82], provides an interesting subsector of string compactification to four dimensions. This model has a low energy limit which is described by $\mathcal{N} = 2$ supergravity coupled to three vector multiplets interacting through the special Kähler manifold $[\text{SL}(2, \mathbb{R})/\text{SO}(2)]^3$. (In the version of [81], the discrete $\text{SL}(2, \mathbb{Z})$ are replaced by a subgroup denoted $\Gamma_0(2)$). The three complex scalars are denoted by the letters S, T and U , hence the name of the model [82, 83]. The remarkable feature that distinguishes it from generic $\mathcal{N} = 2$ supergravities coupled to vectors [84] is its $S - T - U$ triality [82]. There are three different versions with two of the $\text{SL}(2)$ s perturbative symmetries of the Lagrangian and the third a non-perturbative symmetry of the equations of motion. In a fourth version all three are non-perturbative [82, 83]. All four are on-shell equivalent. If there are in addition four hypermultiplets, the STU model is self-mirror. Even though the simplest reduced FTS exhibiting more than one $\Delta > 0$ $\text{Aut}(\mathfrak{F})$ -orbit is given by $\mathfrak{F}(\mathbb{R} \oplus \mathbb{R})$, which corresponds to the ST^2 model of $\mathcal{N} = 2$, $D = 4$ supergravity [80], we will explicitly treat $\mathfrak{F}(\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R})$ corresponding to the slightly larger

³²We will equivalently use $\text{Aut}(\mathfrak{F})$, electric-magnetic duality, or U-duality, even if the first terminology is mathematical, the second more pertaining to Maxwell gravity, and the third one deriving from string theory.

STU model, because it can be considered as a genuine truncation of all (non-degenerate) reduced FTS (with the exception of the T^3 and ST^2 models, which are however particular “degenerations” of the STU model itself), thus covering all such cases.

As determined in [80] (see also the treatment in section F.1 of [75]), in the STU model there are two orbits with $\Delta > 0$, one supersymmetric and one non-supersymmetric (the one with vanishing central charge at the horizon: $Z_H = 0$), and their coset expressions are isomorphic (even if they are $SL(2, \mathbb{R})^{\times 3}$ -disjoint orbits):

$$\mathcal{O}_{\Delta>0,BPS} \cong \frac{SL(2, \mathbb{R})^{\times 3}}{U(1)^{\times 2}} \cong \mathcal{O}_{\Delta>0,non-BPS,Z_H=0}. \quad (8.104)$$

Following the treatment of [73], one can consider a $D0 - D4$ representative (also in the FTS representation [17]) of the orbits $\mathcal{O}_{\Delta>0,BPS}$ and $\mathcal{O}_{\Delta>0,non-BPS,Z_H=0}$:

$$x_{D0-D4,\Delta>0} = (0, p^1, p^2, p^3, q_0, 0, 0, 0)^T = \begin{pmatrix} -q_0 & (p^1, p^2, p^3) \\ (0, 0, 0) & 0 \end{pmatrix}, \quad (8.105)$$

$$\Delta(x_{D0-D4,\Delta>0}) = 4q_0 p^1 p^2 p^3 > 0, \quad (8.106)$$

and determine the further $SL(2, \mathbb{R})^{\times 3}$ -invariant constraints on the sign of q_0^0 , p^1 , p^2 and p^3 defining the orbits $\mathcal{O}_{\Delta>0,BPS}$ and $\mathcal{O}_{\Delta>0,non-BPS,Z_H=0}$ within the $D0 - D4$ configuration (8.105)–(8.110) with $\Delta > 0$ (all other sign combinations pertain to the unique non-BPS $Z_H \neq 0$ orbit $\mathcal{O}_{\Delta<0}$):

- $\frac{1}{2}$ -BPS conditions (triality invariant):

$$(q_0, p^1, p^2, p^3) = (+, +, +, +) \text{ or } (-, -, -, -). \quad (8.107)$$

- non-BPS $Z_H = 0$ conditions (up to triality) :

$$(q_0, p^1, p^2, p^3) = (+, +, -, -) \text{ or } (-, -, +, +). \quad (8.108)$$

Analogously, we can consider (also in the FTS representation [17]) a $D2 - D6$ representative of the orbits $\mathcal{O}_{\Delta>0,BPS}$ and $\mathcal{O}_{\Delta>0,non-BPS,Z_H=0}$:

$$x_{D2-D6,\Delta>0} = (p^0, 0, 0, 0, 0, q_1, q_2, q_3)^T = \begin{pmatrix} 0 & (0, 0, 0) \\ (q_1, q_2, q_3) & p^0 \end{pmatrix}, \quad (8.109)$$

$$\Delta(x_{D2-D6,\Delta>0}) = -4p^0 q_1 q_2 q_3 > 0. \quad (8.110)$$

From the treatment of [80], we can write down the further $SL(2, \mathbb{R})^{\times 3}$ -invariant constraints on the sign of p^0 , q_1 , q_2 and q_3 defining the orbits $\mathcal{O}_{\Delta>0,BPS}$ and $\mathcal{O}_{\Delta>0,non-BPS,Z_H=0}$ within the $D2 - D6$ configuration eqs. (8.109)–(8.110) with $\Delta > 0$ (all other sign combinations pertain to the unique non-BPS $Z_H \neq 0$ orbit $\mathcal{O}_{\Delta<0}$):

- $\frac{1}{2}$ -BPS conditions (triality invariant):

$$(q_0, p^1, p^2, p^3) = (+, -, -, -) \text{ or } (-, +, +, +). \quad (8.111)$$

- non-BPS $Z_H = 0$ conditions (up to triality) :

$$(q_0, p^1, p^2, p^3) = (+, -, +, +) \text{ or } (-, +, -, -). \quad (8.112)$$

As computed in Example 1 of section V.A of [17] (cfr. (198)–(200) therein), the Freudenthal dual of the $D0 - D4$ element eqs. (8.105)–(8.106) yields a particular element of the type $D2 - D6$ type eqs. (8.109)–(8.110) (up to triality; $c(m, n) > 0$):

$$\begin{aligned} x_{D0-D4, \Delta > 0} &= \begin{pmatrix} -n & (n, -m, -m) \\ (0, 0, 0) & 0 \end{pmatrix} \\ \xrightarrow{\sim} x_{\widetilde{D0-D4}, \Delta > 0} &= c(m, n) \begin{pmatrix} 0 & (0, 0, 0) \\ (m, -n, -n) & -m \end{pmatrix}. \end{aligned} \quad (8.113)$$

By using eq. (8.113), one can then compute that the action of Freudenthal duality on the $D0 - D4$ element (8.105)–(8.106) maps the conditions eq. (8.107) and eq. (8.108) respectively into the conditions eq. (8.111) and eq. (8.112): by using the commutativity of $\text{Aut}(\mathfrak{F}(\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R})) = \text{SL}(2, \mathbb{R})^{\times 3}$ and Freudenthal duality $\sim$ [17], one can consequently conclude that the orbit stratification of the $\Delta > 0$ locus of $\mathfrak{F}(\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R})$ is *preserved* under Freudenthal duality $\sim$.

9 Linear realisations of general Freudenthal transformations

As consequence of its definition, Freudenthal duality $\sim$ can only be consistently defined in the *locus* $\Delta \neq 0$ of the FTS $\mathfrak{F}$ itself. In general, the group $\text{Aut}(\mathfrak{F})$ has a non-transitive action over such a *locus*, which undergoes a, (*at least*) twofold stratification, into a (always unique) $\Delta < 0$ orbit and into a $\Delta > 0$ *sub-locus*, which may in turn further stratify into $\text{Aut}(\mathfrak{F})$ -orbits.

While F-duality is a non-linear operation, as discussed in [17] its action can be realised by finite “local/gauged” U-duality transformations $U : \mathfrak{F} \rightarrow \text{Aut}(\mathfrak{F})$, namely, as we will understand throughout the following treatment, that depend on the element of $\mathfrak{F}$ they are applied onto.

More generally, Freudenthal duality, and so GFT, can be mimicked by finite transformations of at least three different kinds, the first two of which are not contained in the U-duality group:

1. Gauged anti-symplectic transformations, as we will discuss, within non-degenerate, reduced FTS’s, for the $\Delta < 0$ orbit in subsection 9.1;
2. Gauged $\text{Sp}(2N, \mathbb{R})$ transformations, where $\dim_{\mathbb{R}} \mathfrak{F} = 2N$. We will discuss these, within the T^3 model ($N = 2$; cfr. (table 1)), for the $\Delta < 0$ orbit in subsection 9.2;
3. Gauged $\text{Aut}(\mathfrak{F})$ transformations, as we will discuss, within the T^3 model, for the $\Delta < 0$ and $\Delta > 0$ orbit in subsections 9.3 and 9.4, respectively.

9.1 Anti-symplectic realisation: $\Delta < 0$

At least within (non-degenerate) reduced FTS's, the quartic polynomial invariant Δ of $\text{Aut}(\mathfrak{F}(J_3)) \approx \text{Conf}(J_3)$ can be written as in eq. (8.4) (see section 8 for this and other expressions) (cfr. e.g. [44, 72, 73]; ($i = 1, \dots, N-1$, $\dim \mathfrak{F} = 2N$)). Let us consider a $D0 - D6$ configuration, studied in subsection 8.2. As we can see there (we refer to eqs. (8.52), (8.53), (8.58) and (8.59)), the action of the Freudenthal duality $\sim$ on x_{D0D6} eq. (8.52) can be represented by a (maximal-rank) $2N \times 2N$ matrix

$$\tilde{x} = -\epsilon \mathbf{O} x, \quad (9.1)$$

$$\mathbf{O} = \begin{pmatrix} 1 & & & \\ & \mathbf{A} & & \\ & & -1 & \\ & & & \mathbf{B} \end{pmatrix}, \quad (9.2)$$

where $\mathbf{A}, \mathbf{B} \in \text{GL}(N-1, \mathbb{R})$. The action of a general Freudenthal transformations is then

$$x_F = ax + b\tilde{x} = (\mathbf{1} - \epsilon \mathbf{O}) x. \quad (9.3)$$

The transformation $\mathbf{O} \in \text{GL}(2N, \mathbb{R})$ is inherently *not unique*. Also, apart from the “ $\mp$ ” branching in (9.1), the whole realisation of Freudenthal duality does not depend on p^0 nor on q_0 , and so it can be (loosely) considered an “ungauged” transformation in $\mathfrak{F}$. In particular, $\mathbf{O}$ is anti-symplectic, namely

$$\mathbf{O}^T \Omega \mathbf{O} = -\Omega \Leftrightarrow \mathbf{A}^T \mathbf{B} = -\mathbf{1} \quad (9.4)$$

where $\mathbf{1}$ is the identity matrix in $N-1$ dimensions. Note, however, that $\mathbf{O}$ is never symplectic (i.e., it always holds that $\mathbf{O}^T \Omega \mathbf{O} \neq \Omega \Leftrightarrow \mathbf{O} \notin \text{Sp}(2N, \mathbb{R})$).

Let us consider a particularly simple anti-symplectic case, for which $\mathbf{A} = -\mathbf{1} \Rightarrow \mathbf{B} = \mathbf{1}$:

$$\mathbf{O}' = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}. \quad (9.5)$$

At least in all reduced FTS's $\mathfrak{F}$'s based on simple and semi-simple rank-3 Jordan algebras (see section 2 and table 1 therein), it can be proved (cfr. appendix D of [85]) that $\mathbf{O}'$ eq. (9.5) realizes an *outer* automorphism of $\text{Aut}(\mathfrak{F}(J_3))$, namely that

$$\mathbf{O}^{-1} \hat{\mathbf{R}}_{\mathfrak{F}}(g) \mathbf{O} \subset \hat{\mathbf{R}}_{\mathfrak{F}}(g), \quad (9.6)$$

$$\mathbf{O} \in \frac{\text{Aut}(\text{Aut}(\mathfrak{F}(J_3)))}{\text{Inn}(\text{Aut}(\mathfrak{F}(J_3)))} \equiv \text{Out}(\text{Aut}(\mathfrak{F}(J_3))), \quad (9.7)$$

where $\hat{\mathbf{R}}_{\mathfrak{F}}(g)$ denotes the $2N \times 2N$ matrix representation of the element g of $\text{Aut}(\mathfrak{F}(J_3))$ acting on $\mathfrak{F}$ itself, and Aut , Inn and Out respectively denote the automorphism group, and its inner resp. outer components. Note that $\mathbf{O}'$ is an involution:

$$\mathbf{O}'^2 = \mathbf{1}, \quad (9.8)$$

where here $\mathbf{1}$ denotes the identity matrix in $2N$ dimensions. However, from eqs. (6.7)–(8.59) it follows that the correct iteration of the Freudenthal duality on the $D0 - D6$ configuration x eq. (8.52) is provided by the application of $\mp \mathbf{O}'$ and then necessarily of $\pm \mathbf{O}'$, thus yielding to $-\mathbf{O}'^2 = -\mathbf{1}$ acting on x , and correctly implying

$$\widetilde{\widetilde{x}} = -x. \quad (9.9)$$

The anti-symplecticity of $\mathbf{O}'$ eq. (9.5) implies that it does not preserve the symplectic structure of $\mathfrak{F}$ (as neither Freudenthal duality does, as well [17]). This is consistent with the fact that $\mathbf{O}'$ realizes an *outer* automorphism of the electric-magnetic U-duality group $\text{Aut}(\mathfrak{F}(J_3))$, which in turn is generally realized in a symplectic way [23, 86, 87]:

$$\widehat{\mathbf{R}}_{\mathfrak{F}}(g)^T \Omega \widehat{\mathbf{R}}_{\mathfrak{F}}(g) = \Omega, \quad \forall g \in \text{Aut}(\mathfrak{F}(J_3)). \quad (9.10)$$

9.1.1 Anti-symplectic symmetries and parity transformations

As observed in [85], *at least* for all automorphism groups of reduced FTS's over simple or semi-simple rank-3 Jordan algebras it holds that (see e.g. [88])

$$\text{Out}(\text{Aut}(\mathfrak{F}(J_3))) \subseteq \mathbb{Z}_2. \quad (9.11)$$

Thus, all non-trivial elements of $\text{Out}(\text{Aut}(\mathfrak{F}(J_3)))$ are implemented by anti-symplectic transformations. In [89] (also cfr. [90]), it was discussed that the global symmetry of the resulting Maxwell-Einstein (super)gravity contains the factor $\mathbb{Z}_2$, which can be offset by a *spatial parity* $\mathbf{P}$ transformation. In particular, from eq. (2.118) of [89], it follows that the global symmetry group G of the resulting Maxwell-Einstein (super)gravity theory is given by

$$G = G_0 \times \mathbb{Z}_2 = \{G_0, G_0 \cdot \mathbf{p}\}, \quad (9.12)$$

where G_0 is the identity-connected, proper electric-magnetic (U -)duality, $\text{Aut}(\mathfrak{F}(J_3))$ -part of G , whereas $\mathbf{p}$ corresponds to an element of G implemented by an *anti-symplectic* transformation.

Interestingly, the above results relate a realisation (*not* the unique one, though! — see subsections 9.2 and 9.3) of the Freudenthal duality on the well-defined representative $D0 - D6$ of the unique, non-BPS (non-supersymmetric) rank-4 $\Delta < 0$ orbit in $\mathfrak{F}$, to spatial parity transformations in the corresponding theory; in fact, anti-symplectic transformations, such as $\mathbf{O}'$ eq. (9.5) are symmetries of the theory, provided that they are combined with spatial ($3D$) parity $\mathbf{P}$.

9.2 Gauged symplectic realisation: $\Delta < 0$

It is also possible to find a (non-unique) symplectic transformation realizing the Freudenthal duality transformation eq. (6.7) on the $D0 - D6$ representative of the $\Delta < 0$ $\text{Aut}(\mathfrak{F})$ -orbit of $\mathfrak{F}$. However, this will necessarily be “gauged” in $\mathfrak{F}$, namely it will depend on the element of F it acts upon (i.e., in this case, on the $D0 - D6$ element eq. (8.52)).

Let us exemplify this in the T^3 model ($N = 2$, as from table 1); we look for a finite $\mathrm{Sp}(4, \mathbb{R})$ transformation (realized as a 4×4 matrix $\mathcal{C}$) such that

$$\mathcal{C} \begin{pmatrix} p^0 \\ 0 \\ q_0 \\ 0 \end{pmatrix} = \mathrm{sgn}(p^0 q_0) \begin{pmatrix} -p^0 \\ 0 \\ q_0 \\ 0 \end{pmatrix}. \quad (9.13)$$

Defining $z \equiv q_0/p^0$, long but straightforward algebra yields to the following expression (for $\mathrm{sgn}(p^0 q_0) = \mathrm{sgn}(z) = \pm 1$):

$$\mathcal{C}_{\pm}(z; a, b, c, d, e, f, g) = \begin{pmatrix} \pm(1 + \frac{a}{z}) \mp(bc \mp \frac{de}{z}) & \frac{2}{z} + \frac{a}{z^2} & \mp(bf \mp \frac{dg}{z}) \\ \pm bz & e & b & g \\ a & -(bcz \mp de) & \pm(1 + \frac{a}{z}) & -(bfz \mp dg) \\ d & c & \pm \frac{d}{z} & f \end{pmatrix}_{ef-gc=1}. \quad (9.14)$$

$\mathcal{C}_{\pm}$ eq. (9.14) depends on z , and it is thus “gauged” in $\mathfrak{F}$. Moreover, it depends on 7 real parameters, with a constraint ($ef - gc = 1$): therefore, it realizes a particular, ($D0 - D6$)-dependent finite transformation of $\mathrm{Sp}(4, \mathbb{R})$, which mimicks the action of Freudenthal duality over the $D0 - D6$ element eq. (8.52).

9.3 Gauged $\mathrm{Aut}(\mathfrak{F})$ realisation: $\Delta < 0$

In the $\Delta < 0$ locus of $\mathfrak{F}$ (on which the action of $\mathrm{Aut}(\mathfrak{F})$ is always transitive, thus defining a unique orbit $\mathcal{O}_{\Delta < 0}$ of $\mathfrak{F}$), we deal with the issue of mimicking the action of Freudenthal duality by an $\mathrm{Aut}(\mathfrak{F})$ transformation (which will generally be local in $\mathfrak{F}$), and consider the following (commutative) diagram:

$$\begin{array}{ccc} x & \xrightarrow{U} & x_{\mathrm{can}} \\ \downarrow \sim & & \downarrow \sim \\ \tilde{x} = M_x x & \xrightarrow{U'} & \tilde{x}_{\mathrm{can}} = M_{x_{\mathrm{can}}} x_{\mathrm{can}} \end{array} \quad (9.15)$$

where $x, x_{\mathrm{can}} \in \mathcal{O}_{\Delta < 0}$. Here, x_{can} denotes a convenient “canonical” representative that can be defined in a uniform manner for all relevant FTS as in [45]. The corresponding $\mathrm{Aut}(\mathfrak{F})$ transformations taking x and $\tilde{x}$ to x_{can} and $\tilde{x}_{\mathrm{can}}$, respectively, are denoted U and U' . Similarly, M_x and $M_{x_{\mathrm{can}}}$ are the gauged $\mathrm{Aut}(\mathfrak{F})$ transformations that send x and x_{can} to $\tilde{x}$ and $\tilde{x}_{\mathrm{can}}$, respectively. Since the square commutes we free to pick a convenient canonical representative.

Generally, *at least* for (non-degenerate) reduced FTS’s, the homogeneous space $\mathcal{O}_{\Delta < 0}$ can be written as

$$\mathcal{O}_{\Delta < 0} = \frac{\mathrm{Conf}(J_3)}{\mathrm{Str}_0(J_3)}, \quad (9.16)$$

where we recall that $\text{Aut}(\mathfrak{F}(J_3)) \simeq \text{Conf}(J_3)$, $\text{Conf}(J_3)$ and $\text{Str}_0(J_3)$ respectively denote the conformal and reduced structure groups of the cubic Jordan algebra J_3 .

For simplicity's sake, let us assume $\text{Str}_0(J_3) = Id$ (namely, there is no continuous nor discrete stabiliser for $\mathcal{O}_{\Delta < 0}$, which thus is a group manifold: $\mathcal{O}_{\Delta < 0} \cong \text{Aut}(\mathfrak{F})$). Actually, this only holds for the T^3 model of $\mathcal{N} = 2$, $D = 4$ supergravity, associated to the simplest example of (non-degenerate) reduced FTS [75, 80]. Let us also choose a convenient representative of $\mathcal{O}_{\Delta < 0}$. An obvious choice is given by the $D0 - -D6$ configuration eq. (8.52), $x_{\text{can}} = x_{D0D6}$, which makes the $\text{Str}_0(J_3)$ stabiliser of $\mathcal{O}_{\Delta < 0}$ manifest for all reduced FTS.

The assumption $\text{Str}_0(J_3) = Id$ implies $U' = U$. From eq. (9.15) we have

$$\begin{aligned} \widetilde{x}_{D0D6} &= \widetilde{U}x \\ &= U' M_x x = U' \widetilde{x} = \widetilde{U'}x; \end{aligned} \quad (9.17)$$

where the last step of the second line follows from the fact that $\text{Aut}(\mathfrak{F})$ and Freudenthal duality commute [17]. Thus, applying Freudenthal duality to (9.17) one obtains

$$Ux = U'x \Rightarrow x = U'^{-1}Ux \Leftrightarrow \left. \begin{array}{l} U'^{-1}U \\ U^{-1}U' \end{array} \right\} \in \text{Stab}(x) = Id \Leftrightarrow U = U' \quad \square \quad (9.18)$$

Consequently, any reasoning involving the diagram eq. (9.15) is independent of the choice of x_{can} ; indeed, the $\text{Aut}(\mathfrak{F}(J_3))$ transformation connecting any two elements of $\mathcal{O}_{\Delta < 0} = \text{Aut}(\mathfrak{F}(J_3))$ (say x_{can} and x'_{can}) will be unique, since $\text{Aut}(\mathfrak{F}(J_3))$ is free on the orbit $\mathcal{O}_{\Delta < 0}$ by assumption.

In order to determine the gauged $\text{Aut}(\mathfrak{F}(J_3))$ transformation $M_{x_{D0D6}}$ mimicking Freudenthal duality acting on $x_{\text{can}} = x_{D0D6}$, i.e. such that $\widetilde{x}_{D0D6} = M_{x_{D0D6}} x_{D0D6}$ (cfr. eq. (9.15)), we will use the “ T^3 degeneration” of the quantum information symplectic frame of the STU model (see e.g. [36, 75, 80, 91, 92]). By recalling that the usual parametrization of an element $x \in \mathfrak{F}(J_3)$ as a formal 2×2 matrix reads [17]

$$x = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}, \quad A \in J_3, \quad B \in \overline{J_3}, \quad (9.19)$$

the relation between the quantum information symplectic frame $x = a$, with $a_{abc} = a_{(abc)}$ ($a, b, c = 0, 1$), the $4D/5D$ special coordinates' symplectic frame $x = (p^0, p^1, q_0, q_1)^T$ and the FTS parametrization eq. (9.19) for the T^3 model ($J_3 = \mathbb{R}$) reads as follows:

$$\begin{array}{lll} \text{QIT frame:} & a_{000} & a_{111} \quad a_{001} \quad a_{110}, \\ 4D/5D \text{ frame:} & p^0 & q_0 \quad -p^1 \quad q_1/3, \\ \text{FTS frame:} & \beta & -\alpha \quad -A \quad B/3, \end{array} \quad (9.20)$$

which is consistent with the “ T^3 degeneration” of (175) and (183), table VII of [17].

Thus, we are searching for a 2×2 matrix $\mathbf{M}$ such that

$$\widetilde{a_{(abc)D0D6}} = \mathbf{M}_a^{a'} \mathbf{M}_b^{b'} \mathbf{M}_c^{c'} a_{(a'b'c')D0D6} \equiv M_{x_{D0D6}} \begin{pmatrix} -q_0 & 0 \\ 0 & p^0 \end{pmatrix}. \quad (9.21)$$

Straightforward algebra yields the unique solution:

$$\mathbf{M}(z) = \text{sgn}(z) \begin{pmatrix} 0 & -z^{-1/3} \\ z^{1/3} & 0 \end{pmatrix}, \quad z = q_0/p^0. \quad (9.22)$$

Note that if and only if $z = \pm 1 \Leftrightarrow p^0 = \pm q_0$, the matrix $\mathbf{M}(z = \pm 1)$ eq. (9.22) (and thus, through (9.21), $M_{x_{D0D6}}$) belongs to $\text{SL}(2, \mathbb{Z})$:

$$\mathbf{M}(z = \pm 1) = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \pm \varepsilon \in \text{SL}(2, \mathbb{Z}), \quad (9.23)$$

where ε is nothing but the symplectic 2×2 metric $\Omega_{2 \times 2}$:

$$\varepsilon \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \equiv \Omega_{2 \times 2}. \quad (9.24)$$

Note that for $p^0, q_0 = \pm 1$ the black hole is *projective*, cfr. the treatment in the quantized charge regime presented in [17]). The integral automorphism group $\text{Aut}(\mathfrak{F}_{\mathbb{Z}})$ acts transitively on projective charge configurations.

With the generalisation to arbitrary reduced FTS in mind, it is useful to reexpress $M_{x_{D0D6}}(z)$, as defined by eq. (9.21), through the elementary $\text{Aut}(\mathfrak{F})$ transformations defined in [14, 93] for generic reduced FTS

$$M_{x_{D0D6}}(z) = \psi \left(-|z|^{-1/3} \right) \circ \varphi \left(|z|^{1/3} \right) \circ \psi \left(-|z|^{-1/3} \right), \quad (9.25)$$

where

$$\begin{aligned} \varphi(C) : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} &\mapsto \begin{pmatrix} \alpha + (B, C) + (A, C^\sharp) + \beta N(C) & A + \beta C \\ B + A \times C + \beta C^\sharp & \beta \end{pmatrix}; \\ \psi(D) : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} &\mapsto \begin{pmatrix} \alpha & A + B \times D + \alpha D^\sharp \\ B + \alpha D & \beta + (A, D) + (B, D^\sharp) + \alpha N(D) \end{pmatrix}; \\ T(\tau) : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} &\mapsto \begin{pmatrix} \lambda^{-1} \alpha & \tau A \\ {}^t \tau^{-1} B & \lambda \beta \end{pmatrix}; \end{aligned} \quad (9.26)$$

where $C, D \in J_3$ and $\tau \in \text{Str}(J_3)$ s.t. $N(\tau A) = \lambda N(A)$. In this form it is straightforward to generalise to arbitrary cubic Jordan algebra as follows. Consider the F-dual pair given by

$$w = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \tilde{w} = \text{sgn}(\alpha\beta) \begin{pmatrix} \alpha & 0 \\ 0 & -\beta \end{pmatrix}. \quad (9.27)$$

For base point E and $c, d \in \mathbb{R}$ we find

$$w' = \psi(dE) \circ \varphi(cE) \circ \psi(dE)(w) \quad (9.28)$$

is given by

$$w' = \begin{pmatrix} (cd+1)^3 \alpha + c^3 \beta & (cd+1) (\alpha d^2 (cd+2)^2 + \beta c (cd+1)) E \\ (cd+1) (\beta c^2 + \alpha d (c^2 d^2 + 3cd+2)) E & d^3 (cd+2)^3 \alpha + (cd+1)^3 \beta \end{pmatrix}. \quad (9.29)$$

Hence, on setting

$$d = -\text{sgn}(\alpha\beta)(\beta/\alpha)^{1/3} = -|\beta/\alpha|^{1/3}$$

and

$$c = \text{sgn}(\alpha\beta)(\alpha/\beta)^{1/3} = |\beta/\alpha|^{1/3}$$

we obtain

$$w' = \text{sgn}(\alpha\beta) \begin{pmatrix} \alpha & 0 \\ 0 & -\beta \end{pmatrix} = \tilde{w}, \quad (9.30)$$

as required. Using eq. (9.15) this gives an explicit realisation of a U-duality relating any F-dual pair $x, \tilde{x}$ with $\Delta(x) < 0$. Explicitly, setting $x_{\text{can}} = w$ using U and then applying $M_w = \psi(dE) \circ \varphi(cE) \circ \psi(dE)$ we have

$$\tilde{x} = U'^{-1} M_w U x. \quad (9.31)$$

Note, since any x in a $\Delta(x) < 0$ orbit has stabiliser $\text{Str}_0(J_3)$ the U-duality transformations are generically non-unique,

$$\tilde{x} = S_{\tilde{x}} U'^{-1} S_{\tilde{w}} M_w S_w U S_x x \quad (9.32)$$

for arbitrary $S_y \in \text{Stab}(y) \subset \text{Aut}(\mathfrak{F})$.

It is also straightforward to define a $\hat{w} \in \mathfrak{F}$ such that

$$\exp\left[\frac{\pi}{2}\Upsilon_{\hat{w}}\right] w = \tilde{w}. \quad (9.33)$$

where $\hat{w}$ is determined by w . Recall, for $x = (\alpha, \beta, A, B)$, $y = (\gamma, \delta, C, D)$, the Freudenthal product

$$\wedge : \mathfrak{F} \otimes \mathfrak{F} \rightarrow \text{Hom}(\mathfrak{F})$$

is defined by,

$$x \wedge y \equiv \Phi_{(\phi, X, Y, \nu)}, \quad \text{where} \quad \begin{cases} \phi = -\frac{1}{2}(A \vee D + C \vee B), \\ X = -\frac{1}{4}(B \times D - \alpha C - \gamma A), \\ Y = \frac{1}{4}(A \times C - \beta D - \delta B), \\ \nu = \frac{1}{8}(\text{Tr}(A, D) + \text{Tr}(C, B) - 3(\alpha\delta + \gamma\beta)), \end{cases} \quad (9.34)$$

and $A \vee B \in \mathfrak{Str}J$ is defined by

$$(A \vee B)C \equiv \frac{1}{2} \text{Tr}(B, C)A + \frac{1}{6} \text{Tr}(A, B)C - \frac{1}{2}B \times (A \times C).$$

The action of $\Phi : \mathfrak{F} \rightarrow \mathfrak{F}$ is given by

$$\Phi_{(\phi, X, Y, \nu)} \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} = \begin{pmatrix} \alpha\nu + (X, B) & \phi A - \frac{1}{3}\nu A + Y \times B + \beta X \\ -{}^t\phi B + \frac{1}{3}\nu B + X \times A + \alpha Y & -\beta\nu + (Y, A) \end{pmatrix}. \quad (9.35)$$

The set of all such homomorphisms yields the automorphism Lie algebra,

$$\mathfrak{Aut}(\mathfrak{F}) = \{\Phi(\phi, X, Y, \nu) \in \text{Hom}_{\mathbb{R}}(\mathfrak{F}) | \phi \in \mathfrak{Str}_0(\mathfrak{J}), X, Y \in \mathfrak{J}, \nu \in \mathbb{R}\}, \quad (9.36)$$

where the Lie bracket

$$[\Phi(\phi_1, X_1, Y_1, \nu_1), \Phi(\phi_2, X_2, Y_2, \nu_2)] = \Phi(\phi, X, Y, \nu) \quad (9.37)$$

is given by

$$\begin{aligned} \phi &= [\phi_1, \phi_2] + 2(X_1 \vee Y_2 - X_2 \vee Y_1), \\ X &= (\phi_1 + \frac{2}{3}\nu_1)X_2 - (\phi_2 + \frac{2}{3}\nu_2)X_1, \\ Y &= ({}^t\phi_2 + \frac{2}{3}\nu_2)Y_1 - ({}^t\phi_1 + \frac{2}{3}\nu_1)Y_2, \\ \nu &= \text{Tr}(X_1, Y_2) - \text{Tr}(Y_1, X_2). \end{aligned} \quad (9.38)$$

Noting that

$$\varphi(C) = \exp[\Phi(0, C, 0, 0)], \quad \psi(D) = \exp[\Phi(0, 0, D, 0)] \quad (9.39)$$

and, from eq. (9.38), that

$$e \equiv \Phi(0, E, 0, 0), \quad f \equiv \Phi(0, 0, E, 0), \quad h \equiv \Phi(0, 0, 0, 3), \quad (9.40)$$

generates an $\mathfrak{sl}(2, \mathbb{R})$ subalgebra,

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f, \quad (9.41)$$

we find

$$\psi(dE) \circ \varphi(-E/d) \circ \psi(dE) = \exp\left[\frac{\pi}{2}\Phi(0, -E/d, dE, 0)\right], \quad (9.42)$$

which on setting $d = -|\beta/\alpha|^{1/3}$ gives the F-dual transformation eq. (9.30).

Using

$$4x \wedge x(y) = 3T(x, x, y) + \{x, y\}x \quad (9.43)$$

and for $\hat{w} = 2(0, 0, \sqrt{d}E, E/\sqrt{d})$

$$\hat{w} \wedge w = \Phi(0, -1/dE, dE, 0) \quad (9.44)$$

we have

$$\exp\left[\frac{\pi}{2}\Upsilon_{\hat{w}}\right] w = \tilde{w}. \quad (9.45)$$

9.4 Gauged $\text{Aut}(\mathfrak{F})$ realisation: $\Delta > 0$

As for the $\Delta(x) < 0$ case treated above, when $\Delta(x) > 0$ the action of Freudenthal duality on x can generally be realised by linear gauged $\text{Sp}(2N, \mathbb{R})$ (recalling that $\dim_{\mathbb{R}} \mathfrak{F} = 2N$) or $\text{Aut}(\mathfrak{F})$ transformations. For simplicity's sake, we confine ourselves here to the study of the gauged $\text{Aut}(\mathfrak{F})$ transformations.³³

³³At least in (non-degenerate) reduced FTS's, a similar treatment along the non-BPS $Z_H = 0$ orbit ($\Delta > 0$) can be given. Within (non-degenerate) reduced FTS's, the smallest model exhibiting a non-BPS $\Delta > 0$ orbit is the ST^2 model [80].

Concerning the $\Delta > 0$ locus of $\mathfrak{F}$, it is generally stratified in two or more orbits under the non-transitive action of $\text{Aut}(\mathfrak{F})$. In the following treatment, we will consider the particularly simple case of the T^3 model, in which such a stratification does not take place, and thus the $\Delta > 0$ locus of $\mathfrak{F}(\mathbb{R})$ corresponds to a unique $\text{Aut}(\mathfrak{F}) = \text{SL}(2, \mathbb{R})$ supersymmetric (1/2-BPS) orbit [75, 80]:

$$\mathfrak{F}(\mathbb{R})_{\Delta>0} = \mathcal{O}_{\Delta>0} \cong \text{SL}(2, \mathbb{R})/\mathbb{Z}_3. \quad (9.46)$$

The non-trivial, discrete, stabiliser of the $\mathcal{O}_{\Delta>0}$ orbits for the T^3 model is up to conjugation given by the $\mathbb{Z}_3 \subset \text{SO}(2) \subset \text{SL}(2, \mathbb{R})$, generated by [75]

$$\hat{M} \equiv \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \quad \det \hat{M} = 1, \quad \hat{M}^{-1} = \hat{M}^T. \quad (9.47)$$

As computed below, the Freudenthal dual of the $D0 - D4$ representatives of the orbits $\mathcal{O}_{\Delta>0}$ are given by $D2 - D6$ elements

$$\tilde{x}_{D0D4} = \frac{1}{\sqrt{q_0 p^3}} (-p^3, 0, 0, 3q_0 p^2). \quad (9.48)$$

In order to determine the gauged $\text{Aut}(\mathfrak{F}(J_3))$ transformation $M_{x_{D0D4}}$ mimicking, along $\mathcal{O}_{\Delta>0}$, Freudenthal duality acting on $x_{\text{can}} = x_{D0D4}$, i.e. such that $\tilde{x}_{D0D4} = M_{x_{D0D4}} x_{D0D6}$ (cfr. eq. (9.15)), we will again use the “ T^3 degeneration” of the quantum information symplectic frame of the STU model, namely we search for a 2×2 matrix $\mathbb{M}$ such that

$$\tilde{a}_{(abc)D0D4} = \mathbb{M}_a^{a'} \mathbb{M}_b^{b'} \mathbb{M}_c^{c'} a_{(a'b'c')D0D4} \equiv M_{x_{D0D4}} x_{D0D4}, \quad (9.49)$$

where x_{D0D4} in the FTS parametrization (following the conventions of [75]) reads

$$x_{D0D4} = \begin{pmatrix} -q_0 & p \\ 0 & 0 \end{pmatrix}. \quad (9.50)$$

Long but straightforward algebra yields the twofold solution ($y = q_0/p > 0$):

$$\mathbb{M}_{\pm}(y) = \begin{pmatrix} \pm \frac{\sqrt{3}}{2} & \frac{1}{2} y^{-1/2} \\ -\frac{1}{2} y^{1/2} & \pm \frac{\sqrt{3}}{2} \end{pmatrix}. \quad (9.51)$$

$M_{x_{D0D4}}(y)$ (defined by eq. (9.49)) can be realized in terms of the $\text{Aut}(\mathfrak{F}(\mathbb{R}))$ transformations eq. (9.26) as follows:

$$M_{x_{D0D4}}(y) = T \left(\pm \frac{\sqrt{3}}{2} \right) \circ \varphi \left(\mp \frac{\sqrt{3}}{4} y^{1/2} \right) \circ \psi \left(\pm \frac{1}{\sqrt{3}} y^{-1/2} \right). \quad (9.52)$$

It is interesting to note that for $y = 1 \Leftrightarrow q_0 = p$, the matrix $\mathbb{M}_{\pm}(y = 1)$ eq. (9.51) (and thus, through eq. (9.49), $M_{x_{D0D4}}$) does not belong to $\text{SL}(2, \mathbb{Z})$:

$$\mathbb{M}(y = 1) = \begin{pmatrix} \pm \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \pm \frac{\sqrt{3}}{2} \end{pmatrix} \notin \text{SL}(2, \mathbb{Z}). \quad (9.53)$$

However, remarkably, only for $y = 1$ another solution to eq. (9.49) can be found (the subscript “add” stands for additional; recall eq. (9.23)):

$$\mathbb{M}_{\text{add}}^{y=1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \varepsilon = \mathbf{M}(z=1) \in \text{SL}(2, \mathbb{Z}), \quad (9.54)$$

with ε defined by eq. (6.21). Due to the existence of the additional solution eq. (9.54), the integral (projective) case is obtained for $y = 1$ (by further setting $p \in \mathbb{Z}$) (cfr. the treatment in the quantized charge regime, presented in [17]).

Recall, since we are assuming $\Delta(x) > 0$, we can use the Freudenthal rotation in $\text{Aut}(\mathfrak{F})$,

$$\exp \left[\frac{\pi}{2} \overline{\Upsilon}_x \right] x = \tilde{x}. \quad (9.55)$$

Specialising to a generic³⁴ element in the $D0D4$ system

$$w = \begin{pmatrix} \alpha & A \\ 0 & 0 \end{pmatrix}, \quad \tilde{w} = \frac{1}{\sqrt{-\alpha N(A)}} \begin{pmatrix} 0 & 0 \\ \alpha A^\sharp & N(A) \end{pmatrix}, \quad (9.56)$$

and using

$$\overline{\Upsilon}_x = \frac{4}{3\sqrt{\Delta(x)}} x \wedge x, \quad (9.57)$$

we have

$$\overline{\Upsilon}_w = \frac{1}{3\sqrt{-\alpha N(A)}} \Phi(0, \alpha A, A^\sharp, 0), \quad (9.58)$$

so that

$$\exp \left[\frac{\pi}{6\sqrt{-\alpha N(A)}} \Phi(0, \alpha A, A^\sharp, 0) \right] w = \tilde{w}. \quad (9.59)$$

Further restricting to the T^3 model and setting $\alpha = -q_0$, $A = p$ we find,

$$\frac{\pi}{6\sqrt{-\alpha N(A)}} \Phi(0, \alpha A, A^\sharp, 0) = \frac{\pi}{6} \Phi(0, -y^{1/2}, y^{-1/2}, 0) \quad (9.60)$$

and

$$\exp \left[\frac{\pi}{6} \Phi(0, -y^{1/2}, y^{-1/2}, 0) \right] = T \left(\frac{\sqrt{3}}{2} \right) \circ \varphi \left(-\frac{\sqrt{3}}{4} y^{1/2} \right) \circ \psi \left(\frac{1}{\sqrt{3}} y^{-1/2} \right) \quad (9.61)$$

as given in eq. (9.52).

³⁴Generically, this is actually a larger class of charge configurations, but in the special cases of the T^3 , ST^2 , STU models it is precisely the $D0D4$ subsector. For $\mathcal{N} = 8$, imposing A is diagonal restricts to the $D0D4$ subsector.

9.5 Non-trivial orbit stabilizers

Here, generalizing the reasoning at the end of subsection 9.3, we want to reconsider the diagram eq. (9.15), and generalize the treatment to the case in which (regardless of the sign of $\Delta(x)$) the $\text{Aut}(\mathfrak{F})$ -orbit to which $x \in \mathfrak{F}$ belongs is endowed with a non-trivial stabilizer $\mathcal{H}$, such that the corresponding homogeneous (generally non-symmetric) manifold can be written as

$$x \in \mathcal{O} = \frac{\text{Aut}(\mathfrak{F})}{\mathcal{H}} \subset \mathfrak{F}. \quad (9.62)$$

Let us deal with the issue of mimicking the action of Freudenthal duality by an $\text{Aut}(\mathfrak{F})$ finite transformation (which will generally be “gauged” in $\mathfrak{F}$), and consider again the (commutative) diagram eq. (9.15). In general³⁵ ($U, U' \in \frac{\text{Aut}(\mathfrak{F})}{\mathcal{H}}$), it holds that

$$\begin{aligned} \widetilde{x}_{\text{can}} &= \widetilde{U}x; \\ \widetilde{x}_{\text{can}} &= U'\widetilde{x} = \widetilde{U'}x, \end{aligned} \quad (9.63)$$

where in the last step of the second line we used the commutativity of Freudenthal duality and $\text{Aut}(\mathfrak{F})$. Thus, eq. (9.63) implies

$$x = U'^{-1}Ux \Leftrightarrow U'^{-1}U = Z_x \in \text{Stab}(x) \quad (9.64)$$

On the other hand:

$$\widetilde{x} = \widetilde{U^{-1}x_{\text{can}}}, \quad \widetilde{x} = U'^{-1}\widetilde{x}_{\text{can}} = \widetilde{U'^{-1}x_{\text{can}}}, \quad (9.65)$$

where again in the last step we used the commutativity of Freudenthal duality and $\text{Aut}(\mathfrak{F})$. Thus, (9.65) implies

$$x_{\text{can}} = U'U^{-1}x_{\text{can}} \Leftrightarrow U'U^{-1} = Z_{x_{\text{can}}} \in \text{Stab}(x_{\text{can}}). \quad (9.66)$$

Let us observe that

$$\text{Stab}(x) = \text{Stab}(\widetilde{x}), \quad (9.67)$$

since for $Z_x \in \text{Stab}(x)$ and $Z_{\widetilde{x}} \in \text{Stab}(\widetilde{x})$ we have

$$Z_{\widetilde{x}}\widetilde{x} = \widetilde{x} = \widetilde{Z_x x} = Z_x \widetilde{x}, \quad (9.68)$$

and vice versa, implying eq. (9.67).

By virtue of results eq. (9.64), eq. (9.66) and eq. (9.67), one can write

$$U'x = Z_{x_{\text{can}}}U Z_x x. \quad (9.69)$$

Moreover, it is here worth commenting that, in presence of non-trivial $\mathcal{H}$, any reasoning involving the diagram eq. (9.15) is actually dependent from the actual choice of x_{can} ; indeed, the $\text{Aut}(\mathfrak{F}(J_3))$ transformation connecting any two elements of $\mathcal{O}$ (say x_{can} and x'_{can}) will not be unique, for the reasons highlighted above.

³⁵If U and U' are more general finite transformations, such as “gauged” symplectic (not belonging to $\text{Aut}(\mathfrak{F})$) or anti-symplectic transformations, things become more complicated, since, for instance, finite transformations of the pseudo-Riemannian non-compact non-symmetric coset $\text{Sp}(2n, \mathbb{R})/\text{Aut}(\mathfrak{F})$ do not generally preserve the $\text{Aut}(\mathfrak{F})$ -orbit structure of $\mathfrak{F}$, and thus they generally do not commute with Freudenthal duality.

10 Summary, concluding and further remarks

The purpose of this work has been to present, extend and clarify old and new results concerning general Freudenthal transformations, which generalise Freudenthal duality, while filling in necessary details not yet appearing in the mathematics or physics literature.

We begin with a detailed and self-contained treatment of FTS, groups of type E_7 , F-duality and GFT in sections 2–5, laying the groundwork for the subsequent sections that apply the formalism to Einstein-Maxwell-Scalar (super)gravity theories.

In section 6 we study the entropy properties of $\mathcal{N} = 2$, $D = 4$ pure supergravity from the point of view of FTS formalism, where it is given by a non-reduced FTS with positive-definite quartic invariant. In this case, general Freudenthal transformations are nothing other than the familiar $U(1)$ electromagnetic duality and Freudenthal duality is an anti-involutive duality transformation.

In section 7 we considered the axion-dilaton model, an $\mathcal{N} = 2, d = 4$ supergravity minimally coupled to one vector multiplet, which can be considered a consistent truncation of $\mathcal{N} = 4$ supergravity. The mathematical structure is in this case a two dimensional FTS with negative-semi-definite quartic invariant. The corresponding extremal BH is non-BPS. A $SO_0(1, 1)$ subgroup of the GFTs leave invariant the entropy. Freudenthal duality reduces in this case to a U-duality transformation. This model, as the pure supergravity studied before, cannot be uplifted to $D = 5$, which corresponds directly to the fact that the associated FTS is not reduced.

In section 8 we proceeded to the analysis of $\mathcal{N} = 2, D = 4$ supergravities with a $D = 5$ origin. The mathematical structure of these models is that of a FTS derived from a cubic Jordan Algebra. In first place we study the T^3 model, or in Freudenthal terminology a $\mathfrak{F}(J_3 = \mathbb{R})$ structure. This model is represented by the unique reduced FTS with $\dim \mathfrak{F} = 4$ as vector space. The automorphism group is a four dimensional representation of $SL(2, \mathbb{R})$ which can be decomposed into $SO(1, 1)$ representations, making the $D = 5$ origin manifest. The full space is split into two two-dimensional planes orthogonal (with respect to the symplectic bilinear form), which can be identified respectively with purely $D6 - D0$ or $D4 - D2$ configurations. Other identifications are, however, possible, for example a $D4 - D0/D6 - D2$ decomposition. General Freudenthal transformations leave invariant the entropy for configurations chosen in each of the planes for every decomposition. Any $D6 - D0$ configuration has a negative value for the quartic invariant, corresponding to extremal non-BPS BHs. Meanwhile, the $D4 - D2$ configurations correspond to BHs with either positive (BPS or non-BPS) or negative (non-BPS) quartic invariant. Each non-BPS $D6 - D0$ BH can be put in correspondence with a BPS $D4 - D2$ one, with the same entropy. Mathematically this correspondence is performed by a Υ_x map, which exponentiates to an element of $\text{Aut}(\mathfrak{F})$. It is also possible to consider an initial family of only $D4 - D0$ configurations. This family includes both BPS and non-BPS BHs (respectively configurations with positive and negative quartic invariants). The application of general Freudenthal transformations to any member of this family generates a two-parameter subset of the full $D6 - D4 - D0 - D2$ configuration space. The configurations with $D6$ or $D4$ charges alone are “small” ($\Delta = 0$) BH. They are related among themselves by general

Freudenthal transformations forming a $D6 - D4$ plane, which is Υ -mapped to a $D2 - D0$ plane. Similar results are obtained in general theories beyond T^3 containing an arbitrary number of charges, they are studied in full detail in subsection 8.2 and subsection 8.3.

General Freudenthal transformations, and in particular Freudenthal duality $\sim$, will preserve the orbit structure for $\Delta < 0$. This is trivially true since, at least for (non-degenerate) reduced FTS, $\text{Aut}(\mathfrak{F})$ has a transitive action on elements with a given $\Delta < 0$. The situation is more complicated for the locus of $\Delta > 0$ configurations, for which, in general (with known exceptions) are stratified in two or more automorphism orbits. In subsection 8.4 we study this question and conclude that the orbit stratification of the $\Delta > 0$ locus of $\mathfrak{F}(\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R})$ is *preserved* under Freudenthal duality $\sim$, and by extension by General Freudenthal transformations. This result generalises to all cases with (non-degenerate) reduced FTS, as they can either be inductively embedded in, or truncated to, the STU model.

In section 9 we show, in different examples, how the action of General Freudenthal transformations, and, in particular, Freudenthal duality can be mimicked/undone by finite U-duality ($\text{Aut}(\mathfrak{F})$), transformations which are gauged in that they depend on the element of $\mathfrak{F}$ they are applied onto. We restrict to two situations. First, for configurations within the $\Delta < 0$ locus of $\mathfrak{F}$, the action of $\text{Aut}(\mathfrak{F})$ is always transitive on elements of a given Δ , thus defining a unique one-parameter family of orbits $\{\mathcal{O}\}_{\Delta(x)<0}$ of $\mathfrak{F}$. The $\Delta > 0$ locus of $\mathfrak{F}$ is generally stratified in two or more orbits for any fixed $\Delta > 0$ under the non-transitive action of $\text{Aut}(\mathfrak{F})$. However it is possible to find particular cases where this stratification does not take place. For example, the T^3 model has a unique orbit for all $\Delta > 0$ under the $\text{SL}(2, \mathbb{R})$ automorphism group. Hence, for the T^3 model all BH with $\Delta > 0$ are $(1/2)$ -BPS. This example is studied in subsection 9.4.

The entropy of a linear superposition of configurations x, y is given by

$$\frac{S_{BH}^2(x+y)}{4\pi} = |\Delta(x+y)|, \quad (10.1)$$

which is not the sum of the entropies of the individual constituents

$$\Delta(x+y) = \Delta(x) + \Delta(y) + 3\{x', y\} + 3\{x, y'\} + \{y, \Upsilon_x(y)\}. \quad (10.2)$$

However for linear combinations of the form given by a General Freudenthal transformation, $x_F = ax + b\tilde{x}$, the entropy of the composite object is simply related to that one of x

$$\frac{S_{BH}^2(x_F)}{4\pi} = |a^2 + \epsilon b^2| \frac{S_{BH}^2(x)}{4\pi}. \quad (10.3)$$

Thus there is a family of configurations for which the entropy is the same to the entropy of x , those with $a^2 + \epsilon b^2 = \pm 1$, with $\epsilon = \text{sgn } \Delta(x)$.

We can use these results to show that it is possible to construct asymptotically “small” (zero entropy) interacting black holes from an initial non-trivial configuration. First, when $\Delta(x) > 0$ ($\epsilon = 1$), the elements defined by (see also eq. (B.36)) $x_{F\pm} = x \pm \tilde{x}$ are rank-4 element of $\mathfrak{F}_x \subset \mathfrak{F}$ (see eq. (B.37) and table 2). However, when $\Delta(x) < 0$ ($\epsilon = -1$) it

follows that $x_{F\pm}$ are null elements³⁶

$$\Delta(x_{F\pm}) \equiv \Delta(x \pm \tilde{x}) = 0. \quad (10.6)$$

with vanishing Bekenstein-Hawking entropy

$$S_{BH}(x \pm \tilde{x}) = 0. \quad (10.7)$$

This suggests the existence of a class of “two-centered black hole solutions” where each centre is “large” non-BPS ($\Delta(x) = \Delta(\pm\tilde{x}) < 0$), they are interacting since $\{\tilde{x}, x\} \neq 0$, yet asymptotically their Bekenstein-Hawking entropy vanishes, so the total system (*before* crossing a line of marginal stability) belongs to a small *nilpotent*³⁷ orbit. The physical or geometric significance of such configurations remains unclear.

Alternatively, small BH solutions can be constructed by the application of the properties of $\Upsilon_x \in \mathfrak{aut}(\mathfrak{F})$ maps. The behaviour of S_{BH} (or Δ) on the $\mathfrak{F}$ or $\mathfrak{F}_{y \perp x}$ planes is similar (see subsection 4.1). The null elements of $\mathfrak{F}_x$ and any $\mathfrak{F}_{y \perp x}$ are “aligned”. The locus of null entropy, is given by the same eq. (4.29) which is independent of y . For $\Delta(x) < 0, y \in \mathfrak{F}_x^\perp$, any element of the form

$$z_\pm = \sqrt{|\Delta(x)|} y \pm \Upsilon_x(y) \quad (10.8)$$

is null, $\Delta(z_\pm) = 0$. This describes another class of two-centred black hole configurations, which are interacting since $\{y, \Upsilon_x y\} \neq 0$, yet asymptotically their Bekenstein-Hawking entropy vanishes

$$S_{BH}(y \pm \bar{\Upsilon}_x(y)) = 0. \quad (10.9)$$

Further work on such small BHs will be presented elsewhere.

The extension of these results to systems with quantized charges is challenging. In this case the requirement that the set of charge vectors $x_f = ax + b\tilde{x}$ belongs to the charge lattice is extremely restrictive. Let us recall that for the case of Freudenthal duality, demanding that $x, \tilde{x}$ are integers restricts us to a open subset of black holes where the entropy is necessarily an integer multiple of π . The complete characterisation of discrete U-duality invariants, which may or may not also be F- and GFT invariant, remains an open question and, hence, so does the F-dual invariance of higher order corrections to the entropy.

Let us just present a simple result. In the case of $\mathcal{N} = 8, D = 4$, the automorphism group is $E_{7(7)}(\mathbb{Z})$ and $\Delta(x)$ is quantised [48], $\Delta(x) = 0_{[4]}$ or $\Delta(x) = 1_{[4]}$, where $n_{[4]} \equiv n \bmod 4$. The requirement that both $x, \tilde{x}$ are integer restricts us to the subset of black holes for which $\pm\Delta$ is a perfect square among other conditions [48]. Let us explore what happens

³⁶The elements $x_{F\pm}$ are actually rank-1 element of $\mathfrak{F}_x \subset \mathfrak{F}$. In this case (eq. (B.37), table 2)

$$(x_{F\pm})' \equiv (x \pm \tilde{x})' = 0, \quad (10.4)$$

$$\Upsilon_{x_{F\pm}}(y) = 0, \quad \forall y \in \mathfrak{F}. \quad (10.5)$$

In particular it does not have a well defined $\tilde{F}$ -dual.

³⁷In the $D = 3$ language; cfr. e.g. the nilpotent orbits of $\mathfrak{so}(4, 4)$ acting on its adjoint irrep. **28** in [96].

for integral GFT. Noting that the F-dual scales linearly $(\tilde{n}x) = n\tilde{x}$, $n \in \mathbb{Z}$, according to eq. (3.13), we find,

$$\Delta(nx + m\tilde{x}) = (n^2 + \epsilon m^2)^2 \Delta(x), \quad n, m \in \mathbb{Z}.$$

From the structure of this expression it is obvious that the requirement the entropy being a perfect square is automatically preserved under a GFT.

Finally, we note that the charges of five dimensional stringy black holes may be described in the context of (cubic) Jordan algebras. The cubic norm defining the relevant algebra determines the BH entropy to lowest order. The Jordan dual (introduced in [48]) is related to the Freudenthal dual of the corresponding 4D model. The generalization of this 4D/5D correspondence from a general Freudenthal transformation to the corresponding putative “General Jordan Transformations” (GJT), of which the Jordan dual is a particular case, will be treated elsewhere [94].

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A Freudenthal triples: assorted properties

A summary of some FTS definitions, notation and properties used through this work. See ref. [14] for additional ones and proofs.

$$\{T(x, y, z), w\} = 2\Delta(x, y, z, w), \tag{A.1}$$

$$x' \equiv T(x, x, x) \equiv T(x), \tag{A.2}$$

$$\Delta(x) \equiv \Delta(x, x, x, x), \tag{A.3}$$

$$3\{T(x, x, y), y'\} = 2\{x, y\}\Delta(x, y, y, y), \quad (\text{A.4})$$

$$6\{T(x, w, y), y'\} = \{x, y\}2\Delta(w, y, y, y) + \{w, y\}2\Delta(x, y, y, y), \quad (\text{A.5})$$

$$6\{T(x, y, y), y'\} = \{x, y\}2\Delta(y), \quad (\text{A.6})$$

$$9T((T(x, y, y), y, y) = -2\Delta(x, y, y, y)y - \Delta(y)x - \{x, y\}y', \quad (\text{A.7})$$

$$T(x', x, x) = -\frac{1}{3}\Delta(x)x, \quad (\text{A.8})$$

$$T(x', x', x) = \frac{1}{3}\Delta(x)x', \quad (\text{A.9})$$

$$(x')' \equiv T(x', x', x') = -\Delta^2(x)x, \quad (\text{A.10})$$

$$\Delta(x', x, x, x) = 0, \quad (\text{A.11})$$

$$\Delta(x', x', x, x) = \frac{1}{3}\Delta^2(x), \quad (\text{A.12})$$

$$\Delta(x', x', x', x) = 0, \quad (\text{A.13})$$

$$\Delta(x', x', x', x') = \Delta^3(x). \quad (\text{A.14})$$

Some additional properties related to eq. (3.12):

$$\text{For } \Delta(x) = 0 : \quad T(ax + bx') = a^3x', \quad (\text{A.15})$$

$$, \quad T(x') = 0, \quad (\text{A.16})$$

$$\Delta(x) < 0 : \quad T(\sqrt{|\Delta(x)|} x \pm x') = 0, \quad (\text{A.17})$$

$$\Delta(x) > 0 : \quad T(\sqrt{\Delta(x)} x \pm x') = \mp 2\Delta(x)^{3/2} \left(\sqrt{\Delta(x)} x \mp x' \right). \quad (\text{A.18})$$

Moreover

$$\Delta(x) < 0 : \quad T(x \pm \tilde{x}) = 0, \quad (\text{A.19})$$

$$\Delta(x) > 0 : \quad T(x \pm \tilde{x}) = \mp 2\sqrt{\Delta(x)} (x \mp \tilde{x}). \quad (\text{A.20})$$

B More $\mathfrak{T}$ -plane properties

B.1 Complexification of real $\mathfrak{F}$'s and $\mathfrak{T}$ -planes

The square of the Υ_x map, for a fixed x , is proportional to the identity ($\Delta_x \equiv \Delta(x)$, $\epsilon = \text{sgn } \Delta(x)$)

$$\Upsilon_x^2(y) = -2\Delta_x y$$

for any $y \in \mathfrak{F}_{y \perp x}$ (see eq. (4.17)). For $\epsilon = 1$ this map defines then a local (depending on the point) complex structure,³⁸ allowing one to endow, the originally real, $\mathfrak{F}$ with the structure of a complex vector space. We may define its complexification by extension to

$$\mathfrak{F}^{\mathbb{C}} = \mathfrak{F} \otimes_{\mathbb{R}} \mathbb{C}. \quad (\text{B.1})$$

since the algebraic closure of $\mathbb{C}$, the map Υ_x is guaranteed to have eigenvalues, proportional to $\pm i$ in the $\epsilon = 1$ case.. The space $\mathfrak{F}_{y \perp x}$ can be then split into one dimensional eigenspaces of Υ_x . We write

$$\mathfrak{F}_{y \perp x} = (\mathfrak{F}_{y \perp x})_- \oplus (\mathfrak{F}_{y \perp x})_+$$

³⁸It defines a split complex structure in the $\epsilon = -1$ case.

where, for any $y_{\pm} \in (\mathfrak{F}_{y \perp x})_{\pm}$,

$$\Upsilon_x(y_{\pm}) = \pm i \sqrt{\Delta_x} y_{\pm}. \quad (\text{B.2})$$

Let us choose normalized eigenvectors (the $\mathfrak{F}_{y \perp x}$ “light-cone” or “null” basis) as follows

$$y_{\pm} = \pm i \sqrt{\Delta_x} y + \Upsilon_x(y). \quad (\text{B.3})$$

It is clear that $\{y_-, y_+\} = 0$. In addition we have the following properties (see [14], eqs. (18,19)),

$$\Delta(y_{\pm}) = 0, \quad (\text{B.4})$$

$$\Delta(y_-, y_+, y_+, y_+) = \Delta(y_+, y_-, y_-, y_-) = 0, \quad (\text{B.5})$$

$$\{(y_{\pm})', y_{\mp}\} = 0. \quad (\text{B.6})$$

Any element $u \in \mathfrak{F}_{y \perp x}$, such that $u = ay + b\Upsilon_x(y)$, can be written as $u = \alpha_+ y_+ + \alpha_- y_-$, with the coordinates in both basis related by $2\alpha_{\mp} = b \pm ia/\sqrt{\Delta(x)}$. This null basis will be useful in what follows.

We will study now the behaviour of the quadrilinear map Δ and the antisymmetric form in this plane $\mathfrak{F}_{y \perp x}$.

For any two vectors $u_1, u_2 \in \mathfrak{F}_{y \perp x}$ we have the following properties ($u_i = a_i y + b_i \Upsilon_x(y) = \alpha_{i+} y_+ + \alpha_{i-} y_-$, $i = 1, 2$) ($2\alpha_{i\mp} = b_i \pm ia_i/\sqrt{\Delta(x)}$),

$$\{u_1, u_2\} = (b_1 a_2 - a_1 b_2) \{\Upsilon_x(y), y\} \quad (\text{B.7})$$

$$= (b_1 a_2 - a_1 b_2) 6\Delta(x, x, y, y), \quad (\text{B.8})$$

$$\{\Upsilon_x(u_1), \Upsilon_x(u_2)\} = \Delta(x) \{u_1, u_2\}, \quad (\text{B.9})$$

$$\Delta(u) = \frac{1}{4\Delta_x^2} (a^2 + b^2 \Delta_x)^2 \Delta(y_-, y_-, y_+, y_+), \quad (\text{B.10})$$

$$\Delta(y_-, y_-, y_+, y_+) = 4\Delta(x)^2 \Delta(y) = 4\Delta(\Upsilon_x(y)). \quad (\text{B.11})$$

The first two expressions are obtained by direct computation. To get eq. (B.10), we note that

$$\Delta(u) = a_+^4 \Delta(y_+) + a_-^4 \Delta(y_-) + 3\alpha_+^3 \alpha_- \Delta(y_-, y_+, y_+, y_+) + \quad (\text{B.12})$$

$$3\alpha_+ \alpha_-^3 \Delta(y_+, y_-, y_-, y_-) + 4\alpha_+^2 \alpha_-^2 \Delta(y_-, y_-, y_+, y_+) \quad (\text{B.13})$$

$$= 4\alpha_+^2 \alpha_-^2 \Delta(y_-, y_-, y_+, y_+) \quad (\text{B.14})$$

$$= \frac{1}{4\Delta_x^2} (a^2 + b^2 \Delta_x)^2 \Delta(y_-, y_-, y_+, y_+) \quad (\text{B.15})$$

where we have used eqs. (B.4), (B.5). Finally eq. (B.11) is a particular case of eq. (B.10).

Eq. (B.9) can also be written as (for $\Delta(x) \neq 0$),

$$\left\{ \frac{\Upsilon_x(u_1)}{\sqrt{|\Delta(x)|}}, \frac{\Upsilon_x(u_2)}{\sqrt{|\Delta(x)|}} \right\} = \epsilon \{u_1, u_2\}. \quad (\text{B.16})$$

As consequence, for $\epsilon = 1$, the map

$$\tilde{\Upsilon}_x \equiv \frac{1}{\sqrt{\Delta_x}} \Upsilon_x$$

preserves the bilinear antisymmetric form $\{, \}$ in each $\mathfrak{F}_{y \perp x}$ plane. This implies necessarily that $\tilde{\Upsilon}_x$ is a symplectic transformation on the plane.

B.2 Maximal rank $\mathfrak{T}$ -planes are disjoint

Let us have two non-degenerate (generated by maximal rank elements) planes $\mathfrak{T}_{x_0}, \mathfrak{T}_{x_1}$ generated by distinct elements x_0, x_1 ($x_0 \neq x_1$). We will show that the two planes are, or the same, or disjoint.

Suppose we can find a common element $y \in \mathfrak{T}_{x_0} \cap \mathfrak{T}_{x_1}$, this implies that a) the signs of $\Delta(x_0), \Delta(x_1)$ are the same and b) also $y' \equiv T(y) \in \mathfrak{T}_{x_0} \cap \mathfrak{T}_{x_1}$. Then, we can find coefficients $a_i, b_i, \alpha_i, \beta_i$ such that

$$y = a_0 x_0 + b_0 x'_0 = a_1 x_1 + b_1 x'_1, \quad (\text{B.17})$$

$$y' = \alpha_0 x_0 + \beta_0 x'_0 = \alpha_1 x_1 + \beta_1 x'_1. \quad (\text{B.18})$$

The coefficients α_i, β_i are given in terms of the a_i, b_i by eq. (3.4). Inserting the values of these coefficients we can see that these equations are invertible as long as

$$\Delta(y)/\Delta(x_1) = \Delta(y)/\Delta(x_0) = (a_0^2 + b_0^2 \Delta_0)^2 = (a_1^2 + b_1^2 \Delta_1)^2 \neq 0.$$

If $\Delta(y) \neq 0$ the equations are invertible: we can write any pair (x_0, x'_0) or (x_1, x'_1) as linear combinations of y, y' . The existence of such linear combinations shows that $\mathfrak{T}_y = \mathfrak{T}_{x_0} = \mathfrak{T}_{x_1}$.

If the common element is not of maximal rank then the situation is different. If $\Delta(y) = 0$, the equations are not invertible but then necessarily both $\Delta(x_0) < 0, \Delta(x_1) < 0$. y is a “null” or “light-cone” element, it can be seen it is then proportional to one of the two vectors

$$\pm \sqrt{|\Delta(x_0)|} x_0 + x'_0$$

and one of the two

$$\pm \sqrt{|\Delta(x_1)|} x_1 + x'_1.$$

We can use these last expressions to arrive to:

$$\pm \sqrt{|\Delta(x_0)|} x_0 + x'_0 \propto \pm \sqrt{|\Delta(x_1)|} x_1 + x'_1. \quad (\text{B.19})$$

“Some” elements of $\mathfrak{T}_{x_0}$ (a one dimensional subset of them) can be written as linear combination of those of $\mathfrak{T}_{x_1}$.

In the case of degenerate planes (generated by non-maximal rank elements), one can show in the same way that the T-“lines” $\mathfrak{T}_{x_0}, \mathfrak{T}_{x_1}$, corresponding to elements x_0, x_1 with $\Delta(x_0) = \Delta(x_1) = 0$ are either identical (if $x'_0 = x'_1 = 0$ or $x'_0 \propto x'_1$) or disjoint (if $x'_0 \not\propto x'_1$).

$Rk(x_0)$		Cases		Rank(x)
4	$\Delta_0 > 0$	$\Delta(x) > 0$		4
4	$\Delta_0 < 0$	$\Delta(x) < 0$		4
		$\Delta(x) = 0$	$x' = 0, x \propto x_{\pm}, \Upsilon_{x+}(x_-) \neq 0,$	2
3	$\Delta_0 = 0, x'_0 \neq 0$	$x \not\propto x'_0$	$x' \propto x'_0,$	3
		$x \propto x'_0$	$T(x') = 0,$	1, 2
1, 2	$\Delta_0 = 0, x'_0 = 0$	$\Delta(x) = 0$	$x' = 0, x \propto x_0, \Upsilon_x \propto \Upsilon_{x_0},$	$= Rank(x_0)$

Table 2. Relation of $rank(x)$, the rank of any element x in the $\mathfrak{T}$ -plane $\mathfrak{T}_0$ generated by x_0 .

B.3 Rank on the $\mathfrak{T}/\mathfrak{F}$ -plane

Let it be a generic vector x_0 of a given rank and its associated (possible degenerate) $\mathfrak{T}$ -plane $\mathfrak{T}_{x_0} \equiv \mathfrak{T}_0$ (we write $\Delta_0 = \Delta(x_0)$). We are interested in studying what it can be said about the rank of any element $x \in \mathfrak{T}_0$ on the plane. Let us consider the different cases.

Let us assume then a nonzero rank and consider the following possibilities:

- A) If $\Delta_0 > 0$ then, by eq. (3.9), $\Delta(x) > 0$, and $Rank(x) = Rank(x_0) = 4, \forall x \in \mathfrak{T}_0$.
- B) If $\Delta_0 < 0$ then, by eq. (3.9), we have two possibilities: $\Delta(x) < 0$ and then $Rank(x) = 4$ or $\Delta(x) = 0$.

If $\Delta(x) = 0$ then x is a null vector, it belongs to the light-cone generated by $x_{\pm} = \pm \sqrt{|\Delta(x_0)|} x_0 + x'_0$, then, by eq. (3.4), $x' = T(x) = 0$. For the sake of concreteness, let us take $x = x_+$. We can find at least one element, its associated element x_- , such that

$$\Upsilon_{x_+}(x_-) \propto -8 |\Delta_0| \sqrt{|\Delta(x_0)|} x_+ \neq 0;$$

similarly would happen if we start by assuming $x = x_-$. We arrive to the conclusion that $Rank(x) = 2$ in this case.

- C) If $\Delta_0 = 0$ ($Rank(x_0) = 1, 2, 3$) then, by eq. (3.9), $\Delta(x) = 0$. We are confronted with two self-excluding possibilities:

C1) If $x'_0 \neq 0$ ($Rank(x_0) = 3$) then $x' = a^3 x'_0 \neq 0$ ($a \neq 0$) and $Rank(x) = 3$. If $a = 0$ then $T(x') = 0$ and the rank of x is $R = 1, 2$.

C2) If $x'_0 = 0$ ($Rank(x_0) = 1, 2$) then $x' \propto x'_0 = 0, x \propto x_0$. $\Upsilon_x(y) \propto \Upsilon_{x_0}(y)$ and $Rank(x) = Rank(x_0)$ in this case.

The results are summarized in table 2. Similar results are obtained for the $\mathfrak{F}$ -plane.

B.4 The $\mathfrak{T}$ - or $\mathfrak{F}$ -plane as a (quadratic) sub-FTS system: Euclidean and hyperbolic planes

Let us fix a maximal rank element x_0 ($\Delta(x_0) \neq 0$) and its $\mathfrak{T}$ -plane $\mathfrak{T}_{x_0} \equiv \mathfrak{T}_0$. As a consequence of eq. (3.9), the restriction of the quartic map Δ to $\mathfrak{T}_0$ can be written in terms of a symmetric bilinear form I_2 as follows

$$\Delta_{\mathfrak{T}_0}(x) = \frac{\epsilon_0}{4} I_2^2(x), \quad (\forall x \in \mathfrak{T}_0), \quad (\text{B.20})$$

where $\epsilon_0 = \text{sgn } \Delta(x_0)$ (we call ϵ_0 the “signature of $\mathfrak{T}_0$ ”, it does not depend on the plane maximal rank element chosen as generator).

Choosing a basis on the plane and coordinates with respect it ($x = x^I e_I$, ($I = 1, 2$), $e_1 \equiv x_0, e_2 \equiv x'_0$), the bilinear form is given by

$$I_2(x, y) \equiv x^I g_{IJ} y^J = 2\sqrt{|\Delta_0|} (x^1 y^1 + x^2 y^2 \Delta_0). \quad (\text{B.21})$$

Then the quadratic form

$$I_2(x) = 2\sqrt{|\Delta_0|} ((x^1)^2 + (x^2)^2 \Delta_0).$$

The full quadrilinear map restricted to the plane is then of the form (for generic vectors $x, y, z, w \in \mathfrak{T}_0$)

$$\Delta_{T_0}(x, y, z, w) = \frac{\epsilon_0}{12} (I_2(x, y)I_2(z, w) + I_2(x, z)I_2(y, w) + I_2(x, w)I_2(y, z)). \quad (\text{B.22})$$

To arrive to this expression we have used the multi-linearity of Δ and the properties $\Delta(x, x, x, x') = \Delta(x, x', x', x') = 0$ (see properties in section A). In the case of $\Delta_0 = 0$ the expressions reduce trivially to $I_2 \equiv 0$.

We can quickly convince us that the $\mathfrak{T}_0$ plane is itself a two dimensional FTS of quadratic (or degenerate) type (see [58]) with a characteristic “signature “ ϵ_0 , and whose symmetric quadrilinear and antisymmetric bilinear forms are those inherited from the original FTS.

Restricted to the $\mathfrak{T}$ -plane the quadrilinear and bilinear forms acquire simple expressions. The antisymmetric bracket is

$$\{x, y\}_{\mathfrak{T}_0} \equiv x^I \omega_{IJ} y^J = \omega_{12} (x^1 y^2 - x^2 y^1). \quad (\text{B.23})$$

with

$$\omega_{12} = \{x_0, x'_0\} = -2\Delta_0.$$

We define the matrix $S = (S_{IJ})$ (see [58]) , such that

$$S^I_J = (g^{-1})^{IK} \omega_{KJ},$$

then

$$S^2 = -\epsilon_0 I.$$

The explicit form of the matrix S is

$$(S^I_J) = \epsilon_0 \begin{pmatrix} 0 & -\sqrt{|\Delta_0|} \\ \frac{1}{\sqrt{|\Delta_0|}} & 0 \end{pmatrix}, \quad (\text{B.24})$$

The trilinear T map, the F-dual maps restricted to $\mathfrak{T}_0$ are given by, in component form [58]

$$T(x, y, z) = \frac{1}{6} \left(I_2(x, y) \hat{S}(z) + I_2(x, z) \hat{S}(y) + I_2(y, z) \hat{S}(x) \right), \quad (\text{B.25})$$

$$x' = \frac{1}{2} I_2(x) \hat{S}(x), \quad (\text{B.26})$$

$$\tilde{x} = \eta(x) \hat{S}(x). \quad (\text{B.27})$$

where it has been used the linear map defined by $\hat{S}(x) \equiv S^I{}_J x^J e_I$ and where

$$\eta(x) \equiv \text{sgn}(I_2(x)) = \text{sgn}((x^1)^2 + \epsilon_0(x^2)^2 \mid \Delta_0 \mid).$$

If $\epsilon_0 = 1$ then $\eta(x) = 1$, $\forall x \in \mathfrak{T}_0$.

In summary any non-degenerate T (or F) plane can be considered as a two-dimensional quadratic FTS system, the three FTS axioms, see section 2, are trivially satisfied by the $\mathfrak{T}_0$ -restricted maps $\Delta_T, \{\}_T, T_T$ ($\Delta_F, \{\}_F, T_F$).

Euclidean, hyperbolic $\mathfrak{T}$ -planes. The signature of the I_2 bilinear form coincides with the signature of the $\mathfrak{T}_0$ plane, the sign of $\Delta(x_0)$ with x_0 any maximal rank element in the plane. Thus I_2 defines an Euclidean or a Minkowskian $R^{1,1}$ (hyperbolic or split-complex) structure on the $\mathfrak{T}$ -plane according to it.

Let us focus on the second case. Endowed by the metric I_2 , $\mathfrak{T}_0$ becomes a Minkowski plane. The set of all transformations of the hyperbolic plane which preserve the I_2 form is the group $O(1, 1)$. This group consists of the hyperbolic subgroup $SO^+(1, 1)$, combined with four discrete reflections given by

$$x \rightarrow \pm x, x' \rightarrow \pm x'.$$

Using standard notation, we say that a non-zero vector $x \in \mathfrak{T}_0$ is spacelike if $I_2(x) > 0$, lightlike, or null, if $I_2(x) = 0$ and timelike if $I_2(x) < 0$.

The T operation, $x \rightarrow x'$, change the character of the vector. The vector x' is “ I_2 -orthogonal” (or “simultaneous events”) to x . One can check that [58],

$$I_2(x, x') = \frac{1}{2} I_2(x) I_2(x, \hat{S}x) \tag{B.28}$$

$$= \frac{1}{2} I_2(x) \{x, x\} = 0. \tag{B.29}$$

The relation $x' = \pm x$ is true if and only if x is lightlike; x' is timelike (resp. spacelike) if x is spacelike (resp. timelike):

$$x : \text{lightlike} \longleftrightarrow x' : \text{lightlike}, \quad x' = \pm x, \tag{B.30}$$

$$x : \text{timelike (spacelike)} \longleftrightarrow x' : \text{spacelike (timelike)}. \tag{B.31}$$

B.5 Coordinates in the $\mathfrak{T}$ -plane. Orbits

On the “Euclidean” $\mathfrak{F}$ -plane, by eq. (5.1), the orbits of the exponential of the Υ map for $\epsilon = 1$ are closed circles. Let $e_i = (e, \tilde{e})$ be the orthonormal basis given by $e \equiv x/|\Delta(x)|^{1/4}$. Then

$$\Delta(e) = \epsilon, \quad \{e_i, e_j\} = -2\varepsilon_{ij}, \tag{B.32}$$

For a generic $u \in \mathfrak{F}_x$ expressed in this basis, $u = u^1 e + u^2 \tilde{e}$, we arrive to, applying eq. (5.6),

$$\Delta(x) > 0 : e^{\bar{\Upsilon}_x}(u) = (\cos(\theta)u^1 - \sin(\theta)u^2)e + (\sin(\theta)u^1 + \cos(\theta)u^2)\tilde{e}. \tag{B.33}$$

Whereas, on a “Minkowskian” plane, with $\epsilon = -1$, the same orbits are hyperbolic, they are given by

$$\Delta(x) < 0 : e^{\bar{\Upsilon}_x}(u) = (\cosh(\theta)u^1 - \sinh(\theta)u^2)e + (-\sinh(\theta)u^1 + \cosh(\theta)u^2)\tilde{e}. \quad (\text{B.34})$$

The case $\epsilon = -1$ with $u^1 = \pm u^2$ corresponds to null vectors with orbits made by $\pi/4$ rays:

$$\Delta(x) < 0, u^1 = \pm u^2 : e^{\bar{\Upsilon}_x}(u) = \pm e^{\mp\theta}u^2(e \pm \tilde{e}). \quad (\text{B.35})$$

B.6 “Light-cone” coordinates in the $\mathfrak{I}$ -plane

It is of interest to define a basis formed by the vectors

$$e_{\pm} = x \pm \tilde{x}. \quad (\text{B.36})$$

For any $y \in F$ we define coordinates $\alpha_{\pm}$ such that

$$y = \alpha_+ e_+ + \alpha_- e_-.$$

In terms of these coordinates the quadrilinear map is given by

$$\Delta(y) = ((\alpha_+ + \alpha_-)^2 + \epsilon(\alpha_+ - \alpha_-)^2)\Delta(x). \quad (\text{B.37})$$

For $\Delta(x) < 0$ ($\epsilon = -1$ in the previous expression) the basis thus defined is a null basis ($\Delta(e_{\pm}) = 0$), $\alpha_{\pm}$ are “null” or “light-cone” coordinates, and then

$$\Delta(y) = (4\alpha_+\alpha_-)^2\Delta(x), \quad (\epsilon = -1), \quad (\text{B.38})$$

$$\Delta(y) = 4(\alpha_+ + \alpha_-)^2\Delta(x), \quad (\epsilon = +1). \quad (\text{B.39})$$

Similar coordinates will be defined on a $\mathfrak{I}$ -plane.

B.7 The general exponential map

Let us consider now the action on the orthogonal complement of a given element $x \in \mathfrak{F}$. For any $y \in \mathfrak{I}_x^{\perp}$ (that means $\{x, y\} = 0, \{x', y\} = 0$, see eq. (4.2)), we have also seen that $\Upsilon_x(y) \in \mathfrak{I}_x^{\perp}$. In conclusion, for any $y \in \mathfrak{I}_x^{\perp}$ the action of successive applications of Υ_x is restricted to lie on $\mathfrak{F}_{y \perp x}$, $\Upsilon_x^n(y)$. The orbit of any $y \in \mathfrak{I}_x^{\perp}$ under $\sigma_x(\theta)$ lies completely on $\mathfrak{F}_{y \perp x}$. This can be seen from eq. (4.18) and eq. (4.19). These equations allow to explicitly compute the exponential map by summing an exponential series, for any rank-4 x ,

$$(\exp \theta \Upsilon_x)(y) = \sum_{k=0}^{\infty} \frac{(\sqrt{-\Delta_x} \theta)^{2k}}{(2k)!} y + \frac{1}{\sqrt{-\Delta_x}} \sum_{k=0}^{\infty} \frac{(\sqrt{-\Delta_x} \theta)^{2k+1}}{(2k+1)!} \Upsilon_x(y). \quad (\text{B.40})$$

A similar series is obtained for $(\exp \theta \Upsilon_x)(\Upsilon_x(y))$.

Summing the series, the exponential of the “normalized” map in the orthogonal plane is fully determined by the expressions ($\bar{\Upsilon}_x \equiv \Upsilon_x / (\sqrt{|\Delta(x)|}, \epsilon = \text{sgn } \Delta(x))$)

$$\left(\exp \theta \bar{\Upsilon}_x \right) (y) = \cos(\sqrt{\epsilon} \theta) y + \epsilon \sqrt{\epsilon} \sin(\sqrt{\epsilon} \theta) \bar{\Upsilon}_x(y), \quad (\text{B.41})$$

$$\left(\exp \theta \bar{\Upsilon}_x \right) \left(\bar{\Upsilon}_x(y) \right) = -\sqrt{\epsilon} \sin(\sqrt{\epsilon} \theta) y + \cos(\sqrt{\epsilon} \theta) \bar{\Upsilon}_x(y). \quad (\text{B.42})$$

The geometrical character of the orbits of the exponential of the Υ_x map in the $\mathfrak{F}_x^\perp$ plane solely depends on ϵ , the sign of $\Delta(x)$, and not, for example, on the signature of y . They are closed (circles or ellipses) or hyperbolic, respectively for $\epsilon = 1$ or -1 .

It can be explicitly checked that

$$\{e^{\theta\bar{\Upsilon}_x}(y), e^{\theta\bar{\Upsilon}_x}(\bar{\Upsilon}_x(y))\} = \{y, \bar{\Upsilon}_x(y)\}. \quad (\text{B.43})$$

We have also, according to eq. (4.27)

$$\Delta(\tilde{\Upsilon}_x^n(y)) = \Delta(y). \quad (\text{B.44})$$

Let us compute now, for a fixed element x , the exponential map $\exp \theta \bar{\Upsilon}_x$ on a generic FTS element z , not necessarily on the orthogonal complement $\mathfrak{F}_x^\perp$. For that purpose, first we decompose the element on its $\mathfrak{F}_x$ parallel and orthogonal components

$$z = z_{\parallel} + z_{\perp}.$$

Without loss of generality we can assume that $z_{\parallel} = x$ (if it is not so, we simply realign the $\mathfrak{F}$ -plane by choosing $z_{\parallel}$ as the defining element of the plane: $\mathfrak{F}_x \equiv \mathfrak{F}_{z_{\parallel}}$). Then $z = x + z_{\perp}$. The action of any power of Υ_x on z is, by linearity,

$$\Upsilon_x^n(x + z_{\perp}) = \Upsilon_x^n(x) + \Upsilon_x^n(z_{\perp}) \quad (\text{B.45})$$

with $\Upsilon_x^n(x) \in \mathfrak{F}_x$ and $\Upsilon_x^n(z_{\perp}) \in \mathfrak{F}_x^\perp$.

As a consequence, the exponential of the Υ_x (or $\bar{\Upsilon}_x$) is of the form (for $z = x + z_{\perp}$):

$$(\exp \theta \bar{\Upsilon}_x)(z) = (\exp \theta \bar{\Upsilon}_x)(x) + (\exp \theta \bar{\Upsilon}_x)(z_{\perp}) \quad (\text{B.46})$$

where any of summands is computed independently using the corresponding relations (eqs. (5.6) and (5.7) for $\mathfrak{F}_x$, or, (B.41) and (B.42) for $\mathfrak{F}_x^\perp$). Putting together these relations, one arrives to

$$\begin{aligned} e^{\theta\bar{\Upsilon}_x}(z) &= \cos\left(\sqrt{\epsilon}\frac{\theta}{3}\right) z_{\perp} + 3\epsilon\sqrt{\epsilon}\sin\left(\sqrt{\epsilon}\frac{\theta}{3}\right) \bar{\Upsilon}_x(z_{\perp}) \\ &\quad + \cos(\sqrt{\epsilon}\theta) x + \epsilon\sqrt{\epsilon}\sin(\sqrt{\epsilon}\theta) \tilde{x}. \end{aligned} \quad (\text{B.47})$$

In particular, for any $z_{\perp}$ in the orthogonal space, the vector

$$w = z_{\perp} \pm 3\epsilon\sqrt{-\epsilon}\bar{\Upsilon}_x(z_{\perp}) \quad (\text{B.48})$$

of the exponential map (strictly, we have to deal with a complexified FTS for $\epsilon = -1$, see subsection B.1):

$$(\exp \theta \bar{\Upsilon}_x)w = e^{\mp\sqrt{-\epsilon}\frac{\theta}{3}} w. \quad (\text{B.49})$$

The map $\exp(\theta\bar{\Upsilon}_x)$ over a generic element y in the FTS can be obtained from the previous formula performing a suitable rotation in the $\mathfrak{F}_x$ plane bringing x to a generic $y_{\parallel}$,

and thus $\tilde{x}$ to $\widetilde{y_{\parallel x}}$:

$$\begin{aligned} \exp(\theta \bar{\Upsilon}_x)(y) &= \cos\left(\sqrt{\varepsilon}\frac{\theta}{3}\right)y + 3\varepsilon\sqrt{\varepsilon}\sin\left(\sqrt{\varepsilon}\frac{\theta}{3}\right)\bar{\Upsilon}_x(y) \\ &\quad + \left[\cos(\sqrt{\varepsilon}\theta) - \cos\left(\sqrt{\varepsilon}\frac{\theta}{3}\right)\right]y_{\parallel x} \\ &\quad + \sqrt{\varepsilon}\left[\sin(\sqrt{\varepsilon}\theta) - 3\eta\sin\left(\sqrt{\varepsilon}\frac{\theta}{3}\right)\right]\widetilde{y_{\parallel x}} \end{aligned} \quad (\text{B.50})$$

where $\eta = \text{sgn}\left(\{\tilde{x}, y\}^2 + \varepsilon\{x, y\}^2\right)$. Explicitly, for $\epsilon = 1$, one obtains

$$\begin{aligned} e^{\theta \bar{\Upsilon}_x}(y) &= \cos\left(\frac{\theta}{3}\right)y + 3\sin\left(\frac{\theta}{3}\right)\bar{\Upsilon}_x(y) \\ &\quad + \left[\cos(\theta) - \cos\left(\frac{\theta}{3}\right)\right]y_{\parallel x} + \left[\sin(\theta) - 3\sin\left(\frac{\theta}{3}\right)\right]\widetilde{y_{\parallel x}}, \end{aligned} \quad (\text{B.51})$$

and similarly for $\epsilon = -1$. Here we have introduced the projections of the generic vector y and its F-dual.³⁹ By reordering the terms in eq. (B.51), one can also write

$$\begin{aligned} e^{\theta \bar{\Upsilon}_x}(y) &= \cos\left(\frac{\theta}{3}\right)y_{\perp} + 3\sin\left(\frac{\theta}{3}\right)(\bar{\Upsilon}_x(y) - \widetilde{y_{\parallel x}}) \\ &\quad + \cos(\theta)y_{\parallel x} + \sin(\theta)\widetilde{y_{\parallel x}}. \end{aligned} \quad (\text{B.52})$$

We see that the orbit of any generic element of the FTS under a $\sigma_x(\theta)$, for fixed x , lies on a 4-dimensional hyperplane spanned by (taking into account that $x' \sim \Upsilon_x(x)$)

$$(x, \Upsilon_x(x), z_{\perp}, \Upsilon_x(z_{\perp}))$$

or

$$\exp \theta \Upsilon_x : \mathfrak{F}_x \oplus \mathfrak{F}_{(z_{\perp}, \Upsilon_x(z_{\perp}))} \rightarrow \mathfrak{F}_x \oplus \mathfrak{F}_{(z_{\perp}, \Upsilon_x(z_{\perp}))} \quad (\text{B.53})$$

or

$$\exp \theta \Upsilon_x : \begin{cases} \mathfrak{F}_x \rightarrow \mathfrak{F}_x, \\ \mathfrak{F}_{(z_{\perp}, \Upsilon_x(z_{\perp}))} \rightarrow \mathfrak{F}_{(z_{\perp}, \Upsilon_x(z_{\perp}))}. \end{cases} \quad (\text{B.54})$$

It is easy to show some explicit examples of the application of eq. (B.47). For $y = x + y_{\perp}$, $w = \exp \theta \bar{\Upsilon}_x(y)$ ($\epsilon = 1$) where it is obvious the 6π periodicity behaviour of the exponential map for this signature:

$$\theta = 0, \quad w = y \quad (\text{B.55})$$

$$\theta = 3\pi/2, \quad w = -\tilde{x} + 3\bar{\Upsilon}_x(y_{\perp}) \quad (\text{B.56})$$

$$\theta = 3\pi, \quad w = -y \quad (\text{B.57})$$

$$\theta = 9\pi/2, \quad w = \tilde{x} - 3\bar{\Upsilon}_x(y_{\perp}) \quad (\text{B.58})$$

$$\theta = 6\pi, \quad w = y. \quad (\text{B.59})$$

One can explicitly check that $\Delta(w) = \Delta(y)$ in all the cases.

³⁹ Note that, comparing with eqs. (4.6)–(4.11), $P_z(y) \equiv y_{\parallel z} = \frac{1}{2\sqrt{|\Delta(z)|}} [\{\tilde{z}, y\}z - \{z, y\}\tilde{z}]$, $Q_z(y) \equiv \widetilde{y_{\parallel z}} = \frac{\eta}{2\sqrt{|\Delta(z)|}} [\{\tilde{z}, y\}\tilde{z} + \varepsilon\{z, y\}z]$.

C FTS Darboux canonical form: a foliation on $\mathfrak{F}_{y \perp x}$ planes

In the previous sections we have seen how it is natural to define structures on the FTS space as $\mathfrak{F}$ -planes and $\mathfrak{F}_x$, their orthogonal complement $\mathfrak{F}_x^\perp$. Within any $\mathfrak{F}_x^\perp$ it results also natural to define planes $\mathfrak{F}_{y \perp x}$ closed under the action of the Υ_x map. This decomposition of the $\mathfrak{F}_x^\perp$ space can be performed in a systematic way providing a natural canonical form for any FTS, similar to the Darboux canonical form of any symplectic space.

The orthogonal space $\mathfrak{F}_x^\perp$ can be further decomposed in 2-dimensional subspaces orthogonal with respect to the antisymmetric bilinear form $\{, \}$.

Given a fixed initial element x_0 of maximal rank, let us first define for convenience the shorthand notation

$$\dot{x} \equiv \Upsilon_{x_0}(x),$$

for the fixed element x_0 . In particular $\dot{x}_0 = 3x'_0 = 3T(x_0)$. We will construct on continuation a series of mutually orthogonal vectors iterating the procedure used before eq. (4.6) in a sort of modified Gram-Schmidt procedure. Let us initially assume a number of pairs, formed by some vectors and their transforms, $(x_0, \dot{x}_0), (x_1, \dot{x}_1), \dots (x_{n-1}, \dot{x}_{n-1})$, which are already mutually orthogonal, that means (for $i, j = 0, n-1$)

$$\{x_i, x_j\} = 0, \quad (\text{C.1})$$

$$\{x_i, \dot{x}_j\} = c_i \delta_{ij}. \quad (\text{C.2})$$

Where c_i are nonzero constants. We now extend this set of pairs by iteration. We show that it is possible to find a pair $(x_n, \dot{x}_n)$ orthogonal to the previous ones. Let us take an arbitrary vector z and decompose in parallel and orthogonal parts with respect all these vectors, $z = z_{\parallel} + z_{\perp}$. The parallel part is easily computed, it is the sum of the parallel parts to each of the individual pairs. It is given by eq. (4.6)

$$z_{\parallel} = \sum_{i=0, n-1} z_{\parallel x_i} = \sum_{i=0, n-1} \frac{1}{\{\dot{x}_i, x_i\}} \begin{vmatrix} x_i & \dot{x}_i \\ \{x_i, z\} & \{\dot{x}_i, z\} \end{vmatrix}. \quad (\text{C.3})$$

Obviously, the $z_{\parallel}$ defined in this way is on the subspace generated by $x_i, \dot{x}_i, (i = 0, n-1)$. The vector $z_{\perp} = z - z_{\parallel}$ is orthogonal by construction to all the subspace,

$$\{x_i, z_{\perp}\} = 0, \quad (\text{C.4})$$

$$\{\dot{x}_i, z_{\perp}\} = 0. \quad (\text{C.5})$$

It is also straightforward to show that $(z_{\perp})' \equiv \Upsilon_{x_0}(z_{\perp}) = 3T(x_0, x_0, z_{\perp})$ is also orthogonal to the full set:

$$\{\Upsilon_{x_0}(z_{\perp}), x_i\} = \{3T(x_0, x_0, z_{\perp}), x_i\} = \{3T(x_0, x_0, x_i), z_{\perp}\} \quad (\text{C.6})$$

$$= \{\Upsilon_{x_0}(x_i), z_{\perp}\} = 0, \quad (\text{C.7})$$

$$\{\Upsilon_{x_0}(z_{\perp}), \dot{x}_i\} = 9\{T(x_0, x_0, z_{\perp}), T(x_0, x_0, x_i)\} = -C_i \Delta(x_0) \{x_0, z_{\perp}\} = 0. \quad (\text{C.8})$$

In the last line $C_0 = 3, C_i = 1, (i = 1, n - 1)$. So $(z_\perp, (\dot{z}_\perp))$ is the pair we were looking for, we redefine

$$x_n \equiv z_\perp, \quad (C.9)$$

$$\dot{x}_n \equiv 3T(x_0, x_0, z_\perp) = (\dot{z}_\perp). \quad (C.10)$$

The process is iterated as long as we exhaust the dimensionality of the vector space ($n = N$) or we cannot find vectors with non trivial pairs, $(x_i, \dot{x}_i) \neq 0$.

In this way we reduce the symplectic form to a canonical Darboux form. In the basis formed by the vectors $(x_0, \dot{x}_0, x_1, \dot{x}_1, \dots)$ the the symplectic form is expressed by the matrix

$$\Omega = \begin{bmatrix} 0 & \lambda_1 & & 0 \\ -\lambda_1 & 0 & & \\ & & \ddots & \\ 0 & & 0 & \lambda_N \\ & & -\lambda_N & 0 \end{bmatrix}, \quad (C.11)$$

where

$$\lambda_i = \{x_i, \dot{x}_i\}.$$

D The reduced $\mathfrak{F}(J_3)$ case: explicit expressions

We present here some explicit formulas used in section 8. By exploiting the results in appendix D of [97], one can compute the components of $T(x, y, z)_M$ eq. (8.9) in the $4D/5D$ special coordinates' symplectic frame, characterizing every *reduced* FTS. Using obvious notation (see section 8), these components read:

$$\begin{aligned} T(x, y, z)_0 &= -\frac{1}{3}d^{ijk}x_i y_j z_k - \frac{1}{3}(x^0 y_0 z_0 + x_0 y^0 z_0 + x_0 y_0 z^0) \\ &\quad - \frac{1}{6}[(x^i y_i + x_i y^i)z_0 + (x^i z_i + x_i z^i)y_0 + (y^i z_i + y_i z^i)x_0]; \end{aligned} \quad (D.1)$$

$$\begin{aligned} T(x, y, z)_i &= \frac{1}{3}d_{ijk}(x_0 y^j z^k + x^j y_0 z^k + x^j y^k z_0) + \frac{1}{3}d_{ijm}d^{mkl}(x^j y_k z_l + x_k y^j z_l + x_k y_l z^j) \\ &\quad - \frac{1}{6}[(x^j y_j + x_j y^j)z_i + (x^j z_j + x_j z^j)y_i + (y^j z_j + y_j z^j)x_i] \\ &\quad - \frac{1}{6}[(x^0 y_0 + x_0 y^0)z_i + (x^0 z_0 + x_0 z^0)y_i + (y^0 z_0 + y_0 z^0)x_i]; \end{aligned} \quad (D.2)$$

$$\begin{aligned} T(x, y, z)^0 &= \frac{1}{3}d_{ijk}x^i y^j z^k - \frac{1}{3}(x^0 y^0 z_0 + x^0 y_0 z^0 + x_0 y^0 z^0) \\ &\quad - \frac{1}{6}[(x^i y_i + x_i y^i)z^0 + (x^i z_i + x_i z^i)y^0 + (y^i z_i + y_i z^i)x^0]; \end{aligned} \quad (D.3)$$

$$\begin{aligned} T(x, y, z)^i &= -\frac{1}{3}d^{ijk}(x^0 y_j z_k + x_j y^0 z_k + x_j y_k z^0) + \frac{1}{3}d_{klm}d^{mij}(x_j y^k z^l + x^k y_j z^l + x^k y^l z_j) \\ &\quad - \frac{1}{6}[(x^j y_j + x_j y^j)z^i + (x^j z_j + x_j z^j)y^i + (y^j z_j + y_j z^j)x^i] \\ &\quad - \frac{1}{6}[(x^0 y_0 + x_0 y^0)z^i + (x^0 z_0 + x_0 z^0)y^i + (y^0 z_0 + y_0 z^0)x^i]. \end{aligned} \quad (D.4)$$

Let us recall that in this symplectic frame the $2N \times 2N$ symplectic metric is given by eq. (6.9). In particular, one can compute the various components of the linear map⁴⁰ $\Upsilon_x(y)$ defined in eq. (2.6) to read in components

$$\Upsilon_x(y)^0 = -3T(x, x, y)^0 + (-x^0 y_0 - x^j y_j + x_0 y^0 + x_j y^j) x^0; \quad (\text{D.5})$$

$$\Upsilon_x(y)^i = -3T(x, x, y)^i + (-x^0 y_0 - x^j y_j + x_0 y^0 + x_j y^j) x^i; \quad (\text{D.6})$$

$$\Upsilon_x(y)_0 = 3T(x, x, y)_0 + (-x^0 y_0 - x^j y_j + x_0 y^0 + x_j y^j) x_0; \quad (\text{D.7})$$

$$\Upsilon_x(y)_i = 3T(x, x, y)_i + (-x^0 y_0 - x^j y_j + x_0 y^0 + x_j y^j) x_i, \quad (\text{D.8})$$

where

$$T(x, x, y)^0 = \frac{1}{3} d_{ijk} x^i x^j y^k - \frac{1}{3} \left[2(x^0)^2 y_0 + 2x^0 x_0 y^0 \right] - \frac{1}{3} [x^i x_i y^0 + (x^i y_i + x_i y^i) x^0];$$

$$\begin{aligned} T(x, x, y)^i &= -\frac{1}{3} d^{ijk} (2x^0 x_j y_k + x_j x_k y^0) + \frac{1}{3} d_{klm} d^{mij} (2x_j x^k y^l + x^k x^l y_j) \\ &\quad - \frac{1}{3} [x^j x_j y^i + (x^j y_j + x_j y^j) x^i] - \frac{1}{3} [x^0 x_0 y^i + (x^0 y_0 + x_0 y^0) x^i]; \end{aligned}$$

$$T(x, x, y)_0 = -\frac{1}{3} d^{ijk} x_i x_j y_k - \frac{1}{3} [2x^0 x_0 y_0 + (x_0)^2 y^0] - \frac{1}{3} [x^i x_i y_0 + (x^i y_i + x_i y^i) x_0];$$

$$\begin{aligned} T(x, x, y)_i &= \frac{1}{3} d_{ijk} (2x_0 x^j y^k + x^j x^k y_0) + \frac{1}{3} d_{ijm} d^{mkl} (2x^j x_k y_l + x_k x_l y^j) \\ &\quad - \frac{1}{3} [x^j x_j y_i + (x^j y_j + x_j y^j) x_i] - \frac{1}{3} [x^0 x_0 y_i + (x^0 y_0 + x_0 y^0) x_i]. \end{aligned}$$

The following quantity, x - and y - dependent function, plays an important role

$$\{y, \Upsilon_x(y)\} = 3\{y, T(x, x, y)\} - \{x, y\}^2 \quad (\text{D.9})$$

$$\begin{aligned} &= -3[-y^0 T(x, x, y)_0 - y^i T(x, x, y)_i + y_0 T(x, x, y)^0 + y_i T(x, x, y)^i] \\ &\quad - (-x^0 y_0 - x^j y_j + x_0 y^0 + x_j y^j)^2. \end{aligned} \quad (\text{D.10})$$

where $T(x, x, y)_0$, $T(x, x, y)_i$, $T(x, x, y)^0$ and $T(x, x, y)^i$ are given above.

Further expressions. Within the $4D/5D$ special coordinates' symplectic frame of reduced FTS's and within the choices eq. (8.52) resp. eq. (8.63) of the rank-4 element $x \in \mathfrak{F}$ (with $\Delta(x) < 0$) and of the rank-4 element $y \in \mathfrak{F}_x^\perp = \mathfrak{F}/\mathfrak{F}_x$ (with $\Delta(y) \geq 0$), we present here some further formulas useful for section 8. In order to determine the condition of closure of $\mathfrak{F}_{y \perp x}$ under T , we have to explicitly compute $T(r) \equiv T(r, r, r)$ for a generic element $r = ay + b\Upsilon_x(y) \in \mathfrak{F}_{y \perp x}$, which is given by

$$\begin{aligned} T(r) &\equiv T(r, r, r) = T(ay + b\Upsilon_x(y), ay + b\Upsilon_x(y), ay + b\Upsilon_x(y)) \\ &= a^3 T(y, y, y) + 3a^2 b T(y, y, \Upsilon_x(y)) + 3a^2 b T(\Upsilon_x(y), \Upsilon_x(y), y) \\ &\quad + b^3 T(\Upsilon_x(y), \Upsilon_x(y), \Upsilon_x(y)). \end{aligned}$$

⁴⁰Note that when $y \in \mathfrak{F}_x^\perp$, then $\{x, y\} = 0$, and the second term in eq. (2.6) is missing [14].

Let us then start and compute the components of $T(r)_M$. In first place, the $T(y, y, y)_M$ quantities are given by

$$T(y, y, y)_0 = -\frac{1}{3}d^{ijk}y_i y_j y_k; \quad (\text{D.11})$$

$$T(y, y, y)_i = \left(d_{mij}d^{mkl}y^j y_k y_l - y^j y_j y_i \right); \quad (\text{D.12})$$

$$T(y, y, y)^0 = \frac{1}{3}d_{ijk}y^i y^j y^k; \quad (\text{D.13})$$

$$T(y, y, y)^i = \left(d_{klm}d^{mij}y_j y^k y^l - y^j y_j y^i \right); \quad (\text{D.14})$$

similarly

$$T(y, y, \Upsilon_x(y))_0 = \frac{1}{3}x^0 x_0 d^{ijk}y_i y_j y_k; \quad (\text{D.15})$$

$$T(y, y, \Upsilon_x(y))_i = -\frac{1}{3}x^0 x_0 \left(d_{mij}d^{mkl}y^j y_k y_l - y^j y_j y_i \right); \quad (\text{D.16})$$

$$T(y, y, \Upsilon_x(y))^0 = \frac{1}{3}x^0 x_0 d_{ijk}y^i y^j y^k; \quad (\text{D.17})$$

$$T(y, y, \Upsilon_x(y))^i = \frac{1}{3}x^0 x_0 \left(d_{klm}d^{mij}y_j y^k y^l - y^j y_j y^i \right); \quad (\text{D.18})$$

and

$$T(\Upsilon_x(y), \Upsilon_x(y), y)_0 = -\frac{1}{3}(x^0 x_0)^2 d^{ijk}y_i y_j y_k; \quad (\text{D.19})$$

$$T(\Upsilon_x(y), \Upsilon_x(y), y)_i = -\frac{1}{3}(x^0 x_0)^2 \left(d_{mij}d^{mkl}y^j y_k y_l - y^j y_j y_i \right); \quad (\text{D.20})$$

$$T(\Upsilon_x(y), \Upsilon_x(y), y)^0 = \frac{1}{3}(x^0 x_0)^2 d_{ijk}y^i y^j y^k; \quad (\text{D.21})$$

$$T(\Upsilon_x(y), \Upsilon_x(y), y)^i = -\frac{1}{3}(x^0 x_0)^2 \left(d_{klm}d^{mij}y_j y^k y^l - y^j y_j y^i \right); \quad (\text{D.22})$$

finally

$$T(\Upsilon_x(y), \Upsilon_x(y), \Upsilon_x(y))_0 = \frac{1}{3}(x^0 x_0)^3 d^{ijk}y_i y_j y_k; \quad (\text{D.23})$$

$$T(\Upsilon_x(y), \Upsilon_x(y), \Upsilon_x(y))_i = (x^0 x_0)^3 \left(d_{mij}d^{mkl}y^j y_k y_l - y^j y_j y_i \right); \quad (\text{D.24})$$

$$T(\Upsilon_x(y), \Upsilon_x(y), \Upsilon_x(y))^0 = \frac{1}{3}(x^0 x_0)^3 d_{ijk}y^i y^j y^k; \quad (\text{D.25})$$

$$T(\Upsilon_x(y), \Upsilon_x(y), \Upsilon_x(y))^i = -(x^0 x_0)^3 \left(d_{klm}d^{mij}y_j y^k y^l - y^j y_j y^i \right). \quad (\text{D.26})$$

Therefore, putting together all the computations, the various components of $T(r)_M$ reads

$$\begin{aligned} T(r)_0 &= (a - x^0 x_0 b)^3 T(y, y, y)_0; \\ T(r)_i &= (a - x^0 x_0 b)^2 (a + x^0 x_0 b) T(y, y, y)_i; \\ T(r)^0 &= (a + x^0 x_0 b)^3 T(y, y, y)^0; \\ T(r)^i &= (a - x^0 x_0 b) (a + x^0 x_0 b)^2 T(y, y, y)^i. \end{aligned} \quad (\text{D.27})$$

Then

$$T(r)^M = \Omega^{MN} T(r)_N = (-T(r)^0, -T(r)^i, T(r)_0, T(r)_i)^T, \quad (\text{D.28})$$

where $T(r)^0$, $T(r)^i$, $T(r)_0$ and $T(r)_i$ are given by (8.75). Final expressions are given in the text, see section 8.

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