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A fuzzy bipolar celestial sphere

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ABSTRACT: We introduce a non-commutative deformation of the algebra of bipolar spherical harmonics supporting the action of the full Lorentz algebra. Our construction is close in spirit to the one of the non-commutative spherical harmonics associated to the fuzzy sphere and, as such, it leads to a maximal value of the angular momentum. We derive the action of Lorentz boost generators on such non-commutative spherical harmonics and show that it is compatible with the existence of a maximal angular momentum.

KEYWORDS: Non-Commutative Geometry, Space-Time Symmetries

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1 Introduction

The discovery of the connection between soft theorems in quantum field theory, the memory effect and asymptotic symmetries has revealed an unexpected richness in the infrared structure of gauge theories [1–3]. In gravity, the corner of this infrared triangle represented by the symmetries of asymptotically flat spacetimes has been subject of a revived interest, mainly due to the potential role of the BMS (Bondi-Metzner-Sachs) group [4] in the resolution of the black hole information paradox [5, 6]. In this context, the existence of an infinite number of conserved charges associated with BMS symmetries [7–9] can equip the black hole with the *soft hair* [10, 11] needed to support correlations between the interior of the black hole and the emitted Hawking quanta. To date, however, the exact mechanism from which the information can be recovered through the BMS charges is not known. Connected to this line of thought is the possibility that modes of a near-horizon BMS symmetry might provide the degrees of freedom needed to microscopically reproduce the Bekenstein-Hawking entropy [12, 13]. One of the obstacles in making such identification concrete is that the actual degrees of freedom which can be associated to BMS charges are *too many*, in fact infinite, while the Bekenstein-Hawking entropy, albeit large, is finite and proportional to the black hole area divided by the Planck length squared. This is already evident in the simplest formulation of the BMS group as the semidirect product of the Lorentz group and the abelian group of supertranslations. The latter are indexed

by the angular momentum of spherical harmonics on the celestial sphere and are infinite in number since one can have infinite angular resolution on such sphere.

In this note we explore the possibility of constructing a non-commutative deformation of the algebra of spherical harmonics supporting an action of the Lorentz algebra and exhibiting a maximal angular resolution. We show how such task *cannot* be accomplished using only *one* set of non-commutative spherical harmonics similar to the one used in the literature to describe a non-commutative analogue of the two-sphere, the so-called *fuzzy sphere* [14]. We find, however, that a matrix generalization of the algebra of *bipolar spherical harmonics* [15] can be constructed, exhibiting a cut-off in the angular modes and carrying a representation of the full Lorentz algebra.

In the standard picture, Lorentz boosts acting on the celestial sphere do not commute with the total angular momentum operator and hence they connect spherical harmonics with different values of the angular momentum. One remarkable aspect of our construction is that the action of Lorentz boosts on the algebra of non-commutative bipolar spherical harmonics is found to be compatible with the existence of a maximal angular momentum and cannot produce harmonics labelled with an arbitrarily high angular momentum.

In the next three sections we recall some basic facts about the asymptotic structure of Minkowski spacetime, showing the action of the Poincaré and the BMS algebra on the celestial sphere. We then review the fuzzy sphere and in particular we will focus on the mapping of ordinary spherical harmonics to the so-called *fuzzy spherical harmonics*, characterized by a maximal angular momentum. Finally we extend such construction in order to introduce an action of the full Lorentz algebra which is consistent with the existence of a maximal value of the angular momentum. We close with a short summary and an outline for future developments.

2 The celestial sphere

We start by recalling the notion of celestial sphere focusing for simplicity on Minkowski spacetime [16, 17], but keeping in mind that the same definition can be given for any asymptotically flat spacetime, since it relies only on asymptotic properties. Given the Minkowski line element in cartesian coordinates

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \quad (2.1)$$

we first pass to ordinary spherical coordinates (r, θ, ϕ) ,

$$x^1 \pm ix^2 = r e^{\pm i\phi} \sin \theta, \quad x^3 = r \cos \theta, \quad (2.2)$$

and then switch from the inertial time coordinate x^0 to the retarded time $u = x^0 - r$. Consider now an observer emitting a light ray at $x^0 = 0$ and $r = 0$ in a direction (θ, ϕ) . We can assign to any point at finite distance along that ray the coordinates (u, r, θ, ϕ) . Here r is just an affine parameter along the geodesic representing the null ray and can be thought as a measure of the distance between the emitter and the particular point considered, in the frame of the emitter. Notice that u is constant along that ray (and is

Figure 1. The celestial sphere $\mathcal{S}^+$ at $u = 0$ of an observer emitting a ray at $x^0 = 0$ and $r = 0$, obtained as the intersection of the null cone $\mathcal{N}$ and $\mathcal{S}^+$.

always equal to 0 for the particular ray considered). The set (u, r, θ, ϕ) is called a retarded Bondi coordinate system.

Future null infinity $\mathcal{S}^+$ can be defined as the asymptotic null region obtained by sending $r, x^0 \rightarrow \infty$ while keeping the retarded time $u = x^0 - r$ constant. In such limit a light ray will intersect $\mathcal{S}^+$ in a point, which we label by (u, θ, ϕ) . By sending light rays in all possible directions one can cover the entire future null cone $\mathcal{N}$. At null infinity this cone will intersect $\mathcal{S}^+$ on a sphere $\mathcal{S}^+$, spanned by the coordinates θ and ϕ . Similarly, it is possible to define a coordinate system on the past null cone, using ingoing null rays and the advanced time $v = x^0 + r$ which is constant along them. The Minkowski conformal diagram is represented in figure 1.

For any fixed value of the retarded time u , the points of $\mathcal{S}^+$ are spheres $\mathcal{S}^+$ of infinite radius, called *celestial spheres*. They are the spheres of all directions towards which an observer at $r = 0$ can look. Alternatively, it is possible to give a definition of celestial sphere [18], which does not rely on a particular choice of coordinates. It can be defined as the set of future-directed null directions passing through a point, i.e. the complex projective line $\mathbb{CP}^1 \simeq \mathbb{S}^2$. This will be very useful in the following. Notice that for any asymptotically flat spacetime future null infinity is always a 3-dimensional $\mathbb{S}^2 \times \mathbb{R}$ manifold, whose $\mathbb{S}^2$ component is the celestial sphere $\mathcal{S}^+$.

3 Lorentz transformations of the celestial sphere

The connected component of the Lorentz group consists of transformations $x'^\mu = \Lambda^\mu_\nu x^\nu$ which relate the coordinates of two different inertial observers. They are isometries of the line element (2.1). In order to describe the action of such transformations on the celestial sphere, we start by introducing complex stereographic coordinates $(\zeta, \bar{\zeta})$ for each point of the sphere.

$$\zeta = e^{i\phi} \tan \frac{\theta}{2} = \frac{x^1 + ix^2}{r + x^3}, \quad \bar{\zeta} = \zeta^*. \quad (3.1)$$

Figure 2. The representation of a direction (θ, ϕ) in the sky $\mathcal{S}^+$ as a stereographic coordinate ζ on the complex plane $\mathbb{C}_\zeta$.

Then, it turns out [19] that any Lorentz transformation on $\mathcal{S}^+$ for the stereographic coordinates is given by a Möbius map,

$$\zeta' = \frac{a\zeta + b}{c\zeta + d}, \quad ad - bc = 1, \quad (3.2)$$

while the retarded time transforms as

$$u' = \frac{1 + |\zeta|^2}{|a\zeta + b|^2 + |c\zeta + d|^2} u \equiv K(\zeta, \bar{\zeta})u, \quad |\zeta|^2 \equiv \zeta\bar{\zeta}, \quad (3.3)$$

i.e. u undergoes an angle-dependent rescaling. Equations (3.2) and (3.3) express the relations between the coordinates on the celestial sphere associated to two different inertial observers. Note that the Möbius maps of (3.2) are just the group of complex projective transformations of the above mentioned complex projective line $\mathbb{CP}^1$ and that the induced metric on the celestial sphere undergoes a conformal transformation. Indeed, the line element of the unit sphere is

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2 = \frac{4}{(1 + |\zeta|^2)^2} d\zeta d\bar{\zeta}, \quad (3.4)$$

under (3.2) transforms as

$$\begin{aligned} ds'^2 &= \frac{4}{(1 + |\zeta'|^2)^2} d\zeta' d\bar{\zeta}' = \left(\frac{1 + |\zeta|^2}{|a\zeta + b|^2 + |c\zeta + d|^2} \right)^2 ds^2 \\ &= K^2(\zeta, \bar{\zeta}) ds^2, \end{aligned} \quad (3.5)$$

i.e. a conformal rescaling with conformal factor $K(\zeta, \bar{\zeta})$. Equations (3.2), (3.3) tell us that Lorentz transformations on the celestial sphere are described by $\text{SL}(2, \mathbb{C})/\mathbb{Z}_2$ matrices

$$A = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1. \quad (3.6)$$

Figure 3. The point P_2 on $\mathcal{S}^+$ is obtained from P_1 by means of a rotation about the x^3 axis. The corresponding orbit in $\mathbb{C}_\zeta$ is a circle.

In particular, a general rotation of an angle φ and of a boost of rapidity χ about an axis $\hat{n} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)$ are described by the following $\text{SL}(2, \mathbb{C})/\mathbb{Z}_2$ matrices

$$L_{\hat{n}}(\varphi) = \pm \begin{pmatrix} \cos \frac{\varphi}{2} - i \cos \theta \sin \frac{\varphi}{2} & -i \sin \theta \sin \frac{\varphi}{2} e^{-i\phi} \\ -i \sin \theta \sin \frac{\varphi}{2} e^{i\phi} & \cos \frac{\varphi}{2} + i \cos \theta \sin \frac{\varphi}{2} \end{pmatrix}, \quad (3.7)$$

$$R_{\hat{n}}(\chi) = \pm \begin{pmatrix} \cosh \frac{\chi}{2} - \cos \theta \sinh \frac{\chi}{2} & -\sin \theta \sinh \frac{\chi}{2} e^{-i\phi} \\ -\sin \theta \sinh \frac{\chi}{2} e^{i\phi} & \cosh \frac{\chi}{2} + \cos \theta \sinh \frac{\chi}{2} \end{pmatrix}. \quad (3.8)$$

Notice that $L_{\hat{n}}(\varphi)$ is an $\text{SU}(2)$ transformation while $R_{\hat{n}}(\chi)$ is not. Indeed, for any rotation the conformal factor is $K(\zeta, \bar{\zeta}) = 1$, because rotations are pure isometries of the 2-sphere, while the boosts are only conformal symmetries.

For example a rotation about the axis x^3 of an angle φ is expressed by $\zeta' = e^{-i\varphi}\zeta$, i.e. a rotation on $\mathbb{C}_\zeta$ (see figure 3). Two observers that are rotated about the axis x^3 see the same celestial sphere, but their coordinates are rotated. Rotations of the celestial sphere map into rotations of the complex plane.

On the other hand, a boost of rapidity χ along the x^3 axis is given by $\zeta' = e^{-\chi}\zeta$. In this case, the two inertial observers still see the same celestial sphere, but the points of the celestial sphere of the boosted observer are dragged away from the south pole and come closer to the north pole as χ increases. On the complex plane this corresponds to a contraction, as shown in figure 4.

Furthermore, it is possible to show [17] that the conformal factor of boosts is related to the Lorentz factor γ through the following relation

$$K^2(\zeta, \bar{\zeta}) = \frac{1}{[\gamma(1 - \frac{\mathbf{v} \cdot \hat{\mathbf{r}}}{c})]^2}. \quad (3.9)$$

If all the stars in one observer's sky are thought of as projected onto its celestial sphere, two boosted observer see a different night sky [20]. This is the classical phenomenon of stellar aberration.

Figure 4. The action of a boost along the x^3 direction on points on the celestial sphere, and the corresponding action on $\mathbb{C}_\zeta$. The two radii are such that $|\zeta'_i|/|\zeta_i| = e^{-\chi}$.

The infinitesimal transformations of (3.2), (3.3) are described by the following vector fields on $\mathcal{I}^+$,

$$L_x = i \left(\sin \varphi \frac{\partial}{\partial \theta} + \cot \theta \cos \varphi \frac{\partial}{\partial \phi} \right), \quad (3.10)$$

$$L_y = i \left(-\cos \varphi \frac{\partial}{\partial \theta} + \cot \theta \sin \varphi \frac{\partial}{\partial \phi} \right), \quad (3.11)$$

$$L_z = -i \frac{\partial}{\partial \phi}, \quad (3.12)$$

$$R_x = -i \left(\cos \theta \cos \varphi \frac{\partial}{\partial \theta} - \frac{\sin \varphi}{\sin \theta} \frac{\partial}{\partial \phi} - u \sin \theta \cos \varphi \frac{\partial}{\partial u} \right), \quad (3.13)$$

$$R_y = -i \left(\cos \theta \sin \varphi \frac{\partial}{\partial \theta} + \frac{\cos \varphi}{\sin \theta} \frac{\partial}{\partial \phi} - u \sin \theta \sin \varphi \frac{\partial}{\partial u} \right), \quad (3.14)$$

$$R_z = i \left(\sin \theta \frac{\partial}{\partial \theta} + u \cos \theta \frac{\partial}{\partial u} \right), \quad (3.15)$$

and it is easy to prove [19] that they are a representation of the Lorentz algebra on $\mathcal{I}^+$ and on the celestial sphere, having fixed the value of u :

$$[L_i, L_j] = i\epsilon_{ijk}L_k, \quad [R_i, R_j] = -i\epsilon_{ijk}L_k, \quad [L_i, R_j] = i\epsilon_{ijk}R_k. \quad (3.16)$$

The celestial sphere, as a smooth manifold, is a 2-sphere S^2 and the commutative algebra of smooth functions defined on it, which will be denoted by $C(S^2)$, is generated by the spherical harmonics $\{Y_{lm}(\theta, \phi)\}$, which provide an orthonormal and complete basis with inner product given by

$$\int d\Omega Y_{l_1 m_1}^*(\theta, \varphi) Y_{l_2 m_2}(\theta, \varphi) = \delta_{l_1 l_2} \delta_{m_1 m_2}, \quad (3.17)$$

Thus, any smooth function $f(\theta, \phi) \in C(S^2)$ can be expanded as

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l f_{lm} Y_{lm}(\theta, \phi), \quad (3.18)$$

with the components of the expansion given by

$$f_{lm} = \int d\Omega Y_{lm}^*(\theta, \varphi) f(\theta, \phi). \quad (3.19)$$

The product of two spherical harmonics can be expressed in terms of a linear combination of spherical harmonics using the Clebsch-Gordan coefficients:

$$Y_{l_1 m_1} Y_{l_2 m_2} = \sum_{l=|l_1-l_2|}^{l_1+l_2} \sum_{m=-l}^l \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}} C_{l_1 0 l_2 0}^{l 0} C_{l_1 m_1 l_2 m_2}^{l m} Y_{lm}. \quad (3.20)$$

Note that such product is commutative, since $C_{l_1 0 l_2 0}^{l 0} C_{l_1 m_1 l_2 m_2}^{l m} = C_{l_2 0 l_1 0}^{l 0} C_{l_2 m_2 l_1 m_1}^{l m}$ and that the maximum value of the angular momentum l is given by $l_{\max} = l_1 + l_2$.

Let us now consider the ladder operators

$$L_+ = L_x + iL_y = e^{i\varphi} \left(\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right), \quad (3.21)$$

$$L_- = L_x - iL_y = -e^{-i\varphi} \left(\frac{\partial}{\partial \theta} - i \cot \theta \frac{\partial}{\partial \phi} \right), \quad (3.22)$$

$$L_z = -i \frac{\partial}{\partial \phi}, \quad (3.23)$$

$$R_+ = R_x + iR_y = -ie^{i\varphi} \left(\cos \theta \frac{\partial}{\partial \theta} + \frac{i}{\sin \theta} \frac{\partial}{\partial \phi} - u \sin \theta \frac{\partial}{\partial u} \right), \quad (3.24)$$

$$R_- = R_x - iR_y = -ie^{-i\varphi} \left(\cos \theta \frac{\partial}{\partial \theta} - \frac{i}{\sin \theta} \frac{\partial}{\partial \phi} - u \sin \theta \frac{\partial}{\partial u} \right), \quad (3.25)$$

$$R_z = i \left(\sin \theta \frac{\partial}{\partial \theta} + u \cos \theta \frac{\partial}{\partial u} \right). \quad (3.26)$$

Their action on spherical harmonics $Y_{lm}(\theta, \phi)$ is given by [15]:

$$L_+(Y_{lm}) = \sqrt{l(l+1) - m(m+1)} Y_{l, m+1}, \quad (3.27)$$

$$L_-(Y_{lm}) = \sqrt{l(l+1) - m(m-1)} Y_{l, m-1}, \quad (3.28)$$

$$L_z(Y_{lm}) = m Y_{lm}, \quad (3.29)$$

$$\begin{aligned} R_+(Y_{lm}) = & -il \sqrt{\frac{(l+m+1)(l+m+2)}{(2l+1)(2l+3)}} Y_{l+1, m+1} \\ & - i(l+1) \sqrt{\frac{(l-m-1)(l-m)}{4l^2-1}} Y_{l-1, m+1}, \end{aligned} \quad (3.30)$$

$$\begin{aligned} R_-(Y_{lm}) = & il \sqrt{\frac{(l-m+1)(l-m+2)}{(2l+1)(2l+3)}} Y_{l+1, m-1} \\ & + i(l+1) \sqrt{\frac{(l+m-1)(l+m)}{4l^2-1}} Y_{l-1, m-1}, \end{aligned} \quad (3.31)$$

$$\begin{aligned} R_z(Y_{lm}) = & il \sqrt{\frac{(l-m+1)(l+m+1)}{(2l+1)(2l+3)}} Y_{l+1, m} \\ & - i(l+1) \sqrt{\frac{(l+m)(l-m)}{4l^2-1}} Y_{l-1, m}. \end{aligned} \quad (3.32)$$

Notice that since the total angular momentum L^2 does not commute with the boosts R_i , the action of a boost on a spherical harmonic changes in general its total angular momentum l .

4 Aside: translations and supertranslations on the celestial sphere

So far, we have discussed what is the effect of Lorentz transformations on the celestial sphere. The isometries of Minkowski space however comprise also the four-translations $x'^\mu = x^\mu + \delta x^\mu$. In this section we describe their effect on the celestial sphere. Any infinitesimal time translation $x'^0 = x^0 + \delta x^0$ clearly maps u into $u' = u + \delta x^0$. It means that the first observer will see the same celestial sphere of the second after a proper time interval δx^0 . The two celestial spheres are just shifted in time by δx^0 . A displacement by an infinitesimal spatial vector $\delta \vec{x}$ induces the transformation $u' = u + \frac{\vec{x} \cdot \delta \vec{x}}{r}$ and thus we can write an infinitesimal four-translation δx^μ of the retarded time using spherical harmonics as

$$\begin{aligned} u' &= u + \delta x^0 + \delta x^1 \cos \phi \sin \theta + \delta x^2 \sin \phi \sin \theta + \delta x^3 \cos \theta \\ &\equiv u + \sum_{l \in \{0,1\}} \sum_{m=-l}^l \alpha_{lm} Y_{lm}(\theta, \phi), \end{aligned} \quad (4.1)$$

where

$$\begin{aligned} \alpha_{00} &= \sqrt{4\pi} \delta x^0, & \alpha_{10} &= -\sqrt{\frac{4\pi}{3}} \delta x^3, \\ \alpha_{1,-1} &= -\sqrt{\frac{2\pi}{3}} (\delta x^1 + i \delta x^2), & \alpha_{11} &= -\sqrt{\frac{2\pi}{3}} (-\delta x^1 + i \delta x^2). \end{aligned} \quad (4.2)$$

While $\text{SL}(2, \mathbb{C})/\mathbb{Z}_2$ is a symmetry group at null infinity both for Minkowski spacetime and for asymptotically flat spacetimes, the picture for the four-translations is dramatically different in the two cases. In fact, the boundary conditions of asymptotically flat spacetimes allow a larger class of transformations, known as *supertranslations*, which generalize (4.1) to arbitrary values of l :

$$u' = u + \sum_{l=0}^{\infty} \sum_{m=-l}^l \alpha_{lm} Y_{lm}(\theta, \phi), \quad (4.3)$$

with α_{lm} complex numbers satisfying $\alpha_{lm} = (-1)^m \alpha_{l,-m}^*$. The generators of these transformations are the vector fields

$$P_{lm} = Y_{lm}(\theta, \phi) \frac{\partial}{\partial u}, \quad (4.4)$$

which span the abelian algebra of supertranslations. The vector fields (4.4), together with (3.10)–(3.15), form the BMS algebra found by Sachs [4], which contains the Poincaré algebra as a subalgebra. This shows that the asymptotic symmetry group of asymptotically flat spacetimes at null infinity is not the Poincaré group, but the BMS group [21, 22] which is infinite dimensional instead, and it is the semi-direct product $\text{SL}(2, \mathbb{C})/\mathbb{Z}_2 \ltimes S$, where S is the abelian group of supertanslations.

5 Non-commutative spherical harmonics: the fuzzy sphere

The first step in order to obtain a non-commutative deformation of the celestial sphere will be to deform the algebra of spherical harmonics (3.20). This essentially boils down to the introduction of *fuzzy spherical harmonics* [23–26] which can be thought of as the algebra of functions on a non-commutative space known as the *fuzzy sphere* [14, 27–37]. This deformation of the algebra of spherical harmonics is concretely realized in terms of a “quantization map” between the commutative algebra of functions on the two-sphere $C(S^2)$ and the algebra of $N \times N$ complex matrices $M_N(\mathbb{C})$,

$$\Omega_N : C(S^2) \rightarrow M_N(\mathbb{C}); \quad \Omega_N[Y_{lm}(\theta, \varphi)] = \begin{cases} \hat{Y}_{lm}^{(N)} & l < N \\ 0 & l \geq N \end{cases} \quad (5.1)$$

where the mapping between the spherical harmonics $Y_{lm}(\theta, \phi)$ and the matrices $\hat{Y}_{lm}^{(N)}$ is explicitly realized as:

$$\hat{Y}_{lm}^{(N)} = \frac{2^l}{l!} \left[\frac{N(N-1-l)!}{(N+l)!} \right]^{\frac{1}{2}} (\mathbf{J}^{(N)} \cdot \nabla)^l \left(r^l Y_{lm}(\theta, \varphi) \right), \quad (5.2)$$

with $\mathbf{J}^{(N)} = (J_x^{(N)}, J_y^{(N)}, J_z^{(N)})$ and $J_i^{(N)}$ are the N -dimensional spin matrices with spin j_N

$$[J_i^{(N)}, J_j^{(N)}] = i\epsilon_{ijk} J_k^{(N)}, \quad J^{(N)2} = j_N(j_N + 1)\mathbb{I}^{(N)}, \quad 2j_N + 1 = N. \quad (5.3)$$

The fuzzy spherical harmonics are irreducible tensor operators of rank l and are proportional to the polarization tensors $\hat{Y}_{lm}^{(N)} = \sqrt{\frac{N}{4\pi}} T_{lm}^{(N)}$. We thus have that, given the ladder operators $J_{\pm}^{(N)} = J_x^{(N)} \pm iJ_y^{(N)}$, their adjoint action $\triangleright$ on the fuzzy spherical harmonics is given by

$$J_{\pm}^{(N)} \triangleright \hat{Y}_{lm}^{(N)} \equiv [J_{\pm}^{(N)}, \hat{Y}_{lm}^{(N)}] = \sqrt{(l \mp m)(l \pm m + 1)} \hat{Y}_{l, m \pm 1}^{(N)}, \quad (5.4)$$

$$J_z^{(N)} \triangleright \hat{Y}_{lm}^{(N)} \equiv [J_z^{(N)}, \hat{Y}_{lm}^{(N)}] = m \hat{Y}_{lm}^{(N)}. \quad (5.5)$$

Furthermore,

$$J^{(N)2} \triangleright \hat{Y}_{lm}^{(N)} = \left[J_+^{(N)}, \left[J_-^{(N)}, \hat{Y}_{lm}^{(N)} \right] \right] + \left[J_z^{(N)}, \left[J_z^{(N)}, \hat{Y}_{lm}^{(N)} \right] \right] \quad (5.6)$$

$$- \left[J_z^{(N)}, \hat{Y}_{lm}^{(N)} \right] = l(l+1) \hat{Y}_{lm}^{(N)} \equiv \Delta \hat{Y}_{lm}^{(N)}, \quad (5.7)$$

where we have introduced the fuzzy Laplacian Δ . This is the non-commutative analogue of the ordinary angular Laplacian and its eigenmatrices are the fuzzy harmonics. Its spectrum is truncated at $l = l_{\max} = 2j_N = N - 1$. Note that the operation $\triangleright$ is a derivation, that is the non-commutative analogue of a vector field.

The product of $\hat{Y}_{l_1 m_1}^{(N)}$ and $\hat{Y}_{l_2 m_2}^{(N)}$ can be expanded as a linear combination of $\hat{Y}_{lm}^{(N)}$ using 6j-symbols [15]

$$\hat{Y}_{l_1 m_1}^{(N)} \hat{Y}_{l_2 m_2}^{(N)} = \sum_{l=0}^{2j_N} (-1)^{2j_N+l} \sqrt{\frac{(2l_1+1)(2l_2+1)(2j_N+1)}{4\pi}} \begin{Bmatrix} l_1 & l_2 & l \\ j_N & j_N & j_N \end{Bmatrix} C_{l_1 m_1 l_2 m_2}^{lm} \hat{Y}_{lm}^{(N)}. \quad (5.8)$$

Notice that the 6j-symbols of (5.8) automatically vanish if the triangular conditions $|l_1 - l_2| < l < l_1 + l_2$ and $0 < l < 2j_N + 1$ are not satisfied. It means that l can assume values up to $l_{\max} = 2j_N = N - 1$, in contrast to what happens in the product of ordinary spherical harmonics (3.20). From the product above we can write the commutator

$$\begin{aligned} [\hat{Y}_{l_1 m_1}^{(N)}, \hat{Y}_{l_2 m_2}^{(N)}] &= \sum_{l=0}^{2j_N} (-1)^{2j_N+l} \sqrt{\frac{(2l_1+1)(2l_2+1)(2j_N+1)}{4\pi}} \\ &\quad \times \left\{ \begin{matrix} l_1 & l_2 & l \\ j_N & j_N & j_N \end{matrix} \right\} C_{l_1 m_1 l_2 m_2}^{lm} \hat{Y}_{lm}^{(N)} [1 - (-1)^{l_1+l_2-l}]. \end{aligned} \quad (5.9)$$

Using the product rule (5.8) and the asymptotic behaviour of the 6j symbols [38] for large values of N

$$\left\{ \begin{matrix} l_1 & l_2 & l \\ j_N & j_N & j_N \end{matrix} \right\} \approx \frac{(-1)^{2j+l}}{\sqrt{(2l+1)(2j_N+1)}} C_{l_1 0 l_2 0}^{l0}, \quad (5.10)$$

we have that

$$\lim_{N \rightarrow \infty} \Omega_N^{-1} \left(\hat{Y}_{l_1 m_1}^{(N)} \hat{Y}_{l_2 m_2}^{(N)} \right) = Y_{l_1 m_1}(\theta, \phi) Y_{l_2 m_2}(\theta, \phi).$$

and thus the commutator (5.9) vanishes in the large- N limit leading to the usual commutative algebra of spherical harmonics. On $M_N(\mathbb{C})$ we can introduce the following scalar product

$$\left(\hat{Y}_{l_1 m_1}^{(N)}, \hat{Y}_{l_2 m_2}^{(N)} \right)_{(N)} = \frac{4\pi}{N} \text{Tr} \left(\hat{Y}_{l_1 m_1}^{(N)\dagger} \hat{Y}_{l_2 m_2}^{(N)} \right) = \delta_{l_1 l_2} \delta_{m_1 m_2}. \quad (5.11)$$

Since there are $\sum_{l=0}^{2j_N} (2l+1) = N^2$ independent fuzzy spherical harmonics the set $\{\hat{Y}_{lm}^{(N)}\}$, equipped with (5.11) is a orthonormal basis in $M_N(\mathbb{C})$. Any element $\hat{f}^{(N)} \in M_N(\mathbb{C})$ can thus be expanded as

$$\hat{f}^{(N)} = \sum_{l=0}^{2j_N} \sum_{m=-l}^l \left(\hat{Y}_{lm}^{(N)\dagger}, \hat{f}^{(N)} \right)_{(N)} \hat{Y}_{lm}^{(N)}. \quad (5.12)$$

Again, note that this expansion is truncated at $l_{\max}$, in contrast to what happens in (3.18). The quantization map (5.1) can be extended by linearity to arbitrary functions of (θ, ϕ)

$$\Omega_N : f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l f_{lm} Y_{lm}(\theta, \phi) \rightarrow \hat{f}^{(N)} = \sum_{l=0}^{N-1} \sum_{m=-l}^l f_{lm} \hat{Y}_{lm}^{(N)}. \quad (5.13)$$

The set $C_N(S^2) \subset C(S^2)$ of truncated functions on the 2-sphere, i.e. the set of functions whose expansion in terms of the spherical harmonics includes only terms with $l < N$ as $f^{(N)}(\theta, \phi) = \sum_{l=0}^{2j_N} \sum_{m=-l}^l f_{lm} Y_{lm}(\theta, \phi)$ is a vector space, but not an algebra with the standard definition of pointwise product of two functions, since the product of two spherical harmonics of order say $N - 1$ has spherical components of order larger than $N - 1$, as remarked before. However, we can equip this vector space with a non-commutative $\star$ -product via the Weyl-Wigner map:

$$\left(f^{(N)} \star g^{(N)} \right) (\theta, \phi) = \sum_{l=0}^{2j_N} \sum_{m=-l}^l \left(\hat{Y}_{lm}^{(N)\dagger}, \hat{f}^{(N)} \hat{g}^{(N)} \right)_{(N)} Y_{lm}(\theta, \phi), \quad (5.14)$$

turning $C_N(S^2)$ into a non-commutative algebra. This non-commutative algebra of functions can be interpreted as functions on the fuzzy sphere. An important feature introduced by the non-commutativity is that we now have a cut-off on the allowed values of the angular momentum $l_{\max}$ in a way which is compatible with the multiplicative structure on the space of non-commutative spherical harmonics. In what follows we will see how the non-commutative deformation of spherical harmonics we just presented can be extended in order to include an action of the Lorentz algebra which, together with the new multiplicative structure, is compatible with the presence of a maximal allowed value of the angular momentum.

6 Fuzzy bipolar spherical harmonics

In order to construct a non-commutative generalization of angular mode functions which supports an action of the full Lorentz algebra we look at the finite dimensional representations of the latter. Every finite-dimensional irreducible representation of the Lorentz algebra with dimension $N = N_1 N_2$ can be constructed in terms of spin matrices as

$$L_i^{(N)} = J_i^{(N_1)} \otimes \mathbb{I}^{(N_2)} + \mathbb{I}^{(N_1)} \otimes J_i^{(N_2)}, \quad (6.1)$$

$$R_i^{(N)} = i \left(J_i^{(N_1)} \otimes \mathbb{I}^{(N_2)} - \mathbb{I}^{(N_1)} \otimes J_i^{(N_2)} \right). \quad (6.2)$$

It is easy to check that these matrices close the Lorentz Lie algebra (3.16). For both sets of spin matrices $J_i^{(N_1)}$ and $J_i^{(N_2)}$ we can construct their associated fuzzy spherical harmonics $\hat{Y}_{lm}^{(N_1)}$ and $\hat{Y}_{lm}^{(N_2)}$ which are $N_1 \times N_1$ and $N_2 \times N_2$ matrices, respectively, that satisfy all the properties discussed in the previous section. In particular, using (5.1), one can construct, for any fixed $l < N_i$ the complete set of fuzzy harmonics as

$$\hat{Y}_l^{(N_i)} \propto \left(J_+^{(N_i)} \right)^l, \quad (6.3)$$

which implies

$$\hat{Y}_{lm}^{(N_i)} \propto \left(J_-^{(N_i)} \right)^{l-m} \triangleright \hat{Y}_l^{(N_i)} \quad (6.4)$$

up to normalization factors. This is shown in appendix A. This procedure automatically stops when $l = N_i$ since $\left(J_+^{(N_i)} \right)^{N_i} = 0$ for the spin matrices. The most straightforward attempt at generalizing this procedure for the representation (6.1) would be thus to use the the generator L_+ in place of J_+ . However, the matrices constructed using this strategy do not provide a basis for $M_N(\mathbb{C})$. Indeed writing the n -th power of the generator L_+ as

$$\left(L_+^{(N)} \right)^n = \sum_{k=0}^n \binom{n}{k} \left(J_+^{(N_1)} \right)^{n-k} \otimes \left(J_+^{(N_2)} \right)^k. \quad (6.5)$$

Setting $n = N_1 + h$ in the above sum the terms with $k \leq h$ are always 0 because $\left(J_+^{(N_1)} \right)^{N_1} = 0$. The term with $k = h + 1$ is

$$\left(J_+^{(N_1)} \right)^{N_1-1} \otimes \left(J_+^{(N_2)} \right)^{h+1}$$

When $h+1 = N_2$ and hence $n = N_1 + N_2 - 1$ we have that $(L_+^{(N)})^n = 0$. We can thus only construct $\sum_{l=0}^{N_1+N_2-2} (2l+1) = (N_1 + N_2 - 1)^2$ independent matrices. But for $N_1, N_2 \neq 1$ we always have $N_1 + N_2 - 1 < N_1 N_2$, and hence we cannot construct a basis of $(N_1 N_2)^2$ matrices for the space of complex matrices $M_N(\mathbb{C})$.

A resolution of this problem is found if we notice that the first equation in (6.1) is just the statement that L is the sum of two angular momenta. From angular momentum theory, if we construct the matrices

$${}^{l_1 l_2} \hat{Y}_{LM}^{(N)} = \sum_{\substack{m_1 \\ m_2}} C_{l_1 m_1 l_2 m_2}^{LM} \hat{Y}_{l_1 m_1}^{(N_1)} \otimes \hat{Y}_{l_2 m_2}^{(N_2)}, \quad (6.6)$$

where $C_{l_1 m_1 l_2 m_2}^{LM}$ are the Clebsh-Gordan coefficients, we automatically have that

$$L_{\pm}^{(N)} \triangleright {}^{l_1 l_2} \hat{Y}_{LM}^{(N)} = \sqrt{(L \mp M)(L \pm M + 1)} {}^{l_1 l_2} \hat{Y}_{L, M \pm 1}^{(N)}, \quad (6.7)$$

$$L_z^{(N)} \triangleright {}^{l_1 l_2} \hat{Y}_{LM}^{(N)} = M {}^{l_1 l_2} \hat{Y}_{LM}^{(N)}, \quad (6.8)$$

$$L^{(N)2} \triangleright {}^{l_1 l_2} \hat{Y}_{LM}^{(N)} = L(L+1) {}^{l_1 l_2} \hat{Y}_{LM}^{(N)}. \quad (6.9)$$

The matrices ${}^{l_1 l_2} \hat{Y}_{LM}^{(N)}$ are irreducible tensors of rank L and are eigenmatrices of $J^{(N_1)2} \otimes \mathbb{I}^{(N_2)}$ and $\mathbb{I}^{(N_1)} \otimes J^{(N_2)2}$ with eigenvalues $l_1(l_1+1)$ and $l_2(l_2+1)$, respectively. The allowed values of the total angular momentum are $L = l_{\min}, \dots, l_{\max}$, and $M = m_1 + m_2 = -L, \dots, L$ with $l_{\min} = |l_1 - l_2|$ and $l_{\max} = l_1 + l_2$ as follows from the rules for the addition of two angular momenta. Note that, since $l_{1\max} = N_1 - 1$ and $l_{2\max} = N_2 - 1$ the value of L is never greater than $L_{\max} = N_1 + N_2 - 2$. The set $\{{}^{l_1 l_2} \hat{Y}_{LM}^{(N)}\}$ is an orthonormal basis in $M_N(\mathbb{C})$ with a scalar product analogous to the one of the fuzzy spherical harmonics, given by

$$\left({}^{l_1 l_2} \hat{Y}_{L_1 M_1}^{(N)}, {}^{l'_1 l'_2} \hat{Y}_{L_2 M_2}^{(N)} \right)_{(N)} = \frac{(4\pi)^2}{N} \text{Tr} \left({}^{l_1 l_2} \hat{Y}_{L_1 M_1}^{(N)\dagger} {}^{l'_1 l'_2} \hat{Y}_{L_2 M_2}^{(N)} \right) = \delta_{L_1 L_2} \delta_{M_1 M_2} \delta_{l_1 l'_1} \delta_{l_2 l'_2},$$

We would now like to obtain the explicit form for the action of boost generators on ${}^{l_1 l_2} \hat{Y}_{LM}^{(N)}$. Using the expression for $R_+^{(N)}$ (6.2) and ${}^{l_1 l_2} \hat{Y}_{LM}^{(N)}$ (6.6)

$$\begin{aligned} R_+^{(N)} \triangleright {}^{l_1 l_2} \hat{Y}_{LM}^{(N)} = i \sum_{\substack{m_1 \\ m_2}} C_{l_1 m_1 l_2 m_2}^{LM} \Big(\sqrt{(l_1 - m_1)(l_1 + m_1 + 1)} \hat{Y}_{l_1, m_1+1}^{(N_1)} \otimes \hat{Y}_{l_2 m_2}^{(N_2)} \\ - \sqrt{(l_2 - m_2)(l_2 + m_2 + 1)} \hat{Y}_{l_1 m_1}^{(N_1)} \otimes \hat{Y}_{l_2, m_2+1}^{(N_2)} \Big). \end{aligned}$$

Our goal is to express the right hand side of the action above as a linear combination of

the basis matrices $\{l_1 l_2 \hat{Y}_{LM}^{(N)}\}$. In order to do so one can evaluate the matrix elements

$$\begin{aligned} & \left(l'_1 l'_2 Y_{L'M'}^{(N)}, R_+^{(N)} \triangleright l_1 l_2 \hat{Y}_{LM}^{(N)} \right)_{(N)} \\ &= i \frac{(4\pi)^2}{N} \sum_{\substack{m_1 m'_1 \\ m_2 m'_2}} C_{l_1 m_1 l_2 m_2}^{LM} C_{l'_1 m'_1 l'_2 m'_2}^{L'M'} \\ & \quad \times \left[\mu_+(l_1, m_1) \text{Tr} \left(\hat{Y}_{l'_1 m'_1}^{(N_1)\dagger} \hat{Y}_{l_1, m_1+1}^{(N_1)} \right) \text{Tr} \left(\hat{Y}_{l'_2 m'_2}^{(N_2)\dagger} \hat{Y}_{l_2, m_2+1}^{(N_2)} \right) \right. \\ & \quad \left. - \mu_+(l_2, m_2) \text{Tr} \left(\hat{Y}_{l'_1 m'_1}^{(N_1)\dagger} \hat{Y}_{l_1, m_1}^{(N_1)} \right) \text{Tr} \left(\hat{Y}_{l'_2 m'_2}^{(N_2)\dagger} \hat{Y}_{l_2, m_2+1}^{(N_2)} \right) \right] \end{aligned} \quad (6.10)$$

$$= i \sum_{\substack{m_1 \\ m_2}} C_{l_1 m_1 l_2 m_2}^{LM} \left(\mu_+(l_1, m_1) C_{l'_1 m_1+1 l'_2 m_2}^{L'M'} - \mu_+(l_2, m_2) C_{l'_1 m_1 l'_2 m_2+1}^{L'M'} \right) \delta_{l_1 l'_1} \delta_{l_2 l'_2}, \quad (6.11)$$

where we used the shorthand notation

$$\mu_{\pm}(l, m) = \sqrt{(l \mp m)(l \pm m + 1)}.$$

Notice that these matrix elements are non-vanishing only if $l_i = l'_i$. The reader will find the details of the calculation in appendix B. The final expression for the action of the boost $R_+^{(N)}$ on our fuzzy harmonics is

$$\begin{aligned} & R_+^{(N)} \triangleright l_1 l_2 \hat{Y}_{LM}^{(N)} \\ &= \frac{i}{L} \sqrt{\frac{(L-M)(L-M-1)[L^2 - (l_{\min})^2][(l_{\max}+1)^2 - L^2]}{(4L^2-1)}} l_1 l_2 \hat{Y}_{L-1, M+1}^{(N)} \\ & \quad + i \frac{l_{\min}(l_{\max}+1)}{L(L+1)} \sqrt{(L-M)(L+M+1)} l_1 l_2 \hat{Y}_{L, M+1}^{(N)} \\ & \quad - \frac{i}{(L+1)} \sqrt{\frac{(L+M+1)(L+M+2)[(L+1)^2 - (l_{\min})^2][(l_{\max}+1)^2 - (L+1)^2]}{(2L+1)(2L+3)}} l_1 l_2 \hat{Y}_{L+1, M+1}^{(N)}. \end{aligned} \quad (6.12)$$

The action of $R_-^{(N)}$ and $R_z^{(N)}$ can be calculated similarly and are given by

$$\begin{aligned} & R_-^{(N)} \triangleright l_1 l_2 \hat{Y}_{LM}^{(N)} \\ &= -\frac{i}{L} \sqrt{\frac{(L+M)(L+M-1)[L^2 - (l_{\min})^2][(l_{\max}+1)^2 - L^2]}{(4L^2-1)}} l_1 l_2 \hat{Y}_{L-1, M-1}^{(N)} \\ & \quad + i \frac{l_{\min}(l_{\max}+1)}{L(L+1)} \sqrt{(L+M)(L-M+1)} l_1 l_2 \hat{Y}_{L, M-1}^{(N)} \\ & \quad + \frac{i}{(L+1)} \sqrt{\frac{(L-M+1)(L-M+2)[(L+1)^2 - (l_{\min})^2][(l_{\max}+1)^2 - (L+1)^2]}{(2L+1)(2L+3)}} l_1 l_2 \hat{Y}_{L+1, M-1}^{(N)}, \end{aligned} \quad (6.13)$$

and

$$\begin{aligned} & R_z^{(N)} \triangleright l_1 l_2 \hat{Y}_{LM}^{(N)} \\ &= \frac{i}{L} \sqrt{\frac{(L+M)(L-M)[L^2 - (l_{\min})^2][(l_{\max}+1)^2 - L^2]}{(4L^2-1)}} l_1 l_2 \hat{Y}_{L-1, M}^{(N)} \\ & \quad + i M \frac{l_{\min}(l_{\max}+1)}{L(L+1)} l_1 l_2 \hat{Y}_{LM}^{(N)} \\ & \quad + \frac{i}{(L+1)} \sqrt{\frac{(L+M+1)(L-M+1)[(L+1)^2 - (l_{\min})^2][(l_{\max}+1)^2 - (L+1)^2]}{(2L+1)(2L+3)}} l_1 l_2 \hat{Y}_{L+1, M}^{(N)}. \end{aligned} \quad (6.14)$$

While these results might appear at first sight not very illuminating they are in fact remarkable. Indeed, unlike the case of commutative spherical harmonics on the celestial sphere, we now have a *maximum value* of the angular momentum L . Moreover the coefficients of the ${}^{l_1 l_2} \hat{Y}_{L+1M+q}^{(N)}$ terms automatically vanish if L equals $l_{\max}$ and thus the action of boosts, which in the standard case always maps the harmonic with given l to one with $l+1$, is now compatible with the existence of a cut-off in the value of L . Thus the actions (6.7), (6.8), (6.9) and (6.12), (6.13), (6.14) could be thought as the non-commutative analogue of (3.27)–(3.32).

As a final step let us write explicitly the algebra of the matrices (6.6). Using the summation rule [15]

$$\sum_{\beta\gamma\epsilon\varphi} C_{b\beta c\gamma}^{a\alpha} C_{ee f\varphi}^{d\delta} C_{ee b\beta}^{g\eta} C_{f\varphi c\gamma}^{j\mu} = \Pi_{adj} \sum_{\rho\sigma} C_{g\eta j\mu}^{\rho\sigma} C_{d\delta a\alpha}^{\rho\sigma} \begin{Bmatrix} c & b & a \\ f & e & d \\ j & g & k \end{Bmatrix},$$

where $\Pi_{ab\dots c} = \sqrt{(2a+1)(2b+1)\dots(2c+1)}$ and $\begin{Bmatrix} c & b & a \\ f & e & d \\ j & g & k \end{Bmatrix}$ are 9j-symbols one finds that such product is given by

$$\begin{aligned} & {}^{l'_1 l'_2} \hat{Y}_{L'M'}^{(N)} {}^{l''_1 l''_2} \hat{Y}_{L''M''}^{(N)} \\ &= \sum_{\substack{LM \\ l_1 l_2}} \frac{\sqrt{N}}{4\pi} \sqrt{(2l_1+1)(2l'_1+1)(2l''_1+1)(2L'+1)(2l_2+1)(2l'_2+1)(2l''_2+1)(2L''+1)} \\ &\quad \times \begin{Bmatrix} l'_1 & l''_1 & l_1 \\ j_{N_1} & j_{N_1} & j_{N_1} \end{Bmatrix} \begin{Bmatrix} l'_2 & l''_2 & l_2 \\ j_{N_2} & j_{N_2} & j_{N_2} \end{Bmatrix} (-1)^{2j_{N_1}+2j_{N_2}+l_1+l_2} C_{L'M'L''M''}^{LM} \begin{Bmatrix} l'_1 & l'_2 & L' \\ l''_1 & l''_2 & L'' \\ l_1 & l_2 & L \end{Bmatrix} {}^{l_1 l_2} \hat{Y}_{LM}^{(N)}. \end{aligned} \quad (6.15)$$

For large values of $N = N_1 N_2$ we have

$$\begin{Bmatrix} l'_1 & l''_1 & l_1 \\ j_{N_1} & j_{N_1} & j_{N_1} \end{Bmatrix} \begin{Bmatrix} l'_2 & l''_2 & l_2 \\ j_{N_2} & j_{N_2} & j_{N_2} \end{Bmatrix} \approx \frac{(-1)^{j_{N_1}+j_{N_2}+l_1+l_2}}{\sqrt{N(2l_1+1)(2l_2+1)}} C_{l'_1 0 l''_1 0}^{l_1 0} C_{l'_2 0 l''_2 0}^{l_2 0}, \quad (6.16)$$

so that the algebra becomes

$$\begin{aligned} & {}^{l'_1 l'_2} \hat{Y}_{L'M'}^{(N)} {}^{l''_1 l''_2} \hat{Y}_{L''M''}^{(N)} \approx \sum_{\substack{LM \\ l_1 l_2}} \frac{1}{4\pi} \sqrt{(2l'_1+1)(2l''_1+1)(2L'+1)(2l'_2+1)(2l''_2+1)(2L''+1)} \\ &\quad \times C_{l'_1 0 l''_1 0}^{l_1 0} C_{l'_2 0 l''_2 0}^{l_2 0} C_{L'M'L''M''}^{LM} \begin{Bmatrix} l'_1 & l'_2 & L' \\ l''_1 & l''_2 & L'' \\ l_1 & l_2 & L \end{Bmatrix} {}^{l_1 l_2} \hat{Y}_{LM}^{(N)}, \end{aligned} \quad (6.17)$$

which is exactly the algebra closed by the bipolar spherical harmonics (see e.g. [15]), as one would expect. For these reason, the matrices of (6.7) can be thought of as *fuzzy bipolar spherical harmonics*. The ordinary bipolar spherical harmonics form a basis in the algebra of functions on the manifold $S^2 \times S^2$ and hence the fuzzy bipolar spherical

harmonics of (6.7) can be thought as a realization of a non-commutative $S^2 \times S^2$ space. Their commutator is given by

$$\begin{aligned}
& \left[l'_1 l'_2 \hat{Y}_{L'M'}^{(N)}, l''_1 l''_2 \hat{Y}_{L''M''}^{(N)} \right] \\
&= \sum_{\substack{LM \\ l_1 l_2}} \frac{\sqrt{N}}{4\pi} \sqrt{(2l_1+1)(2l'_1+1)(2l''_1+1)(2L'+1)(2l_2+1)(2l'_2+1)(2l''_2+1)(2L''+1)} \\
&\quad \times \left\{ \begin{matrix} l'_1 & l''_1 & l_1 \\ j_{N_1} & j_{N_1} & j_{N_1} \end{matrix} \right\} \left\{ \begin{matrix} l'_2 & l''_2 & l_2 \\ j_{N_2} & j_{N_2} & j_{N_2} \end{matrix} \right\} (-1)^{2j_{N_1}+2j_{N_2}+l_1+l_2} C_{L'M'L''M''}^{LM} \left\{ \begin{matrix} l'_1 & l'_2 & L' \\ l''_1 & l''_2 & L'' \\ l_1 & l_2 & L \end{matrix} \right\} \\
&\quad \times [1 - (-1)^{l_1+l_2+l'_1+l'_2+l''_1+l''_2}] l_1 l_2 \hat{Y}_{LM}^{(N)}. \tag{6.18}
\end{aligned}$$

These equations define our non-commutative algebra of fuzzy bipolar spherical harmonics.

7 Conclusions and outlook

We have shown how the algebra of spherical harmonics on the celestial sphere can be generalized to a non-commutative algebra in order to accommodate a maximal value of the angular momentum. In particular, we derived an action of Lorentz boosts which is consistent with the existence of a maximal angular momentum. Our construction is based on a matrix realization of angular mode functions and uses basic techniques of non-commutative geometry. These results suggest that, since the generators of supertranslations of the BMS group are proportional to the spherical harmonics on the celestial sphere, it could be possible to construct a generalization of the BMS algebra¹ characterized by a *non-abelian* sub-algebra of supertranslations having a *finite* number of generators. These would give a finite number of conserved supertranslation charges and thus non-commutativity, or the fuzziness of the angular mode functions, could be the ingredient needed to provide a consistent cut-off mechanism for soft modes. It is tempting to speculate that a similar mechanism could be used to provide the missing link between soft hair and the Bekenstein-Hawking entropy for black holes.

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A Construction of the fuzzy spherical harmonics

An explicit way to construct the fuzzy spherical harmonics can be done by using directly the Weyl-Wigner map of (5.2) and equation (5.4). The scalar product $\mathbf{J}^{(N)} \cdot \nabla$ in spherical components reads as

$$\mathbf{J}^{(N)} \cdot \nabla = J_i^{(N)} \nabla_i = -J_{+1}^{(N)} \nabla_- + J_0^{(N)} \nabla_0 - J_{-1}^{(N)} \nabla_+, \tag{A.1}$$

¹For a recent attempt at generalizing the BMS algebra using quantum group techniques see [39].

where the spherical components of a vector $\mathbf{A}$ are defined as usual, $A_{\pm 1} = \mp \frac{1}{\sqrt{2}}(A_x \pm iA_y)$ and $A_0 = A_z$. Hence, the contact with the notation we used previously is

$$J_{+1}^{(N)} = -\frac{1}{\sqrt{2}}J_+^{(N)}, \quad J_{-1}^{(N)} = \frac{1}{\sqrt{2}}J_-^{(N)}, \quad J_0^{(N)} = J_z^{(N)}. \quad (\text{A.2})$$

Furthermore the followig identities hold [15]

$$\nabla_0[r^l Y_{lm}(\theta, \phi)] = \sqrt{\frac{l^2 - m^2}{(2l+1)(2l-1)}}(2l+1)r^{l-1}Y_{l-1m}(\theta, \phi), \quad (\text{A.3})$$

$$\nabla_{\pm 1}[r^l Y_{lm}(\theta, \phi)] = -\sqrt{\frac{(l \mp m - 1)(l \mp m)}{2(2l-1)(2l+1)}}(2l+1)r^{l-1}Y_{l-1m-1}(\theta, \phi). \quad (\text{A.4})$$

Suppose we want to construct $\hat{Y}_l^{(N)}$. We must apply (A.1) l times to $Y_{lm}(\theta, \phi)$. The first time we apply it, only the term proportional to ∇_- in (A.1) contributes, producing $-\sqrt{\frac{2l(2l+1)}{2}}r^{l-1}Y_{l-1m-1}(\theta, \phi)$. In general, everytime we apply the operator (A.1) only the term proportional to ∇_- will contribute. Acting n times we have

$$(-J_{+1}^{(N)}\nabla_-)^n(r^l Y_{lm}(\theta, \phi)) = \left(J_{+1}^{(N)}\right)^n \sqrt{\frac{(2l+1)(2l)\dots(2l-2n+2)}{2^n}}r^{l-n}Y_{l-n,m-n}(\theta, \phi). \quad (\text{A.5})$$

For $n = l$ we get, for $\hat{Y}_l^{(N)}$

$$\hat{Y}_l^{(N)} = \frac{2^l}{l!} \left[\frac{N(N-1-l)!}{(N+l)!} \frac{(2l+1)2l(2l-2)\dots 2}{2^l} \right]^{\frac{1}{2}} \frac{1}{2\sqrt{\pi}} \left(J_{+1}^{(N)}\right)^l \propto \left(J_+^{(N)}\right)^l, \quad (\text{A.6})$$

as claimed in (6.3). By acting on $Y_{lm}^{(N)}$ with the lowering operator $J_-^{(N)}$ it is possible to construct all the $2l+1$ fuzzy spherical harmonics at fixed l .

B Derivation of the action of Lorentz boosts on bipolar fuzzy spherical harmonics

From the following recursion formula for the Clebsch-Gordan coefficients

$$\mu_-(L', M')C_{l_1 m_1 l_2 m'_2}^{L' M' -1} = \mu_-(l_1, m_1 + 1)C_{l_1 m_1 + 1 l_2 m_2}^{L' M'} + \mu_-(l_2, m_2 + 1)C_{l_1 m_1 l_2 m_2 + 1}^{L' M'},$$

we have that

$$C_{l_1 m_1 + 1 l_2 m_2}^{L' M'} = \frac{\mu_-(L', M')}{\mu_-(l_1, m_1 + 1)}C_{l_1 m_1 l_2 m'_2}^{L' M' -1} - \frac{\mu_-(l_2, m_2 + 1)}{\mu_-(l_1, m_1 + 1)}C_{l_1 m_1 l_2 m_2 + 1}^{L' M'}.$$

Plugging this expression in (6.10) we obtain

$$\begin{aligned} & \left(t_{1l_2} Y_{L'M'}^{(N)}, R_+^{(N)} \triangleright^{l_1 l_2} \hat{Y}_{LM}^{(N)} \right)_{(N)} \\ &= i \sum_{\substack{m_1 \\ m_2}} C_{l_1 m_1 l_2 m_2}^{LM} \left[\frac{\mu_+(l_1, m_1) \mu_-(L', M')}{\mu_-(l_1, m_1 + 1)} C_{l_1 m_1 l_2 m'_2}^{L'M'-1} \right. \\ & \quad \left. - \left(\frac{\mu_+(l_1, m_1) \mu_-(l_2, m_2 + 1)}{\mu_-(l_1, m_1 + 1)} + \mu_+(l_2, m_2) \right) C_{l_1 m_1 l_2 m_2 + 1}^{L'M'} \right] \end{aligned} \quad (\text{B.1})$$

$$= i \sum_{\substack{m_1 \\ m_2}} \left[\mu_-(L', M') C_{l_1 m_1 l_2 m_2}^{LM} C_{l_1 m_1 l_2 m'_2}^{L'M'-1} - 2\mu_+(l_2, m_2) C_{l_1 m_1 l_2 m_2}^{LM} C_{l_1 m_1 l_2 m_2 + 1}^{L'M'} \right], \quad (\text{B.2})$$

where we have used

$$\mu_+(l, m) = \mu_-(l, m + 1).$$

From the orthogonality of the Clebsh-Gordan coefficients

$$\sum_{\substack{m_1 \\ m_2}} C_{l_1 m_1 l_2 m_2}^{LM} C_{l_1 m_1 l_2 m_2}^{L'M'} = \delta_{LL'} \delta_{MM'},$$

we have that the first term of the matrix element (B.1) is

$$i\mu_-(L', M') \delta_{LL'} \delta_{M'M+1}. \quad (\text{B.3})$$

For the second term

$$-2i \sum_{\substack{m_1 \\ m_2}} \sqrt{(l_2 - m_2)(l_2 + m_2 + 1)} C_{l_1 m_1 l_2 m_2}^{LM} C_{l_1 m_1 l_2 m_2 + 1}^{L'M'}, \quad (\text{B.4})$$

we use the following recursion relation [15]

$$\begin{aligned} & \sqrt{(l_2 - m_2)(l_2 + m_2 + 1)} C_{l_1 m_1 l_2 m_2}^{LM} \\ &= -\frac{1}{2L} \sqrt{\frac{(L-M)(L-M-1)[L^2 - (l_2 - l_1)^2][(l_2 + l_1 + 1)^2 - L^2]}{(4L^2 - 1)}} C_{l_1 m_1 l_2 m_2 + 1}^{L-1, M+1} \\ & \quad + \frac{1}{2L(L+1)} [(l_2(l_2+1) - l_1(l_1+1) + L(L+1))] \sqrt{(L-M)(L+M+1)} C_{l_1 m_1 l_2 m_2 + 1}^{LM+1} \\ & \quad + \frac{1}{2(L+1)} \sqrt{\frac{(L+M+1)(L+M+2)[(L+1)^2 - (l_2 - l_1)^2][(l_1 + l_2)^2 - L^2 + 2(l_1 + l_2 - L)]}{(2L+1)(2L+3)}} \\ & \quad \times C_{l_1 m_1 l_2 m_2 + 1}^{L+1, M+1}. \end{aligned}$$

Substituting and using again the orthogonality condition for the Clebsh-Gordan coefficients we have that the second term in (B.4) can be written as

$$\begin{aligned} & \frac{i}{L} \sqrt{\frac{(L-M)(L-M-1)[L^2 - (l_2 - l_1)^2][(l_2 + l_1 + 1)^2 - L^2]}{(4L^2 - 1)}} \delta_{L', L-1} \delta_{M', M+1} \\ & \quad - \frac{i}{L(L+1)} [(l_2(l_2+1) - l_1(l_1+1) + L(L+1))] \sqrt{(L-M)(L+M+1)} \delta_{L', L} \delta_{M', M+1} \\ & \quad - \frac{i}{(L+1)} \sqrt{\frac{(L+M+1)(L+M+2)[(L+1)^2 - (l_2 - l_1)^2][(l_1 + l_2)^2 - L^2 + 2(l_1 + l_2 - L)]}{(2L+1)(2L+3)}} \\ & \quad \times \delta_{L', L+1} \delta_{M', M+1}. \end{aligned}$$

The term proportional to $\delta_{L',L}\delta_{M',M+1}$ in the previous expression, together with (B.3), can be written as

$$\begin{aligned} & i \frac{[l_1(l_1+1) - l_2(l_2+1)]}{L(L+1)} \sqrt{(L-M)(L+M+1)} \\ &= i \frac{(l_1-l_2)(l_1+l_2+1)}{L(L+1)} \sqrt{(L-M)(L+M+1)} \\ &= \pm i \frac{l_{\min}(l_{\max}+1)}{L(L+1)} \sqrt{(L-M)(L+M+1)}. \end{aligned} \quad (\text{B.5})$$

Using similar procedures it is possible to obtain the action of $R_-^{(N)}$ and of $R_z^{(N)}$ on ${}^{l_1 l_2} \hat{Y}_{LM}^{(N)}$.

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