A minimalistic pure spinor sigma-model in AdS

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Abstract: The b-ghost of the pure spinor formalism in a general curved background is not holomorphic. For such theories, the construction of the string measure requires the knowledge of the action of diffeomorphisms on the BV phase space. We construct such an action for the pure spinor sigma-model in $AdS_5 \times S^5$. From the point of view of the BV formalism, this sigma-model belongs to the class of theories where the expansion of the Master Action in antifields terminates at the quadratic order. We show that it can be reduced to a simpler degenerate sigma-model, preserving the AdS symmetries. We construct the action of the algebra of worldsheet vector fields on the BV phase space of this minimalistic sigma-model, and explain how to lift it to the original model.

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Contents

1 Introduction

The *b*-ghost of the pure spinor formalism in a general curved background is only holomorphic up to a Q-exact expression [1]. The construction of the string measure for such theories was suggested in [2, 3]. It requires the knowledge of the action of the group of worldsheet diffeomorphisms on the BV phase space. For a vector field ξ on the worldsheet (= infinitesimal diffeomorphism) let Φ_ξ be the BV Hamiltonian generating the action of ξ on the BV phase space. Then, the string measure is, schematically:

$$\exp\left(S_{\mathrm{BV}} + \sigma + \Phi_F\right) \tag{1.1}$$

where:

- S_{BV} is the worldsheet Master Action

- σ is the generating function of the variations of the Lagrangian submanifold (for the standard choice of the family, this is just the usual $\int \mu_{\bar z}^z b_{zz} + \mu_z^{\bar z} b_{\bar z \bar z}$)

- F is the curvature of the connection on the equivalence class of worldsheet theories, considered as a principal bundle over the space of theories modulo diffeomorphisms

It is not completely trivial to construct Φ_ξ for the pure spinor superstring in AdS. One of the complications is the somewhat unusual form of the pure spinor part of the action. Schematically:

$$S_{\lambda w} = \int w_{L-}(\partial_- + A_-)\lambda_L \ + \ w_{R-}(\partial_+ + A_+)\lambda_R \ + \ S w_{L+}\lambda_L w_{R-}\lambda_R \tag{1.2}$$

where S is a linear combination of Ramond-Ramond field strengths. Notice that the conjugate momenta w_L and w_R only enter through their $(1,0)$ and $(0,1)$ component, respectively. We can try to integrate out w, ending up with a "standard" kinetic term for ghosts:

$$\frac{(\partial_- + A_-)\lambda_L \ (\partial_+ + A_+)\lambda_R}{S \lambda_L \lambda_R} \tag{1.3}$$

Notice that S landed in the denominator. It would seem that the theory depends quite irregularly on the Ramond-Ramond field, but this is not true. All physics sits at $\lambda = 0$, and the $w\lambda w\lambda$ term is in some sense subleading.

In this paper we will show, closely following [4, 5], that the pure spinor terms (1.2) can actually be removed by reduction to a smaller BV phase space, keeping intact all the symmetries of $AdS_5 \times S^5$. The resulting action is degenerate, and therefore can not be immediately used for quantization. On the other hand, it is simpler than the original action. In particular, the action of worldsheet diffeomorphisms in this reduced BV phase space is rather transparent, although the explicit expression eq. (4.25) is somewhat involved. We then explain how to lift this action to an action on some quantizable theory which is basically the same as the original pure spinor sigma-model of [7].

For the case of flat spacetime, the formal expressions are somewhat more complicated. The construction of the action of diffeomorphisms is a work in progress with Renann Lipinski [6].

Formal application of BV formalism. Here, as in [3], we formally apply the formalism of odd symplectic manifolds in the infinite-dimensional case (the field space of two-dimensional sigma-models). This should be proven in perturbation theory, but in this paper we restrict ourselves with purely formal manipulations. We believe that supersymmetry will play crucial role in controlling quantum anomalies; therefore it is important that our constructions preserve supersymmetries (see section 4.1).

Plan of the paper. We begin in section 2 with the general discussion of the reduction procedure when a BV Master Action is a quadratic-linear functional of antifields. In section 3 we apply this to the case of pure spinor superstring in $AdS_5 \times S^5$. In sections 4 we construct the action of diffeomorphisms in the minimalistic sigma-model. Then in section 5 we construct the action of diffeomorphisms on the BV phase space of the non-degenerate theory, which is essentially equivalent (quasiisomorphic) to the original sigma-model. Sections 6 and 7 contain summary and generalizations, and section 8 open problems.

2 Master Actions quadratic-linear in antifields

Suppose that the BV phase space is an odd cotangent bundle, i.e. is of the form $\Pi T^* N$ for some supermanifold N (the "field space"). If ϕ^a are coordinates on N, then $\phi_a^\star$ are coordinates on $\Pi T^* N$, and "Π" means that the statistics of $\phi_a^\star$ is opposite to the statistics of ϕ^a. There is an odd Poisson bracket (the "BV bracket"):

$$\{\phi_a^\star, \phi^b\} = \delta_a^b \tag{2.1}$$

This bracket is geometrically well-defined, in a sense that the bracket of two functions $\{F, G\}$ is actually independent of how the coordinates ϕ^a on N are choosen. Equivalently, there is an odd symplectic form[1] (which, as any differential form, can be considered a function on $\Pi T(\Pi T^* N)$):

$$\omega_{\mathrm{BV}} = \sum_a (-1)^{\bar a} d\phi^a \, d\phi_a^\star \tag{2.2}$$

(As a slight overuse of Einstein notations, we will omit the summation sign Σ_a in such cases.) Suppose that the Master Action is of the form:

$$S_{\mathrm{BV}} \;=\; S_{\mathrm{cl}}(\phi) + Q^a(\phi)\phi_a^\star + \frac{1}{2}\phi_a^\star \pi^{ab}(\phi)\phi_b^\star \tag{2.3}$$

(writing $\phi_a^\star \pi^{ab}(\phi)\phi_b^\star$ rather than $\pi^{ab}(\phi)\phi_a^\star \phi_b^\star$ simplifies some signs later).

We will assume that S_{BV} satisfies the *classical Master Equation*:

$$\{S_{\mathrm{BV}}, S_{\mathrm{BV}}\} = 0 \tag{2.4}$$

If N is purely even, we can think of functions on $\Pi T^* N$ as polyvector fields on N. For example, $Q^a(\phi)\phi_a^\star$ corresponds to the vector field $Q^a(\phi)\frac{\partial}{\partial \phi^a}$, and $\frac{1}{2}\pi^{ab}(\phi)\phi_a^\star \phi_b^\star$ corresponds to a Poisson bivector $\pi^{ab}(\phi)\frac{\partial}{\partial \phi^a} \wedge \frac{\partial}{\partial \phi^b}$. The odd Poisson bracket $\{\_,\_\}$ corresponds to the Schouten bracket of polyvector fields.

If N is a *super*-manifold, then this polyvector picture does not seem to be very illuminating. However, one can still apply the intuition of Hamiltonian mechanics. The linear function $Q = Q^a(\phi)\phi_a^\star$ still defines a vector field; the derivative of a function $f \in C^\infty(N)$ along it is: $\{Q^a(\phi)\phi_a^\star, f(\phi)\} = Q^a\partial_a f$. The quadratic function $\pi = \frac{1}{2}\phi_a^\star\pi^{ab}(\phi)\phi_b^\star$ still defines a map from functions on N to vector fields on N:

$$f \mapsto \{\pi, f\} \tag{2.5}$$

The Master Equation (2.4) implies, order by order in expansion in $\phi^\star$:

$$\{Q, S_{\mathrm{cl}}\} = 0 \tag{2.6}$$

$$\{Q, Q\} + 2\{\pi, S_{\mathrm{cl}}\} = 0 \tag{2.7}$$

$$\{Q, \pi\} = 0 \tag{2.8}$$

$$\{\pi, \pi\} = 0 \tag{2.9}$$

It follows from eq. (2.9) that vector fields of the form $\{\pi, f\}$, $f \in C^\infty(N)$, form a closed subalgebra in the algebra of vector fields. They are all tangent to a family of submanifolds of N which can be called "symplectic leaves of π". As a slight abuse of notations, the letter Q will denote both the BRST transformation $Q^a\partial_a$ and the function $Q^a\phi_a^\star$ on ΠT^*N. Eq. (2.7) says that generally speaking the BRST operator Q is only nilpotent on-shell [8].

We will show that under some conditions, this theory can be reduced to a simpler theory which has BRST operator nilpotent off-shell (and therefore its Master Action has no quadratic terms $\phi^\star\phi^\star$).

The case when π is non-degenerate. Let us first consider the case when the Poisson bivector π^{ab} is nondegenerate. Eq. (2.8) implies that an odd function $\psi \in \mathrm{Fun}(N)$ locally exists, such that $Q = \{\pi, \psi\}$. Suppose that ψ is also defined globally. Let us consider the canonical transformation of the Darboux coordinates generated by ψ:

$$\begin{aligned}
(\phi, \phi^\star) &\to (\tilde{\phi}, \tilde{\phi}^\star) \\
\phi^a &= \tilde{\phi}^a \\
\phi_a^\star &= \tilde{\phi}_a^\star + \frac{\partial}{\partial\tilde{\phi}^a}\psi(\tilde{\phi})
\end{aligned} \tag{2.10}$$

More geometrically: $\tilde{\phi}$ and $\tilde{\phi}^\star$ (functions on ΠT^*N) are pullbacks of ϕ and $\phi^\star$ by the flux of the Hamiltonian vector field $\{\psi, \_\}$ by the time 1. (The flux integrates to eqs. (2.10) because ψ only depends on ϕ, and therefore the velocity of $\phi^\star$ is $\phi^\star$-independent.)

In the new coordinates:

$$S = \widetilde{S}_{\mathrm{cl}} + \frac{1}{2}\phi_a^\star\pi^{ab}(\phi)\phi_b^\star \tag{2.11}$$

$$\text{where} \qquad \widetilde{S}_{\mathrm{cl}} = S_{\mathrm{cl}} + \frac{1}{2}\partial_a\psi\pi^{ab}\partial_b\psi \tag{2.12}$$

The $\phi^\star$-linear term is gone! The Master Equation implies that $\{\widetilde{S}_{\mathrm{cl}}, \pi\} = 0$. Since we assumed that π is nondegenerate, this implies:

$$\widetilde{S}_{\mathrm{cl}} = \mathrm{const.} \tag{2.13}$$

The case of degenerate π. We are actually interested in the case when π is degenerate. Let $\mathcal{P} \subset TN$ be the distribution tangent to symplectic leaves of π:

$$\mathcal{P} = \operatorname{im} \pi \subset TN \tag{2.14}$$

This distribution is integrable because π satisfies the Jacobi identity. We also assume that Q is transverse to $\mathcal{P}$:

$$Q \notin \mathcal{P} \tag{2.15}$$

Let us also consider the distribution $\mathcal{P} + \mathcal{Q}$ which is generated by elements of $\mathcal{P}$ and by Q. Eqs. (2.8) and (2.9) imply that $\mathcal{P} + \mathcal{Q}$ is also integrable. Let us *assume* the existence of a 2-form[2] ω on each integrable surface[3] of $\mathcal{P} + \mathcal{Q}$ and a function $\psi \in \operatorname{Fun}(N)$ which satisfy:

$$\pi \omega \pi = \pi \tag{2.16}$$

$$\omega \pi \omega = \omega \tag{2.17}$$

$$d\omega|_{\mathcal{P}+\mathcal{Q}} = 0 \tag{2.18}$$

$$(\iota_Q \omega - d\psi)|_{\mathcal{P}} = 0 \tag{2.19}$$

where $\pi \omega \pi$ and $\omega \pi \omega$ are defined as follows:

$$\phi_a^\star (\pi \omega \pi)^{ab} \phi_b^\star = \phi_a^\star \pi^{aa'} \omega_{a'b'} \pi^{b'b} \phi_b^\star \tag{2.20}$$

$$d\phi^a (\omega \pi \omega)_{ab} d\phi^b = d\phi^a \omega_{aa'} \pi^{a'b'} \omega_{b'b} d\phi^b \tag{2.21}$$

Existence of ψ satisfying eq. (2.19) locally follows from eqs. (2.16) and (2.18), because they imply $d(\iota_Q \omega)|_{\mathcal{P}} = 0$. But we also require this ψ to be a *globally* well-defined function on N. Contracting $\iota_Q \omega - d\psi$ with $\pi \omega$ we find that:

$$Q - \{\pi, \psi\} \in \ker (\omega|_{\mathcal{P}+\mathcal{Q}}) \tag{2.22}$$

Let us define the new odd vector field:

$$\widetilde{Q} = Q - \{\pi, \psi\} \tag{2.23}$$

Eq. (2.18) implies that $\ker (\omega|_{\mathcal{P}+\mathcal{Q}})$ is an integrable distribution inside an integral surface of $\mathcal{P} + \mathcal{Q}$. Therefore eq. (2.22) implies that $\widetilde{Q}^2$ is proportional to $\widetilde{Q}$, i.e. there exists a function ζ such that: $\widetilde{Q}^2 = \zeta \widetilde{Q}$. In fact $\zeta = 0$, since $\widetilde{Q}^2 \in \mathcal{P}$ and $\widetilde{Q} \notin \mathcal{P}$. We conclude:

$$\widetilde{Q}^2 = 0 \tag{2.24}$$

Let us consider the canonical transformation (2.10) of Darboux coordinates generated by ψ. With these new Darboux coordinates:

$$S_{\mathrm{BV}} = S_{\mathrm{cl}} - \frac{1}{2}\omega(Q, Q) + (Q - \{\pi, \psi\})^a \tilde{\phi}_a^\star + \frac{1}{2}\tilde{\phi}_a^\star \pi^{ab} \tilde{\phi}_b^\star \tag{2.25}$$

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[2]This ω is *even*; it should not be confused with the odd symplectic form of ΠT^*N.

[3]It is enough to define ω on each integrable surface of $\mathcal{P} + \mathcal{Q}$; it does not have to be defined on the whole N.

Notice that the new "classical action":

$$\tilde{S}_{\mathrm{cl}} = S_{\mathrm{cl}} - \frac{1}{2}\omega(Q,Q) \tag{2.26}$$

is automatically constant on symplectic leaves of π. Also, it follows that $\widetilde{Q}$ consistently defines an odd nilpotent vector field on the moduli space of symplectic leaves of π. These facts follow from $\{S_{\mathrm{BV}}, S_{\mathrm{BV}}\} = 0$. To summarize:

$$S_{\mathrm{BV}} = \tilde{S}_{\mathrm{BV}} + \frac{1}{2}\tilde{\phi}_a^\star \pi^{ab}(\phi)\tilde{\phi}_b^\star \tag{2.27}$$

$$\text{where} \qquad \tilde{S}_{\mathrm{BV}} = \tilde{S}_{\mathrm{cl}}(\chi) + \tilde{Q}(\chi)^m \chi_m^\star \tag{2.28}$$

where χ is coordinates on the space of symplectic leaves of π. We therefore constructed a new, simpler theory, on the space of symplectic leaves of π.

This theory can be interpreted as the result of integrating out[4] some antifields. More precisely, let us define a submanifold $N_0 \subset N$ by picking one point from each symplectic leaf. Fibers of the odd conormal bundle[5,6] $\Pi T^* N_0$ are isotropic submanifolds in $\Pi T^* N$, and we can integrate them out as described in [3]. In this paper the coordinates in these fibers will be called $w^\star$ (and integrated out).

Oversimplified example. We will now illustrate the relation by a toy sigma-model (we will actually run the procedure "in reverse"). Let Σ be a two-dimensional worldsheet. Let us start with:

$$S_{\mathrm{BV}} = S_{\mathrm{cl}} + \int_\Sigma \lambda\theta^\star \tag{2.29}$$

where S_{cl} does not depend neither on the fermionic field θ^a nor on the bosonic field λ^a. (It depends on some other fields ϕ^μ.) We postulate the odd symplectic form so that our fields are Darboux coordinates[7] [3], as in eq. (2.2):

$$\omega_{\mathrm{BV}} = \int_\Sigma d\lambda^\star d\lambda - d\theta^\star d\theta + \sum_{\mu \in \left\{ \substack{\text{other} \\ \text{fields}} \right\}} (-1)^{\bar{\mu}} d\phi_\mu^\star d\phi^\mu \tag{2.30}$$

This action is highly degenerate; the path integral $\int [d\lambda][d\theta][d\phi] e^{S_{\mathrm{cl}}(\phi)}$ is undefined (infinity from integrating over λ times zero from integrating over θ). To regularize $\infty \times 0$, let us introduce a new field-antifield pair $w, w^\star$, where w is a bosonic 1-form on the worldsheet and $w^\star$ is a fermionic 1-form on the worldsheet:

$$w = w_+ dz + w_- d\bar{z} \tag{2.31}$$

$$w^\star = w_+^\star dz + w_-^\star d\bar{z} \tag{2.32}$$

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[4]http://andreimikhailov.com/math/bv/transfer/Partial_Integration.html

[5]http://andreimikhailov.com/math/bv/BRST-formalism/Family_of_Lagrangian_submanifolds.html#(part._.Conormal_bundle)

[6]The fiber of the conormal bundle of $N_0 \subset N$ at the point $\phi \in N_0$ consists of those elements of $T_\phi^* N$ which vanish on $T_\phi N_0 \subset T_\phi N$.

[7]http://andreimikhailov.com/math/bv/BV-formalism/Odd_symplectic_manifolds.html

The total odd symplectic form is postulated as follows:

$$\omega_{\text{BV}}^{\text{tot}} = \omega_{\text{BV}} + \int_\Sigma dw^\star \wedge dw \tag{2.33}$$

(where d is the field space differential, *not* the worldsheet differential). Let us add $(w^\star)^2$ to the BV action:

$$S_{\text{BV}} = S_{\text{cl}} + \int \lambda \theta^\star + \int w^\star \wedge w^\star \tag{2.34}$$

(Notice that this $\int w^\star \wedge w^\star$ does not involve the worldsheet metric.) This corresponds to:

$$\omega = \int_\Sigma dw \wedge dw \tag{2.35}$$

(again, d is the field space differential, *not* the worldsheet differential). In this case $\mathcal{P}$ is the subspace of the tangent space generated by $\frac{\partial}{\partial w}$, and $\mathcal{Q}$ is generated by $\lambda \frac{\partial}{\partial \theta}$. Then, shift the Lagrangian submanifold by a gauge fermion:

$$\Psi = \int_\Sigma w \wedge d\theta \tag{2.36}$$

This results in the new classical action:

$$S_{\text{cl}}^{\text{new}} = S_{\text{cl}} + \int_\Sigma w \wedge d\lambda + \int_\Sigma d\theta \wedge d\theta \tag{2.37}$$

$$S_{\text{BV}} = S_{\text{cl}}^{\text{new}} + \int \lambda \theta^\star + \int d\theta \, w^\star + \int w^\star \wedge w^\star \tag{2.38}$$

$$Q^{\text{new}} = \lambda \frac{\partial}{\partial \theta} + d\theta \frac{\partial}{\partial w} \tag{2.39}$$

Here we have run the procedure of section 2 "in reverse". That is, eq. (2.37) is an example of the S_{cl} of eq. (2.3), and eq. (2.34) is an example of the "split" eq. (2.11). Notice that π is degenerate, as it does not involve $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial \lambda}$. Because of that, the S_{cl} of eq. (2.34) is not constant as in eq. (2.26), but just independent of w. The vector field $\{\pi, \Psi\}$ is the $d\theta \frac{\partial}{\partial w}$-part of Q^{new}, as in eq. (2.22).

This is, still, not a quantizable action (the kinetic term for θ is a total derivative). One particular way of choosing a Lagrangian submanifold leading to quantizable action is to treat w_+ and w_- asymmetrically (pick a worldsheet complex structure), see section on A-model in AKSZ [9] and section 5.3 of this paper. This requires more than one flavour of w.

3 Pure spinor superstring in $AdS_5 \times S^5$

3.1 Notations

We follow the notations in [10]. The superconformal algebra $\mathbf{g} = \mathbf{psu}(2,2|4)$ has $\mathbf{Z}_4$-grading:

$$\mathbf{g} = \mathbf{g}_{\bar{0}} + \mathbf{g}_{\bar{1}} + \mathbf{g}_{\bar{2}} + \mathbf{g}_{\bar{3}} \tag{3.1}$$

Bars over subindices are to remind that they are mod 4. Geometrically, $\mathbf{g}_{\bar{2}}$ can be identified with the tangent space to the bosonic $AdS_5 \times S^5$, which is the direct sum of the tangent space to AdS_5 and the tangent space to S^5:

$$T(AdS_5 \times S^5) = T(AdS_5) \oplus T(S^5) \tag{3.2}$$

Therefore elements of $\mathbf{g}_{\bar{2}}$ are vectors from this tangent space. We can also consider the tangent space to the full superspace M:

$$M = \mathrm{super}(AdS_5 \times S^5) = \frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(1,4) \times \mathrm{SO}(5)} \tag{3.3}$$

$$T\left(\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(1,4) \times \mathrm{SO}(5)}\right) = \mathbf{g}_{\bar{1}} \oplus \mathbf{g}_{\bar{2}} \oplus \mathbf{g}_{\bar{3}} \tag{3.4}$$

this is a direct sum of three vector bundles. We parametrize a point in M by $g \in \mathrm{PSU}(2,2|4)$ modulo the equivalence relation:

$$g \simeq hg \quad \text{for all} \quad h \in \mathrm{SO}(1,4) \times \mathrm{SO}(5) \tag{3.5}$$

We are identifying representations of $\mathbf{g}_{\bar{0}} = \mathrm{Lie}(\mathrm{SO}(1,4) \times \mathrm{SO}(5))$, such as $\mathbf{g}_{\bar{1}}$, $\mathbf{g}_{\bar{2}}$, $\mathbf{g}_{\bar{3}}$, with the corresponding vector bundles over the coset space (3.3). In fact, the worldsheet field λ_L takes values in the fibers of $\mathbf{g}_{\bar{3}}$ and λ_R takes values in the fibers of $\mathbf{g}_{\bar{1}}$. The pure spinor conditions define the cones C_L and C_R:

$$C_L \; : \; \{\lambda_L, \lambda_L\} = 0 \tag{3.6}$$

$$C_R \; : \; \{\lambda_R, \lambda_R\} = 0 \tag{3.7}$$

Here $\{\_, \_\}$ denotes the anticommutator (the Lie superalgebra operation) of elements of $\mathbf{g}$. It should not be confused with neither the odd Poisson bracket, nor the even Poisson bracket corresponding to π^{ab} of section 2. Again, we identify C_L and C_R as bundles over super-AdS. (They are not *vector* bundles, because their fibers are cones and not linear spaces.) We will denote:

$$PS \, AdS_5 \times S^5 \; = \; \frac{C_L \times C_R \times \mathrm{PSU}(2,2|4)}{\mathrm{SO}(1,4) \times \mathrm{SO}(5)} \tag{3.8}$$

where the prefix PS on the l.h.s. stands for "Pure spinors" (and on the r.h.s. for "Projective" and "Special").

In appendix A we construct $\mathrm{PSU}(2,2|4)$-invariant surjective maps of bundles ("projectors"):

$$\mathbf{P}_{31} \; : (\mathbf{g}_{\bar{3}} \times C_L) \to TC_L \tag{3.9}$$

$$\mathbf{P}_{13} \; : (\mathbf{g}_{\bar{1}} \times C_R) \to TC_R \tag{3.10}$$

They are rational functions of λ_L and λ_R.

3.2 Standard action

The action of the AdS sigma-model has the following form [7]:

$$S_0 = \int dz\, d\bar{z}\, \mathrm{Str}\left(\frac{1}{2}J_{2+}J_{2-} + \frac{3}{4}J_{1+}J_{3-} + \frac{1}{4}J_{3+}J_{1-}\right. \tag{3.11}$$

$$\left. + w_{1+}D_{0-}\lambda_3 + w_{3-}D_{0+}\lambda_1 - N_{0+}N_{0-}\right)$$

where J_n are the $\mathbf{g}_{\bar{n}}$-components of $J = -dgg^{-1} = J_+dz + J_-d\bar{z}$. We write λ_3 instead of λ_L and λ_1 instead of λ_R, just to highlight the $\mathbf{Z}_4$-grading.(And also because neither λ_L is strictly speaking left-moving, nor is λ_R right-moving.) The covariant derivative $D_{0\pm}$ is defined as follows:

$$D_{0\pm} = \partial_\pm + [J_{0\pm}, \_] \tag{3.12}$$

Since λ_3 and λ_1 both satisfy the pure spinor constraints, the corresponding conjugate momenta are defined up to "gauge transformations":

$$\delta_{v_2}w_{1+} = [v_{2+}, \lambda_3] \tag{3.13}$$

$$\delta_{u_2}w_{3-} = [u_{2-}, \lambda_1] \tag{3.14}$$

where v_2 and u_2 are arbitrary sections of the pullback to the worldsheet of $\mathbf{g}_{\bar{2}}$. The BRST transformations are defined up to gauge transformations corresponding to the equivalence relation (3.5). It is possible to fix this ambiguity[8] so that:

$$Q\lambda_L = Q\lambda_R = 0 \tag{3.15}$$

$$Qg = (\lambda_L + \lambda_R)g \tag{3.16}$$

$$Qw_{1+} = -J_{1+}\,, \qquad Qw_{3-} = -J_{3-} \tag{3.17}$$

The first line in eq. (3.11) is by itself *not* BRST invariant. Modulo total derivatives, its BRST variation is:

$$Q\int d\tau\, d\sigma\, \mathrm{Str}\left(\frac{1}{2}J_{2+}J_{2-} + \frac{3}{4}J_{1+}J_{3-} + \frac{1}{4}J_{3+}J_{1-}\right)$$

$$= \int d\tau\, d\sigma\, \mathrm{Str}\left(-D_{0+}\lambda_1\, J_{3-} - D_{0-}\lambda_3\, J_{1+}\right) \tag{3.18}$$

This cancels with the BRST variation of the second line in eq. (3.11).

3.3 New action

On the other hand, we observe that:

$$Q\,\mathrm{STr}\left(J_{1+}\mathbf{P}_{31}J_{3-}\right) = \mathrm{STr}\left(-D_{0+}\lambda_1\, J_{3-} - D_{0-}\lambda_3\, J_{1+}\right) \tag{3.19}$$

Notice that the projector drops out on the r.h.s. because $D_{0\pm}\lambda$ is automatically tangent to the cone. Comparing this to (3.18) we see that the following expression:

$$S_0' = \int d\tau\, d\sigma\, \mathrm{STr}\left(\frac{1}{2}J_{2+}J_{2-} + \frac{3}{4}J_{1+}J_{3-} + \frac{1}{4}J_{3+}J_{1-} - J_{1+}\mathbf{P}_{31}J_{3-}\right) \tag{3.20}$$

is BRST invariant. It does not contain neither derivatives of pure spinors, nor their conjugate momenta.

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[8]http://andreimikhailov.com/slides/talk_Perimeter/LiftOfQ.html

3.4 The b-ghost

We define:

$$b_{++} = \frac{\mathrm{STr}\left(\left(\{J_{3+}, \lambda_3\} - \{J_{1+}, \lambda_1\}\right)\overline{J_{2+}}\right)}{\mathrm{STr}(\lambda_3\lambda_1)} = \frac{\mathrm{Tr}\left(\left(\{J_{3+}, \lambda_3\} - \{J_{1+}, \lambda_1\}\right)J_{2+}\right)}{\mathrm{STr}(\lambda_3\lambda_1)} \quad (3.21)$$

$$b_{--} = \text{same but with } + \text{ replaced with } - \quad (3.22)$$

(See appendix A for notations. We use the fact that $\mathrm{Str}(A_2\overline{B_2}) = \mathrm{Str}(A_2 B_2 \Sigma) = \mathrm{Tr}(A_2 B_2)$.) These expressions satisfy (appendix B):

$$Qb_{++} = T_{++} \quad \text{and} \quad Qb_{--} = T_{--} \quad (3.23)$$

$$\text{where} \quad T_{++} = \mathrm{Str}\left(\frac{1}{2}J_{2+}J_{2+} + J_{1+}(\mathbf{1} - \mathbf{P}_{31})J_{3+}\right)$$

$$T_{--} = \mathrm{Str}\left(\frac{1}{2}J_{2-}J_{2-} + J_{1-}(\mathbf{1} - \mathbf{P}_{31})J_{3-}\right)$$

Notice that:

$$S_0' = S_0'' + QB \quad (3.24)$$

$$\text{wher} \quad B = \int d\tau d\sigma \frac{\mathrm{Tr}\left(\left(\{J_{3+}, \lambda_3\} - \{J_{1+}, \lambda_1\}\right)J_{2-} + (+ \leftrightarrow -)\right)}{\mathrm{STr}(\lambda_3\lambda_1)} \quad (3.25)$$

$$S_0'' = \int \mathrm{STr}\left(J_1 \wedge (\mathbf{1} - \mathbf{P}_{31})J_3 - J_1 \wedge \mathbf{P}_{31}J_3\right) \quad (3.26)$$

and S_0'' is diffeomorphism-invariant (and therefore degenerate!). The BRST invariance of S_0'' can be verified explicitly as follows:

$$QS_0'' = \int \mathrm{STr}\left([\lambda_3, J_2] \wedge J_3 - [\lambda_1, J_2] \wedge J_1 - D_0\lambda_1 \wedge J_3 + D_0\lambda_3 \wedge J_1\right)$$

$$= \int d\,\mathrm{STr}(\lambda_3 J_1 - \lambda_1 J_3) = 0 \quad (3.27)$$

3.5 Gauge fixing $\mathbf{SO}(1,4) \times \mathbf{SO}(5)$

Consider the action of the BRST operator given by Eq. (3.16) on g. It is nilpotent only up to the $\mathbf{g}_0$-gauge transformation by $\{\lambda_3, \lambda_1\}$. We have so far worked on the factorspace by gauge transformations. This means that we think of the group element g and pure spinors λ as defined only modulo the gauge transformation:

$$(g, \lambda) \simeq (hg, h\lambda h^{-1}) \quad (3.28)$$

It turns out that the action of these gauge transformations on the BV phase space is somewhat nontrivial, see section 5.4. We will now just fix the gauge, postponing the discussion of gauge transformations to section 5.4. Let us parametrize the group element $g \in \mathrm{PSU}(2,2|4)$ by u, x, θ:

$$g = e^u e^{x+\theta} \quad (3.29)$$

where $u \in \mathbf{g}_0$, $x \in \mathbf{g}_2$ and $\theta \in \mathbf{g}_3 + \mathbf{g}_1$, and impose the following gauge fixing condition:

$$u = 0 \tag{3.30}$$

Since eq. (3.30) does not contain derivatives, this gauge is "ghostless", the Faddeev-Popov procedure is not needed.[9] In this gauge fixed formalism, the BRST operator includes the gauge fixing term (*cp.* eqs. (3.15), (3.16), (3.17)):

$$Qg = (\lambda_3 + \lambda_1 + A_0)g \tag{3.31}$$

$$Q\lambda_3 = [A_0, \lambda_3] \,, \qquad\qquad Q\lambda_1 = [A_0, \lambda_1] \tag{3.32}$$

$$Qw_{1+} = -J_{1+} + [A_0, w_{1+}] \,, \qquad Qw_{3-} = -J_{3-} + [A_0, w_{3-}] \tag{3.33}$$

where $A_0 \in \mathbf{g}_{\bar{0}}$ is some function of θ, λ and x, defined by eqs. (3.31) and (3.30); schematically $A_0 = \{\theta_L, \lambda_1\} + \{\theta_R, \lambda_3\} + \dots$ This A_0 is usually called "the compensating gauge transformation". It automatically satisfies:

$$QA_0 = -\{\lambda_3, \lambda_1\} + \frac{1}{2}[A_0, A_0] \tag{3.34}$$

Gauge fixing is only possible locally in $AdS_5 \times S^5$. In order for our constructions to work globally, we will cover $AdS_5 \times S^5$ with patches and gauge-fix over each patch. Then we have to glue overlapping patches. We will explain how to do this in section 5.4.

3.6 In BV language

We will now show that the difference between the original action and the action (3.20) can be interpreted in the BV formalism as a particular case of the construction outlined in section 2.

The BRST symmetry of the pure spinor superstring in $AdS_5 \times S^5$ is nilpotent only on-shell. More precisely, the only deviation from the nilpotence arises when we act on the conjugate momenta of the pure spinors:

$$Q^2 w_{1+} = \frac{\delta S_0}{\delta w_{3-}} \tag{3.35}$$

$$Q^2 w_{3-} = \frac{\delta S_0}{\delta w_{1+}} \tag{3.36}$$

(while the action of Q^2 on the *matter* fields is zero even off-shell). This means that the BV Master Action contains a term quadratic in the antifields:

$$S_{\mathrm{BV}} = S_0 + \int (QZ^i) Z_i^\star + \int (Q\lambda)\lambda^\star + \int (Qw)w^\star + \int \mathrm{Str}\left(w_{1+}^\star w_{3-}^\star\right) \tag{3.37}$$

In this formula Z and $Z^\star$ stand for matter fields (x and θ) and their antifields, and S_0 is given by eq. (3.11). The matter fields Z are essentially x and θ where $J = -dgg^{-1}$ with $g = e^{x+\theta}$, $x \in \mathbf{g}_2$, $\theta \in \mathbf{g}_3 \oplus \mathbf{g}_1$:

$$Z = x \text{ and } \theta \tag{3.38}$$

---

[9]The Faddeev-Popov procedure in such cases leads to ghost action of the form $\int f(\phi)\bar{c}c$ where $f(\phi)$ is some function of the fields. Integration out c and $\bar{c}$ leads to *local* expressions (in fact, proportional to $\delta(0)$) which are absorbed by counterterms. Similar topics were discussed in [11, 12].

Their BRST transformation QZ^i is read from eq. (3.31). We observe that the action is of the same type as described in section 2. The Poisson bivector is:

$$\pi = \int \text{Str} \left(\frac{\partial}{\partial w_{1+}} \wedge \frac{\partial}{\partial w_{3-}} \right) \tag{3.39}$$

The 2-form ω discussed in section 2 can be choosen as follows:

$$\omega = \int \text{Str} \left(dw_{1+} \wedge \mathbf{P}_{31} dw_{3-} \right) \tag{3.40}$$

The projector $\mathbf{P}_{31}$ is needed to make ω invariant with respect to the gauge transformations (3.13) and (3.14). We take the following generating function ψ satisfying eq. (2.19):

$$\psi = \int \text{Str} \; (w_{1+}\mathbf{P}_{31}J_{3-} + w_{3-}\mathbf{P}_{13}J_{1+} + w_{1+}[A_0, w_{3-}]) \tag{3.41}$$

The new "classical action" $\widetilde{S}_{\text{cl}}$ is given by Eq. (3.20). (We will provide more details for a slightly more general calculation in section 5.) It is, indeed, constant along the symplectic leaves of π, as the fields $w_\pm$ are not present in this new Lagrangian at all. The new BV action is:

$$\widetilde{S}_{\text{BV}} \;=\; \int d\tau \, d\sigma \, \text{Str}\bigg(\frac{1}{2}J_{2+}J_{2-} + \frac{3}{4}J_{1+}J_{3-} + \frac{1}{4}J_{3+}J_{1-} - J_{1+}\mathbf{P}_{31}J_{3-}$$
$$+ \sum_{Z \in \{x,\theta,\lambda\}} (QZ)Z^\star \bigg) \tag{3.42}$$

where Z^i runs over θ, x, λ and the action of Q on Z^i is the same as it was in the original σ-model. The new BV phase space is smaller, it only contains $\theta, x, \lambda, \theta^\star, x^\star, \lambda^\star$. The BRST operator is now nilpotent off-shell; the dependence of the BV action on the antifields is linear. The fields $\lambda_{L|R}$ enter only through their combination invariant under local rescalings (they enter through $\mathbf{P}_{31}$). This in particular implies that the BRST symmetry Q is now a *local* symmetry.

Of course, the new action (3.20) is degenerate.

4 Action of diffeomorphisms

4.1 Formulation of the problem

Let $L_2^\star$ be the BV Hamiltonian generating the left shift by elements of $\mathbf{g}_{\bar{2}}$; if f is any function of g, then:

$$\{\text{Str}(A_2 L_2^\star), f\}_{\text{BV}}(g) \;=\; \left.\frac{d}{dt}\right|_{t=0} f\left(e^{tA_2}g\right) \tag{4.1}$$

The $L_0^\star$, $L_1^\star$ and $L_3^\star$ are defined similarly. In particular:

$$\{\widetilde{S}_{\text{BV}}, \_\} \;=\; \int \text{Str} \left(\lambda_3 L_1^\star + \lambda_1 L_3^\star\right) \tag{4.2}$$

With these notations, when X and Y are two even elements of $\mathbf{g}$,

$$\left\{\int \mathrm{Str}(XL^\star), \int \mathrm{Str}(YL^\star)\right\}_{\mathrm{BV}} = -\int \mathrm{Str}([X,Y]L^\star) \tag{4.3}$$

(Even elements are generators of $\mathbf{g}_2$ and $\mathbf{g}_0$, and also the generators of $\mathbf{g}_3$ and $\mathbf{g}_1$ multiplied by a Grassmann odd parameter.)

The infinitesimal action of diffeomorphisms is generated by the following BV Hamiltonian V_ξ:

$$V_\xi = \int \mathrm{Str}\Big((\iota_\xi D_0\lambda_3)\lambda_1^\star + (\iota_\xi D_0\lambda_1)\lambda_3^\star - (\iota_\xi J_3)L_1^\star - (\iota_\xi J_1)L_3^\star - (\iota_\xi J_2)L_2^\star\Big) \tag{4.4}$$

where $\quad D_0\lambda = d\lambda + [J_0,\lambda] \tag{4.5}$

In this section we will construct Φ_ξ such that:

$$V_\xi = \{\widetilde{S}_{\mathrm{BV}}, \Phi_\xi\}_{\mathrm{BV}} \tag{4.6}$$

It is very easy to construct such Φ_ξ if we don't care about the global symmetries of $AdS_5 \times S^5$. (Something like $\Phi_\xi = \frac{\theta^\alpha}{\lambda^\alpha}V_\xi$.) But we will construct a Φ_ξ invariant under the supersymmetries of $AdS_5 \times S^5$, i.e. invariant under the right shifts of g. We believe that such an invariant construction has better chance of satisfying the equivariance conditions of [2, 3] at the quantum level, because supersymmetries restrict quantum corrections. In particular, the equivariance condition must require that the Φ_ξ correspond, in some sense, to a primary operator.

Comment on gauge transformations. In this section we discuss vector fields on the factorspace $PS\,AdS$ defined by eq. (3.8). They are the same as $\mathrm{SO}(1,4) \times \mathrm{SO}(5)$-invariant vector fields on $C_L \times C_R \times \mathrm{PSU}(2,2|4)$ modulo $\mathrm{SO}(1,4) \times \mathrm{SO}(5)$-invariant vertical vector fields. All the formulas here are modulo vertical $\mathrm{SO}(1,4) \times \mathrm{SO}(5)$-invariant vector fields.

4.2 Subspaces associated to a pair of pure spinors

We use the notations of section A.5. For $X_3 \in [\mathbf{g}_{2L}, \lambda_1]$ and $X_1 \in [\mathbf{g}_{2R}, \lambda_3]$, let $\mathbf{T}_2(X_1+X_2)$ denote the map:

$$\mathbf{T}_2 : [\lambda_1, \mathbf{g}_{2L}] \oplus [\lambda_3, \mathbf{g}_{2R}] \longrightarrow \mathbf{g}_{2L} \oplus \mathbf{g}_{2R} \tag{4.7}$$

$$\mathbf{T}_2([\lambda_1, v_{2L}] + [\lambda_3, v_{2R}]) = v_{2L} + v_{2R} \tag{4.8}$$

(This is a direct sum of two completely independent linear maps.) For a pair $I_3 \oplus I_1 \in T^\perp C_R \oplus T^\perp C_L$ we decompose:[10]

$$I_3 \oplus I_1 = (I_3^{\mathrm{split}} \oplus I_1^{\mathrm{split}}) + (I_3^{\mathrm{ker}} \oplus I_1^{\mathrm{ker}}) + (I_3^{\mathrm{coker}} \oplus I_1^{\mathrm{coker}}) \tag{4.9}$$

where $\quad I_3^{\mathrm{split}} \oplus I_1^{\mathrm{split}} \in [\lambda_1, \mathbf{g}_{2L}] \oplus [\lambda_3, \mathbf{g}_{2R}] \tag{4.10}$

$$I_3^{\mathrm{ker}} \oplus I_1^{\mathrm{ker}} \in \ker\left[T^\perp C_R \oplus T^\perp C_L \xrightarrow{(+)\circ(\{\lambda_3,\text{-}\}\oplus\{\lambda_1,\text{-}\})} \mathbf{g}_2\right] \tag{4.11}$$

$$I_3^{\mathrm{coker}} \oplus I_1^{\mathrm{coker}} \in \mathrm{coker}\left[\mathbf{g}_2 \xrightarrow{\{\lambda_3,\text{-}\}+\{\lambda_1,\text{-}\}} T^\perp C_R \oplus T^\perp C_L\right] \tag{4.12}$$

---

[10]For example, I_3^{split} denotes the component of I_3 which belongs to $[\lambda_1, \mathbf{g}_{2L}]$; the label "split" is because we could not invent any better notation.

where we must use a special representative of the cokernel:

$$I_3^{\text{ker}} = \frac{1}{2}\frac{\text{Tr}(\lambda_3 I_3 - \lambda_1 I_1)}{\text{Tr}\left([\lambda_1,\lambda_3]_{\text{STL}}\right)^2}\left[\lambda_1,[\lambda_3,\lambda_1]_{\text{STL}}\right] \tag{4.13}$$

$$I_1^{\text{ker}} = \frac{1}{2}\frac{\text{Tr}(\lambda_3 I_3 - \lambda_1 I_1)}{\text{Tr}\left([\lambda_1,\lambda_3]_{\text{STL}}\right)^2}\left[\lambda_3,[\lambda_3,\lambda_1]_{\text{STL}}\right] \tag{4.14}$$

$$I_3^{\text{coker}} = \frac{1}{2}\frac{\text{Tr}(\lambda_3 I_3 + \lambda_1 I_1)}{\text{Tr}\left([\lambda_1,\lambda_3]_{\text{STL}}\right)^2}\left[\lambda_1,[\lambda_3,\lambda_1]_{\text{STL}}\right] \tag{4.15}$$

$$I_1^{\text{coker}} = -\frac{1}{2}\frac{\text{Tr}(\lambda_3 I_3 + \lambda_1 I_1)}{\text{Tr}\left([\lambda_1,\lambda_3]_{\text{STL}}\right)^2}\left[\lambda_3,[\lambda_3,\lambda_1]_{\text{STL}}\right] \tag{4.16}$$

Similarly, any $I_2 \in \mathbf{g}_2$ (assumed to be both TL and STL) can be decomposed:

$$I_2 = I_2^{\text{L}} + I_2^{\text{R}} + I_2^{\text{ker}} + I_2^{\text{coker}} \tag{4.17}$$

$$\text{where} \qquad I_2^{\text{L}} \in \mathbf{g}_{2L} \tag{4.18}$$

$$I_2^{\text{R}} \in \mathbf{g}_{2R} \tag{4.19}$$

$$I_2^{\text{ker}} \in \ker\left[\mathbf{g}_2 \xrightarrow{\{\lambda_3,\text{-}\}+\{\lambda_1,\text{-}\}} T^{\perp}C_L \oplus T^{\perp}C_R\right] \tag{4.20}$$

$$I_2^{\text{coker}} \in \text{coker}\left[T^{\perp}C_R \oplus T^{\perp}C_L \xrightarrow{(+)\circ(\{\lambda_3,\text{-}\}\oplus\{\lambda_1,\text{-}\})} \mathbf{g}_2\right] \tag{4.21}$$

Explicitly:

$$I_2^{\text{ker}} = \frac{\text{Tr}(I_2\overline{[\lambda_3,\lambda_1]})}{\text{Tr}([\lambda_3,\lambda_1]_{\text{STL}})^2}\overline{[\lambda_3,\lambda_1]}_{\text{TL}} \tag{4.22}$$

$$I_2^{\text{coker}} = \frac{\text{Tr}(I_2[\lambda_3,\lambda_1])}{\text{Tr}([\lambda_3,\lambda_1]_{\text{STL}})^2}[\lambda_3,\lambda_1]_{\text{STL}} \tag{4.23}$$

4.3 Construction of Φ_ξ

The generating function V_ξ of the infinitesimal worldsheet diffeomorphisms (= vector fields) $\xi = \xi^\tau\partial_\tau + \xi^\sigma\partial_\sigma$, given by eq. (4.4), is BV-exact:

$$V_\xi = \{\widetilde{S}_{\text{BV}},\Phi_\xi\} \tag{4.24}$$

$$\Phi_\xi = -\int \text{Str}\Big((\mathbf{P}_{31}\iota_\xi J_3)\lambda_1^\star + (\mathbf{P}_{13}\iota_\xi J_1)\lambda_3^\star$$
$$+ \left(\mathbf{T}_2\left(\iota_\xi J_3^{\text{split}} + \iota_\xi J_1^{\text{split}}\right) + \mathcal{A}[\lambda_3,\lambda_1]_{\text{STL}}\right)L_2^\star$$
$$+ \mathcal{B}\left([\lambda_1,[\lambda_3,\lambda_1]_{\text{STL}}]L_3^\star - [\lambda_3,[\lambda_3,\lambda_1]_{\text{STL}}]L_1^\star\right)\Big) \tag{4.25}$$

$$\text{where} \qquad \iota_\xi J = -\xi^\alpha\partial_\alpha g g^{-1}$$

$$\mathcal{A} = \frac{1}{2}\frac{\text{Tr}\left(\lambda_3(1-\mathbf{P}_{31})\iota_\xi J_3 - \lambda_1(1-\mathbf{P}_{13})\iota_\xi J_1\right)}{\text{Tr}([\lambda_3,\lambda_1]_{\text{STL}})^2}$$

$$\mathcal{B} = \frac{\text{STr}([\lambda_3,\lambda_1]\iota_\xi J_2)}{\text{STr}(\lambda_3\lambda_1)\,\text{Tr}([\lambda_3,\lambda_1]_{\text{STL}})^2} \tag{4.26}$$

The coefficients $\mathcal{A}$ and $\mathcal{B}$ satisfy:

$$(Q\mathcal{A})[\lambda_3, \lambda_1]_{\mathrm{STL}} = \iota_\xi J_2^{\mathrm{coker}} \tag{4.27}$$

$$\mathcal{A}\left([\lambda_1, [\lambda_3, \lambda_1]_{\mathrm{STL}}] + [\lambda_3, [\lambda_3, \lambda_1]_{\mathrm{STL}}]\right) = \iota_\xi J_3^{\mathrm{ker}} + \iota_\xi J_1^{\mathrm{ker}} \tag{4.28}$$

$$(Q\mathcal{B})\left([\lambda_1, [\lambda_3, \lambda_1]_{\mathrm{STL}}] - [\lambda_3, [\lambda_3, \lambda_1]_{\mathrm{STL}}]\right) = \iota_\xi J_1^{\mathrm{coker}} + \iota_\xi J_3^{\mathrm{coker}} \tag{4.29}$$

$$\mathcal{B}\left(\{\lambda_3, [\lambda_1, [\lambda_3, \lambda_1]_{\mathrm{STL}}]\} - \{\lambda_1, [\lambda_3, [\lambda_3, \lambda_1]_{\mathrm{STL}}]\}\right) = \iota_\xi J_2^{\mathrm{ker}} \tag{4.30}$$

Eq. (4.24) follows from:

$$\begin{aligned}
\iota_\xi J &= \mathbf{P}_{31}\iota_\xi J_3 + \mathbf{P}_{13}\iota_\xi J_1 \\
&\quad + \iota_\xi J_3^{\mathrm{split}} + \iota_\xi J_1^{\mathrm{split}} + \iota_\xi J_3^{\mathrm{ker}} + \iota_\xi J_1^{\mathrm{ker}} \\
&\quad + (Q\mathcal{B})\left([\lambda_1, [\lambda_3, \lambda_1]_{\mathrm{STL}}] - [\lambda_3, [\lambda_3, \lambda_1]_{\mathrm{STL}}]\right) \\
&\quad + \iota_\xi J_2^{\mathrm{split}} + \iota_\xi J_2^{\mathrm{ker}} + (Q\mathcal{A})[\lambda_3, \lambda_1]_{\mathrm{STL}}
\end{aligned} \tag{4.31}$$

Some useful identities.

$$\mathrm{STr}\left([\lambda_3, [\lambda_3, \lambda_1]_{\mathrm{STL}}](1 - \mathbf{P}_{31})\iota_\xi J_3\right) = -\frac{\mathrm{STr}(\lambda_3\lambda_1)}{2}\mathrm{Tr}\left(\lambda_3(1-\mathbf{P}_{31})\iota_\xi J_3\right) \tag{4.32}$$

$$\begin{aligned}
\{\lambda_1, [\lambda_3, [\lambda_3, \lambda_1]_{\mathrm{STL}}]\} &= -\{\lambda_3, [\lambda_1, [\lambda_3, \lambda_1]_{\mathrm{STL}}]\} \\
&= \frac{1}{2}\overline{[\lambda_1, \lambda_3]}_{\mathrm{TL}}\mathrm{Str}(\lambda_3\lambda_1) + \frac{1}{8}\left((\mathrm{Str}(\lambda_3\lambda_1))^2 - 2\mathrm{Tr}[\lambda_1, \lambda_3]^2\right)\mathbf{1} \\
&= \frac{1}{2}\overline{[\lambda_1, \lambda_3]}_{\mathrm{TL}}\mathrm{Str}(\lambda_3\lambda_1) - \frac{1}{4}\mathrm{Tr}([\lambda_3, \lambda_1]_{\mathrm{STL}})^2\,\mathbf{1}
\end{aligned} \tag{4.33}$$

$$\begin{aligned}
\mathrm{STr}\left([\lambda_1, [\lambda_3, \lambda_1]_{\mathrm{STL}}][\lambda_3, [\lambda_3, \lambda_1]_{\mathrm{STL}}]\right) &= -\frac{1}{2}\mathrm{Str}(\lambda_3\lambda_1)\mathrm{Str}\left([\lambda_1, \lambda_3]_{\mathrm{STL}}\overline{[\lambda_1, \lambda_3]}_{\mathrm{TL}}\right) \\
&= -\frac{1}{2}\mathrm{Str}(\lambda_3\lambda_1)\mathrm{Tr}([\lambda_3, \lambda_1]_{\mathrm{STL}})^2
\end{aligned} \tag{4.34}$$

Notice that we have $\mathrm{Tr}([\lambda_3, \lambda_1]_{\mathrm{STL}})^2$ in denominators. At the same time:

$$\mathrm{STr}([\lambda_3, \lambda_1]_{\mathrm{STL}})^2 = \mathrm{STr}([\lambda_3, \lambda_1])^2 = 0 \tag{4.35}$$

5 Regularization

The "minimalistic action" (3.42) cannot be regularized in a way that would preserve the symmetries of $AdS_5 \times S^5$; it is impossible to choose a $PSU(2, 2|4)$-invariant Lagrangian submanifold so that the restriction of the Master Action of eq. (3.42) to it be non-degenerate. Let us therefore return to the original action of eqs. (3.11), (3.37), but in a way preserving the worldsheet diffeomorphisms. The construction is somewhat similar to the description of the topological A-model in [9].

5.1 Adding more fields

Add a pair of bosonic 1-form fields ω_3 and ω_1, taking values in $\mathbf{g}_3$ and $\mathbf{g}_1$, respectively, and their antifields $\omega_1^\star$ and $\omega_3^\star$, also 1-forms:

$$\omega_{\mathrm{BV}} = \int \mathrm{STr}\left(d\omega_3^\star \wedge d\omega_1 + d\omega_1^\star \wedge d\omega_3\right) \tag{5.1}$$

(where d is the differential in the field space, not on the worldsheet!). In other words, for any "test 1-forms" f_1 and g_3:

$$\left\{ \int \operatorname{Str} f_1 \wedge \omega_3^\star \, , \, \int \operatorname{Str} \omega_1 \wedge g_3 \right\} \; = \; \int \operatorname{Str} f_1 \wedge g_3 \tag{5.2}$$

$$\left\{ \int \operatorname{Str} g_3 \wedge \omega_1 \, , \, \int \operatorname{Str} \omega_3^\star \wedge f_1 \right\} \; = \; \int \operatorname{Str} g_3 \wedge f_1 \tag{5.3}$$

We define the BV Master Action as follows:

$$\widetilde{S}_{\mathrm{BV}}^+ = \widetilde{S}_{\mathrm{BV}} + \int \operatorname{STr}(\omega_3^\star \wedge \omega_1^\star) \tag{5.4}$$

and the BV Hamiltonian for the action of diffeomorphisms as follows:

$$\widehat{V}_\xi \; = \{ \widetilde{S}_{\mathrm{BV}}^+ \, , \, \widehat{\Phi}_\xi \}_{\mathrm{BV}} \tag{5.5}$$

$$\widehat{\Phi}_\xi \; = \Phi_\xi + \int \operatorname{STr}(\omega_3 \wedge \mathcal{L}_\xi \omega_1) \tag{5.6}$$

where $\mathcal{L}_\xi$ is the Lie derivative.

The expression $\int \operatorname{STr}(\delta\omega_3 \wedge \delta\omega_1)$ defines a symplectic structure on the space of 1-forms with values in $\mathbf{g}_{\mathrm{odd}}$. The expression $\int \operatorname{STr}(\omega_3^\star \wedge \omega_1^\star)$ is the corresponding Poisson bivector.[11] The Lie derivative preserves this (even) symplectic structure, and $\int \operatorname{STr}(\omega_3 \wedge \mathcal{L}_\xi \omega_1)$ is the corresponding Hamiltonian.

5.2 A canonical transformation

Let us do the canonical transformation by a flux of the following odd Hamiltonian:

$$\Psi_{(0)} \; = \; \int \operatorname{STr} [A_0, \omega_3] \wedge \omega_1 \; = \; - \int \operatorname{STr} [A_0, \omega_1] \wedge \omega_3 \tag{5.7}$$

This is the Hamiltonian of $[A_0, \_]$ in the same sense as $\int \operatorname{STr}(\omega_3 \wedge \mathcal{L}_\xi \omega_1)$ is the Hamiltonian of $\mathcal{L}_\xi$; we again use the same procedure of passing from eq. (2.3) to eq. (2.25), actually in reverse.

The effect of the flux of $\Psi_{(0)}$ on the BV Master Action $\widetilde{S}_{\mathrm{BV}}^+$ of eq. (5.4) is:

$$\widetilde{S}_{\mathrm{BV}}^+ = \widetilde{S}_{\mathrm{BV}} + \int \operatorname{STr} \omega_3^\star \wedge \omega_1^\star$$

$$\text{becomes} \quad \widetilde{S}_{\mathrm{BV}}' = \widetilde{S}_{\mathrm{BV}} + \int \operatorname{STr} \left[\left(\{\lambda_3, \lambda_1\} - \frac{1}{2}[A_0, A_0] \right), \omega_3 \right] \wedge \omega_1$$

$$+ \int \operatorname{STr}(\omega_3^\star + [A_0, \omega_3]) \wedge (\omega_1^\star - [A_0, \omega_1])$$

$$= \widetilde{S}_{\mathrm{BV}}^+ + \int \operatorname{STr} \{\lambda_3, \lambda_1\}\{\omega_3, \wedge\omega_1\} \tag{5.8}$$

$$+ \int \operatorname{STr}\Big([A_0, \omega_3] \wedge \omega_1^\star + [A_0, \omega_1] \wedge \omega_3^\star \Big)$$

---

[11]http://andreimikhailov.com/math/bv/omega/Duistermaat-Heckman_formula.html

Notice that the terms of the form $A^2\omega^2$ cancelled. This is automatic, because such terms would contradict the Master Equation (the bracket $\{\omega^\star\omega^\star, A^2\omega^2\}$ would have nothing to cancel against).

The purpose of this canonical transformation was, essentially, to introduce the compensator term $[A, \omega]$ into the action of Q on ω, *cp.* eq. (3.31). We will discuss this in a more general context in section 5.4.

We are now ready to construct the Lagrangian submanifold.

5.3 Constraint surface and its conormal bundle

The configuration space X of this new theory is parametrized by $g, \lambda_3, \lambda_1, \omega_{3\pm}$ and $\omega_{1\pm}$. Let us consider a subspace $Y \subset X$ defined by the constraints:

$$
\begin{aligned}
(1 - \mathbf{P}_{13})\omega_{1+} &= 0 \\
(1 - \mathbf{P}_{31})\omega_{3-} &= 0 \\
\omega_{1-} &= 0 \\
\omega_{3+} &= 0
\end{aligned}
\tag{5.9}
$$

Consider the odd conormal bundle[12] $\Pi T^\perp Y$ of $Y \subset X$ in the BV phase space $\Pi T^* X$. As any conormal bundle, this is a Lagrangian submanifold. The restriction of $\widetilde{S}'_{\mathrm{BV}}$ on this Lagrangian submanifold is still degenerate. But let us deform it by the following generating function:

$$
\Psi = \int \mathrm{STr}\Big(\omega_{3-}\mathbf{P}_{13}J_{1+} + \omega_{1+}\mathbf{P}_{31}J_{3-}\Big)
\tag{5.10}
$$

The restriction of $\widetilde{S}'_{\mathrm{BV}}$ to this deformed Lagrangian submanifold is equal to:

$$
\begin{aligned}
\int \mathrm{STr}\Big(\frac{1}{2}J_{2+}J_{2-} &+ \frac{3}{4}J_{1+}J_{3-} + \frac{1}{4}J_{3+}J_{1-} \\
&+ w_{1+}D_{0-}\lambda_3 + w_{3-}D_{0+}\lambda_1 + N_{0+}N_{0-} + \omega^\star_{3+}\omega^\star_{1-} + \omega^\star_{3-}\omega^\star_{1+}\Big)
\end{aligned}
\tag{5.11}
$$

where $N_{0+} = \{w_{1+}, \lambda_3\}$, $N_{0-} = \{w_{3-}, \lambda_1\}$,

$$
w_{1+} = \mathbf{P}_{13}\omega_{1+} \texttt{ and } w_{3-} = \mathbf{P}_{31}\omega_{3-}
\tag{5.12}
$$

Notice that the terms:

$$
\begin{aligned}
\int \mathrm{STr}\Big([A_0, \lambda_3]\lambda^\star_3 &+ [A_0, \lambda_1]\lambda^\star_1 \\
&+ [A_0, \omega_{3+}]\omega^\star_{1-} + [A_0, \omega_{3-}]\omega^\star_{1+} + [A_0, \omega_{1+}]\omega^\star_{3-} + [A_0, \omega_{1-}]\omega^\star_{3+}\Big)
\end{aligned}
\tag{5.13}
$$

vanish on $T^\perp Y$. Indeed, the vector field:

$$
\begin{aligned}
[A_0, \lambda_3]\frac{\partial}{\partial\lambda_3} &+ [A_0, \lambda_1]\frac{\partial}{\partial\lambda_1} \\
&+ [A_0, \omega_{3+}]\frac{\partial}{\partial\omega_{3+}} + [A_0, \omega_{3-}]\frac{\partial}{\partial\omega_{3-}} + [A_0, \omega_{1+}]\frac{\partial}{\partial\omega_{1+}} + [A_0, \omega_{1-}]\frac{\partial}{\partial\omega_{1-}}
\end{aligned}
\tag{5.14}
$$

---

[12]http://andreimikhailov.com/math/bv/BRST-formalism/Family_of_Lagrangian_submanifolds.html
#(part._Conormal_bundle)

is tangent to the constraint surface (5.9); the conormal bundle, by definition, consists of those one-forms which vanish on such vectors. The term $\omega_{3+}^{\star}\omega_{1-}^{\star} + \omega_{3-}^{\star}\omega_{1+}^{\star}$ computes the contrubution to the action from the fiber $\Pi T^{\perp}Y$. The coordinates of the fiber enter without derivatives, and decouple.

We therefore return to the original $AdS_5 \times S^5$ action of eq. (3.11).

But now we understand how the worldsheet diffeomorphisms act, at the level of the BV phase space.

5.4 Gluing charts

In our construction we used a lift of $AdS_5 \times S^5$ to PSU(2, 2|4) (section 3.5). This is only possible locally. Therefore, we have to explain how to glue together overlapping patches. This is a particular case of a general construction, which we will now describe.

The idea: is to build a theory which is *locally* (on every patch of $AdS_5 \times S^5$) a direct product of two theories $S_{(\phi)}$ and $S_{(w)}$:

$$S_{\text{tot}} = S_{(\phi)} + S_{(w)} = S_{\text{cl}}(\phi) + Q^{\mu}(\phi)\phi_{\mu}^{\star} + \frac{1}{2}w_a^{\star}(\omega^{-1})^{ab}w_b^{\star} \tag{5.15}$$

but transition functions between overlapping patches mix ϕ and w.

Consider the following data, consisting of two parts. The first part is a Lie group H and a principal H-bundle E with base B. Suppose that B comes with a nilpotent vector field $Q \in \text{Vect}(B)$ and a Q-invariant action $S_{\text{cl}} \in \text{Fun}(B)$. Then $S_B(\phi, \phi^{\star}) = S_{\text{cl}}(\phi) + Q^{\mu}(\phi)\phi_{\mu}^{\star}$ satisfies the Master Equation on the BV phase space $\Pi T^* B$. The second part of the data is a symplectic vector space W which is a representation of H. This means that W is equipped with an *even* H-invariant symplectic form ω.

Let us cover B with charts $\{U_i | i \in \mathcal{I}\}$ and trivialize E over each chart:

$$p^{-1}(U_i) \simeq U_i \times H \tag{5.16}$$

At the intersection $U_i \cap U_j$ we identify $(\phi, h_i) \in U_i \times H$ with $(\phi, h_j) \in U_j \times H$ if

$$h_j = u_{ji}(\phi)h_i \tag{5.17}$$

All this comes from $E \xrightarrow{H} B$. We will now construct a new odd symplectic manifold, which is locally $\Pi T^* U_j \times \Pi T^* W$, with some transition functions, which we will now describe.

Technical assumption: in this section we assume that all w are bosons, and that H is a "classical" (i.e. not super) Lie group. This is enough for our considerations.

Transition functions: let $\mathbf{h}$ be the Lie algebra of H. For each $\alpha \in \text{Map}(B, \mathbf{h})$ consider the following BV Hamiltonian:

$$\chi_\alpha = \{S_{\text{tot}}, F_\alpha\} \tag{5.18}$$

$$\text{where} \qquad F_\alpha = -\frac{1}{2}w^b \rho_*(\alpha(\phi))_b^a \, \omega_{ac} \, w^c \tag{5.19}$$

Here ρ_* is the representation of the Lie algebra corresponding to the representation ρ of the group, and ω is the symplectic form of W. Eq. (5.19) defines F_α as the Hamiltonian of the infinitesimal action of α on w, i.e. the "usual" (even) moment map. (Here we use our assumption that ω is H-invariant.) The explicit formula for χ_α is:

$$\chi_\alpha = \rho_*(\alpha(\phi))_b^a w^b w_a^\star - \frac{1}{2} w^b \rho_*(Q\alpha(\phi))_b^a \omega_{ac} w^c \tag{5.20}$$

Notice that:

$$\{\chi_{\alpha_1}, F_{\alpha_2}\} = -F_{[\alpha_1,\alpha_2]} \tag{5.21}$$

The flux of the BV-Hamiltonian vector field $\{\chi_\alpha, \_\}$ is a canonical transformation, and eq. (5.18) implies that this canonical transformation is a symmetry of S_{tot}. This canonical transformation does not touch ϕ^μ, it only acts on $\phi^\star, w, w^\star$. We identify $(\phi, \phi_i^\star, w_i, w_i^\star)$ on chart $U_{(i)}$ with $(\phi, \phi_j^\star, w_j, w_j^\star)$ on chart $U_{(j)}$ when $(\phi_j^\star, w_j, w_j^\star)$ is the flux of $(\phi_i^\star, w_i, w_i^\star)$ by the time 1 along the vector field $\{\chi_{\alpha_{ji}}, \_\}$ where α_{ji} is the log of u_{ji}, i.e. $u_{ji} = e^{\alpha_{ji}}$. Explicitly:

$$w_j^a = \rho\,(u_{ji})_b^a\, w_i^b \tag{5.22}$$

$$w_{ja}^\star = \rho\left(u_{ji}^{-1}\right)_a^b\, w_{ib}^\star - \omega_{ab}\, Q\rho\,(u_{ji})_c^b\, w_i^c \tag{5.23}$$

$$\phi_{j\mu}^\star = \phi_{i\mu}^\star - w_{ja}^\star \rho_* \left(u_{ji}\, \frac{\overleftarrow{\partial}}{\partial\phi^\mu}\, u_{ji}^{-1}\right)_b^a w_j^b - \frac{1}{2} w_j^a \omega_{ab} \frac{\partial}{\partial\phi^\mu} \rho_* \left(Qu_{ji} u_{ji}^{-1}\right)_c^b w_j^c \tag{5.24}$$

These gluing rules are consistent on triple intersections because of eq. (5.21).

Lagrangian submanifold. Eqs. (5.23) and (5.24) look somewhat unusual. In particular, the "standard" Lagrangian submanifold[13] $\phi^\star = w^\star = 0$ is not well-defined, because it is incompatible with our transition functions. One simple example of a well-defined Lagrangian submanifold is $w = \phi^\star = 0$. We will now give another example, which repairs the ill-defined $w^\star = \phi^\star = 0$.

The construction requires a choice of a *connection* in the principal bundle $E \xrightarrow{H} B$. To specify a connection, we choose on every chart U_i some **h**-valued 1-form $A_{i\mu}$, with the following identifications on the intersection $U_i \cap U_j$:

$$\frac{\partial}{\partial\phi^\mu} + A_{j\mu}(\phi) = u_{ji}(\phi) \left(\frac{\partial}{\partial\phi^\mu} + A_{i\mu}(\phi)\right) (u_{ji}(\phi))^{-1} \tag{5.25}$$

and in particular:

$$Q\rho(u_{ji})_b^a + \rho_*(Q^\mu A_{j\mu})_c^a \rho\,(u_{ji})_b^c - \rho(u_{ji})_c^a \rho_*(Q^\mu A_{i\mu})_b^c = 0 \tag{5.26}$$

---

[13]In BV formalism, there is no such thing as the standard Lagrangian submanifold. We invented this notion to denote the one where all antifields (w.r.to some Darboux coordinates) are zero. This is often a useful starting point to construct Lagrangian submanifolds.

On every chart, let us pass to a new set of Darboux coordinates, by doing the canonical transformation with the following gauge fermion:

$$\Psi_i = \frac{1}{2} w_i^a \, \omega_{ab} \, Q^\mu(\phi)\rho_*(A_{i\mu}(\phi))_c^b \, w_i^c \tag{5.27}$$

Notice that Ψ_i does not depend on antifields; therefore this canonical transformation is just a shift:

$$\tilde{w}_i^a = w_i^a \tag{5.28}$$

$$\tilde{w}_{ia}^\star = w_{ia}^\star - \omega_{ab} \, Q^\nu(\phi)\rho_*(A_{i\nu}(\phi))_c^b \, w_i^c \tag{5.29}$$

$$\tilde{\phi}_{i\mu}^\star = \phi_{i\mu}^\star - \frac{1}{2}\frac{\partial}{\partial\phi^\mu}\Big[w_i^a \, \omega_{ab} \, Q^\nu(\phi)\rho_*(A_{i\nu}(\phi))_c^b \, w_i^c \Big] \tag{5.30}$$

This canonical transformation does not preserve S_{BV}, therefore the *expression* for the action will be different in different charts, see eq. (5.8). In particular, it will contain the term $\tilde{w}^\star Q^\mu \rho_*(A_{i\mu})\tilde{w}$, which means that the action of the BRST operator on $\tilde{w}$ involves the connection. On the other hand, the transition functions simplify:

$$\tilde{w}_j^a = \rho\,(u_{ji}(\phi))_b^a \, \tilde{w}_i^b \tag{5.31}$$

$$\tilde{w}_{ja}^\star = \rho\,(u_{ji}(\phi)^{-1})_a^b \, \tilde{w}_{ib}^\star \tag{5.32}$$

$$\tilde{\phi}_{j\mu}^\star = \tilde{\phi}_{i\mu}^\star - \tilde{w}_{ic}^\star\rho\,(u_{ji}(\phi)^{-1})_a^c \left(\rho\,(u_{ji}(\phi))_b^a \, \frac{\overleftarrow{\partial}}{\partial\phi^\mu} \right) \tilde{w}_i^b \tag{5.33}$$

These are the usual transition functions of the odd cotangent bundle $\Pi T^*\mathcal{W}$, where $\mathcal{W}$ is the vector bundle with the fiber W, associated to the principal vector bundle $E \xrightarrow{H} B$.

In particular, the "standard" Lagrangian submanifold $\tilde{w}^\star = \tilde{\phi}^\star = 0$ is compatible with gluing. The corresponding BRST operator is defined by the part of the BV action linear in the antifields:

$$Q_{\text{BRST}} = Q^\mu\frac{\partial}{\partial\phi^\mu} + Q^\nu\rho_*(A_\nu)_b^a\tilde{w}^b\frac{\partial}{\partial\tilde{w}^a} \tag{5.34}$$

After this canonical transformation of eqs. (5.28), (5.29) and (5.30), the new S_{cl} is such that this Q_{BRST} is nilpotent on-shell.

Gluing together $\widehat{\Phi}_\xi$: let us consider the relation between the functions $\widehat{\Phi}_\xi$ defined by eq. (5.6) on two overlapping charts. It is enough to consider the case of infinitesimal transition function, i.e. $u_{ji} = \mathbf{1} + \epsilon\alpha_{ji}$, where ϵ is infinitesimally small. With F_α defined in eq. (5.19), the difference between $\widehat{\Phi}_\xi$ on two coordinate charts is:

$$\delta_{ji}\widehat{\Phi}_\xi = \{\{S_{\text{tot}}, F_{\alpha_{ji}}\}, \widehat{\Phi}_\xi\} = -\{\{S_{\text{tot}}, \widehat{\Phi}_\xi\}, F_{\alpha_{ji}}\} + \{S_{\text{tot}}, \{F_{\alpha_{ji}}, \widehat{\Phi}_\xi\}\} \tag{5.35}$$

The first term on the r.h.s. is zero:

$$\{\{S_{\text{tot}}, \widehat{\Phi}_\xi\}, F_{\alpha_{ji}}\} = 0 \tag{5.36}$$

since F_α is diffeomorphism-invariant. Let us study the second term. We have:

$$\begin{aligned}
\{F_{\alpha_{ji}}, \widehat{\Phi}_\xi\} = \{F_{\alpha_{ji}}, \Phi_\xi\} &= \frac{1}{2} w^a \omega_{ab} \Phi_\xi^\mu \frac{\partial}{\partial \phi^\mu} (\alpha_{ji})_c^b w^c \\
&= -\frac{1}{2} \left(w^a \omega_{ab} \Phi_\xi^\mu (\delta_{ji} A_\mu - [\alpha_{ji}, A_\mu])_c^b w^c \right) \\
&= -F_{\Phi_\xi^\mu \delta_{ji} A_\mu} + \frac{1}{2} w^a \omega_{ab} \Phi_\xi^\mu [\alpha_{ji}, A_\mu]_c^b w^c \\
&= -F_{\Phi_\xi^\mu \delta_{ji} A_\mu} - \{\{S_{\text{tot}}, F_{\alpha_{ji}}\}, F_{\Phi^\mu A_\mu}\}
\end{aligned} \tag{5.37}$$

where A_μ is any connection, transforming as in eq. (5.25). Therefore the following expression:

$$\widehat{\Phi}'_\xi = \widehat{\Phi}_\xi + \{S_{\text{tot}}, F_{\Phi_\xi^\mu A_\mu}\} \tag{5.38}$$

is consistent on intersections of patches.

The correcting term $\{S_{\text{tot}}, F_{\Phi_\xi^\mu A_\mu}\}$ is the infinitesimal gauge transformation (see eqs. (5.18) and (5.19)) with the parameter $\Phi_\xi^\mu A_\mu$.

Back to $AdS_5 \times S^5$: in our case B is the pure spinor bundle over super-$AdS_5 \times S^5$; the coordinates ϕ are functions from the worldsheet to $PS\ AdS_5 \times S^5$ (defined in eq. (3.8)). The total space E is the space of maps from the worldsheet to $C_L \times C_R \times \mathrm{PSU}(2,2|4)$. Notice that $C_L \times C_R \times \mathrm{PSU}(2,2|4)$ is a principal H-bundle over $PS\ AdS_5 \times S^5$. It has a natural $\mathrm{PSU}(2,2|4)$-invariant connection, which for every tangent vector:

$$(\dot{\lambda}_L, \dot{\lambda}_R, \dot{g}) \in T\left(C_L \times C_R \times \mathrm{PSU}(2,2|4)\right) \tag{5.39}$$

declares its vertical component to be $(\dot{g} g^{-1})_{\bar{0}}$, i.e. the projection of $\dot{g}$ on the denominator of (3.3) using the Killing metric. This defines, pointwise, the connection on the space of maps.

It is natural to use this connection as A_μ in eq. (5.38).

Notice that we do *not* need a connection to write the BV Master Action (eq. (5.4)). But the connection *is* needed to construct $\widehat{\Phi}'_\xi$ (and also in our construction of the Lagrangian submanifold).

6 Taking apart the AdS sigma model

The standard action given by eq. (3.11) depends on the worldsheet complex structure and is polynomial in the pure spinor variables. In the BV formalism, it corresponds to a specific choice of the Lagrangian submanifold. We can change the action to a physically equivalent one, by adding BRST quartets and/or deforming the Lagrangian submanifold. We can ask ourselves, what is the simplest formulation of the theory, in the BV language, preserving the symmetries of $AdS_5 \times S^5$? (Of course, the notion of "being the simplest" is somewhat

subjective.) In this paper we gave an example of such a "minimalistic" formulation:

$$
\begin{aligned}
S_{\mathrm{BV}} &= S_{(g,\lambda)} + S_{(\omega)} \\
&= \int \mathrm{STr}\,(J_1 \wedge (1 - \mathbf{P}_{31})J_3 - J_1 \wedge \mathbf{P}_{31}J_3) \\
&\quad + \int \mathrm{STr}\,(\lambda_3 L_1^\star + \lambda_1 L_3^\star) + \int \frac{1}{2}\mathrm{STr}\,(\omega_3^\star \wedge \omega_1^\star)
\end{aligned}
\tag{6.1}
$$

Here $L^\star$ are the BV Hamiltonians of the left shift, eq. (4.1). The relation of eq. (6.1) to the original BV action (3.37) is through adding BRST quartet (section 5) and canonical transformations (eqs. (3.24), (5.7), (5.10)). Subjectively, eq. (6.1) is the simplest way of presenting the worldsheet Master Action for $AdS_5 \times S^5$.

The Master Action (6.1) does not depend on the worldsheet metric. The dependence on the worldsheet metric (through the complex structure) comes later when we choose the Lagrangian submanifold.

The way eq. (6.1) is written, it seems that w is completely decoupled from g and λ. But the transition functions on overlapping charts, described in section 5.4, do mix the two sets of fields.

The Master Action (6.1) is non-polynomial in λ, because of $\mathbf{P}_{31}$.

7   Generalization

Consider a sigma-model whose target space is some supermanifold $\mathcal{X}$. Suppose that $\mathcal{X}$ is equipped with a nilpotent odd vector field $Q \in \mathrm{Vect}(\mathcal{X})$, generating a *gauge symmetry* of the sigma-model. In minimalistic sigma-models *the BRST operator is just an odd nilpotent vector field on the target space.*

This means that the field configuration $X(\sigma,\tau)$ has the same action as $e^{\epsilon(\sigma,\tau)Q}X(\sigma,\tau)$ for an arbitrary odd gauge parameter function ϵ on the worldsheet:

$$
S[X] = S[e^{\epsilon Q}X]
\tag{7.1}
$$

Locally and away from the fixed points of Q this implies that one of the target space fermionic coordinates completely decouples from the action (the action does not depend on it). In case of pure spinor sigma-model, this gauge symmetry does not account for *all* degeneracy of the action. All directions in the θ space tangent to the pure spinor cones are degenerate directions of the quadratic part of the action.

Let us add an additional scalar field on the worldsheet $\Lambda(\sigma,\tau)$ and consider the following solution of the Master Equation:

$$
S_{\mathrm{BV}} = S + \int \Lambda Q^A(X) X_A^\star
\tag{7.2}
$$

In the pure spinor case $\mathcal{X}$ is parametrized by $g \in \mathrm{PSU}(2,2|4)$ and λ_L, λ_R modulo rescaling (i.e. projective pure spinors).

In Type II pure spinor theory, there are actually *two* anticommuting BRST symmetries, Q_L and Q_R, and the term in S_{BV} linear in antifields is

$$
\int \Lambda_L Q_L^A(X) X_A^\star + \Lambda_R Q_R^A(X) X_A^\star
\tag{7.3}
$$

The action S is given by eq. (3.20). Such a theory requires regularization.

The minimalistic sigma-model action is written in terms of the target space metric $\mathbf{G}$ and the B-field $\mathbf{B}$. For example, the action of eq. (3.20) corresponds to:

$$\mathbf{G} = \mathrm{Str}\left(\frac{1}{2}J_2 J_2 + J_1(\mathbf{1} - \mathbf{P}_{31})J_3\right) \tag{7.4}$$

$$\mathbf{B} = \mathrm{Str}\left(\frac{1}{2}J_1 \wedge J_3 - J_1 \wedge \mathbf{P}_{31}J_3\right) \tag{7.5}$$

The existence of the b-ghost is equivalent to the metric being the Lie derivative along Q of some symmetric tensor $\mathbf{b}$:

$$\mathbf{G} = \mathcal{L}_Q \mathbf{b} \tag{7.6}$$

where $\mathcal{L}_Q$ is the Lie derivative along the vector field Q. In our case (appendix B):

$$\mathbf{b} = \frac{\mathrm{Tr}\left((\{J_3, \lambda_3\} - \{J_1, \lambda_1\})J_2\right)}{\mathrm{STr}(\lambda_3 \lambda_1)} \tag{7.7}$$

As in section 3.4, the part of the action involving the target space metric $\mathbf{G}$ is BRST exact.

8 Open problems

We did not verify that $\widehat{\Phi}'_\xi$ of eq. (5.38) satisfies the conditions[14] formulated in [2, 3]. In particular, we may hope for $\{\widehat{\Phi}'_\xi, \widehat{\Phi}'_\xi\} = 0$, but more complicated scenarios are also possible. We believe that the invariance of our construction under the symmetries of $AdS_5 \times S^5$ is important to satisfy those conditions at the quantum level.

We did not explicitly calculate the restriction of the $\widehat{\Phi}'_\xi$ to the standard family of Lagrangian submanifolds, corresponding to the integration over the space of metrics. It can probably be expressed in terms of $\mathcal{O}$ where $\overline{\partial}b = Q\mathcal{O}$ as calculated in [1]. In any case, it is most likely nonzero, and therefore the string measure of eq. (1.1) is not just the product of Beltrami differentials, but involves also the curvature terms $\widehat{\Phi}'_F$.

A The projector

A.1 Definition

Let π_A and π_S denote the projectors:

$$\pi_A \; : \; T(AdS_5 \times S^5) \to T(AdS_5) \text{ projector along } T(S^5) \tag{A.1}$$

$$\pi_S \; : \; T(AdS_5 \times S^5) \to T(S^5) \text{ projector along } T(AdS_5) \tag{A.2}$$

$$\pi_A(v) + \pi_S(v) = \; v \tag{A.3}$$

---

[14]http://andreimikhailov.com/math/bv/omega/Equivariant_Form.html

For any vector v, we will denote by $\overline{v}$ the difference of its AdS_5 and S^5 components:

$$\overline{v} \stackrel{\text{def}}{=} \pi_A(v) - \pi_S(v) \tag{A.4}$$

The projector $\mathbf{P}_{13} : \mathbf{g}_1 \to \mathbf{g}_1$ was defined in [10] as follows:

$$\mathbf{P}_{13}A_1 = A_1 + [S_2, \lambda_3] \tag{A.5}$$

$$\{\lambda_1 , \mathbf{P}_{13}A_1\} = 0 \tag{A.6}$$

where $S_2 \in \mathbf{g}_2$ is adjusted to satisfy (A.6). In fact $\mathbf{P}_{13}$ is the projection to the tangent space TC_R along the space $T^\perp C_L$ which is orthogonal to TC_L with respect to the metric defined by Str:

$$(\mathbf{1} - \mathbf{P}_{13})A_1 \in T^\perp C_L \tag{A.7}$$

In other words, for generic λ_3 and λ_1 we have an exact sequence:

$$0 \longrightarrow T^\perp C_L \stackrel{i}{\longrightarrow} \mathbf{g}_1 \stackrel{\mathbf{P}_{13}}{\longrightarrow} TC_R \longrightarrow 0 \tag{A.8}$$

In section A.3 we will give an explicit formula for $\mathbf{P}_{13}$ following [1].

A.2 Matrix language

It turns out that computations can often be streamlined by thinking about elements of $\mathbf{g}$ literally as 4|4-matrices. In fact $\mathbf{g}$ is a *factorspace* of $sl(4|4)$ modulo a subspace generated by the unit matrix. Therefore, when talking about a matrix corresponding to an element of $\mathbf{g}$, we have to explain every time how we choose a representative. The $\mathbf{Z}_4$ grading of $psl(4|4)$ can be extended to $sl(4|4)$; the unit matrix has grade two. Therefore, the ambiguity of adding a unit matrix only arises for representing elements of $\mathbf{g}_2$. To deal with this problem, we introduce some notations. Given a matrix X of grade two, we denote by X_{TL} the corresponding traceless matrix:

$$X_{\text{TL}} = X - \frac{\text{Tr}(X)}{8}\mathbf{1} \tag{A.9}$$

(The subscript "TL" is an abbreviation for "traceless".) Also, it is often useful to consider 4|4-matrices with nonzero supertrace. Such matrices do not correspond to any elements of $\mathbf{g}$. For a 4|4-matrix Y we define:

$$Y_{\text{STL}} = Y - \frac{\text{STr}(Y)}{8}\Sigma \tag{A.10}$$

$$\text{where} \quad \Sigma = \text{diag}(1,1,1,1,-1,-1,-1,-1) \tag{A.11}$$

In particular:

$$(Y_{\text{TL}})_{\text{STL}} = (Y_{\text{STL}})_{\text{TL}} = Y - \frac{\text{Tr}(Y)}{8}\mathbf{1} - \frac{\text{STr}(Y)}{8}\Sigma \tag{A.12}$$

We also define, for any even matrix Y:

$$\overline{Y} = Y\Sigma = \Sigma Y \tag{A.13}$$

This definition agrees with eq. (A.4).

A.3 Explicit formula for the projector

In fact S_2 is given by the following expression:

$$S_2 = \frac{2}{\mathrm{Str}(\lambda_1 \lambda_3)} \overline{\{\lambda_1, A_1\}}_{\mathrm{STL}} \tag{A.14}$$

Notice that $\overline{\{\lambda_1, A_1\}}_{\mathrm{STL}}$ is actually both super-traceless *and* traceless; it is the same as $\overline{\{\lambda_1, A_1\}}_{\mathrm{TL}}$ (with the overline extending over "TL"). We have to prove that the S_2 defined this way satisfies (A.6). Indeed, we have:

$$[S_2, \lambda_3] = \frac{2}{\mathrm{Str}(\lambda_1 \lambda_3)} \left[\overline{\{\lambda_1, A_1\}}_{\mathrm{STL}}, \lambda_3 \right] \tag{A.15}$$

and we have to prove eq. (A.6). We have:

$$
\begin{aligned}
\left\{ \lambda_1, \left[\overline{\{\lambda_1, A_1\}}_{\mathrm{STL}}, \lambda_3 \right] \right\} &= \{ \lambda_1, [\Sigma \{\lambda_1, A_1\}_{\mathrm{TL}}, \lambda_3] \} \\
&= -\Sigma [\lambda_1, \{\{\lambda_1, A_1\}_{\mathrm{TL}}, \lambda_3\}] \\
&= -\Sigma \{[\lambda_1, \lambda_3], \{\lambda_1, A_1\}_{\mathrm{TL}}\}
\end{aligned} \tag{A.16}
$$

Both $\{\lambda_1, A_1\}$ and $[\lambda_1, \lambda_3]$ have $\mathbf{Z}_4$-grading two. Let us use:

$$[\lambda_1, \lambda_3] = \frac{1}{4} \mathrm{Str}(\lambda_1 \lambda_3) \Sigma + [\lambda_1, \lambda_3]_{\mathrm{STL}} \tag{A.17}$$

For all grade $\bar{2}$ matrices A_2 and B_2 such that $\mathrm{Tr} A_2 = \mathrm{Tr} B_2 = \mathrm{STr} A_2 = \mathrm{Str} B_2 = 0$ the following identity holds:[15]

$$\{A_2, B_2\} = A_2 B_2 + B_2 A_2 = \frac{1}{4} \left(\mathrm{Str}(A_2 B_2) \Sigma + \mathrm{Tr}(A_2 B_2) \mathbf{1} \right) \tag{A.18}$$

Therefore:

$$\left\{ \lambda_1, \left[\overline{\{\lambda_1, A_1\}}, \lambda_3 \right] \right\} \texttt{ mod } \mathbf{1} = -\frac{1}{2} \mathrm{Str}(\lambda_1 \lambda_3) \{\lambda_1, A_1\}_{\mathrm{TL}} \tag{A.19}$$

(where "$\texttt{mod1}$" means "modulo the center of $psl(4|4)$", i.e. up to a multiple of the unit matrix). This proves (A.6).

The central part of $\{\lambda_1, [\overline{\{\lambda_1, A_1\}}, \lambda_3]\}$ is generally speaking nonzero:

$$
\begin{aligned}
\mathrm{Tr} \left\{ \lambda_1, \left[\overline{\{\lambda_1, A_1\}}, \lambda_3 \right] \right\} &= 2 \mathrm{Tr} \left(\lambda_1 \left[\overline{\{\lambda_1, A_1\}}, \lambda_3 \right] \right) \\
&= -2 \mathrm{Tr} \left([\lambda_1, \lambda_3]_{\mathrm{STL}} \Sigma \{\lambda_1, A_1\} \right) \\
&= -2 \mathrm{STr} \left([\lambda_1, \lambda_3]_{\mathrm{STL}} \{\lambda_1, A_1\} \right)
\end{aligned} \tag{A.20}
$$

In Γ-matrix notations, $[\lambda_1, \lambda_3]_{\mathrm{STL}}$ is $(\lambda_1, \overline{\Gamma}^m \lambda_3)$ and $\{\lambda_1, A_1\}$ is $(\lambda_1, \Gamma^m A_1)$.

---

[15]http://andreimikhailov.com/math/pure-spinor-formalism/AdS5xS5/Symmetries.html

Let us define (*cp.* eq. (A.14)):

$$\mathbf{S}_{2/1} : \mathbf{g}_1 \to \mathbf{g}_2$$
$$\mathbf{S}_{2/1}A_1 = \frac{2}{\mathrm{Str}(\lambda_1\lambda_3)}\overline{\{\lambda_1, A_1\}}_{\mathrm{STL}} \tag{A.21}$$

$$\mathbf{S}_{2/3} : \mathbf{g}_3 \to \mathbf{g}_2$$
$$\mathbf{S}_{2/3}A_3 = \frac{2}{\mathrm{Str}(\lambda_3\lambda_1)}\overline{\{\lambda_3, A_3\}}_{\mathrm{STL}} \tag{A.22}$$

so that:

$$\mathbf{P}_{13}A_1 = A_1 + [\mathbf{S}_{2/1}A_1, \lambda_3] \tag{A.23}$$
$$\mathbf{P}_{31}A_3 = A_3 + [\mathbf{S}_{2/3}A_3, \lambda_1] \tag{A.24}$$

A.4   Properties of $\mathbf{P}_{13}$ and $\mathbf{P}_{31}$

It follows from the definition, that for any $v_2 \in \mathbf{g}_{\bar{2}}$ we have

$$\mathbf{P}_{13}[v_2, \lambda_3] = 0 \tag{A.25}$$

Let us verify this explicitly using the definition (A.5) with the explicit expression for S_2 given by (A.14). We have:

$$\mathbf{P}_{13}[v_2, \lambda_3] = [v_2, \lambda_3] + \frac{2}{\mathrm{Str}(\lambda_1\lambda_3)}\left[\overline{\{\lambda_1, [v_2, \lambda_3]\}}, \lambda_3\right] \tag{A.26}$$

Consider the expression $[\overline{\{\lambda_1, [v_2, \lambda_3]\}}, \lambda_3]$:

$$\left[\overline{\{\lambda_1, [v_2, \lambda_3]\}}, \lambda_3\right] = [\Sigma\{\lambda_1, [v_2, \lambda_3]\}, \lambda_3] \tag{A.27}$$

$$= -[\Sigma\{[\lambda_1, \lambda_3], v_2\}, \lambda_3] + [\Sigma\{\lambda_1, v_2\}, \lambda_3], \lambda_3] \tag{A.28}$$

Let us consider the first expression on the r.h.s. of (A.27). Using (A.17) we rewrite:

$$-[\Sigma\{[\lambda_1, \lambda_3], v_2\}, \lambda_3] = -\frac{1}{4}\mathrm{Str}(\lambda_1\lambda_3)[\Sigma\{\Sigma, v_2\}, \lambda_3] = -\frac{1}{2}\mathrm{Str}(\lambda_1\lambda_3)[v_2, \lambda_3] \tag{A.29}$$

This cancels with the first term on the r.h.s. of (A.26). And the second expression on the r.h.s. of (A.27) is zero:

$$[\Sigma[\{\lambda_1, v_2\}, \lambda_3], \lambda_3] = [\{\Sigma\{\lambda_1, v_2\}, \lambda_3\}, \lambda_3] = 0 \tag{A.30}$$

A.5   Subspaces of $\mathbf{g}$ associated to pure spinors

Consider the decomposition:

$$\mathbf{g}_2 = \mathbf{g}_{2L} \oplus \mathbf{g}_{2R} \oplus \mathbf{C}[\lambda_3, \lambda_1]_{\mathrm{STL}} \oplus \mathbf{C}\overline{[\lambda_3, \lambda_1]}_{\mathrm{TL}} \tag{A.31}$$

Here $\mathbf{g}_{2L}$ is a 4-dimensional subspace Tr-orthogonal to $\mathbf{C}\overline{[\lambda_3, \lambda_1]}_{\mathrm{TL}}$ and commuting with λ_3, and $\mathbf{g}_{2R}$ is Tr-orthogonal to $\mathbf{C}\overline{[\lambda_3, \lambda_1]}_{\mathrm{TL}}$ and commuting with λ_1.

Similarly we can refine $T^\perp C_R$ and $T^\perp C_L$:

$$\mathbf{g}_3 \supset T^\perp C_R = [\mathbf{g}_{2L}, \lambda_1] \oplus \mathbf{C}[[\lambda_3, \lambda_1]_{\mathrm{STL}}, \lambda_1] \tag{A.32}$$
$$\mathbf{g}_1 \supset T^\perp C_L = [\mathbf{g}_{2R}, \lambda_3] \oplus \mathbf{C}[[\lambda_3, \lambda_1]_{\mathrm{STL}}, \lambda_3] \tag{A.33}$$

B BRST variation of the *b*-tensor

Here we will prove:

$$(Q_L + Q_R)\frac{\text{Tr}\{J_1, \lambda_1\}J_2}{\text{Str}(\lambda_3 \lambda_1)} = -\text{Str}\left(\frac{1}{4}J_2 J_2 + \frac{1}{2}J_1(\mathbf{1} - \mathbf{P}_{31})J_3\right) \tag{B.1}$$

(Remember that $\text{Tr} \ldots = \text{Str}(\ldots \Sigma)$.) In fact, only Q_L contributes; the action of Q_R is zero:

$$Q_R \,\text{Str}\left(\{J_1, \lambda_1\}J_2 \Sigma\right) = -\text{Str}\left(\{J_1, \lambda_1\}\{J_1, \lambda_1\}_{\text{TL}}\right) = 0 \tag{B.2}$$

because J_1 is a fermion. Let us compute the action of Q_L:

$$\begin{aligned}
Q_L \,\text{Str}\left(\{J_1, \lambda_1\}J_2 \Sigma\right) = & -\text{Str}\left((\{[J_2, \lambda_3], \lambda_1\}J_2 + \{J_1, \lambda_1\}\{J_3, \lambda_3\}_{\text{TL}})\,\Sigma\right) \\
= & -\text{Tr}\left(\{J_2, [J_2, \lambda_3]\}\lambda_1\right) + \text{Str}\left(\overline{\{J_1, \lambda_1\}}_{\text{STL}}\{J_3, \lambda_3\}\right) \\
= & -\frac{1}{4}\text{Str}(\lambda_3 \lambda_1)\text{Str}(J_2^2) + \text{Str}\left(J_3 \left[\overline{\{J_1, \lambda_1\}}_{\text{STL}}, \lambda_3\right]\right) \\
= & -\frac{1}{4}\text{Str}(\lambda_3 \lambda_1)\text{Str}(J_2^2) - \frac{1}{2}\text{Str}(\lambda_3 \lambda_1)\text{Str}\left(J_3(\mathbf{1} - \mathbf{P}_{13})J_1\right) \tag{B.3}
\end{aligned}$$

References

[1] N. Berkovits and L. Mazzucato, *Taming the b antighost with Ramond-Ramond flux*, *JHEP* **11** (2010) 019 [`arXiv:1004.5140`] [INSPIRE].

[2] A. Mikhailov and A. Schwarz, *Families of gauge conditions in BV formalism*, *JHEP* **07** (2017) 063 [`arXiv:1610.02996`] [INSPIRE].

[3] A. Mikhailov, *Integration over families of Lagrangian submanifolds in BV formalism*, *Nucl. Phys.* **B 928** (2018) 107 [`arXiv:1611.04978`] [INSPIRE].

[4] N. Berkovits, *Simplifying and extending the $AdS_5 \times S^5$ pure spinor formalism*, *JHEP* **09** (2009) 051 [`arXiv:0812.5074`] [INSPIRE].

[5] M. Tonin, *On semiclassical equivalence of Green-Schwarz and pure spinor strings in $AdS_5 \times S^5$*, *J. Phys.* **A 46** (2013) 245401 [`arXiv:1302.2488`] [INSPIRE].

[6] R. Lipinski and A. Mikhailov, to appear.

[7] N. Berkovits, *Super Poincaré covariant quantization of the superstring*, *JHEP* **04** (2000) 018 [`hep-th/0001035`] [INSPIRE].

[8] N. Berkovits and C. Vafa, *Towards a worldsheet derivation of the Maldacena Conjecture*, *JHEP* **03** (2008) 031 [`arXiv:0711.1799`] [INSPIRE].

[9] M. Alexandrov, A. Schwarz, O. Zaboronsky and M. Kontsevich, *The geometry of the master equation and topological quantum field theory*, *Int. J. Mod. Phys.* **A 12** (1997) 1405 [`hep-th/9502010`] [INSPIRE].

[10] O.A. Bedoya, L.I. Bevilaqua, A. Mikhailov and V.O. Rivelles, *Notes on β-deformations of the pure spinor superstring in $AdS_5 \times S^5$*, *Nucl. Phys.* **B 848** (2011) 155 [`arXiv:1005.0049`] [INSPIRE].

[11] K.M. Apfeldorf and C. Ordonez, *Field redefinition invariance in quantum field theory*, *Nucl. Phys.* **B 479** (1996) 515 [`hep-th/9408100`] [INSPIRE].

[12] D. Kreimer and A. Velenich, *Field diffeomorphisms and the algebraic structure of perturbative expansions*, *Lett. Math. Phys.* **103** (2013) 171 [`arXiv:1204.3790`] [INSPIRE].

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