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Radiative leptonic decay $B \to \gamma\ell\nu_\ell$ with subleading power corrections

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Abstract: We reconsider the QCD predictions for the radiative decay $B \to \gamma\ell\nu_\ell$ with an energetic photon in the final state by taking into account the $1/E_\gamma, 1/m_b$ power-suppressed hard-collinear and soft corrections from higher-twist B-meson light-cone distribution amplitudes (LCDAs). The soft contribution is estimated through a dispersion relation and light-cone QCD sum rules. The analysis of theoretical uncertainties and the dependence of the decay form factors on the leading-twist LCDA $\phi_+(\omega)$ shows that the latter dominates. The radiative leptonic decay is therefore well suited to constrain the parameters of $\phi_+(\omega)$, including the first inverse moment, $1/\lambda_B$, from the expected high-statistics data of the BELLE II experiment.

Keywords: Heavy Quark Physics, Perturbative QCD, Resummation

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1 Introduction

When the energy E_γ of the photon is large compared to the strong interaction scale Λ, the radiative leptonic decay $B \to \gamma \ell \nu_\ell$ of the charged B meson is the simplest decay that probes the light-cone structure of the B meson, relevant to QCD factorization of exclusive B decays [1]. In this respect, this decay represents the analogue of the $\gamma\gamma^* \to \pi^0$ form factor for mesons with a heavy quark, in which case the mass m_b of the quark sets the scale of the hard interaction. Factorization at leading power in an expansion of the decay amplitude in Λ/E_γ and Λ/m_b has been established [2, 3] to all orders in the strong coupling α_s. In this approximation, the branching fraction depends only on the leading-twist B-meson light-cone distribution amplitude (LCDA) $\phi_+(\omega)$ [4, 5]. More precisely, it is proportional to the square of the inverse moment $1/\lambda_B^2$, which is the most important B-meson LCDA parameter in exclusive decays. Yet, λ_B remains uncertain by a large factor with estimates ranging from 200 MeV favoured by non-leptonic decays [6, 7] to 460 ± 110 MeV from QCD sum rules [8]. The radiative leptonic decay has therefore been suggested as a measurement of λ_B [9]. Including next-to-leading logarithmic resummed radiative corrections, known next-to-leading power effects and an estimate of an unknown next-to-leading power form factor $\xi(E_\gamma)$, the partial branching fractions $\mathrm{Br}(B \to \gamma \ell \nu_\ell, E_\gamma > E_{\mathrm{cut}})$ have been predicted in [9] and have been employed by the BELLE collaboration to provide a constraint on λ_B from their complete data set [10]. The main limitation of this method is due to Λ/E_γ and Λ/m_b power corrections.

In this paper we attempt to quantify the leading power-suppressed effects. A factorization analysis of the radiative leptonic decay in next-to-leading power would be desirable and interesting by itself, but this can presently not be done with rigour comparable to leading power due to a lack of understanding of "endpoint contributions" in the LCDAs, where the spectator partons in the B meson carry an anomalously small momentum fraction $\omega \ll \Lambda$. We therefore resort to the light-cone sum rule technique [11], which expresses the contribution of the endpoint region in the partonic calculation through a dispersion relation in terms of hadronic resonance parameters and B-meson LCDAs. This technique was originally applied to the analogous problem for the $\gamma\gamma^* \to \pi^0$ form factor [12, 13] and for the problem at hand in [14] in the tree-level and leading-twist approximation for the B-meson LCDAs. The one-loop correction to the leading-twist approximation for the dispersive representation of the soft contribution was added in [15]. The reanalysis [15] of the predicted branching fraction including these new contributions led to a considerable weakening of the bounds on λ_B. The purpose of the present paper is twofold: first, we focus on power corrections from higher-twist B-meson LCDAs using the complete parametrization of these LCDAs from [16]. Second, we perform an extensive analysis of the model dependence by quantifying the uncertainty through different families of B-meson LCDA models with a consistent implementation of the equation-of-motion constraints. Taken together, this results in a more reliable assessment of the potential of radiative leptonic decay for determining the inverse-moment parameter λ_B than in previous work [9, 15].

The outline of the paper is as follows. In section 2 we provide the relevant definitions, kinematics and notation. The subsequent two sections 3 and 4 contain the results for the power-suppressed hard-collinear contributions to the form factor and the dispersive representation of the soft endpoint contributions, respectively. Section 5 presents the numerical analysis of the form factors including the above results in several B-meson LCDA models. We summarize in section 6. Appendix A collects formulae for and relations between the two- and three-particle B-meson LCDAs up to twist four employed in this work.

2 Kinematics and notation

The radiative leptonic B-meson decay amplitude[1]

$$A(B^- \to \gamma\ell\bar{\nu}_\ell) = \frac{G_F V_{ub}}{\sqrt{2}} \langle \ell\bar{\nu}_l\gamma|\bar{\ell}\,\gamma^\nu(1-\gamma_5)\nu_\ell\bar{u}\gamma_\nu(1-\gamma_5)b|B^-\rangle \tag{2.1}$$

can be written in terms of two form factors, F_V and F_A, defined through the Lorentz decomposition of the hadronic tensor

$$\begin{aligned} T_{\mu\nu}(p,q) &= -i\int d^4x\, e^{ipx}\langle 0|T\{j_\mu^{em}(x)\,\bar{u}(0)\gamma_\nu(1-\gamma_5)b(0)\}|B^-(p+q)\rangle \\ &= \epsilon_{\mu\nu\tau\rho}p^\tau v^\rho F_V + i\big[-g_{\mu\nu}(pv) + v_\mu p_\nu\big]F_A - i\frac{v_\mu v_\nu}{(pv)} f_B m_B + p_\mu\text{-terms}\,. \end{aligned} \tag{2.2}$$

Here p and q are the photon and lepton-pair momenta, respectively, so that $p+q = m_B v$ is the B-meson momentum in terms of its four-velocity. In the above $j^\mu_{\rm em} = \sum_q e_q\bar{q}\gamma_\mu q$ is the electromagnetic current. The $v_\mu v_\nu$ term is fixed by the Ward identity [9, 17]

$$p^\mu T_{\mu\nu} = -if_B m_B v_\nu \tag{2.3}$$

[1]In the following, ℓ may refer to the electron or muon. The muon mass is set to zero in the kinematic expressions below.

and the terms proportional to p_μ contract to zero with the photon polarization vector, see [9] for more details.

The form factors can be written as functions of the lepton-pair invariant mass squared q^2, or, equivalently, of the photon energy $E_\gamma = vp$ in the B-meson rest frame:

$$q^2 = (m_B v - p)^2 = m_B^2 + p^2 - 2m_B E_\gamma \,. \tag{2.4}$$

For a real photon, $p^2 = 0$ and

$$E_\gamma = \frac{m_B^2 - q^2}{2m_B}\,, \qquad 0 \le E_\gamma \le \frac{m_B}{2}\,, \qquad 0 \le q^2 \le m_B^2\,. \tag{2.5}$$

The differential decay width is given by

$$\frac{d\Gamma}{dE_\gamma} = \frac{\alpha_{\rm em} G_F^2 |V_{ub}|^2}{6\pi^2} m_B E_\gamma^3 \left(1 - \frac{2E_\gamma}{m_B}\right) \left(\Big|F_V\Big|^2 + \Big|F_A + \frac{e_\ell f_B}{E_\gamma}\Big|^2 \right), \tag{2.6}$$

where, following [9],[2] the contact term in (2.2) is included in the axial form factor.

For large photon energies the form factors can be written as [9]

$$\begin{aligned} F_V(E_\gamma) &= \frac{e_u f_B m_B}{2E_\gamma \lambda_B(\mu)} R(E_\gamma,\mu) + \xi(E_\gamma) + \Delta\xi(E_\gamma)\,, \\ F_A(E_\gamma) &= \frac{e_u f_B m_B}{2E_\gamma \lambda_B(\mu)} R(E_\gamma,\mu) + \xi(E_\gamma) - \Delta\xi(E_\gamma)\,. \end{aligned} \tag{2.7}$$

The first term is equal in both expressions and represents the leading-power contribution in the heavy-quark expansion (HQE). It originates only from photon emission from the light spectator quark in B meson (figure 1). In the above, f_B is the decay constant of B meson, and the quantity λ_B is the first inverse moment of the B-meson LCDA,

$$\frac{1}{\lambda_B(\mu)} = \int_0^\infty \frac{d\omega}{\omega}\, \phi_+(\omega,\mu)\,. \tag{2.8}$$

The factor $R(E_\gamma,\mu)$ in (2.7) takes into account radiative corrections (see [9] for details) and equals one at the tree level.

The remaining terms in (2.7) are the power-suppressed, $1/m_b$ and $1/(2E_\gamma)$, corrections. They are written as a sum of the "symmetry-preserving" part, i.e. the same for the both form factors F_V and F_A, and the "symmetry-breaking" part which has opposite sign. The leading contributions to the symmetry-breaking part are [9]

$$\Delta\xi(E_\gamma) = \frac{e_b f_B m_B}{2E_\gamma m_b} + \frac{e_u f_B m_B}{(2E_\gamma)^2}\,. \tag{2.9}$$

The equality of the two form factors at leading power in the heavy-quark and large photon energy ($E_\gamma \sim m_b$) expansion is a consequence of the left-handedness of the weak interaction current and helicity-conservation of the quark-gluon interaction in the high-energy limit. In terms of the helicity form factors $F_\mp \equiv (F_V \pm F_A)/2$, the above implies

[2]Note the change of notation: F_A is denoted by $\hat{F}_A$ in [9].

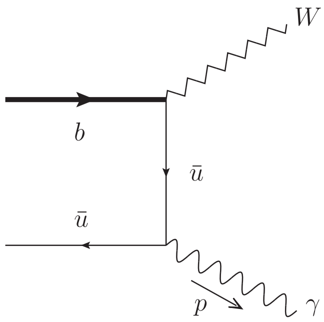

Figure 1. Leading contribution to $B \to \gamma \ell \nu_\ell$.

that $F_+ = \Delta\xi$ vanishes at leading power, while ξ represents the power correction to the non-vanishing helicity form factor F_-. Our aim is to provide improved estimates of $\xi(E_\gamma)$ and $\Delta\xi(E_\gamma)$, for which currently factorization formulae are not available. We split the calculation into "higher-twist corrections" of order Λ/E_γ and Λ/m_b from the region where the currents in (2.2) are separated by a small light-cone distance $x^2 \sim 1/(m_b\Lambda)$, and the soft or endpoint corrections. In case of the former, the virtuality of the quark propagator that joins the weak and electromagnetic vertex (see figure 1) is hard-collinear, that is of order $m_b\Lambda$. The latter arise when the light-cone projection ω of the spectator-quark momentum become anomalously small $\omega \ll \Lambda$, such that the quark propagator virtuality $E_\gamma\omega$ enters the soft region, Λ^2. Note that in this work "soft" always refers to the virtuality of the propagator, and should be distinguished from the "soft region" in the factorization treatments [2, 3, 9] where it refers to the region $\omega \sim \Lambda$, which defines the B-meson LCDA.

3 Higher-twist corrections

The higher-twist corrections can be accessed via the light-cone expansion of the weak-electromagnetic current product at $x^2 \to 0$.

At tree level one can replace the b-quark by the effective heavy quark field

$$\bar{q}\gamma_\mu b = \bar{q}\gamma_\mu h_v + \frac{1}{2m_b}\bar{q}\gamma_\mu i\vec{\not{D}} h_v + \dots \tag{3.1}$$

where $\vec{\not{D}} = \not{D} - (v\cdot D)\not{v}$. Then for the u-quark contribution (figure 1) we get

$$\begin{aligned} T^{(u)}_{\mu\nu}(p,q) &= -ie_u\sqrt{m_B}\int d^4x\, e^{ipx}\langle 0|T\{\bar{u}(x)\gamma_\mu u(x)\,\bar{u}(0)\gamma_\nu(1-\gamma_5)h_v(0)\}|B(v)\rangle \\ &\quad -\frac{ie_u\sqrt{m_B}}{2m_b}\int d^4x\, e^{ipx}\langle 0|T\{\bar{u}(x)\gamma_\mu u(x)\,\bar{u}(0)\gamma_\nu(1-\gamma_5)i\vec{\not{D}}h_v(0)\}|B(v)\rangle + \dots \end{aligned} \tag{3.2}$$

up to terms $\mathcal{O}(1/m_b^2)$ which will be neglected.

We should note that (3.1) does not represent the correct heavy-quark/large-energy expansion of the weak current when the covariant derivative contains a collinear gluon field. In this case the tree-level expansion of the current in soft-collinear effective theory must be

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used (see [18]). However, we shall make use of the above expression only to compute the light-cone expansion of the u-quark propagator in a soft-gluon background. Whenever a diagram involves a hard-collinear gluon propagator (as is the case for radiative corrections and the soft factorizable four-particle contribution discussed in the following section), we compute the corresponding contribution with the QCD current and quark-gluon vertex rather than the effective ones.

For the contribution in the first line of (3.2), using the light-cone expansion of the quark propagator [19]

$$\overline{u(x)\bar{u}}(0) = \frac{i}{2\pi^2}\frac{\not{x}}{x^4} - \frac{1}{8\pi^2 x^2}\int_0^1 du \Big\{ix^\rho g\widetilde{G}_{\rho\sigma}(ux)\gamma^\sigma\gamma_5 + (2u-1)x^\rho g G_{\rho\sigma}(ux)\gamma^\sigma\Big\} + \ldots\,, \tag{3.3}$$

and the definitions of B-meson LCDAs collected in appendix A we obtain

$$\begin{aligned} T^{(u)}_{\mu\nu} &= \frac{ie_u f_B m_B}{2\pi^2}\int d^4x\,\frac{e^{ipx}}{x^4}\left[(vx)g_{\mu\nu} + i\epsilon_{\mu\nu\rho\sigma}x^\rho v^\sigma\right] \\ &\quad\times\left\{\Phi_+(vx) + x^2 G_+(vx) - \frac{x^2}{4}\int_0^1 du\,\Big[(2u-1)\Psi_4 - \widetilde{\Psi}_4\Big](vx,uvx)\right\} + \ldots \\ &= \frac{ie_u f_B m_B}{2\pi^2}\int d^4x\,\frac{e^{ipx}}{x^4}\left[(vx)g_{\mu\nu} + i\epsilon_{\mu\nu\rho\sigma}x^\rho v^\sigma\right] \\ &\quad\times\left\{\Phi_+(vx) + x^2 G_+^{\rm WW}(vx) - \frac{x^2}{2(vx)^2}\Phi_-^{\rm t3}(vx) - \frac{x^2}{4}\int_0^1 du\,\Big[\Psi_4 - \widetilde{\Psi}_4\Big]^{\rm t4}(vx,uvx)\right\} \\ &\quad + \ldots \end{aligned} \tag{3.4}$$

where in the second representation we combined the "genuine" higher-twist contributions to the two-particle DA G_+ with the contributions of three-particle LCDAs. The ellipses stand for the contributions proportional to $p_\mu(p_\nu)$, $v_\mu(v_\nu)$ and terms $\mathcal{O}(1/E_\gamma^3)$. Note that the first term $\Phi_+(vx)$ in the curly bracket produces not only the known leading-order ($R(E_\gamma,\mu) \to 1$ in (2.7)), leading-power expression that we drop here, but also a power correction that has to be retained.

Going over to the momentum space and identifying the two relevant Lorentz structures, we find from this expression

$$\begin{aligned} \xi^{\rm ht}_{1/E_\gamma} &= \frac{e_u f_B m_B}{4E_\gamma^2}\bigg\{1 - 2\frac{\bar\Lambda}{\lambda_B} + 2\int_0^\infty d\omega\,\ln\omega\,\phi_-^{\rm t3}(\omega) \\ &\qquad\qquad + \int_0^\infty\frac{d\omega_1}{\omega_1}\int_0^\infty\frac{d\omega_2}{\omega_1+\omega_2}\left[\psi_4 - \widetilde{\psi}_4\right]^{\rm t4}(\omega_1,\omega_2)\bigg\} \\ &= \frac{e_u f_B m_B}{4E_\gamma^2}\bigg\{-1 + 2\int_0^\infty d\omega\,\ln\omega\,\phi_-^{\rm t3}(\omega) - \int_0^\infty\frac{d\omega_1}{\omega_1}\int_0^\infty\frac{d\omega_2}{\omega_2}\left[\psi_4 + \widetilde{\psi}_4\right](\omega_1,\omega_2)\bigg\} \\ &= \frac{e_u f_B m_B}{4E_\gamma^2}\bigg\{-1 + 2\int_0^\infty d\omega\,\ln\omega\,\phi_-^{\rm t3}(\omega) - 2\int_0^\infty\frac{d\omega_2}{\omega_2}\,\phi_4(0,\omega_2)\bigg\}, \end{aligned} \tag{3.5}$$

$$\Delta\xi^{\rm ht}_{1/E_\gamma} = \frac{e_u f_B m_B}{4E_\gamma^2}\,, \tag{3.6}$$

where $\phi_-^{\rm t3}(\omega)$ is the "genuine" twist-three contribution to the LCDA $\phi_-(\omega)$, cf. (A.6). Interestingly, $\xi^{\rm ht}_{1/E_\gamma}$ only involves the three-particle LCDAs on the line $x=0$ in the (s,x)-representation [16], see (A.32), that are directly related by the equation of motion (EOM) relations (A.23), (A.30). This property allows one to rewrite the answer in several equivalent forms, as shown above. The $\Delta\xi^{\rm ht}_{1/E_\gamma}$-term arises from the first term $\Phi_+(vx)$ in the curly bracket and agrees with the corresponding contribution in (2.9) obtained in [9] by a different method.

From the analysis of the renormalization group behaviour [16] one expects

$$\begin{aligned}
&\phi_-^{\rm t3}(\omega)\sim\omega^0, &&\phi_4(0,\omega_2)\sim\omega_2^2, &&g_+^{\rm WW}(\omega)\sim\omega^2\,,\\
&\phi_3(\omega_1,\omega_2)\sim\omega_1\omega_2^2, &&\psi_4(\omega_1,\omega_2)\sim\widetilde\psi_4(\omega_1,\omega_2)\sim\omega_1\omega_2\,, &&
\end{aligned}\tag{3.7}$$

so that all integrals are endpoint-finite for small ω_i. Hence, to the twist-four accuracy there is no overlap with the soft region. Contributions of higher twist-five, six, etc., are suppressed by extra powers of the photon energy E_γ in the hard-collinear region. These contributions can, however, have power-like endpoint divergences that spoil the power counting. Hence, the soft endpoint contributions from higher-twist terms are not necessarily suppressed by powers of $1/E_\gamma$ relative to the twist-four terms. We will discuss this mechanism in more detail in the next section.

The contribution from the second line in (3.2) can be calculated using the operator identity

$$\bar q(x)\Gamma\,\overrightarrow{D}_\xi\,h_v(0)=\partial_\xi\bar q(x)\Gamma h_v(0)+i\int_0^1du\,\bar u\,\bar q(x)x^\rho gG_{\rho\xi}(ux)\Gamma h_v(0)-\frac{\partial}{\partial x^\xi}\bar q(x)\Gamma h_v(0)\,.\tag{3.8}$$

Since this contribution is suppressed by $1/m_b$, for this case we only need the leading term in the $1/E_\gamma$, expansion. We obtain

$$\begin{aligned}
\xi^{\rm ht}_{1/m_b}&=\frac{e_uf_Bm_B}{4m_bE_\gamma}\biggl\{\frac{\bar\Lambda}{\lambda_B}-2+\int_0^\infty d\omega\,\ln\omega\,\phi_-^{\rm t3}(w)\\
&\qquad+2\int_0^\infty\frac{d\omega_1}{\omega_1}\int_0^\infty\frac{d\omega_2}{\omega_2}\,\phi_3(\omega_1,\omega_2)\left\{1-\frac{\omega_1}{\omega_2}\ln\frac{\omega_1+\omega_2}{\omega_1}\right\}\biggr\},\\
&=\frac{e_uf_Bm_B}{4m_bE_\gamma}\biggl\{\frac{\bar\Lambda}{\lambda_B}-2+2\int_0^\infty\frac{d\omega_1}{\omega_1}\int_0^\infty\frac{d\omega_2}{\omega_1+\omega_2}\,\phi_3(\omega_1,\omega_2)\biggr\},
\end{aligned}\tag{3.9}$$

$$\Delta\xi^{\rm ht}_{1/m_b}=0\,.\tag{3.10}$$

The complete higher-twist corrections $1/E_\gamma, 1/m_b$ are given by the sum of the above two contributions

$$\begin{aligned}
\xi^{\rm ht}&=\xi^{\rm ht}_{1/E_\gamma}+\xi^{\rm ht}_{1/m_b},\\
\Delta\xi^{\rm ht}&=\frac{e_bf_Bm_B}{2E_\gamma m_b}+\frac{e_uf_Bm_B}{(2E_\gamma)^2}\,,
\end{aligned}\tag{3.11}$$

where for $\Delta\xi^{\rm ht}$ we have added the contribution of the photon emission from the b-quark [9]. The second equation of (3.11) agrees with the previous result (2.9). The absence of an endpoint divergence in the higher-twist correction arises as a consequence of non-trivial relations between the various terms in (3.4) and it would be interesting to understand this in the context of a factorization theorem for the $1/E_\gamma$ power corrections, which is, however, beyond the scope of the present paper. A previous attempt [15] to compute $\xi^{\rm ht}_{1/E_\gamma}$ did not include the G_+ term and used an incorrect parametrization of the three-particle matrix element (A.14), resulting in a qualitatively different, endpoint-divergent higher-twist correction.

4 Soft corrections

In addition to higher-twist corrections, the power-suppressed contributions to the form factors can originate from large distances between the currents in (2.2), $x^2 \sim 1/\Lambda^2$, which cannot be accessed in the light-cone expansion. The soft distance between the currents implies that the B-meson LCDA is probed in the endpoint region $\omega \ll \Lambda$. Such contributions may or may not be "visible" through the infrared (endpoint) divergences of the hard-collinear higher-twist contributions, and cannot be factorized in terms of the LCDAs without additional assumptions. We will use the approach suggested in [14] that is based on using dispersion relations and quark-hadron duality. This technique has originally been proposed for the study of the $\gamma^*\gamma \to \pi$ transition form factor [12] and has become the method of choice for this reaction, see e.g. [13, 20] for recent refinements. Our aim is to put the calculation of the radiative leptonic decay form factors on the same level as the $\gamma^*\gamma \to \pi$ transition form factor.

The starting point is the more general process $B \to \gamma^*\ell\nu_\ell$ with a transversely polarized, virtual photon with $p^2 < 0$. If $-p^2 \sim m_B\Lambda$, the correlation function in (2.2) does not receive any soft endpoint contribution and can be calculated (in principle) in terms of the B-meson LCDAs of increasing twist to arbitrary power in the $1/E_\gamma, 1/m_b, 1/p^2$ expansion. The idea is to access the real photon limit $p^2 = 0$ starting from this expansion by using the dispersion relation. In this way, the explicit evaluation of soft contributions is effectively replaced by a certain ansatz (assumption) for the hadronic spectral density in the p^2-channel. The procedure can be understood as the matching of two different representations for the correlation function (2.2) — the QCD calculation in terms of quarks and gluons vs. physical hadrons in the intermediate state — and is usually referred to as light-cone sum rules (LCSR) [11]. The LCSR approach provides a well-motivated model for the yet to be defined matrix elements for the endpoint contribution in the framework of the factorization treatment [2, 3, 9]. Whether or not the soft, endpoint contribution implies an endpoint divergence in the hard-collinear region in the factorization approach, is then related to the renormalization properties of these matrix elements.

On the one hand, one can argue on general grounds that the generalized form factors $F_{B\to\gamma^*}(E_\gamma, p^2)$ ($F_{B\to\gamma^*}$ refers to both, vector and axial, F_V and F_A) satisfy an unsubtracted dispersion relation in the variable p^2 at fixed $2m_B E_\gamma \equiv 2m_B\, vp = m_B^2 + p^2 - q^2$. Separating the contribution of the lowest-lying vector mesons ρ, ω, we write

$$F_{B\to\gamma^*}(E_\gamma, p^2) = \frac{f_\rho F_{B\to\rho}(q^2)}{m_\rho^2 - p^2} + \frac{1}{\pi}\int_{s_0}^\infty ds\, \frac{{\rm Im} F_{B\to\gamma^*}(E_\gamma, s)}{s - p^2}\,, \tag{4.1}$$

where s_0 defines an effective continuum threshold. For simplicity we combined here the ρ and ω contributions in one resonance term assuming $m_\rho \simeq m_\omega$ and the zero-width approximation. In this expression, f_ρ is the usual decay constant of the vector meson and $F_{B\to\rho}(q^2)$ is a generic $B \to \rho(\omega)$ transition form factor, whose explicit definition will not be needed. Since there are no massless states, the real photon limit is recovered by setting $p^2 \to 0$ in (4.1).

On the other hand, the same form factors can be calculated for sufficiently large $-p^2$ using QCD factorization. The result, $F^{\rm QCDF}_{B\to\gamma^*}(E_\gamma, p^2)$, satisfies a similar dispersion relation

$$F^{\rm QCDF}_{B\to\gamma^*}(E_\gamma, p^2) = \frac{1}{\pi}\int_0^\infty ds\, \frac{\mathrm{Im}\, F^{\rm QCDF}_{B\to\gamma^*}(E_\gamma, s)}{s-p^2}\,, \tag{4.2}$$

where the limit $p^2 \to 0$ cannot be taken directly. Singular terms in $1/p^2$ appear (cf. [13] for the case of the $\gamma\gamma^* \to \pi^0$ form factor), signalling that QCD factorization cannot be applied directly to the real photon case $p^2 = 0$ beyond the leading power in $1/m_b$ and $1/E_\gamma$.

The main assumption of the method (quark-hadron duality) is that the physical spectral density above the threshold s_0 coincides with the calculated in QCD spectral density upon averaging with a smooth weight function over a sufficiently broad interval of the energy s:

$$\mathrm{Im}\, F_{B\to\gamma^*}(E_\gamma, s) \simeq \mathrm{Im}\, F^{\rm QCDF}_{B\to\gamma^*}(E_\gamma, s) \qquad \text{for} \quad s > s_0\,. \tag{4.3}$$

For the simplest sum rule, one uses that the QCD factorization calculation must reproduce the "true" form factors $F_{B\to\gamma^*}(E_\gamma, p^2)$ for asymptotically large values of $-p^2$. Equating the two representations (4.1) and (4.2) at $p^2 \to -\infty$ and subtracting the contributions of $s > s_0$ from the both sides one obtains

$$f_\rho F_{B\to\rho}(q^2) = \frac{1}{\pi}\int_0^{s_0} ds\, \mathrm{Im}\, F^{\rm QCDF}_{B\to\gamma^*}(E_\gamma, s)\,. \tag{4.4}$$

In practical applications of this method one uses an additional trick [21] which allows one to reduce the sensitivity to the duality assumption in (4.3) and simultaneously suppress the contributions of higher twists in the light-cone expansion. This is done by passing to the Borel representation of the dispersion relation, which effectively substitutes $1/(s-p^2) \to \exp(-s/M^2)$. The net effect on (4.4) is the appearance of an additional weight factor under the integral:

$$f_\rho F_{B\to\rho}(q^2) = \frac{1}{\pi}\int_0^{s_0} ds\, e^{-(s-m_\rho^2)/M^2}\, \mathrm{Im}\, F^{\rm QCDF}_{B\to\gamma^*}(E_\gamma, s)\,. \tag{4.5}$$

The value of the Borel parameter M^2 corresponds, roughly speaking, to the inverse (Euclidean) distance at which the matching is done between the quark and hadron representations. In ideal case there should be no M^2-dependence so that varying M^2 within a certain window, usually $M^2 = 1 - 2\,\mathrm{GeV}^2$, one obtains an indication of the accuracy of the calculation.

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With this refinement, substituting (4.5) into (4.1) and using (4.3), one obtains for $p^2 \to 0$ [14]

$$
\begin{aligned}
F_{B\to\gamma}(E_\gamma) &= \frac{1}{\pi}\int_0^{s_0} \frac{ds}{m_\rho^2}\,\mathrm{Im}\,F^{\rm QCDF}_{B\to\gamma^*}(E_\gamma,s)\,e^{-(s-m_\rho^2)/M^2} + \frac{1}{\pi}\int_{s_0}^{\infty}\frac{ds}{s}\,\mathrm{Im}\,F^{\rm QCDF}_{B\to\gamma^*}(E_\gamma,s) \\
&= F^{\rm QCDF}_{B\to\gamma}(E_\gamma) + \xi^{\rm soft}_{B\to\gamma}(E_\gamma)\,.
\end{aligned} \tag{4.6}
$$

In passing to the second line we extend the lower limit of the second integral to 0 and subtract the added contribution from the first. In this way the second integral equals $F^{\rm QCDF}_{B\to\gamma}(E_\gamma) = F^{\rm QCDF}_{B\to\gamma^*}(E_\gamma, p^2 = 0)$ calculated using QCD factorization and

$$
\xi^{\rm soft}_{B\to\gamma}(E_\gamma) = \frac{1}{\pi}\int_0^{s_0}\frac{ds}{s}\left[\frac{s}{m_\rho^2}e^{-(s-m_\rho^2)/M^2} - 1\right]\mathrm{Im}\,F^{\rm QCD}_{B\to\gamma^*}(E_\gamma,s) \tag{4.7}
$$

is the soft correction that originates from the nonperturbative modification of the spectral density. Conceptually, the effect of this modification is to create a mass gap in the vector-meson mass spectrum. Separating in (4.7) the contributions that are the same for the form factors F_V and F_A, and those of opposite sign, we can decompose the soft correction in the "symmetry-preserving" part $\xi^{\rm soft}(E_\gamma)$, and the "symmetry-breaking" part $\Delta\xi^{\rm soft}(E_\gamma)$ in the notation of (2.7).

Note that both terms in the first line of (4.6) and hence the full result are finite, whereas the decomposition as the sum of the "pure" QCD factorization expression and the soft correction in the second line can in principle (but not in the above) produce logarithmic and/or power divergences from the $s \to 0$ region. In such cases (see example below), for bookkeeping purposes we will attribute the whole contribution to the soft correction.

In the following we apply (4.6) to the leading-power and higher-twist hard-collinear contributions calculated in [9] and in the previous section. That is, for each hard-collinear contribution to $F^{\rm QCDF}_{B\to\gamma}(E_\gamma)$ we obtain the corresponding $\xi^{\rm soft}_{B\to\gamma}(E_\gamma)$ due to the nonperturbative modification of the spectral function in the soft region.

The soft correction to the leading-order, leading-twist hard-collinear contribution given by the first term in the two equations (2.7) with $R(E_\gamma,\mu)$ set to 1 was considered in [14]. For the form factors at non-vanishing p^2, we obtain,

$$
F_V^{\rm (LO)}(E_\gamma,p^2) = F_A^{\rm (LO)}(E_\gamma,p^2) = e_u f_B m_B\, U_{\rm LL}\int_0^\infty d\omega\,\frac{\phi_+(\omega,\mu)}{2E_\gamma\omega - p^2}\,. \tag{4.8}
$$

Here $U_{\rm LL}$ is the renormalization-group factor $U(E_\gamma,\mu_{h1},\mu_{h2},\mu)$ [9] truncated to the leading-logarithmic approximation,[3] which sums large logarithms from the hard scales $\mu_{h1},\mu_{h2} \sim m_b, E_\gamma$ to a hard-collinear scale of order $\sqrt{-p^2}$. The integral in (4.8) can be converted to the form of a dispersion relation by the change of variables $s = 2E_\gamma\omega$. Following the procedure

[3] See appendix A of [9]. In the leading-logarithmic approximation, the $\alpha_s(\mu_h)$ terms in (A.3) are neglected. We follow the terminology of [9], which implies that LL includes the two-loop cusp and one-loop non-cusp anomalous dimension in the renormalization group equation, NLL three-loop cusp and two-loop non-cusp, and so on.

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described above and changing the integration variable back to $\omega = s/(2E_\gamma)$, we obtain

$$\begin{aligned}\xi^{\rm soft}_{\rm (LO)}(E_\gamma) &= \frac{e_u f_B m_B}{2E_\gamma}\, U_{\rm LL} \int_0^{\frac{s_0}{2E_\gamma}} d\omega \left[\frac{2E_\gamma}{m_\rho^2} e^{-(2E_\gamma\omega - m_\rho^2)/M^2} - \frac{1}{\omega}\right]\phi_+(\omega,\mu)\,,\\ \Delta\xi^{\rm soft}_{\rm (LO)}(E_\gamma) &= 0\,. \end{aligned} \tag{4.9}$$

The soft correction defined by (4.9) comes from the region $\omega < s_0/(2E_\gamma) \sim s_0/m_b$. With $\sqrt{s_0}$ a few times Λ, this is indeed an endpoint spectator-quark contribution corresponding to contribution from a soft distance $1/\Lambda$ between the weak and electromagnetic current in (2.2). For large scales $\mu \sim m_b$ the LCDA $\phi_+(\omega,\mu) \sim \omega$ for $\omega \to 0$, hence one obtains a power correction of the order of $s_0/(2E_\gamma\Lambda)$ for $E_\gamma \sim m_b \to \infty$ with respect to the leading, hard-collinear contribution, in agreement with the usual power counting for the soft form factor $\xi(E_\gamma)$. Since the shape of $\phi_+(\omega,\mu)$ is governed by the QCD scale Λ, while ω is restricted to values smaller than $s_0/(2E_\gamma) \ll \Lambda$ in (4.9), one might be tempted to approximate $\phi_+(\omega,\mu)$ by its asymptotic behaviour $\phi_+(\omega,\mu) \sim \omega$ as $\omega \to 0$. However, this would amount to the first term in an expansion of the integral in powers of $s_0/(E_\gamma\Lambda)$, which for realistic values of $s_0 \approx 1.5\ \text{GeV}^2$ and $E_\gamma \sim 1.5 - 2.5\,\text{GeV}$ is not a valid approximation. We therefore always keep the full functional form of the LCDA in the integrals for the soft contributions.

Applying the same method to the next-to-leading order $\mathcal{O}(\alpha_s)$ correction to the leading-twist contribution requires factorizing the hadronic tensor into a hard matching coefficient $C(E_\gamma, \mu_{h1})$ and a hard-collinear function, which is convoluted with $\phi_+(\omega)$. The hard function is independent of the hard-collinear variable $-p^2$ and is given in [9]. The hard-collinear function calculated for $p^2 = 0$ in [2, 3] must be generalized to $-p^2 \neq 0$. The result can be brought into the form [15]

$$\begin{aligned}\xi^{\rm soft}_{\rm (NLO)}(E_\gamma) &= \frac{e_u f_B m_B}{2E_\gamma}\, C(E_\gamma,\mu_{h1}) K^{-1}(\mu_{h2}) U(E_\gamma,\mu_{h1},\mu_{h2},\mu)\\ &\qquad\times \int_0^{\frac{s_0}{2E_\gamma}} d\omega' \left[\frac{2E_\gamma}{m_\rho^2}\, \mathrm{e}^{-(2E_\gamma\omega' - m_\rho^2)/M^2} - \frac{1}{\omega'}\right]\, \phi_+^{\rm eff}(\omega',\mu)\,,\\ \Delta\xi^{\rm soft}_{\rm (NLO)}(E_\gamma) &= 0\,, \end{aligned} \tag{4.10}$$

where "NLO" is meant to include the LO contribution and the prefactor includes the hard NLO matching correction and next-to-leading-logarithmic resummation as given in [9]. The convolution of the generalized hard-collinear function with $\phi_+(\omega,\mu)$ after applying the dispersive treatment and letting $p^2 \to 0$ at the end, is summarized in

$$\begin{aligned}\phi_+^{\rm eff}(\omega',\mu) = \phi_+(\omega',\mu) + \frac{\alpha_s(\mu)C_F}{4\pi}\Bigg\{ &\left(\ln^2\frac{\mu^2}{2E_\gamma\omega'} + \frac{\pi^2}{6} - 1\right)\phi_+(\omega',\mu)\\ &+ \left(2\ln\frac{\mu^2}{2E_\gamma\omega'} + 3\right)\omega' \int_{\omega'}^\infty d\omega \ln\frac{\omega-\omega'}{\omega'}\frac{d}{d\omega}\frac{\phi_+(\omega,\mu)}{\omega}\\ &- 2\ln\frac{\mu^2}{2E_\gamma\omega'}\int_0^{\omega'} d\omega \ln\frac{\omega'-\omega}{\omega'}\frac{d}{d\omega}\phi_+(\omega,\mu)\\ &+ \int_0^{\omega'} d\omega \ln^2\frac{\omega'-\omega}{\omega'}\frac{d}{d\omega}\left[\frac{\omega'}{\omega}\phi_+(\omega,\mu) + \phi_+(\omega,\mu)\right]\Bigg\}. \end{aligned} \tag{4.11}$$

The hard-collinear NLO contribution [9] can be written in this notation as

$$\frac{J(E_\gamma,\mu)}{\lambda_B(\mu)} = \int_0^\infty \frac{d\omega'}{\omega'}\, \phi_+^{\rm eff}(\omega',\mu)\,. \tag{4.12}$$

The soft correction for the $\mathcal{O}(\alpha_s)$ leading-twist contribution was previously calculated in [15]. We find that the above expression (4.11) can be rewritten into the one given in [15] up to an obvious misprint.

For the higher-twist contributions considered in section 3, the dispersive treatment of the soft contribution corresponding to the hard-collinear terms (3.5), (3.6), (3.9) and (3.10) yields

$$\begin{aligned}
\xi^{\rm soft}_{(tw-3,4)}(E_\gamma) &= \frac{e_u m_B f_B}{4E_\gamma^2}\int_0^{\frac{s_0}{2E_\gamma}} d\omega \left[\frac{2E_\gamma}{m_\rho^2}\, {\rm e}^{-(2E_\gamma\omega-m_\rho^2)/M^2} - \frac{1}{\omega}\right]\Xi_1(\omega) \\
&\quad + \frac{e_u m_B f_B}{4m_b E_\gamma}\int_0^{\frac{s_0}{2E_\gamma}} d\omega \left[\frac{2E_\gamma}{m_\rho^2}\, {\rm e}^{-(2E_\gamma\omega-m_\rho^2)/M^2} - \frac{1}{\omega}\right]\Xi_2(\omega)\,, \qquad (4.13)\\
\Delta\xi^{\rm soft}_{(tw-3,4)}(E_\gamma) &= \frac{e_u m_B f_B}{4E_\gamma^2}\int_0^{\frac{s_0}{2E_\gamma}} d\omega \left[\frac{2E_\gamma}{m_\rho^2}\, {\rm e}^{-(2E_\gamma\omega-m_\rho^2)/M^2} - \frac{1}{\omega}\right]\omega\,\phi_+(\omega)\,. \qquad (4.14)
\end{aligned}$$

As was the case with (3.5), (3.9) the result can be written in several equivalent ways using the EOM, e.g.,

$$\begin{aligned}
\Xi_1(\omega) &= -\int_0^\omega d\omega_1 \int_{\omega-\omega_1}^\infty \frac{d\omega_2}{\omega_2}\frac{\partial}{\partial\omega_1}\big[\psi_4+\widetilde{\psi}_4\big](\omega_1,\omega_2) \\
&\quad -\int_0^\omega d\omega_2 \int_{\omega-\omega_2}^\infty \frac{d\omega_1}{\omega_1}\frac{\partial}{\partial\omega_2}\big[\psi_4+\widetilde{\psi}_4\big](\omega_1,\omega_2) \\
&\quad +2\int_\omega^\infty d\rho\,\phi_-^{\rm t3}(\rho) - 2\omega\phi_-^{\rm WW}(\omega) + 2\omega\phi_+(\omega) + \omega^2\frac{d}{d\omega}\phi_+(\omega)\,, \qquad (4.15)\\
\Xi_2(\omega) &= 2\int_0^\infty \frac{d\omega_2}{\omega_2}\,\phi_3(\omega,\omega_2) - 2\int_0^\omega d\omega_1\int_{\omega-\omega_1}^\infty \frac{d\omega_2}{\omega_2^2}\,\phi_3(\omega_1,\omega_2) + \int_\omega^\infty d\rho\,\phi_-^{\rm t3}(\rho) \\
&\quad + (\bar\Lambda-\omega)\,\phi_+(\omega) - \omega\phi_-^{\rm WW}(\omega)\,. \qquad (4.16)
\end{aligned}$$

In these expressions we used that $\big[\psi_4+\widetilde{\psi}_4\big](\omega_1=0,\omega_2) = \big[\psi_4+\widetilde{\psi}_4\big](\omega_1,\omega_2=0)=0$.

Since the higher-twist contributions in section 3 do not suffer from a soft endpoint divergence for real photon emission, the modification of the spectral density according to (4.6) results in a soft correction (4.13), which is suppressed by an additional power of E_γ and is therefore, strictly speaking, beyond our accuracy. However, the actual suppression factor relative to the leading-power form factor is $\{1/E_\gamma^2, 1/(m_bE_\gamma)\}\times s_0/(E_\gamma\Lambda)$ and since $s_0/(E_\gamma\Lambda)\gg\Lambda/E_\gamma$ such corrections can be numerically significant. We recall that also for the leading-twist contributions (4.9) and (4.10), we keep the full expressions and do not expand the result in powers of $s_0/(E_\gamma\Lambda)$, $M^2/(E_\gamma\Lambda)$, since this expansion converges very

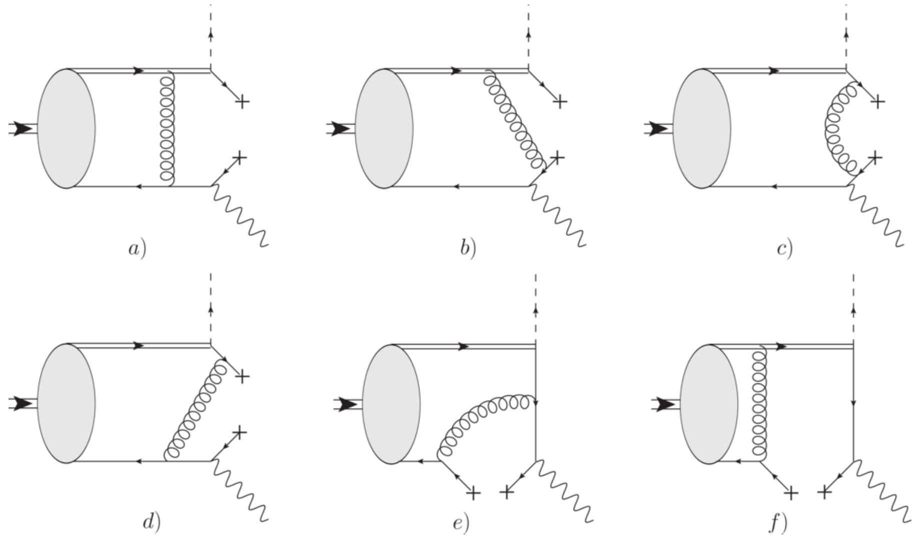

Figure 2. Factorizable higher-twist corrections to $B \to \gamma\ell\nu_\ell$.

slowly for realistic energies $E_\gamma \sim 1.5 - 2.5\,\text{GeV}$. Thus we take the soft corrections due to twist-three and twist-four contributions into account in the numerical analysis.[4]

We stress again that the soft contribution cannot be obtained through the light-cone expansion of the current product and as a consequence the usual hierarchy of the contributions of different (collinear) twist breaks down: B-meson LCDAs of all twists can in principle contribute to the form factor to the same power $1/E_\gamma^2$ in the $1/E_\gamma$ expansion. The basic idea of the light-cone sum rule approach is that higher-twist LCDAs have higher dimension and their contribution to the form factors is, generically, suppressed by increasing powers of the Borel parameter or continuum threshold, $s_0, M^2 \gg \Lambda^2$. Thus one can hope that only the first few terms in this expansion are numerically important.

As noted above, this does not actually happen for the twist-3 and -4 contributions, at least at the tree-level, due to their endpoint finiteness. In the following we consider the simplest higher-twist contribution of this unsuppressed kind, the contribution of twist-five and twist-six four-particle LCDAs in the factorization approximation — as a product of lower-twist LCDAs and the quark condensate (cf. [13]), see the relevant diagrams in

[4]As noted above, the dispersion relation is done in p^2 at fixed $E_\gamma \equiv vp$. Using instead the "canonical" dispersion relation in p^2 for fixed q^2 would lead to the following modifications: first, an extra factor $(1 - \omega/m_B)$ appears in the denominator of the ω-integral in (4.8). Second, the upper limit on the invariant mass is redefined from $2E_\gamma\omega - \omega^2 < s_0$ to $2E_\gamma\omega - \omega^2 < s_0\,(1-\omega/m_B)$. In both versions we expand this constraint assuming $\omega \sim \Lambda_{\text{QCD}} \ll E_\gamma, m_b$ and take into account $\mathcal{O}(\omega/E_\gamma)$, $\mathcal{O}(\omega/m_B)$ terms, which is consistent with twist-four accuracy in the collinear expansion. The resulting difference in the soft correction, which is already suppressed as $1/E_\gamma$ with respect to the leading term, is suppressed by an additional factor $1/m_B$, and should be viewed as an ambiguity of the method. This ambiguity is, in principle, of the same order as the term in Ξ_2 in (4.13). For the other terms it is yet higher order, if we accept that terms $(s_0/(E_\gamma\Lambda))^k$ should be retained whenever possible, whereas higher-order terms in $s_0/(m_b\Lambda)$ can be dropped. We note that different terms in Ξ_1 formally contribute at different order in the $s_0/(E_\gamma\Lambda)$ expansion; e.g. the term $\omega\phi_+(\omega)$ in the expression for Ξ_1 contributes to the form factor only at order $1/E_\gamma^2\,(s_0/(E_\gamma\Lambda))^2$. The symmetry-breaking soft contribution (4.14) is entirely of this order.

figure 2. For the diagram in figure 2 a we obtain after a short calculation

$$T^{\rm Fig1a}_{\mu\nu}(p,q) = \frac{ie_u f_B m_B g_s^2 C_F \langle \bar{u}u\rangle}{48p^2 E_\gamma}\,{\rm Tr}\left[\not{p}\gamma_\mu\gamma_\nu(1-\gamma_5)\not{v}\right]\int_0^\infty d\omega \frac{\phi_-(\omega)}{p^2-2E_\gamma\omega} + \mathcal{O}(1/E_\gamma^3)\,, \tag{4.17}$$

where $\langle \bar{u}u\rangle \approx -(240\,\text{MeV})^3$ (at the scale 1 GeV) is the quark condensate. If $|p^2| \sim E_\gamma\Lambda$ this contribution is suppressed by three powers of the hard-collinear scale in agreement with twist counting. However, the real photon limit $p^2 \to 0$ cannot be taken because of the $1/p^2$ factor, and also the integral of the twist-three DA $\phi_-(\omega)$ becomes logarithmically divergent in this limit. The effect of the dispersion relation improvement is, for the simplest case of a pure pole in p^2, the substitution [22]

$$\frac{1}{-p^2} \mapsto \frac{1}{m_\rho^2 - p^2} \overset{p^2\to 0}{\longrightarrow} \frac{1}{m_\rho^2}\,. \tag{4.18}$$

In this way the contribution to the form factors corresponding to (4.17) remains finite but the power counting changes and we obtain a term $\mathcal{O}(1/E_\gamma^2)$ similar to the hard-collinear contribution of the twist-three and twist-four LCDAs considered in section 3.

The other diagrams in figure 2 can be evaluated in a similar manner.[5] We find that the contributions in figures 2 a, c, e get promoted in the limit $p^2 \to 0$ to a $1/E_\gamma^2$ correction, whereas the contributions in figures 2 b, d, f remain of order $\mathcal{O}(1/E_\gamma^3)$ and can be neglected. We obtain

$$\begin{aligned}\xi^{\rm soft}_{(tw-5,6)}(E_\gamma) &= \frac{e_u g_s^2 C_F\langle\bar{u}u\rangle f_B m_B}{48E_\gamma^2 m_\rho^2}\left\{ e^{m_\rho^2/M^2}\int_0^{\frac{s_0}{2E_\gamma}} \frac{d\omega}{\omega}\left(e^{-2E_\gamma\omega/M^2}-1\right)\phi_-^{\rm WW}(\omega)\right.\\ &\qquad \left. + \int_{\frac{s_0}{2E_\gamma}}^\infty \frac{d\omega}{\omega}\left(\frac{m_\rho^2}{2E_\gamma\omega} - e^{m_\rho^2/M^2}\right)\phi_-^{\rm WW}(\omega) - \frac{5}{\lambda_B}e^{m_\rho^2/M^2}\right\},\\ \Delta\xi^{\rm soft}_{(tw-5,6)}(E_\gamma) &= -\frac{e_u g_s^2 C_F\langle\bar{u}u\rangle f_B m_B}{48E_\gamma^2 m_\rho^2\lambda_B}e^{m_\rho^2/M^2}\,. \end{aligned}\tag{4.19}$$

Note that to our working accuracy one has to substitute $\phi_-(\omega)$ by the "Wandzura-Wilczek contribution" $\phi_-^{\rm WW}(\omega)$ (A.13).

5 Results

In the numerical study presented below we use the NLL resummed result for the leading-power form factors [9] and the power-suppressed contributions $\xi \pm \Delta\xi$ in (2.7) given by the

[5]The calculation of the diagrams in figures 2 b, c, d is straightforward, while figure 2 e can most easily be obtained using the background-field expansion of the quark propagator [19]. Figure 2 f effectively corresponds to a contribution from the two-particle twist-five LCDA $g_-(\omega)$ (see appendix A), which can be factorized into a product of the quark condensate and a lower-twist LCDA. Figures 2 a, b involve a hard-collinear gluon propagator and therefore have to be calculated with the full QCD current and vertex, as mentioned before. It turns out, however, that the difference to using HQET rules appears only at order, $1/(m_b E_\gamma^2)$, beyond the accuracy of our calculation.

μ_0	$1\,\mathrm{GeV}$		
$\Lambda^{(4)}_{\rm QCD}$	0.291552 GeV	$\alpha_s(\mu_0)$	0.348929
μ	(1.5 ± 0.5) GeV	μ_h	$m_b/2 \div 2m_b$
m_b	(4.8 ± 0.1) GeV	$\bar{\Lambda}$	$m_B - m_b$
λ_E^2/λ_H^2	0.5 ± 0.1	$2\lambda_E^2 + \lambda_H^2$	(0.25 ± 0.15) GeV^2
s_0	(1.5 ± 0.1) GeV^2	M^2	(1.25 ± 0.25) GeV^2
$\langle \bar{u}u\rangle(\mu_0)$	$-(240 \pm 15$ MeV$)^3$		
m_B	$5.27929\,\mathrm{GeV}$	m_ρ	$0.77526\,\mathrm{GeV}$
G_F	1.166378×10^{-5} GeV^{-2}	τ_B	$1.638 \times 10^{-12}s$
f_B	(192.0 ± 4.3) MeV [23]	$\lvert V_{ub}\rvert^{\rm excl}$	$(3.70 \pm 0.16) \times 10^{-3}$ [24]

Table 1. Central values and ranges of all parameters used in this study. The four-flavour $\Lambda_{\rm QCD}$ parameter corresponds to $\alpha_s(m_Z) = 0.1180$ with three-loop evolution and decoupling of the bottom quark at the scale m_b.

sum of hard-collinear higher-twist and soft corrections

$$\begin{aligned} \xi &= \xi^{\rm ht}\Big|_{(3.11)} + \xi^{\rm soft}_{\rm (NLO)}\Big|_{(4.10)} + \xi^{\rm soft}_{(tw-3,4)}\Big|_{(4.13)} + \xi^{\rm soft}_{(tw-5,6)}\Big|_{(4.19)}\,, \\ \Delta\xi &= \Delta\xi^{\rm ht}\Big|_{(3.11)} + \Delta\xi^{\rm soft}_{(tw-3,4)}\Big|_{(4.14)} + \Delta\xi^{\rm soft}_{(tw-5,6)}\Big|_{(4.19)}\,. \end{aligned} \tag{5.1}$$

For the reader's convenience we have indicated the corresponding equation numbers.

The nonperturbative inputs in the calculation have to be defined at a certain reference scale, μ_0. As was done in previous work, we use $\mu_0 = 1\,\mathrm{GeV}$. Unless stated otherwise, the values of all scale-dependent hadronic parameters given below refer to this scale. In the calculation of the leading-power contributions to the form factors and the related soft correction $\xi^{\rm soft}_{\rm (NLO)}$ we evolve the inputs to the hard-collinear scale μ, adopting $\mu = 1.5$ GeV as default. In the absence of the two-loop non-cusp anomalous dimension of the twist-2 B-meson LCDA $\phi_+(\omega)$, we perform the evolution in the LL approximation. Higher-twist contributions and the related soft corrections are always evaluated at the scale μ_0. We use three-loop running of the strong coupling with $n_f = 4$ active flavors. The central values and ranges of all parameters are collected in table 1.

The principal input in our analysis is provided by the leading-twist B-meson LCDA $\phi_+(\omega)$. For the leading-power contribution to the form factors, the precise functional form of the LCDA is not important as it can be expressed in terms of the logarithmic moments[6]

$$\widehat{\sigma}_n = \int_0^\infty d\omega\, \frac{\lambda_B}{\omega}\, \ln^n \frac{\lambda_B e^{-\gamma_E}}{\omega}\, \phi_+(\omega) \tag{5.2}$$

[6]Note that our definition of the log-moments differs from those in [8] and [9] by the substitution $\ln \mu/\omega \to \ln(e^{-\gamma_E}\lambda_B/\omega)$ and $\ln \mu_0/\omega \to \ln(e^{-\gamma_E}\lambda_B/\omega)$, respectively. The purpose of this change is to decorrelate the log-moments from the value of λ_B in the models for $\phi_+(\omega)$ considered below.

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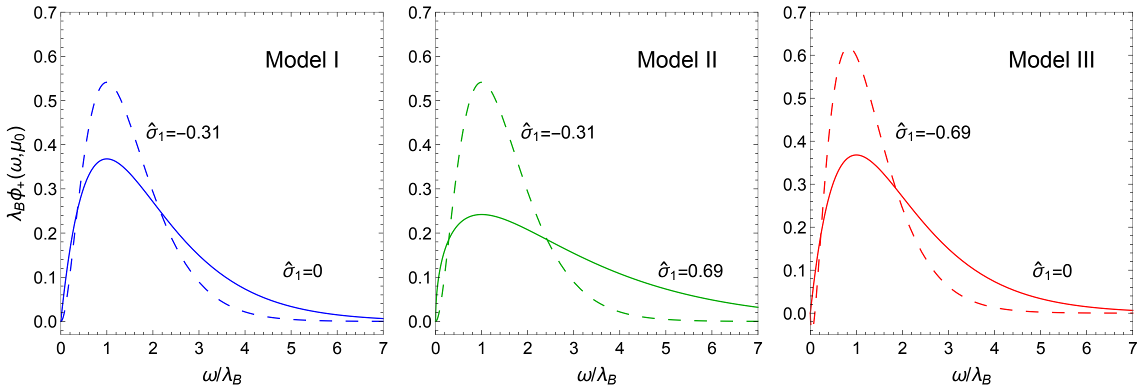

Figure 3. B-meson leading-twist LCDA $\lambda_B\phi_+(\omega,\mu_0)$ for the three models described in the text.

with $\hat\sigma_0 = 1$ defining λ_B. To the NLL accuracy only the values of λ_B, $\hat\sigma_1$ and $\hat\sigma_2$ are needed. The LCDA and the moments are renormalization scale dependent. For brevity, we omit the explicit scale argument μ.

In contrast, the evaluation of the soft (endpoint) contributions requires the full functional form of the LCDAs. We will use three two-parameter families of functions to assess the model dependence of the soft contribution. In the s-space representation (A.6) [25, 26]

$$\eta_+^{(\mathrm{I})}(s) = {}_1F_1(1+2/b, 2/b, -s\omega_0) = \left(1-\frac{1}{2}bs\omega_0\right)e^{-s\omega_0}\,, \qquad 0\le b\le 1\,, \tag{5.3a}$$

$$\eta_+^{(\mathrm{II})}(s) = {}_1F_1(2+a, 2, -s\omega_0)\,, \qquad -0.5<a<1\,, \tag{5.3b}$$

$$\eta_+^{(\mathrm{III})}(s) = {}_1F_1(3/2+a, 3/2, -s\omega_0)\,, \qquad 0<a<0.5\,, \tag{5.3c}$$

corresponding in momentum space to

$$\phi_+^{(\mathrm{I})}(\omega) = \left[(1-b)+\frac{b\,\omega}{2\omega_0}\right]\frac{\omega}{\omega_0^2}\,\mathrm{e}^{-\omega/\omega_0}\,, \tag{5.4a}$$

$$\phi_+^{(\mathrm{II})}(\omega) = \frac{1}{\Gamma(2+a)}\frac{\omega^{1+a}}{\omega_0^{2+a}}\mathrm{e}^{-\omega/\omega_0}\,, \tag{5.4b}$$

$$\phi_+^{(\mathrm{III})}(\omega) = \frac{\sqrt{\pi}}{2\Gamma(3/2+a)}\frac{\omega}{\omega_0^2}\,e^{-\omega/\omega_0}\,U(-a, 3/2-a, \omega/\omega_0)\,, \tag{5.4c}$$

where ${}_1F_1(\alpha,\beta,z)$ is a hypergeometric function, and $U(\alpha,\beta,z)$ the confluent hypergeometric function of the second kind. The above functional forms are assumed to hold at $\mu_0 = 1\,\mathrm{GeV}$.

The three models in (5.4) for the limiting values of the parameters (5.3) are shown in figure 3. They can be viewed as particular cases of the more general three-parameter ansatz

$$\eta_+(s) = {}_1F_1(\alpha,\beta,-s\omega_0)\,, \qquad \alpha,\beta>1\,,$$
$$\phi_+(\omega) = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\frac{\omega}{\omega_0^2}\,e^{-\omega/\omega_0}U(\beta-\alpha, 3-\alpha, \omega/\omega_0)\,. \tag{5.5}$$

For this ansatz

$$\lambda_B = \frac{\alpha - 1}{\beta - 1}\omega_0 \,, \qquad \widehat{\sigma}_1 = \psi(\beta - 1) - \psi(\alpha - 1) + \ln\frac{\alpha - 1}{\beta - 1}\,, \tag{5.6}$$

etc., so that the only dimensionful parameter ω_0 can be traded for λ_B and the logarithmic moments defined in (5.2) depend on the "shape parameters", α and β.

The particular choices (5.3), (5.4) are motivated by the experience in the modelling of the pion LCDA, where especially the endpoint behaviour came under scrutiny in connection with the BaBar and BELLE measurements of the $\gamma^* \to \pi\gamma$ transition form factor, see e.g. [13, 27–29]. The parameter range indicated in (5.3) corresponds to $-0.306853 < \widehat{\sigma}_1 < 0$ for Model I, $-0.306853 < \widehat{\sigma}_1 < 0.693147$ for Model II and $-0.693147 < \widehat{\sigma}_1 < 0$ for Model III, so that, taken together, they cover the range

$$-0.693147 < \widehat{\sigma}_1 < 0.693147 \tag{5.7}$$

for arbitrary λ_B. The value $\widehat{\sigma}_1 = 0$ corresponds to the simple exponential model $\phi_+(\omega) = (\omega/\omega_0^2)\, e^{-\omega/\omega_0}$ suggested in [4].

The large-momentum behaviour of the B-meson LCDA can be studied in perturbation theory in a cutoff scheme [30]. In this way the first moment $\int_0^{\mu_F} d\omega\, \omega\phi_+(\omega)$ is related to a properly defined $\bar{\Lambda}(\mu_F) = m_B - m_b(\mu_F)$ and the second moment, $\int_0^{\mu_F} d\omega\, \omega^2\phi_+(\omega)$, to matrix elements of the quark-gluon operators (A.25), which were estimated with QCD sum rules [4, 31]. However, it was shown in [32] that such relations do not generally provide significant constraints on the logarithmic moments $\widehat{\sigma}_1$, $\widehat{\sigma}_2$, since they can be satisfied by adding a large-momentum "tail" to any given (reasonable) model for $\phi_+(\omega)$. Following this argument, we will assume that the "true" LCDA can be written as

$$\phi_+(\omega, \mu) = \phi_+^{\text{model}}(\omega, \mu) + \delta\phi_+(\omega, \mu)\,, \tag{5.8}$$

where $\phi_+^{\text{model}}(\omega, \mu)$ refers to one of the models specified in (5.4) and the added "tail" is concentrated at large momenta $\omega \gg \lambda_B$. Its role is to ensure that the relations for the first two moments are satisfied to the required accuracy. We assume that this additional term can be chosen in such a way that the first few logarithmic moments are not affected [32]. In this case an explicit expression for $\delta\phi_+(\omega, \mu)$ is not needed as it does not enter any of the three contributions to the form factors: neither (1) the perturbative leading-twist leading-power contribution, as it is expressed in terms of the logarithmic moments, nor (2) the soft corrections, as they originate from small momenta, nor (3) higher-twist corrections, as they are expressed directly in terms of $\bar{\Lambda}$ and higher-twist matrix elements λ_E^2, λ_H^2 (see below).[7]

In ref. [16] several models for the higher-twist LCDAs have been suggested that have the expected low-momentum behaviour and satisfy the (tree-level) EOM constraints. One can show that these models can be obtained as particular cases of the more general

[7]We must assume that $\delta\phi_+$ decreases sufficiently fast at $\omega \to \infty$ so that its first few moments are finite. While this cannot hold true in general due to perturbative radiative corrections [33], the assumption is necessary for consistency of tree-level calculations of higher-twist contributions as performed here. See also the appendix.

ansatz (A.39). For these models one obtains the remarkably simple expression

$$\xi^{\rm ht}(E_\gamma) = -\frac{e_u f_B m_B}{2E_\gamma^2}\left\{\frac{2(\lambda_E^2+2\lambda_H^2)}{6\bar\Lambda^2+2\lambda_E^2+\lambda_H^2}+\frac{1}{2}\right\} + \frac{e_u f_B m_B}{4m_b E_\gamma}\left\{\frac{\bar\Lambda}{\lambda_B}-2+\frac{4(\lambda_E^2-\lambda_H^2)}{6\bar\Lambda^2+2\lambda_E^2+\lambda_H^2}\right\}, \tag{5.9}$$

and the higher-twist correction does not depend on the functional form of the profile function $f(\omega)$ in the ansatz (A.39).

We use the range $m_b = 4.7 \div 4.9\,\text{GeV}$ for the pole mass and define $\bar\Lambda = m_B - m_b$. It has to be mentioned that the derivation of the higher-twist corrections is based on equations of motion at tree level [16, 34]. For consistency, the relations between moments of the LCDA and local matrix elements have to be assumed at tree level as well, eq. (A.24). The scheme-dependence of $\bar\Lambda$ and our result (3.11), (5.9) for the higher-twist correction should be cancelled by a correction proportional to $\mu_F\alpha_s/\pi$ that has not been calculated so far. Before this is done, the numerical value of $\bar\Lambda$ (or, equivalently, of the b-quark pole mass m_b) should be viewed as an educated guess.

The matrix elements λ_E^2 and λ_H^2 are defined in (A.25). The existing QCD sum rule estimates (A.27) fall in the range

$$0.1\ \text{GeV}^2 < 2\lambda_E^2+\lambda_H^2 < 0.4\ \text{GeV}^2, \qquad \lambda_E^2/\lambda_H^2 = 0.5\pm 0.1\,. \tag{5.10}$$

For this range of values, the dependence of the higher-twist correction in (5.9) on λ_E^2 and λ_H^2 is rather weak so that a large uncertainty in the matrix elements does not play a major role, except for large λ_B.

In order to understand the qualitative features of soft corrections let us consider the leading-order twist-two contribution $\xi^{\rm soft}_{\rm (LO)}$ (4.9) as an example. Normalizing to the leading-order QCD result (2.7), and extracting the expected $1/(2E_\gamma)$ suppression factor we define [14]

$$\xi^{\rm soft}_{\rm (LO)}(E_\gamma) = \frac{e_u f_B m_B}{2E_\gamma\lambda_B(\mu)}\, U_{\rm LL}\,\frac{\widehat\xi^{\rm soft}_{\rm (LO)}(E_\gamma)}{2E_\gamma}\,,$$
$$\widehat\xi^{\rm soft}_{\rm (LO)}(E_\gamma) = 2E_\gamma\lambda_B(\mu)\int\limits_0^{s_0/2E_\gamma} d\omega\left[\frac{2E_\gamma}{m_\rho^2}e^{-(2E_\gamma\omega-m_\rho^2)/M^2}-\frac{1}{\omega}\right]\phi_+(\omega,\mu)\,. \tag{5.11}$$

This expression involves two parameters — the continuum threshold s_0 and the Borel parameter M^2 — which we choose in the range

$$1.4\,\text{GeV}^2 < s_0 < 1.6\,\text{GeV}^2, \qquad 1.0\,\text{GeV}^2 < M^2 < 1.5\,\text{GeV}^2. \tag{5.12}$$

The soft correction originating from twist-five and twist-six LCDAs depends in addition on the quark condensate $\langle\bar u u\rangle(1\ \text{GeV}) = -(240\pm 15\ \text{MeV})^3$.

In the asymptotic regime $\mu^2 \sim \Lambda_{\rm QCD}E_\gamma \to \infty$ the LCDA $\phi_+(\omega,\mu)$ is driven by the renormalization group flow to linear behaviour $\phi_+(\omega)\sim\omega\phi_+'(0)$ for $\omega\to 0$ independent on

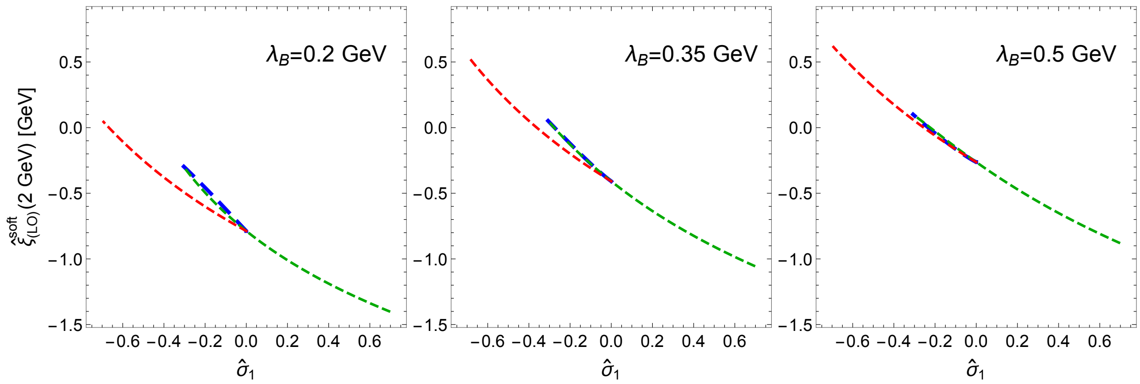

Figure 4. The leading-order soft correction normalized to the corresponding QCD result, $\widehat{\xi}^{\rm soft}_{\rm (LO)}(2\,\text{GeV})$ (5.11), as a function of the first logarithmic moment $\widehat{\sigma}_1$ (5.2) for the three models of the leading-twist B-meson LCDA defined in (5.4a) (blue), (5.4b) (green) and (5.4c) (red), respectively, and for three different values of λ_B as specified on the plots.

the initial condition at low scales. In this case $\widehat{\xi}^{\rm soft}_{\rm (LO)}(E_\gamma) \to \text{const} \cdot \lambda_B \phi'_+(0) + \mathcal{O}(1/E_\gamma)$ so that the soft correction is proportional to the (in this limit finite) derivative of the LCDA at zero momentum. For physically interesting photon momenta $E_\gamma \sim 1.5 - 2.5\,\text{GeV}$ the dominance of the $\omega \to 0$ region does not hold since the integration in (5.11) goes over the momentum region $0 < \omega < 300 \div 500\,\text{MeV}$ that is comparable with the characteristic momentum scale λ_B in the LCDA. Thus the integral is determined by global properties of the LCDA (normalization, width, etc.) rather than the endpoint behaviour. The situation is similar in this respect to the better studied reaction $\gamma^*\gamma \to \pi$ in which case it was shown [13] that an anomalous endpoint behaviour of the pion LCDA cannot explain by itself the strong scaling violation observed by BaBar [35] up to much higher scales $Q^2 \sim 20 \div 30\,\text{GeV}^2$.

As already noticed in [14], the normalized soft correction $\widehat{\xi}^{\rm soft}_{\rm (LO)}(E_\gamma)$ (5.11) depends only weakly on photon energy E_γ (in the relevant range). For illustration we plot $\widehat{\xi}^{\rm soft}_{\rm (LO)}(E_\gamma = 2\,\text{GeV})$ in figure 4 as a function of $\widehat{\sigma}_1$ for three different values of λ_B and central values of the sum rule parameters, $s_0 = 1.5\,\text{GeV}^2$ and $M^2 = 1.25\,\text{GeV}^2$. The blue, green and red curves are obtained using models I, II, and III in (5.4), respectively, with the indicated parameter range. Note that this correction can be quite sizable, e.g., the value $\widehat{\xi}^{\rm soft}_{\rm (LO)} = -1.0\,\text{GeV}$ corresponds to a power-suppressed contribution to the form factors of the order of $(-1.0\,\text{GeV})/(2E_\gamma)$ with respect to the leading-order, leading-twist result. It attracts attention that $\widehat{\xi}^{\rm soft}_{\rm (LO)}(2\,\text{GeV})$ can be both positive and negative, and depends strongly on the value of the first logarithmic moment, $\widehat{\sigma}_1$. For a given $\widehat{\sigma}_1$, the correction is fairly close in all three models (in the regions where there is an overlap). This agreement is trivial for $\widehat{\sigma}_1 = 0$ as all models reduce to the same simple exponential model, but it is not trivial for the whole range. Note also that the precise small-ω behaviour of the LCDA is irrelevant: for model II the derivative $\phi'_+(0)$ is changing from zero to infinity as the parameter a and $\widehat{\sigma}_1$ change sign, with no visible effect on the result.

The relative size of various contributions to the "symmetry-preserving" form factor combination $(F_V + F_A)/2$ (alias the helicity form factor F_-) is illustrated in figure 5 for

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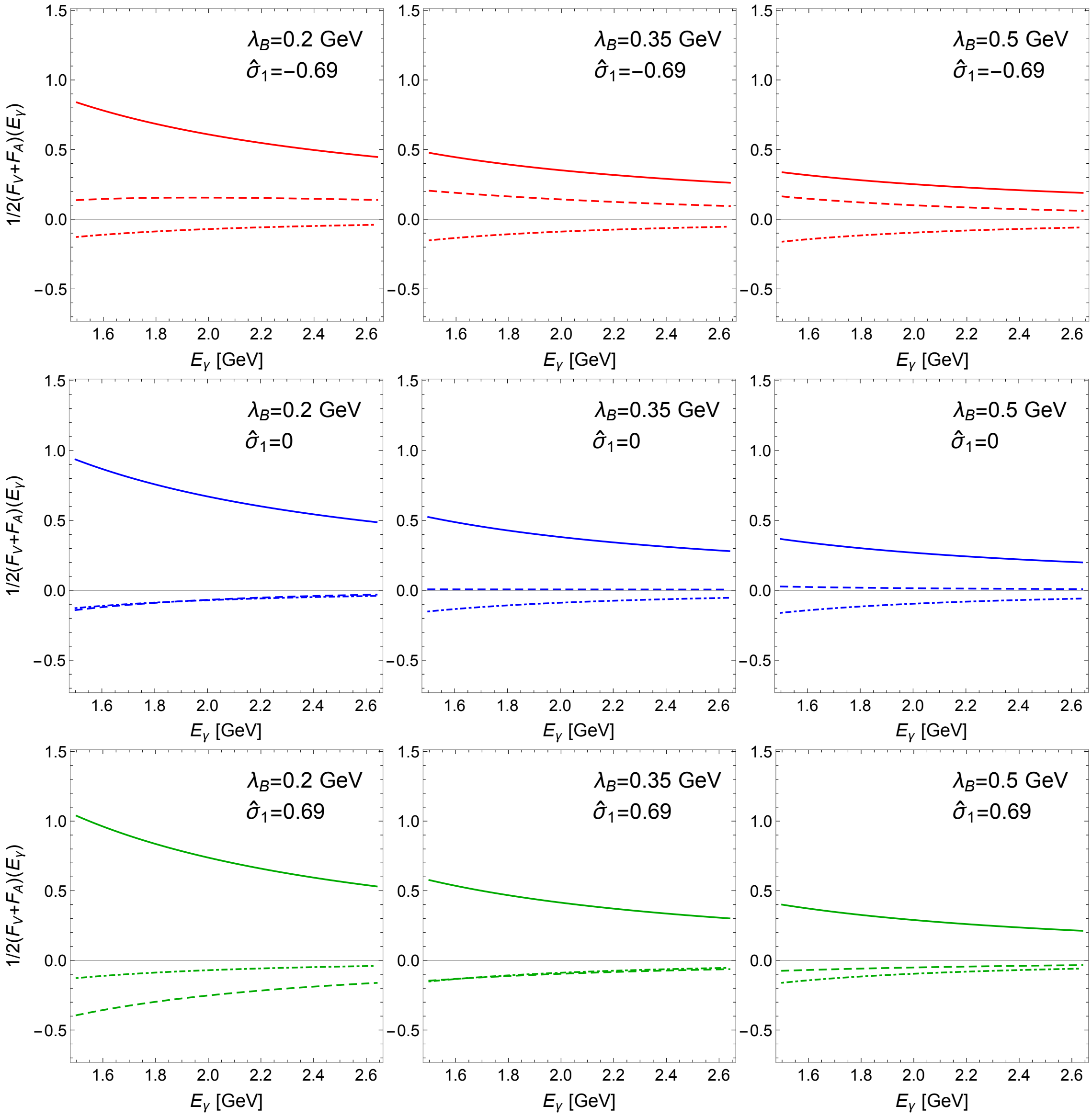

Figure 5. Perturbative (solid curves), soft (dashed curves) and higher-twist (dash-dotted curves) contributions to the form factor $(1/2)(F_V + F_A)$ as functions of photon energy E_γ for different choices of the parameters λ_B and σ_1. The colour coding corresponds to figure 3.

several choices of the parameters λ_B and σ_1. We show the NLL resummed perturbative result [9] (solid curves), the total soft correction $\xi^{\rm soft} = \xi^{\rm soft}_{\rm (NLO)} + \xi^{\rm soft}_{(tw-3,4)} + \xi^{\rm soft}_{(tw-5,6)}$, and the hard-collinear higher-twist correction $\xi^{\rm ht}$ by the solid, dashed and dash-dotted curves, respectively. One sees that the higher-twist correction is negative and relatively small for all cases, whereas the soft correction can be of either sign and for small λ_B becomes rather large. The effect of the soft correction is always to counteract the change of the perturbative contribution due to the variation of λ_B and, in particular, σ_1 so that the sensitivity of the form factor to the model of the LCDA is reduced upon accounting for the soft correction as

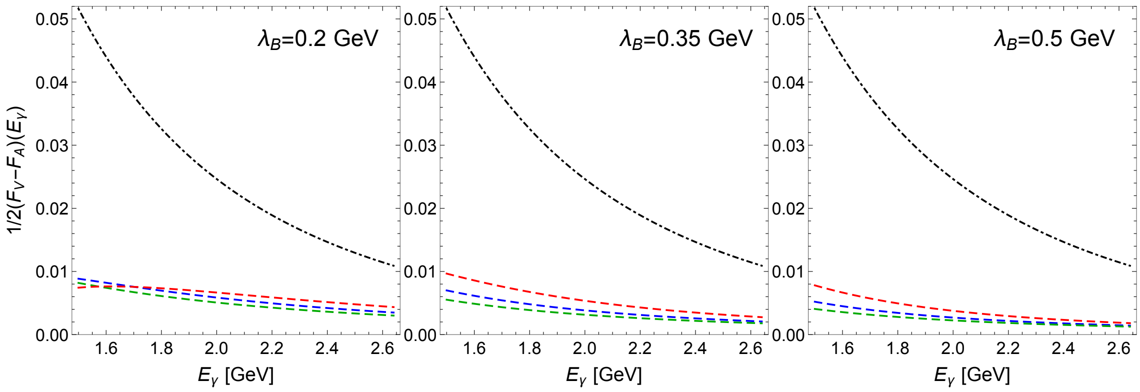

Figure 6. Model-independent higher-twist [9] (black dash-dotted curves) and soft (dashed curves) contributions to the form factor difference $(1/2)(F_V - F_A)$ for three choices of λ_B. The soft corrections are shown for $\widehat{\sigma}_1 = -0.69, 0, +0.69$ in red, green and blue colour, respectively. The colour coding corresponds to figure 3.

compared to the leading-power result alone. Among the different contributions to the soft correction the part related to the leading-twist LCDA, $\xi^{\rm soft}_{\rm (NLO)}$ (4.10), is dominant in all cases; the other two contributions are relatively small. In particular $\xi^{\rm soft}_{(tw-5,6)}$ is at most 6% of the total value. This is reassuring, suggesting that soft contributions related to the LCDAs of even higher twist can be small as well, and also because the approximation leading to (4.19) is rather crude. We also find that the $1/m_b$ power corrections are generally much smaller than the $1/E_\gamma$ corrections, and so are the "genuine" three-particle higher-twist corrections relative to those that can be related to two-particle terms by the equations of motion.

A similar decomposition of the various contributions to the "symmetry-breaking" form factor difference $\Delta\xi = (F_V - F_A)/2$ is shown in figure 6. It is dominated by the model-independent higher-twist correction (2.9) [9] (black dash-dotted curves) whereas the soft contributions (dashed) turn out to be small in all cases. They are shown in three colours corresponding to the choice $\widehat{\sigma}_1 = -0.69, 0, +0.69$ at the boundaries and in the middle of the three models' envelope.

To visualize the relative importance of different uncertainties due to the choice of the parameters in the range specified in table 1, we consider the vector form factor F_V for $E_\gamma = 2\,\text{GeV}$, and $\lambda_B = 0.35\,\text{GeV}$, in the middle of the range of interest, and two extreme values for the first logarithmic moment, $\widehat{\sigma}_1 = \pm 0.693$. We obtain

$$
\begin{aligned}
&F_V(E_\gamma = 2\,\text{GeV}, \lambda_B = 0.35\,\text{GeV}, \widehat{\sigma}_1 = 0.693) \\
&\quad = 0.258 + \begin{pmatrix} +0.012 \\ -0.017 \end{pmatrix}_{m_b} + \begin{pmatrix} +0.000 \\ -0.007 \end{pmatrix}_{\mu} + \begin{pmatrix} +0.006 \\ -0.006 \end{pmatrix}_{\mu_h} + \begin{pmatrix} +0.001 \\ -0.000 \end{pmatrix}_{M^2} + \begin{pmatrix} +0.001 \\ -0.001 \end{pmatrix}_{s_0} \\
&\qquad + \begin{pmatrix} +0.016 \\ -0.013 \end{pmatrix}_{2\lambda_E^2 + \lambda_H^2} + \begin{pmatrix} +0.002 \\ -0.003 \end{pmatrix}_{\lambda_E^2/\lambda_H^2} + \begin{pmatrix} +0.004 \\ -0.003 \end{pmatrix}_{\langle \bar{u}u \rangle} = 0.258^{+0.021}_{-0.024},
\end{aligned}
$$

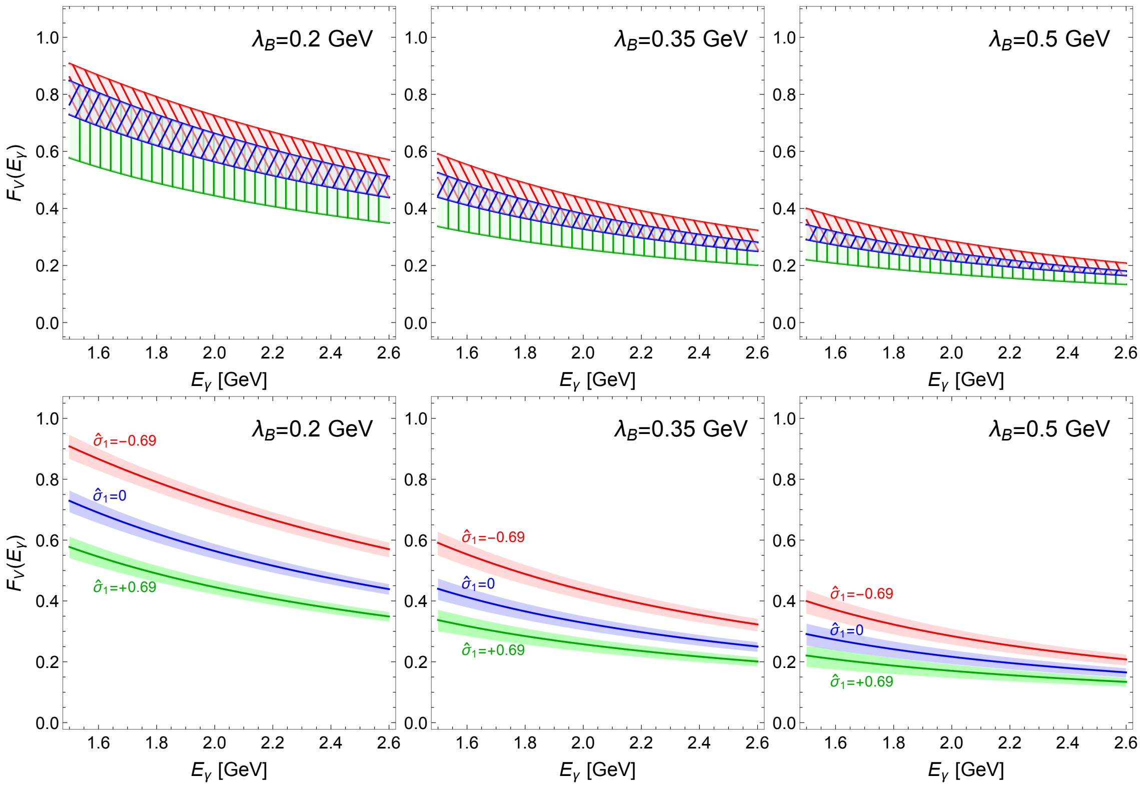

Figure 7. Vector form factor $F_V(E_\gamma)$. The shaded regions on the upper panels show the variation for a given model with the range of parameters specified in (5.3). The uncertainty due to other parameters in the range specified in table 1 is shown on the three lower panels for $\widehat{\sigma}_1 = \pm 0.69$ corresponding to the boundary of the models' envelope in the upper plot, and for $\widehat{\sigma}_1 = 0$ corresponding to the simple exponential ansatz [4]. The colour coding corresponds to figure 3.

$$\begin{aligned} &F_V(E_\gamma = 2\,\text{GeV}, \lambda_B = 0.35\,\text{GeV}, \widehat{\sigma}_1 = -0.693) \\ &\quad = 0.435 + \begin{pmatrix} +0.013 \\ -0.017 \end{pmatrix}_{m_b} + \begin{pmatrix} +0.000 \\ -0.006 \end{pmatrix}_{\mu} + \begin{pmatrix} +0.010 \\ -0.009 \end{pmatrix}_{\mu_h} + \begin{pmatrix} +0.013 \\ -0.018 \end{pmatrix}_{M^2} + \begin{pmatrix} +0.003 \\ -0.004 \end{pmatrix}_{s_0} \\ &\qquad + \begin{pmatrix} +0.014 \\ -0.011 \end{pmatrix}_{2\lambda_E^2+\lambda_H^2} + \begin{pmatrix} +0.002 \\ -0.002 \end{pmatrix}_{\lambda_E^2/\lambda_H^2} + \begin{pmatrix} +0.004 \\ -0.003 \end{pmatrix}_{\langle \bar{u}u \rangle} = 0.435^{+0.025}_{-0.030}\,, \end{aligned} \tag{5.13}$$

where we added the errors in quadrature to arrive at the final numbers. We do not include here the uncertainty due to the B-meson decay constant f_B, cf. table 1, which enters as an overall factor, and can therefore trivially be added. Apart from this, the overall uncertainty is only about 6-9%, with the main contributions from the b-quark mass (alias $\bar{\Lambda}$), the Borel parameter, and the twist-four matrix element $2\lambda_E^2 + \lambda_H^2$. The hard-collinear factorization scale (μ) dependence of $\xi^{\text{soft}}_{(\text{NLO})}$ turns out to be large (up to 30%) but is always anticorrelated with the scale dependence of the leading-power contribution such that the μ-dependence of their sum is reduced compared to that of the leading power term alone.

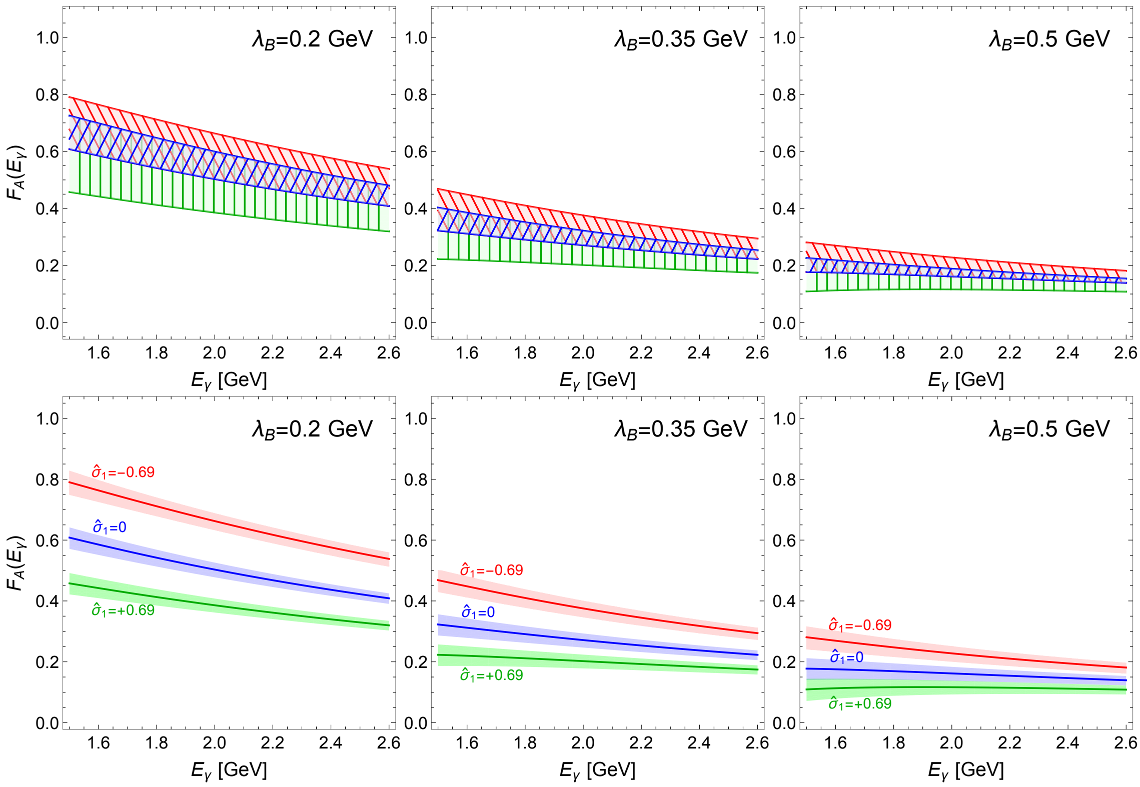

Figure 8. Axial form factor $F_A(E_\gamma)$. The legend follows figure 7.

Our final results for the vector F_V and axial F_A form factors are shown in figures 7 and 8, respectively.[8] The shaded regions on the upper panels in both figures show the variation for a given model with the range of parameters specified in (5.3) and central values for other parameters. The colour coding follows figure 3. The uncertainty from variation of the other parameters in the range specified in table 1 is shown on the lower panels for three cases: $\widehat{\sigma}_1 = \pm 0.69$ corresponding to the boundaries of the three models' envelope, and $\widehat{\sigma}_1 = 0$. For the last value our three models coincide and reduce to the simple exponential model of ref. [4]. This uncertainty is below 15% in all cases.

Two important conclusions can be drawn from these results. First, the uncertainty from all parameters except those of the leading-twist B-meson LCDA $\phi_+(\omega)$ is generally smaller than the dependence on $\phi_+(\omega)$ itself, which is large. This is welcome, since the measurement of the $B \to \gamma \ell \nu_\ell$ process is primarily seen as a means to determine the B-meson LCDA $\phi_+(\omega)$, in particular λ_B. The calculation of the power-suppressed "soft symmetry-preserving form factor" ξ introduced in [9] — performed here within the dispersive sum-rule approach — considerably improves the prediction relative to the agnostic parameterization of [9] and the leading-order calculation of ξ in [14]. Second, the dependence of the form factors on the shape of the B-meson LCDA (which is mostly a dependence on $\widehat{\sigma}_1$) is as strong as on λ_B. Thus any future comparison with experiment should aim at the extrac-

[8]We recall that the contribution of photon emission from the final state lepton is not included in F_A, cf. (2.6).

tion of correlated values for λ_B and "shape parameters" $\widehat{\sigma}_1$, etc., rather than extracting λ_B alone and treating the "shape parameters" as theoretical uncertainty parameters.

We finally calculate the partial branching fraction $\mathrm{BR}(B \to \gamma\ell\nu_\ell, E_\gamma > E_{\min})$, integrating (2.6) over the photon energy interval $E_{\min} < E_\gamma < m_B/2$. The result is shown in figure 9 as a function of λ_B for three values of the photon energy cut, $E_{\min} = 1.0$ GeV, $E_{\min} = 1.5$ GeV and $E_{\min} = 2.0$ GeV with colour coding referring to the three models as discussed above. The band corresponds to the variation of $\widehat{\sigma}_1$ such that the envelope of all three bands reflects the total $\widehat{\sigma}_1$ dependence. For this plot we adopted the exclusive $|V_{ub}|$ average, $|V_{ub}| = (3.70 \pm 0.16) \times 10^{-3}$ [24], but as in the case of f_B, we do not include the theoretical uncertainty, since the dependence on $f_B|V_{ub}|$ can in principle be eliminated by normalizing to another exclusive $b \to u$ decay. The theoretical approach requires the photon energy to be large compared to the strong interaction scale Λ. We find that the power corrections become increasingly large for smaller E_γ such that the expansion cannot be considered reliable below $E_\gamma \sim 1.5\,\mathrm{GeV}$. Given that the first data is statistics-limited [10], it is nevertheless tempting to extrapolate to $E_{\min} = 1.0$ GeV, and we have done so in figure 9 — adding that any conclusions drawn from this plot may at best be indicative.

6 Summary

In anticipation of the forthcoming high-statistics measurements of the radiative decay $B \to \gamma\ell\nu_\ell$ by the BELLE II experiment at KEK we reconsider its QCD calculation. The interest in this decay is mainly due to its distinguished role as the simplest process that probes the light-cone B-meson distribution amplitude, which in turn is an important nonperturbative input in QCD factorization for exclusive processes involving B mesons [1].

The main theoretical issue is to quantify the leading power-suppressed effects in $1/E_\gamma$, $1/m_b$, as the leading-power calculation is well understood [9]. Following the technique used already in [14, 15] we employ dispersion relations and duality to calculate the power-suppressed soft contributions. In this approach soft corrections arise from the modification of the spectral functions of the hard-collinear perturbative contributions in the soft region, guided by the requirement of a mass gap in the hadronic spectrum in the photon channel. A strong feature of this technique is that the result is insensitive to redefinition of the hard-collinear contributions (e.g. by introducing an explicit cutoff for the soft region), which only affects the decomposition of the answer in hard-collinear and soft contributions but leaves their sum intact.

The present work goes beyond previous ones [9, 14, 15] in several directions. On the technical side, first, we present a calculation of power-suppressed higher-twist corrections to the form factors that are due to higher Fock states in the B-meson and to the transverse momentum (virtuality) of the light quark in the valence state. These two effects are related by the equations of motion and in the sum a rather compact expression can be found, which further reduces to a few constants under rather general assumptions on the form of the higher twist LCDAs. The resulting correction to the form factors is negative and not very large, of the order of 10-30% depending on the size of the leading contribution. Second, we calculate the soft corrections due to twist-five and -six B-meson LCDAs in the

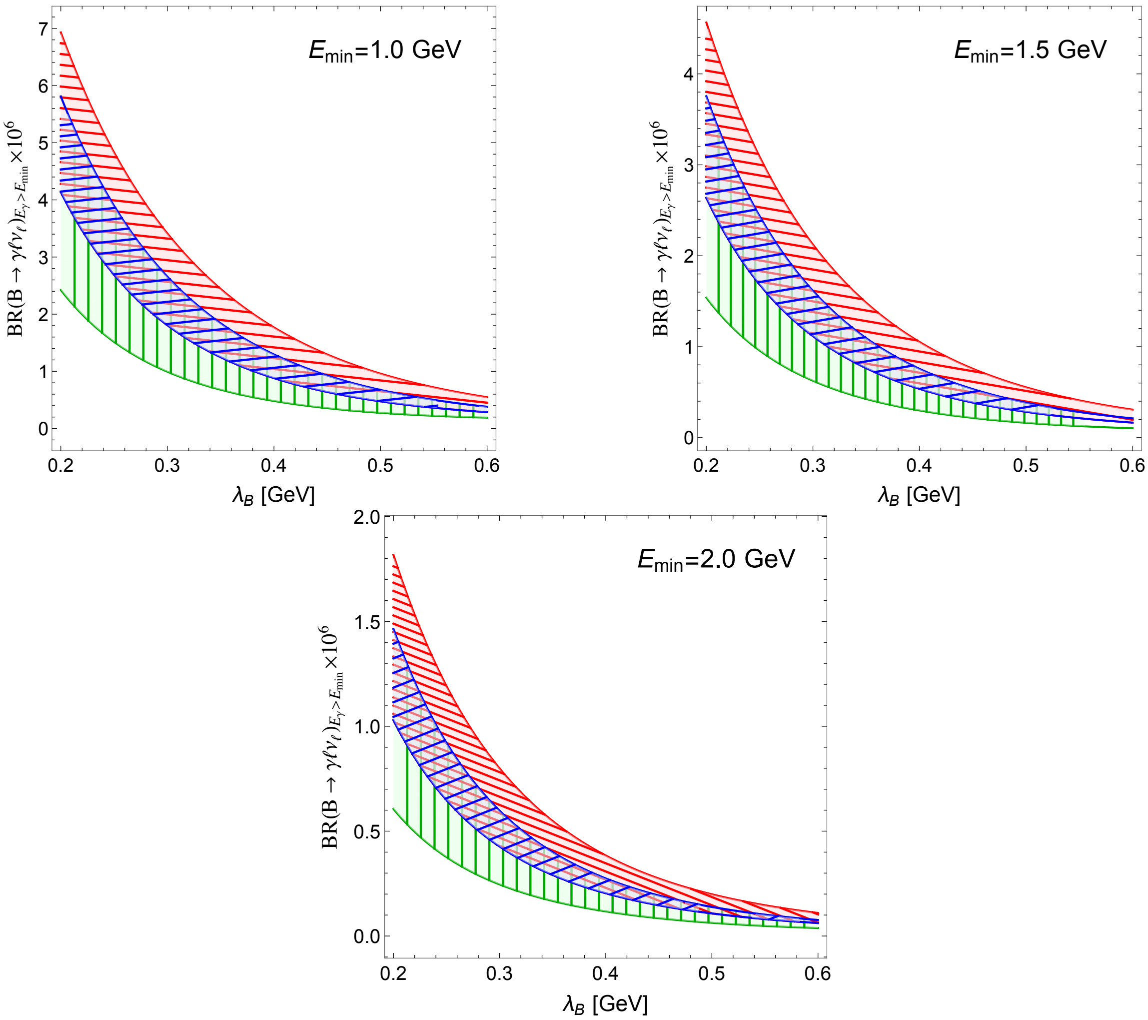

Figure 9. Integrated partial branching fraction $\mathrm{BR}(B \to \gamma \ell \nu_\ell, E_\gamma > E_{\rm min})$ for $E_{\rm min} = 1$ GeV (left), $E_{\rm min} = 1.5$ GeV (right) and $E_{\rm min} = 2$ GeV (lower).

factorization approximation. These terms turn out to be much smaller than soft corrections originating from the lower-twist LCDAs, which is encouraging, since it indicates that the twist expansion for soft corrections is converging.

On the analysis side, aiming to set the stage for the data analysis once more experimental results become available, we present a detailed numerical study of the predictions using a rather general class of models for the leading-twist LCDA, and the corresponding error analysis. We find that the model dependence can be parameterized to a large extent by λ_B and by the value of the first logarithmic moment $\widehat{\sigma}_1$ (which we redefine, see (5.2), compared to previous studies [8, 9] in order to decorrelate it from λ_B). For a given LCDA, the uncertainty of the calculation of the form factors is small, but the dependence of the results on $\widehat{\sigma}_1$ (in addition to the expected dependence on λ_B) is significant. Unless the model space can be constrained otherwise, future data should be analyzed in terms of both parameters, λ_B and $\widehat{\sigma}_1$.

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The most important remaining theory issue in describing $B \to \gamma\ell\nu_\ell$ decay for large photon energy is the consistent implementation of some version of a cutoff scheme [30] together with the rederivation of the equation-of-motion relations between two-particle and three-particle LCDAs in the same framework. This would allow the calculation of power-suppressed effects with well-defined moments in the presence of a radiatively generated tail of $\phi_+(\omega)$ and, hence, to get rid of $\bar{\Lambda}$ and most of the higher-twist matrix elements as independent parameters. The two-loop evolution equation for the leading-twist B-meson LCDA would also be useful for theoretical consistency to match the NLL accuracy of the hard and hard-collinear evolution.

When this paper was being finalized, ref. [36] appeared suggesting a "hybrid" approach where the calculation of the leading-power contribution using QCD factorization [9] is complemented by the calculation of the power-suppressed correction due to photon emission from large distances in terms of photon (rather than B-meson) LCDAs in the LCSR framework. The soft form factor ξ is then entirely independent of the parameters of the B-meson LCDAs. A potential problem of such "hybrid" approaches is that the result is not insensitive to the redefinition of the perturbative, hard-collinear contributions at sub-leading power accuracy (see above). Nevertheless, the validity of this technique and its relation to the approach used in the present work are interesting topics for further study. The $\gamma^*\gamma\pi$ form factor offers itself as a somewhat simpler process where such connections can be investigated.

Acknowledgments

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A B-meson distribution amplitudes

Following [4] we define the B-meson LCDAs as matrix elements of the renormalized non-local operators built of an effective heavy quark field $h_v(0)$ and a light antiquark at a light-like separation,

$$\langle 0|\bar{q}(nz)\Gamma[nz,0]h_v(0)|\bar{B}(v)\rangle = -\frac{i}{2}F_B\,\mathrm{Tr}\left\{\gamma_5\Gamma P_+\left[\Phi_+(z,\mu) - \frac{\not{n}}{2}\Big(\Phi_+(z,\mu)-\Phi_-(z,\mu)\Big)\right]\right\}, \tag{A.1}$$

where

$$[zn,0] \equiv \mathrm{Pexp}\left[ig\int_0^1 du\, n_\mu A^\mu(uzn)\right] \tag{A.2}$$

is the Wilson line factor that ensures gauge invariance. Such factors are always implied.

Here and below v_μ is the heavy quark velocity, n_μ is a light-like vector, $n^2 = 0$, such that $n\cdot v = 1$, $P_+ = \frac{1}{2}(1+\not{v})$, Γ stands for an arbitrary Dirac structure, $|\bar{B}(v)\rangle$ is the $\bar{B}$-meson state in the heavy quark effective theory (HQET) and $F_B(\mu)$ is the scale-dependent

HQET decay constant which is related to the physical B-meson decay constant, to one-loop accuracy, as

$$f_B\sqrt{m_B} = F_B(\mu)K(\mu) = F_B(\mu)\left[1+\frac{C_F\alpha_s}{4\pi}\left(3\ln\frac{m_b}{\mu}-2\right)+\dots\right]. \tag{A.3}$$

The parameter z specifies the light antiquark position on the light cone. To fix the normalization, we assume

$$n_\mu = (1,0,0,1)\,, \qquad \bar n_\mu = (1,0,0-1)\,, \qquad v_\mu = \frac{1}{2}(n_\mu+\bar n_\mu)\,, \qquad n\cdot\bar n = 2\,. \tag{A.4}$$

The functions $\Phi_+(z,\mu)$ and $\Phi_-(z,\mu)$ are the leading- and subleading-twist two-particle B-meson LCDAs [5]. They are analytic functions of z in the lower half-plane $\mathrm{Im}(z)<0$, and are related by Fourier transformation to the momentum-space LCDAs

$$\Phi_\pm(z,\mu) = \int_0^\infty d\omega\, e^{-i\omega z}\,\phi_\pm(\omega,\mu)\,. \tag{A.5}$$

We use upper (lower) case letters for the coordinate-space (momentum-space) distributions.

The coordinate-space LCDAs $\Phi_\pm$ can be written in the form [25, 26, 37][9]

$$\begin{aligned}
\Phi_+(z,\mu) &= -\frac{1}{z^2}\int_0^\infty ds\, s\, e^{is/z}\,\eta_+(s,\mu)\,,\\
\Phi_-(z,\mu) &= -\frac{i}{z}\int_0^\infty ds\, e^{is/z}\left[\eta_+(s,\mu)+\eta_3^{(0)}(s,\mu)\right] = \Phi_-^{\mathrm{WW}}(z,\mu)+\Phi_-^{\mathrm{t3}}(z,\mu),
\end{aligned} \tag{A.6}$$

where $\eta_+(s,\mu)$ and $\eta_3^{(0)}(s,\mu)$ are twist-two and twist-three nonperturbative functions that have autonomous scale dependence:

$$\begin{aligned}
\eta_+(s,\mu) &= U_+(s;\mu,\mu_0)\eta_+(s,\mu_0)\,,\\
\eta_3^{(0)}(s,\mu) &= r^{N_c/\beta_0}U_+(s;\mu,\mu_0)\eta_3^{(0)}(s,\mu_0)\,.
\end{aligned} \tag{A.7}$$

Here $r = \alpha_s(\mu)/\alpha_s(\mu_0)$ and

$$\begin{aligned}
U_+(s;\mu,\mu_0) \;=\; \exp\Bigg\{&-\frac{\Gamma_0}{4\beta_0^2}\left(\frac{4\pi}{\alpha_s(\mu_0)}\left[\ln r - 1 + \frac{1}{r}\right]\right.\\
&\left.-\frac{\beta_1}{2\beta_0}\ln^2 r + \left(\frac{\Gamma_1}{\Gamma_0}-\frac{\beta_1}{\beta_0}\right)[r-1-\ln r]\right)\Bigg\}\;\left(se^{2\gamma_E}\mu_0\right)^{\frac{\Gamma_0}{2\beta_0}\ln r}\; r^{\frac{\gamma_0}{2\beta_0}}\,,
\end{aligned} \tag{A.8}$$

where

$$\gamma_0 = -2C_F\,, \qquad \Gamma_0 = 4C_F\,, \qquad \Gamma_1 = C_F\left[\frac{268}{3}-4\pi^2-\frac{40}{9}n_f\right]. \tag{A.9}$$

The difference with the corresponding expression in [16, 26, 37] is that we included the terms in β_1 and the two-loop cusp anomalous dimension Γ_1, which is consistent with resummation to the leading-logarithmic accuracy. We further replaced $\mu\to\mu e^{\gamma_E}$ to pass from the coordinate-space version of the minimal subtraction scheme used there to conventional $\overline{\mathrm{MS}}$ scheme, cf. [38].

[9] In notation of ref. [25] $s\eta_+(s,\mu)\equiv\rho_+(1/s,\mu)$.

In momentum space the exponential factors are substituted by Bessel functions, in particular

$$\phi_+(\omega,\mu) = \int_0^\infty ds\,\sqrt{\omega s}\,J_1(2\sqrt{\omega s})\eta_+(s,\mu)\,. \tag{A.10}$$

This relation can be inverted to express $\eta_+(s,\mu)$ in terms of $\phi_+(\omega,\mu)$:

$$\eta_+(s,\mu) = \int_0^\infty d\omega\,J_1(2\sqrt{s\omega})\frac{1}{\sqrt{\omega s}}\phi_+(\omega,\mu)\,. \tag{A.11}$$

For the generic ansatz (5.5), an analytic expression for the LCDA $\phi_+(\omega,\mu)$ at arbitrary scale can be found in terms of hypergeometric functions using

$$\begin{aligned}
&\omega_0\int_0^\infty ds\,(\omega_0 s)^p\sqrt{\omega s}J_1(2\sqrt{\omega s})\,{}_1F_1(\alpha,\beta,-\omega_0 s) = \\
&\quad= \frac{\omega}{\omega_0}\frac{\Gamma(\beta)\Gamma(2+p)\Gamma(\alpha-p-2)}{\Gamma(\alpha)\Gamma(\beta-p-2)}\,{}_2F_2(p+2,p+3-\beta;2,p+3-\alpha,-\omega/\omega_0) \\
&\qquad+\left(\frac{\omega}{\omega_0}\right)^{\alpha-p-1}\frac{\Gamma(\beta)\Gamma(p+2-\alpha)}{\Gamma(\beta-\alpha)\Gamma(\alpha-p)}\,{}_2F_2(\alpha,\alpha-\beta+1;\alpha-p-1,\alpha-p,-\omega/\omega_0)\,.
\end{aligned} \tag{A.12}$$

Note that the LCDA $\Phi_-(z,\mu)$ is written as a sum of two terms. The first one, $\Phi_-^{\rm WW}(z,\mu)$, is related to the leading-twist LCDA Φ_+ [5] and is traditionally referred to as the Wandzura-Wilczek (WW) contribution. In momentum space

$$\phi_-^{\rm WW}(\omega,\mu) = \int_\omega^\infty \frac{d\omega'}{\omega'}\phi_+(\omega',\mu)\,. \tag{A.13}$$

The second term, $\Phi_-^{\rm t3}(z,\mu)$, is "genuinely" twist-three and can be expressed in terms of the three-particle LCDA Φ_3 discussed below.

The three-particle quark-gluon matrix element is parametrized by eight invariant functions that can be defined as [16]

$$\begin{aligned}
&\langle 0|\bar q(nz_1)g_sG_{\mu\nu}(nz_2)\Gamma h_v(0)|\bar B(v)\rangle = \\
&= \frac{1}{2}F_B(\mu)\ {\rm Tr}\Big\{\gamma_5\Gamma P_+\Big[(v_\mu\gamma_\nu - v_\nu\gamma_\mu)\big[\Psi_A-\Psi_V\big] - i\sigma_{\mu\nu}\Psi_V - (n_\mu v_\nu - n_\nu v_\mu)X_A \\
&\qquad + (n_\mu\gamma_\nu - n_\nu\gamma_\mu)\big[W+Y_A\big] - i\epsilon_{\mu\nu\alpha\beta}n^\alpha v^\beta\gamma_5\widetilde X_A + i\epsilon_{\mu\nu\alpha\beta}n^\alpha\gamma^\beta\gamma_5\widetilde Y_A \\
&\qquad - (n_\mu v_\nu - n_\nu v_\mu)\not{n}\,W + (n_\mu\gamma_\nu - n_\nu\gamma_\mu)\not{n}\,Z\Big]\Big\}(z_1,z_2;\mu)\,.
\end{aligned} \tag{A.14}$$

We use the standard $(+,-,-,-)$ convention for the metric and $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. The totally antisymmetric Levi-Civita tensor $\epsilon_{\alpha\beta\mu\nu}$ is defined with $\epsilon_{0123}=1$. The covariant derivative is defined as $D_\mu = \partial_\mu - igA_\mu$ and the dual gluon strength tensor as $\widetilde G_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G^{\alpha\beta}$. The momentum space distributions are defined through

$$\Psi_A(z_1,z_2) = \int_0^\infty d\omega_1\int_0^\infty d\omega_2\,e^{-i\omega_1 z_1 - i\omega_2 z_2}\,\psi_A(\omega_1,\omega_2) \tag{A.15}$$

and similarly for the other LCDAs.

The invariant functions appearing in the Lorentz structure decomposition (A.14) can be expanded in contributions of different collinear twist. One finds one LCDA of twist three

$$\Phi_3 = \Psi_A - \Psi_V\,, \tag{A.16}$$

and three twist-four LCDAs [16]

$$\Phi_4 = \Psi_A + \Psi_V\,, \qquad \Psi_4 = \Psi_A + X_A\,, \qquad \widetilde{\Psi}_4 = \Psi_V - \widetilde{X}_A\,. \tag{A.17}$$

Neglecting contributions of four-particle operators of the type $\bar{q}GGh_v$ and $\bar{q}q\bar{q}h_v$ the following relation holds [16]

$$2\frac{d}{dz_1}z_1\Phi_4(z_1,z_2) = \left(\frac{d}{dz_2}z_2+1\right)\Big[\Psi_4(z_1,z_2)+\widetilde{\Psi}_4(z_1,z_2)\Big]\,, \tag{A.18}$$

or, equivalently,

$$[\psi_4+\widetilde{\psi}_4](\omega_1,\omega_2) - \omega_2\frac{\partial}{\partial\omega_2}[\psi_4+\widetilde{\psi}_4](\omega_1,\omega_2) = -2\omega_1\frac{\partial}{\partial\omega_1}\phi_4(\omega_1,\omega_2)\,, \tag{A.19}$$

so that only two of the three twist-four LCDAs are independent.

The analysis of the renormalization-group equations for the relevant operators [16, 37] suggests the following representations:

$$\begin{aligned}
\Phi_3(z_1,z_2,\mu) &= \int_0^\infty ds\Big[\eta_3^{(0)}(s,\mu)\,Y_3^{(0)}(s\,|\,z_1,z_2)+\frac{1}{2}\int_{-\infty}^\infty dx\,\eta_3(s,x,\mu)\,Y_3(s,x\,|\,z_1,z_2)\Big],\\
\Phi_4(z_1,z_2,\mu) &= \frac{1}{2}\int_0^\infty ds\int_{-\infty}^\infty dx\,\eta_4^{(+)}(s,x,\mu)\,Y_{4;1}^{(+)}(s,x\,|z_1,z_2)\,,\\
(\Psi_4+\widetilde{\Psi}_4)(z_1,z_2,\mu) &= -\int_0^\infty ds\int_{-\infty}^\infty dx\,\eta_4^{(+)}(s,x,\mu)\,Y_{4;2}^{(+)}(s,x\,|z_1,z_2)\,,
\end{aligned} \tag{A.20}$$

and

$$(\Psi_4-\widetilde{\Psi}_4)(z_1,z_2,\mu) = (\Psi_4-\widetilde{\Psi}_4)^{\rm t3}(z_1,z_2,\mu) + (\Psi_4-\widetilde{\Psi}_4)^{\rm t4}(z_1,z_2,\mu)\,, \tag{A.21}$$

where

$$\begin{aligned}
(\Psi_4-\widetilde{\Psi}_4)^{\rm t3} &= 2\int_0^\infty ds\left(\frac{iz_2}{s}\right)\Big[\eta_3^{(0)}(s,\mu)\,Y_3^{(0)}(s\,|z_1,z_2)\\
&\qquad\qquad +\frac{1}{2}\int_{-\infty}^\infty dx\,\eta_3(s,x,\mu)\,Y_3(s,x\,|z_1,z_2)\Big]\,,\\
(\Psi_4-\widetilde{\Psi}_4)^{\rm t4} &= -\int_0^\infty ds\int_{-\infty}^\infty dx\,\varkappa_4^{(-)}(s,x,\mu)\,Z_{4;2}^{(-)}(s,x\,|z_1,z_2)\,.
\end{aligned} \tag{A.22}$$

The Y- and Z-functions in these expressions are eigenfunctions of the large-N_c evolutions equations so that the corresponding nonperturbative coefficients $\eta_3^{(0)}(s,\mu)$, $\eta_3(s,x,\mu)$ (twist-three) and $\eta_4^{(+)}(s,x,\mu)$, $\varkappa_4^{(-)}(s,x,\mu)$ (twist-four) have autonomous scale dependence to this accuracy. The function $\eta_3^{(0)}(s,\mu)$ is in fact not independent and can be obtained by analytic continuation of $\eta_3(s,x,\mu)$ to the complex plane, $x\to i/2$, see (A.33) below. Explicit expressions for the eigenfunctions in coordinate and momentum space, and the corresponding anomalous dimensions can be found in [16, 37]. Note that the sum $(\Psi_4+\widetilde{\Psi}_4)$ is purely twist-four, whereas the difference $(\Psi_4-\widetilde{\Psi}_4)$ contains both the twist-three contribution that is related to Φ_3, and the "genuine" twist-four part. The two twist-four nonperturbative functions on the line $x=0$ are related as

$$\left[1+\partial_s s-\partial_s^2 s^2-2s\bar{\Lambda}\right]\,\eta_+(s,\mu)=\pi\sqrt{s}\varkappa_4^{(-)}(s,0,\mu)-\pi\sqrt{s}\eta_4^{(+)}(s,0,\mu)\,. \tag{A.23}$$

This equation presents the nonlocal generalization of the moment relations [4]

$$\int_0^\infty d\omega\,\omega\,\phi_+(\omega)=\frac{4}{3}\bar{\Lambda}\,,\qquad \int_0^\infty d\omega\,\omega^2\phi_+(\omega)=2\bar{\Lambda}^2+\frac{2}{3}\lambda_E^2+\frac{1}{3}\lambda_H^2\,, \tag{A.24}$$

where λ_E^2 and λ_H^2 parametrize the matrix element of the local quark-gluon operator

$$\begin{aligned}&\langle 0|\bar{q}(0)g_sG_{\mu\nu}(0)\Gamma h_v(0)|\bar{B}(v)\rangle=\\&=-\frac{i}{6}F_B\lambda_H^2\,\mathrm{Tr}\left[\gamma_5\Gamma P_+\sigma_{\mu\nu}\right]-\frac{1}{6}F_B\left(\lambda_H^2-\lambda_E^2\right)\mathrm{Tr}\left[\gamma_5\Gamma P_+(v_\mu\gamma_\nu-v_\nu\gamma_\mu)\right].\end{aligned} \tag{A.25}$$

The matrix element can be estimated from QCD sum rules. One obtains

$$\lambda_E^2=0.11\pm0.06\ \mathrm{GeV}^2,\qquad \lambda_H^2=0.18\pm0.07\ \mathrm{GeV}^2,\qquad [4] \tag{A.26}$$

$$\lambda_E^2=0.03\pm0.02\ \mathrm{GeV}^2,\qquad \lambda_H^2=0.06\pm0.03\ \mathrm{GeV}^2,\qquad [31] \tag{A.27}$$

where the second calculation takes into account some NLO corrections. Note that the ratio

$$R=\lambda_E^2/\lambda_H^2\simeq 0.5 \tag{A.28}$$

is almost the same in both calculations and is generally more reliable than the values of the matrix elements themselves as many uncertainties cancel. If the moment relations (A.24) are imposed, then, for a given leading twist LCDA $\phi_+(\omega)$, only this ratio remains a free parameter.

Physical quantities usually involve an integration over the position of the gluon field operator and the resulting expressions often become much simpler, e.g.,

$$\begin{aligned}\int_0^1 du\,\left[\Psi_4-\widetilde{\Psi}_4\right]^{\mathrm{t3}}(z,uz)&=z^{-2}\,\Phi_-^{\mathrm{t3}}(z)=-\frac{i}{z^3}\int_0^\infty ds\,e^{is/z}\,\eta_3^{(0)}(s,\mu)\,,\\ \int_0^1 du\,\left[\Psi_4-\widetilde{\Psi}_4\right]^{\mathrm{t4}}(z,uz)&=-\frac{i}{z^3}\int_0^\infty ds\,\mathrm{e}^{is/z}\,\pi\sqrt{s}\varkappa_4^{(-)}(s,0,\mu)\,,\end{aligned} \tag{A.29}$$

and, using (A.23),

$$\begin{aligned}\int_0^1 du\left[\Psi_4-\widetilde{\Psi}_4\right]^{\mathrm{t4}}(z,uz)=&-\frac{1}{z}\left[\int_0^1 du\,u\,\Phi_+'(uz)+\Phi_+'(z)+2i\bar{\Lambda}\Phi_+(z)\right]\\&-\int_0^1 du\left\{\left[\Psi_4+\widetilde{\Psi}_4\right](z,uz)+\left[\Psi_4+\widetilde{\Psi}_4\right](uz,z)\right\}.\end{aligned} \tag{A.30}$$

A useful representation of the twist-four coefficient functions on the line $x = 0$ in terms of the momentum-space LCDAs reads

$$\eta_4^{(+)}(s,0,\mu) = \frac{\sqrt{s}}{\pi}\int\limits_0^\infty \frac{d\omega_1}{\sqrt{\omega_1}}\int\limits_0^\infty \frac{d\omega_2}{\sqrt{\omega_2}}\int\limits_0^1 \frac{du}{\sqrt{u\bar u}} J_1(2\sqrt{su\omega_1})J_1(2\sqrt{s\bar u\omega_2})[\psi_4+\widetilde{\psi}_4](\omega_1,\omega_2)\,, \tag{A.31}$$

$$\varkappa_4^{(-)}(s,0,\mu) = -\frac{\sqrt{s}}{\pi}\int\limits_0^\infty \frac{d\omega_1}{\sqrt{\omega_1}}\int\limits_0^\infty \frac{d\omega_2}{\sqrt{\omega_2}}\int\limits_0^1 \frac{u\,du}{\sqrt{u\bar u}} J_1(2\sqrt{su\omega_1})J_1(2\sqrt{s\bar u\omega_2})[\psi_4-\widetilde{\psi}_4]^{\mathrm{tw}-4}(\omega_1,\omega_2)\,.$$

The integrals appearing in the higher-twist corrections (3.5), (3.6) and (3.9), (3.10) to $B \to \gamma\ell\nu_\ell$ can be expressed in terms of the LCDAs in the (s, x) representation as follows:

$$\begin{aligned}
\int_0^\infty d\omega\, \ln\omega\, \phi_-^{\mathrm{t3}}(\omega) &= -\int_0^\infty \frac{ds}{s}\,\eta_3^{(0)}(s,\mu)\,,\\
\int_0^\infty \frac{d\omega_1}{\omega_1}\int_0^\infty \frac{d\omega_2}{\omega_1+\omega_2}[\psi_4-\widetilde{\psi}_4]^{\mathrm{t4}}(\omega_1,\omega_2) &= -\pi\int_0^\infty \frac{ds}{\sqrt{s}}\,\varkappa_4^{(-)}(s,0,\,\mu)\,,\\
\int_0^\infty \frac{d\omega_1}{\omega_1}\int_0^\infty \frac{d\omega_2}{\omega_1+\omega_2}\phi_3(\omega_1,\omega_2) &= \frac{1}{2}\int_0^\infty \frac{ds}{s}\,\eta_3^{(0)}(s,\mu) - \pi\int_0^\infty \frac{ds}{s}\,\eta_3^{(1)}(s,\mu)\,,
\end{aligned} \tag{A.32}$$

where $\eta_3^{(0)}(s,\mu)$ (A.6) and $\eta_3^{(1)}(s,\mu)$ are the first two coefficients in the Laurent expansion of $\eta_3(s,x,\mu)$ for $x \to -i/2$,

$$\eta_3(s,x,\mu)\Big|_{x\to -i/2} = -\frac{i}{\pi}\frac{1}{x+i/2}\eta_3^{(0)}(s,\mu) + \eta_3^{(1)}(s,\mu) + \mathcal{O}\Big(x+\frac{i}{2}\Big). \tag{A.33}$$

The leading off-light cone contributions in the current correlation functions can be calculated in terms of the two-particle higher-twist LCDAs by extending the definition from [5] to include $\mathcal{O}(x^2)$ terms as follows:

$$\begin{aligned}
\langle 0|\bar q(x)\Gamma[x,0]h_v(0)|\bar B(v)\rangle &= -\frac{i}{2}F_B\,\mathrm{Tr}\Big[\gamma_5\Gamma P_+\Big]\int_0^\infty d\omega\, e^{-i\omega(vx)}\Big\{\phi_+(\omega)+x^2 g_+(\omega)\Big\}\\
&+\frac{i}{4}F_B\,\mathrm{Tr}\Big[\gamma_5\Gamma P_+\not{x}\Big]\frac{1}{vx}\int_0^\infty d\omega\, e^{-i\omega(vx)}\Big\{[\phi_+-\phi_-](\omega)+x^2[g_+-g_-](\omega)\Big\}.
\end{aligned} \tag{A.34}$$

It is assumed that $|x^2| \ll 1/\Lambda^2$. The two new LCDAs, $g_+(\omega)$ and $g_-(\omega)$ are of twist four and five, respectively. They are not independent and can be calculated in terms of the three-particle LCDAs and the two-particle LCDAs of lower twist [16, 34]. To the tree-level accuracy one obtains in coordinate space

$$2z^2\mathrm{G}_+(z) = -\Big[z\frac{d}{dz}-\frac{1}{2}+iz\bar\Lambda\Big]\Phi_+(z) - \frac{1}{2}\Phi_-(z) - z^2\int_0^1 \bar u du\,\Psi_4(z,uz)\,, \tag{A.35}$$

$$2z^2\mathrm{G}_-(z) = -\Big[z\frac{d}{dz}-\frac{1}{2}+iz\bar\Lambda\Big]\Phi_-(z) - \frac{1}{2}\Phi_+(z) - z^2\int_0^1 \bar u du\,\Psi_5(z,uz)\,, \tag{A.36}$$

where

$$\mathrm{G}_\pm(z,\mu) = \int_0^\infty d\omega\, e^{-i\omega z} g_\pm(\omega,\mu)\,. \tag{A.37}$$

In the present context it is important that the expression for the two-particle LCDA $G_+(z)$ in (A.35) and the constraint in (A.23) are derived under the same assumptions; hence also the relations in (A.24) have to be satisfied.

In practical calculations it proves to be advantageous to write $G_+(z)$ as the sum of the Wandzura-Wilczek part and the remaining twist-three and twist-four contributions

$$\begin{aligned} \mathrm{G}_+^{\mathrm{WW}}(z) &= -\frac{1}{2z^2}\left[z\frac{d}{dz} - \frac{1}{2} + iz\bar{\Lambda}\right]\Phi_+(z) - \frac{1}{4z^2}\Phi_-^{\mathrm{WW}}(z)\,, \\ \mathrm{G}_+^{\mathrm{t3+t4}}(z) &= -\frac{1}{4z^2}\Phi_-^{\mathrm{t3}}(z) - \frac{1}{2}\int_0^1 \bar{u}du\,\Psi_4(z,uz)\,, \end{aligned} \tag{A.38}$$

and to combine the higher-twist terms with the contribution of gluon emission from the hard-collinear quark propagator, see (3.4). In this way, remarkable cancellations occur that partially can be expected as a consequence of Ward identities.

In [16] several models for the higher-twist LCDAs have been suggested that incorporate the correct low-momentum behaviour and satisfy the (tree-level) EOM constraints. One can show that all these models can be obtained as particular cases of a more general ansatz

$$\begin{aligned} \phi_+(\omega) &= \omega\, f(\omega)\,, \\ \phi_3(\omega_1,\omega_2) &= -\frac{1}{2}\varkappa(\lambda_E^2 - \lambda_H^2)\,\omega_1\omega_2^2\, f'(\omega_1+\omega_2)\,, \\ \phi_4(\omega_1,\omega_2) &= \frac{1}{2}\varkappa(\lambda_E^2 + \lambda_H^2)\,\omega_2^2\, f(\omega_1+\omega_2)\,, \end{aligned} \tag{A.39}$$

where $f'(\omega) = df(\omega)/d\omega$ and the normalization constant $\varkappa$ is fixed by the leading-twist LCDA through the EOM relation (A.24) to

$$\varkappa^{-1} = \frac{1}{6}\int_0^\infty d\omega\,\omega^3 f(\omega) = \bar{\Lambda}^2 + \frac{1}{6}\big(2\lambda_E^2 + \lambda_H^2\big). \tag{A.40}$$

Here it is assumed that the function $f(\omega)$ is normalized as $\int_0^\infty d\omega\,\omega\, f(\omega) = 1$ and decreases sufficiently fast at $\omega \to \infty$ so that at least the first three moments $\int_0^\infty d\omega\,\omega^k f(\omega)$, $k = 1,2,3\ldots$ are finite. While this cannot hold true in general due to the large-momentum tail generated by perturbative radiative corrections [33], the assumption is consistent in the context of tree-level calculations of higher-twist contributions as performed here.

From (A.19) one obtains for this ansatz

$$[\psi_4 + \widetilde{\psi}_4](\omega_1,\omega_2) = \varkappa(\lambda_E^2 + \lambda_H^2)\,\omega_1\omega_2\, f(\omega_1+\omega_2)\,, \tag{A.41}$$

but for $[\psi_4 - \widetilde{\psi}_4]$ only the integral (A.30) can be determined for a generic profile function $f(\omega)$ without additional assumptions. Luckily, only this integral is relevant for $B \to \gamma\ell\nu_\ell$.[10] The WW part of ϕ_- and g_+ and the "genuine" twist-three part of ϕ_- can be expressed in

[10] For the special choices of the profile function $f(\omega)$ made in [16] the LCDAs ψ_4 and $\widetilde{\psi}_4$ can be separated. One obtains $\psi_4(\omega_1,\omega_2) = \varkappa\,\lambda_E^2\,\omega_1\omega_2\, f(\omega_1+\omega_2)$ and $\widetilde{\psi}_4(\omega_1,\omega_2) = \varkappa\,\lambda_H^2\,\omega_1\omega_2\, f(\omega_1+\omega_2)$.

terms of $f(\omega)$ in the form

$$
\begin{aligned}
\phi_-^{\rm WW}(\omega) &= \int_\omega^\infty d\rho\, f(\rho)\,,\\
\phi_-^{\rm t3}(\omega) &= \frac{1}{6}\varkappa(\lambda_E^2-\lambda_H^2)\left[\omega^2 f'(\omega)+4\omega f(\omega)-2\int_\omega^\infty d\rho\, f(\rho)\right],\\
g_+^{\rm WW}(\omega) &= -\frac{1}{4}\int_\omega^\infty d\rho\,\Big(\rho\,\phi_-^{\rm WW}(\rho)+2(\bar\Lambda-\rho)\phi_+(\rho)\Big)\\
&= \frac{1}{8}\int_\omega^\infty d\rho\,(\omega^2+3\rho^2-4\bar\Lambda\rho)\,f(\rho)
\end{aligned}
\tag{A.42}
$$

For the particular combinations of the integrated LCDAs entering the higher-twist corrections in (3.5), (3.6) and (3.9), (3.10) we find the remarkably simple results

$$
\begin{aligned}
\int_0^\infty d\omega\, \ln\omega\, \phi_-^{\rm t3}(\omega) &= \frac{1}{6}\varkappa(\lambda_E^2-\lambda_H^2)\,,\\
\int_0^\infty \frac{d\omega_1}{\omega_1}\int_0^\infty \frac{d\omega_2}{\omega_1+\omega_2}\,\phi_3(\omega_1,\omega_2) &= \frac{1}{3}\varkappa(\lambda_E^2-\lambda_H^2)\,,\\
\int_0^\infty \frac{d\omega_1}{\omega_1}\int_0^\infty \frac{d\omega_2}{\omega_2}\big[\psi_4+\widetilde\psi_4\big](\omega_1,\omega_2) = 2\int_0^\infty \frac{d\omega_2}{\omega_2}\phi_4(0,\omega_2) &= \varkappa(\lambda_E^2+\lambda_H^2)\,,
\end{aligned}
\tag{A.43}
$$

which do not depend on the shape of the function $f(\omega)$. Also the auxiliary functions $\Xi_{1,2}(\omega)$ in (4.15), (4.16) can be expressed simply as

$$
\Xi_1(\omega)=\frac{2}{3}\varkappa(\lambda_E^2+2\lambda_H^2)\left[\omega^2 f(\omega)-2\omega\phi_-^{\rm WW}(\omega)\right]-2\omega\phi_-^{\rm WW}(\omega)+3\omega^2 f(\omega)+\omega^3 f'(\omega)\,, \tag{A.44}
$$

$$
\Xi_2(\omega)=-\frac{2}{3}\varkappa(\lambda_E^2-\lambda_H^2)\left[\omega^2 f(\omega)-2\omega\phi_-^{\rm WW}(\omega)\right]+(\bar\Lambda-\omega)\,\omega f(\omega)-\omega\phi_-^{\rm WW}(\omega) \tag{A.45}
$$

with $\phi_-^{\rm WW}(\omega)$ given by (A.42) above.

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