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Attractive strings and five-branes, skew-holomorphic Jacobi forms and moonshine

Miranda C.N. Cheng,[a] John F.R. Duncan,[b] Sarah M. Harrison,[c,d] Jeffrey A. Harvey,[e] Shamit Kachru[f] and Brandon C. Rayhaun[f]

[a] *Institute of Physics and Korteweg-de Vries Institute for Mathematics, University of Amsterdam, Amsterdam, the Netherlands*

[b] *Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322, U.S.A.*

[c] *Center for the Fundamental Laws of Nature, Harvard University, Cambridge, MA 02138, U.S.A.*

[d] *Department of Mathematics and Statistics and Department of Physics, McGill University, Montreal, QC, Canada*

[e] *Enrico Fermi Institute and Department of Physics, University of Chicago, Chicago IL 60637, U.S.A.*

[f] *Stanford Institute for Theoretical Physics, Stanford University, Palo Alto, CA 94305, U.S.A.*

E-mail: chengm@math.jussieu.fr, john.duncan@emory.edu, sarharr@gmail.com, j-harvey@uchicago.edu, skachru@stanford.edu, brandonrayhaun@gmail.com

Abstract: We show that certain BPS counting functions for both fundamental strings and strings arising from fivebranes wrapping divisors in Calabi-Yau threefolds naturally give rise to skew-holomorphic Jacobi forms at rational and attractor points in the moduli space of string compactifications. For M5-branes wrapping divisors these are forms of weight negative one, and in the case of multiple M5-branes skew-holomorphic mock Jacobi forms arise. We further find that in simple examples these forms are related to skew-holomorphic (mock) Jacobi forms of weight two that play starring roles in moonshine. We discuss examples involving M5-branes on the complex projective plane, del Pezzo surfaces of degree one, and half-K3 surfaces. For del Pezzo surfaces of degree one and certain half-K3 surfaces we find a corresponding graded (virtual) module for the degree twelve Mathieu group. This suggests a more extensive relationship between Mathieu groups and complex surfaces, and a broader role for M5-branes in the theory of Jacobi forms and moonshine.

Keywords: Black Holes in String Theory, Discrete Symmetries, M-Theory, Superstrings and Heterotic Strings

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1 Introduction

Jacobi forms and mock Jacobi forms play important roles as counting functions governing black hole entropy in string theory. For a recent comprehensive discussion see [1]. They also play starring roles in studies of moonshine, as in, e.g., [2–4]. Skew-holomorphic Jacobi forms, first introduced by Skoruppa in [5, 6], also play an important role in moonshine. Indeed, the weight one-half modular forms exhibiting moonshine for the Thompson group in [7] can be recast as the theta components of skew-holomorphic Jacobi forms, an observation extended in [8] to obtain a larger family of moonshine phenomena. In this work we promote the idea that BPS counting functions appearing in the theory of strings and wrapped fivebranes at rational and attractor points provide a rich source of such objects and suggest further new possibilities for connections between moonshine, black holes, and BPS state counting.

Our first main observation is that half-BPS state counting functions for the heterotic string on S^1 at rational points in the Narain moduli space lead directly to skew-holomorphic Jacobi forms. Our second main observation is that M5-branes wrapping divisors in Calabi-Yau threefolds, studied in e.g. [9] as giving rise to black strings in M-theory, provide another natural source of skew-holomorphic Jacobi forms. As discussed in [10–13], the modified elliptic genera counting supersymmetric states in these theories are non-holomorphic modular forms of a certain kind. We will see that at suitable moduli these functions can be specialized to skew-holomorphic Jacobi forms. A number of examples of such genera were computed in a closely related setup in [14] (note that many of these do not satisfy the "ampleness" assumption of [9]). We will see, in several cases, that a skew-holomorphic Jacobi form or mock Jacobi form of weight 2 which plays a role in moonshine can be extracted. We will focus on cases where either a single M5-brane is wrapped, or two M5-branes are wrapped. Skew-holomorphic mock Jacobi forms appear in the latter case, due to the presence of bound states of single wrapped M5-branes.

Another important observation concerns the particular example of a single M5-brane wrapping a del Pezzo surface of degree one (i.e. $\mathbb{P}^2$ blown up at eight points). As we explain in section 5.2, the corresponding skew-holomorphic Jacobi form of weight 2 admits an interpretation as a generating function for the graded dimension of a graded virtual module for the sporadic simple group M_{12}. This suggests a non-trivial relationship between M_{12} and del Pezzo surfaces, and a concrete path to begin its exploration. In section 5.3 we give evidence that this relationship can be extended to half-K3 surfaces (i.e. blow-ups of $\mathbb{P}^2$ at nine points) at certain moduli. In addition to this, the form in which the relevant skew-holomorphic Jacobi forms are found points toward a concrete construction in terms of a vertex algebra attached to a certain indefinite lattice.

The plan of this note is as follows. In section 2 we give a brief review of skew-holomorphic Jacobi forms. In section 3 we discuss S^1 heterotic string compactifications at rational points in Narain moduli space and highlight the connection between the BPS counting function and skew-holomorphic Jacobi forms. In section 4 we review the M5-brane elliptic genus, and show that, when evaluated at a relevant attractor point in moduli space, it gives a skew-holomorphic Jacobi form of weight -1. In section 5 we discuss several examples where weight 2 skew-holomorphic (mock) Jacobi forms that are implicated in moonshine appear. The discussion of M_{12} and del Pezzo surfaces appears in section 5.2, and this is extended to half-K3 surfaces in section 5.3. Some further details and supporting data for these relationships appears in appendix A.

2 Skew-holomorphic Jacobi forms

We briefly review skew-holomorphic Jacobi forms in this section, referring to [1] or [15] for more details.

In very general terms, a skew-holomorphic Jacobi form of weight k and index m is a function of the form

$$\varphi(\tau, z) = \sum_{r \bmod 2m} \overline{f_r(\tau)} \theta_{m,r}(\tau, z) \tag{2.1}$$

where the *theta-coefficients* f_r are the components of a holomorphic vector-valued modular form of weight $k - \frac{1}{2}$. In this work we consider $m \in \frac{1}{2}\mathbb{Z}$, and use

$$\theta_{m,r}(\tau, z) := \sum_{\substack{\ell \in \mathbb{Z}+m \\ \ell = r \bmod 2m}} \mathrm{e}(m\ell) y^\ell q^{\frac{\ell^2}{4m}}, \tag{2.2}$$

for $r \in \mathbb{Z}+m$, where $\mathrm{e}(x) := e^{2\pi i x}$ and $y := \mathrm{e}(z)$ and $q := \mathrm{e}(\tau)$. Usually it is required that $f_r(\tau) = O(1)$ as $\Im(\tau) \to \infty$, for all r, and the term *weakly skew-holomorphic* is used when this is relaxed to $f_r(\tau) = O(e^{C\Im(\tau)})$ for some $C > 0$. A skew-holomorphic mock Jacobi form is a function as in (2.1) for which the f_r are mock modular forms in the usual sense (cf. e.g. [1]).

In order to formulate some examples define the *thetanullwerte*

$$\begin{aligned} \theta^0_{m,r}(\tau) &:= \sum_{\substack{\ell\in\mathbb{Z}+m \\ \ell=r \bmod 2m}} \mathrm{e}(m\ell) q^{\frac{\ell^2}{4m}}, \\ \theta^1_{m,r}(\tau) &:= \sum_{\substack{\ell\in\mathbb{Z}+m \\ \ell=r \bmod 2m}} \mathrm{e}(m\ell)\ell q^{\frac{\ell^2}{4m}}. \end{aligned} \tag{2.3}$$

Then for $k \in \{1, 2\}$ and $m \in \frac{1}{2}\mathbb{Z}$ the function

$$t_{k,m}(\tau, z) := \sum_{r \bmod 2m} \overline{\theta^{k-1}_{m,r}(\tau)} \theta_{m,r}(\tau, z) \tag{2.4}$$

is a skew-holomorphic Jacobi form of weight k and index m. These *theta-type* skew-holomorphic Jacobi forms (cf. section 3.1 of [15]) arise as shadows in umbral moonshine. For example, if

$$H^{(2)}(\tau) = -2q^{-\frac{1}{8}} + 90q^{\frac{7}{8}} + 462q^{\frac{15}{8}} + 1540q^{\frac{23}{8}} + \ldots \tag{2.5}$$

is the McKay-Thompson series attached to the identity element of M_{24} by Mathieu moonshine [2] then $\phi^{(2)}(\tau, z) := H^{(2)}(\tau)(\theta_{2,-1}(\tau, z) - \theta_{2,1}(\tau, z))$ is a (weakly holomorphic) mock Jacobi form of weight 1 and index 2, and its shadow is proportional to $t_{2,2}(\tau, z)$.

The half-integral index theta series (2.2), (2.3) include some familiar examples, which will play a role in section 5. For instance, for $m = \frac{1}{2}$ we have

$$\begin{aligned} \theta_{\frac{1}{2},\frac{1}{2}}(\tau, z) &= i \sum_{n\in\mathbb{Z}} y^{n+\frac{1}{2}} q^{\frac{1}{2}(n+\frac{1}{2})^2} \\ &= i y^{\frac{1}{2}} q^{\frac{1}{8}} \prod_{n>0} (1 - y^{-1}q^n)(1 - yq^n)(1 - q^n). \end{aligned} \tag{2.6}$$

So $\theta^0_{\frac{1}{2},\frac{1}{2}}$ vanishes identically, but $\theta^1_{\frac{1}{2},\frac{1}{2}} = i\eta^3$, where η denotes the Dedekind eta function, $\eta(\tau) := q^{\frac{1}{24}} \prod_{n>0}(1 - q^n)$. For $m = \frac{3}{2}$ we have $\theta^0_{\frac{3}{2},\frac{3}{2}} = 0$ and

$$\theta^0_{\frac{3}{2},\pm\frac{1}{2}}(\tau) = \mp i \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{6}(3n\pm\frac{1}{2})^2} = \mp i\eta(\tau). \tag{2.7}$$

Also note the identity $t_{2,\frac{1}{2}}(\tau, z) = \frac{1}{2} t_{2,2}(\tau, \frac{1}{2}z)$, which hints at an index $m = \frac{1}{2}$ formulation of Mathieu moonshine. A broader context for this is given in [16].

From a number theoretic point of view skew-holomorphic Jacobi forms play a complementary role to holomorphic Jacobi forms in a particular formulation of the Shimura correspondence, developed by Skoruppa and Zagier [6, 17, 18]. Consequently there are Waldspurger-type results relating Fourier coefficients of holomorphic and (non theta-type) skew-holomorphic Jacobi forms of weight at least 2 to special values of L-functions of cuspidal modular forms with level (cf. [18]). This mechanism plays an important role in the arithmetic geometry of elliptic curves according to the celebrated Birch-Swinnerton-Dyer conjecture. Applications to moonshine have appeared, for instance, in [15] and [19].

Our focus in section 5 will be on examples of M5-brane configurations that produce (weakly) skew-holomorphic (mock) Jacobi forms of weight 2.

3 Rational heterotic string compactifications

In this section we analyze examples of S^1 compactifications of the heterotic string at points in the Narain moduli space that correspond to rational conformal field theories. By definition these are points at which there is an extended chiral algebra with the CFT containing a finite number of irreducible representations of the chiral algebra. The partition function thus decomposes into a finite sum of the form

$$Z(q) = \sum_{j,\bar{j}} N_{j\bar{j}} \chi_j(q) \bar{\chi}_{\bar{j}}(\bar{q}) \tag{3.1}$$

where the $N_{j\bar{j}}$ are non-negative integers and the χ_j ($\bar{\chi}_{\bar{j}}$) furnish holomorphic (anti-holomorphic) irreducible characters of the extended chiral algebra which is larger than the Virasoro algebra. Of course, the χ_j and $\bar{\chi}_{\bar{j}}$ are in general reducible with respect to the Virasoro algebra and decompose into a possibly infinite sum of its irreducible characters. We show that the half-BPS state counting functions which arise can be written in terms of skew-holomorphic Jacobi forms. See [20] for a general discussion of the relationship between rational CFT and attractor points in the moduli space of string compactifications.

3.1 The rational Gaussian model

The $c = 1$ Gaussian model, corresponding to string compactification on a S^1 of radius R, is defined by an embedding of the unique unimodular even lattice of signature $(1,1)$ into $\mathbb{R}^{1,1}$. We denote the embedded lattice by $\Gamma^{1,1}$ and write lattice vectors and their standard projections as $p = (p_L, p_R)$. More generally, for $r = s \bmod 8$, we will use $\Gamma^{r,s}$ to denote an embedding of the unique unimodular even lattice of signature (r,s) into $\mathbb{R}^{r,s}$. Using conventions in which the inverse string tension is $\alpha' = 2$ we have

$$p_L = \frac{n}{R} + \frac{wR}{2} \tag{3.2}$$

$$p_R = \frac{n}{R} - \frac{wR}{2} \tag{3.3}$$

with $n, w \in \mathbb{Z}$. The moduli space of the $c = 1$ Gaussian model is

$$\mathbb{Z}_2 \backslash O(1,1;\mathbb{R})/O(1) \times O(1) \simeq \mathbb{Z}_2 \backslash \mathbb{R}_+ \tag{3.4}$$

where the $\mathbb{Z}_2$ acts as T-duality, $R \mapsto \frac{2}{R}$. Thus the moduli space is the half line $[\sqrt{2}, \infty)$ parametrized by R.

The model contains holomorphic and anti-holomorphic U(1) currents J, $\bar{J}$ with eigenvalues proportional to p_L, p_R. Introducing chemical potentials $\zeta = (\zeta_L, \zeta_R)$ to keep track of these U(1) charges leads to the partition function

$$Z(\tau,\zeta) = \Theta_{1,1}(R;\tau,\zeta)|\eta(\tau)|^{-2} \tag{3.5}$$

where

$$\Theta_{1,1}(R;\tau,\zeta) := \sum_{p\in\Gamma^{1,1}} q^{\frac{1}{2}p_L^2}\bar{q}^{\frac{1}{2}p_R^2}e^{2\pi i\zeta\cdot p}\,. \tag{3.6}$$

Let $\Gamma_R := \{(0,p_R)\in\Gamma^{1,1}\}$ be the lattice of right-moving momenta. We now consider rational points in the moduli space where $R^2\in\mathbb{Q}$, and say that Γ_R is generated by p_0. In order to facilitate the comparison to skew-holomorphic Jacobi forms using the conventions of the previous section we specialize to the case $\zeta(z) = \bar{z}p_0$ (this corresponds to choosing the normalization of $\bar{J}$ such that the associated charge has integer eigenvalues). We will show that the Siegel-Narain theta function $\Theta_{1,1}$ is the complex conjugate of a weight one skew-holomorphic Jacobi form of theta-type at such rational points.

Consider first the self-dual point $R=\sqrt{2}$. We then have

$$\Theta_{1,1}(\sqrt{2};\tau,\zeta(z)) = \sum_{n,w\in\mathbb{Z}} q^{\frac{(n+w)^2}{4}}\bar{q}^{\frac{(n-w)^2}{4}}\bar{y}^{n-w} \tag{3.7}$$

with $\bar{y} = \mathrm{e}(-\bar{z})$. Breaking the sum into terms with $n+w$ even and $n+w$ odd gives

$$\Theta_{1,1}(\sqrt{2};\tau,\zeta(z)) = \sum_{r \bmod 2} \overline{\theta_{1,r}(\tau,z)}\theta^0_{1,r}(\tau) = \overline{t_{1,1}(\tau,z)}\,, \tag{3.8}$$

which is of the claimed form.

It is not difficult to generalize this to general rational $\frac{R^2}{2}$, a problem which appears as Exercise 10.21 in [21]. We write $R^2 = 2\frac{\kappa'}{\kappa}$ with κ',κ coprime integers. We then have

$$\begin{aligned}\Theta_{1,1}\left(\sqrt{2\frac{\kappa'}{\kappa}};\tau,\zeta(z)\right) &= \sum_{(p_L,p_R)\in\Gamma^{1,1}} q^{\frac{1}{2}p_L^2}\bar{q}^{\frac{1}{2}p_R^2}\bar{y}^{p_R\sqrt{2\frac{\kappa'}{\kappa}}} \\ &= \sum_{n,w} q^{\frac{(n\kappa+w\kappa')^2}{4\kappa\kappa'}}\bar{q}^{\frac{(n\kappa-w\kappa')^2}{4\kappa\kappa'}}\bar{y}^{n\kappa-w\kappa'}.\end{aligned} \tag{3.9}$$

Now define r_0, s_0 to be integers for which $\kappa r_0 - \kappa' s_0 = 1$, which is always possible since κ,κ' are coprime. Define ω_0 and r to be the values of $\kappa r_0 + \kappa' s_0$ and $n\kappa + w\kappa'$ modulo $2\kappa\kappa'$ respectively. Then a short computation shows that $n\kappa - w\kappa' = \omega_0 r \bmod 2\kappa\kappa'$ which allows us to write

$$\begin{aligned}\Theta_{1,1}\left(\sqrt{2\frac{\kappa'}{\kappa}};\tau,\zeta(z)\right) &= \sum_{r \bmod 2\kappa\kappa'}\ \sum_{n=\omega_0 r \bmod 2\kappa\kappa'} \bar{q}^{\frac{n^2}{4\kappa\kappa'}}\bar{y}^n \sum_{m=r \bmod 2\kappa\kappa'} q^{\frac{m^2}{4\kappa\kappa'}} \\ &= \sum_{r \bmod 2\kappa\kappa'} \overline{\theta_{\kappa\kappa',\omega_0 r}(\tau,z)}\theta^0_{\kappa\kappa',r}(\tau)\end{aligned} \tag{3.10}$$

This is almost of the desired form except for the factor of ω_0. This factor can be understood in terms of an automorphism of the fusion rule algebra as discussed in [20, 22] and in the mathematical literature is related to well-known objects, namely the Eichler-Zagier operators which played a prominent role in [3].

To see this, we can perform a trivial rewriting of the previous equation,

$$\Theta_{1,1}\left(\sqrt{2\frac{\kappa'}{\kappa}};\tau,\zeta(z)\right) = \sum_{s,r \bmod 2\kappa\kappa'} \overline{\theta_{\kappa\kappa',s}(\tau,z)}\delta_{s,\omega_0 r}\theta^0_{\kappa\kappa',r}(\tau)\,. \tag{3.11}$$

The matrix with matrix elements $\delta_{s,\omega_0 r}$ is an Eichler-Zagier matrix,

$$\Omega_{\kappa\kappa'}(\kappa)_{sr} = \delta_{s,\omega_0 r}\,, \tag{3.12}$$

see [3] for conventions. Recall that $\Omega_m(n)_{sr} = 1$ if $s+r = 0 \bmod 2n$ and $s-r = 0 \bmod \frac{2m}{n}$, and 0 otherwise. An easy calculation shows that the two conditions required for a matrix element of $\Omega_{\kappa\kappa'}(\kappa)$ to be nonzero are equivalent to $s = \omega_0 r \bmod 2\kappa\kappa'$:

$$\begin{aligned}(s-\omega_0 r) \bmod 2\kappa &= s-(\kappa r_0+\kappa' s_0)r \bmod 2\kappa \\ &= s+(\kappa r_0-\kappa' s_0)r \bmod 2\kappa \\ &= (s+r) \bmod 2\kappa\end{aligned} \tag{3.13}$$

$$\begin{aligned}(s-\omega_0 r) \bmod 2\kappa' &= s-(\kappa r_0+\kappa' s_0)r \bmod 2\kappa' \\ &= s-(\kappa r_0-\kappa' s_0)r \bmod 2\kappa' \\ &= (s-r) \bmod 2\kappa'\end{aligned} \tag{3.14}$$

from which it easily follows that $\Omega_{\kappa\kappa'}(\kappa)_{sr} = \delta_{s,\omega_0 r}$ and thus that (3.11) is the complex conjugate of a skew-holomorphic Jacobi form:

$$\Theta_{1,1}\left(\sqrt{2\frac{\kappa'}{\kappa}};\tau,\zeta(z)\right) = \overline{\theta_{\kappa\kappa'}(\tau,z)}\cdot\Omega_{\kappa\kappa'}(\kappa)\cdot\theta^0_{\kappa\kappa'}(\tau) \tag{3.15}$$

where we have suppressed the vector indices in the above equation.

3.2 Heterotic strings with Wilson lines

We now explain the relevance of this computation to BPS state counting for heterotic strings on S^1. In this case the Narain moduli space has dimension 17, corresponding to the radius of the S^1 and a choice of Wilson lines in the Cartan subalgebra of $E_8\times E_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$. Half-BPS states correspond to right-moving ground states with arbitrary left-moving excitations [23] and have squared mass proportional to p_R^2. The generating function for these BPS states, summed over all p_R^2 and weighted by a chemical potential for p_R is given by[1]

$$Z_{\mathrm{BPS}}(\tau,\zeta) = \Theta_{17,1}(\tau,\zeta)\eta^{-24}(\tau) \tag{3.16}$$

[1]For the purpose of comparing to black hole microstate counts, we comment that the partition function defined here has the same leading asymptotic behavior as the familiar $1/\eta^{24}(\tau)$, receiving only subleading corrections from the theta function. Similar comments apply to the rest of the counting functions considered in this paper.

where now

$$\Theta_{17,1}(\tau,\zeta) := \sum_{p\in\Gamma^{17,1}} q^{\frac{1}{2}p_L^2}\bar{q}^{\frac{1}{2}p_R^2}e^{2\pi i\zeta\cdot p}\,. \tag{3.17}$$

We expect that $Z_{\rm BPS}$ can be written in terms of skew-holomorphic Jacobi forms at rational points in the Narain moduli space

$$\mathcal{N}_{17,1} := O(17,1,\mathbb{Z})\backslash O(17,1,\mathbb{R})/O(17)\times O(1). \tag{3.18}$$

We will show this explicitly for two examples below, and defer comments about the general case to section 3.3.

The first example involves considering points in the moduli space (3.18) where the Wilson lines are turned off. At these points, the embedded lattice $\Gamma^{17,1}$ respects the standard splitting $\mathbb{R}^{17,1} = \mathbb{R}^{16}\oplus\mathbb{R}^{1,1}$ in the sense that $L := \Gamma^{17,1}\cap\mathbb{R}^{16}$ is a positive-definite even unimodular lattice with rank 16 and $\Gamma^{17,1}\cap\mathbb{R}^{1,1}$ is unimodular and even with signature (1,1). If we further specialize to points in the moduli space where the $\Gamma^{1,1}$ corresponds to a rational CFT of radius $R=\sqrt{2\frac{\kappa'}{\kappa}}$ we then find

$$\begin{aligned} Z_{\rm BPS}(\tau,\zeta(z)) &= \Theta_{17,1}(\tau,\zeta(z))\eta^{-24}(\tau)\\ &= \sum_{(p_L,p_R)\in\Gamma^{17,1}} q^{\frac{1}{2}p_L^2}\bar{q}^{\frac{1}{2}p_R^2}\bar{y}^{p_R\sqrt{2\kappa'\kappa}}\\ &= \overline{\theta_{\kappa\kappa'}(\tau,z)}\cdot\Omega_{\kappa\kappa'}(\kappa)\cdot\theta^0_{\kappa\kappa'}(\tau)\Theta_L(\tau)\eta^{-24}(\tau)\end{aligned} \tag{3.19}$$

where Θ_L is the theta-function attached to the lattice L. There are only two even unimodular lattices of rank 16; namely $E_8\oplus E_8$ and D_{16}^+. In both cases Θ_L is the unique weight 8 modular form that satisfies $\Theta_L(\tau)=1+O(q)$ as $\Im(\tau)\to\infty$. The partition function $Z_{\rm BPS}(\tau,\zeta(z))$ is the complex conjugate of a weakly skew-holomorphic Jacobi form of weight -3.

We can obtain a more subtle rational point by utilizing a construction due to Nikulin [24]. Let Λ_L be an even, rank 7, positive-definite lattice which is primitively embedded into an even, unimodular, rank 24 lattice N, thus N is the Leech lattice or one of the 23 Niemeier lattices. Let Λ_R be a negative-definite lattice bijectively isometric to Λ_L (up to an overall minus sign in the quadratic form) and primitively embedded as a sublattice of the (negative-definite) E_8 root lattice. Define $\Gamma_L := \Lambda_L^\perp\cap N$ and $\Gamma_R := \Lambda_R^\perp\cap(-E_8)$ to be the orthogonal complements of Λ_L and Λ_R respectively. Then the lattice

$$\Gamma := \bigoplus_i\left((\Gamma_L,0)\oplus(0,\Gamma_R)+\left(g_L^{(i)},g_R^{(i)}\right)\right) \tag{3.20}$$

is an even, unimodular lattice of signature $(17,1)$. In the above $g_L^{(i)}$ and $g_R^{(i)}$ are glue vectors which run over the non-trivial elements of the discriminant group of Γ_L,Γ_R, such that the map $g_R^{(i)}\to g_L^{(i)}$ is an isometry.

Our second example will use this construction for the choice $\Lambda_L=E_7$ which is embedded in the Niemeier lattice with root system $A_{17}E_7$, and $\Lambda_R=(-E_7)$ which is primitively embedded into $(-E_8)$.[2] Our conventions for the $A_{17}E_7$ root lattice are as follows. Take

[2]For details on the construction of such lattices, see e.g. [25, 26].

$e_1, e_2, \ldots, e_{26}$ to be an orthonormal basis for $\mathbb{R}^{26} = \mathbb{R}^8 \oplus \mathbb{R}^{18}$ and take the E_7 root lattice to be embedded in the first $\mathbb{R}^8$ with simple roots $r_i := e_{i+2} - e_{i+1}$ for $i = 1, 2, \ldots, 6$, and $r_7 := \frac{1}{2}(e_1 + e_2 + e_3 + e_4 - e_5 - e_6 - e_7 - e_8)$. For the A_{17} root system we embed in the $\mathbb{R}^{18}$ factor and take the simple roots to be $r_i := e_i - e_{i+1}$ for $i = 9, 10, \ldots, 26$. Recall the construction of the Niemeier lattice N corresponding to the $A_{17}E_7$ root system. Letting $L^\star$ denote the dual of a lattice L, we have embeddings

$$A_{17}E_7 \subset N \subset (A_{17}E_7)^\star \tag{3.21}$$

which implies that $N/A_{17}E_7$ is a subgroup of $(A_{17}E_7)^\star/A_{17}E_7$. Moreover, since N is an even lattice, it is an isotropic subgroup, meaning that the quadratic form of the discriminant group restricted to $N/A_{17}E_7$ vanishes. Now, the discriminant group of $A_{17}E_7$ is $\mathbb{Z}_2 \times \mathbb{Z}_{18}$ from the E_7 and A_{17} factors respectively.

The $\mathbb{Z}_2$ component of the discriminant group is generated by

$$v := \frac{1}{4}(3e_1 - e_2 - e_3 - e_4 - e_5 - e_6 + 3e_7 - e_8), \tag{3.22}$$

while the $\mathbb{Z}_{18}$ is generated by

$$w := \frac{1}{18}\left(17e_9 - \sum_{i=10}^{26} e_i\right). \tag{3.23}$$

One can check that the quadratic form on the discriminant group vanishes on the isotropic subgroup $\langle v + 3w\rangle \simeq \mathbb{Z}_6$. The Niemeier lattice is obtained as

$$N = \bigcup_{n=0,\ldots,5} (A_{17}E_7 + n(v + 3w)) \tag{3.24}$$

and the orthogonal complement of the E_7 root lattice in N is easily seen to be

$$\Gamma_L = A_{17} \cup (A_{17} + 6w) \cup (A_{17} + 12w). \tag{3.25}$$

One can convince oneself that the discriminant group $\Gamma_L^\star/\Gamma_L = \langle 3w\rangle \simeq \mathbb{Z}_2$.

The even unimodular lattice Γ that we obtain in this way satisfies

$$\Theta_{17,1}(\tau, \zeta(z)) = \Theta_{\Gamma_L}(\tau)\overline{\theta_{1,0}(\tau, z)} + \Theta_{\Gamma_L+3w}(\tau)\overline{\theta_{1,1}(\tau, z)}, \tag{3.26}$$

and an explicit computation of the theta coefficients yields

$$\begin{aligned}\Theta_{\Gamma_L}(\tau) &= 1 + 306q + 55488q^2 + 1161984q^3 + 10054242q^4 \\ &\quad + 53585088q^5 + 210351744q^6 + 668519424q^7 + \cdots,\end{aligned} \tag{3.27}$$

$$\begin{aligned}\Theta_{\Gamma_L+3w}(\tau) &= 1632q^{5/4} + 134912q^{9/4} + 2110176q^{13/4} \\ &\quad + 15898368q^{17/4} + 76968384q^{21/4} + 286866432q^{25/4} + \cdots.\end{aligned} \tag{3.28}$$

We can identify the above theta coefficients further using results in [5]. Skoruppa classifies the weight k index 1 skew-holomorphic Jacobi forms:

$$J^{\mathrm{sk}}_{k,1} = M_{k-1}(\mathrm{SL}_2(\mathbb{Z})) \cdot t_{1,1}(\tau, z) \oplus M_{k-3}(\mathrm{SL}_2(\mathbb{Z})) \cdot U(\tau, z). \tag{3.29}$$

Here, $M_k(\mathrm{SL}_2(\mathbb{Z}))$ is the space of weight k holomorphic modular forms for $\mathrm{SL}_2(\mathbb{Z})$, and

$$U(\tau,z) := \frac{12}{\pi i}\frac{\partial}{\partial\bar\tau}t_{1,1}(\tau,z) + \overline{E_2(\tau)}t_{1,1}(\tau,z), \tag{3.30}$$

$$E_2(\tau) := 1 - 24\sum_{\ell\geq 1}\left(\sum_{d|\ell} d\right)q^\ell. \tag{3.31}$$

Letting $E_k(\tau) = 1 + O(q)$ be the Eisenstein series of weight k, the weight nine skew-holomorphic forms (and in particular the function we found above) should be of the form

$$a\overline{E_4(\tau)^2}t_{1,1}(\tau,z) + b\overline{E_6(\tau)}U(\tau,z). \tag{3.32}$$

One can verify that the theta function we computed earlier corresponds to the choice $a = \frac{5}{6}$ and $b = \frac{1}{6}$,

$$\Theta_{17,1}(\tau,\zeta(z)) = \frac{1}{6}\left(5\overline{E_4(\tau)^2}t_{1,1}(\tau,z) + \overline{E_6(\tau)}U(\tau,z)\right). \tag{3.33}$$

We are then left with the BPS counting function

$$Z_{\mathrm{BPS}}(\tau,\zeta(z)) = \frac{1}{6}\left(5E_4(\tau)^2\overline{t_{1,1}(\tau,z)} + E_6(\tau)\overline{U(\tau,z)}\right)\eta^{-24}(\tau). \tag{3.34}$$

It should not be hard to generalize this analysis to other rational points in the moduli space (3.18) at which the BPS state counting function can be expressed in terms of skew-holomorphic Jacobi forms.

3.3 Rational toroidal compactifications

As a technical aside, we would like to briefly sketch the general construction which underlies the examples of the previous sections. Quite generally, toroidal string compactifications correspond to Narain lattices Γ of signature $(d+8s,d)$. The points in the moduli space of such lattices where the associated CFT becomes rational are specified by triples (Γ_L,Γ_R,ϕ), where we demand that $\phi : \Gamma_R^\star/\Gamma_R \to \Gamma_L^\star/\Gamma_L$ be an isometric bijection of the discriminant groups. The discriminant groups $\Gamma_L^\star/\Gamma_L$ and $\Gamma_R^\star/\Gamma_R$ inherit their norms from the norms on $\Gamma_L^\star$ and $\Gamma_R^\star$ reduced modulo 2. Using ϕ to obtain so-called glue vectors $(\phi(\lambda),\lambda)$, we may construct the full, rational, unimodular lattice from this data as

$$\Gamma := \bigcup_{\lambda\in\Gamma_R^\star/\Gamma_R}\Big(\Gamma_L\oplus\Gamma_R + (\phi(\lambda),\lambda)\Big). \tag{3.35}$$

It easily follows that the Siegel-Narain theta function admits the decomposition

$$\Theta_\Gamma(\tau) = \sum_{\lambda\in\Gamma_R^\star/\Gamma_R}\Theta_{\Gamma_L+\phi(\lambda)}(\tau)\overline{\Theta_{\Gamma_R+\lambda}(\tau)} \tag{3.36}$$

where we have defined

$$\Theta_{L+\lambda}(\tau) := \sum_{\gamma\in L+\lambda} q^{\frac{\gamma^2}{2}} \tag{3.37}$$

for an arbitrary positive-definite, even lattice L. In this construction, $\Gamma_L := \{(p_L, 0) \in \Gamma\}$ is the lattice of purely left-moving momenta, and similarly for Γ_R. See section 10.2 of [20] for a more detailed discussion.

In the previous sections, we specialized to $d = 1$ and exploited the fact that the right-moving momentum lattice must be of the form $\Gamma_R \simeq \sqrt{2m}\mathbb{Z}$ with associated theta-function

$$\Theta_{\sqrt{2m}\mathbb{Z}+r}(\tau) = \theta^0_{m,r}(\tau) \tag{3.38}$$

for r in $\Gamma_R^\star/\Gamma_R \simeq \mathbb{Z}_{2m}$. Indeed, upon flavoring by an additional $\bar{J}$ quantum number, we find that the points in the moduli space with $\Gamma_R \simeq \sqrt{2m}\mathbb{Z}$ recovered (complex conjugates of) index m skew-holomorphic Jacobi forms.

In this language, the $c = 1$ Gaussian model with radius $R = \sqrt{2\frac{\kappa'}{\kappa}}$ corresponds to the triple $(\sqrt{2\kappa'\kappa}\mathbb{Z}, \sqrt{2\kappa'\kappa}\mathbb{Z}, r \to \omega_0 r)$, with the gluing of left and right-moving momentum lattices specified by the isometry "multiplication by ω_0." The different choices of isometries give rise to different Eichler-Zagier matrices $\Omega_{\kappa\kappa'}(\kappa)$ which commute with the action of the modular group on the thetanullwerte. Similar comments should apply to the problem of classifying the rational points in the moduli space of the heterotic string with Wilson lines, as well as the skew-holomorphic Jacobi forms which arise.

We now turn to a richer source of strings — those arising from wrapped M5-branes — and show that their associated elliptic genera can also be expressed in terms of skew-holomorphic Jacobi forms.

4 The M5-brane elliptic genus

Here we review basic facts about the worldsheet theory on a wrapped M5-brane.

4.1 Multiplets

The M5-brane wrapping a divisor in a Calabi-Yau threefold gives rise, at low-energies, to an effective string, sometimes called an "MSW string," with (0,4) worldsheet supersymmetry. This theory was studied in detail from various viewpoints in, e.g., [9–11, 27]. Suppose the M5-brane is wrapping a divisor $\mathcal{P}$ in a Calabi-Yau threefold X. Then the low-energy theory on the effective string (arising from dimensional reduction of the M5-brane worldvolume fields) is as follows.

Consider the inclusion map

$$i : \mathcal{P} \to X\,.$$

This naturally gives rise to a pullback map $i^* : H^2(X, \mathbb{Z}) \to H^2(\mathcal{P}, \mathbb{Z})$. We define Λ to be $i^*(H^2(X, \mathbb{Z}))$ equipped with the bilinear form given by $(A|B) := -\int_{\mathcal{P}} A \wedge B$. The pullback two-forms $i^*\alpha \in H^2(\mathcal{P}, \mathbb{Z})$ can be associated with chiral worldvolume fields in the $(0, 4)$ worldsheet σ-model as follows.

- Self-dual two-forms on $\mathcal{P}$ that extend non-trivially to X give rise to left-moving scalars on the worldsheet.
- Anti self-dual two-forms on $\mathcal{P}$ that extend non-trivially to X give rise to right-moving scalars on the worldsheet.

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In fact, for a Calabi-Yau threefold X, the Kähler form is the only two-form that pulls back to an anti self-dual form on $\mathcal{P}$. As a result there are $b^2(X) - 1$ left-moving scalars and 1 right-moving scalar coming from these sources.

It is important to remember that the worldsheet fields include universal worldsheet multiplets arising from (super) Goldstone modes. This gives three additional non-chiral scalars that can translate the effective string. The total of four right-moving bosons (including the one arising from the pullback of the Kähler form) have four Fermi superpartners arising from the (0,4) supersymmetry. Zero modes of these fermions lead to a modification of the definition of the M5-brane ellliptic genus relative to the conventional elliptic genus (see (4.1)) as the conventional quantity would vanish in this circumstance.

In a model-dependent way, there are also additional fields present in the generic wrapped M5-brane theory. These parametrize the moduli space of motions of the wrapped divisor in the Calabi-Yau space X. Although our subsequent discussion will be independent of these fields it should be mentioned that in the limit of large central charge, where the effective string can sometimes be related to a weakly curved black string, they constitute the most numerous degrees of freedom.

4.2 The index

For a fixed M5 theory the worldsheet elliptic genus can be defined as follows. First define

$$Z'(\tau,\zeta) := \mathrm{tr}_R \left(F^2(-1)^F e^{\pi i p\cdot Q} e^{2\pi i\tau\left(L_0-\frac{c_L}{24}\right)} e^{-2\pi i\bar\tau\left(\bar L_0-\frac{c_R}{24}\right)} e^{2\pi i\zeta\cdot Q}\right). \tag{4.1}$$

Here the ζ^a are chemical potentials and the Q_a are charges under the $b_2(X)$ abelian currents associated with the chiral bosons; i.e. (b_2-1) left-moving currents, and a single right-moving current. The p^a parametrize the (discrete) choice of divisor in $H_4(X,\mathbb{Z})$ that the M5-brane wraps. The fermion number is defined in the usual way as twice the charge of the U(1) generator in the $\mathrm{SU}(2)_R$ R-symmetry which exists in the $\mathcal{N}=4$ superconformal algebra. The extra factor of F^2 as compared to the conventional elliptic genus is present in order to absorb the fermion zero modes mentioned above.

This quantity isn't quite the one we want to work with, as it includes information about the momenta in the $\mathbb{R}^3$ transverse to the effective string in the non-compact directions of space. Instead, the *generalized elliptic genus* $Z(\tau,\zeta)$ is defined by requiring that

$$Z'(\tau,\zeta) = Z(\tau,\zeta) \int d^3\vec\pi\, (e^{2\pi i\tau} e^{-2\pi i\bar\tau})^{\frac{1}{2}\vec\pi^2} = Z(\tau,\zeta)\,(2\,\mathrm{Im}\tau)^{\frac{3}{2}}. \tag{4.2}$$

It is easy to see [10–12] that Z has weight $\left(-\frac{3}{2},\frac{1}{2}\right)$, in the sense that

$$Z\left(\frac{a\tau+b}{c\tau+d},\frac{\zeta}{c\tau+d}\right)\frac{(c\tau+d)^2}{|c\tau+d|}\,\mathrm{e}\left(m\frac{cz^2}{c\bar\tau+d}\right) = \chi\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) Z(\tau,\zeta) \tag{4.3}$$

for some m, and some multiplier $\chi : SL_2(\mathbb{Z}) \to \mathbb{C}$, when $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. This is what we would expect from (the complex conjugate of) a skew-holomorphic Jacobi form of weight -1 (with a multiplier system). The shift in charges under a large gauge transformation states that the generalized index admits a decomposition

$$Z(\tau,\zeta)=\sum_{\mu\in\Lambda^*/\Lambda}\Theta_\mu(\tau,\zeta)\,h_\mu(\tau) \tag{4.4}$$

into Siegel-Narain theta functions

$$\Theta_\mu(\tau,\zeta):=\sum_{Q\in\mu+\Lambda+\frac{p}{2}}\mathrm{e}\left(\frac{\tau}{2}(Q_+|Q_+)+\frac{\bar\tau}{2}(Q_-|Q_-)+\left(Q|\zeta+\frac{p}{2}\right)\right), \tag{4.5}$$

where the splitting $Q=Q_++Q_-$ depends on the Grassmannian $\frac{O(b_2-1,1)}{O(b_2-1)\times O(1)}$. In the sequel we will always set the chemical potentials conjugate to the left-moving currents to zero, and only keep track of the right-moving chemical potential.

Next we will show that, at certain points in the moduli space, the generalized elliptic genus of an MSW string is naturally a skew-holomorphic (mock) Jacobi form. First, recall that in the presence of an MSW string the Calabi-Yau moduli which are vector multiplets in the low-energy supergravity undergo an "attractor flow." That is, they flow to certain specific values at the horizon of the related black string, independent of their values at infinity in $\mathbb{R}^5$. This 'attractor mechanism' gives a natural preferred choice of moduli. In M-theory on X, the vector multiplet moduli are the Kähler moduli of X (excepting the overall volume, which transforms in a hypermultiplet). At the attractor point in moduli space, the Kähler form J on X satisfies $J\sim p$. As a result one can find the right-moving chiral U(1) current and its associated charge to be

$$Q_-=\frac{p\cdot Q}{p^2}p\,. \tag{4.6}$$

As we already know that $Z(\tau,\zeta)$ transforms as a weight $(-\frac{3}{2},\frac{1}{2})$ modular form, what remains to check is that $\overline{Z(\tau,\zeta)}$, for a specific choice of $\zeta=\zeta(z)$, satisfies the elliptic transformation

$$\overline{Z(\tau,\zeta(z+\lambda\tau+\mu))}\,\mathrm{e}\left(m(\lambda^2\tau+2\lambda z+\lambda+\mu)\right)=\overline{Z(\tau,\zeta(z))} \tag{4.7}$$

for $\lambda,\mu\in\mathbb{Z}$. This will imply that $\overline{Z}$ admits a decomposition as in (2.1).

Let $\zeta=\bar zp$. At the attractor moduli the Siegel-Narain theta function $\Theta_\mu(\tau,\bar\tau,\zeta)$ becomes

$$\tilde\theta_\mu(\tau,\bar z):=\sum_{Q\in\mu+\Lambda+\frac{p}{2}}(-1)^{p\cdot Q}q^{\frac{1}{2}Q_+^2}\bar q^{\frac{1}{2}\frac{(p\cdot Q)^2}{p^2}}e^{2\pi i\bar z p\cdot Q} \tag{4.8}$$

where $q=\mathrm{e}(\tau)$ and $\bar q=\mathrm{e}(-\bar\tau)$, and we used (4.6) in writing the power of $\bar q$. We can show that

$$\tilde\theta_\mu(\tau,\bar z+n\bar\tau+m)=(-1)^{p^2(m+n)}\,\mathrm{e}\left(\frac{1}{2}p^2n^2\right)\tilde\theta_\mu(\tau,\bar z) \tag{4.9}$$

by a shift $Q\mapsto Q+pn$ in the sum. This verifies that, at the attractor point in moduli space, $\overline{Z(\tau,p\bar z)}$ is a skew-holomorphic Jacobi form of index $\frac{1}{2}p^2$ with elliptic variable z.

An interesting question for future work would be to determine if there are other (non-attractor) moduli where the M5-brane elliptic genus reduces to a skew-holomorphic Jacobi form.

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5 Examples

We now discuss several examples of M5-brane elliptic genera computed in [14]. In each case we find a natural relation to a weakly skew-holomorphic Jacobi form of weight 2 that plays a role in a moonshine.

5.1 The projective plane

The elliptic genus for one M5-brane wrapping $\mathbb{P}^2$ can be written as

$$Z^{(1)}_{\mathbb{P}^2}(\tau,z) = (-i)\theta_{\frac{1}{2},\frac{1}{2}}(-\bar{\tau},-z)\eta^{-3}(\tau) \tag{5.1}$$

(cf. (2.6)) thanks to work of Göttsche [28]. In comparison with section 4 we have kept only the chemical potential for the right-moving U(1) charge, which we henceforth denote by z.

So the function $\overline{Z^{(1)}_{\mathbb{P}^2}}(\tau,z)$ is a skew-holomorphic Jacobi form of weight -1 and index $\frac{1}{2}$, and since $\theta^1_{\frac{1}{2},\frac{1}{2}} = i\eta^3$ (cf. (2.6)) we may write

$$Z^{(1)}_{\mathbb{P}^2}(\tau,z) = \overline{\varphi^{(1)}_{\mathbb{P}^2}(\tau,\bar{z})}\eta^{-6}(\tau) \tag{5.2}$$

where $\varphi^{(1)}_{\mathbb{P}^2}(\tau,z) = t_{2,\frac{1}{2}}(\tau,z) = \frac{1}{2}t_{2,2}(\tau,\frac{1}{2}z)$, and $t_{2,2}$ is the weight 2, index 2 skew-holomorphic Jacobi form that appears as a shadow in Mathieu moonshine (cf. (2.4)).

The connection to Mathieu groups becomes stronger when we consider two M5-branes wrapping $\mathbb{P}^2$. To explain this let $H(n)$ denote the Hurwitz class number of binary quadratic forms of discriminant $-n$ when $n > 0$, and set $H(0) := -\frac{1}{12}$. Then $\mathscr{H}(\tau) := \sum_{n\geq 0} H(n)q^n$ is a mock modular form of weight $\frac{3}{2}$ for $\Gamma_0(4)$ with shadow (proportional to) $\theta^0_{1,0}$ (cf. (2.3)). This was first discovered by Zagier [29]. Very recent work [30] proves that

$$24\mathscr{H}(\tau) = -2 + 8q^3 + 12q^4 + 24q^7 + 24q^8 + \dots \tag{5.3}$$

is the graded dimension of a graded virtual module for the sporadic Mathieu group M_{11}, and $48\mathscr{H}(\tau) = -4 + 16q^3 + 24q^4 + \dots$ is the graded dimension of a graded virtual module for M_{23}.

Now set $\hat{f}_j(\tau) := 3\hat{h}_j(\tau)\eta^{-6}(\tau)$ for $j \in \{0,1\}$, where $\hat{h}_j$ is the completion of the mock modular form

$$h_j(\tau) := \sum_{n=0}^{\infty} H(4n+3j)q^{n+\frac{3j}{4}}. \tag{5.4}$$

Then the elliptic genus of two M5-branes wrapping $\mathbb{P}^2$ is given [31–33] by

$$Z^{(2)}_{\mathbb{P}^2}(\tau,z) = \hat{f}_0(\tau)\theta_{1,1}(-\bar{\tau},-z) - \hat{f}_1(\tau)\theta_{1,0}(-\bar{\tau},-z).$$

Similar to (5.2) we may write

$$Z^{(2)}_{\mathbb{P}^2}(\tau,z) = \overline{\varphi^{(2)}_{\mathbb{P}^2}(\tau,\bar{z})}\eta^{-6}(\tau) \tag{5.5}$$

where $\varphi^{(2)}_{\mathbb{P}^2}(\tau,z) := 3\overline{\hat{h}_0(\tau)}\theta_{1,0}(\tau,z) - 3\overline{\hat{h}_1(\tau)}\theta_{1,1}(\tau,z)$ is a skew-holomorphic mock Jacobi form of weight 2 and index 1 that exhibits moonshine for the Mathieu groups M_{11} and M_{23}

according to [30]. Thus M5-branes on $\mathbb{P}^2$ give a starting point from which we may pursue a geometric understanding of Mathieu moonshine for (rescaled) Hurwitz class numbers.

It is interesting to note that the generating function $\mathscr{H}(\tau)$ also arises as an example of a function counting BPS jumping loci of maximal rank for $K3\times T^2$, or equivalently, counting attractor black holes, in the precise sense described in [35]. Also, the theta-coefficients of φ in (5.5) recur in the elliptic genus for two M5-branes wrapping the Hirzebruch surface $\mathbb{F}_1$ (see section 4.2 of [14]). In both these settings, and of course for two M5-branes wrapping $\mathbb{P}^2$, it would be interesting to compare geometric twinings with the functions coming from the analysis of [30].

5.2 Degree one del Pezzo surfaces

Next we consider M5-branes wrapping a del Pezzo surface of degree 1 (i.e. $\mathbb{P}^2$ blown up at 8 points). The elliptic genus was first described in [10]. Start with the Fermat quintic $\{\sum_i x_i^5 = 0\} \subset \mathbb{P}^4$ and quotient by the $\mathbb{Z}_5$ action $x_i \to \omega^i x_i$ where $\omega := \mathrm{e}(\frac{1}{5})$. The hyperplane section $\mathcal{P}$ of the resulting orbifold has $\chi(\mathcal{P}) = 11$ and is rigid with $b_2^+ = 1$. It has $H^2(\mathcal{P},\mathbb{Z}) = \mathbb{Z}\oplus(-E_8)$ and is thus a del Pezzo surface of degree 1.

For a single M5-brane wrapping $\mathcal{P}$ we have $Z^{(1)}_{\mathrm{dP}_8}(\tau,z) = \overline{\varphi^{(1)}_{\mathrm{dP}_8}(\tau,\bar{z})}\eta^{-6}(\tau)$ for the elliptic genus, where

$$\begin{aligned}\varphi^{(1)}_{\mathrm{dP}_8}(\tau,z) &:= \overline{E_4(\tau)\eta^{-8}(\tau)}t_{2,\frac{1}{2}}(\tau,z)\\ &= \overline{E_4(\tau)\eta^{-5}(\tau)}(-i)\theta_{\frac{1}{2},\frac{1}{2}}(\tau,z).\end{aligned} \tag{5.6}$$

This is a weakly skew-holomorphic Jacobi form of weight 2 with a multiplier, and may be compared to (5.2).

Inspired by the discussion in section 5.1 we consider the possibility that the coefficients of the anti-holomorphic factor in (5.6) also admit interpretations in terms of representations of Mathieu groups. Observe that

$$f^{(1)}_{\mathrm{dP}_8}(\tau) := E_4(\tau)\eta^{-5}(\tau) \tag{5.7}$$

is the unique modular form of weight $\frac{3}{2}$ for $SL_2(\mathbb{Z})$ that has the same multiplier as η^{-5} and satisfies $f^{(1)}_{\mathrm{dP}_8}(\tau) = q^{-\frac{5}{24}} + O(q^{\frac{19}{24}})$ as $\Im(\tau)\to\infty$. By considering analogous functions for the congruence subgroups $\Gamma_0(n) < SL_2(\mathbb{Z})$ we are led to a family $f^{(1)}_{\mathrm{dP}_8,nZ}$ of modular forms of weight $\frac{3}{2}$ with various levels which achieves this goal for the sporadic simple Mathieu group M_{12}. That is, the $f^{(1)}_{\mathrm{dP}_8,nZ}$ serve as trace functions

$$f^{(1)}_{\mathrm{dP}_8,[g]}(\tau) = \sum_{d\in\mathbb{Z}+\frac{19}{24}} \mathrm{tr}\left(g|W^{(1)}_{\mathrm{dP}_8,d}\right)q^d \tag{5.8}$$

for a graded virtual M_{12}-module $W^{(1)}_{\mathrm{dP}_8} = \bigoplus_d W^{(1)}_{\mathrm{dP}_8,d}$ with graded dimension given by (5.7).

Details on the modular forms $f^{(1)}_{\mathrm{dP}_8,nZ}$ are given in appendix A, including the first few coefficients in their Fourier expansions (see tables 2–3) and the decompositions of the corresponding $W^{(1)}_{\mathrm{dP}_8,d}$ into irreducible modules for M_{12} (see tables 4–8). From that

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information alone it is not immediate that the virtual M_{12}-module $W^{(1)}_{\mathrm{dP}_8}$ satisfying (5.8) exists, but we can verify this using arguments very similar to those appearing in recent literature on moonshine in weight $\frac{3}{2}$, including [19, 30]. So we refrain from reproducing the details here.

The reader will note that $f^{(1)}_{\mathrm{dP}_8}$ is η^3 times the graded dimension of the basic representation V_{E_8} of the affine Lie algebra of type E_8. This space naturally admits an action by the adjoint Lie group $E_8(\mathbb{C})$, so it is natural to ask if the twining functions $f^{(1)}_{\mathrm{dP}_8,[g]}$ are related to this action. Here we note that M_{12} is not a subgroup of $E_8(\mathbb{C})$ according to [34], so the virtual M_{12}-module $W^{(1)}_{\mathrm{dP}_8}$ cannot be recovered in a simple way from V_{E_8}.

We obtain an assignment of weakly skew-holomorphic Jacobi forms of weight 2 and index $\frac{1}{2}$ to elements of M_{12} simply by setting

$$\varphi^{(1)}_{\mathrm{dP}_8,nZ}(\tau,z) := (-i)\overline{f^{(1)}_{\mathrm{dP}_8,nZ}(\tau)}\theta_{\frac{1}{2},\frac{1}{2}}(\tau,z). \tag{5.9}$$

These forms in turn define twinings

$$Z^{(1)}_{\mathrm{dP}_8,nZ}(\tau,z) := \overline{\varphi^{(1)}_{\mathrm{dP}_8,nZ}(\tau,\bar{z})}\eta^{-6}(\tau) \tag{5.10}$$

of the M5-brane elliptic genus $Z^{(1)}_{\mathrm{dP}_8}$. As a result, it is natural to ask how the twining functions (5.10) are related to the symmetries of M5-brane theory on $\mathcal{P}$, and whether this relationship between M_{12} and the del Pezzo surface of degree 1 is connected in some way to the original Mathieu moonshine [2], which relates M_{24} to the K3 elliptic genus. It would be interesting to gain a physical or geometric understanding of the twining functions $Z^{(1)}_{\mathrm{dP}_8,nZ}$.

For a pair of M5-branes on $\mathcal{P}$ we have $Z^{(2)}_{\mathrm{dP}_8}(\tau,z) = \overline{\varphi^{(2)}_{\mathrm{dP}_8}(\tau,\bar{z})}\eta^{-6}(\tau)$ where

$$\varphi^{(2)}_{\mathrm{dP}_8}(\tau,z) := \overline{E_4^2(\tau)\eta^{-16}(\tau)}\left(\overline{\hat{h}_0(\tau)}\theta_{1,0}(\tau,z) - \overline{\hat{h}_1(\tau)}\theta_{1,1}(\tau,z)\right) \tag{5.11}$$

(cf. (5.4)). In light of the discussions above and in section 5.1 it seems likely that naturally defined Mathieu group twinings of $Z^{(2)}_{\mathrm{dP}_8}$ also exist. Are there naturally defined twinings of $Z^{(n)}_{\mathrm{dP}_8}$ by $g \in M_{12}$ for all n? What do they tell us about M5-branes on $\mathcal{P}$?

5.3 Half-K3 surfaces

In this final section we consider the elliptic genus for a single M5-brane wrapping a half-K3 surface (i.e. $\mathbb{P}^2$ blown up at 9 points). Such surfaces play an important role in the study of E-strings via geometric engineering.

To compute the genus in question we first discuss the cohomology group $H^2(\frac{1}{2}\mathrm{K3},\mathbb{Z})$. Geometrically it is generated by the hyperplane class corresponding to the hyperplane intersection with $\mathbb{P}^2$, denoted by H, and the nine blow-ups c_i, for $i = 1,\ldots,9$. The quadratic form inherited from the intersection form is then given by $\mathrm{diag}(1,-1,\ldots,-1)$. In fact, this lattice is isomorphic to $\mathbb{Z} \oplus (-\mathbb{Z}) \oplus (-E_8)$, and in particular is unimodular (but not even). The corresponding basis is given [36] by

$$\begin{aligned} b_1 &:= 3H + \sum_{i=1}^{8} c_i\,, \quad b_2 := -c_9, \\ e_8 &:= H + \sum_{i=6}^{8} c_i\,, \quad e_i := c_i - c_{i+1} \text{ for } i = 1, \dots 7. \end{aligned} \tag{5.12}$$

Since the lattice is unimodular there is only one term in the decomposition (4.4) of the genus into theta functions. For the case at hand, the class $\mathcal{P}$ of the surface wrapped by the $M5$ brane is given by the anti-canonical class

$$\mathcal{P} = K_{\frac{1}{2}\mathrm{K3}} = b_1 - b_2. \tag{5.13}$$

The theta function we are interested in will depend on the moduli of the half-K3, and one such modulus is given by the size of the elliptic fiber, denoted here by $\frac{1}{R}$. Taking the shift by $\frac{1}{2}\mathcal{P}$ into account, the Siegel-Narain theta function (4.5) is $\Theta(R;\tau,z) = E_4(\tau)\Theta^{\mathrm{odd}}_{1,1}(R;\tau,z)$, where

$$\begin{aligned} &\Theta^{\mathrm{odd}}_{1,1}(R;\tau,z) := \\ &\sum_{a,b\in\mathbb{Z}} (-1)^{a+b} q^{\frac{1}{2R^2}\left(R^2\frac{(a-b)}{2}+\frac{a+b+1}{2}\right)^2} \bar{q}^{\frac{1}{2R^2}\left(R^2\frac{(b-a)}{2}+\frac{a+b+1}{2}\right)^2} \bar{y}^{\left(R^2\frac{(b-a)}{2}+\frac{a+b+1}{2}\right)}. \end{aligned} \tag{5.14}$$

The elliptic genus is given by

$$Z^{(1)}_{\frac{1}{2}\mathrm{K3}}(R;\tau,z) = E_4(\tau)\Theta^{\mathrm{odd}}_{1,1}(R;\tau,z)\eta^{-12}(\tau). \tag{5.15}$$

The question of moonshine-type phenomena is potentially richer in this setting due to the dependence on the parameter R. In this work we refrain from a full analysis and restrict ourselves to some special cases. In preparation for this note that (5.14) specializes to theta-type skew-holomorphic Jacobi forms of half-integral index (cf. (2.4)) at special values of R. Indeed, by a similar analysis to that given for $\Theta_{1,1}$ in section 3.1 we obtain

$$\Theta^{\mathrm{odd}}_{1,1}(\sqrt{2m};\tau,z) = \overline{t_{1,m}(\tau,z)} \tag{5.16}$$

when $m \in \mathbb{Z} + \frac{1}{2}$ and $m > 0$.

Motivated by the discussions in section 5.1 and section 5.2 we now consider the decomposition $Z^{(1)}_{\frac{1}{2}\mathrm{K3}}(\sqrt{2m};\tau) = \overline{\varphi^{(1)}_{\frac{1}{2}\mathrm{K3},m}(\tau,z)\eta^{-6}(\tau)}$, where by (5.16) we have

$$\varphi^{(1)}_{\frac{1}{2}\mathrm{K3},m}(\tau,z) = \overline{E_4(\tau)\eta^{-6}(\tau)}t_{1,m}(\tau,z), \tag{5.17}$$

which is a skew-holomorphic Jacobi form of weight 2 and index m.

The first case to consider is $m = \frac{1}{2}$, but $t_{1,\frac{1}{2}}$ vanishes identically (cf. (2.6)), so we set this case aside for the moment. The next case is $m = \frac{3}{2}$, where, after applying (2.7) we find that

$$\varphi^{(1)}_{\frac{1}{2}\mathrm{K3},\frac{3}{2}}(\tau,z) = \overline{E_4(\tau)\eta^{-5}(\tau)}(i)\left(\theta_{\frac{3}{2},\frac{1}{2}}(\tau,z) - \theta_{\frac{3}{2},-\frac{1}{2}}(\tau,z)\right). \tag{5.18}$$

Observe that the anti-holomorphic factor in (5.18) is precisely the same as that which appears in $\varphi^{(1)}_{\mathrm{dP}_8}$ (cf. (5.6)), in connection with del Pezzo surfaces of degree 1. So from the discussion in section 5.2 we naturally obtain twinings

$$\varphi^{(1)}_{\frac{1}{2}\mathrm{K3},\frac{3}{2},[g]}(\tau,z) := \overline{f^{(1)}_{\mathrm{dP}_8,[g]}(\tau)}(i)\left(\theta_{\frac{3}{2},\frac{1}{2}}(\tau,z)-\theta_{\frac{3}{2},-\frac{1}{2}}(\tau,z)\right) \tag{5.19}$$

(cf. (5.8)) of the weight 2 skew-holomorphic Jacobi form (5.18) by $g \in M_{12}$. This in turn leads to twinings

$$Z^{(1)}_{\frac{1}{2}\mathrm{K3},[g]}(\sqrt{3};\tau,z) := \overline{\varphi^{(1)}_{\frac{1}{2}\mathrm{K3},\frac{3}{2},[g]}(\tau,z)}\eta^{-6}(\tau) \tag{5.20}$$

by $g \in M_{12}$ of the single M5-brane elliptic genus for half-K3 surfaces at the modulus $R=\sqrt{3}$.

The vanishing of (5.17) at $m=\frac{1}{2}$ suggests that we modify the elliptic genus by introducing a fermion number operator. This amounts to replacing $t_{1,m}$ with $t_{2,m}$ in (5.17). Indeed, if we define

$$\widetilde{Z}^{(1)}_{\frac{1}{2}\mathrm{K3}}(R;\tau,z) := E_4(\tau)\widetilde{\Theta}^{\mathrm{odd}}_{1,1}(R;\tau,z)\eta^{-12}(\tau) \tag{5.21}$$

where

$$\begin{aligned}\widetilde{\Theta}^{\mathrm{odd}}_{1,1}(R;\tau,z) := \sum_{a,b\in\mathbb{Z}}(-1)^{a+b}\left(R^2\frac{(a-b)}{2}+\frac{a+b+1}{2}\right)q^{\frac{1}{2R^2}\left(R^2\frac{(a-b)}{2}+\frac{a+b+1}{2}\right)^2} \\ \times\,\bar{q}^{\frac{1}{2R^2}\left(R^2\frac{(b-a)}{2}+\frac{a+b+1}{2}\right)^2}\bar{y}^{\left(R^2\frac{(b-a)}{2}+\frac{a+b+1}{2}\right)}\end{aligned} \tag{5.22}$$

then $\widetilde{\Theta}^{\mathrm{odd}}_{1,1}(\sqrt{2m};\tau,z) = \overline{t_{2,m}(\tau,z)}$ when $m \in \mathbb{Z}+\frac{1}{2}$ and $m>0$. In this setting we consider the decomposition $\widetilde{Z}^{(1)}_{\frac{1}{2}\mathrm{K3}}(\sqrt{2m};\tau) = \overline{\widetilde{\varphi}^{(1)}_{\frac{1}{2}\mathrm{K3},m}(\tau,z)}\eta^{-4}(\tau)$ so that

$$\widetilde{\varphi}^{(1)}_{\frac{1}{2}\mathrm{K3},m}(\tau,z) = \overline{E_4(\tau)\eta^{-8}(\tau)}t_{2,m}(\tau,z) \tag{5.23}$$

is the associated weakly skew-holomorphic Jacobi form of weight 2 and index m.

Now comparing with (5.6) we find that (5.23) at $m=\frac{1}{2}$ is precisely the skew-holomorphic Jacobi form of weight 2 and index $\frac{1}{2}$ that appeared in section 5.2 in connection with del Pezzo surfaces of degree 1. So it is natural to define

$$\widetilde{\varphi}^{(1)}_{\frac{1}{2}\mathrm{K3},\frac{1}{2},[g]}(\tau,z) := \varphi^{(1)}_{\mathrm{dP}_8,[g]}(\tau,z) \tag{5.24}$$

for $g \in M_{12}$. We then obtain twinings

$$\widetilde{Z}^{(1)}_{\frac{1}{2}\mathrm{K3},[g]}(1;\tau) := \overline{\widetilde{\varphi}^{(1)}_{\frac{1}{2}\mathrm{K3},\frac{1}{2},[g]}(\tau,z)}\eta^{-4}(\tau) \tag{5.25}$$

of the modified single M5-brane elliptic genus for half-K3 surfaces by $g \in M_{12}$ when $R=1$.

We conclude this section with four remarks. Firstly, it is natural to ask if twinings by M_{12} of the elliptic genera (5.15) and (5.21) for half-K3 surfaces can be defined for all R. Are there special values of R for which this hidden symmetry extends beyond M_{12}? Secondly, it would be interesting to compare the physical twinings of the elliptic genera (5.15) and (5.21) with the series (5.20) and (5.25) that arise from M_{12} in the manner we have just described. Thirdly, note that the given expressions (5.15) and (5.21) for the elliptic genera considered in this section point to explicit realizations in terms of the vertex algebra attached to the lattice $\mathbb{Z} \oplus (-\mathbb{Z}) \oplus (-E_8)$. It would be interesting to determine if the twinings (5.20) and (5.25) by elements of M_{12} can also be realized using this structure. In view of the well-known Mathieu moonshine connection between M_{24} and K3 surfaces [2], it is appealing that the Euler characteristic of a half-K3 surface is 12. So finally we ask, to what extent is the connection between half-K3 surfaces and M_{12} described in this section related to the original Mathieu moonshine?

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A New Mathieu moonshine

Here we present numerical data in support of the discussions of section 5.2 and section 5.3 relating the M5-brane elliptic genera for degree one del Pezzo surfaces and half-K3 surfaces to the Mathieu group M_{12}. Since the relationship to half-K3 surfaces is formulated in terms of the functions that appear in section 5.2 we employ the notation of section 5.2 in what follows.

The character table of M_{12} is table 1, wherein $b_{11} := -\frac{1}{2} + \frac{\sqrt{-11}}{2}$. Tables 2–3 give the coefficients of q^d in the McKay-Thompson series $f^{(1)}_{\mathrm{dP}_8,nZ}$ up to $d = \frac{2275}{24}$. The naming of the conjugacy classes is as in table 1. Tables 4–8 give the multiplicity generating functions for irreducible characters in the graded virtual M_{12}-module $W^{(1)}_{\mathrm{dP}_8,d}$ of (5.8). That is, for χ an irreducible character of M_{12}, the coefficient of q^d in $f^{(1)}_{\mathrm{dP}_8,\chi}$ denotes the multiplicity

$[g]$	1A	2A	2B	3A	3B	4A	4B	5A	6A	6B	8A	8B	10A	11A	11B
$n \vert h$	$1\vert 1$	$2\vert 2$	$2\vert 1$	$3\vert 1$	$3\vert 3$	$4\vert 1$	$4\vert 1$	$5\vert 1$	$6\vert 6$	$6\vert 1$	$8\vert 1$	$8\vert 1$	$10\vert 2$	$11\vert 1$	$11\vert 1$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	11	-1	3	2	-1	-1	3	1	-1	0	-1	1	-1	0	0
χ_3	11	-1	3	2	-1	3	-1	1	-1	0	1	-1	-1	0	0
χ_4	16	4	0	-2	1	0	0	1	1	0	0	0	-1	b_{11}	$\overline{b_{11}}$
χ_5	16	4	0	-2	1	0	0	1	1	0	0	0	-1	$\overline{b_{11}}$	b_{11}
χ_6	45	5	-3	0	3	1	1	0	-1	0	-1	-1	0	1	1
χ_7	54	6	6	0	0	2	2	-1	0	0	0	0	1	-1	-1
χ_8	55	-5	7	1	1	-1	-1	0	1	1	-1	-1	0	0	0
χ_9	55	-5	-1	1	1	3	-1	0	1	-1	-1	1	0	0	0
χ_{10}	55	-5	-1	1	1	-1	3	0	1	-1	1	-1	0	0	0
χ_{11}	66	6	2	3	0	-2	-2	1	0	-1	0	0	1	0	0
χ_{12}	99	-1	3	0	3	-1	-1	-1	-1	0	1	1	-1	0	0
χ_{13}	120	0	-8	3	0	0	0	0	0	1	0	0	0	-1	-1
χ_{14}	144	4	0	0	-3	0	0	-1	1	0	0	0	-1	1	1
χ_{15}	176	-4	0	-4	-1	0	0	1	-1	0	0	0	1	0	0

Table 1. Character table of M_{12}.

of χ in the (virtual) M_{12}-module $W^{(1)}_{\mathrm{dP}_8,d}$. In tables 4–8 the characters are named by their dimensions, and appear in the same order as in table 1.

The modular forms $f^{(1)}_{\mathrm{dP}_8,nZ}$ may be realized as Rademacher sums. Specifically, consider the degree 24 permutation representation of M_{12} that arises by restricting the defining permutation representation of M_{24}. The corresponding character is $2\chi_1 + \chi_2 + \chi_3$ in the notation of table 1. If $g \in M_{12}$ has order n and h is the minimal length of a cycle in the cycle shape of g (in this permutation representation) then $f^{(1)}_{\mathrm{dP}_8,[g]}$ is the Rademacher sum of weight $\frac{3}{2}$ for $\Gamma_0(n)$ with polar part $q^{-\frac{5}{24}}$ (at the infinite cusp), and multiplier system given by $\gamma \mapsto \mathrm{e}(\frac{cd}{nh})\epsilon^{-5}(\gamma)$ for $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_0(n)$, where ϵ is the multiplier system of η. The values of n and h for each conjugacy class $[g] \subset M_{12}$ are given in table 1. We refer to [37] for details of the Rademacher sum construction.

Note that since the factor $\mathrm{e}(\frac{cd}{nh})$ is trivial on $\Gamma_0(nh)$ the function $f^{(1)}_{\mathrm{dP}_8,[g]}\eta^5$ is a holomorphic modular form of weight 4 for $\Gamma_0(nh)$. This, together with the Fourier coefficients in tables 2–3 gives an alternative method for reconstructing the $f^{(1)}_{\mathrm{dP}_8,nZ}$. For example, we find in this way that

$$\begin{aligned} f^{(1)}_{\mathrm{dP}_8,\mathrm{2B}}(\tau) &= \frac{1}{15}(16E_4(2\tau) - E_4(\tau))\eta^{-5}(\tau), \\ f^{(1)}_{\mathrm{dP}_8,\mathrm{3A}}(\tau) &= \frac{1}{80}(81E_4(3\tau) - E_4(\tau))\eta^{-5}(\tau), \\ f^{(1)}_{\mathrm{dP}_8,\mathrm{2A}}(\tau) &= f^{(1)}_{\mathrm{dP}_8,2B}(\tau) + 32\eta^3(\tau)\eta(4\tau)^8\eta^{-8}(2\tau). \end{aligned} \tag{A.1}$$

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24d	1A	2A	2B	3A	3B	4AB	5A	6A	6B	8AB	10A	11AB
-5	1	1	1	1	1	1	1	1	1	1	1	1
19	245	21	−11	2	−7	5	−5	−3	−2	1	1	3
43	3380	−44	52	−22	14	4	5	−2	−2	0	1	−8
67	22385	113	−143	29	11	−15	10	11	1	1	−2	0
91	110110	−322	286	31	−77	−18	−15	−1	7	−6	−7	0
115	438746	602	−550	−112	77	26	−4	−7	8	−6	2	0
139	1531985	−1071	1105	113	77	17	−15	−3	−11	1	−1	4
163	4804910	1870	−2002	71	−253	−66	35	−17	−13	−6	−5	0
187	13914285	−3283	3245	−381	231	−19	35	11	−13	13	7	11
211	37674325	5525	−5291	334	145	117	−50	29	10	13	0	7
235	96580627	−8621	8723	277	−704	51	2	16	29	−5	4	0
259	236144545	13377	−13663	−911	637	−143	−80	−15	17	13	12	−23
283	554578570	−20790	20618	811	469	−86	70	−39	−1	−14	10	−4
307	1256789730	31458	−31006	534	−1617	226	105	−21	−46	−14	−7	−2
331	2760379655	−46073	46343	−2173	1364	135	−95	4	−25	7	−13	0
355	5894771883	67179	−67925	1824	924	−373	8	48	16	−13	4	0
379	12275038600	−97752	97416	1249	−3575	−168	−150	33	45	20	−12	0
403	24982062560	139584	−138528	−4420	2915	528	185	−9	36	20	−11	−7
427	49794727675	−196133	196603	3595	2002	235	175	−62	−29	−1	7	0
451	97369902630	274022	−275418	2376	−6930	−698	−245	−46	−72	14	7	11
475	187076653120	−381024	380224	−8738	5698	−400	−5	42	−62	−44	1	38
499	353616436085	524277	−522379	7079	3641	949	−290	81	35	−43	12	0
523	658376681690	−713734	714714	4727	−13300	490	315	80	123	6	11	−31
547	1208616966765	966093	−968851	−16155	10593	−1379	390	−51	89	−31	−22	32
571	2189664565985	−1302879	1301729	12908	7004	−575	−390	−144	−40	57	−24	−27
595	3918286118747	1744155	−1740453	8147	−23947	1851	−3	−111	−141	51	5	0
619	6930554664880	−2316048	2317744	−29150	18865	848	−495	45	−98	−24	−13	0
643	12125024699095	3062199	−3066921	22825	11935	−2361	595	183	45	43	−21	0
667	20994476982895	−4034001	4031599	14815	−42092	−1201	645	132	175	−57	29	−24
691	35997712990855	5283495	−5277305	−50087	32956	3095	−770	−60	133	−45	20	0
715	61152257741861	−6880923	6883877	39062	21098	1477	−14	−234	−106	13	2	11
739	102971911295570	8927122	−8935342	24530	−71194	−4110	−930	−134	−250	−38	32	42
763	171940936021855	−11542209	11538527	−84881	55132	−1841	980	120	−193	83	36	0
787	284816074366495	14858431	−14847713	65539	34552	5359	1120	280	127	75	−24	−54
811	468201092136435	−19041645	19046643	41319	−118503	2499	−1190	213	339	−1	−20	31
835	764062908896885	24320821	−24334219	−138991	90923	−6699	10	−125	233	45	−4	−51
859	1238199118586430	−30972578	30966078	106548	57057	−3250	−1445	−407	−96	−118	−43	0
883	1993162922073180	39301308	−39284388	66213	−191268	8460	1680	−264	−423	−112	−32	0
907	3187894582604875	−49688021	49696075	−224249	146146	4027	1750	118	−269	31	34	0
931	5067385763330905	62640249	−62662055	170881	90349	−10903	−1970	477	157	−75	44	−8
955	8007296783387517	−78760899	78751101	106299	−304920	−4899	17	324	507	149	1	0
979	12580745731759205	98730149	−98702747	−353800	231308	13701	−2420	−196	352	125	44	55
1003	19657853187268125	−123390883	123403293	268005	143418	6205	2500	−574	−243	−43	32	93
1027	30553454673736995	153823299	−153856989	164577	−475629	−16845	2870	−429	−639	111	−46	0
1051	47245393529211705	−191308807	191292729	−550938	359007	−8039	−3045	299	−462	−175	−67	−96
1075	72695798459621870	237317774	−237275922	414887	219776	20926	−5	752	267	−150	−1	38
1099	111322113952614145	−293640095	293659649	255274	−732809	9777	−3605	559	818	37	−45	−57
1123	169685042685799025	362516273	−362568591	−842182	550550	−26159	4025	−310	558	−119	−67	0

Table 2. McKay-Thompson series $f^{(1)}_{\mathrm{dP}_8,nZ}$, part 1.

$24d$	1A	2A	2B	3A	3B	4AB	5A	6A	6B	8AB	10A	11AB
1147	257489647589909735	−446610297	446586855	631943	337337	−11721	4235	−963	−273	195	63	0
1171	389036935666670005	548983797	−548919371	384994	−1110824	32213	−4745	−648	−1010	189	57	0
1195	585321427870953137	−673325583	673354929	−1274347	830753	14673	12	333	−699	−39	−8	1
1219	877051946642599585	824194401	−824272735	951751	505153	−39167	−5540	1125	383	121	56	0
1243	1308987868412927175	−1006964089	1006927559	579426	−1666434	−18265	5925	770	1178	−273	81	66
1267	1946143827213299420	1227840124	−1227744804	−1900084	1241051	47660	6545	−413	852	−248	−71	80
1291	2882640700620305625	−1494261447	1494305241	1413606	753753	21897	−6875	−1347	−534	45	−57	0
1315	4254293388075663716	1815256100	−1815372956	855485	−2465848	−58428	−34	−964	−1451	−180	0	−151
1339	6256460810550333175	−2201441577	2201389047	−2806274	1830877	−26265	−8075	573	−1026	347	−87	34
1363	9169285487540907855	2665121487	−2664979633	2080578	1105104	70927	8855	1656	602	303	−73	−105
1387	13393282867503977295	−3220900017	3220964175	1257615	−3616767	32079	9420	1173	1839	−97	68	0
1411	19499387894072616035	3886321059	−3886491037	−4096807	2675288	−84989	−10340	−708	1265	235	104	0
1435	28299145776890952502	−4681963594	4681885494	3027835	1613458	−39050	2	−2050	−645	−362	6	0
1459	40942900182985321885	5631643773	−5631439203	1820005	−5246318	102285	−11990	−1398	−2151	−319	88	−32
1483	59056800250327174295	−6763471561	6763564951	−5930560	3868865	46695	12670	713	−1460	99	94	0
1507	84933501203533915790	8110930190	−8111177074	4368671	2321594	−123442	13790	2474	791	−242	−110	121
1531	121796962409191079920	−9713117456	9713006576	2623747	−7548422	−55440	−14705	1666	2519	448	−111	184
1555	174169266048720253146	11615262618	−11614967078	−8500497	5550897	147770	21	−891	1759	386	−7	0
1579	248377592707632513150	−13870493250	13870626430	6245646	3326730	66590	−16850	−2850	−1094	−58	−100	−206
1603	353253324430055413910	16541557942	−16541909098	3734813	−10760596	−175578	18410	−1988	−3067	274	−118	106
1627	501093959691581866120	−19701491768	19701331592	−12095267	7891807	−80088	19495	1231	−2191	−576	147	−160
1651	708983776598335157920	23434722304	−23434304608	8862610	4711630	208848	−21080	3418	1250	−524	124	0
1675	1000603206531275493740	−27839878548	27840067948	5292041	−15230776	94700	−10	2424	3709	124	2	0
1699	1408702624118535140395	33032707435	−33033205205	−17064938	11141053	−248885	−24230	−1379	2530	−365	160	0
1723	1978477667607266537870	−39147551762	39147329166	12473543	6641696	−111298	25620	−4172	−1317	674	148	−7
1747	2772165465354178800390	46339376966	−46338787066	7423860	−21384528	294950	27765	−2872	−4360	606	−139	0
1771	3875291180776298187905	−54788436927	54788701825	−23916919	15607823	132449	−29345	1491	−2951	−183	−137	121
1795	5405141185164404628126	64705171230	−64705866082	17441601	9274839	−347426	1	4887	1649	462	5	226
1819	7522234988258726535800	−76332974536	76332659832	10365428	−29834728	−157352	−33575	3284	5076	−768	−191	0
1843	10445828606352623315780	89952081860	−89951262908	−33278401	21724703	409476	36155	−1885	3559	−652	−165	−261
1867	14474828224085044461275	−105887271301	105887640027	24216572	12892502	184363	38150	−5626	−2112	167	154	124
1891	20015952136902352017085	124515746461	−124516712515	14350090	−41334620	−483027	−41165	−3956	−5998	−495	191	−203
1915	27621587036170435738716	−146272523812	146272092252	−46027323	30036996	−215780	−34	2336	−4179	892	−12	0
1939	38040588686504509649390	171657529358	−171656394514	33423194	17775758	567422	−46735	6710	2378	810	173	0
1963	52286338577292264775330	−201247917374	201248425378	19779820	−56935802	254002	49455	4666	7156	−162	171	0
1987	71727767470164471985970	235711210322	−235712537038	−63253216	41292251	−663358	53095	−2629	4868	558	−213	−75
2011	98210898877425806480580	−275815152540	275814555588	45846324	24403500	−298476	−56045	−7932	−2592	−1104	−215	0
2035	134220886793221405680657	322440005713	−322438454767	27064422	−77947254	775473	32	−5402	−8362	−999	−12	187
2059	183097700277061507239685	−376598728955	376599424773	−86460161	56425369	347909	−63565	2905	−5649	229	−205	328
2083	249322773322218559282605	439458518285	−439460332883	62548884	33271029	−907299	67980	9221	3160	−703	−240	0
2107	338899391506589069707460	−512358042588	512357234116	36873998	−106155721	−404236	71960	6279	9562	1288	252	−399
2131	459856714942619762629680	596828898736	−596826783696	−117510219	76706784	1057520	−76945	−3464	6585	1104	241	203
2155	622916642534098871009415	−694628315129	694629258375	84864729	45168123	471623	40	−10613	−3831	−313	−4	−329
2179	842374854253122861597685	807773972437	−807776428811	49926859	−143781275	−1228187	−86815	−7307	−11201	825	247	0
2203	1137263164627672249293665	−938574065919	938572965473	−158926840	103724390	−550223	91665	4242	−7780	−1427	301	0
2227	1532880862104682603160505	1089663943033	−1089661090375	114584796	60954696	1426329	97755	12388	4364	−1247	−257	0
2251	2062809388122724224605390	−1264058425906	1264059700430	67322939	−193818955	637262	−103235	8561	13199	302	−261	−67
2275	2771559316831560463943735	1465208427543	−1465211739849	−213852964	139592453	−1656153	−15	−4851	9012	−901	−7	0

Table 3. McKay-Thompson series $f^{(1)}_{\mathrm{dP}_8,nZ}$, part 2.

$24d$	1	11	$11'$	16	$16'$	45	54	55
-5	1	0	0	0	0	0	0	0
19	0	0	0	−1	−1	1	1	−2
43	−1	1	1	2	2	0	4	4
67	2	0	0	7	7	14	8	8
91	−4	20	20	9	9	35	62	83
115	5	35	35	89	89	239	251	222
139	20	201	201	236	236	694	881	950
163	36	528	528	833	833	2323	2702	2665
187	156	1655	1655	2311	2311	6485	7918	8235
211	413	4263	4263	6423	6423	18051	21397	21498
235	1020	11382	11382	16087	16087	45358	54939	56379
259	2455	26994	26994	40021	40021	112341	134076	135889
283	5879	64617	64617	93003	93003	261869	315210	322155
307	13169	144938	144938	212050	212050	596095	713920	725468
331	29087	320286	320286	464053	464053	1305434	1568719	1600075
355	61991	680939	680939	993468	993468	2793599	3348812	3407543
379	129169	1422770	1422770	2064719	2064719	5808170	6975047	7109138
403	262777	2888505	2888505	4208325	4208325	11833971	14193562	14449220
427	524286	5767282	5767282	8379616	8379616	23570078	28293712	28827745
451	1023996	11264435	11264435	16396496	16396496	46112567	55321988	56332449
475	1968752	21659408	21659408	31488517	31488517	88564546	106295853	108283778
499	3720441	40917665	40917665	59539940	59539940	167451476	200915361	204609370
523	6927917	76215745	76215745	110825425	110825425	311704255	374082188	381045664
547	12715863	139866126	139866126	203487898	203487898	572297901	686707842	699375765
571	23041098	253459390	253459390	368608354	368608354	1036726053	1244135671	1267242033
595	41225542	453471967	453471967	659672220	659672220	1855310757	2226288378	2267426223
619	72924892	802191793	802191793	1166723003	1166723003	3281429787	3937829833	4010870330
643	127575605	1403299073	1403299073	2041300549	2041300549	5741128020	6889198936	7016622004
667	220904792	2429997813	2429997813	3534354858	3534354858	9940416134	11928705034	12149811849
691	378758456	4166294843	4166294843	6060311787	6060311787	17044567168	20453213655	20831707181
715	643445607	7077944811	7077944811	10294877321	10294877321	28954418447	34745644272	35389433549
739	1083447396	11917871022	11917871022	17335484390	17335484390	48755959253	58506711305	59589713385
763	1809154794	19900794975	19900794975	28946097935	28946097935	81411011737	97693785348	99503517664
787	2996789379	32964531410	32964531410	47949080384	47949080384	134856640526	161827223016	164823268101
811	4926375742	54190323007	54190323007	78821410398	78821410398	221685419135	266023467395	270950794599
835	8039356619	88432717409	88432717409	128630532359	128630532359	361773104706	434126499529	442164641760
859	13028224735	143310698119	143310698119	208450511320	208450511320	586267397566	703522419579	716552193823
883	20971787728	230689383931	230689383931	335549960886	335549960886	943733861399	1132478688778	1153448509388
907	33542713964	368970251977	368970251977	536681774077	536681774077	1509417984175	1811304051250	1844849248586
931	53318392379	586501744668	586501744668	853096286376	853096286376	2399332671290	2879196063095	2932511326216
955	84251935429	926772031772	926772031772	1348028436453	1348028436453	3791330807137	4549600936442	4633856811066
979	132373064244	1456102857556	1456102857556	2117972326676	2117972326676	5956796105864	7148150369696	7280518493936
1003	206837818945	2275216938197	2275216938197	3309400831213	3309400831213	9307691151506	11169235537586	11376079524539
1027	321479780918	3536276449763	3536276449763	5143681782673	5143681782673	14466603424870	17359916463414	17681388557332
1051	497110820190	5468220598525	5468220598525	7953766796046	7953766796046	22369971051653	26843974790631	27341095178372
1075	764896637498	8413860876702	8413860876702	12238353855121	12238353855121	34420367790300	41304429463648	42069314230101
1099	1171318829840	12884509771652	12884509771652	18741091700903	18741091700903	52709323491410	63251202946890	64422536456281
1123	1785406214518	19639465309775	19639465309775	28566511591466	28566511591466	80343309988087	96411953800436	98197341895753
1147	2709277080802	29802051397724	29802051397724	43348418062984	43348418062984	121917430520635	146300938919330	149010238333467

Table 4. Multiplicity generating functions $f^{(1)}_{\mathrm{dP}_8,\chi}$, part 1.

$24d$	$55'$	$55''$	66	99	120	144	176
-5	0	0	0	0	0	0	0
19	0	0	0	0	0	2	0
43	3	3	1	5	2	1	9
67	12	12	21	21	36	35	37
91	65	65	72	116	130	170	207
115	244	244	303	450	572	668	809
139	910	910	1053	1625	1892	2299	2843
163	2743	2743	3380	4946	6152	7328	8873
187	8105	8105	9602	14577	17408	21007	25847
211	21733	21733	26243	39149	47808	57171	69639
235	56011	56011	66950	100712	121602	146250	178995
259	136438	136438	164139	245788	298688	357966	437126
283	321282	321282	384883	578131	699412	839873	1027282
307	726800	726800	873253	1308380	1588170	1904874	2326881
331	1598171	1598171	1916117	2876429	3483272	4181523	5112694
355	3410318	3410318	4094710	6139138	7445828	8932518	10914961
379	7105048	7105048	8522970	12788159	15494804	18597231	22733170
403	14455051	14455051	17350401	26020434	31548542	37853674	46261017
427	28819592	28819592	34577010	51873536	62864134	75443098	92215429
451	56343865	56343865	67622177	101421034	122953390	147535322	180310064
475	108267895	108267895	129908352	194879528	236191550	283442660	346445082
499	204631232	204631232	245574876	368340362	446507564	535790948	654835854
523	381015906	381015906	457195844	685822008	831254198	997529608	1219228018
547	699415924	699415924	839330695	1258957772	1526071004	1831253012	2238164410
571	1267187750	1267187750	1520582135	2280927045	2764674464	3317651311	4054954916
595	2267499033	2267499033	2721057746	4081511489	4947403906	5936828205	7256056407
619	4010773890	4010773890	4812849666	7219375201	8750602126	10500800283	12834400672
643	7016749493	7016749493	8420202262	12630174462	15309503684	18371299544	22453696426
667	12149643643	12149643643	14579440062	21869323134	26508010882	31809749824	38878728441
691	20831927400	20831927400	24998485648	37497515848	45451874944	54542074762	66662346131
715	35389146944	35389146944	42466747340	63700406402	77212161966	92654819516	113245032965
739	59590085251	59590085251	71508403663	107262225204	130015413134	156018202222	190688575918
763	99503036747	99503036747	119403255079	179105374690	217096655922	260516372674	318409337999
787	164823887402	164823887402	197789160389	296683119323	359616877840	431539751281	527436921384
811	270950001188	270950001188	325139371484	487709837991	591162203858	709395286805	867039376965
835	442165654782	442165654782	530599589906	795898389892	964726901324	1157671471836	1414930915206
859	716550903166	716550903166	859860053460	1289791366536	1563381441138	1876058749736	2292961837333
883	1153450147408	1153450147408	1384141494283	2076210582947	2516621491560	3019944492097	3691041793857
907	1844847178517	1844847178517	2213814945093	3320724518828	4025117339308	4830142466496	5903509327159
931	2932513935701	2932513935701	3519018815116	5278525605052	6398216968080	7677858253874	9384046650468
955	4633853529026	4633853529026	5560621623978	8340935682923	10110219936042	12132266566371	14828328689896
979	7280522608177	7280522608177	8736630398824	13104941535681	15884784057350	19061737583669	23297675655094
1003	11376074383581	11376074383581	13651285150689	20476932857057	24820516574124	29784623972565	36403433861254
1027	17681394966175	17681394966175	21217679111305	31826512201395	38577600704766	46293115745065	56580469055649
1051	27341087206947	27341087206947	32809298242028	49213955409660	59653266656898	71583926371636	87491472717522
1075	42069324119087	42069324119087	50483196856446	75724785382753	91787634227288	110145153118297	134621845007103
1099	64422524221421	64422524221421	77307019308615	115960541117888	140558212466154	168669864792640	206152067770248
1123	98197356999292	98197356999292	117836840441407	176755245669203	214248806330278	257098555519575	314231554534832
1147	149010219724356	149010219724356	178812248794508	268218391773495	325113172828678	390135822209555	476832688109889

Table 5. Multiplicity generating functions $f^{(1)}_{\mathrm{dP}_8,\chi}$, part 2.

$24d$	1	11	$11'$	16	$16'$
1171	4093401506990	45027412291973	45027412291973	65494442717998	65494442717998
1195	6158685759395	67745548890523	67745548890523	98538949769247	98538949769247
1219	9228239349225	101510625677235	101510625677235	147651856584014	147651856584014
1243	13773021513812	151503245506811	151503245506811	220368311165802	220368311165802
1267	20477101283532	225248103749002	225248103749002	327633661514469	327633661514469
1291	30330817057006	333638999586028	333638999586028	485293022305906	485293022305906
1315	44763186057643	492395032232149	492395032232149	716211038247564	716211038247564
1339	65829766710070	724127452040606	724127452040606	1053276194195726	1053276194195726
1363	96478169516795	1061259841697878	1061259841697878	1543650799851638	1543650799851638
1387	140922592375703	1550148543933051	1550148543933051	2254761371678042	2254761371678042
1411	205170323111458	2256873521699226	2256873521699226	3282725299689321	3282725299689321
1435	297760377206316	3275364187275260	3275364187275260	4764165877605326	4764165877605326
1459	430796502686742	4738761484014951	4738761484014951	6892744232195987	6892744232195987
1483	621388898570251	6835277940289662	6835277940289662	9942222151968754	9942222151968754
1507	893660568325681	9830266182559709	9830266182559709	14298569361270602	14298569361270602
1531	1281533705334823	14096870841750772	14096870841750772	20504538963932962	20504538963932962
1555	1832589066704092	20158479636409529	20158479636409529	29321425454783979	29321425454783979
1579	2613400611330406	28747406838047288	28747406838047288	41814409315423788	41814409315423788
1603	3716891022406435	40885801111701610	40885801111701610	59470256913313545	59470256913313545
1627	5272453300096449	57996986464667060	57996986464667060	84359252145641326	84359252145641326
1651	7459846110139069	82058307013061490	82058307013061490	119357538538247375	119357538538247375
1675	10528232420636209	115810556863115569	115810556863115569	168451717806641647	168451717806641647
1699	14822207710919004	163044284544165030	163044284544165030	237155324477163395	237155324477163395
1723	20817315567358978	228990471562943319	228990471562943319	333077047765997253	333077047765997253
1747	29168407625477909	320852483499759068	320852483499759068	466694523558355316	466694523558355316
1771	40775370225169032	448529072932153077	448529072932153077	652405921777788429	652405921777788429
1795	56872276713083413	625595043298945690	625595043298945690	909956429556887554	909956429556887554
1819	79148095494043006	870629051078933309	870629051078933309	1266369525369451301	1266369525369451301
1843	109909812683600080	1209007938767976703	1209007938767976703	1758557005937509432	1758557005937509432
1867	152302485633902688	1675327342846829765	1675327342846829765	2436839766599162279	2436839766599162279
1891	210605556865399279	2316661124493547173	2316661124493547173	3369688914010001224	3369688914010001224
1915	290631176879715729	3196942946893625371	3196942946893625371	4650098825202784367	4650098825202784367
1939	400258719166874651	4402845909393277271	4402845909393277271	6404139512372754278	6404139512372754278
1963	550150869077258932	6051659561542349276	6051659561542349276	8802413898544613901	8802413898544613901
1987	754711357815895642	8301824934007779454	8301824934007779454	12075381732915932852	12075381732915932852
2011	1033363835278555220	11367002190347074914	11367002190347074914	16533821355238048767	16533821355238048767
2035	1412256805146747435	15534824853948167213	15534824853948167213	22596108893118543439	22596108893118543439
2059	1926533042028023191	21191863465442002954	21191863465442002954	30824528659900088909	30824528659900088909
2083	2623345678439652253	28856802459153067204	28856802459153067204	41973530869649280580	41973530869649280580
2107	3565860601400413225	39224466619703622922	39224466619703622922	57053769605360276336	57053769605360276336
2131	4838559710474916833	53224156810243842644	53224156810243842644	77416955387499037845	77416955387499037845
2155	6554257603147504112	72096833640381599416	72096833640381599416	104868121627158155768	104868121627158155768
2179	8863371782121759020	97497089596648487456	97497089596648487456	141813948540918808370	141813948540918808370
2203	11966152827447202656	131627681109732904436	131627681109732904436	191458445207879722486	191458445207879722486
2227	16128796948623520740	177416766425737654861	177416766425737654861	258060751214233038172	258060751214233038172
2251	21704644236610653731	238751086613304015784	238751086613304015784	347274307743692860669	347274307743692860669
2275	29162029847290462732	320782328307975295104	320782328307975295104	466592477605502723278	466592477605502723278
2299	39120827378017906356	430329101172286761258	430329101172286761258	625933237991633696014	625933237991633696014

Table 6. Multiplicity generating functions $f^{(1)}_{\mathrm{dP}_8,\chi}$, part 3.

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$24d$	45	54	55	$55'$	$55''$
1171	184203114462622	221043710022343	225137084071984	225137106948035	225137106948035
1195	277140803111755	332568997319530	338727716741557	338727688687159	338727688687159
1219	415270838158915	498324964525296	507553162674521	507553197014225	507553197014225
1243	619785885692343	743743113342304	757516185209007	757516143250948	757516143250948
1267	921469660027838	1105763530523634	1126240570403383	1126240621565028	1126240621565028
1291	1364886640954779	1637864043787277	1668194935551870	1668194873292078	1668194873292078
1315	2014343526237095	2417212140959443	2461975236268978	2461975311902653	2461975311902653
1339	2962339318825550	3554807292476658	3620637169265217	3620637077537819	3620637077537819
1363	4341517847042590	5209821283094568	5306299319337420	5306299430386979	5306299430386979
1387	6341516391505723	7609819831217510	7750742584630229	7750742450426761	7750742450426761
1411	9232664864481335	11079197642782316	11284367771598830	11284367933524970	11284367933524970
1435	13399216579858759	16079060129761174	16376820741075612	16376820545992293	16376820545992293
1459	19385843094557620	23263011432427915	23693807653507039	23693807888163762	23693807888163762
1483	27962499872265004	33555000184510544	34176389421242631	34176389139433773	34176389139433773
1507	40214726244543434	48257671087653672	49151331250463847	49151331588413804	49151331588413804
1531	57669015937475566	69202819611374765	70484353802379119	70484353397662854	70484353397662854
1555	82466508969735861	98959810182387002	100792398668290829	100792399152265772	100792399152265772
1579	117603026344789245	141123632306950527	143737033611788997	143737033033854903	143737033033854903
1603	167260097396709360	200712116050021046	204429006245393365	204429006934618721	204429006934618721
1627	237260396863613020	284712477220607098	289984931505798416	289984930684900175	289984930684900175
1651	335693076895668315	402831691102628332	410291536040979311	410291537017434228	410291537017434228
1675	473770456621652456	568524549339514438	579052783152120837	579052781992128638	579052781992128638
1699	666999349746211386	800399218042669045	815221424102015049	815221425478366554	815221425478366554
1723	936779197250978703	1124135038657868408	1144952356182632432	1144952354551480412	1144952354551480412
1747	1312578347025836554	1575094014116214816	1604262419424649864	1604262421355471132	1604262421355471132
1771	1834891655568344765	2201869989419897616	2242645362384455501	2242645360101610419	2242645360101610419
1795	2559252457456586760	3071102945711666490	3127975219189577966	3127975221885611857	3127975221885611857
1819	3561664290897644899	4273997152896785912	4353145252207517405	4353145249026968359	4353145249026968359
1843	4945941578258894303	5935129889410954110	6045039697596848087	6045039701344867210	6045039701344867210
1867	6853611844665722902	8224334218891965090	8376636709820185138	8376636705408223928	8376636705408223928
1891	9477250069357366762	11372700077007117472	11583305627646850394	11583305632834988246	11583305632834988246
1915	13078402947401857383	15694083544192819487	15984714728386214908	15984714722291519041	15984714722291519041
1939	18011642376763874898	21613970843532086552	22014229554115942440	22014229561268362549	22014229561268362549
1963	24756789091754815561	29708146920173901712	30258297799313493059	30258297790928171326	30258297790928171326
1987	33962011121364327166	40754413333847397958	41509124679877898687	41509124689699169994	41509124689699169994
2011	46501372564484998594	55801647091170275277	56835010940239663338	56835010928747353467	56835010928747353467
2035	63551556258539592867	76261867494133421282	77674124283157975065	77674124296593010152	77674124296593010152
2059	86693986859883218954	104032784250684215406	105959317311542087994	105959317295850490724	105959317295850490724
2083	118050555566317547234	141660666657604661458	144284012314071615021	144284012332382347583	144284012332382347583
2107	160463727020416092994	192556472450127762662	196122333077146178381	196122333055797906898	196122333055797906898
2131	217735187021112290780	261282224395485799571	266120784076119006487	266120784100986919412	266120784100986919412
2155	294941592083627118072	353929910535079510346	360484168172958312466	360484168144015486852	360484168144015486852
2179	398851730262924276402	478622076275134635183	487485448016868002166	487485448050525203783	487485448050525203783
2203	538476877156922474149	646172252635225208162	658138405509601252323	658138405470493978887	658138405470493978887
2227	725795862778692020557	870955035279941337060	887083832174081304741	887083832219484026698	887083832219484026698
2251	976708990542309495285	1172050788713993804393	1193755433013807220005	1193755432961138141154	1193755432961138141154
2275	1312291343250192222140	1574749611826956036768	1603911641600986492613	1603911641662036771525	1603911641662036771525
2299	1760437231869163874840	2112524678327840972437	2151645505790711677538	2151645505720001135999	2151645505720001135999

Table 7. Multiplicity generating functions $f^{(1)}_{\mathrm{dP}_8,\chi}$, part 4.

$24d$	66	99	120	144	176
1171	270164546681156	405246797027805	491208274997242	589449911767971	720438760605286
1195	406473203907733	609709834093878	739042178780400	886850636998028	1083928581424780
1219	609063863910983	913595761484332	1107388855916070	1328866599517240	1624170257738622
1243	909019338414137	1363529049387265	1652762418194392	1983314935496409	2424051624958728
1267	1351488786694107	2027233129147441	2457252358138100	2948702788864913	3603970030037952
1291	2001833798159548	3002750759449688	3639697792134294	4367637400215692	5338223544445360
1315	2954370434912214	4431555576448250	5371582636389448	6445899103301566	7878321058779942
1339	4344764419512895	6517146721381044	7899571638612220	9479486039749018	11586038574903016
1363	6367559405323185	9551338996863518	11577380777243530	13892856843630246	16980158265658175
1387	9300890833306805	13951336383760152	16910710557156234	20292852776170360	24802375734252748
1411	13541241649561517	20311862312968942	24620439421902434	29544527176781278	36109977517072127
1435	19652184499179137	29478276943727693	35731244472881150	42877493523182575	52405825590524819
1459	28432569653735348	42648854245377598	51695581273866940	62034697341246448	75820185430196005
1483	41011666741538202	61517500394937483	74566666700591406	89480000266235449	109364445021070861
1507	58981598176550861	88472396926681358	107239269534589732	128687123170645012	157284261352456668
1531	84581223753771591	126871836034520310	153784043041385876	184540851973883398	225549930549289886
1555	120950879369401488	181426318571322984	219910689938939870	263892827539670898	322535677674888309
1579	172484439178360861	258726659345247778	313608071022950170	376329685689204628	459958505244752664
1603	245314808873442099	367972212619696564	446026925474876674	535232310019095576	654172822742952752
1627	347981916164512698	521972875069314411	632694392728415992	759233271930954787	927951777535669872
1651	492349845202179646	738524766826508033	895181537085930972	1074217843720999295	1312932919235178135
1675	694863337463226605	1042295007353137208	1263387885874730544	1516065463978597964	1852968901447872354
1699	978265711674229209	1467398566137307066	1778664930818030794	2134397915880732844	2608708562632879662
1723	1373942824157073997	2060914237866252831	2498077861510042094	2997693435115570067	3663847533257402807
1747	1925114907172125283	2887672358825009115	3500208922832935414	4200250705856191083	5133639749883610819
1771	2691174430294308510	4036761647727631051	4893044417887901864	5871653303292074233	7176465150500213001
1795	3753570268419902966	5630355399933081775	6824673216288466950	8189607857387351047	10009520712187446446
1819	5223774296289292241	7835661447611118530	9497771446641673624	11397325738516222932	13930064794343989332
1843	7254047644610345604	10881071463172141658	13189177537019635416	15827013041425604314	19344127047303818283
1867	10051964042960644068	15077946068852005519	18276298258323001986	21931557913514503889	26805237453771981804
1891	13899966763554443537	20849950140138817382	25272666844712164358	30327200209506516810	37066578029225470889
1915	19181657661871572164	28772486498908357454	34875741201188675430	41850889446302706172	51151087106459775710
1939	26417075479244374467	39625613211712943497	48031046328498284852	57637255588472376133	70445534601774047231
1963	36309957342408061267	54464936021991049526	66018104255873659084	79221725113760100204	96826552924265854344
1987	49810949635492675761	74716424443426330769	90565362977197226288	108678435564780334215	132829199014898120022
2011	68202013105303745330	102303019669446105608	124003660187280231114	148804392233925001344	181872034962788778272
2035	93208949166662983780	139813423736550819498	169470816671544273222	203364979995109897828	248557197759850986121
2059	127151180742462470212	190726771129397283986	231183964980593334826	277420757989266397324	339069815334173156022
2083	173140814813508567478	259711222201949509299	314801481485761998530	377761777768258741081	461708839478260064519
2107	235346799649883728800	353020199496161859438	427903272082932960552	513483926516604492932	627591465761482246451
2131	319344940941071942661	479017411386756381538	580627165356451864750	696752598407849494756	851586509143058978448
2155	432581001749665552605	648871502653437926113	786510912261590705900	943813094737053749501	1153549338037678618915
2179	584982537687562696957	877473806497670520373	1063604613989623332136	1276325536760630740267	1559953433788616761379
2203	789766086533298317643	1184649129839076522389	1435938339137237812658	1723126006995973066501	2106042897474304114574
2227	1064500598699704522077	1596750898004149803311	1935455634015967754356	2322546760782826588291	2838668263138648397534
2251	1432506519511239076275	2148759779319506061861	2604557308183097498046	3125468769861864014025	3820017385433520294132
2275	1924693970043272902043	2887040955003888251120	3499443581919069498760	4199332298254045573934	5132517253367370460268
2299	2581974606807435465725	3872961910281857647683	4694499285078708148956	5633399142151001036453	6885265618247405253843

Table 8. Multiplicity generating functions $f^{(1)}_{\mathrm{dP}_8,\chi}$, part 5.

References

[1] A. Dabholkar, S. Murthy and D. Zagier, *Quantum Black Holes, Wall Crossing and Mock Modular Forms*, arXiv:1208.4074 [INSPIRE].

[2] T. Eguchi, H. Ooguri and Y. Tachikawa, *Notes on the K3 Surface and the Mathieu Group M24*, Exper. Math. **20** (2011) 91.

[3] M.C.N. Cheng, J.F.R. Duncan and J.A. Harvey, *Umbral Moonshine and the Niemeier Lattices*, Res. Math. Sci. **1** (2014) 3 [arXiv:1307.5793] [INSPIRE].

[4] M.C.N. Cheng, J.F.R. Duncan and J.A. Harvey, *Weight One Jacobi Forms and Umbral Moonshine*, J. Phys. **A 51** (2018) 104002 [arXiv:1703.03968] [INSPIRE].

[5] N.-P. Skoruppa, *Developments in the theory of Jacobi forms*, in *Automorphic functions and their applications*, Khabarovsk (1988), pp. 167–185.

[6] N.-P. Skoruppa, *Explicit formulas for the Fourier coefficients of Jacobi and elliptic modular forms*, Invent. Math. **102** (1990) 501.

[7] J.A. Harvey and B.C. Rayhaun, *Traces of Singular Moduli and Moonshine for the Thompson Group*, Commun. Num. Theor. Phys. **10** (2016) 23 [arXiv:1504.08179] [INSPIRE].

[8] J.F.R. Duncan, J.A. Harvey and B.C. Rayhaun, *Skew-Holomorphic Jacobi Forms and Moonshine*, in preparation.

[9] J.M. Maldacena, A. Strominger and E. Witten, *Black hole entropy in M-theory*, JHEP **12** (1997) 002 [hep-th/9711053] [INSPIRE].

[10] D. Gaiotto, A. Strominger and X. Yin, *The M5-Brane Elliptic Genus: Modularity and BPS States*, JHEP **08** (2007) 070 [hep-th/0607010] [INSPIRE].

[11] J. de Boer, M.C.N. Cheng, R. Dijkgraaf, J. Manschot and E. Verlinde, *A Farey Tail for Attractor Black Holes*, JHEP **11** (2006) 024 [hep-th/0608059] [INSPIRE].

[12] F. Denef and G.W. Moore, *Split states, entropy enigmas, holes and halos*, JHEP **11** (2011) 129 [hep-th/0702146] [INSPIRE].

[13] J. Manschot and G.W. Moore, *A Modern Farey Tail*, Commun. Num. Theor. Phys. **4** (2010) 103 [arXiv:0712.0573] [INSPIRE].

[14] M. Alim, B. Haghighat, M. Hecht, A. Klemm, M. Rauch and T. Wotschke, *Wall-crossing holomorphic anomaly and mock modularity of multiple M5-branes*, Commun. Math. Phys. **339** (2015) 773 [arXiv:1012.1608] [INSPIRE].

[15] M.C.N. Cheng and J.F.R. Duncan, *Optimal Mock Jacobi Theta Functions*, arXiv:1605.04480 [INSPIRE].

[16] M.C.N. Cheng and J.F.R. Duncan, *Meromorphic Jacobi Forms of Half-Integral Index and Umbral Moonshine Modules*, arXiv:1707.01336 [INSPIRE].

[17] N.-P. Skoruppa and D. Zagier, *Jacobi forms and a certain space of modular forms*, Invent. Math. **94** (1988) 113.

[18] N.-P. Skoruppa, *Heegner cycles, modular forms and Jacobi forms*, *J. Théor. Nombres Bordeaux* **3** (1991) 93.

[19] J. Duncan, M. Mertens and K. Ono, *O'Nan moonshine and arithmetic*, arXiv:1702.03516.

[20] G.W. Moore, *Arithmetic and attractors*, hep-th/9807087 [INSPIRE].

[21] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal Field Theory*, Springer-Verlag, New York (1997).

[22] G.W. Moore and N. Seiberg, *Naturality in Conformal Field Theory*, *Nucl. Phys.* **B 313** (1989) 16 [INSPIRE].

[23] A. Dabholkar and J.A. Harvey, *Nonrenormalization of the Superstring Tension*, *Phys. Rev. Lett.* **63** (1989) 478 [INSPIRE].

[24] V.V. Nikulin, *Integral Symmetric Bilinear Forms and Some of their Applications*, *Math. USSR Izv.* **14** (1980) 103 [*Izv. Akad. Nauk SSSR Ser. Mat.* **43** (1979) 111].

[25] J.H. Conway and N.J.A. Sloane. *Sphere packings, lattices and groups*, vol. 290, Springer Science & Business Media (2013).

[26] W. Ebeling, *Lattices and Codes. A course partially based on lectures by Friedrich Hirzebruch*, Advanced Lectures in Mathematics, Springer Spektrum (2013).

[27] R. Minasian, G.W. Moore and D. Tsimpis, *Calabi-Yau black holes and* $(0,4)$ σ*-models*, *Commun. Math. Phys.* **209** (2000) 325 [hep-th/9904217] [INSPIRE].

[28] L. Göttsche, *The Betti numbers of the Hilbert schemes of points on a smooth projective surface*, *Math. Ann.* **286** (1990) 193.

[29] D. Zagier, *Nombres de classes et formes modulaires de poids 3/2*, *Sém. Théor. Nombres Bordeaux* **4** (1975) 1.

[30] M. Cheng, J. Duncan and M. Mertens, *Class Number Moonshine*, in preparation.

[31] K. Yoshioka, *The Betti numbers of the moduli space of stable sheaves of rank* 2 *on* $\mathbb{P}^2$, *J. Reine Angew. Math.* **453** (1994) 193.

[32] K. Yoshioka, *The Betti numbers of the moduli space of stable sheaves of rank* 2 *on a ruled surface*, *Math. Ann.* **302** (1995) 519.

[33] C. Vafa and E. Witten, *A Strong coupling test of S duality*, *Nucl. Phys.* **B 431** (1994) 3 [hep-th/9408074] [INSPIRE].

[34] R. Griess and A. Ryba, *Finite simple groups which projectively embed in an exceptional Lie group are classified!*, *Bull. Am. Math. Soc.* **36** (1999) 75.

[35] S. Kachru and A. Tripathy, *Black Holes and Hurwitz Class Numbers*, *Int. J. Mod. Phys.* **D 26** (2017) 1742003 [arXiv:1705.06295] [INSPIRE].

[36] J.A. Minahan, D. Nemeschansky, C. Vafa and N.P. Warner, *E strings and* $N = 4$ *topological Yang-Mills theories*, *Nucl. Phys.* **B 527** (1998) 581 [hep-th/9802168] [INSPIRE].

[37] M.C.N. Cheng and J.F.R. Duncan, *Rademacher Sums and Rademacher Series*, *Contrib. Math. Comput. Sci.* **8** (2014) 143 [arXiv:1210.3066] [INSPIRE].

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