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The non-Abelian tensor multiplet

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ABSTRACT: We assume the existence of a background vector field that enables us to make an ansatz for the superconformal transformations for the non-Abelian 6d $(1,0)$ tensor multiplet. Closure of supersymmetry on generators of the conformal algebra and the R-symmetry, requires that the vector field is Abelian, has scaling dimension minus one and that the supersymmetry parameter as well as all the fields in the tensor multiplet have vanishing Lie derivatives along this vector field. We couple the tensor multiplet to an adjoint hypermultiplet, and present a Lagrangian for the combined system that has enhanced $(2,0)$ superconformal symmetry. We also obtain the off-shell supersymmetry variations for both the tensor and the hypermultiplets.

KEYWORDS: Conformal Field Theory, Field Theories in Higher Dimensions, M-Theory, Supersymmetric Gauge Theory

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1 Introduction

There might have been speculations that the non-Abelian 6d $(2,0)$ tensor multiplet theory could be a non-Abelian gerbe theory [12], although so far no successful such theory has been found. The simplest example of a non-Abelian gerbe is where one introduces a non-Abelian two-form together with a flat one-form gauge field [3]. Let us assume that a classical field

theory description for the non-Abelian tensor multiplet exists, and that we just have not found it yet. Let us refer to it as the theory $X[G]$ where G denotes the gauge group. The theory $X[G]$ might be characterized by a classical Lagrangian, or classical equations of motion, perhaps making use of some kind of non-Abelian gerbe construction.

Not much may be known about the theory $X[G]$, but upon circle compactification and dimensional reduction down to 5d, it shall reduce to 5d MSYM with gauge group G .

If we just compactify one circle direction but do not perform the dimensional reduction, then there will appear modes that are the Kaluza-Klein (KK) modes if we rewrite the theory $X[G]$ in terms of 5d fields. The zeroth KK mode gives the 5d MSYM.

The conjecture of [16, 17] says that 6d (2, 0) theory compactified, but not dimensionally reduced, on a circle is actually equivalent with the 5d maximally supersymmetric Yang-Mills (MSYM) that one gets by performing dimensional reduction along that circle.

But as we just compactify the classical field theory $X[G]$, we get all the higher KK modes in addition to the zero modes. We conclude that the two statements

1. The conjecture of [16, 17] holds.
2. The classical field theory $X[G]$ exists,

can not be simultaneously true.

In this paper, we will assume that at least one of the two statements above is correct. But as they can not both be correct, we see that either one of the two statements must be correct. But as we will argue, which one of the two statements 1 and 2 is realized, is not universal. It depends on the choice of gauge group G among other things.

If the gauge group $G = \text{U}(1)$ is Abelian, then the theory $X[\text{U}(1)]$ exists. We have a 6d classical field theory description in terms of the 6d (2, 0) Abelian tensor multiplet fields. This means that the conjecture can not hold when $G = \text{U}(1)$, as was also pointed out in [22].

We may deform the Abelian theory by making space noncommutative. This gives the Abelian 5d MSYM theory the structure reminiscent of a non-Abelian theory, due to a noncommutative star-product between the fields that live on a noncommutative space. In this deformed case, there does not seem to exist a classical field theory in 6d.¹ Indeed then it seems that the conjecture becomes true. It has been shown that the noncommutative 5d MSYM theory has noncommutative instanton particle solutions that capture all the missing KK modes [18, 21] by comparing with Abelian 6d (2, 0) theory that should be a very good approximation for a very small noncommutativity parameter.

We have thus far argued that if the conjecture of [16, 17] is correct, then no 6d theory exists that upon dimensional reduction gives 5d SYM. But there may be a loophole in our argument if the 6d theory is such that upon dimensional reduction it gives rise to 5d SYM

¹A no-go theorem to constructing a noncommutative M5 brane was found in [13]. This no-go theorem was circumvented in [15], but the star 3-product used there to circumvent the no-go theorem has no application to the noncommutative M5 brane. As the noncommutativity becomes small, the Abelian tensor multiplet theory becomes a good approximation. But it is not an exact description as long as the noncommutativity parameter is non-zero.

without the additional KK modes, or if the KK modes cancel out. While this scenario seems unlikely if the 6d theory is a local field theory, perhaps it can be possible if the 6d theory is a nonlocal theory that reduces to a local theory in 5d [28].

Since we are lacking a Lagrangian formulation for the non-Abelian 6d (2,0) theory, the best we can do is to consider the Abelian 6d (2,0) theory on a six-manifold that is a circle-bundle over a five-manifold and perform dimensional reduction along the circle to get a 5d SYM theory that lives on that five-manifold. In 5d we can then find the non-Abelian generalization. In this paper we will show that we do not need to perform the explicit dimensional reduction along the circle fiber, in the sense that we can reformulate the 5d SYM theory as a non-Abelian 6d theory, but where we impose as a constraint that all fields have vanishing Lie derivative along the Killing vector field that generates the circle-fiber. Although putting the Lie derivative to zero and performing the explicit dimensional reduction are equivalent operations, the former operation allows us to express the metric in whatever coordinates we want, while the latter explicit dimensional reduction, fixes the metric to be of the circle bundle form

$$ds^2 = -f(x^m)(dt + \kappa_m dx^m)^2 + G_{mn} dx^m dx^n$$

if, say, the dimensional reduction is performed along the timelike vector field $V = \partial/\partial t$. The explicit dimensional reduction breaks the 6d diffeomorphism invariance by selecting a particular 5d base-manifold, here spanned by the coordinates x^m . It may be advantageous to be able to work in a formulation where we do not have to choose a base-manifold. An example would be S^6 , where the base-manifold will be a very asymmetric manifold that makes the nice symmetries of S^6 obscure. The explicit dimensional reduction also makes the 6d superconformal symmetry obscure. In 5d SYM it is not obvious that the Yang-Mills coupling constant should be given an interpretation as the length of the vector field V as measured by the 6d metric. Also it is not so obvious how the conformal transformations act in the 5d theory.

Suppose that we have two distinct vector fields in the six-manifold that span a two-torus. They may be generators for the a and b cycles of that two-torus. We can pick our vector field as $V = ma + nb$ where (m, n) are two coprime integers. In our 6d formulation, the Lagrangian will be of the same form regardless of the choice of (m, n) . On the other hand, the corresponding 5d Lagrangians to which the 6d Lagrangian will reduce, will be very different.

Our work can be seen as a no-go result for constructing a local non-Abelian 6d theory with (2,0) supersymmetry within a certain ansatz. To generalize the Abelian tensor multiplet supersymmetry variations to a non-Abelian gauge group in a local fashion, one is led to introducing a vector field. This is because one needs an extra gamma matrix to match chiralities in the supersymmetry variation of the fermion — the fermion spinor field and the supersymmetry parameters have opposite 6d chiralities. That extra gamma matrix has to be contracted by something that carries a single vector index, and since there is no vector field in the tensor multiplet, the only option is to introduce an auxiliary vector field for this purpose [14]. Now, if one makes such an ansatz for the supersymmetry variations, one finds that they close on-shell only if one requires that all the fields have

vanishing Lie derivatives along that vector field. Thus the vanishing of the Lie derivative is not an assumption but a consequence of supersymmetry within the ansatz that we make. We start with a very general 3-algebra formulation with a vector field v^M valued in the 3-algebra. We demand gauge symmetry and supersymmetry, and from there everything else follows, and we end up with a 6d formulation that is equivalent with 5d SYM.

In section 2 we introduce a non-Abelian two-form gauge field B_{MN} and a vector field v^M , both of which are taking values in a 3-algebra. We construct a vector gauge field $A_M = \langle B_{MN}, v^N \rangle$ where the bracket maps two 3-algebra elements into a Lie algebra element. We show that gauge symmetry requires v^M to satisfy $\langle v^M, v^N \rangle = 0$ and the two-form gauge field to have vanishing Lie derivative along the vector field, $\mathcal{L}_v B_{MN} = 0$. In section 3 we review the non-Abelian $(2, 0)$ and $(1, 0)$ systems of [14, 24] but where we relax their assumption that v^M shall transform as a dynamical field in the tensor multiplet under the supersymmetry variations. In section 4 we show that by relaxing this assumption, the $(1, 0)$ tensor multiplet closes on-shell on the conformal transformations and an R-symmetry rotation, thus realizing the superconformal algebra. In section 5 we close the $(1, 0)$ hypermultiplet off-shell while keeping the $(1, 0)$ tensor multiplet on-shell. In section 6 we present an explicit solution to the constraint $\langle v^M, v^N \rangle = 0$ using Lorentzian 3-algebra. In section 7 we use this explicit solution for v^M to completely eliminate the non-Abelian tensor gauge field from the formulation by only keeping the components $F_{MN,a} = H_{MNP,a} V^P$ where a denotes a 3-algebra index (that also gets the interpretation of a Lie algebra index) in the Lorentzian 3-algebra, $v^M = V^M T^+$ and T^+ is one of the lightlike 3-algebra generators in the Lorentzian 3-algebra. In this section we also present the superconformal Lagrangian with $(2, 0)$ supersymmetry. In the formulation we can reduce the supersymmetry down to $(1, 0)$ supersymmetry. But unless we couple to a supergravity background R-gauge field, the supersymmetry for a theory with one adjoint hyper becomes automatically enhanced to $(2, 0)$ supersymmetry [24].

2 Gauge symmetry

We will not introduce a one-form gauge field A_M as an independent field from the two-form B_{MN} . Instead we will follow [14] and assume the existence of a vector field v^M . We will assume that B_{MN} and v^M both take values in a set $\mathcal{A}_3$ and that there exists a multiplication on $\mathcal{A}_3$ that we denote as $\langle \bullet, \bullet \rangle$ that maps two elements of $\mathcal{A}_3$ into a Lie algebra $\mathfrak{g}$ of some gauge group. This should be the gauge group that appears upon dimensional reduction on a circle down to 5d SYM. We define the $\mathfrak{g}$ -valued gauge field from B_{MN} as

$$A_M = \langle B_{MN}, v^N \rangle$$

For the gauge parameters, we assume the following relation,

$$\Lambda = \langle \Lambda_M, v^M \rangle$$

between the $\mathfrak{g}$ -valued gauge parameter Λ , and the $\mathcal{A}_3$ -valued gauge parameters Λ_M . If $\varphi, v^M \in \mathcal{A}_3$, then we assign the gauge variation rules

$$\begin{aligned}\delta\varphi &= -(\Lambda, \varphi) \\ \delta v^M &= -(\Lambda, v^M)\end{aligned}$$

where we use the round bracket to map elements in $\mathfrak{g} \times \mathcal{A}_3$ to an element in $\mathcal{A}_3$. We also use the standard square bracket for the commutator of two elements in $\mathfrak{g}$. If we multiply two fields φ_1 and φ_2 that are valued in $\mathcal{A}_3$, then we get a field that is valued in $\mathfrak{g}$. Let us assume that this composite field is in the adjoint representation. Then we shall have the gauge variation

$$\delta(\langle\varphi_1, \varphi_2\rangle) = -[\Lambda, \langle\varphi_1, \varphi_2\rangle]$$

On the other hand, the gauge variation should satisfy the Leibniz rule,

$$\delta(\langle\varphi_1, \varphi_2\rangle) = \langle\delta\varphi_1, \varphi_2\rangle + \langle\varphi_1, \delta\varphi_2\rangle$$

Putting these two ingredients together, we find the following Jacobi identity,

$$\langle(\Lambda, \varphi_1), \varphi_2\rangle + \langle\varphi_1, (\Lambda, \varphi_2)\rangle = [\Lambda, \langle\varphi_1, \varphi_2\rangle]$$

We will take the product $\langle\bullet, \bullet\rangle : \mathcal{A}_3 \times \mathcal{A}_3 \rightarrow \mathfrak{g}$ to be antisymmetric. That antisymmetry does not follow from gauge symmetry, and indeed we will introduce another gauge invariant product that we will denote as $\bullet \cdot \bullet : \mathcal{A}_3 \times \mathcal{A}_3 \rightarrow \mathbb{C}$ that is hermitian

$$\varphi_1 \cdot \varphi_2 = (\varphi_2 \cdot \varphi_1)^*$$

and gauge invariant

$$(\Lambda, \varphi_1) \cdot \varphi_2 + \varphi_1 \cdot (\Lambda, \varphi_2) = 0$$

This is the inner product that we will use for writing a gauge invariant action.

The infinitesimal gauge variation for A_M is

$$\delta A_M = D_M \Lambda \tag{2.1}$$

The gauge covariant derivative is

$$\begin{aligned}D_M \Lambda &= \partial_M \Lambda + [A_M, \Lambda] \\ D_M \varphi &= \partial_M \varphi + (A_M, \varphi)\end{aligned}$$

where it acts on a $\mathfrak{g}$ -valued field Λ and on an $\mathcal{A}_3$ valued field φ respectively. Let us expand (2.1),

$$\begin{aligned}\delta A_M &= \partial_M \langle\Lambda_N, v^N\rangle + [A_M, \langle\Lambda_N, v^N\rangle] \\ &= \langle\partial_M \Lambda_N, v^N\rangle + \langle\Lambda_N, \partial_M v^N\rangle + [\langle B_{MN}, v^N\rangle, \Lambda] \\ &= \langle\partial_M \Lambda_N - \partial_N \Lambda_M, v^N\rangle + [\langle B_{MN}, v^N\rangle, \Lambda] \\ &\quad + \langle\partial_N \Lambda_M, v^N\rangle + \langle\Lambda_N, \partial_M v^N\rangle\end{aligned}$$

We use

$$\delta A_M = \langle \delta B_{MN}, v^N \rangle + \langle B_{MN}, \delta v^N \rangle$$

and the Jacobi identity in the form

$$\langle (\Lambda, v^N), B_{MN} \rangle - \langle (\Lambda, B_{MN}), v^N \rangle = [\Lambda, \langle v^N, B_{MN} \rangle]$$

Consistency requires that the first term $\langle (\Lambda, v^N), B_{MN} \rangle$ is vanishing. But we do not want this to impose a constraint on B_{MN} nor on Λ_P , but we want this to be a constraint on v^M only. For this to be possible, we assume that we have associativity

$$(\langle \Lambda_P, v^P \rangle, v^N) = (\Lambda_P, \langle v^P, v^N \rangle)$$

As the product is associative, we can introduce a 3-bracket notation. For any three elements $a, b, c \in \mathcal{A}_3 \times \mathcal{A}_3 \times \mathcal{A}_3$, we define

$$[a, b, c] := (\langle a, b \rangle, c) = (a, \langle b, c \rangle)$$

We see that the consistency condition that we shall impose on v^M is the Abelian constraint

$$\langle v^M, v^N \rangle = 0$$

We also define

$$\langle \mathcal{L}_v, \Lambda_M \rangle := \langle v^N, \partial_N \Lambda_M \rangle + \langle \partial_M v^N, \Lambda_N \rangle$$

Using this, we get

$$\delta A_M = \langle \delta B_{MN}, v^N \rangle + \langle B_{MN}, \delta v^N \rangle - \langle \mathcal{L}_v, \Lambda_M \rangle$$

where

$$\begin{aligned} \delta B_{MN} &= \partial_M \Lambda_N - \partial_N \Lambda_M - (\Lambda, B_{MN}) \\ \delta v^N &= -(\Lambda, v^N) \end{aligned}$$

We obtain the consistency condition

$$\langle \mathcal{L}_v, \Lambda_M \rangle = 0$$

Let us move on to the field strength that we expand as

$$\begin{aligned} F_{MN} &= \partial_M A_N - \partial_N A_M + [A_M, A_N] \\ &= \langle \partial_M B_{NP}, v^P \rangle + \langle B_{NP}, \partial_M v^P \rangle \\ &\quad - \langle \partial_N B_{MP}, v^P \rangle - \langle B_{MP}, \partial_N v^P \rangle \\ &\quad + [\langle B_{MP}, v^P \rangle, \langle B_{NQ}, v^Q \rangle] \end{aligned}$$

Using the Jacobi identity

$$[A_M, \langle B_{NP}, v^P \rangle] = \langle (A_M, B_{NP}), v^P \rangle - \langle (A_M, v^P), B_{NP} \rangle$$

together with the fact that

$$(A_N, v^P) = [B_{NQ}, v^Q, v^P] = 0$$

by the Abelian constraint on v^M , we get

$$\begin{aligned} F_{MN} = & \langle \partial_M B_{NP} + \partial_N B_{PM} + \partial_P B_{MN}, v^P \rangle \\ & - \langle \partial_P B_{MN}, v^P \rangle + \langle B_{PN}, \partial_M v^P \rangle + \langle B_{MP}, \partial_N v^P \rangle \\ & + \langle (A_M, B_{NP}), v^P \rangle \end{aligned}$$

If we note that²

$$\langle (A_P, B_{MN}), v^P \rangle = 0$$

then we may write this result as

$$F_{MN} = \langle H_{MNP}, v^P \rangle \quad (2.2)$$

where

$$\begin{aligned} H_{MNP} = & 3\partial_{[M} B_{NP]} + \frac{3}{2}(A_{[M}, B_{NP]}) + C_{MNP} \\ \langle C_{MNP}, v^P \rangle = & 0 \end{aligned}$$

provided that we impose the constraint

$$\mathcal{L}_v B_{MN} = 0$$

where

$$\mathcal{L}_v B_{MN} = \langle v^P, \partial_P B_{MN} \rangle + \langle \partial_M v^P, B_{PN} \rangle + \langle \partial_N v^P, B_{MP} \rangle$$

We have no explicit expression for C_{MNP} , but we can derive its gauge variation by requiring that H_{MNP} transforms homogeneously. We compute the gauge variation as

$$\delta H_{MNP} = 3\partial_M \delta B_{NP} + \frac{3}{2}(\delta A_M, B_{NP}) + \frac{3}{2}(A_M, \delta B_{NP}) + \delta C_{MNP}$$

We make the ansatz

$$\delta C_{MNP} = -(\Lambda, C_{MNP}) + \delta' C_{MNP}$$

for the gauge variation of C_{MNP} . We compute each term in turn,

$$\begin{aligned} 3\partial_M \delta B_{NP} = & -3(\partial_M \Lambda, B_{NP}) - 3(\Lambda, \partial_M B_{NP}) \\ \frac{3}{2}(\delta A_M, B_{NP}) = & \frac{3}{2}(\partial_M \Lambda, B_{NP}) + \frac{3}{2}([A_M, \Lambda], B_{NP}) \\ \frac{3}{2}(A_M, \delta B_{NP}) = & 3(A_M, \partial_N \Lambda_P) - \frac{3}{2}(A_M, (\Lambda, B_{NP})) \end{aligned}$$

²Here is a proof that uses the 3-bracket language and the fundamental identity,

$$\begin{aligned} & [[B_{PQ}, B_{MN}, v^Q], v^P, \bullet] + [v^Q, [B_{PQ}, B_{MN}, v^P], \bullet] \\ & = [B_{PQ}, B_{MN}, [v^Q, v^P, \bullet]] - [v^Q, v^P, [B_{PQ}, B_{MN}, \bullet]] = 0. \end{aligned}$$

We then find the gauge variation,³

$$\begin{aligned}\delta H_{MNP} &= -3(\Lambda, \partial_M B_{NP}) + \frac{3}{2}([A_M, \Lambda], B_{NP}) - \frac{3}{2}(A_M, (\Lambda, B_{NP})) + (\Lambda, C_{MNP}) \\ &= -3(\Lambda, H_{MNP})\end{aligned}$$

if we assume that

$$\delta' C_{MNP} = \frac{3}{2}(\partial_M \Lambda, B_{NP}) - 3(A_M, \partial_N \Lambda_P)$$

Let us summarize what we have found so far. Consistency with gauge symmetry of A_M together with the minimal assumption that $A_M = \langle B_{MN}, v^N \rangle$ implies that the following conditions must be satisfied:

- The Lie derivatives along v^M vanish, $\mathcal{L}_v B_{MN} = 0$ and $\mathcal{L}_v \Lambda_M = 0$.
- We have the constraint $\langle v^M, v^N \rangle = 0$.
- We have the constraint $F_{MN} = \langle H_{MNP}, v^P \rangle$.
- We have the 3-algebra Jacobi identities

$$\langle (A, a), b \rangle - \langle (A, b), a \rangle = [A, \langle a, b \rangle] \quad (2.3)$$

$$(A, (B, b)) - (B, (A, a)) = ([A, B], a) \quad (2.4)$$

$$[[A, B], C] - [[A, C], B] = [A, [B, C]] \quad (2.5)$$

for elements $a, b, \dots \in \mathcal{A}_3$ and $A, B, \dots \in \mathfrak{g}$.

The Jacobi identity (2.5) is the closure relation of the gauge algebra acting on a Lie algebra value field. The 3-algebra Jacobi identity (2.4) is the closure relation when acting on a 3-algebra valued field,

$$[\delta_{\Lambda'}, \delta_{\Lambda}] \varphi = \delta_{[\Lambda, \Lambda']} \varphi$$

and the 3-algebra Jacobi identity (2.3) is the Leibniz rule for gauge transformations,

$$\delta_{\Lambda} \langle \varphi_1, \varphi_2 \rangle = \langle \delta_{\Lambda} \varphi_1, \varphi_2 \rangle + \langle \varphi_1, \delta_{\Lambda} \varphi_2 \rangle$$

This 3-algebra was first applied in a gauge theory in [5–7]. In [7] it was shown that the three Jacobi identities (2.3), (2.4), (2.5) are equivalent with the so-called fundamental identity, which is a generalized Jacobi identity satisfied by the 3-bracket $[a, b, c]$.

³Here we use the Jacobi identity

$$([A_M, \Lambda], B_{NP}) = (A_M, (\Lambda, B_{NP})) - (\Lambda, (A_M, B_{NP})).$$

3 Review of 6d (2, 0) classical field theory

We begin by reviewing the 6d theory of [14, 24], but where we drop the assumption that v^M is a dynamical field that belongs to the tensor multiplet. We will instead assume that v^M is a background vector field that does not belong to the tensor multiplet. As we will see below, without making this assumption, we can not extend the Poincare supersymmetry in [14, 24] to the full superconformal symmetry. Following [14], we assume that $\phi^A, \Psi, H_{MNP}, v^M \in \mathcal{A}_3$ and $A_M \in \mathfrak{g}$. Here $M = 0, 1, 2, 3, 4, 5$ is a spacetime vector index and $A = 1, 2, 3, 4, 5$ is an $\text{SO}(5)$ R-symmetry vector index. One may object to taking v^M as a background field and at the same time assuming it is in $\mathcal{A}_3$ and hence transforming under gauge transformations as

$$\delta v^M = -(\Lambda, v^M)$$

where $\Lambda = \langle \Lambda_N, v^N \rangle$. But this means that by the Abelian constraint $\langle v^M, v^N \rangle = 0$, we actually have that v^M transforms trivially under gauge transformations,

$$\delta v^M = 0$$

so there is no contradiction in assuming that v^M is a background field.

In 11d spinors have 32 components. We break this down to 16 components by imposing the 6d Weyl projections $\Gamma\Psi = \Psi$ and $\Gamma\epsilon = -\epsilon$ where $\Gamma = \Gamma^{012345}$. We use 11d gamma matrices that we denote as Γ^M and $\hat{\Gamma}^A$.⁴ In 11d notation, these are satisfying the Clifford algebra

$$\begin{aligned} \{\Gamma^M, \Gamma^N\} &= 2\eta^{MN} \\ \{\hat{\Gamma}^A, \hat{\Gamma}^B\} &= 2\delta^{AB} \\ \{\Gamma^M, \hat{\Gamma}^A\} &= 0 \end{aligned}$$

where the spacetime metric is $\eta_{MN} = \text{diag}(-1, 1, 1, 1, 1, 1)$. The (2, 0) Poincare supersymmetry variations on $\mathbb{R}^{1,5}$ read [14]

$$\begin{aligned} \delta\phi^A &= i\bar{\epsilon}\hat{\Gamma}^A\Psi \\ \delta\Psi &= \frac{1}{12}\Gamma^{MNP}\epsilon H_{MNP} + \Gamma^M\hat{\Gamma}^A\epsilon D_M\phi^A - \frac{1}{2}\Gamma^M\hat{\Gamma}^{AB}\epsilon[\phi^A, \phi^B, v_M] \\ \delta H_{MNP} &= 3i\bar{\epsilon}\Gamma_{NP}D_M\Psi + i\bar{\epsilon}\hat{\Gamma}^A\Gamma_{MNPQ}[\Psi, v^Q, \phi^A] \\ \delta A_M &= i\bar{\epsilon}\Gamma_{MN}\langle\Psi, v^N\rangle \end{aligned}$$

We notice that the 3-bracket term in δH_{MNP} vanishes when this is contracted by v^P . This follows from the Abelian constraint together with the Jacobi identity $\langle(A, v^Q), v^P\rangle - \langle(A, v^P), v^Q\rangle = [A, \langle v^Q, v^P\rangle] = 0$ where we take $A = \langle\Psi, \phi^A\rangle$. By using this, the variation of H_{MNP} implies the following variation for $F_{MN} = \langle H_{MNP}, v^P\rangle$,

$$\delta F_{MN} = 2i\bar{\epsilon}\Gamma_{NP}\langle D_M\Psi, v^P\rangle + i\bar{\epsilon}\Gamma_{MN}\langle D_P\Psi, v^P\rangle$$

⁴The hat notation is used just to distinguish say Γ^1 from $\hat{\Gamma}^1$. We let $M = 0, 1, \dots, 5$ be the spacetime vector index and $A = 1, \dots, 5$ be the R-symmetry index.

On the other hand, the variation of A_M implies the variation $\delta F_{MN} = 2D_M \delta A_N$, which is given by

$$\delta F_{MN} = 2i\bar{\epsilon}\Gamma_{NP}\langle D_M\Psi, v^P\rangle + 2i\bar{\epsilon}\Gamma_{NP}\langle \Psi, D_M v^P\rangle$$

Consistency requires the two expressions agree, which is the case only if

$$\mathcal{L}_v(\bar{\epsilon}\Gamma_{MN}\psi) = 0$$

Here this Lie derivative is given by

$$\mathcal{L}_v(\bar{\epsilon}\Gamma_{MN}\psi) = \langle v^P, \bar{\epsilon}\Gamma_{MN}D_P\psi\rangle + \langle D_M v^P, \bar{\epsilon}\Gamma_{PN}\psi\rangle + \langle D_N v^P, \bar{\epsilon}\Gamma_{MP}\psi\rangle$$

To close these supersymmetry variations on-shell, we need the following constraints

$$\begin{aligned} F_{MN} &= \langle H_{MNP}, v^P\rangle \\ \langle \mathcal{L}_v, \text{any field in the tensor multiplet}\rangle &= 0 \\ \langle v^M, v^N\rangle &= 0 \end{aligned}$$

Then these supersymmetry variations close on-shell. In particular we note that the only constraint we need to impose on v^M is the Abelian constraint. As was noted in [14], there are two ways we can solve the second constraint. If the 3-algebra is nontrivial, that is, if the gauge group is non-Abelian, then we need the Lie derivative of all the fields to vanish. But if the 3-algebra is trivial so that all 3-brackets vanish and the gauge group is Abelian, then we do not need any Lie derivatives to vanish and we get back the usual Abelian $(2,0)$ tensor multiplet. So the Abelian case is included in this formulation, which is a nice property to have for a theory that should be a generalization of the Abelian theory.

We have the following fermionic equation of motion,

$$\Gamma^M D_M \Psi + \Gamma^M \hat{\Gamma}^A[\Psi, v_M, \phi^A] = 0$$

that can be derived from the Lagrangian

$$\mathcal{L}_\Psi = \frac{i}{2}\bar{\Psi} \cdot \Gamma^M D_M \Psi + \frac{i}{2}\bar{\Psi} \cdot \Gamma^M \hat{\Gamma}^A[\Psi, v_M, \phi^A] \quad (3.1)$$

3.1 Reducing down to $(1,0)$ supersymmetry

To reduce down to $(1,0)$ supersymmetry we let the supersymmetry parameter be subject to the Weyl projection [24]

$$\hat{\Gamma}\epsilon = -\epsilon$$

where $\hat{\Gamma} = \hat{\Gamma}^{1234}$. If we assume a sign convention such that $\hat{\Gamma}^5 = \Gamma\hat{\Gamma}$, then

$$\hat{\Gamma}^5\epsilon = \epsilon \quad (3.2)$$

The $(2,0)$ tensor multiplet separates into one $(1,0)$ tensor multiplet and one $(1,0)$ hypermultiplet. For the tensor multiplet fermions ψ , we have

$$\begin{aligned} \hat{\Gamma}\psi &= -\psi \\ \hat{\Gamma}^5\psi &= -\psi \end{aligned}$$

and for the hypermultiplet fermions χ , we have

$$\begin{aligned}\widehat{\Gamma}\chi &= \chi \\ \widehat{\Gamma}^5\chi &= \chi\end{aligned}$$

A consequence of this Weyl projection is the $\text{SO}(5)_R$ symmetry is broken to $\text{SU}(2)_R \times \text{SU}(2)_F$. We note that $\widehat{\Gamma}^{ij}$ acting on ψ belongs to the same $\text{SU}(2)$ as when it acts on ϵ since both ψ and ϵ have the same $\widehat{\Gamma}$ chirality, so this is in $\text{SU}(2)_R$, the R-symmetry. Whereas $\widehat{\Gamma}^{ij}$ when acting on χ is an element in the flavor symmetry $\text{SU}(2)_F$. Thus $\text{SO}(5)_R$ R-symmetry is reduced to $\text{SU}(2)_R$ R-symmetry, and in addition there is a global $\text{SU}(2)_F$ symmetry. We split the index $A = \{i, 5\}$ where $i = 1, 2, 3, 4$ and we will use the notation $\sigma = \phi^5$. The supersymmetry variations are, for the tensor multiplet,

$$\begin{aligned}\delta\sigma &= -i\bar{\epsilon}\psi \\ \delta\psi &= \frac{1}{12}\Gamma^{MNP}\epsilon H_{MNP} + \Gamma^M\epsilon D_M\sigma - \frac{1}{2}\Gamma^M\widehat{\Gamma}^{ij}\epsilon[\phi^i, \phi^j, v_M] \\ \delta H_{MNP} &= 3i\bar{\epsilon}\Gamma_{NP}D_M\psi - i\bar{\epsilon}\Gamma_{MNPQ}[\psi, v^Q, \sigma] \\ \delta A_M &= i\bar{\epsilon}\Gamma_{MN}\langle\psi, v^N\rangle\end{aligned}$$

and for the hypermultiplet,

$$\begin{aligned}\delta\phi^i &= i\bar{\epsilon}\widehat{\Gamma}^i\chi \\ \delta\chi &= \Gamma^M\widehat{\Gamma}^i\epsilon D_M\phi^i - \Gamma^M\widehat{\Gamma}^i\epsilon[\phi^i, \sigma, v_M]\end{aligned}$$

We get the fermionic equations of motion

$$\begin{aligned}\Gamma^M D_M\psi + \Gamma^M\widehat{\Gamma}^i[\chi, v_M, \phi^i] - \Gamma^M[\psi, v_M, \sigma] &= 0 \\ \Gamma^M D_M\chi + \Gamma^M\widehat{\Gamma}^i[\psi, v_M, \phi^i] + \Gamma^M[\chi, v_M, \sigma] &= 0\end{aligned}$$

which can be integrated up to the Lagrangian

$$\begin{aligned}\mathcal{L}_\Psi &= \frac{i}{2}\bar{\chi} \cdot \Gamma^M D_M\chi + \frac{i}{2}\bar{\chi} \cdot \Gamma^M[\chi, v_M, \sigma] \\ &\quad + \frac{i}{2}\bar{\psi} \cdot \Gamma^M D_M\psi - \frac{i}{2}\bar{\psi} \cdot \Gamma^M[\psi, v_M, \sigma] \\ &\quad + i\bar{\psi} \cdot \Gamma^M\widehat{\Gamma}^i[\chi, v_M, \phi^i]\end{aligned}$$

The last term is a coupling term between tensor and hypermultiplet fermions, which makes it hard to separate this Lagrangian into one Lagrangian for the tensor multiplet and another for the hypermultiplet.

4 Superconformal symmetry

We now ask ourselves if the theories of [14], [24] are superconformal at the classical level. Our starting point is to assume that we have a supersymmetry parameter that is a conformal Killing spinor ϵ_f satisfying

$$D_M\epsilon_f = \Gamma_M\eta \tag{4.1}$$

Since we do not have a formulation in terms of a 2-form gauge potential B_{MN} , it is not apriori clear how we shall generalize the supersymmetry variation for H_{MNP} in [14], [24] to the case when ϵ_f is not a constant spinor. We will make the following ansatz for the $(1,0)$ tensor multiplet,

$$\begin{aligned}\delta H_{MNP} &= 3i\bar{\epsilon}_f \Gamma_{[NP} D_M] \psi - 3i\bar{\eta} \Gamma_{MNP} \psi - i\bar{\epsilon}_f \Gamma_{MNPQ} [\psi, v^Q, \sigma] \\ \delta A_M &= i\bar{\epsilon}_f \Gamma_{MN} \langle \psi, v^N \rangle \\ \delta \sigma &= -i\bar{\epsilon}_f \psi \\ \delta \psi &= \frac{1}{12} \Gamma^{MNP} \epsilon_f H_{MNP} + \Gamma^M \epsilon_f D_M \sigma + 4\eta \sigma\end{aligned}\tag{4.2}$$

Here a coupling term to the hypermultiplet has been put to zero for simplicity. This ansatz for the supersymmetry variations is consistent with the relation⁵

$$\delta F_{MN} = \langle \delta H_{MNP}, v^P \rangle + i \langle \mathcal{L}_v, \bar{\epsilon} \Gamma_{MN} \psi \rangle\tag{4.3}$$

where

$$\langle \mathcal{L}_v, \bar{\epsilon} \Gamma_{MN} \psi \rangle := \langle v^P, D_P (\bar{\epsilon} \Gamma_{MN} \psi) \rangle + \langle D_M v^P, \bar{\epsilon} \Gamma_{PN} \psi \rangle + \langle D_N v^P, \bar{\epsilon} \Gamma_{MP} \psi \rangle$$

Thus, in order for the ansatz to be compatible with the constraint (2.2), we need to impose the constraints

$$\begin{aligned}\langle \mathcal{L}_v, \psi \rangle &= 0 \\ \mathcal{L}_v \epsilon_f &= 0\end{aligned}$$

where

$$\begin{aligned}\langle \mathcal{L}_v, \psi \rangle &= \langle v^P, D_P \psi \rangle + \frac{1}{4} \langle D_M v_N, \Gamma^{MN} \psi \rangle \\ \mathcal{L}_v \epsilon_f &= v^P D_P \epsilon_f + \frac{1}{4} D_M v_N \Gamma^{MN} \epsilon_f\end{aligned}$$

In flat $\mathbb{R}^{1,5}$ we can solve the conformal Killing spinor equation (4.1). The solution is given by

$$\epsilon_f = \epsilon + \Gamma_M \eta x^M$$

where ϵ gives the Poincare supersymmetry and η gives the special conformal supersymmetry. In this paper, we will assume that ϵ_f and all associated supersymmetry parameters (such as ϵ and η) are commuting spinors. This means that the variation δ_{ϵ_f} is anticommuting. When closing the supersymmetry variations, it is then enough to just compute $\delta_{\epsilon_f}^2$, and this will provide for us the most general supersymmetry closure relations. Let us spend some lines on explaining this point in some detail here. If we consider the perhaps

⁵To see this, we use the gamma matrix identity

$$3\Gamma_{MNP} = 2\Gamma_{[M} \Gamma_{N]P} + \Gamma_P \Gamma_{MN}.$$

more familiar situation with say two anticommuting Poincare supersymmetry parameters ϵ and ϵ' , then we will compute the commutator $[\delta_\epsilon, \delta_{\epsilon'}]$ in order to check the most general closure of these supersymmetry variations. If we instead have commuting supersymmetry parameters, we should correspondingly compute the anticommutator $\{\delta_\epsilon, \delta_{\epsilon'}\}$ between two different supersymmetry variations, and not just the square δ_ϵ^2 of some supersymmetry variation. But as long as the supersymmetry variations are linear in the supersymmetry parameter, we may indeed equally well just compute the squares of supersymmetry variations. The reason why this will be sufficient, is because of linearity

$$\delta_{\epsilon+\epsilon'} = \delta_\epsilon + \delta_{\epsilon'}$$

and because we have the identity

$$\{\delta_\epsilon, \delta_{\epsilon'}\} = \delta_{\epsilon+\epsilon'}^2 - \delta_\epsilon^2 - \delta_{\epsilon'}^2 \quad (4.4)$$

where on the right-hand side only appears perfect squares. So by just computing δ_ϵ^2 for a general parameter ϵ , we can extract the most general anticommutator closure relations by using (4.4).

Inspired by the closure computation for 3d superconformal transformations in [23], we will perform the same type of closure computation here for our 6d (1,0) theory. We will find that the above supersymmetry transformations close into the generators of the conformal group up to a gauge transformation,

$$\begin{aligned} \delta A_M &= D_M \Lambda \\ \delta \sigma &= -(\Lambda, \sigma) \\ \delta \psi &= -(\Lambda, \psi) \end{aligned}$$

But the closure relation is not very clear to us for a generic 6d manifold, so in the end we specialize to flat $\mathbb{R}^{1,5}$ where interpretations in terms of generators of the superconformal algebra is clear.

Let us introduce the quantity

$$S^M := \bar{\epsilon}_f \Gamma^M \epsilon_f \quad (4.5)$$

By using (4.1) we get the derivatives

$$\begin{aligned} D_M S_N &= 2\bar{\epsilon}_f \eta g_{MN} - 2\bar{\epsilon}_f \Gamma_{MN} \eta \\ D^M S_M &= 12\bar{\epsilon}_f \eta \end{aligned}$$

In flat $\mathbb{R}^{1,5}$ we have the expansion

$$S^M \mathcal{P}_M = \bar{\epsilon} \Gamma^M \epsilon \mathcal{P}_M + 2\bar{\epsilon} \eta \mathcal{D} + \bar{\epsilon} \Gamma^{MN} \eta \mathcal{L}_{MN} - \bar{\eta} \Gamma^M \eta \mathcal{K}_M$$

where we define

$$\begin{aligned} \mathcal{P}_M &= -i\partial_M \\ \mathcal{D} &= -ix^M \partial_M \\ \mathcal{L}_{MN} &= i(x_M \partial_N - x_N \partial_M) \\ \mathcal{K}_M &= i(-2x_M x^N \partial_N + |x|^2 \partial_M) \end{aligned}$$

As we will show below, we find the following closure relation when we act on any of the fields in the tensor multiplet in flat space,

$$\delta^2 = \bar{\epsilon}\Gamma^M \epsilon P_M + 2\bar{\epsilon}\eta D + \bar{\epsilon}\Gamma^{MN} \eta L_{MN} - \bar{\eta}\Gamma^M \eta K_M - 2\bar{\epsilon}\hat{\Gamma}^{ij} \eta S^{ij}$$

where the generators of the conformal group are given by [26]

$$\begin{aligned} P_M &= \mathcal{P}_M \\ D &= \mathcal{D} - i\Delta \\ L_{MN} &= \mathcal{L}_{MN} + S_{MN} \\ K_M &= \mathcal{K}_M - 2i\Delta x_M - S_{MN}x^N \end{aligned}$$

Here Δ is the scaling dimension of the field, and S_{MN} is the spin part of the Lorentz rotation, which we normalize such that

$$\begin{aligned} S_{MN} &= \frac{i}{2}\Gamma_{MN} \\ S_{MN}^{PQ} &= 2i\delta_{MN}^{PQ} \end{aligned}$$

when acting on a spinor and a vector field respectively. The last generator S^{ij} is a generator for the R-symmetry $SU(2)_R$ and it acts nontrivially only on the tensor multiplet fermions and the hypermultiplet scalars.

We will view v^M as a background field, perhaps a bit similar to the background metric tensor field g_{MN} . In supersymmetric field theory, as opposed to supergravity, we normally do not include g_{MN} in the supermultiplet and demand its supersymmetry variation is vanishing,

$$\delta g_{MN} = 0$$

But let us assume that we do this anyway, just to see where this can lead us. Then we shall require supersymmetric closure on the metric tensor,

$$\delta^2 g_{MN} = -i\mathcal{L}_S g_{MN}$$

We then reach the conclusion that S^M shall be a Killing vector in order for supersymmetry closure,

$$\mathcal{L}_S g_{MN} = 0 \tag{4.6}$$

which shows that extension to superconformal symmetry where S^M is a conformal Killing vector,

$$\mathcal{L}_S g_{MN} = \frac{1}{3}g_{MN}D^P S_P \tag{4.7}$$

is not possible. We note that (4.1) implies (4.7) but not necessarily (4.6) which is too restrictive. Similarly, if we include v^M in the supersymmetry variations, we can not generalize Poincare supersymmetry to the full superconformal symmetry. The closure relation we would need to require is given by

$$\delta^2 v_M = -i\mathcal{L}_S v_M$$

and what we get from assuming the supersymmetry variation

$$\delta v_M = 0$$

is $\delta^2 v_M = 0$ and so we would have to require that the Lie derivative vanishes,

$$S^N D_N v_M + (D_M S^N) v_N = 0$$

If we expand this out on $\mathbb{R}^{1,5}$, we get

$$\bar{\epsilon} \Gamma^N \epsilon D_N v_M + 2\bar{\epsilon} \eta (-ix^N D_N + 1) v_M + \dots = 0$$

Since $\bar{\epsilon} \Gamma^N \epsilon$, $\bar{\epsilon} \eta$, $\dots$ are all independent, each term must vanish separately. The vanishing of the first term leads to the constraint

$$D_N v_M = 0$$

which is a constraint that was found in [17]. The vanishing of the second term leads to $\eta = 0$. That is, only the Poincare supersymmetry can be realized.

Since we want the tensor multiplet to be superconformal, we are led to assume that v^M shall be a background field that does not belong to the tensor multiplet.

We will now verify that we have the appropriate closure relations when we act on anyone of the fields in the tensor multiplet. But this check does not include the vector field v^M , which therefore requires a separate treatment.

Closure on the scalar field. For the scalar field σ we get

$$\delta^2 \sigma = -i S^M D_M \sigma - \frac{i}{3} (D^M S_M) \sigma$$

We can also write this in the form

$$\delta^2 \sigma = -i S^M \partial_M \sigma - \frac{i}{3} (D^M S_M) \sigma - (\Lambda, \sigma)$$

where

$$\Lambda = i S^M (A_M + \langle v_M, \sigma \rangle)$$

Expanding this out, we get

$$\begin{aligned} \delta^2 \sigma &= \bar{\epsilon} \Gamma^M \epsilon \mathcal{P}_M \sigma \\ &+ 2\bar{\epsilon} \eta (\mathcal{D} - 2i) \sigma \\ &+ \bar{\epsilon} \Gamma^{PQ} \eta \mathcal{L}_{PQ} \sigma \\ &- \bar{\eta} \Gamma^P \eta (\mathcal{K}_P - 4ix_P) \sigma \end{aligned}$$

Thus we get closure, and the second line is telling us that the scaling dimension of σ is $\Delta = 2$.

Closure on the gauge field. For the gauge field A_M we get

$$\begin{aligned}\delta^2 A_M &= -iS^R \langle H_{RMN}, v^N \rangle + D_M \Lambda' \\ &\quad + \frac{i}{2} S^R \langle H_{RMN}^-, v^N \rangle \\ &\quad + iS_M \langle \mathcal{L}_v, \sigma \rangle \\ &\quad + i \langle \sigma, \mathcal{L}_v S_M \rangle\end{aligned}$$

where

$$\begin{aligned}\langle \mathcal{L}_v, \sigma \rangle &:= \langle D_N \sigma, v^N \rangle \\ \mathcal{L}_v S_M &:= v^N D_N S_M + (D_M v^N) S_N\end{aligned}$$

and

$$\Lambda' = iS^M \langle v_M, \sigma \rangle$$

Selfduality of H_{MNP}^+ is connected with Weyl projection of ϵ_f , such that

$$\Gamma^{MNP} \epsilon H_{MNP}^- = 0$$

With our convention,

$$H_{MNP}^\pm = \frac{1}{2} \left(H_{MNP} \pm \frac{1}{6} \epsilon_{MNP}{}^{RST} H_{RST} \right)$$

To get the closure relation, we have also noted that

$$2\bar{\epsilon}_f \Gamma_{MN} \eta = -D_M S_N + D_N S_M$$

We thus find the closure relation

$$\delta^2 A_M = -iS^R F_{RM} + D_M \Lambda'$$

if we impose the constraints

$$\begin{aligned}H_{MNP}^- &= 0 \\ \langle \mathcal{L}_v, \sigma \rangle &= 0 \\ \mathcal{L}_v S_M &= 0\end{aligned}$$

We can write this as

$$\delta^2 A_M = -i\mathcal{L}_S A_M + D_M \Lambda$$

where the gauge parameter is the same as before when we closed supersymmetry on σ ,

$$\Lambda = iS^M (A_M + \langle v_M, \sigma \rangle)$$

and the Lie derivative is given by

$$\mathcal{L}_S A_M = S^N \partial_N A_M + (\partial_M S^N) A_N$$

If we expand out this Lie derivative explicitly, we get

$$\begin{aligned}
 -i\mathcal{L}_S A_M &= \bar{\epsilon}\Gamma^P \epsilon \mathcal{P}_P A_M \\
 &\quad + 2\bar{\epsilon}\eta (\mathcal{D} - i) A_M \\
 &\quad + \bar{\epsilon}\Gamma^{PQ} \eta (\mathcal{L}_{PQ} \delta_M^N + (S_{PQ})_{MN}) A_N \\
 &\quad - \bar{\eta}\Gamma^P \eta (\mathcal{K}_P \delta_M^N - 2(S_{PQ})_M{}^N - 2ix_P \delta_M^N) A_N
 \end{aligned}$$

Thus we closure and the second line is telling us that the scaling dimension of A_M is $\Delta = 1$.

Closure on the fermions. For the fermions ψ we get

$$\begin{aligned}
 \delta^2 \psi &= i \left(\frac{1}{4} \Gamma^{MNP} \epsilon_f \bar{\epsilon}_f \Gamma_{NP} - \Gamma^M \epsilon_f \bar{\epsilon}_f \right) D_M \psi \\
 &\quad + i \left(-\frac{1}{4} \Gamma^{MNP} \epsilon_f \bar{\eta} \Gamma_{MNP} + \Gamma^M \epsilon_f \bar{\eta} \Gamma_M - 4\eta \bar{\epsilon}_f \right) \psi \\
 &\quad + i \left(-\frac{1}{12} \Gamma^{MNP} \epsilon_f \bar{\epsilon}_f \Gamma_{MNPQ} + \Gamma^M \epsilon_f \bar{\epsilon}_f \Gamma_{MQ} \right) [\psi, v^Q, \sigma]
 \end{aligned}$$

We have the following Fierz identities,

$$\begin{aligned}
 \epsilon_f \bar{\epsilon}_f &= \frac{1}{4} (1 - \Gamma)(1 - \hat{\Gamma}) \left(c^M \Gamma_M + c^{MNP,ij} \Gamma_{MNP} \hat{\Gamma}^{ij} \right) \\
 \epsilon_f \bar{\eta} &= \frac{1}{4} (1 - \Gamma)(1 - \hat{\Gamma}) \left(c + c^{MN} \Gamma_{MN} + c^{ij} \hat{\Gamma}^{ij} + c^{MN,ij} \Gamma_{MN} \hat{\Gamma}^{ij} \right) \\
 \eta \bar{\epsilon}_f &= \frac{1}{4} (1 + \Gamma)(1 - \hat{\Gamma}) \left(-c + c^{MN} \Gamma_{MN} + c^{ij} \hat{\Gamma}^{ij} - c^{MN,ij} \Gamma_{MN} \hat{\Gamma}^{ij} \right)
 \end{aligned}$$

where

$$\begin{aligned}
 c^M &= \frac{1}{8} \bar{\epsilon}_f \Gamma^M \epsilon_f \\
 c^{MNP,ij} &= \frac{1}{96} \bar{\epsilon}_f \Gamma^{RST} \hat{\Gamma}^{ij} \epsilon_f \\
 c &= -\frac{1}{8} \bar{\epsilon}_f \eta \\
 c_+^{ij} &= -\frac{1}{32} \bar{\epsilon}_f \hat{\Gamma}^{ij} \eta \\
 c^{MN} &= -\frac{1}{16} \bar{\epsilon}_f \Gamma^{MN} \eta \\
 c^{MN,ij} &= -\frac{1}{64} \bar{\epsilon}_f \Gamma^{MN} \hat{\Gamma}^{ij} \eta
 \end{aligned}$$

We have indicated with a subscript $+$ the fact that c_+^{ij} is selfdual. We separate $\delta^2 \psi$ into three parts,

$$\begin{aligned}
 (\delta^2 \psi)_{\text{abel}} &= -8ic^M D_M \psi + 2ic^Q \Gamma_Q \Gamma^M D_M \psi \\
 (\delta^2 \psi)_{\text{curv}} &= \left(40ic - 8i\Gamma_{MN} c^{MN} + 32ic^{ij} \hat{\Gamma}^{ij} \right) \psi \\
 (\delta^2 \psi)_{\text{comm}} &= - \left(8ic^M + 2ic^R \Gamma_R \Gamma^M \right) [\psi, v_M, \sigma]
 \end{aligned}$$

Let us first rewrite

$$(\delta^2\psi)_{\text{abel}} = -8ic^M \partial_M \psi + 2ic^Q \Gamma_Q \Gamma^M D_M \psi - (\Lambda'', \psi)$$

where

$$\Lambda'' = 8ic^M A_M$$

Then we expand out the sum of the three terms in components to get

$$\begin{aligned} \delta^2\psi &= \bar{\epsilon}\Gamma^P \epsilon \mathcal{P}_P \psi \\ &+ 2\bar{\epsilon}\eta \left(\mathcal{D} - \frac{5i}{2} \right) \psi \\ &+ \bar{\epsilon}\Gamma^{PQ} \eta (\mathcal{L}_{PQ} + S_{PQ}) \psi \\ &- \bar{\eta}\Gamma^P \eta (\mathcal{K}_P - 2S_{PQ}x^Q - 5ix_P) \psi \\ &- 2\bar{\epsilon}\hat{\Gamma}^{ij} \eta S^{ij} \psi \\ &- (\Lambda, \psi) \\ &+ 2ic^Q \Gamma_Q (\Gamma^M D_M \psi - \Gamma^M [\psi, v_M, \sigma]) \end{aligned}$$

where the gauge parameter is

$$\Lambda = iS^M (A_M + \langle v_M, \sigma \rangle)$$

Thus we have closure and the second line is telling us that the scaling dimension of ψ is $\Delta = 5/2$.

Closure on the tensor field. For the tensor field H_{MNP} we get

$$(\delta^2 H_{MNP})_{\text{abel}} = \frac{i}{4} D_M (\bar{\epsilon}_f \Gamma_{NP} \Gamma^{RST} \epsilon_f H_{RST}) \quad (4.8)$$

$$+ 3i\bar{\epsilon}_f \Gamma_{NP} \Gamma^R \epsilon_f D_M D_R \sigma \quad (4.9)$$

$$+ 12i\bar{\epsilon}_f \Gamma_{NP} \eta D_M \sigma \quad (4.10)$$

$$(\delta^2 H_{MNP})_{\text{curv}} = 3i\bar{\epsilon}_f (\Gamma_{NP} \Gamma^R \Gamma_M - \Gamma^R \Gamma_{MNP}) \eta D_R \sigma \quad (4.11)$$

$$(\delta^2 H_{MNP})_{\text{comm}} = -\frac{i}{12} \bar{\epsilon}_f \Gamma_{MNPQ} \Gamma^{RST} \epsilon_f [H_{RST}, v^Q, \sigma] \quad (4.12)$$

$$- i\bar{\epsilon}_f \Gamma_{MNPQ} \Gamma^R \epsilon_f [D_R \sigma, v^Q, \sigma] \quad (4.13)$$

$$- 3[\bar{\epsilon}_f \Gamma_{NP} \psi, \bar{\epsilon}_f \Gamma_{MQ} \psi, v^Q] \quad (4.14)$$

We use

$$\Gamma_{NP} \Gamma^R \Gamma_M = -\frac{2}{3} \Gamma_{MNP} \Gamma^R + \frac{1}{3} \Gamma^R \Gamma_{MNP}$$

to rewrite

$$(\delta^2 H_{MNP})_{\text{curv}} = -12i\bar{\epsilon}_f \Gamma_{MN} \eta D_P \sigma$$

which then cancels against (4.10). We next look at the commutator terms. We have

$$\bar{\epsilon}_f \Gamma_{MNPQ} \Gamma^{RST} \epsilon_f = 12\bar{\epsilon}_f \left(\delta_{MNP}^{RST} \Gamma_Q - 3\delta_{Q[MN}^{RST} \Gamma_{P]} \right) \epsilon_f$$

This then enables us to rewrite (4.12) as

$$-(\Lambda'', H_{MNP}) + 3iS_{[P}[H_{MN]Q}, v^Q, \sigma] \quad (4.15)$$

where

$$\Lambda'' = iS^M \langle v_M, \sigma \rangle$$

Let us return to the abelian type of terms,

$$\begin{aligned} (\delta^2 H_{MNP})_{\text{abel}} = & -i(S^Q D_Q H_{MNP} + 3(D_M S^Q) H_{NPQ}) \\ & - 4iS^Q D_{[M} H_{NPQ]} \\ & - 3iS_{[P}(F_{MN]}, \sigma) \end{aligned}$$

The last term cancels against (4.15) if we assume that

$$F_{MN} = \langle H_{MNP}, v^P \rangle$$

The remaining terms in $(\delta^2 H_{MNP})_{\text{comm}}$ cancel by modifying the Bianchi identity [14, 24]. Let us now summarize the closure relation that we have got,

$$\delta^2 H_{MNP} = -i(S^Q D_Q H_{MNP} + 3(D_M S^Q) H_{NPQ}) - (\Lambda'', H_{MNP})$$

We can now extract the Lie derivative,

$$\delta^2 H_{MNP} = -i\mathcal{L}_S H_{MNP} - (\Lambda, H_{MNP})$$

where

$$\begin{aligned} \mathcal{L}_S H_{MNP} &= S^Q \nabla_Q H_{MNP} + 3(\nabla_M S^Q) H_{NPQ} \\ &= S^Q \partial_Q H_{MNP} + 3(\partial_M S^Q) H_{NPQ} \end{aligned}$$

where we have used the fact that $D_M S^Q = \nabla_M S^Q$ since S^Q is a gauge singlet, so only the Christoffel symbol in D_M acts on it nontrivially. We thus decompose $D_M = \nabla_M + A_M$.

If we finally expand out the Lie derivative, we can see the conformal generators and the scaling dimension explicitly,

$$\begin{aligned} -i\mathcal{L}_S H_{MNP} = & \bar{\epsilon} \Gamma^Q \epsilon \mathcal{P}_Q H_{MNP} \\ & + 2\bar{\epsilon} \eta (\mathcal{D} - 3i) H_{MNP} \\ & + \bar{\epsilon} \Gamma^{PQ} \eta (\mathcal{L}_{PQ} + S_{PQ}) H_{MNP} \\ & - \bar{\eta} \Gamma^P \eta (\mathcal{K}_P - 2S_{PQ} x^Q - 6ix_P) H_{MNP} \end{aligned}$$

Thus we have closure and the second line is telling us that the scaling dimension of H_{MNP} is $\Delta = 3$.

This finishes our on-shell closure computation for the $(1, 0)$ tensor multiplet. It shows that the vector field v^M shall be constrained by two relations,

$$\langle v^M, v^N \rangle = 0 \quad (4.16)$$

$$\mathcal{L}_v \epsilon_f = 0 \quad (4.17)$$

in order to have superconformal symmetry.

Let us now return to our vector field v^M . Since scaling dimension for H_{MNP} is $\Delta = 3$ and for F_{MN} it is $\Delta = 2$, in order for the constraint $F_{MN} = \langle H_{MNP}, v^P \rangle$ to transform covariantly under conformal transformations, we shall assign v^M scaling dimension $\Delta = -1$ and spin one.

5 Off-shell supersymmetry for the 6d hypermultiplet

Off-shell supersymmetry for 10d SYM was first found in [2] and later this was used in [4] for localization computations. There are 9 off-shell components for the gauge potential after using the Gauss law constraint, 16 off-shell components for the spinor. So we need 7 auxiliary bosonic field components in order to make the number of off-shell bosonic degrees of freedom match with the number of off-shell fermionic degrees of freedom. If we dimensionally reduce to 5d we expect to find an off-shell tensor multiplet (with 3 auxiliary fields) coupled to an off-shell hypermultiplet (with 4 off-shell fields). But instead of following such an approach, there was a direct construction of an off-shell 5d tensor multiplet coupled to an off-shell 5d hypermultiplet appearing in [20].

One may ask whether there exists a 6d uplift of this off-shell supersymmetry. For the 6d hypermultiplet we can have off-shell supersymmetry since we can get this multiplet by reducing 10d $\mathcal{N} = 1$ SYM down to $\mathcal{N} = (1, 1)$ SYM in 6d [27]. Reducing supersymmetry further down to $\mathcal{N} = (1, 0)$, we get a 6d vector multiplet coupled to a 6d hypermultiplet. So in 6d we can have an off-shell hypermultiplet as well as an off-shell vector multiplet, but by such a construction, none of these multiplets will be conformal. The 6d vector multiplet can not be conformal. But as we will show, the 6d hypermultiplet can become superconformal. This is intuitively clear because not only can we get the $\mathcal{N} = (1, 0)$ hypermultiplet from reducing the nonconformal $\mathcal{N} = (1, 1)$ 6d vector multiplet down to $\mathcal{N} = (1, 0)$, but also from the reducing the superconformal 6d $\mathcal{N} = (2, 0)$ tensor multiplet down to $\mathcal{N} = (1, 0)$.

5.1 Abelian gauge group and Poincare supersymmetry

We make the following ansatz for the off-shell supersymmetry variations,

$$\begin{aligned}\delta\phi^i &= i\bar{\epsilon}\hat{\Gamma}^i\chi \\ \delta\chi &= \Gamma^M\hat{\Gamma}^i\epsilon\partial_M\phi^i + \sum_{\alpha=1}^n\nu^\alpha F^\alpha \\ \delta F^\alpha &= -i\bar{\nu}^\alpha\Gamma^M\partial_M\chi\end{aligned}$$

From now, we will suppress the summation symbol when α is contracted, and always assume Einstein summation convention. For closure on ϕ^i , we compute

$$\delta^2\phi^i = -i\bar{\epsilon}\hat{\Gamma}^M\epsilon\partial_M\phi^i + i\bar{\epsilon}\hat{\Gamma}^i\nu^\alpha F^\alpha$$

Thus closure on ϕ^i requires that

$$\bar{\epsilon}\hat{\Gamma}^i\nu^\alpha = 0 \tag{5.1}$$

Next,

$$\delta^2 F^\alpha = -i\bar{\nu}^\alpha \Gamma^M \nu^\beta \partial_M F^\beta$$

Thus closure on F^α requires in addition that

$$\bar{\nu}^\alpha \Gamma^M \nu^\beta = \delta^{\alpha\beta} \bar{\epsilon} \Gamma^M \epsilon \quad (5.2)$$

Finally, for the fermion, we get

$$\delta^2 \chi = i \left(\Gamma^M \hat{\Gamma}^i \bar{\epsilon} \hat{\Gamma}^i - \nu^\alpha \bar{\nu}^\alpha \Gamma^M \right) \partial_M \chi$$

Using Fierz identities and assuming (5.6) holds, we get

$$\delta^2 \chi = i \frac{S^Q}{8} (-4\Gamma^M \Gamma^Q - n\Gamma^Q \Gamma^M) \partial_M \chi$$

and thus by taking $n = 4$ we get off-shell closure

$$\delta^2 \chi = -iS^Q \partial_Q \chi$$

but we also need to satisfy the constraints

$$\bar{\nu}^\alpha \Gamma^{MNP} \hat{\Gamma}^{ij} \nu^\alpha = 0 \quad (5.3)$$

We have found the constraints

$$\bar{\epsilon} \hat{\Gamma}^i \nu^\alpha = 0 \quad (5.4)$$

$$\bar{\nu}^\alpha \Gamma^M \nu^\beta = \delta^{\alpha\beta} \bar{\epsilon} \Gamma^M \epsilon \quad (5.5)$$

$$\bar{\nu}^\alpha \Gamma^{MNP} \hat{\Gamma}^{ij} \nu^\alpha = 0 \quad (5.6)$$

Since we need $n = 4$, we can try to make the ansatz

$$\nu^i = \hat{\Gamma}^i \nu$$

in order to solve them. The index α has now been identified as the index i . Plugging in this ansatz, we automatically solve (5.6) since $\hat{\Gamma}^i \hat{\Gamma}^{kl} \hat{\Gamma}^i = 0$, and the constraints reduce to

$$\bar{\epsilon} \nu = 0$$

$$\bar{\epsilon} \hat{\Gamma}^{ij} \nu = 0$$

$$\bar{\epsilon} \Gamma^M \epsilon = \bar{\nu} \Gamma^M \nu$$

To understand how the last constraint appears, we compute

$$\begin{aligned} \bar{\nu}^i \Gamma^M \nu^j &= -\bar{\nu} \hat{\Gamma}^i \Gamma^M \hat{\Gamma}^j \nu \\ &= \delta^{ij} \bar{\nu} \Gamma^M \nu \end{aligned}$$

Up to a minus sign that depends on the convention we use for the charge conjugation matrix, these are now the same constraints,⁶ as those that appeared in 5d in [20] and so these constraints can be solved as it was shown there.

⁶More precisely, they are equivalent to, but we have not yet written them in the same form as they appeared in [20]. For now, let us just note that we have 1 + 3 constraints $\bar{\epsilon} \nu = 0$, $\bar{\epsilon} \hat{\Gamma}^{ij} \nu = 0$ while [20] has 2×2 constraints of the form $\bar{\epsilon}_{IJ} \nu_J = 0$. We present all details how to convert between the two languages below.

5.2 Abelian gauge group and superconformal symmetry

We make the ansatz

$$\begin{aligned}\delta\phi^i &= i\bar{\epsilon}_f \hat{\Gamma}^i \chi \\ \delta\chi &= \Gamma^M \hat{\Gamma}^i \epsilon_f \partial_M \phi^i + \nu_f^\alpha F^\alpha - 4\hat{\Gamma}^i \eta \phi^i \\ \delta F^\alpha &= -i\bar{\nu}^\alpha \Gamma^M D_M \chi\end{aligned}$$

where we assume that

$$\begin{aligned}D_M \epsilon_f &= \Gamma_M \eta \\ D_M \nu_f &= \Gamma_M \rho\end{aligned}$$

In flat space we solve these equations as

$$\begin{aligned}\epsilon_f &= \epsilon + \Gamma_M \eta x^M \\ \nu_f &= \nu + \Gamma_M \rho x^M\end{aligned}$$

We now proceed to check the closure relations. For the scalars we get

$$\delta^2 \phi^i = -iS^M \partial_M \phi^i - 4i\bar{\epsilon}_f \eta \phi^i - 4i\bar{\epsilon}_f \hat{\Gamma}^{ij} \eta \phi^j$$

provided that we satisfy the constraint

$$\bar{\epsilon}_f \hat{\Gamma}^i \nu_f^\alpha = 0 \tag{5.7}$$

In flat space the closure relation becomes

$$\begin{aligned}\delta^2 \phi^i &= \bar{\epsilon} \Gamma^M \epsilon \mathcal{P}_M \phi^i \\ &+ 2\bar{\epsilon} \eta (\mathcal{D} - 2i) \phi^i \\ &+ \bar{\epsilon} \Gamma^{PQ} \eta \mathcal{L}_{PQ} \phi^i \\ &- \bar{\eta} \Gamma^P \eta (\mathcal{K}_P - 4ix_P) \phi^i \\ &- 2\bar{\epsilon} \hat{\Gamma}^{kl} \eta (S^{kl})^{ij} \phi^j\end{aligned}$$

Thus we have closure and the second line is telling us that the scaling dimension of ϕ^i is $\Delta = 2$. We note that the last line is the SU(2) R-symmetry rotation with the same coefficient as we found on ψ in the tensor multiplet.

The constraint equations (5.7) are algebraic and can be solved at each point in space by the same argument as we used for the Poincare supersymmetry parameters. In flat space (5.7) decomposes into the following constraints,

$$\begin{aligned}\bar{\epsilon} \hat{\Gamma}^i \nu^\alpha &= 0 \\ \bar{\eta} \hat{\Gamma}^i \Gamma_M \nu &= \bar{\epsilon} \hat{\Gamma}^i \Gamma_M \rho \\ \bar{\eta} \hat{\Gamma}^i \rho &= 0\end{aligned}$$

Next we consider closure on the auxiliary fields,

$$\delta^2 F^\alpha = -i\bar{\nu}_f^\alpha \Gamma^M \Gamma^N \hat{\Gamma}^i \Gamma_M \eta D_N \phi^i \quad (5.8)$$

$$-i\bar{\nu}_f^\alpha \hat{\Gamma}^i \epsilon_f D_M^2 \phi^i \quad (5.9)$$

$$-i\bar{\nu}_f^\alpha \Gamma^M D_M \nu_f^\beta F^\beta \quad (5.10)$$

$$-i\bar{\nu}_f^\alpha \Gamma^M \nu_f^\beta D_M F^\beta \quad (5.11)$$

$$+4i\bar{\nu}_f^\alpha \Gamma^M \hat{\Gamma}^i D_M \eta \phi^i \quad (5.12)$$

$$+4i\bar{\nu}_f^\alpha \Gamma^M \hat{\Gamma}^i \eta D_M \phi^i \quad (5.13)$$

We now analyze each line in turn. By using the identity $\Gamma^M \Gamma^N \Gamma_M = -4\Gamma^N$ we see that (5.8) cancels against (5.13). Line (5.9) vanishes by using the constraint (5.4). We decompose (5.10) in a symmetric and an antisymmetric piece

$$i\bar{\nu}_f^\alpha \Gamma^M D_M \nu_f^\beta F^\beta = i\bar{\nu}_f^{(\alpha} \Gamma^M D_M \nu_f^{\beta)} F^\beta + i\bar{\nu}_f^{[\alpha} \Gamma^M D_M \nu_f^{\beta]} F^\beta$$

and write the symmetric piece as

$$i\bar{\nu}_f^{(\alpha} \Gamma^M D_M \nu_f^{\beta)} F^\beta = -\frac{i}{2} (D^M S_M) F^\alpha$$

The antisymmetric piece gives an internal rotation,

$$i\bar{\nu}_f^{[\alpha} \Gamma^M D_M \nu_f^{\beta]} F^\beta = R^{\alpha\beta} F^\beta$$

in the 4d space of auxiliary fields. While this is a new sort of transformation, it is a symmetry of the theory. In the Lagrangian we shall have the term $F^\alpha F^\alpha$, which is invariant. Eq. (5.11) is the translation term,

$$-iS^M D_M F^\alpha$$

Eq. (5.12) is vanishing by using (5.7). We summarize the result as

$$\delta^2 F^\alpha = -iS^M D_M F^\alpha - 6i\bar{\epsilon}_f \eta F^\alpha + R^{\alpha\beta} F^\beta$$

We can now recognize this as the closure relation that gives us the scaling dimension for F^α as $\Delta = 3$.

We finally consider closure on the fermions

$$\begin{aligned} \delta^2 \chi &= i \left(\Gamma^M \hat{\Gamma}^i \epsilon_f \bar{\epsilon}_f \hat{\Gamma}^i - i\nu_f^\alpha \bar{\nu}_f^\alpha \Gamma^M \right) D_M \chi \\ &\quad - i \left(\Gamma^M \hat{\Gamma}^i \epsilon_f \bar{\eta} \Gamma_M \hat{\Gamma}^i + 4\hat{\Gamma}^i \eta \bar{\epsilon}_f \hat{\Gamma}^i \right) \chi \end{aligned}$$

The first line has exactly the same structure as we had in flat Poincare superspace, we just need to attach subscripts f to the supersymmetry parameters and then we can borrow the flat space result. Let us therefore consider the second line. We use

$$\begin{aligned} \epsilon_f \bar{\eta} &= c + c^{RS} \Gamma_{RS} + \dots \\ \eta \bar{\epsilon}_f &= -c + c^{RS} \Gamma_{RS} + \dots \end{aligned}$$

and we do not need to consider the dots, because of the identity $\widehat{\Gamma}^i \widehat{\Gamma}^{kl} \widehat{\Gamma}^i$. We then get that the second line is equal to

$$-5i\bar{\epsilon}_f \eta \chi + \frac{i}{2} \bar{\epsilon}_f \Gamma_{RS} \eta \Gamma^{RS} \chi$$

which gives the scaling dimension 5/2. We have no contribution from the other terms thanks to the identity $\widehat{\Gamma}^i \widehat{\Gamma}^{kl} \widehat{\Gamma}^i = 0$. Let us then summarize this,

$$\delta^2 \chi = -i\mathcal{L}_S \chi - 5i\bar{\epsilon}_f \eta \chi$$

where things nicely combine into the Lie derivative

$$\begin{aligned} \mathcal{L}_S \chi &= S^M D_M \chi + \frac{1}{4} D_M S_N \Gamma^{MN} \chi \\ &= S^M D_M \chi - \frac{1}{2} \bar{\epsilon}_f \Gamma_{MN} \eta \Gamma^{MN} \chi \end{aligned}$$

This completes the closure computation for Abelian gauge group.

5.3 Non-Abelian gauge group and Poincare supersymmetry

For the non-Abelian case, we have two options as far as Poincare supersymmetry concerns. Either we can couple the hypermultiplet to a (1, 0) vector multiplet, or we can couple it to a (1, 0) tensor multiplet. Here we will consider the latter case, and make the ansatz

$$\begin{aligned} \delta \phi^i &= i\bar{\epsilon} \widehat{\Gamma}^i \chi \\ \delta \chi &= \Gamma^M \widehat{\Gamma}^i \epsilon D_M \phi^i + \Gamma^M \widehat{\Gamma}^i \epsilon [v_M, \sigma, \phi^i] + \nu^\alpha F^\alpha \\ \delta F^\alpha &= -i\bar{\nu}^\alpha \left(\Gamma^M D_M \chi + \Gamma^M \widehat{\Gamma}^i [\psi, v_M, \phi^i] + \Gamma^M [\chi, v_M, \sigma] \right) \end{aligned}$$

where ψ and σ are the tensor multiplet fields whose supersymmetry variations we will take to be the on-shell variations.

We may assume that we have the constraints (5.4), (5.5) and (5.6) satisfied. There is no way that these constraints can get modified by making the gauge group non-Abelian since the supersymmetry parameters do not carrying any gauge group index.

We begin by closure on the scalars. We get

$$\delta^2 \phi^i = -iS^M D_M \phi^i - (\Lambda, \phi^i)$$

where the gauge parameter is

$$\Lambda = iS_M \langle v^M, \sigma \rangle$$

Next we consider closure on the fermions,

$$\delta^2 \chi = i \left(\Gamma^M \widehat{\Gamma}^i \epsilon \bar{\epsilon} \widehat{\Gamma}^i + \nu^\alpha \bar{\nu}^\alpha \Gamma^M \right) D_M \chi \quad (5.14)$$

$$+ i \left(\Gamma_N \widehat{\Gamma}^i \epsilon \bar{\epsilon} \Gamma^{NM} + \Gamma^M \widehat{\Gamma}^i \epsilon \bar{\epsilon} - \nu^\alpha \bar{\nu}^\alpha \Gamma^M \widehat{\Gamma}^i \right) [\psi, v_M, \phi^i] \quad (5.15)$$

$$+ i \left(\Gamma^M \widehat{\Gamma}^i \epsilon \bar{\epsilon} \widehat{\Gamma}^i - \nu^\alpha \bar{\nu}^\alpha \right) [\chi, v_M, \sigma] \quad (5.16)$$

Line (5.16) gives the gauge transformation term

$$(5.16) = -(\Lambda, \chi)$$

with

$$\Lambda = iS^M \langle v_M, \sigma \rangle$$

We find that

$$(5.15) = 0$$

by using a Fierz identity and the identities

$$\begin{aligned} -\Gamma_N \Gamma_Q \Gamma^{NM} - \Gamma^M \Gamma_Q - 4\Gamma_Q \Gamma^M &= 0 \\ -\Gamma_N \Gamma_{RST} \Gamma^{NM} - \Gamma^M \Gamma_{RST} &= 0 \end{aligned}$$

Line (5.14) is of the same form as for the Abelian case, and thus we have got the off-shell closure relation,

$$\delta^2 \chi = -iS^M D_M \chi - (\Lambda, \chi)$$

Finally we turn to closure on F^α . By using the on-shell variations for the tensor multiplet fields σ and ψ , we get

$$\delta F^\alpha = -\frac{i}{2} \bar{\nu}^\alpha \Gamma^{MN} \hat{\Gamma}^i \epsilon (F_{MN}, \phi^i) \quad (5.17)$$

$$-i \bar{\nu}^\alpha \hat{\Gamma}^i \Gamma^{MN} \epsilon [D_M v_N, \sigma, \phi^i] \quad (5.18)$$

$$-i \bar{\nu}^\alpha \Gamma^{MN} \hat{\Gamma}^i \epsilon [v_N, D_M \sigma, \phi^i] \quad (5.19)$$

$$-i \bar{\nu}^\alpha \Gamma^{MN} \hat{\Gamma}^i \epsilon [v_N, \sigma, D_M \phi^i] \quad (5.20)$$

$$-i \bar{\nu}^\alpha \Gamma^M \nu^\beta D_M F^\beta \quad (5.21)$$

$$+ \frac{i}{12} \bar{\nu}^\alpha \Gamma_M \Gamma^{PQR} \hat{\Gamma}^i \epsilon [H_{PQR}, v^M, \phi^i] \quad (5.22)$$

$$-i \bar{\nu}^\alpha \Gamma^{MN} \hat{\Gamma}^i \epsilon [v_M, D_N \sigma, \phi^i] \quad (5.23)$$

$$-i \bar{\nu}^\alpha \Gamma^M \nu^\beta [v_M, \sigma, F^\beta] \quad (5.24)$$

$$+ i \bar{\nu}^\alpha \Gamma^{MN} \hat{\Gamma}^i \epsilon [v_M, D_N \phi^i, \sigma] \quad (5.25)$$

$$+ \frac{i}{2} \bar{\nu}^\alpha \Gamma_M \hat{\Gamma}^i \Gamma^Q \hat{\Gamma}^{kl} \epsilon [v^M, \phi^i, [v_Q, \phi^k, \phi^\ell]] \quad (5.26)$$

$$-i \bar{\nu}^\alpha \Gamma^M \Gamma^N \hat{\Gamma}^i \epsilon [v_M, \sigma, [v_N, \sigma, \phi^i]] \quad (5.27)$$

Lines (5.21) and (5.24) give the translation and the gauge transformation respectively,

$$\delta^2 F^\alpha = -iS^M D_M F^\alpha - (\Lambda, F^\alpha)$$

which is the off-shell closure relation that we want. It now remains to show that all the remaining terms vanish. Lines (5.17) and (5.22) cancel if we assume that

$$F_{MN} = \langle H_{MNP}, v^P \rangle$$

Lines (5.19) and (5.25) cancel. Lines (5.20) and (5.25) cancel. Each of the double commutators in lines (5.26) and (5.27) vanish by themselves, by applying the fundamental identity in the form

$$[\alpha, \beta, [\gamma, \delta, \bullet]] - [\gamma, \delta, [\alpha, \beta, \bullet]] = [[\alpha, \beta, \gamma], \delta, \bullet] + [\gamma, [\alpha, \beta, \delta], \bullet]$$

and by assuming that

$$\langle v_M, v_N \rangle = 0$$

Line (5.18) survives, unless we impose the constraint

$$\Gamma^{MN} \epsilon D_M v_N = 0$$

In the next subsection we will see that the left hand side is part of the Lie derivative $\mathcal{L}_v \epsilon$, which we can not see now as we assume that ϵ is a constant Poincare supersymmetry parameter.

This is not end of the story since we have further terms that are bilinear in fermionic fields. These terms are

$$(\delta^2 F^i)_{\text{bilin}} = i\bar{\nu} \hat{\Gamma}^i \Gamma^M (\delta A_M, \chi) - i\bar{\nu} \hat{\Gamma}^i \Gamma^M \hat{\Gamma}^j [\psi, v_M, \delta \phi^j] - i\bar{\nu} \hat{\Gamma}^i \Gamma^M [\chi, v_M, \delta \sigma]$$

These terms are algebraic in the sense that they do not involve any derivatives, so they share exactly the same gamma matrix and Fierz identity structures as the corresponding terms that arise in the 5d closure computation in [20], and so these terms shall cancel out here as well. This relation to [20] becomes clear if one chooses the Abelian vector field v_M to lie in say the 0-th direction and the generalization to arbitrary vector field then follows by the fact that this contribution is algebraic, together with the fact that $\delta A_0 \sim \Gamma_{00} = 0$ so that we are effectively left with only the 5d contribution. In appendix E we show how to translate our spinor language into the spinor language of [20].

By taking away the auxiliary field F^α and closing these variations on-shell, we obtain the fermionic equation of motion

$$\Gamma^M D_M \chi + \Gamma^M \hat{\Gamma}^i [\psi, v_M, \phi^i] + \Gamma^M [\chi, v_M, \sigma] = 0$$

We see that it is this equation of motion that reappears in δF^α when we close supersymmetry off-shell.

5.4 Non-Abelian gauge group and superconformal symmetry

We make the ansatz

$$\begin{aligned} \delta \phi^i &= i\bar{\epsilon}_f \hat{\Gamma}^i \chi \\ \delta \chi &= \Gamma^M \hat{\Gamma}^i \epsilon_f D_M \phi^i + \Gamma^M \hat{\Gamma}^i \epsilon_f [v_M, \sigma, \phi^i] + \nu^\alpha F^\alpha - 4\hat{\Gamma}^i \eta \phi^i \\ \delta F^\alpha &= -i\bar{\nu}^\alpha \left(\Gamma^M D_M \chi + \Gamma^M \hat{\Gamma}^i [\psi, v_M, \phi^i] + \Gamma^M [\chi, v_M, \sigma] \right) \end{aligned}$$

New type of terms in the closure computation arise when 3-brackets mix and curvature corrections mix. Such a mixing does not arise when we vary ϕ^i or χ twice, but it arises at many places when we vary F^α twice.

We can use the previous computations to argue that we have closure on ϕ^i with the gauge parameter $\Lambda = iS^M \langle v_M, \sigma \rangle$. When we vary ϕ^i twice we get a 3-bracket term plus a curvature term. For both terms we have already shown closure, although for the 3-bracket term we now need to substitute ϵ in the previous computation with ϵ_f here. For this case, a direct computation is also very easy to carry out. In the same way we can argue that we have closure on χ up to the same gauge transformation. Although here we need to vary some fields that sit inside 3-brackets, all those fields are bosonic and for the variations of the bosonic fields there are no curvature corrections and so we find no mixing between curvature corrections and 3-brackets, and so we have closure on χ .

New contributions arise when we make a second supersymmetry variation of δF^α . We find two types of new terms. One type of new terms arise from the 3-brackets that contain a fermion field in one of the entries. The variation of fermion fields have curvature correction terms, which produce new terms. The other type of contribution arises from when the derivative hits ϵ_f . All such terms were encountered in the case of Abelian gauge group, except for the 3-bracket in $\delta\chi$ that now produces the second type of new term. We thus only need to examine these new kind of terms. They are given by

$$(\delta^2 F^\alpha)_{\text{new}} = - (i\bar{\nu}^\alpha \Gamma^M D_M (\delta\chi)_{\text{comm}})_{\text{new}} \\ - i\bar{\nu}^\alpha \Gamma^M \hat{\Gamma}^i [(\delta\psi)_{\text{curv}}, v_M, \phi^i] - i\bar{\nu}^\alpha \Gamma^M [(\delta\chi)_{\text{curv}}, v_M, \sigma]$$

where

$$(\delta\chi)_{\text{comm}} = \Gamma^M \hat{\Gamma}^i \epsilon_f [v_M, \sigma, \phi^i] \\ (\delta\chi)_{\text{curv}} = -4\hat{\Gamma}^i \eta \phi^i \\ (\delta\psi)_{\text{curv}} = 4\eta \sigma$$

In the first term we will pick out the contributions when the derivative hits ϵ_f and v_M , since previously we have obtained the contributions from when the derivative hits the other fields. This is why we indicate this by the subscript *new* in the first term.

Plugging in these variations, we get

$$(\delta^2 F^\alpha)_{\text{new}} = -4i\bar{\nu}^\alpha \hat{\Gamma}^i (D_M \epsilon_f) [v^M, \sigma, \phi^i] - i\bar{\nu}^\alpha \hat{\Gamma}^i \Gamma^{MN} \epsilon_f [D_M v_N, \sigma, \phi^i]$$

These two terms nicely combine into

$$(\delta^2 F^\alpha)_{\text{new}} = -4i\bar{\nu}^\alpha \hat{\Gamma}^i [\mathcal{L}_v \epsilon_f, \sigma, \phi^i]$$

where

$$\mathcal{L}_v \epsilon_f = v^M D_M \epsilon_f + \frac{1}{4} D_M v_N \Gamma^{MN} \epsilon_f$$

To get closure on F^α we need to impose the constraint

$$\mathcal{L}_v \epsilon_f = 0 \tag{5.28}$$

that we saw already for the tensor multiplet. So it is not a surprise to also find this constraint again here for the hypermultiplet. While Lie derivatives of bosonic fields can be defined for any vector field, the Lie derivative on a spinor can be introduced only if that vector field is a Killing vector field. So the constraint (5.28) implies in particular that the 6d manifold has at least one isometry direction and that $\mathcal{L}_v g_{MN} = 0$.

6 Solving two constraints in Lorentzian 3-algebra

We will now restrict ourselves to Lorentzian 3-algebras [8–11] for which we have generators $T^\pm$ together with Lie algebra generators T^a that satisfy

$$[T^a, T^b] = f^{ab}_c T^c$$

The generator $X = X_a T^a + X_+ T^+ + X_- T^-$ has the norm

$$X \cdot X = h^{ab} X_a X_b + 2X_+ X_-$$

The 3-algebra relations are

$$\begin{aligned} [T^a, T^b, T^+] &= f^{ab}_c T^c \\ [T^a, T^b, T^-] &= 0 \\ [T^a, T^b, T^c] &= f^{abc} \end{aligned}$$

where $f^{abc} = f^{ab}_d h^{dc}$ and h^{ab} is a gauge invariant metric, so that f^{abc} is totally antisymmetric. Similarly, let us also introduce the 3-algebra structure constants $f^{ab+}_c = f^{ab}_c$ which have the properties that f^{abc+} is totally antisymmetric in all four entries $a, b, c, +$. We solve the Abelian constraint by taking [14]

$$v^M = V^M T^+$$

where we use the notation $V^M := v^M_+$. Since all interaction terms are mediated by the coupling vector V^M , we see that 3-algebra valued fields multiplying $T^\pm$ will amount to two free tensor multiplets. Let us from now on discard these, and only retain the interacting 3-algebra valued fields, which are those that take values in the Lie algebra generated by the T^a 's. Thus we expand any non-Abelian 3-algebra valued field as $\varphi = \sigma_a T^a$ where we discard the free field part $\varphi_+ T^+ + \varphi_- T^-$. Let us now clarify the map from the 3-algebra notation to the Lie algebra notation. First we notice that if we have two fields φ_1 and φ_2 whose free field parts have been discarded, the 3-algebra inner product reduces to the inner product on the Lie algebra,

$$\varphi_1 \cdot \varphi_2 = h^{ab} \varphi_{1a} \varphi_{2b}$$

and so we do not need to use a separate notation for the inner product on the Lie algebra. Let us next consider the covariant derivative. We expand out $(A_M, \sigma) = [B_{MN}, v^N, \sigma] = B_{MN,a} v^N_+ [T^a, T^+, \sigma] = -B_{MN,a} V^N [T^a, \sigma]$. Thus if we define

$$A_M = A_{M,a} T^a = B_{MN,a} V^N T^a$$

then we will have the covariant derivative

$$D_M \sigma = \partial_M \sigma - [A_M, \sigma]$$

and we will have the constraint

$$F_{MN,a} = H_{MNP,a} V^P$$

To solve this constraint, we make the ansatz

$$H_{MNP,a} = F_{MN,a}\kappa_P + F_{NP,a}\kappa_M + F_{PM,a}\kappa_N + C_{MNP,a}$$

where $C_{MNP,a}V^P = 0$. If we plug this ansatz into the constraint, it reduces to

$$F_{MN,a} = F_{MN,a}\kappa_P V^P + 2F_{P[M,a}\kappa_{N]}V^P$$

This leads to the constraints

$$\kappa_{[M}V_{N]} = 0 \tag{6.1}$$

$$\kappa_P V^P = 1 \tag{6.2}$$

$$F_{MN,a}V^N = 0 \tag{6.3}$$

The constraints (6.1) and (6.2) are solved by taking

$$\begin{aligned} \kappa_M &= \frac{V_M}{g^2}, \\ g^2 &= V^M V_M \end{aligned}$$

The constraint (6.3) says that

$$\mathcal{L}_V A_M + [\Lambda, A_M] = 0$$

where $\mathcal{L}_V A_M = V^N \partial_N A_M + (\partial_M V^N) A_N$ and $\Lambda = V^M A_M$. We will impose the gauge fixing condition that puts $\Lambda = 0$.

The on-shell supersymmetry variations for the (1, 0) tensor multiplet in the Lie algebra formulation read

$$\begin{aligned} \delta A_M &= i\bar{\epsilon}_f \Gamma_{MN} \psi V^N \\ \delta H_{MNP} &= 3i\bar{\epsilon}_f \Gamma_{[NP} D_{M]} \psi - 3i\bar{\eta} \Gamma_{MNP} \psi + i\bar{\epsilon}_f \Gamma_{MNPQ} [\psi, \sigma] V^Q \\ \delta \sigma &= -i\bar{\epsilon}_f \psi \\ \delta \psi &= \frac{1}{12} \Gamma^{MNP} \epsilon_f H_{MNP} + \Gamma^M \epsilon_f D_M \sigma + 4\eta \sigma \end{aligned} \tag{6.4}$$

We shall remember the sign flip in the covariant derivatives, $D_M \varphi = \partial_M \varphi - [A_M, \varphi]$ in the Lie algebra formulation.

7 Tensor multiplet without the tensor field

When we close supersymmetry variations on H_{MNP} , we find a modified Bianchi identity with the following structure [14]⁷

$$D_{[Q} H_{MNP]} = \frac{1}{4} \epsilon_{MNPQ}{}^{RS} [\sigma, D_R \sigma] V_S + \frac{i\lambda}{8} \epsilon_{MNPQ}{}^{RS} [\bar{\psi}, \Gamma_R \psi] V_S \tag{7.1}$$

⁷Here λ is a constant that can be determined with some efforts, as was done in [14].

If we contract by V^P , the left-hand side of this modified Bianchi identity trivializes,

$$(4D_{[Q}H_{MNP]})V^P = 3D_{[Q}F_{MN]} - \mathcal{L}_V H_{QMN} = 0$$

and so does the right-hand side as well, thanks to the Abelian constraint, or more simply, $V_{[M}V_{N]} = 0$.

Since only the selfdual part of H_{MNP} enters in the supersymmetry variation of ψ , one may expect one can use only the components $F_{MN,a} = H_{MNP,a}V^P$ since the other components are related by selfduality. Indeed we find that the following non-Abelian (2, 0) tensor multiplet Lagrangian

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4g^2}F^{MN} \cdot F_{MN} - \frac{1}{2}D^M\phi^A \cdot D_M\phi^A + \frac{i}{2}\bar{\Psi} \cdot \Gamma^M D_M\Psi - \frac{R}{10}\phi^A \cdot \phi^A \\ & - \frac{i}{2}\bar{\Psi} \cdot \Gamma^M \hat{\Gamma}^A[\Psi, \phi^A]V_M - \frac{g^2}{4}[\phi^A, \phi^B] \cdot [\phi^A, \phi^B] \end{aligned}$$

has the variation

$$\begin{aligned} \delta\mathcal{L} = & -\frac{i}{2}\bar{\Psi}\Gamma^{MNPQ}\epsilon \cdot D_M(F_{NP}\kappa_Q) \\ & + \frac{i}{2g^2}\mathcal{L}_V(\bar{\Psi}\Gamma^{MN}\epsilon) \cdot F_{MN} \\ & - \left(\frac{i}{2}\mathcal{L}_V(\bar{\Psi}\hat{\Gamma}^{AB}\epsilon) + 2i\bar{\Psi}\hat{\Gamma}^{AB}\mathcal{L}_V\epsilon \right) \cdot [\phi^A, \phi^B] \end{aligned}$$

under the (2, 0) supersymmetry variations

$$\begin{aligned} \delta\phi^A &= i\bar{\epsilon}_f\hat{\Gamma}^A\Psi \\ \delta\Psi &= \Gamma^M\hat{\Gamma}^A\epsilon_f D_M\phi^A + \frac{1}{2}\Gamma^{MNP}\epsilon_f F_{MN}\kappa_P - 4\hat{\Gamma}^A\eta\phi^A - \frac{1}{2}\Gamma^M\hat{\Gamma}^{AB}\epsilon_f[\phi^A, \phi^B]V_M \\ \delta A_M &= i\bar{\epsilon}_f\Gamma_{MN}\Psi V^N \end{aligned}$$

where

$$g^2 := V^M V_M$$

Thus the action becomes supersymmetric if we impose the constraints

$$\begin{aligned} \mathcal{L}_V(\text{fields}) &= 0 \\ \mathcal{L}_V\epsilon &= 0 \end{aligned}$$

and if we add a graviphoton term $\mathcal{L}_{\text{graviphoton}}$ whose variation is

$$\delta\mathcal{L}_{\text{graviphoton}} = \frac{i}{2}\bar{\psi}\Gamma^{MNPQ}\epsilon \cdot D_M(F_{NP}\kappa_Q)$$

In this graviphoton term where all vector indices are antisymmetrized, no Christoffel symbol survives and in fact the term is topological. However, this graviphoton term can be explicitly expressed only if we switch to the 5d language. To this end, we start by writing the 6d metric in the form

$$ds^2 = g(x^\mu)^2(dy + \kappa_\mu dx^\mu)^2 + G_{\mu\nu}(x^\mu)dx^\mu dx^\nu \quad (7.2)$$

We identify our Killing vector field as

$$V = \frac{\partial}{\partial y}$$

and $\kappa_M = (\kappa_\mu, 1)$ where $x^M = (x^\mu, y)$. Then we have

$$\begin{aligned} \delta \mathcal{L}_{\text{graviphoton}} &= \frac{i}{2} \bar{\psi} \Gamma^{\mu\nu\lambda\tau} \epsilon \cdot D_\mu (F_{\nu\lambda} \kappa_\tau) \\ &\quad + \frac{i}{2} \bar{\psi} \Gamma^{\mu\nu\lambda y} \epsilon \cdot D_\mu F_{\nu\lambda} \\ &\quad + \frac{i}{2} \bar{\psi} \Gamma^{y\nu\lambda\tau} \epsilon \cdot D_y (F_{\nu\lambda} \kappa_\tau) \end{aligned}$$

The second term vanishes by the Bianchi identity $D_{[\mu} F_{\nu\lambda]} = 0$. There are no terms involving $F_{My} = F_{MN} V^N$ since this is zero by the constraints $\mathcal{L}_V(\text{fields}) = 0$ together with the gauge fixing condition $A_y = A_M V^M = 0$. Finally we notice that $\mathcal{L}_V(F_{\nu\lambda} \kappa_\tau) = \partial_y(F_{\nu\lambda} \kappa_\tau)$ but we got D_y in place of ∂_y in the variation above. But then we notice that the metric plays no role in this variation which is a topological term, so we may right away take away the Christoffel symbols from all the covariant derivatives. Second, $A_y = 0$. After this, we have $D_y = \partial_y$ and thus

$$\delta \mathcal{L}_{\text{graviphoton}} = \frac{i}{2} \bar{\psi} \Gamma^{\mu\nu\lambda\tau} \epsilon \cdot D_\mu (F_{\nu\lambda} \kappa_\tau)$$

This is the variation of the graviphoton term [19]

$$\mathcal{L}_{\text{graviphoton}} = \frac{1}{8} \epsilon^{\mu\nu\lambda\kappa\tau} F_{\mu\nu} \cdot F_{\lambda\kappa} \kappa_\tau$$

Since in the end we want to obtain off-shell supersymmetry, we will split this Lagrangian into a sum

$$\mathcal{L} = \mathcal{L}_T + \mathcal{L}_H$$

consisting of one tensor multiplet part

$$\begin{aligned} \mathcal{L}_T &= -\frac{1}{4g^2} F^{MN} \cdot F_{MN} - \frac{1}{2} D^M \sigma \cdot D_M \sigma + \frac{i}{2} \bar{\psi} \cdot \Gamma^M D_M \psi - \frac{R}{10} \sigma \cdot \sigma \\ &\quad + \frac{i}{2} \bar{\psi} \cdot \Gamma^M [\psi, \sigma] V_M \end{aligned}$$

and one hypermultiplet part

$$\begin{aligned} \mathcal{L}_H &= -\frac{1}{2} D^M \phi^i \cdot D_M \phi^i + \frac{i}{2} \bar{\chi} \cdot \Gamma^M D_M \chi - \frac{R}{10} \phi^i \cdot \phi^i \\ &\quad - \frac{i}{2} \bar{\chi} \cdot \Gamma^M [\chi, \sigma] V_M - i \bar{\psi} \cdot \Gamma^M \hat{\Gamma}^i [\chi, \phi^i] V_M \\ &\quad - \frac{g^2}{2} [\phi^i, \sigma] \cdot [\phi^i, \sigma] - \frac{g^2}{4} [\phi^i, \phi^j] \cdot [\phi^i, \phi^j] \end{aligned}$$

Corresponding supersymmetry variations are

$$\begin{aligned} \delta \sigma &= -i \bar{\epsilon}_f \psi \\ \delta A_M &= i \bar{\epsilon}_f \Gamma_{MN} \psi V^N \\ \delta \psi &= \frac{1}{2} \Gamma^{MNP} \epsilon_f F_{MN} \kappa_P + \Gamma^M \epsilon_f D_M \sigma - \frac{1}{2} \Gamma^M \hat{\Gamma}^{ij} \epsilon_f [\phi^i, \phi^j] V_M + 4\eta \sigma \end{aligned}$$

and

$$\begin{aligned}\delta\phi^i &= i\bar{\epsilon}_f \hat{\Gamma}^i \chi \\ \delta\chi &= \Gamma^M \hat{\Gamma}^i \epsilon_f D_M \phi^i - \Gamma^M \hat{\Gamma}^i \epsilon_f [\phi^i, \sigma] V_M - 4\hat{\Gamma}^i \eta \phi^i\end{aligned}$$

As we now have reduced supersymmetry from $(2, 0)$ down to $(1, 0)$, we should ask ourselves if there could be a free parameter which is not completely fixed by $(1, 0)$ supersymmetry. By making the most general ansatz with arbitrary coefficients, we have shown that $(1, 0)$ supersymmetry fixes all the coefficients uniquely. Hence one $(1, 0)$ tensor multiplet coupled to one adjoint $(1, 0)$ hypermultiplet will automatically become $(2, 0)$ supersymmetric. There is no free parameter, other than the vector field V_M that has to be a Killing vector field, but which is otherwise our free choice. To get a genuinely $(1, 0)$ supersymmetric theory, we may consider just the $(1, 0)$ tensor multiplet in isolation, with no hypermultiplet, or we may consider several hypermultiplets in different representations. Our result is in agreement with that of [24], although in that reference one free parameter b appeared. However, that parameter can be absorbed into the hypermultiplet fields by a simple field rescaling, which means that this is not a genuine parameter that parametrizes physically different theories. We may also turn on supergravity background fields, such as the R-gauge field, which can have the effect of both reducing the supersymmetry down to $(1, 0)$ as well as bringing in an extra free parameter.

Before addressing the invariance of the action, we consider the closure relations for the $(1, 0)$ tensor multiplet. Let us drop the coupling term $-\frac{1}{2}\Gamma^M \hat{\Gamma}^{ij} \epsilon_f [\phi^i, \phi^j] V_M$ to the hypermultiplet for simplicity. (Keeping it, will only have the effect of modifying the tensor multiplet fermionic equation of motion by an extra coupling term to the hypermultiplet.) Then for the scalar we get the same kind of closure relation as before,

$$\delta^2 \sigma = -iS^M D_M \sigma - 4i\bar{\epsilon}_f \eta \sigma$$

For the gauge field A_M , we get

$$\begin{aligned}\delta^2 A_M &= -iS^T F_{TM} + D_M \Lambda + i\mathcal{L}_V (S_M \sigma) \\ \Lambda &= -iS_N V^N \sigma\end{aligned}$$

by using

$$\begin{aligned}F_{MN} V^N &= 0 \\ \kappa_{[M} V_{N]} &= 0\end{aligned}$$

For the fermions we get⁸

$$\begin{aligned}\delta^2 \psi &= -i\mathcal{L}_S \psi - [\psi, \Lambda] + \dots \\ &\quad + \frac{3i}{8} S^Q (\Gamma^Q - 2V_Q \kappa_P \Gamma^P) (\Gamma^M D_M \psi + \Gamma^M [\psi, \sigma] V_M)\end{aligned}$$

⁸The dots contain the R-rotation, the scaling dimension and all those terms we saw before in the 3-algebra language.

with

$$\Lambda = -iS^M V_M \sigma$$

To see this requires a small trick where we remove a certain Lie derivative $\mathcal{L}_V$ by using constraints. The tricky term is

$$(\delta^2 \psi)_{\text{tricky}} = \frac{1}{2} \Gamma^{MNP} \epsilon_f \delta F_{MN} \kappa_P$$

We use the identity

$$\delta F_{MN} = \delta H_{MNP} V^P - i \mathcal{L}_V (\bar{\epsilon}_f \Gamma_{MN} \psi)$$

and the trick is to note that by our constraints, we can drop the second term. Then we get

$$(\delta^2 \psi)_{\text{tricky}} = \frac{1}{2} \Gamma^{MNP} \epsilon_f \delta H_{MNQ} V^Q \kappa_P$$

Expanding it out, we get a sum of two terms

$$(\delta^2 \psi)_{\text{tricky}} = (\delta^2 \psi)_{\text{tricky},1} + (\delta^2 \psi)_{\text{tricky},2}$$

where

$$\begin{aligned} (\delta^2 \psi)_{\text{tricky},1} &= \frac{3i}{2} \Gamma^{MNP} \epsilon_f \bar{\epsilon}_f \Gamma_{[NQ} D_M] \psi V^Q \kappa_P \\ (\delta^2 \psi)_{\text{tricky},2} &= -\frac{3i}{2} \Gamma^{MNP} \epsilon_f \bar{\eta} \Gamma_{MNQ} \psi V^Q \kappa_P \end{aligned}$$

It is the term $(\delta^2 \psi)_{\text{tricky},2}$, that along with some other terms, will give us the correct scaling dimension $\Delta = 5/2$ for the fermion. For this part, the computation is completely analogous to our previous computation in 3-algebra language.

Let us now study the first term. This term can be separated into two terms

$$(\delta^2 \psi)_{\text{tricky},1} = (\delta^2 \psi)_{\text{tricky},1a} + (\delta^2 \psi)_{\text{tricky},1b}$$

where

$$\begin{aligned} (\delta^2 \psi)_{\text{tricky},1a} &= i \Gamma^{MNP} \epsilon_f \bar{\epsilon}_f \Gamma_{NQ} D_M \psi V^Q \kappa_P \\ (\delta^2 \psi)_{\text{tricky},1b} &= \frac{i}{2} \Gamma^{MNP} \epsilon_f \bar{\epsilon}_f \Gamma_{MN} D_Q \psi V^Q \kappa_P \end{aligned}$$

The first term $(\delta^2 \psi)_{\text{tricky},1a}$ is what we would like to refer to as a basic term. That is, a term that also would arise in 5d Poincare supersymmetry variations. It will combine with other equally basic terms into a basic variation,

$$(\delta^2 \psi)_{\text{basic}} = i \left(\Gamma^{MNP} \epsilon_f \bar{\epsilon}_f \Gamma_{NR} V^R \kappa_P - \Gamma^M \epsilon_f \bar{\epsilon}_f \right) D_M \psi - i \Gamma^M \epsilon_f \bar{\epsilon}_f \Gamma_{MN} [\psi, \sigma] V^N$$

Let us now compute the contribution from the first term in the Fierz identity,

$$\epsilon_f \bar{\epsilon}_f = c_Q \Gamma^Q + \dots$$

We note the following identity

$$\Gamma^{MNP}\Gamma^Q\Gamma_{NR}V^R\kappa_P = 6V^Q\kappa^M - 6\Gamma^P\Gamma^M\kappa_P V^Q + 6\kappa^Q V^M - 2\Gamma^Q\Gamma^P\kappa_P V^M - 6g^{MQ} + 2\Gamma^Q\Gamma^M$$

Then we get

$$\begin{aligned} & -iS^Q D_Q \psi \\ & + \frac{3i}{8} S^Q (\Gamma_Q - 2V^Q \kappa_P \Gamma^P) (\Gamma^M D_M \psi + \Gamma^M [\psi, \sigma] V_M) \\ & + \frac{i}{8} S_Q (12\kappa^Q - 2\Gamma^Q \Gamma^P \kappa_P) V^M D_M \psi \end{aligned} \quad (7.3)$$

On the other hand, we get

$$\begin{aligned} (\delta^2 \psi)_{\text{tricky,1b}} &= \frac{i c_Q}{2} \Gamma^{MNP} \Gamma^Q \Gamma_{MN} D_R \psi V^R \kappa_P \\ &= -\frac{i}{8} S_Q (12\kappa^Q - 2\Gamma^Q \Gamma^P \kappa_P) V^M D_M \psi \end{aligned}$$

precisely canceling the second line in (7.3).

Let us now turn to the invariance of the tensor multiplet Lagrangian. The most nontrivial part is the one that involves F_{MN} . Those variations come from two places,

$$\begin{aligned} (\delta \mathcal{L}_T)_1 &:= \delta \left(-\frac{1}{4g^2} F^{MN} \cdot F_{MN} \right) \\ &= i\bar{\epsilon}_f \Gamma_{NP} \psi V^P D_M \left(\frac{1}{g^2} F^{MN} \right) \end{aligned}$$

and⁹

$$(\delta \mathcal{L}_T)_2 := -i\bar{\psi} \Gamma^M D_M (\delta \psi)_{F_{MN}} = -\frac{i}{2} \bar{\psi} \Gamma^M \Gamma^{RST} \epsilon_f D_M (F_{RS} \kappa_T)$$

Now we use $\kappa_M = V_M/g^2$ and

$$\Gamma^M \Gamma^{RST} = 3g^{M[R} \Gamma^{ST]} + \Gamma^{MRST}$$

and we get

$$\begin{aligned} (\delta \mathcal{L}_T)_2 &= -i\bar{\epsilon}_f \Gamma_{NP} \psi D_M \left(\frac{1}{g^2} F^{MN} \right) V^P \\ &\quad + \frac{i}{2} \mathcal{L}_V (\bar{\epsilon}_f \Gamma_{NP} \psi) \frac{1}{g^2} F^{NP} \\ &\quad - \frac{i}{2} \bar{\psi} \Gamma^{MRST} \epsilon_f D_M (F_{RS} \kappa_T) \end{aligned}$$

We put the Lie derivative to zero by constraints on ϵ_f and ψ . We cancel the first term against $(\delta \mathcal{L}_T)_1$ and we are left with the last term

$$(\delta \mathcal{L}_T)_1 + (\delta \mathcal{L}_T)_2 = -\frac{i}{2} \bar{\psi} \Gamma^{MRST} \epsilon_f D_M (F_{RS} \kappa_T)$$

⁹Here we do not include the term that we get from letting D_M act on ϵ_f .

This variation is canceled by adding $\mathcal{L}_{\text{graviphoton}}$ to the Lagrangian, as we have discussed above.

Let us next illustrate the absence of a modified Bianchi identity. This can be seen from varying the following terms in the $(1, 0)$ tensor multiplet Lagrangian

$$\mathcal{L}_T = \dots - \frac{1}{2} D^M \sigma \cdot D_M \sigma + \frac{i}{2} \bar{\psi} \cdot \Gamma^M D_M \psi + \frac{i}{2} \bar{\psi} \cdot \Gamma^M [\psi, \sigma] V_M$$

Retaining only the terms that could be of relevance to get a possible modified Bianchi identity, we find

$$\begin{aligned} \delta \mathcal{L}_T = & \dots - \frac{i}{2} \bar{\psi} \Gamma^{MRST} \epsilon D_M (F_{RS} \kappa_T) \\ & + (1 - 1) i \bar{\epsilon} \Gamma^{MN} \psi \cdot [D_M \sigma, \sigma] V_N \\ & - \frac{1}{2} \bar{\epsilon} \Gamma_N \Gamma_M \psi \cdot [\bar{\psi}, \Gamma^M \psi] V^N + \frac{1}{2} (1 - 1) \bar{\epsilon} \psi \cdot [\bar{\psi}, \Gamma_N \psi] V^N \end{aligned}$$

Here all brackets act on Lie-algebra indices only, so for example $[\psi, \Gamma^M \psi] := \bar{\psi}^a \Gamma^M \psi^b [T_a, T_b]$. By using the Fierz identity

$$\psi^a \bar{\psi}^b = -\frac{1}{8} \bar{\psi}^b \Gamma^M \psi^a P_+ \hat{P}_- \Gamma_M - \frac{1}{96} \bar{\psi}^b \Gamma^{MNP} \hat{\Gamma}^{ij} \psi^a \Gamma_{MNP} \hat{\Gamma}_{ij}$$

we get the identity

$$\Gamma_N \Gamma_M \psi^a \bar{\psi}^b \Gamma^M \psi^c f_{abc} = -\frac{1}{2} \Gamma_N \Gamma_M \psi^a \bar{\psi}^b \Gamma^M \psi^c f_{abc}$$

which means that this cubic term is zero. This computation is analogous with that of SYM theory [1]. Thus we see that all the terms that could have potentially have contributed to a modified Bianchi identity, all either cancel out (as we have indicated by writing $1 - 1$), or vanishes identically, and the remaining term is canceled by the variation of $\mathcal{L}_{\text{graviphoton}}$. If the terms had added up (so that instead of $1 - 1$ we had had $1 + 1$), then we would have had a modified Bianchi identity.

We are left with checking many more terms in the Lagrangian, but most of those terms are analogous with that of 5d SYM theory. The new ingredient here is superconformal symmetry. In the appendix we present a detailed derivation for superconformal symmetry of the actions when the gauge group is Abelian. The remaining contributions that are nonstandard are coming from variations that mix commutators and the superconformal added contribution $\delta' \psi = 4\eta \sigma$. There is only one such a nonstandard contribution. This is coming from the variation of the commutator,

$$-i \bar{\psi} \cdot \Gamma^M [\delta' \psi, \sigma] V_M$$

but this vanishes because of $[\sigma, \sigma] = 0$.

7.1 Taking the tensor multiplet off-shell

We have not been able to take the Abelian tensor multiplet off-shell. However, the non-Abelian¹⁰ tensor multiplet should be, as we have argued, an uplift from the 5d vector

¹⁰By this we mean no $U(1)$ factors in the gauge group.

multiplet to 6d, and it is a standard result that the 5d vector multiplet can be taken off-shell. So we should expect that its 6d uplift where we impose the constraints $\mathcal{L}_V = 0$ on everything, also has an off-shell generalization.

These off-shell supersymmetry variations should be given by¹¹

$$\begin{aligned}\delta\sigma &= -i\bar{\epsilon}_f\psi \\ \delta A_M &= i\bar{\epsilon}_f\Gamma_{MN}\psi V^N \\ \delta\psi &= \frac{1}{2}\Gamma^{MNP}\epsilon_f F_{MN}\kappa_P + \Gamma^M\epsilon_f D_M\sigma + 4\eta\sigma + \frac{1}{2}\hat{\Gamma}^{ij}\Gamma^M\epsilon_f D^{ij}\kappa_M \\ \delta D^{ij} &= \frac{i}{2}\bar{\epsilon}_f\Gamma^P\hat{\Gamma}^{ij}(\Gamma^M D_M\psi + \Gamma^M[\psi, \sigma]V_M)V_P\end{aligned}$$

where D^{ij} are three selfdual auxiliary fields with scaling dimensions $\Delta = 2$. That scaling dimension can be read off from $\delta^2 D^{ij} = \dots - 4i\bar{\epsilon}_f\eta D^{ij}$, but we can also see that the scaling dimensions

$$\begin{aligned}[\epsilon_f] &= L^{1/2} \\ [V_M] &= L \\ [\kappa_M] &= L^{-1} \\ [\psi] &= L^{-5/2} \\ [D^{ij}] &= L^{-2} \\ [\sigma] &= L^{-2} \\ [D_M] &= L^{-1}\end{aligned}$$

are consistent with these variations. These variations should close off-shell on the generators of the conformal group plus an R-symmetry rotation that acts on D^{ij} and ψ . The non-Abelian generalization for the Poincare part of the supersymmetry should hold since we know that the existence of off-shell supersymmetry for the 5d vector multiplet and this is essentially just a trivial 6d uplift of that. It remains to check the mixed contributions when we combine non-Abelian with superconformal generalization. There is only one such mixed term that can arise here. This is from $\delta^2 D^{ij} = \dots - (i/2)\bar{\epsilon}_f\Gamma^P\hat{\Gamma}^{ij}\Gamma^M[\delta'\psi, \sigma]V_M V_P$, where $\delta'\psi = 4\eta\sigma$. But that term is zero by $[\sigma, \sigma] = 0$. To the Lagrangian we need to add the term $\frac{1}{2g^2}D^{ij} \cdot D^{ij}$.

8 Discussion

We have obtained non-Abelian tensor and hypermultiplet theories with off-shell 6d $(1, 0)$ superconformal symmetry where we found that we had to impose the constraint $\mathcal{L}_V = 0$ on all fields. The vector field V_M has the interpretation of a gauge coupling constant, though it is a vector field and not just a constant parameter. It has length dimension one and transforms as a spin-one field under conformal transformations. The vector field is

¹¹For maximal simplicity, we have dropped the coupling to hypermultiplet fields again.

constrained by

$$\begin{aligned}\mathcal{L}_V \epsilon_f &= 0 \\ D_M \epsilon_f &= \Gamma_M \eta\end{aligned}$$

Our 6d theory might find uses in localization computations. This might be an easier approach in some cases, than performing dimensional reduction to 5d and carrying out localization computations in 5d. For a situation where the fiber length $2\pi g$ is not constant, it may be an advantage to be able to work directly in a 6d covariant formulation. We may keep the 6d symmetries of the problem all the way until we implement the constraint $\mathcal{L}_V = 0$, which is where we break the symmetry, and which is where we may encounter the biggest difficulties in the computation. In practice this may amount to selecting harmonic modes of some Laplace operator which have vanishing $\mathcal{L}_V$, which may be a difficult problem in some cases. However, we expect that one also in 6d will have to consider an instanton factor. Instanton-branes in 6d are instantons located in four transverse directions and extended along the time direction and the vector field V^M . By taking these nonperturbative effects into account, we expect that V^M will not be an observable vector field and so one may ask whether implementing $\mathcal{L}_V = 0$ and taking into account nonperturbative instanton sectors, could in fact be equivalent with dropping both the constraint $\mathcal{L}_V = 0$ and all the nonperturbative sectors. If that is true, then that could lead to simpler localization computations directly in 6d. One may reformulate the YM term in terms of a selfdual H_{MNP} and also try to find a Lagrangian whose coupling constant is fixed by selfduality. Such a Lagrangian may be found if one uses for instance the $6 = 3 + 3$ formalism [25]. We may still encounter V^M in cubic interaction terms. However, in localization we only need to be concerned with the quadratic terms. If those quadratic terms can be rephrased such that they are all independent of g and V^M , then the localization computation may become tractable even when g is a field that depends on the spacetime position. The difficult step would then be to implement the constraint $\mathcal{L}_V = 0$ on all the fields in the path integral. But as we have speculated above, maybe we do not really need to impose $\mathcal{L}_V = 0$ in the computation of the one-loop determinants, and in that way we could hope to get the nonperturbative contribution for free. Typically the one-loop determinant involves the determinant of a Laplace-like operator. We expect full 6d covariance at the full quantum level. The most naive way to achieve 6d covariance at the quantum level would be to drop $\mathcal{L}_V = 0$ in the perturbative determinants, and at the same time drop the nonperturbative contributions. It remains to be seen whether that naive prescription can give the correct result. To be able to put the 6d theory on a wider class of six-manifolds, one may also couple the 6d theory to supergravity background fields.

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A Gamma matrix identities

For the closure on ψ we have used the identities

$$\begin{aligned}
 \Gamma^{MNP}\Gamma_R\Gamma_{NP} &= -24\delta_R^M + 4\Gamma_R\Gamma^M \\
 \Gamma^{MNP}\Gamma_{RST}\Gamma_{NP} &= 4\Gamma^M\Gamma_{RST} \\
 \Gamma^M\Gamma_R &= 2\delta_R^M - \Gamma_R\Gamma^M \\
 \Gamma^{MNP}\Gamma_{MNP} &= -120 \\
 \Gamma^{MNP}\Gamma_{RS}\Gamma_{MNP} &= 24\Gamma_{RS} \\
 \Gamma^M\Gamma_M &= 6 \\
 \Gamma^M\Gamma_{RS}\Gamma_M &= 2\Gamma_{RS} \\
 \Gamma^{MNP}\Gamma_R\Gamma_{MNPQ} &= 60g_{RQ} - 12\Gamma_{RQ} \\
 \Gamma^{MNP}\Gamma_{RST}\Gamma_{MNPQ} &= 12\Gamma_Q\Gamma_{RST} \\
 \Gamma^M\Gamma_R\Gamma_{MQ} &= -2g_{TQ} - 3\Gamma_R\Gamma_Q \\
 \Gamma^M\Gamma_{RST}\Gamma_{MQ} &= \Gamma_Q\Gamma_{RST}
 \end{aligned}$$

For the closure on A_M we have used

$$\{\Gamma_{MN}, \Gamma^{RST}\} = -12\delta_{MN}^{RS}\Gamma^T + 2\Gamma_{MN}^{RST}$$

For the closure on ψ in the Lie algebra formulation, we have used

$$\begin{aligned}
 \Gamma^{MNP}\Gamma_Q\Gamma_{NR} &= \Gamma^{MP}{}_Q\Gamma_{NR} + 3\Gamma^{MP}{}_Qg_{NR} - 2\Gamma^M{}_Q\delta_R^P + 2\Gamma^P{}_Q\delta_R^M \\
 &\quad + 3\Gamma^M{}_R\delta_Q^P - 3\Gamma^P{}_R\delta_Q^M - 4\delta_Q^M\delta_R^P + 4\delta_Q^P\delta_R^M
 \end{aligned}$$

B The conformal Killing spinor equation

Here we note some implications of the conformal Killing spinor equation,

$$D_M\epsilon_f = \Gamma_M\eta$$

We define

$$\begin{aligned}
 [D_M, D_N]\epsilon &= \frac{1}{4}R_{MNPQ}\Gamma^{PQ}\epsilon \\
 R_{MN} &= g^{PQ}R_{MPNQ} \\
 R &= g^{MN}R_{MN}
 \end{aligned}$$

We have the identities

$$5D^M D_M\epsilon = \Gamma^{MN}D_M D_N\epsilon = -\frac{R}{4}\epsilon$$

They follows from

$$\Gamma^N\Gamma^M D_M D_N\epsilon = \Gamma^N\Gamma^M\Gamma_N D_M\eta = -4\Gamma^M D_M\eta$$

and

$$D^M D_M \epsilon = \Gamma^M D_M \eta$$

We have

$$\Gamma^M D_M \eta = -\frac{R}{20} \epsilon$$

and then we get

$$\Gamma^M [D_N, D_M] \epsilon = 4D_N \eta - \frac{R}{20} \Gamma_N \epsilon$$

so

$$4D_N \eta = \frac{1}{4} R_{NMPQ} \Gamma^M \Gamma^{PQ} \epsilon + \frac{R}{20} \Gamma_N \epsilon$$

Finally we use $R_{N[MPQ]} = 0$ and get

$$4D_N \eta = -\frac{1}{2} R_{NQ} \Gamma^Q \epsilon + \frac{R}{20} \Gamma_N \epsilon$$

We may check the consistency of this relation by contracting by Γ^N .

C Conformal actions, Abelian case

Here we show that the Abelian actions

$$\mathcal{L}_T = -\frac{1}{24} H^{MNP} H_{MNP} - \frac{1}{2} D^M \sigma D_M \sigma + \frac{i}{2} \bar{\psi} \Gamma^M D_M \psi - \frac{R}{10} \sigma^2$$

and

$$\mathcal{L}_H = -\frac{1}{2} (D_M \phi^i)^2 + \frac{i}{2} \bar{\chi} \Gamma^M D_M \chi - \frac{R}{10} \phi^i \phi^i$$

are separately superconformal. All that we need to assume here, is that the supersymmetry parameter satisfies

$$D_M \epsilon = \Gamma_M \eta$$

C.1 The tensor multiplet

We begin with the Abelian tensor multiplet and split the Lagrangian into two parts,

$$\begin{aligned} \mathcal{L}_0 &= -\frac{1}{24} H^{MNP} H_{MNP} - \frac{1}{2} D^M \sigma D_M \sigma + \frac{i}{2} \bar{\psi} \Gamma^M D_M \psi \\ \mathcal{L}_1 &= -\frac{\mu}{2} \sigma^2 \end{aligned}$$

Likewise we split the variation into two parts,

$$\begin{aligned} \delta_0 B_{MN} &= i\bar{\epsilon} \Gamma_{MN} \psi \\ \delta_0 \sigma &= -i\bar{\epsilon} \psi \\ \delta_0 \psi &= \frac{1}{12} \Gamma^{MNP} \epsilon H_{MNP} + \Gamma^M \epsilon D_M \sigma \end{aligned}$$

and

$$\delta_1 \psi = \lambda \Gamma^M (D_M \epsilon) \sigma$$

We note the following identity

$$\bar{\psi}\Gamma^{MN}D_M\epsilon D_N\sigma + 5\bar{\psi}D_M\epsilon D^M\sigma = 0$$

To prove this identity, we use $D_M\epsilon = \Gamma_M\eta$ to rewrite

$$\bar{\psi}\Gamma^M\Gamma^N D_M\epsilon D_N\sigma = -\frac{2}{3}\bar{\psi}\Gamma^N\Gamma^M D_M\epsilon D_N\sigma$$

from which the identity immediately follows. By using this identity, we find

$$\delta_0\mathcal{L}_0 = 4i\bar{\psi}D_M\epsilon D^M\sigma$$

The sign is corresponding to assuming that δ is anticommuting. Also note that there is no contribution coming from the variation of H_{MNP} thanks to the identity $\Gamma^Q\Gamma_{MNP}\Gamma_Q = 0$. We get

$$\delta_0\mathcal{L}_1 = -i\mu\sigma\bar{\psi}\epsilon$$

assuming ϵ is commuting. Clearly the action is not invariant unless we also consider δ_1 variations. We find

$$\begin{aligned}\delta_1\mathcal{L}_0 &= -6i\lambda\bar{\psi}D_M\epsilon D^M\sigma + i\lambda\frac{3R}{10}\bar{\psi}\epsilon\sigma \\ \delta_1\mathcal{L}_1 &= 0\end{aligned}$$

Thus by taking

$$\begin{aligned}\lambda &= \frac{2}{3} \\ \mu &= \frac{R}{5}\end{aligned}$$

we see that the action becomes superconformal.

C.2 The hypermultiplet

We decompose the Lagrangian as $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ where

$$\begin{aligned}\mathcal{L}_0 &= -\frac{1}{2}(D_M\phi^i)^2 + \frac{i}{2}\bar{\chi}\Gamma^M D_M\chi \\ \mathcal{L}_1 &= -\frac{R}{10}\phi^i\phi^i\end{aligned}$$

Likewise we decompose the variations as $\delta = \delta_0 + \delta_2$ where

$$\begin{aligned}\delta_0\phi^i &= i\widehat{\epsilon}\Gamma^i\chi \\ \delta_0\chi &= \Gamma^M\widehat{\Gamma}^i\epsilon D_M\phi^i \\ \delta_1\chi &= -4\widehat{\Gamma}^i\eta\phi^i\end{aligned}$$

We get the variations

$$\begin{aligned}\delta_0\mathcal{L}_0 &= -D^M\phi^i D_M\delta\phi^i - i\bar{\chi}\Gamma^M D_M\delta\chi \\ &= 4i\bar{\chi}\widehat{\Gamma}^i\Gamma^M\eta D_M\phi^i\end{aligned}$$

and

$$\begin{aligned}\delta_1 \mathcal{L}_0 &= -i\bar{\chi}\Gamma^M D_M \delta_1 \chi \\ &= -4i\bar{\chi}\hat{\Gamma}^i \Gamma^M \eta D_M \phi^i - 4i\bar{\chi}\hat{\Gamma}^i \Gamma^M D_M \eta \phi^i\end{aligned}$$

We now need

$$\Gamma^M D_M \eta = -\frac{R}{20}\epsilon$$

We now get

$$\begin{aligned}\delta \mathcal{L}_0 &= \frac{R}{5} i\bar{\chi}\hat{\Gamma}^i \epsilon \phi^i \\ \delta \mathcal{L}_1 &= -\frac{R}{5} i\bar{\chi}\hat{\Gamma}^i \epsilon \phi^i\end{aligned}$$

and hence $\delta \mathcal{L} = 0$.

D Off-shell 10d SYM supersymmetry

A nice overview of the lower dimensional off-shell supersymmetric theories can be found by starting with the 10d SYM theory [2, 4]. We assume 10d gamma matrices with the chirality matrix $\Gamma^{(10)}$ and spinor and Poincare supersymmetry parameters that are 10d Weyl

$$\begin{aligned}\Gamma^{(10)}\psi &= \psi \\ \Gamma^{(10)}\epsilon &= \epsilon\end{aligned}$$

and Majorana, $\bar{\psi} = \psi^T C$ where C can be taken as the same as for our 11d spinor conventions. (In 11d we just include $\Gamma^{(10)}$ as the 11th gamma matrix.) In particular then $C^T = -C$. In this appendix, and only here, we will let $M = 0, 1, \dots, 9$. The supersymmetry variations are [4]

$$\begin{aligned}\delta A_M &= \bar{\epsilon}\Gamma_M \psi \\ \delta \psi &= \frac{1}{2}\Gamma^{MN}\epsilon F_{MN} + \nu^i K^i \\ \delta K^i &= -\bar{\nu}^i \Gamma^M D_M \psi\end{aligned}$$

These variations close off-shell for ν^i being 7 parameters subject to the constraints

$$\begin{aligned}\bar{\epsilon}\Gamma_M \nu_i &= 0 \\ \bar{\nu}^i \Gamma_M \nu^j &= -\delta^{ij} \bar{\epsilon}\Gamma_M \epsilon\end{aligned}$$

One of the most intricate places in this closure computation, is the term in $\delta^2 K^i$ that is bilinear in fermions. This term is given by

$$\begin{aligned}(\delta^2 K^i)_{\text{bilin}} &= \bar{\nu}^i \Gamma^M [\delta A_M, \psi] \\ &= [\bar{\psi}\Gamma_M \epsilon, \bar{\nu}^i \Gamma^M \psi]\end{aligned}$$

We now note the Fierz identity

$$\epsilon \bar{\nu}^i = c_M \Gamma^M + c_{MNP} \Gamma^{MNP} + c_{MNPQR} \Gamma^{MNPQR}$$

The last term drops out because $\Gamma_M \Gamma^{RSTUV} \Gamma^M = 0$. The second term drops out since it is symmetric, $\bar{\psi}^a \Gamma^{MNP} \psi^b$ is symmetric in the gauge indices a, b . Only the first term may contribute, but due to the constraint $\bar{\epsilon} \Gamma_M \nu^i = 0$ the coefficient c_M vanishes itself. Hence $(\delta^2 K^i)_{\text{bilin}} = 0$.

We can dimensionally reduce on T^4 down to 6d where we find the $\mathcal{N} = (1, 1)$ vector multiplet. By further imposing the additional Weyl projection on the supersymmetry parameter $\widehat{\Gamma} \epsilon = \epsilon$ where $\widehat{\Gamma} := \Gamma^{6789}$, we split the 6d (1,1) vector multiplet into one 6d (1,0) vector multiplet and one 6d (1,0) hyper multiplet. Further reduction to 5d gives the following off-shell hypermultiplet supersymmetry variations

$$\begin{aligned} \delta \phi^i &= \bar{\epsilon} \widehat{\Gamma}^i \chi \\ \delta \chi &= \Gamma^\mu \widehat{\Gamma}^i \epsilon D_\mu \phi^i + \Gamma^5 \widehat{\Gamma}^i \epsilon [\sigma, \phi^i] + \nu^\alpha F^\alpha \\ \delta F^\alpha &= -\bar{\nu}^\alpha \left(\Gamma^\mu D_\mu \chi + \Gamma^5 [\sigma, \chi] + \widehat{\Gamma}^i [\phi^i, \psi] \right) \end{aligned}$$

where $\widehat{\Gamma} \psi = \psi$ and $\widehat{\Gamma} \chi = -\chi$ are vector and hypermultiplet fermions in 5d respectively, and where α now runs over four values for the hyper. Here $\mu = 0, 1, 2, 3, 4$ and $\sigma = A_5$ and $\phi^i = A_i$. These variations can be uplifted and related to the 6d variations that we have obtained in the main text, by relating the 10 Weyl projection to the 6d Weyl projection by the following unitary rotation of the spinors,

$$\begin{aligned} \psi &= U^\dagger \psi' \\ \nu^\alpha &= U^\dagger \nu'^\alpha \\ \epsilon &= U \epsilon' \end{aligned}$$

where $U = \frac{1}{\sqrt{2}}(1 + \Gamma^{01234})$. The rotated spinors are no longer subject to the 10d Weyl projections, but instead to the following 6d Weyl projections,

$$\begin{aligned} \Gamma^{01234(10)} \psi' &= \psi' \\ \Gamma^{01234(10)} \epsilon' &= -\epsilon' \end{aligned}$$

where the M-theory circle (usually taken as the 5th direction) now is along the 10th direction.

E Many hypermultiplets

In 5d the hypermultiplet was presented in a different language in [20] that enables for generalization to r hypermultiplets, labeled by a gauge index $A = 1, 2, \dots, 2r$. In this appendix we present the map from our language into their language. But since such a map is available only for $r = 1$, we will assume that $r = 1$.

We realize 11d gamma matrices as follows. We split the 11d vector index into $M = 0, 1, 2, 3, 4, 5$ extended along the M5 brane, and the remaining indices are labeled $A = 1, 2, 3, 4, 5$. Then we let

$$\begin{aligned}\Gamma^M &= \Sigma^M \otimes 1 \\ \hat{\Gamma}^A &= \Sigma \otimes \hat{\gamma}^A\end{aligned}$$

where the hat does not mean anything. We assume the convention

$$\hat{\gamma}^{12345} = 1$$

We let $M = \{0, m\}$ for $m = 1, 2, 3, 4, 5$ and

$$\begin{aligned}\Sigma^0 &= i\sigma^2 \otimes 1 \\ \Sigma^m &= \sigma^1 \otimes \gamma^m\end{aligned}$$

We assume the convention

$$\gamma^{12345} = 1$$

We define the 6d chirality matrices

$$\begin{aligned}\Gamma &= \Gamma^{012345} \\ \Sigma &= \Sigma^{012345} \\ &= \sigma^3 \otimes 1 \\ \Gamma &= \Sigma \otimes 1 \\ \hat{\Gamma} &= \hat{\Gamma}^{1234} \\ &= I \otimes \hat{\gamma}^{1234}\end{aligned}$$

such that

$$\Gamma \hat{\Gamma} \hat{\Gamma}^5 = 1$$

We let $A = \{i, 5\}$ and we realize

$$\hat{\gamma}^i = \begin{pmatrix} 0 & \sigma_{IA}^i \\ \tilde{\sigma}^{iBJ} & 0 \end{pmatrix}$$

where we take

$$\begin{aligned}\sigma^i &= (\sigma^{1,2,3}, -i) \\ \tilde{\sigma}^i &= (\sigma^{1,2,3}, i)\end{aligned}$$

We then get

$$\hat{\gamma} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\delta_I^J & 0 \\ 0 & \delta_B^A \end{pmatrix}$$

that is acting on a Dirac spinor with the indices

$$\Psi = \begin{pmatrix} \psi_I \\ \chi^A \end{pmatrix}$$

We note that for the Pauli sigma matrices we have

$$(\sigma^{1,2,3})^T = -\epsilon \sigma^{1,2,3} \epsilon^{-1}$$

where

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The 11d charge conjugation matrix is represented as

$$C_{11d} = \epsilon \otimes C \otimes \hat{C}$$

such that

$$\begin{aligned} C^T &= -C \\ \hat{C}^T &= -\hat{C} \\ C_{11d}^T &= -C_{11d} \end{aligned}$$

We have

$$\tilde{\sigma}^{iAI} = -\epsilon^{IJ} \sigma_{JB}^i \epsilon^{BA}$$

We use the conventions

$$\hat{C} = \begin{pmatrix} \epsilon^{IJ} & 0 \\ 0 & \epsilon_{AB} \end{pmatrix}$$

and

$$\hat{C}^{-1} = \begin{pmatrix} \epsilon_{IJ} & 0 \\ 0 & \epsilon^{AB} \end{pmatrix}$$

where

$$\begin{aligned} \epsilon^{IJ} &= \epsilon^{+-} = 1 \\ \epsilon_{AB} &= \epsilon_{+-} = 1 \end{aligned}$$

and the inverses of these are denoted as

$$\begin{aligned} \epsilon_{IJ} &= \epsilon_{+-} = -1 \\ \epsilon^{AB} &= \epsilon^{+-} = -1 \end{aligned}$$

Let us introduce the full index notation of an 11d spinor,

$$\Psi = \begin{pmatrix} \psi_I^{a\alpha} \\ \psi^{a\alpha A} \end{pmatrix}$$

The M5 brane spinor is 6d Weyl, and decomposes into two tensor multiplet spinors

$$\psi_I^{+\alpha}$$

labeled by the $SU(2)_R$ index $I = +, -$, and two hypermultiplet spinors

$$\chi^{+\alpha A}$$

labeled by the $SU(2)_F$ index $A = +, -$. The two supersymmetry parameters are 6d anti-Weyl,

$$\epsilon_I^{-\alpha}$$

and labeled by the R-symmetry index $I = +, -$.

We define the Dirac conjugate as

$$\begin{aligned}\bar{\epsilon}_{+\alpha}^I &= \epsilon_J^{-\beta} \epsilon_{-+} C_{\beta\alpha} \epsilon^{JI} \\ \bar{\nu}_{-\alpha}^I &= \nu_J^{+\beta} \epsilon_{+-} C_{\beta\alpha} \epsilon^{JI}\end{aligned}$$

This notation is consistent with the Weyl projections

$$\begin{aligned}\Gamma \epsilon &= -\epsilon \\ \bar{\epsilon} \Gamma &= \bar{\epsilon}\end{aligned}$$

if we represent the 6d chirality matrix when acting on the first index as

$$\Gamma = (\sigma^3)^a{}_b$$

Namely then,

$$\begin{aligned}(\sigma^3)^-_- \epsilon^- &= -\epsilon^- \\ \bar{\epsilon}_+ (\sigma^3)^+_+ &= \bar{\epsilon}^+\end{aligned}$$

We define

$$\epsilon_{ab} = \epsilon_{+-} = 1$$

which means that the Dirac conjugate becomes

$$\begin{aligned}\bar{\epsilon}_{+\alpha}^I &= -\epsilon_J^{-\beta} C_{\beta\alpha} \epsilon^{JI} \\ \bar{\nu}_{-\alpha}^I &= \nu_J^{+\beta} C_{\beta\alpha} \epsilon^{JI}\end{aligned}$$

We use the gamma matrix realization

$$\begin{aligned}\Gamma^m &= (\sigma^1)^a{}_b (\gamma^m)^\alpha{}_\beta \begin{pmatrix} \delta_I^J & 0 \\ 0 & \delta^A{}_B \end{pmatrix} \\ \hat{\Gamma}^i &= (\sigma^3)^a{}_b \delta^\alpha{}_\beta \begin{pmatrix} 0 & \sigma_{IA}^i \\ \tilde{\sigma}^{iBJ} & 0 \end{pmatrix}\end{aligned}$$

Let us now translate

$$\delta\phi^i = i\bar{\epsilon}\hat{\Gamma}^i\chi$$

into the new language. First, by writing out all indices, we have

$$\delta\phi^i = -i\bar{\epsilon}_{+\alpha}^I (\sigma^3)^+_+ \delta_\beta^\alpha \sigma_{IA}^i \chi^{+\beta A}$$

that is simplified to

$$\delta\phi^i = -i\bar{\epsilon}_{+\alpha}^I \sigma_{IA}^i \chi^{+\beta A}$$

Then we contract both sides with $\tilde{\sigma}^{iBJ}$ and use the identity

$$\sigma_{IA}^i \tilde{\sigma}^{iBJ} = 2\delta_I^J \delta_A^B$$

We then get

$$\delta\phi^{AI} = -2i\tilde{\epsilon}_{+\alpha}^I \chi^{+\beta A}$$

where we define

$$\phi^{AI} = \phi^i \tilde{\sigma}^{iAI}$$

Next

$$\delta\chi = \Gamma^m \hat{\Gamma}^i \epsilon \partial_m \phi^i$$

gives

$$\delta\chi^{+\alpha A} = -(\gamma^m)^\alpha{}_\beta \epsilon_I^{-\beta} \partial_m \phi^{AI}$$

Let us move on to the auxiliary field. Let us consider the term

$$\delta\chi = \hat{\Gamma}^i \nu F^i$$

Writing out indices, this becomes

$$\delta\chi^{+\alpha A} = \tilde{\sigma}^{iAI} \nu_I^{+\alpha} F^i$$

and defining

$$F^{AI} = t\sigma^{iAI} F^i$$

we have

$$\delta\chi^{+\alpha A} = \nu_I^{+\alpha} F^{AI}$$

It is now clear that we can not identify ν_I with ϵ_I since they live in opposite 6d Weyl projection spaces.

Finally we consider

$$\delta F^i = i\bar{\nu} \hat{\Gamma}^i \Gamma^m \partial_m \chi$$

This becomes

$$\delta F^i = i\bar{\nu}_{-\alpha}^I (\gamma^m)^\alpha{}_\beta \sigma_{IA}^i \partial_m \chi^{+\beta A}$$

so if we contract both sides by $\tilde{\sigma}^{iBJ}$ we get

$$\delta F^{AI} = 2i\bar{\nu}_{-\alpha}^I (\gamma^m)^\alpha{}_\beta \partial_m \chi^{+\beta A}$$

Let us now summarize what we have got in the new language,

$$\begin{aligned} \delta\phi^{AI} &= -2i\tilde{\epsilon}^I \chi^A \\ \delta\chi^A &= -\gamma^m \epsilon_I \partial_m \phi^{AI} + \nu_I F^{AI} \\ \delta F^{AI} &= 2i\bar{\nu}^I \gamma^m \partial_m \chi^A \end{aligned}$$

By remembering how we defined the Dirac conjugates and by noting that our definition differs from [20] by signs, we can see that we have got an exact agreement with [20].

The generalization to non-Abelian case reads

$$\begin{aligned} \phi^{AI} &= -2i\tilde{\epsilon}^I \chi^A \\ \delta\chi^A &= -\gamma^m \epsilon_I D_m \phi^{AI} + \nu_I F^{AI} - \gamma^m \epsilon_I [v_m, \sigma, \phi^{AI}] - 4\eta_I \phi^{AI} \\ \delta F^{AI} &= 2i\bar{\nu}^I \gamma^m D_m \chi^A + 2i\bar{\nu}^I \gamma^m [\psi_J, v_m, \phi^{AJ}] + 2i\bar{\nu}^I \gamma^m [\chi^A, v_m, \sigma] \end{aligned}$$

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