

# On a family of $\alpha'$ -corrected solutions of the Heterotic Superstring effective action

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**ABSTRACT:** We compute explicitly the first-order in  $\alpha'$  corrections to a family of solutions of the Heterotic Superstring effective action that describes fundamental strings with momentum along themselves, parallel to solitonic 5-branes with Kaluza-Klein monopoles (Gibbons-Hawking metrics) in their transverse space. These solutions correspond to 4-charge extremal black holes in 4 dimensions upon dimensional reduction on  $T^6$ . We show that some of the  $\alpha'$  corrections can be cancelled by introducing solitonic  $SU(2) \times SU(2)$  Yang-Mills fields, and that this family of  $\alpha'$ -corrected solutions is invariant under  $\alpha'$ -corrected T-duality transformations. We study in detail the mechanism that allows us to compute explicitly these  $\alpha'$  corrections for the ansatz considered here, based on a generalization of the 't Hooft ansatz to hyperKähler spaces.

**KEYWORDS:** Black Holes in String Theory, Superstrings and Heterotic Strings, Superstring Vacua

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## 1 Introduction

Although all the supersymmetric solutions of the Heterotic Superstring effective action have been classified in refs. [1, 2], there are many interesting particular solutions yet to be constructed in detail and studied.

Typically, the construction of the solutions of this theory is made using an ansatz for  $H$ , the 3-form field strength of the Kalb-Ramond 2-form  $B$ , and its Bianchi identity has to be solved together with the equations of motion of all the fields. The preferred way of doing this at first order in  $\alpha'$  is to use the analogue of the Green-Schwarz anomaly-cancellation mechanism and choose a gauge field strength  $F$  such that

$$\alpha' \text{Tr} [F \wedge F + R_{(-)} \wedge R_{(-)}] = 0, \tag{1.1}$$

where  $R_{(-)}$  is the curvature 2-form of the torsionful spin connection  $\Omega_{(-)}$  (See e.g. section 2). Then, solving the Bianchi identity

$$dH - 2\alpha' \text{Tr} [F \wedge F + R_{(-)} \wedge R_{(-)}] = 0, \tag{1.2}$$

reduces to the much simpler problem of finding a closed 3-form  $H$ .

This mechanism constrains the gauge field to be essentially identical to, at least, certain components of the torsionful spin connection. Thus, one may wish to relax as much as possible this condition so that the gauge field can have other values or even not be present at all. However, except in some simple cases, it was not known how to solve the Bianchi identity without using this mechanism.

In ref. [3] we observed that in certain cases the instanton number density  $\text{Tr } F \wedge F$  takes the form of the Laplacian of a function in 4-dimensional Euclidean space ( $\mathbb{E}^4$ ) times the volume 4-form. Therefore, if  $H$  is assumed to be of the form  $H \sim \star_{(4)} d\mathcal{Z}_0$  (up to a closed 3-form on  $\mathbb{E}^4$ ) for some function  $\mathcal{Z}_0$  defined on the same space, the first two terms in the above Bianchi identity become the Laplacian of a linear combination of functions with constant coefficients. Almost magically, the third term turns out to be another Laplacian over the same space and the Bianchi identity is solved by equating the argument of the Laplacian to zero, up to a harmonic function on  $\mathbb{E}^4$ . In the case considered in ref. [3] it was possible to choose the gauge field (a BPST instanton) so as to achieve the above cancellation, but this was not completely necessary and one could study the first-order  $\alpha'$  corrections to the solution consisting in the harmonic function alone.

The configuration considered in ref. [3] corresponds, after compactification on  $T^5$ , to a single, spherically symmetric, 3-charge, extremal 5-dimensional black hole.<sup>1</sup> The modification in the zeroth-order solution introduced by the gauge field was already known from non-Abelian gauged 5-dimensional supergravity [4–6]. The torsionful spin connection behaves as just another gauge field and, quite remarkably, its contribution to the  $\alpha'$  corrections had to be similar to that of the instanton, at least in the above Bianchi identity.

From experience, the simplest generalization one can make to this kind of solutions is to extend the ansatz to multicenter solutions, allowing the functions occurring in the metric to be arbitrary functions of the  $\mathbb{E}^4$  coordinates. In the case of the gauge field, this requires the use of the so-called 't Hooft ansatz that can describe many BPST instantons, and is reviewed and generalized in appendix A. Perhaps not so surprisingly, allowing the function  $\mathcal{Z}_0$  to have arbitrary dependence on the  $\mathbb{E}^4$  coordinates automatically forces some components of the torsionful spin connection to take the form of the 't Hooft ansatz. Then, one can show that the instanton density 4-forms are, once again, Laplacians, and the Bianchi identity can be solved in exactly the same way.

It is natural to wonder if this result can be extended further. An interesting generalization is obtained by replacing  $\mathbb{E}^4$  with a 4-dimensional hyperKähler space that has a curvature with the same selfduality properties as the gauge field. It is well known that the simplest 4-dimensional black holes one can construct in Heterotic Superstring theory include a Kaluza-Klein monopole, which is one of the simplest hyperKähler spaces with one triholomorphic isometry (a Gibbons-Hawking space [7, 8]). The additional isometry is necessary to obtain a 4-dimensional solution by compactification on  $T^6$ . Therefore, this generalization could be used to compute  $\alpha'$  corrections to 4-dimensional black holes such as those considered in refs. [9, 10], which also contain non-Abelian gauge fields.

First of all, one needs to generalize the 't Hooft ansatz to an arbitrary hyperKähler space and show that, again, one gets the Laplacian of some function in that space. We have done this in appendix A. Now, from the torsionful spin connection we get terms with the form of this ansatz, which lead to the same result, and other terms corresponding to the spin connection of the 4-dimensional hyperKähler manifold. Fortunately, the self-duality properties of these two contributions are opposite and they do not mix. However, the

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<sup>1</sup>On top of the function  $\mathcal{Z}_0$ , its fields are described with another two functions,  $\mathcal{Z}_+$  and  $\mathcal{Z}_-$ .

contribution of the latter to the instanton number density might not necessarily take the form of the Laplacian of some function.

At this stage one could try to add a second  $SU(2)$  gauge field whose instanton number density cancels that of the hyperKähler manifold. This is the standard use of the anomaly-cancellation mechanism and has been used, for this kind of solutions<sup>2</sup> in ref. [11]. However, it turns out that, if we restrict ourselves to Gibbons-Hawking spaces, the connection can also be written in an 't Hooft ansatz-like form that we have called *twisted 't Hooft ansatz* (see appendix B) and we get, yet once again, a combination of Laplacians. Adding a second  $SU(2)$  gauge field is optional but convenient if we want to cancel the  $\alpha'$  corrections.

Thus, for the ansatz we are going to make, we are able to solve the Bianchi identity of  $H$  without invoking the anomaly-cancellation mechanism.

It is somewhat surprising that the equations of motion can be solved as well in these conditions and there may be another interesting explanation for it. At any rate, the class of solutions that we find includes all the static, extremal, (supersymmetric) 4-dimensional black holes of Heterotic Superstring theory and their first-order in  $\alpha'$  corrections, a result that deserves to be studied and exploited in more detail elsewhere [12]. In this work we will just obtain the general solution and we will explain, to the best of our knowledge, why it can be obtained at all.

### 1.1 Self-dual connections and the Atiyah-Hitchin-Singer theorem

Before closing this introduction, it is amusing to think about the relation between the 't Hooft ansatz that we use for the Yang-Mills fields and which arises in the torsionful spin connection and the Atiyah-Hitchin-Singer theorem ref. [13] on self-duality in Riemannian geometry.<sup>3</sup> The theorem deals with 4-dimensional Riemannian manifolds and the decomposition of the components of their Levi-Civita spin connection 1-forms into self- and anti-self-dual combinations according to the well-known local isomorphism  $\mathfrak{so}(4) \cong \mathfrak{su}(2)^+ \times \mathfrak{su}(2)^-$ . We will denote the two terms corresponding to this decomposition by  $\omega^{+mn}$ , respectively  $\omega^{-mn}$ . On the one hand, the theorem states about  $\omega^{+mn}$  that

The curvature 2-form of  $\omega^{+mn}$  is self-dual if and only if the manifold is Ricci flat.

This statement applies, in particular, to hyperKähler manifolds, which are Ricci flat and, therefore, for them,  $\omega^{+mn}$  has self-dual curvature. Moreover, since these have special holonomy  $SU(2)$ ,  $\omega^{-mn} = 0$ .

On the other hand, the theorem also says that

The curvature 2-form of  $\omega^{-mn}$  is self-dual if and only if the Ricci scalar vanishes and the manifold is conformal to another one with self-dual curvature 2-form.

This can be used to construct self-dual  $SU(2)$  instantons: consider the metric

$$ds^2 = P^2 d\sigma^2, \tag{1.3}$$

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<sup>2</sup>Without the additional two functions that our class of solutions contains.

<sup>3</sup>The theorem is reviewed and applied to the construction of self-dual Yang-Mills instantons on Gibbons-Hawking spaces in [14, 15]. These instanton solutions were first obtained and studied in references [16, 17].

where  $d\sigma^2$  is a hyperKähler metric and where  $P$  is some function defined on it. The Ricci scalar of the full metric is proportional to the Laplacian of  $P$  in the hyperKähler space and vanishes if  $P$  is harmonic on the hyperKähler metric, so in this case the second part of the theorem applies. If we choose the Vierbein basis  $e^m = Pv^m$  where  $v^m$  is a Vierbein basis of the hyperKähler manifold, the first Cartan structure equation  $de^m + \omega^{mn} \wedge e^n = 0$  leads to

$$d \log P \wedge v^m - \varpi^{mn} \wedge v^n + \omega^{mn} \wedge v^n = 0, \quad \Rightarrow \quad \omega^{mn} = \varpi^{mn} - \partial_{[m} \log P \delta_{n]p} v^p, \quad (1.4)$$

where we have used the same equation for the hyperKähler spin connection  $dv^m + \varpi^{mn} \wedge v^n = 0$ . We can now project the above equation onto the anti-self-dual part of  $\mathfrak{so}(4)$ , i.e.  $\mathfrak{su}(2)^-$ , with the matrices  $(\mathbb{M}_{mn}^-)^{pq}$  defined in eq. (A.5),

$$\omega^{-pq} = (\mathbb{M}_{nm}^-)^{pq} \partial_m \log P v^n, \quad (1.5)$$

and, then, the theorem tells us that the expression in the r.h.s. is a connection with self-dual curvature 2-form, or, equivalently, a  $SU(2)$  gauge connection with self-dual field strength, i.e. an instanton connection. We prove this fact explicitly in appendix A. This provides a justification for the generalized 't Hooft ansatz that we are using, albeit it does not let one suspect that the instanton number density will be proportional to a Laplacian.

On the other hand, if we consider the part of the 10-dimensional metric ansatz conformal to the 4-dimensional hyperKähler manifold, which reads

$$ds^2 = \mathcal{Z}_0 d\sigma^2, \quad (1.6)$$

where, at zeroth-order in  $\alpha'$ ,  $\mathcal{Z}_0$  is a harmonic function in the hyperKähler manifold. Now the Ricci scalar does not vanish, because there is a missing factor of 2 in the exponent of  $\mathcal{Z}_0$ , and the theorem does not apply. This is, nevertheless, the metric associated to solitonic 5-branes, and we cannot change it at will. If we repeat the above calculation we get

$$\omega^{-pq} = \frac{1}{2} (\mathbb{M}_{nm}^-)^{pq} \partial_m \log \mathcal{Z}_0 v^n, \quad (1.7)$$

but now the curvature 2-form of this connection will not be self-dual. Moreover,  $\omega^{+pq}$  contains the spin connection of the hyperKähler manifold  $\varpi^{mn}$  and some additional terms, which spoil self-duality in the  $\mathfrak{su}(2)^+$  part as well.

This is where the magic of the Heterotic Superstring comes to our rescue because, now, the object of interest is not the Levi-Civita connection, but the torsionful spin connection 1-form  $\Omega_{(-)}^{mn} \equiv \omega^{mn} - \frac{1}{2} H_p^{mn} e^p$ , and the contribution of the torsion is such that

$$\Omega_{(-)}^{-pq} = (\mathbb{M}_{nm}^-)^{pq} \partial_m \log \mathcal{Z}_0 v^n, \quad \Omega_{(-)}^{+pq} = \varpi^{mn}. \quad (1.8)$$

Then,  $\Omega_{(-)}^{-pq}$  and  $\varpi^{mn}$  are both Yang-Mills self-dual instantons. The curvature 2-form of these connections will, therefore, be automatically self-dual.

Therefore, in this kind of Heterotic Superstring configurations, the same kind of objects come up naturally in both the Yang-Mills and in the torsionful spin connection, via

the Atiyah-Hitchin-Singer theorem or via a different construction which, perhaps, can be related to a generalization of that theorem. An interesting recent result from ref. [18], which considers the case of compact spaces, sheds light on this direction. It states that given two instantons on a given background that satisfies the equations of motion of the heterotic theory at zeroth order in  $\alpha'$ , it is always possible to rescale this background to obtain a solution of first order in  $\alpha'$ .

The rest of the paper is organized as follows: in section 2 we give a quick review of the low-energy field theory effective action of the Heterotic Superstring in order to set up the problem and fix conventions. In section 3 we introduce the ansatz we will work with, although the details of the (generalized) 't Hooft ansatz for the gauge fields and its relation with the spin connection of Gibbons-Hawking spaces are to be found in the appendices. In section 4 we show that all the field configurations corresponding to our ansatz preserve 1/4 of the 16 possible supersymmetries, irrespectively of whether they solve the equations of motion or not. In section 5 we plug the ansatz into and solve the equations of motion to first-order in  $\alpha'$ , using the above mechanism and which is explained in more detail in the appendices. In section 6 we study the behavior of the solution under  $\alpha'$ -corrected T-duality transformations in the direction in which the strings lie and the waves propagate (thereby interchanging them), as well as in the isometric direction of the Gibbons-Hawking space. Finally, in section 7 we make some general considerations on the validity of these solutions to higher orders in  $\alpha'$ .

## 2 The Heterotic Superstring effective action to $\mathcal{O}(\alpha')$

In order to describe the Heterotic Superstring effective action to  $\mathcal{O}(\alpha')$  as given in ref. [19] (but in string frame), we start by defining the zeroth-order 3-form field strength of the Kalb-Ramond 2-form  $B$ :

$$H^{(0)} \equiv dB, \tag{2.1}$$

and constructing with it the zeroth-order torsionful spin connections

$$\Omega_{(\pm)}^{(0) a b} = \omega^a_b \pm \frac{1}{2} H_{\mu}^{(0) a b} dx^{\mu}, \tag{2.2}$$

where  $\omega^a_b$  is the Levi-Civita spin connection 1-form.<sup>4</sup> With them we define the zeroth-order Lorentz curvature 2-form and Chern-Simons 3-forms

$$R_{(\pm)}^{(0) a b} = d\Omega_{(\pm)}^{(0) a b} - \Omega_{(\pm)}^{(0) a c} \wedge \Omega_{(\pm)}^{(0) c b}, \tag{2.3}$$

$$\omega_{(\pm)}^{L(0)} = d\Omega_{(\pm)}^{(0) a b} \wedge \Omega_{(\pm)}^{(0) b a} - \frac{2}{3} \Omega_{(\pm)}^{(0) a b} \wedge \Omega_{(\pm)}^{(0) b c} \wedge \Omega_{(\pm)}^{(0) c a}. \tag{2.4}$$

Next, we introduce the gauge fields. We will only activate a  $SU(2) \times SU(2)$  subgroup and we will denote by  $A^{A_{1,2}}$  ( $A_{1,2} = 1, 2, 3$ ) the components. The gauge field strength and

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<sup>4</sup>We follow the conventions of ref. [20] for the spin connection, the curvature and the gamma matrices.

the Chern-Simons 3-form for each SU(2) factor are defined by

$$F^A = dA^A + \frac{1}{2}\epsilon^{ABC}A^B \wedge A^C, \quad (2.5)$$

$$\omega^{\text{YM}} = dA^A \wedge A^A + \frac{1}{3}\epsilon^{ABC}A^A \wedge A^B \wedge A^C. \quad (2.6)$$

Then, we are ready to define recursively

$$\begin{aligned} H^{(1)} &= dB + 2\alpha' \left( \omega^{\text{YM}} + \omega_{(-)}^{\text{L}(0)} \right), \\ \Omega_{(\pm)}^{(1)ab} &= \omega^a{}_b \pm \frac{1}{2}H_{\mu}^{(1)a}{}_b dx^{\mu}, \\ R_{(\pm)}^{(1)ab} &= d\Omega_{(\pm)}^{(1)ab} - \Omega_{(\pm)}^{(1)a}{}_c \wedge \Omega_{(\pm)}^{(1)c}{}_b, \\ \omega_{(\pm)}^{\text{L}(1)} &= d\Omega_{(\pm)}^{(1)ab} \wedge \Omega_{(\pm)}^{(1)b}{}_a - \frac{2}{3}\Omega_{(\pm)}^{(1)ab} \wedge \Omega_{(\pm)}^{(1)bc} \wedge \Omega_{(\pm)}^{(1)ca}. \\ H^{(2)} &= dB + 2\alpha' \left( \omega^{\text{YM}} + \omega_{(-)}^{\text{L}(1)} \right), \end{aligned} \quad (2.7)$$

and so on.

In practice only  $\Omega_{(\pm)}^{(0)}, R_{(\pm)}^{(0)}, \omega_{(\pm)}^{\text{L}(0)}, H^{(1)}$  will occur to the order we want to work at, but, often, it is simpler to work with the higher-order objects ignoring the terms of higher order in  $\alpha'$  when necessary. Thus we will suppress the  $(n)$  upper indices.

Finally, we define three “ $T$ -tensors” associated to the  $\alpha'$  corrections

$$\begin{aligned} T^{(4)} &\equiv 6\alpha' \left[ F^A \wedge F^A + R_{(-)}^a{}_b \wedge R_{(-)}^b{}_a \right], \\ T^{(2)}{}_{\mu\nu} &\equiv 2\alpha' \left[ F^A{}_{\mu\rho} F^A{}_{\nu\rho} + R_{(-)\mu\rho}^a{}_b R_{(-)\nu\rho}^b{}_a \right], \\ T^{(0)} &\equiv T^{(2)}{}_{\mu}{}^{\mu}. \end{aligned} \quad (2.8)$$

In terms of all these objects, the Heterotic Superstring effective action in the string frame and to first-order in  $\alpha'$  can be written as

$$S = \frac{g_s^2}{16\pi G_N^{(10)}} \int d^{10}x \sqrt{|g|} e^{-2\phi} \left\{ R - 4(\partial\phi)^2 + \frac{1}{2 \cdot 3!} H^2 - \frac{1}{2} T^{(0)} \right\}, \quad (2.9)$$

where  $G_N^{(10)}$  is the 10-dimensional Newton constant, whose precise value will not concern us here,  $\phi$  is the dilaton field and the vacuum expectation value of  $e^{\phi}$  is the Heterotic Superstring coupling constant  $g_s$ .  $R$  is the Ricci scalar of the string-frame metric  $g_{\mu\nu}$ .

The equations of motion are very complicated, but, following section 3 of ref. [21], we separate the variations with respect to each field into those corresponding to occurrences via  $\Omega_{(-)}^a{}_b$ , that we will call *implicit*, and the rest, that we will call *explicit*:

$$\begin{aligned} \delta S &= \frac{\delta S}{\delta g_{\mu\nu}} \delta g_{\mu\nu} + \frac{\delta S}{\delta B_{\mu\nu}} \delta B_{\mu\nu} + \frac{\delta S}{\delta A^{A_i}{}_{\mu}} \delta A^{A_i}{}_{\mu} + \frac{\delta S}{\delta \phi} \delta \phi \\ &= \frac{\delta S}{\delta g_{\mu\nu}} \Big|_{\text{exp.}} \delta g_{\mu\nu} + \frac{\delta S}{\delta B_{\mu\nu}} \Big|_{\text{exp.}} \delta B_{\mu\nu} + \frac{\delta S}{\delta A^{A_i}{}_{\mu}} \Big|_{\text{exp.}} \delta A^{A_i}{}_{\mu} + \frac{\delta S}{\delta \phi} \delta \phi \\ &\quad + \frac{\delta S}{\delta \Omega_{(-)}^a{}_b} \left( \frac{\delta \Omega_{(-)}^a{}_b}{\delta g_{\mu\nu}} + \frac{\delta \Omega_{(-)}^a{}_b}{\delta B_{\mu\nu}} \delta B_{\mu\nu} + \frac{\delta \Omega_{(-)}^a{}_b}{\delta A^{A_i}{}_{\mu}} \delta A^{A_i}{}_{\mu} \right). \end{aligned} \quad (2.10)$$

We can then apply a lemma proven in ref. [19]:  $\delta S/\delta\Omega_{(-)}^{ab}$  is proportional to  $\alpha'$  and to the zeroth-order equations of motion of  $g_{\mu\nu}$ ,  $B_{\mu\nu}$  and  $\phi$  plus terms of higher order in  $\alpha'$ .

The upshot is that, if we consider field configurations which solve the zeroth-order equations of motion<sup>5</sup> up to terms of order  $\alpha'$ , the contributions to the equations of motion associated to the implicit variations are at least of second order in  $\alpha'$  and we can safely ignore them here.

If we restrict ourselves to this kind of field configurations, the equations of motion reduce to

$$R_{\mu\nu} - 2\nabla_\mu\partial_\nu\phi + \frac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} - T^{(2)}{}_{\mu\nu} = 0, \tag{2.11}$$

$$(\partial\phi)^2 - \frac{1}{2}\nabla^2\phi - \frac{1}{4\cdot 3!}H^2 + \frac{1}{8}T^{(0)} = 0, \tag{2.12}$$

$$d\left(e^{-2\phi}\star H\right) = 0, \tag{2.13}$$

$$\alpha'e^{2\phi}\mathfrak{D}_{(+)}\left(e^{-2\phi}\star F^{A_i}\right) = 0, \tag{2.14}$$

where  $\mathfrak{D}_{(+)}$  stands for the exterior derivative covariant with respect to each  $SU(2)$  subgroup and with respect to the torsionful connection  $\Omega_{(+)}$ : suppressing the subindices 1, 2 that distinguish the two subgroups

$$e^{2\phi}d\left(e^{-2\phi}\star F^A\right) + \epsilon^{ABC}A^B\wedge\star F^C + \star H\wedge F^A = 0. \tag{2.15}$$

If the ansatz is given in terms of the 3-form field strength we will need to solve the Bianchi identity

$$dH - \frac{1}{3}T^{(4)} = 0, \tag{2.16}$$

as well.

### 3 The ansatz

It is convenient to describe our ansatz for each field separately, starting with the metric, which is assumed to take the general form

$$ds^2 = \frac{2}{\mathcal{Z}_-}du\left[ dv - \frac{1}{2}\mathcal{Z}_+du \right] - \mathcal{Z}_0d\sigma^2 - dy^i dy^i, \tag{3.1}$$

where

$$d\sigma^2 = h_{mn}dx^m dx^n, \quad m, n = \sharp, 1, 2, 3, \tag{3.2}$$

is the metric of a 4-dimensional hyperKähler space and  $\mathcal{Z}_+$ ,  $\mathcal{Z}_-$ ,  $\mathcal{Z}_0$  are functions on that 4-dimensional space. Thus, the metric is independent of the light-cone coordinates  $u, v$  and of the 4 spatial coordinates  $y^i$ ,  $i, j = 1, 2, 3, 4$ . The hyperKähler metric is characterized by the self-duality of its spin connection 1-form  $\varpi^{mn}$  with respect to the orientation  $\varepsilon^{\sharp 123} = +1$  in an appropriate Vierbein basis  $v^m$

$$h_{\underline{mn}} = v^{\underline{p}}{}_{\underline{m}}v^{\underline{p}}{}_{\underline{n}}. \tag{3.3}$$

In order to be able to solve the Bianchi identity of the 3-form  $H$  to first order in  $\alpha'$ , in section 5 we will find it convenient to restrict ourselves to Gibbons-Hawking (GH) spaces.

<sup>5</sup>These can be obtained from eqs. (2.11)–(2.14) by setting  $\alpha' = 0$ . This eliminates the Yang-Mills fields, the  $T$ -tensors and the Chern-Simons terms in  $H$ .



The 3-form field strength is assumed to take the form

$$H = d\mathcal{Z}_-^{-1} \wedge du \wedge dv + \star_{(4)} d\mathcal{Z}_0, \tag{3.4}$$

where  $\star_{(4)}$  is the Hodge operator in the 4-dimensional hyperKähler metric  $d\sigma^2$  with the above choice of orientation.

The dilaton field is given by

$$e^{-2\phi} = e^{-2\phi_\infty} \frac{\mathcal{Z}_-}{\mathcal{Z}_0}, \tag{3.5}$$

where  $\phi_\infty$  is a constant that, in spaces which asymptote to some vacuum solution, can be identified with the vacuum expectation value, i.e.  $e^{\phi_\infty} = g_s$ .

Finally, we will assume each of the  $SU(2)$  field strengths to live and be self-dual in the 4-dimensional hyperKähler space with the same orientation above:

$$F^{A_{1,2}} = + \star_{(4)} F^{A_{1,2}}. \tag{3.6}$$

In order to solve explicitly the equations of motion and, especially, the Bianchi identity of the 3-form  $H$  to first order in  $\alpha'$ , it is necessary to know explicitly the 1-form connections. Thus, we are going to propose an ansatz for them which, as shown in appendix A, automatically gives self-dual 2-form field strengths in hyperKähler manifolds and which has other advantages that will be discussed later. This ansatz is most naturally written using pairs of antisymmetric, self-dual,  $SO(4)$  indices  $mn$  as adjoint  $SU(2)$  indices:

$$A_{1,2}^{mn} = (\mathbb{M}_{pq}^{(-)})^{mn} \partial_q \log P_{1,2} v^p, \tag{3.7}$$

where  $\mathbb{M}_{pq}^{(-)}$  are the self-dual generators of  $\mathfrak{so}(4)$ , defined in eq. (A.5),  $\partial_q = v_q^m \partial_m$ , and the functions  $P_1$  and  $P_2$  are harmonic in the hyperKähler space

$$\nabla_{(4)}^2 P_{1,2} = 0. \tag{3.8}$$

In this article we only consider gauge fields (3.7) with finite instanton number. In particular the harmonic functions that characterize them are positive-definite and have at most pointlike singularities in the 4-dimensional hyperKähler space, see [22]. In the case in which the hyperKähler metric is a Gibbons-Hawking space, this implies that any pole of the harmonic functions characterizing the Yang-Mills instanton must coincide with a pole of the Gibbons-Hawking harmonic function.

This ansatz generalizes the one recently considered in ref. [3] for a single, static, 3-charge plus non-Abelian instanton black hole in three respects:

1. No spherical symmetry is assumed: the ansatz can describe multicenter configurations.
2. The  $\mathbb{R}^4$  space transverse to the S5-branes has been replaced by an arbitrary hyperKähler space.
3. A second  $SU(2)$  gauge field has been added to the theory. We will show that it can be used to suppress  $\alpha'$  corrections associated to the non-trivial hyperKähler space, just as the first  $SU(2)$  gauge field can compensate the  $\alpha'$  corrections associated to the S5-brane.

## 4 Supersymmetry of the ansatz

All the configurations encompassed by our ansatz preserve 1/4 of the 16 possible supersymmetries, no matter whether they solve the equations of motion or not. The Killing spinor equations associated to the local supersymmetry transformations of the gravitino, dilatino and gaugino are, respectively

$$\nabla_{\mu}^{(+)}\epsilon \equiv \left( \partial_{\mu} - \frac{1}{4}\mathcal{Q}_{(+)\mu} \right) \epsilon = 0, \quad (4.1)$$

$$\left( \not{\partial}\phi - \frac{1}{12}\mathcal{H} \right) \epsilon = 0, \quad (4.2)$$

$$-\frac{1}{4}\alpha' F^{A1,2}\epsilon = 0. \quad (4.3)$$

and, using the results of appendix C it is easy to see that the above equations take the same form as in section 2.1 of ref. [3], except for the  $m$  components of the first equation, which receives a contribution from the spin connection of the 4-dimensional hyperKähler space and the “doubling” of the last equation, owed to the presence of a second SU(2) gauge field.

Since the contribution of the spin connection of the 4-dimensional hyperKähler space is self-dual, just as the contribution coming from the conformal factor  $\mathcal{Z}_0$ , the  $m$  component of the equation simply gets another term containing the chirality projector  $\frac{1}{2}(1 - \tilde{\Gamma})$  where  $\tilde{\Gamma} \equiv \Gamma^{2345}$  is the chirality matrix in the 4-dimensional hyperKähler space. Since the two SU(2) gauge fields have self-dual field strengths, the two associated equations (4.3) contain the same chirality projector  $\frac{1}{2}(1 - \tilde{\Gamma})$  acting on  $\epsilon$ .

In order to make the paper more self-contained, we write below all the components of the Killing spinor equations in the frame specified in appendix C

$$\left[ \partial_{+} + \frac{1}{4} \frac{\mathcal{Z}_{-}\partial_m\mathcal{Z}_{+}}{\mathcal{Z}_0^{1/2}} \Gamma^m \Gamma^{+} \right] \epsilon = 0, \quad (4.4)$$

$$\left[ \partial_{-} + \frac{1}{2} \frac{\partial_m \log \mathcal{Z}_{-}}{\mathcal{Z}_0^{1/2}} \Gamma^m \Gamma^{+} \right] \epsilon = 0, \quad (4.5)$$

$$\left\{ \partial_m + \frac{1}{8\mathcal{Z}_0^{1/2}} [\partial_q \log H(\mathbb{N}_{np}^+)_{qm} + \partial_q \log \mathcal{Z}_0(\mathbb{M}_{qm}^+)_{np}] \Gamma^{np} (1 - \tilde{\Gamma}) \right\} \epsilon = 0, \quad (4.6)$$

$$\partial_i \epsilon = 0, \quad (4.7)$$

$$-\frac{1}{2\mathcal{Z}_0^{1/2}} \Gamma^m \left[ \partial_m \log \mathcal{Z}_{-} \Gamma^{-} \Gamma^{+} - \partial_m \log \mathcal{Z}_0 (1 - \tilde{\Gamma}) \right] \epsilon = 0, \quad (4.8)$$

$$-\frac{1}{8}\alpha' F^{A1,2} (1 - \tilde{\Gamma}) \epsilon = 0. \quad (4.9)$$

We conclude that the Killing spinor equations are solved by constant spinors satisfying the constraints

$$\tilde{\Gamma}\epsilon = +\epsilon, \quad \Gamma^{+}\epsilon = 0, \quad (4.10)$$

exactly as in the solution studied in ref. [3].

## 5 Solving the equations of motion

Since our ansatz is given in terms of the 3-form field strength, it is convenient to start by solving its Bianchi identity eq. (2.16). The fact that it can be solved is one of our main results.

Due to the structure of our ansatz for  $H$ ,  $dH$  is just a Laplacian in the 4-dimensional hyperKähler space:

$$dH = d \star_{(4)} d\mathcal{Z}_0 = -\nabla_{(4)}^2 \mathcal{Z}_0 |v| d^4x. \quad (5.1)$$

The  $T^{(4)}$  tensor has three different pieces:

$$T^{(4)} = 6\alpha' \left[ F^{A_1} \wedge F^{A_1} + F^{A_2} \wedge F^{A_2} + R_{(-)}^{a_b} \wedge R_{(-)}^{b_a} \right]. \quad (5.2)$$

Since we are using a 't Hooft ansatz for the  $SU(2)$  gauge fields, we can directly use the result in eq. (A.30). Furthermore, since our hyperKähler space is, by assumption, a GH space, we can use the result in eq. (B.12) and, substituting these partial results in eq. (2.16), we get

$$\nabla_{(4)}^2 \left\{ \mathcal{Z}_0 + 2\alpha' \left[ (\partial \log P_1)^2 + (\partial \log P_2)^2 - (\partial \log \mathcal{Z}_0)^2 - (\partial \log H)^2 \right] \right\} |v| d^4x = \mathcal{O}(\alpha'^2), \quad (5.3)$$

which is solved exactly to this order by<sup>6</sup>

$$\mathcal{Z}_0 = \mathcal{Z}_0^{(0)} - 2\alpha' \left[ (\partial \log P_1)^2 + (\partial \log P_2)^2 - (\partial \log \mathcal{Z}_0^{(0)})^2 - (\partial \log H)^2 \right] + \mathcal{O}(\alpha'^2), \quad (5.4)$$

with

$$\nabla_{(4)}^2 \mathcal{Z}_0^{(0)} = 0. \quad (5.5)$$

Some regular gauge fields, when written in the gauge associated to the 't Hooft ansatz, have singularities that can be removed by a gauge transformation. However, these unphysical singularities end up contributing to the instanton number densities  $F^A \wedge F^A$  and  $R_{(-)}^{a_b} \wedge R_{(-)}^{b_a}$  as  $\delta$ -functions, basically because one is taking derivatives at points in which the local form of gauge field we are using becomes singular. In virtue of the removable singularity theorem of Uhlenbeck ref. [22], it is possible to perform a local gauge transformation that precisely removes those singularities from the evaluation of the instanton number densities and, in the preceding expressions this should carefully be done in the terms inside the squared brackets. Thus, if the gauge fields are indeed regular, and one has eliminated those singularities, the only  $\delta$ -function singularities that remain are those associated to the harmonic functions  $\mathcal{Z}^{(0)}$  and these singularities will be associated to the presence of branes which source the fields at the locations of those  $\delta$ -functions. These delocalized contributions associated to the instantons correspond, precisely, to the non-singular terms in brackets.

The removal of the singularities is a very subtle problem, because, in the end, the hyperKähler space is not part of the physical space, which is the one that dictates where the physical singularities are and we will not deal with it here. However, this is an important

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<sup>6</sup>The equations are solved everywhere except at the singularities of the harmonic function  $\mathcal{Z}_0^{(0)}$ , which, in general, will give  $\delta$ -function singularities that, in general, indicate the presence of solitonic 5-branes.

issue from the physical point of view which should be discussed in more depth on a case by case basis. We will make some further comments concerning this point in section 6.

Let us now move to the equations of motion (2.11)–(2.14).

The ansatz automatically solves the Yang-Mills equation (2.14)–(2.15).

The Kalb-Ramond field equation (2.13) reduces to a Laplace equation in the hyperKähler space

$$\nabla_{(4)}^2 \mathcal{Z}_- = 0. \tag{5.6}$$

Using the expressions above it is straightforward to conclude that the  $(++)$  component<sup>7</sup> of the Einstein equations (which is the only non-trivial equation for our ansatz) gives

$$\mathcal{Z}_+ = \mathcal{Z}_+^{(0)} + \mathcal{O}(\alpha'), \quad \text{with} \quad \nabla_{(4)}^2 \mathcal{Z}_+^{(0)} = 0, \tag{5.7}$$

with the  $\mathcal{O}(\alpha')$  corrections vanishing identically for Heterotic supergravity. In order to add the stringy corrections one has to evaluate the  $(++)$  component of the  $T^{(2)}$  tensor:

$$T_{++}^{(2)} = -2\alpha' R_{(-)+abc} R_{(-)+}{}^{abc} = -2\alpha' \frac{\mathcal{Z}_-}{\mathcal{Z}_0} \nabla_{(4)}^2 \left( \frac{\partial_n \mathcal{Z}_+^{(0)} \partial_n \mathcal{Z}_-}{\mathcal{Z}_0^{(0)} \mathcal{Z}_-} \right) + \mathcal{O}(\alpha'^2). \tag{5.8}$$

Then

$$\mathcal{Z}_+ = \mathcal{Z}_+^{(0)} - 4\alpha' \left( \frac{\partial_n \mathcal{Z}_+^{(0)} \partial_n \mathcal{Z}_-}{\mathcal{Z}_0^{(0)} \mathcal{Z}_-} \right) + \mathcal{O}(\alpha'^2). \tag{5.9}$$

Obviously, the same comments concerning the removal of spurious singularities applies to this  $\alpha'$  correction.

The dilaton equation (2.12) is automatically solved in these conditions and needs not to be checked explicitly. Then, given a solution to  $\mathcal{O}(\alpha^0)$  of the form of our ansatz, which is completely determined by the harmonic functions  $\mathcal{Z}_{+,-,0}^{(0)}$  and  $H^{(0)}$ , the most general  $\alpha'$ -corrected solution of the same form will be determined by the corrected functions

$$\mathcal{Z}_+ = \mathcal{Z}_+^{(0)} - 4\alpha' \left( \frac{\partial_n \mathcal{Z}_+^{(0)} \partial_n \mathcal{Z}_-^{(0)}}{\mathcal{Z}_0^{(0)} \mathcal{Z}_-} \right) + \mathcal{O}(\alpha'^2), \tag{5.10}$$

$$\mathcal{Z}_- = \mathcal{Z}_-^{(0)} + \mathcal{O}(\alpha'^2), \tag{5.11}$$

$$\mathcal{Z}_0 = \mathcal{Z}_0^{(0)} - 2\alpha' \left[ (\partial \log P_1^{(0)})^2 + (\partial \log P_2^{(0)})^2 - (\partial \log \mathcal{Z}_0^{(0)})^2 - (\partial \log H^{(0)})^2 \right] + \mathcal{O}(\alpha'^2), \tag{5.12}$$

$$H = H^{(0)} + \mathcal{O}(\alpha'^2), \tag{5.13}$$

$$P_{1,2} = P_{1,2}^{(0)} + \mathcal{O}(\alpha'^2). \tag{5.14}$$

This is the main result of our paper. To get a better understanding of this family of solutions, we are going to study, first, their behavior under T-duality transformations.

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<sup>7</sup>We use the frame specified in equation (C.1).

## 6 $\alpha'$ -corrected T-duality

As we have discussed in section 3, the solutions we have found are a generalization of those studied in ref. [3] with a very similar structure but more non-trivial harmonic functions that can be interpreted as describing more extended objects.  $\mathcal{Z}_{-,+,0}$ , present in the solution of ref. [3], are associated, respectively, to fundamental strings (F1), momentum along the strings (W) and Neveu-Schwarz (solitonic) 5-branes (S5).  $P_{1,2}$  are associated to gauge 5-branes sourced by the instantons. The qualitatively new feature is the non-trivial hyperKähler space which, generically, describes gravitational instantons, and the additional (triholomorphic) isometry of this space, which reduces the possible hyperKähler spaces to be of GH type. These are completely determined by a harmonic function,  $H$ . The typical choice  $H = 1 + 1/r$  corresponds to a Kaluza-Klein (KK) monopole, also called (Euclidean) Taub-NUT space.

In ref. [3] we studied how T-duality acts in the direction of propagation and winding of the F1 in the presence of first-order  $\alpha'$  corrections which affect  $\mathcal{Z}_+$  but not  $\mathcal{Z}_-$ . At zeroth order, the standard Buscher rules would simply interchange the complete  $\mathcal{Z}_+$  and  $\mathcal{Z}_-$  functions, including the  $\alpha'$  corrections. When first-order corrections are included, this would be wrong since the dualized solution belongs to the same ansatz and only the transformed  $\mathcal{Z}'_+$  can receive  $\alpha'$  corrections.

Somewhat extraordinarily, using the  $\alpha'$ -corrected Buscher rules proposed in ref. [23], we showed that the  $\alpha'$  corrections of the transformed solution only occur where they should and, therefore, the solutions, as a family, are self-T-dual, as it happens at zeroth-order in  $\alpha'$ . This is a highly non-trivial test for both the solutions and the T-duality rules.

The existence of a second non-trivial isometry in the GH space transverse to the S5-branes provides us with another non-trivial test. At zeroth order in  $\alpha'$ , the single S5-brane solution and the KK monopole are T-dual, and T-duality simply interchanges their associated harmonic functions  $\mathcal{Z}_0$  and  $H$ . Now, only the former has  $\alpha'$  corrections and T-duality should leave them there. The solutions we have found should be self-T-dual as a family.

If we perform a T-duality transformation in the direction  $x$ , the  $\alpha'$ -corrected T-duality rules proposed in ref. [23] read ( $\mu, \nu \neq \underline{x}$ )

$$\begin{aligned}
 g'_{\mu\nu} &= g_{\mu\nu} + [g_{\underline{x}\underline{x}}G_{\underline{x}\mu}G_{\underline{x}\nu} - 2G_{\underline{x}\underline{x}}G_{\underline{x}(\mu}g_{\nu)\underline{x}}] / G_{\underline{x}\underline{x}}^2, \\
 B'_{\mu\nu} &= B_{\mu\nu} - G_{\underline{x}[\mu}G_{\nu]\underline{x}} / G_{\underline{x}\underline{x}}, \\
 g'_{\underline{x}\mu} &= -g_{\underline{x}\mu} / G_{\underline{x}\underline{x}} + g_{\underline{x}\underline{x}}G_{\underline{x}\mu} / G_{\underline{x}\underline{x}}^2, & B'_{\underline{x}\mu} &= -B_{\underline{x}\mu} / G_{\underline{x}\underline{x}} - G_{\underline{x}\mu} / G_{\underline{x}\underline{x}}, \\
 g'_{\underline{x}\underline{x}} &= g_{\underline{x}\underline{x}} / G_{\underline{x}\underline{x}}^2, & e^{-2\phi'} &= e^{-2\phi} |G_{\underline{x}\underline{x}}|, \\
 A'^A_{\underline{x}} &= -A^A_{\underline{x}} / G_{\underline{x}\underline{x}}, & A'^A_{\mu} &= A^A_{\mu} - A^A_{\underline{x}} G_{\underline{x}\mu} / G_{\underline{x}\underline{x}},
 \end{aligned} \tag{6.1}$$

where  $G_{\mu\nu}$  (for all the possible values of the indices  $\mu, \nu$  including  $\underline{x}$ ) is defined by

$$G_{\mu\nu} \equiv g_{\mu\nu} - B_{\mu\nu} - 2\alpha' \left\{ A^A_{\mu} A^A_{\nu} + \Omega_{(-)\mu}{}^a{}_{\nu} \Omega_{(-)\nu}{}^b{}_{\mu} \right\}. \tag{6.2}$$

The use of these rules requires the explicit knowledge of the components of the Kalb-Ramond 2-form  $B$ , which are gauge-dependent. It is natural to use the gauge of the 't Hooft

ansatz in which the Chern-Simons terms take the forms computed in eqs. (A.29) and (C.8), which we reproduce here for convenience<sup>8</sup>

$$\omega^{YM} = -\star d [(\partial \log P_1)^2 + (\partial \log P_2)^2] + \mathcal{O}(\alpha'^2), \quad (6.3)$$

$$\omega_{(-)}^L = \star_{(4)} d [(\partial \log H)^2 + (\partial \log \mathcal{Z}_0)^2] + \mathcal{O}(\alpha'^2). \quad (6.4)$$

Then,

$$dB = H - 2\alpha'(\omega^{YM} + \omega_{(-)}^L) = \star_{(4)} d\mathcal{Z}_0^{(0)} + d\frac{1}{\mathcal{Z}_-} \wedge du \wedge dv + \mathcal{O}(\alpha'^2), \quad (6.5)$$

and

$$B = \xi_0 + \frac{1}{\mathcal{Z}_-} du \wedge dv + \mathcal{O}(\alpha'^2), \quad (6.6)$$

where  $\xi_0 = \frac{1}{2}\xi_{0mn}v^m \wedge v^n$  is a 2-form on the hyperKähler space such that

$$d\xi_0 = \star_{(4)} d\mathcal{Z}_0^{(0)}. \quad (6.7)$$

The integrability condition of this equation is the harmonicity of  $\mathcal{Z}_0^{(0)}$  in the hyperKähler space, which guarantees the existence of  $\xi_0$ .

In order to apply the Buscher T-duality rules, one needs to compute the tensor  $G_{\mu\nu}$  defined above in eq. (6.2). In 10-dimensional flat indices, its non-vanishing components are<sup>9</sup>

$$G_{++} = -4\alpha' \frac{\partial_m \mathcal{Z}_+ \partial_m \mathcal{Z}_-}{\mathcal{Z}_0}, \quad (6.8)$$

$$G_{-+} = 2, \quad (6.9)$$

$$G_{ij} = -\delta_{ij}, \quad (6.10)$$

$$\begin{aligned} G_{mn} = & -\delta_{mn} - \frac{\xi_{0mn}}{\mathcal{Z}_0} - \frac{2\alpha'}{\mathcal{Z}_0} \left\{ \delta_{mn} \left[ (\partial \log P_1)^2 + (\partial \log P_2)^2 - (\partial \log H)^2 - (\partial \log \mathcal{Z}_0^{(0)})^2 \right] \right. \\ & - \partial_m \log P_1 \partial_n \log P_1 - \partial_m \log P_2 \partial_n \log P_2 + \partial_m \log H \partial_n \log H + \partial_m \log \mathcal{Z}_0^{(0)} \partial_n \log \mathcal{Z}_0^{(0)} \\ & \left. + 2\partial_m \log \mathcal{Z}_- \partial_n \log \mathcal{Z}_- \right\}. \end{aligned} \quad (6.11)$$

If one makes the coordinate transformation  $X = Au + Bv$ ,  $Y = Cu + Dv$ , with  $AD - BC = 1$  and then T-dualizes along  $X$ , we get the T-dual solution

$$ds^{2'} = \frac{2}{\mathcal{Z}'_-} dX \left( dY - \frac{1}{2} \mathcal{Z}'_+ dX \right) - \mathcal{Z}_0 h_{mn} dx^m dx^n - dy^i dy^i, \quad (6.12)$$

$$A^{A1,2'} = A^{A1,2}, \quad (6.13)$$

$$B' = \xi_0 + \left( \frac{B}{D} + \frac{1}{\mathcal{Z}'_-} \right) dX \wedge dY, \quad (6.14)$$

$$e^{2\phi'} = e^{2\phi_\infty} \frac{\mathcal{Z}_0}{\mathcal{Z}'_-}, \quad (6.15)$$

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<sup>8</sup>According to the discussion in the previous section, in certain cases at least, we should eliminate the spurious singularities from these Chern-Simons terms. In general, this should simply result in a shift by a harmonic function of  $\mathcal{Z}_0$  that can be absorbed in  $\mathcal{Z}_0^{(0)}$ .

<sup>9</sup>Observe that some of these components have singularities associated to the 't Hooft ansatz gauge.  $G_{\mu\nu}$  is not a gauge-invariant quantity, though, and T-duality does not commute with gauge or Lorentz transformations. Therefore, it is not clear at all whether these singularities should and can be removed. Again, this is a problem to be studied on a case by case basis and we will not discuss it here any further.

with

$$\mathcal{Z}'_- = D \left( 2C + D\mathcal{Z}_+^{(0)} \right), \quad (6.16)$$

$$\mathcal{Z}'_+ = \mathcal{Z}_- - 4\alpha' \frac{D\partial_n \mathcal{Z}_- \partial_n \mathcal{Z}_+^{(0)}}{\mathcal{Z}_0^{(0)}(2C + D\mathcal{Z}_+^{(0)})} = \mathcal{Z}_- - 4\alpha' \frac{\partial_n \mathcal{Z}_- \partial_n \mathcal{Z}'_-}{\mathcal{Z}_0^{(0)} \mathcal{Z}'_-}. \quad (6.17)$$

Choosing  $C = 0, A = D = 1$  we preserve the asymptotic behavior of the harmonic functions and, calling  $Y \equiv v'$  and  $X \equiv u'$  it is immediate to see that the T-dual solution belongs to the same family as the original. This is the result obtained in ref. [3] extended to the presence of a hyperKähler transverse space.

If  $\mathcal{Z}_{+,-,0}, P_{1,2}$  are independent of the coordinate adapted to the triholomorphic isometry of the GH metric,  $z$ , (as  $H$  is), then the isometry of the GH space is also an isometry of the full solution and one can T-dualize it along  $z$ . In this case, the harmonic functions are harmonic with respect to 3-dimensional Euclidean space  $\mathbb{E}^3$  and eq. (6.7) can be rewritten as

$$d\xi_0 = (dz + \chi) \wedge \star_{(3)} d\mathcal{Z}_0^{(0)} \equiv (dz + \chi) \wedge d\chi_0, \quad \text{where} \quad \begin{cases} d\chi = \star_{(3)} dH, \\ d\chi_0 \equiv \star_{(3)} d\mathcal{Z}_0^{(0)}, \end{cases} \quad (6.18)$$

This implies that, up to a closed 2-form,

$$\xi_0 = \chi_0 \wedge (dz + \chi) + \tilde{\xi}_0, \quad (6.19)$$

where  $\tilde{\xi}_0$  is a 2-form on  $\mathbb{E}^3$  such that

$$d\tilde{\xi}_0 = d\chi \wedge \chi_0. \quad (6.20)$$

Observe that  $\tilde{\xi}_0$  does not have any  $z$  components.

Then, the original solution, written in coordinates adapted to the isometry we want to T-dualize with respect to, is

$$ds^2 = \frac{2}{\mathcal{Z}_-} du \left( dv - \frac{1}{2} \mathcal{Z}_+ du \right) - \mathcal{Z}_0 \left[ \frac{1}{H} (dz + \chi)^2 + H dx^r dx^r \right] - dy^i dy^i, \quad (6.21)$$

$$\begin{aligned} A_{1,2} &= \mathbb{M}_{mn}^- \partial_n \log P_{1,2} v^m \\ &= H^{-1} \mathbb{M}_{\#r}^- \partial_r \log P_{1,2} (dz + \chi) + \mathbb{M}_{sr}^- \partial_r \log P_{1,2} dx^s, \end{aligned} \quad (6.22)$$

$$B = \chi_0 \wedge (dz + \chi) + \tilde{\xi}_0 + \frac{1}{\mathcal{Z}_-} du \wedge dv, \quad (6.23)$$

$$e^{2\phi} = e^{2\phi_\infty} \frac{\mathcal{Z}_0}{\mathcal{Z}_-}, \quad (6.24)$$

and the T-dual solution is

$$ds^{2'} = \frac{2}{\mathcal{Z}'_-} du \left( dv - \frac{1}{2} \mathcal{Z}'_+ du \right) - \mathcal{Z}'_0 \left[ \frac{1}{\mathcal{Z}'_0^{(0)}} (dz + \chi_0)^2 + \mathcal{Z}'_0 dx^r dx^r \right] - dy^i dy^i, \quad (6.25)$$

$$\begin{aligned} A'_{1,2} &= \mathbb{M}'_{mn} \tilde{\partial}_n \log P_{1,2} \tilde{v}^m \\ &= \mathcal{Z}'_0{}^{-1} \mathbb{M}'_{\#r} \partial_r \log P (dz + \chi_0) + \mathbb{M}'_{sr} \partial_r \log P dx^s, \end{aligned} \quad (6.26)$$

$$B' = \chi_0 \wedge (dz + \chi) + \tilde{\xi}'_0 + \frac{1}{\mathcal{Z}'_-} du \wedge dv, \quad (6.27)$$

$$e^{2\phi} = e^{2\phi_\infty} \frac{\mathcal{Z}'_0}{\mathcal{Z}'_-}, \quad (6.28)$$

where

$$\mathcal{Z}'_0 = H - 2\alpha' \left[ (\tilde{\partial} \log P_1)^2 + (\tilde{\partial} \log P_2)^2 - (\tilde{\partial} \log \mathcal{Z}_0^{(0)})^2 - (\tilde{\partial} \log H)^2 \right], \quad (6.29)$$

$\tilde{\xi}'_0$  is a 2-form on  $\mathbb{E}^3$  defined by

$$d\tilde{\xi}'_0 = d\chi_0 \wedge \chi, \quad (6.30)$$

and where  $\tilde{\partial}_m$  and  $\tilde{v}^m$  are derivatives in flat indices and Vierbein associated with the new GH-space obtained substituting  $H \rightarrow \mathcal{Z}_0^{(0)}$  and, correspondingly  $\chi \rightarrow \chi_0$ .

The T-dual solution clearly belongs to the same family as the original and the net effect of the  $\alpha'$ -corrected T-duality transformation is the interchange between the harmonic functions associated to S5-branes and KK monopoles  $\mathcal{Z}_0^{(0)}$  and  $H$  everywhere, including the  $\alpha'$  corrections. This interchange necessarily has to be accompanied by the interchange of associated 1-forms  $\chi_0$  and  $\chi$ .

This is a highly non-trivial simultaneous test of these  $\alpha'$ -corrected solutions and T-duality rules.

## 7 Range of validity of the solutions

Since the class of solutions that we are presenting is very wide, not much can be said in full detail about the range of validity of the solutions. It is, however, clear that the same mechanism used in ref. [3] to cancel the  $\alpha'$  corrections in  $\mathcal{Z}_0$  can be used here: it is enough to choose  $P_1 = H$  and  $P_2 = \mathcal{Z}_0^{(0)}$  to do it. The result is that all the S5-branes become symmetric 5-branes. Actually, the solution studied in ref. [24] must be a particular example of this class of solutions with no  $\alpha'$  corrections given by  $\mathcal{Z}_\pm = 1$  and  $\mathcal{Z}_0 = H$ .

The first-order  $\alpha'$  corrections in  $\mathcal{Z}_+$  cannot be cancelled in the same fashion, at least with the kind of Yang-Mills fields we have used for our ansatz. The arguments used in ref. [3] suggest that the second and higher  $\alpha'$  corrections can be made arbitrary small or vanishing if we use the above mechanism to cancel the first-order corrections of  $\mathcal{Z}_0$ .

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## A Generalized 't Hooft ansatz in 4d hyperKähler spaces

The 6 generators of the Lie algebra  $\mathfrak{so}(4)$  in the defining (vector) representation can be labeled by a pair of antisymmetric indices  $m, n = \sharp, 1, 2, 3$ ,<sup>10</sup>

$$(\mathbb{M}_{mn})^{pq} \equiv 2\delta_{mn}{}^{pq}, \quad (\text{A.1})$$

and their commutators are given by

$$[\mathbb{M}_{mn}, \mathbb{M}_{pq}] = -2\mathbb{M}_{[m|r}(\mathbb{M}_{pq})^r{}_{|n]}. \quad (\text{A.2})$$

These labels are very convenient but they introduce a twofold redundancy, as each generator appears twice: once as  $\mathbb{M}_{\sharp 1}$ , for instance, and once as  $\mathbb{M}_{1\sharp}$ . Thus, if we want to sum once over all the independent generators and we sum over these labels, we must introduce additional factors of  $1/2$ . For instance, the structure constants have to be defined by

$$[\mathbb{M}_{mn}, \mathbb{M}_{pq}] \equiv \frac{1}{2}f_{mn\ pq}{}^{rs}\mathbb{M}_{rs}, \quad (\text{A.3})$$

and, comparing with the above commutators, we get

$$f_{mn\ pq}{}^{rs} = -4(\mathbb{M}_{pq})^r{}_{[m}\delta_n]{}^s. \quad (\text{A.4})$$

We can define the self- and anti-self-dual combinations

$$\mathbb{M}_{mn}^{\pm} \equiv \frac{1}{2} \left( \mathbb{M}_{mn} \pm \frac{1}{2}\varepsilon_{mn}{}^{pq}\mathbb{M}_{pq} \right), \quad \frac{1}{2}\varepsilon_{mn}{}^{pq}\mathbb{M}_{pq}^{\pm} = \pm\mathbb{M}_{mn}^{\pm}, \quad (\text{A.5})$$

which are explicitly given by<sup>11</sup>

$$(\mathbb{M}_{mn}^{\pm})^{pq} = \delta_{mn}{}^{pq} \pm \frac{1}{2}\varepsilon_{mn}{}^{pq} = (\mathbb{M}_{pq}^{\pm})^{mn}, \quad (\text{A.6})$$

and which must generate two independent subalgebras because they satisfy the commutation relations

$$[\mathbb{M}_{mn}^{\pm}, \mathbb{M}_{pq}^{\pm}] = -2\mathbb{M}_{[m|r}^{\pm}(\mathbb{M}_{pq}^{\pm})^r{}_{|n]}, \quad (\text{A.7})$$

$$[\mathbb{M}_{mn}^{+}, \mathbb{M}_{pq}^{-}] = 0. \quad (\text{A.8})$$

The (anti-)self-duality properties imply that only three of each kind are independent and we can pick representatives  $\mathbb{M}_{\sharp i}^{\pm}$ ,  $i = 1, 2, 3$  at the expense of losing manifest  $\text{SO}(4)$  covariance. When we work with an antisymmetric pair of  $\text{SO}(4)$  indices, their fourfold redundancy has to be taken into account introducing factors of  $1/4$ :

$$[\mathbb{M}_{mn}^{\pm}, \mathbb{M}_{pq}^{\pm}] \equiv \frac{1}{4}f_{mn\ pq}^{\pm}{}^{rs}\mathbb{M}_{rs}^{\pm}, \quad \Rightarrow \quad f_{mn\ pq}^{\pm}{}^{rs} = 4(\mathbb{M}_{pq}^{\pm})^x{}_{[m}(\mathbb{M}_{n]x}^{\pm})^{rs}. \quad (\text{A.9})$$

<sup>10</sup>Upper and lower indices are identical. The positions of the indices are chosen for the sake of clarity.

<sup>11</sup>Due to the interchange property, their self-duality properties hold in both sets of indices.

In order to identify the two 3-dimensional Lie subalgebras, it is convenient to use the representatives. From the above commutation relations, and with the convention  $\varepsilon_{\sharp 123} = +1$ , we find

$$[\mathbb{M}_{\sharp i}^{\pm}, \mathbb{M}_{\sharp j}^{\pm}] = \mp \varepsilon_{ijk} \mathbb{M}_{\sharp k}^{\pm}. \quad (\text{A.10})$$

Therefore, they are two  $\mathfrak{su}(2)$  subalgebras that we are going to denote by  $\mathfrak{su}_{\pm}(2)$ . This corresponds to the well known Lie algebra isomorphism  $\mathfrak{so}(4) = \mathfrak{su}_{+}(2) \oplus \mathfrak{su}_{-}(2)$ .

The (anti-)self-dual combinations can be used in different ways. To start with, they can be used as a hypercomplex structure in a hyperKähler space in the basis in which the components are constant.<sup>12</sup> To fix our conventions and get rid of an excess of  $\pm$  and  $\mp$  symbols, we are only going to use anti-self-dual hypercomplex structures and we are going to define

$$J_{mn}^i \equiv 2(\mathbb{M}_{\sharp i}^{-})^{mn}. \quad (\text{A.11})$$

Then, the preservation of the hypercomplex structure by the hyperKähler space's Levi-Civita connection 1-form  $\varpi_{mn}$ ,

$$\nabla_m J_{np}^i = 0, \quad (\text{A.12})$$

implies

$$[\varpi, J^i] = 0, \quad \Rightarrow \quad \varpi = \varpi^+, \quad (\text{A.13})$$

so the Levi-Civita connection is self-dual in the  $\mathfrak{so}(4)$  indices. The integrability condition of the preservation equation

$$[\nabla_m, \nabla_n] J_{pq}^i = 0, \quad (\text{A.14})$$

implies

$$[R, J^i] = 0, \quad \Rightarrow \quad R = R^+, \quad (\text{A.15})$$

and the Riemann tensor is also self-dual in the  $\mathfrak{so}(4)$  indices. This property combined with the Bianchi identity  $\varepsilon^{mnpq} R_{npqr} = 0$  leads to one of the main properties of hyperKähler spaces: their Ricci flatness

$$R_{mn} = R_{mpn}{}^p = 0. \quad (\text{A.16})$$

The second use of the hypercomplex structures we are interested in is the construction of anti-self-dual  $SU(2)$  instantons through the so-called *'t Hooft ansatz*, since they can also be seen as generators of the  $\mathfrak{su}(2)$  algebra. In this context they are usually called *'t Hooft symbols* and the following notation is commonly used

$$\eta_{pq}^i \equiv 2(\mathbb{M}_{\sharp i}^{(+)})^{pq}, \quad \bar{\eta}_{pq}^i \equiv 2(\mathbb{M}_{\sharp i}^{(-)})^{pq} = J_{pq}^i. \quad (\text{A.17})$$

In this case however, we will stick to the  $SO(4)$ -covariant notation, in terms of which the *'t Hooft Ansatz* for  $SU(2)$  connection 1-forms reads

$$A^{mn} = (\mathbb{M}_{pq}^{\pm})^{mn} V^q v^p, \quad (\text{A.18})$$

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<sup>12</sup>This basis may not always exist. In that case, one may use the non-constant hypercomplex structure to define the ansatz, although some calculations would be more complicated to carry out. We thank G. Papadopoulos for discussions on this point.

for some  $SO(4)$  vector field  $V^m(x)$  and some basis of 1-forms in the hyperKähler space  $v^m = v^m_{\underline{n}} dx^n$ , related to the Levi-Civita 1-form connection by

$$dv^m + \varpi^{mn} \wedge v^n = 0, \tag{A.19}$$

in our conventions. In order to compute the corresponding field strength, of which we will demand self-duality in the spatial indices, we must compute

$$dA = \nabla_m (\mathbb{M}_{np}^{\pm} V^p) v^m \wedge v^n, \tag{A.20}$$

where we are omitting the  $SU(2) \subset SO(4)$  indices, and, in order to simplify the computations we are going to assume that

$$\nabla_m \mathbb{M}_{np}^{\pm} = 0, \tag{A.21}$$

where only the lower indices of  $\mathbb{M}^{\pm}$  are taken into account in the covariant derivative.

Thus, except for Euclidean space, whose connection is both self- and anti-self-dual simultaneously, we can only use one of the two hypercomplex structures, which will lead to only one kind of instanton field. Since we have assumed that it is the anti-self-dual hypercomplex structure the one which is preserved by the connection, we use only that one

$$A = \mathbb{M}_{mp}^{-} V^p v^m. \tag{A.22}$$

With this ansatz, taking into account the commutation relations of the representatives  $\mathbb{M}_{0i}^{-}$  in eq. (A.10), the definition for the field strength which leads to the standard  $SU(2)$  Yang-Mills field strength

$$F^i = dA^i + \frac{1}{2} \varepsilon^{ijk} A^j \wedge A^k, \tag{A.23}$$

is

$$F^{mn} = dA^{mn} + A^{mp} \wedge A^{pn}, \tag{A.24}$$

and a simple calculation gives

$$F = - \left\{ \frac{1}{2} \mathbb{M}_{mn}^{-} V^p V^p + \mathbb{M}_{mp}^{-} (\nabla_n V^p - V_n V^p) \right\} v^m \wedge v^n. \tag{A.25}$$

Demanding now self-duality

$$F_{mn} = + \frac{1}{2} \varepsilon_{mnpq} F_{pq}, \quad \Rightarrow \quad \nabla_{[m} V_{n]} = 0, \quad \text{and} \quad \nabla_m V^m + V_m V^m = 0, \tag{A.26}$$

which is solved by

$$V_m = \partial_m \log P, \quad \text{where} \quad \nabla^2 P = 0, \tag{A.27}$$

so  $P$  is a harmonic function on the hyperKähler space. Observe that the  $SU(2)$  connection and field strengths are both anti-self-dual in the  $SO(4)$ -type gauge indices (which are not shown). However, in the  $SO(4)$  tangent space indices, the field strength is self-dual. There is no chance that the components  $F_{mn}{}^{pq}$  can be interpreted as the components of a Riemann curvature tensor because, as we have just remarked,  $F_{mn}{}^{pq} \neq F^{pq}{}_{mn}$ . We could have made that interpretation if we had demanded anti-self-duality of the field strength, which leads to more complicated equations for  $V^m$ .

The SU(2) Yang-Mills Chern-Simons 3-form, defined in this case by<sup>13</sup>

$$\omega^{\text{YM}} \equiv - \left( dA^{mn} \wedge A^{nm} + \frac{2}{3} A^{mn} \wedge A^{np} \wedge A^{pm} \right), \quad (\text{A.28})$$

takes for this connection the value

$$\omega^{\text{YM}} = - \star dV^2 = - \star d(\partial \log P)^2, \quad (\text{A.29})$$

where  $V^2 = V^m V^m$ . The instanton number density is, then, given by

$$F^A \wedge F^A = d\omega^{\text{YM}} = -d \star d(\partial \log P)^2 = \nabla^2 [(\partial \log P)^2] |v| d^4x, \quad (\text{A.30})$$

where  $|v|$  is the determinant of the Vierbein or the square root of the determinant of the metric. In this and other calculations one should be extremely careful to subtract, in the end, any spurious, non-physical singularities arising from the singularities of the 't Hooft ansatz, as explained in section 5.

The Lorentz Chern-Simons 3-form of a SO(4) connection  $\Omega^{mn}$  in a 4-dimensional manifold is defined in this case by<sup>14</sup>

$$\omega^{\text{L}} \equiv d\Omega^{mn} \wedge \Omega^{nm} + \frac{2}{3} \Omega^{mn} \wedge \Omega^{np} \wedge \Omega^{pm}. \quad (\text{A.31})$$

If the connection  $\Omega$  takes the form of the 't Hooft ansatz in a hyperKähler space

$$\Omega = \mathbb{M}_{mp}^- W^p v^m, \quad W_m = \partial_m \log K, \quad \text{where } \nabla^2 K = 0, \quad (\text{A.32})$$

then,

$$\omega^{\text{L}} = \star dW^2 = \star d(\partial \log K)^2, \quad (\text{A.33})$$

and

$$R^{mn} \wedge R^{nm} = d\omega^{\text{L}} = d \star d(\partial \log K)^2 = -\nabla^2 [(\partial \log K)^2] |v| d^4x. \quad (\text{A.34})$$

## B The twisted 't Hooft ansatz in Gibbons-Hawking spaces

The metric of hyperKähler spaces admitting a triholomorphic isometry (Gibbons-Hawking spaces) can always be written in the form<sup>15</sup>

$$d\sigma^2 = H^{-1} (d\eta + \chi)^2 + H dx^x dx^x, \quad \partial_{\underline{x}} H = \varepsilon_{xyz} \partial_{\underline{y}} \chi_{\underline{z}}. \quad (\text{B.1})$$

In the frame

$$\begin{aligned} v^{\sharp} &= H^{-\frac{1}{2}} [d\eta + \chi_{\underline{x}} dx^x], & v_{\sharp} &= H^{\frac{1}{2}} \partial_{\eta} \equiv \partial_{\sharp}, \\ v^x &= H^{\frac{1}{2}} dx^x, & v_x &= H^{-\frac{1}{2}} [\partial_{\underline{x}} - \chi_{\underline{x}} \partial_{\eta}] = \partial_x, \end{aligned} \quad (\text{B.2})$$

<sup>13</sup>Observe that the trace implies sum over pairs of indices  $\sharp i, \sharp i$ , which can be reexpressed as sums over pairs  $mn, nm$  with a global minus sign. The latter form allows us to use all the machinery we have developed.

<sup>14</sup>Observe that now the trace directly implies sum over pairs  $mn, nm$ , which leads to a different global sign.

<sup>15</sup>Here  $\eta = x^{\sharp}$  and we are using the 3-dimensional, curved, indices  $x, y, z = 1, 2, 3$  which should not be mistaken with coordinates.

the non-vanishing components of the Levi-Civita connection eq. (A.19) are given by

$$\begin{aligned}\varpi_{\sharp\sharp x} &= -\frac{1}{2}\partial_x \log H, & \varpi_{x\sharp y} &= -\frac{1}{2}\epsilon_{xyz}\partial_z \log H, \\ \varpi_{\sharp xy} &= -\frac{1}{2}\epsilon_{xyz}\partial_z \log H, & \varpi_{xyz} &= \delta_{x[y}\partial_{z]}\log H,\end{aligned}\tag{B.3}$$

and they look very similar to those of a SO(4) connection based on the 't Hooft ansatz eq. (A.18). As we have explained, the 't Hooft ansatz does not give a spin connection that can be associated to a Vierbein, or a proper Riemann tensor and a careful inspection indeed shows that not all signs of the above components match with that ansatz.

It is possible to *twist* the 't Hooft ansatz to adapt it to the above spin connection 1-form, at the expense of breaking the manifest SO(4) invariance of the ansatz, which is in agreement with the existence of an isometric direction in the space. This requires the introduction of a new set of self- and anti-self-dual SO(4) generators

$$\mathbb{N}_{mn}^\pm = \pm \frac{1}{2}\epsilon_{mnpq}\mathbb{N}_{pq}^\pm,\tag{B.4}$$

whose representation matrices  $(\mathbb{N}_{mn}^\pm)^{pq}$  have the opposite self-duality properties, that is

$$(\mathbb{N}_{mn}^\pm)^{pq} = \mp \frac{1}{2}\epsilon_{pqrs}(\mathbb{N}_{mn}^\pm)^{rs}.\tag{B.5}$$

These matrices can be constructed using the  $\mathbb{M}_{mn}^\pm$  matrices and a metric  $\eta_{mn} = \text{diag}(-+++)$

$$(\mathbb{N}_{mn}^\pm)^{pq} \equiv \eta_{mr}\eta_{ns}(\mathbb{M}_{rs}^\mp)^{pq} \quad \Rightarrow \quad (\mathbb{N}_{mn}^\pm)^{pq} = (\mathbb{N}_{pq}^\mp)^{mn},\tag{B.6}$$

and satisfy the algebra

$$[\mathbb{N}_{mn}^\pm, \mathbb{N}_{pq}^\pm] = -2\mathbb{N}_{[m|r}^\pm (\mathbb{N}_{pq}^\pm)^{st}\eta_{sr}\eta_{t|n]} = -2\mathbb{N}_{[m|r}^\pm (\mathbb{M}_{pq}^\pm)^r{}_{|n]},\tag{B.7}$$

$$[\mathbb{N}_{mn}^+, \mathbb{N}_{pq}^-] = 0.\tag{B.8}$$

Then, in terms of these matrices, the above spin connection can be rewritten in the form

$$\varpi_{mn} = (\mathbb{N}_{mn}^+)_{pq}\partial_q \log H v^p \equiv (\mathbb{N}_{mn}^+)_{pq}V^q v^p,\tag{B.9}$$

with curvature

$$R^{mn} = -\left\{\frac{1}{2}(\mathbb{N}_{mn}^+)_{rs}V^pV^p + (\mathbb{N}_{mn}^+)_{rp}(\nabla_s V^p - V_s V^p)\right\}v^r \wedge v^s.\tag{B.10}$$

The Chern-Simons 3-form is given by

$$\omega^{\text{LHK}} = \star_{(4)}d(\partial \log H)^2,\tag{B.11}$$

and, therefore

$$R^{mn} \wedge R^{nm} = d\omega^{\text{LHK}} = d\star_{(4)}d(\partial \log H)^2 = -\nabla^2 [(\partial \log H)^2] |v|d^4x.\tag{B.12}$$

## C Connections and curvatures

In this appendix we are going to compute the Levi-Civita and torsionful spin connections and their associated Chern-Simons terms and curvatures for our ansatz, which is described in section 3.

A simple choice of Zehnbein is

$$e^+ = \frac{du}{\mathcal{Z}_-}, \quad e^- = dv - \frac{1}{2}\mathcal{Z}_+ du, \quad e^m = \mathcal{Z}_0^{1/2} v^m, \quad e^i = dy^i, \quad (\text{C.1})$$

where  $v^m = v^m_{\underline{n}} dx^{\underline{n}}$  is a Vierbein of the four-dimensional hyper-Kähler space defined in eq. (3.3). The inverse basis is

$$e_+ = \mathcal{Z}_- \left( \partial_u + \frac{1}{2}\mathcal{Z}_+ \partial_v \right), \quad e_- = \partial_v, \quad e_m = \mathcal{Z}_0^{-1/2} \partial_m, \quad e_i = \partial_i, \quad (\text{C.2})$$

where  $\partial_m \equiv v^m_{\underline{n}} \partial_{\underline{n}}$  is the inverse basis in the hyperKähler space and any other  $m, n$  index will be a flat index in the hyperKähler space and will be raised and lowered with  $+\delta_{mn}$ .

Using the structure equation  $de^a = \omega^a_b \wedge e^b$  we find that the non-vanishing components of the spin connection are given by

$$\begin{aligned} \omega_{-+m} = \omega_{+-m} = \omega_{m+-} &= \frac{1}{2\mathcal{Z}_0^{1/2}} \partial_m \log \mathcal{Z}_-, & \omega_{++m} &= \frac{\mathcal{Z}_-}{2\mathcal{Z}_0^{1/2}} \partial_m \mathcal{Z}_+, \\ \omega_{mnp} &= \mathcal{Z}_0^{-1/2} \left[ \varpi_{mnp} + \frac{1}{2} (\mathbb{M}_{mq})_{np} \partial_q \log \mathcal{Z}_0 \right], \end{aligned} \quad (\text{C.3})$$

where  $\varpi_{mnp}$  are the components of the spin connection on the hyperKähler space defined with the convention eq. (A.19).<sup>16</sup> We assume they satisfy the properties eq. (A.12)–(A.16) with the conventions we use.

In order to compute the components of the torsionful spin connections, we need the components of the 3-form field strength. From eq. (3.4), in the above Zehnbein basis they are given by

$$H_{m+-} = -\mathcal{Z}_0^{-1/2} \partial_m \log \mathcal{Z}_-, \quad H_{mnp} = \mathcal{Z}_0^{-1/2} \varepsilon_{mnpq} \partial_q \log \mathcal{Z}_0. \quad (\text{C.4})$$

Then, the non-vanishing flat components of the torsionful spin connection  $\Omega_{(-)abc} \equiv \omega_{abc} - \frac{1}{2} H_{abc}$  are

$$\begin{aligned} \Omega_{(-)+-m} = \Omega_{(-)m+-} &= \mathcal{Z}_0^{-1/2} \partial_m \log \mathcal{Z}_-, & \Omega_{(-)++m} &= \frac{1}{2} \mathcal{Z}_- \mathcal{Z}_0^{-1/2} \partial_m \mathcal{Z}_+, \\ \Omega_{(-)mnp} &= \mathcal{Z}_0^{-1/2} \left[ \varpi_{mnp} + (\mathbb{M}_{mq}^-)_{np} \partial_q \log \mathcal{Z}_0 \right], \end{aligned} \quad (\text{C.5})$$

and those of  $\Omega_{(+)abc} \equiv \omega_{abc} + \frac{1}{2} H_{abc}$  are given by

$$\begin{aligned} \Omega_{(+) -+m} &= \mathcal{Z}_0^{-1/2} \partial_m \log \mathcal{Z}_-, & \Omega_{(+) ++m} &= \frac{1}{2} \mathcal{Z}_- \mathcal{Z}_0^{-1/2} \partial_m \mathcal{Z}_+, \\ \Omega_{(+) mnp} &= \mathcal{Z}_0^{-1/2} \left[ \varpi_{mnp} + (\mathbb{M}_{mq}^+)_{np} \partial_q \log \mathcal{Z}_0 \right], \end{aligned} \quad (\text{C.6})$$

where the  $4 \times 4$  matrices  $\mathbb{M}_{np}^{\pm}$  are defined in eq. (A.5).

<sup>16</sup>These 4-dimensional tangent-space indices are raised and lowered with  $+\delta_{mn}$  and there is no difference between them, beyond an esthetic one.

The Lorentz-Chern-Simons 3-form  $\omega_{(-)}^L$  reduces to the Chern-Simons 3-form of the  $\text{SO}(4)$  connection  $\Omega_{(-)mn}$

$$\begin{aligned} \omega_{(-)}^L &\equiv d\Omega_{(-)}^a{}_b \wedge \Omega_{(-)}^b{}_a - \frac{2}{3}\Omega_{(-)}^a{}_b \wedge \Omega_{(-)}^b{}_c \wedge \Omega_{(-)}^c{}_a \\ &= d\Omega_{(-)mn} \wedge \Omega_{(-)nm} + \frac{2}{3}\Omega_{(-)mn} \wedge \Omega_{(-)np} \wedge \Omega_{(-)pm}, \end{aligned} \tag{C.7}$$

which, in its turn, is just the sum of the Chern-Simons 3-forms of the self-dual and anti-self-dual pieces of  $\Omega_{(-)mn}$ : the self-dual spin connection of the hyperKähler manifold and the anti-self-dual 1-form  $(\mathbb{M}_{mq}^-)_{np} \partial_q \log \mathcal{Z}_0$ . The latter has the form of the 't Hooft ansatz eq. (A.18) discussed in appendix A and, therefore, its Chern-Simons term takes the value computed in eq. (A.33) with  $K$  replaced by  $\mathcal{Z}_0$ . The Chern-Simons 3-form of the spin connection of the hyperKähler manifold has to be computed case by case, except when it is a Gibbons-Hawking space. In that case, there is a general expression for it (See e.g. eq. (B.11)) and for its total derivative which are particularly convenient for us because the Bianchi identity of the 3-form field strength  $H$  becomes a linear combination of Laplacians on the Gibbons-Hawking space that can be solved exactly.

Then, in these conditions, we have

$$\omega_{(-)}^L = \star_{(4)} d [(\partial \log H)^2 + (\partial \log \mathcal{Z}_0)^2], \tag{C.8}$$

and

$$\text{Tr}(R_{(-)} \wedge R_{(-)}) = d\omega_{(-)}^L = -\nabla^2 [(\partial \log H)^2 + (\partial \log \mathcal{Z}_0)^2]. \tag{C.9}$$

Clearly, it would be extremely interesting to find other hyperKähler spaces with no tri-holomorphic isometry that still enjoy the same property. The Atiyah-Hitchin hyperKähler space [25], which has been considered before in the context of supergravity solutions in refs. [26, 27], might provide an explicit example. We leave this study for future work. Interestingly, for arbitrary self-dual  $\text{SU}(2)$  instanton fields on  $\mathbb{R}^4$ , and not just for those in the 't Hooft ansatz, this *Laplacian property* was proven in ref. [28] using the ADHM construction [29, 30]. Our results suggest that this property could also hold in hyperKähler backgrounds and, therefore, for the spin connections of the hyperKähler spaces themselves, as it happens in Gibbons-Hawking spaces.

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