

# From 4d Yang-Mills to 2d $\mathbb{C}\mathbb{P}^{N-1}$ model: IR problem and confinement at weak coupling

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**ABSTRACT:** We study four-dimensional  $SU(N)$  Yang-Mills theory on  $\mathbb{R} \times \mathbb{T}^3 = \mathbb{R} \times S_A^1 \times S_B^1 \times S_C^1$ , with a twisted boundary condition by a  $\mathbb{Z}_N$  center symmetry imposed on  $S_B^1 \times S_C^1$ . This setup has no IR zero modes and hence is free from IR divergences which could spoil trans-series expansion for physical observables. Moreover, we show that the center symmetry is preserved at weak coupling regime. This is shown by first reducing the theory on  $\mathbb{T}^2 = S_A \times S_B$ , to connect the model to the two-dimensional  $\mathbb{C}\mathbb{P}^{N-1}$ -model. Then, we prove that the twisted boundary condition by the center symmetry for the Yang-Mills is reduced to the twisted boundary condition by the  $\mathbb{Z}_N$  global symmetry of  $\mathbb{C}\mathbb{P}^{N-1}$ . There are  $N$  classical vacua, and fractional instantons connecting those  $N$  vacua dynamically restore the center symmetry. We also point out the presence of singularities on the Borel plane which depend on the shape of the compactification manifold, and comment on its implications.

**KEYWORDS:** Confinement, Nonperturbative Effects, Solitons Monopoles and Instantons, Wilson, 't Hooft and Polyakov loops

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**1 Introduction**

**1.1 Confinement and resurgence**

It has been a long-standing problem to show that a non-trivial quantum Yang-Mills theory exists on the flat space  $\mathbb{R}^4$ , and that it has a mass gap. This is a challenging problem since Yang-Mills theory, while asymptotic free in the UV, is intrinsically strongly coupled in the IR.

One possible approach is to start with the perturbative expansion of the theory: this is the best-understood part of the theory, can also be formulated mathematically [1]. One might then hope that a suitable re-summation of such a perturbative series might lead to a complete theory [2].

Of course, Yang-Mills theory on  $\mathbb{R}^4$  is strongly-coupled in the IR and hence the perturbative expansion breaks down. One might nevertheless try to avoid this problem by compactifying the theory onto e.g. a circle  $S^1$  of radius  $L$ . When the radius is smaller than the dynamical scale  $\Lambda$  of the theory ( $L \ll \Lambda^{-1}$ ), the renormalization-group-running of the gauge coupling constant stops at the scale  $1/L$ , at which the gauge coupling is small. One then hopes to do a reliable weak coupling computation, perturbatively in the gauge coupling constant. Once we have done this, one might hope to adiabatically continue back to the flat space.

The hope is that such an adiabatic continuation can be achieved by the theory of Borel-Écalle re-summation and resurgence [3, 4] (see e.g. [5, 6] for recent exposition). This states that the perturbative series (say of the vacuum expectation value of an operator  $\mathcal{O}$ ) should be thought of as part of the so-called trans-series expansion, containing both perturbative and non-perturbative corrections:

$$\langle \mathcal{O} \rangle = \sum_{k=0}^{\infty} c_{0,k} g^k + \sum_I e^{-\frac{S_I}{g^2}} \left( \sum_{k=0}^{\infty} c_{I,k} g^k \right) + \dots, \tag{1.1}$$

where  $I$  is a label for the saddle points of the action  $S$ , and  $S_I$  denotes the value of the action at the  $I$ -th saddle point; the coefficients  $c_{I,k}$  represent the perturbative expansions around the  $I$ -th saddle point. Assuming that we know all the saddle points contributing to the path integral, all the coefficients  $c_{0,k}$  and  $c_{I,k}$  can be computed by perturbative methods, and we obtain a trans-series as in (1.1). Now one hopes to turn the trans-series into a well-defined function  $\langle \mathcal{O} \rangle(g)$  as a function of the coupling constant with the help of the resurgence theory,<sup>1</sup> and if that function is indeed resurgent (as might be expected from resurgence theory), i.e. endlessly continuable, we can adiabatically continue back to the large value of the coupling constant along a suitable choice of path in the complex plane.

**Challenges in resurgence.** There are problems with such an optimistic scenario.

First, such a compactification of the theory can dramatically change the physics; for example, if we compactify the theory on a temporal circle we expect to encounter the confinement-deconfinement transition, and thus the weak coupling computation is possible only in the deconfined phase. One way to avoid such a phase transition is to ensure the existence of the center symmetry (more on this later in this paper). This can be achieved, for example, by twisted boundary conditions or by suitable deformation of the Lagrangian. Such a continuity from small to large size of the compactified direction is

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<sup>1</sup>The word resurgence in the physics literature sometimes refers to a stronger statement. Namely, large-order asymptotic growth (as  $k$  large) of the perturbative coefficients  $c_{0,k}$  around the trivial saddle point contains the information of the non-perturbative saddle points (inside the same topological sector). This has been discussed in the literature since long ago [7, 8]. See e.g., [9–14] for various points of view on this issue.

called adiabatic continuity. It was considered in physics independently and later combined with the mathematical idea of resurgence: see e.g., [15–34]. (See also literature on large  $N$  twisted Eguchi-Kawai model [35–37]).

Second, in order to cancel the ambiguity of Borel re-summation of perturbative series, we need to have a corresponding semi-classical saddle point for a singularity of the Borel plane on the positive real axis. In particular, it was argued that there is the renormalon pole [2], whose action is of order  $1/N$  of that of the instanton in the large  $N$  limit. Recently there has been huge progress in this respect (together with the adiabatic continuity), in the context of recent connection with resurgence. The activities in this field has been triggered by the works of [23, 24] on 2d  $\mathbb{C}\mathbb{P}^{N-1}$ -model and [21, 22] for 4d adjoint QCD.<sup>2</sup>

While these points are of relevance to this paper, our main focus here is yet another, although related, source of trouble for the resurgence program, associated with IR divergences. The problem of IR divergences is already stressed in the context of resurgence in [23], but we discuss this problem again by emphasizing the point that they spoil the formal trans-series expansion of physical observables.

## 1.2 The problem of the infrared divergences

In this paper we point out and study the subtlety which spoils the resurgence program: the issue of IR divergence. In short, the problem is that the perturbative (or more generally trans-series) expansion (1.1) in itself is ill-defined if the observable suffers from IR divergences.

**Instanton computation of vacuum energy.** To illustrate the issue of IR divergence, let us study the theta-angle dependence of the vacuum energy of the Yang-Mills theory on  $\mathbb{R}^4$ . Instanton computation gives the formula for the  $\theta$ -dependence of vacuum energy  $E(\theta)$  as [39]

$$E(\theta) \sim - \int_0^\infty \frac{d\rho}{\rho^5} (\mu\rho)^{b_1} e^{-\frac{8\pi^2}{g^2(\mu)}} \cos \theta, \tag{1.2}$$

where  $\mu$  is an arbitrary renormalization scale,  $b_1$  is the coefficient of the one-loop beta function,  $\rho$  is the size modulus of the instanton, and  $g(\mu)$  is the running gauge coupling constant at the scale  $\mu$ . This integral is divergent in the IR region  $\rho \rightarrow \infty$ . Still, one might hope that the formula gives qualitatively right answer by introducing naive IR cutoff, which we may take to be the dynamical scale  $\Lambda$  of the theory. This gives the answer<sup>3</sup>

$$E(\theta) \sim -\Lambda^4 \cos \theta. \tag{1.3}$$

It turns out, however, that this is not the correct result. Witten argued [41, 42] (see the first section of [42] for a beautiful summary; see also [43–45] for further support) that in

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<sup>2</sup>However, see [38] which argued that the usual renormalon diagrams do not give any factorial growth of the coefficients of perturbative series in the setup of [21, 22].

<sup>3</sup>The minus sign is chosen for consistency with the result of Vafa and Witten [40] that the lowest energy is realized at  $\theta = 0$ .

the large  $N$  limit, there are infinitely-many metastable vacua labelled by an integer  $\mathbf{e} \in \mathbb{Z}$ , and the vacuum energy of each of the metastable vacua are given by

$$E_{\mathbf{e}}(\theta) \rightarrow \Lambda^4(\theta - 2\pi\mathbf{e})^2 \quad (N \rightarrow \infty). \quad (1.4)$$

Note that each  $E_{\mathbf{e}}(\theta)$  is not even a periodic function of  $\theta$ ! We can recover the periodicity by considering the energy of the true vacuum, namely minimum of all the  $E_{\mathbf{e}}$ 's:

$$E(\theta) = \min_{\mathbf{e}} E_{\mathbf{e}}(\theta), \quad (1.5)$$

however this is discontinuous at  $\theta = \pm\pi$ . (See [46] and references therein for the physics at  $\theta = \pm\pi$ .) The actual situation is that each vacuum labeled by  $\mathbf{e} \in \mathbb{Z}$  is metastable (in the large  $N$  limit), and the true stable vacuum  $\mathbf{e} = 0$  at  $\theta = 0$  is adiabatically continued to a metastable vacuum under the monodromy  $\theta \rightarrow \theta + 2\pi$ . This is very different from the naive answers expected from the instanton calculus (1.2), even at the qualitative level.

**Linde's problem.** Let us next consider the thermal free energy  $F$  defined by

$$\exp(-F \cdot \text{Vol.}) = \text{Tr} e^{-\beta H} = \int_{S^1_{\beta} \times \mathbb{R}^3} [DA] e^{-S}, \quad (1.6)$$

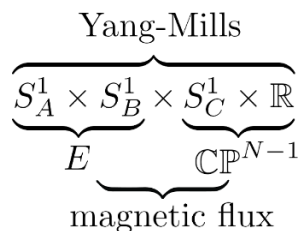
where  $\beta = T^{-1}$  is the inverse temperature and Vol. is the space-time volume, which we regularize to be a finite number by introducing some IR cutoff. After the compactification on the thermal circle  $S^1_{\beta}$ , the 4d Yang-Mills is reduced to 3d Yang-Mills up to Kaluza-Klein (KK) modes with the 3d gauge coupling given by  $g_3^2 = Tg^2$ . The contribution of 3d Yang-Mills to the free energy is expanded by  $g_3^2 = Tg^2$ , but it has a mass dimension. Thus the expansion is of the form

$$\beta F \sim [\text{KK contribution}] + \sum_{n=0}^{\infty} a_n \frac{(Tg^2)^{n+3}}{(m_{\text{IR}})^n}, \quad (1.7)$$

where [KK contribution] means the contributions from nonzero modes on  $S^1_{\beta}$ , and the expansion by  $Tg^2$  comes from 3d Yang-Mills. In naive perturbation theory,  $m_{\text{IR}} = 0$  and the above expansion has severe infrared problem. In improved perturbation theory,  $m_{\text{IR}}$  may be taken to be the thermal mass of the fields. However, for the 3d gauge fields, the thermal mass is known to be at most  $m_{\text{thermal}} \lesssim Tg^2$ . Therefore, in the above expansion, higher loops do not give higher powers of the coupling  $g$  and hence all the higher loops give the fixed order  $\mathcal{O}(g^6)$  of the coupling expansion (in the most optimistic case of  $m_{\text{thermal}} \sim Tg^2$ ). The perturbative expansion, and hence the (trans)series expansion, is therefore not well-defined [47].

The naive cutoff prescription of the IR divergence does not work in each of the two cases discussed above. Of course, some observables (IR safe observables) do not depend on the details of IR regularizations, but many observables of physical interest are sensitive to such subtleties.

We stress that the IR problems mentioned above are conceptually different from (although related to) the problem of whether the coupling  $g$  is large or small and we can



**Figure 1.** We consider Yang-Mills theory on the geometry  $\mathbb{R} \times S_A^1 \times S_B^1 \times S_C^1$ . When the torus  $E = S_A^1 \times S_B^1$  is small, we obtain the two-dimensional  $\mathbb{CP}^{N-1}$ -model in the remaining directions  $\mathbb{R} \times S_C^1$ . We include a unit 't Hooft magnetic flux (i.e., twisting by the center symmetry) along  $S_B^1 \times S_C^1$ , which plays rather crucial roles in this paper.

numerically trust the weak coupling results or not.<sup>4</sup> The issue is whether or not the coefficient in the perturbative (or trans-series) expansion is well-defined. Even if  $g$  is very large, the resurgence might give a sensible answer as long as the trans-series expansion is well-defined. But if there is IR divergences, there is no way to do resurgence from the beginning.

The question is then to find a concrete setup in which the trans-series expansion is well-defined. Doing this for the pure  $SU(N)$  Yang-Mills theory<sup>5</sup> is the motivation for the next section.

## 2 Our setup

In this paper we study the pure Yang-Mills theory with gauge group  $SU(N)$ , with gauge coupling constant  $g$  and theta angle  $\theta$ . Our setup is summarized in figure 1.

### 2.1 Yang-Mills theory in a box

In this paper the pure Yang-Mills theory is put into a box, namely compactified on a spatial three-torus

$$\mathbb{R} \times \mathbb{T}^3 = \mathbb{R} \times S_A^1 \times S_B^1 \times S_C^1, \tag{2.1}$$

where the non-compact direction  $\mathbb{R}$  is the temporal direction. Here  $S_{A,B,C}^1$  are spatial directions. We will denote their circumferences of three spatial directions by  $L_{A,B,C}$ , respectively. These circumferences play the role of the IR cutoff, and the limit to the flat space  $\mathbb{R}^4$  is achieved by  $L_{A,B,C} \rightarrow \infty$  with their ratios kept finite.

In this paper we will primarily study the parameter region

$$L_A, L_B \ll L_C, \tag{2.2}$$

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<sup>4</sup>For example, even for weakly coupled QED, there are IR divergences due to soft photons. Therefore the problem of IR divergences are conceptually different from the strong dynamics. However, in the case of soft photons, physical treatment of IR divergences are not so difficult.

<sup>5</sup>One can also add adjoint fermions, and also fundamental fermions whose flavor number is a multiple of  $N$ .

with the relative size of  $L_A$  and  $L_B$  kept finite. Note this breaks the symmetries between  $S_{A,B}^1$  and  $S_C^1$ .

The hierarchy in scales (2.2) means that below energy scales  $1/L_A$  and  $1/L_B$  the theory is effectively described by the two-dimensional theory on  $\mathbb{R} \times S_C^1$ . We show that the resulting two-dimensional theory is given by the two-dimensional sigma model whose target space is  $\mathbb{C}\mathbb{P}^{N-1}$ . This connection with the two-dimensional physics will be rather useful for understanding the physics of four-dimensional Yang-Mills theory.

Notice that the parameter region (2.2) is different from the decompactification limit of  $\mathbb{R} \times \mathbb{T}^3$  into  $\mathbb{R}^3 \times S^1$ , where one might encounter a version of Linde’s problem:

$$L_A, L_B \gg L_C . \tag{2.3}$$

However, one can hope that after a suitable resurgence analysis in the region (2.2) we may extrapolate the answer to the region (2.3). We will give some comments on this point in section 6.

## 2.2 Twist by 1-form $\mathbb{Z}_N$ symmetry

As another crucial ingredient for our construction, we consider twisted boundary condition along  $S_B^1 \times S_C^1$ . The four-dimensional pure Yang-Mills theory has a center symmetry, or in modern language a global “electric” one-form  $\mathbb{Z}_N$ -symmetry [48]. In the Language of [48], the one-form symmetry twist along  $S_B^1 \times S_C^1$  used in this paper is generated by the surface operator supported on  $(\text{pt}) \times S_A^1 \times \mathbb{R}$ , where  $(\text{pt}) \in S_B^1 \times S_C^1$  is a point. (See appendix A for quick summary of one-form symmetry). As we will see later, this global symmetry acts on the Wilson lines along the cycle  $S_B^1$ .

This twist is crucial for the following reason. First, we will see that this ensures that

- gauge symmetry is completely broken at classical vacua around which we perform perturbation, hence there is no massless modes in the IR.

We therefore do not encounter a subtle issue of IR divergences. Second, we will see later in this paper (section 5.3) that while the one-form center symmetry is broken classically, it is restored after quantum tunneling effects are taken into account:

- center symmetry is preserved.

Since the presence of the one-form center symmetry is proposed as a characterization of the confinement phase [48], we conclude that our system confines, at least in the parameter regions (2.2). We expect that this is true for all  $L_A, L_B$ , and  $L_C$  as long as their ratios are kept finite. (The case of taking  $L_{A,B}/L_C \rightarrow \infty$  is commented in section 6.)

Such a dynamical restoration of center symmetry is shown by reducing the theory onto the  $\mathbb{R} \times S_C^1$ -directions, and matching the resulting theory with the two-dimensional  $\mathbb{C}\mathbb{P}^{N-1}$ -model with  $\mathbb{Z}_N$ -twisted boundary condition studied recently in connection with resurgence [23, 24, 34]. Namely, we show that the following two twists are the same after the reduction from four to two dimensions:

- the twist in the  $SU(N)$  Yang-Mills theory by the one-form symmetry on  $S_B^1 \times S_C^1$ ,
- the twist in the  $\mathbb{C}\mathbb{P}^{N-1}$  model by the  $\mathbb{Z}_N$  global symmetry on  $S_C^1$ .

For this reason we will spend the next two sections exploring the connection between four-dimensional Yang-Mills theory and the two-dimensional  $\mathbb{CP}^{N-1}$ -model.

### 3 From $SU(N)$ Yang-Mills to $\mathbb{CP}^{N-1}$

Let us first consider the compactification of the pure  $SU(N)$  Yang-Mills theory on the two-dimensional torus (elliptic curve)  $E = \mathbb{T}^2 = S_A^1 \times S_B^1$  down to two dimensions.

When the Yang-Mills theory is compactified on  $E$  and the Kaluza-Klein modes are integrated out, the light degrees of freedom on  $E$  are the zero-energy configuration of gauge fields, i.e. flat connections on  $E$ , except near singular points of the moduli space of flat connections where there is an enhanced symmetry and W-bosons become light. The moduli space of flat  $SU(N)$  connections on  $E$  is known to be the projective space  $\mathbb{CP}^{N-1}$  at the holomorphic level [49–51]. Here we review and spell out some details of this relation between flat connections and  $\mathbb{CP}^{N-1}$ .

#### 3.1 Yang-Mills flat connections on a torus

We choose a complex coordinate of the torus  $E = \mathbb{T}^2$  to be

$$z \sim z + 1 \sim z + \tau, \tag{3.1}$$

where  $\tau$  is a complex parameter (modulus of torus) with  $\text{Im}(\tau) > 0$ . The torus  $E$  has the flat metric

$$ds^2 = L_E^2 |dz|^2, \tag{3.2}$$

where  $L_E$  is the size of the torus. In the notation of the previous section, these parameters are given as  $\tau = i(L_B/L_A)$  and  $L_E = L_A$ . But it is not difficult to take  $\tau$  to be more general complex numbers.

We call the cycles on  $E$  corresponding to  $z \rightarrow z + 1$  and  $z \rightarrow z + \tau$  as the A-cycle and B-cycle, respectively. In our previous notation these are  $S_A^1$  and  $S_B^1$ , respectively.

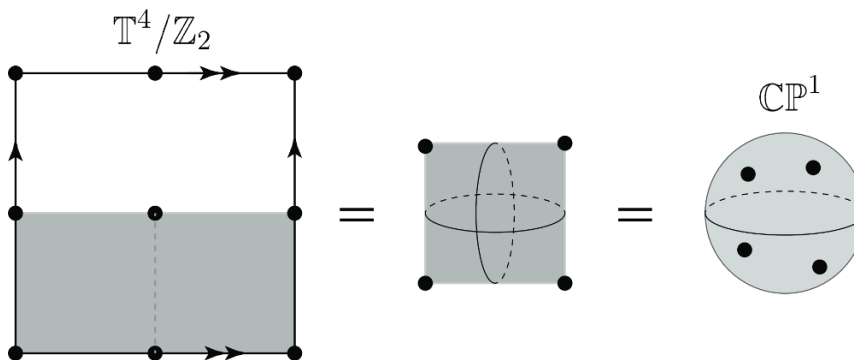
The gauge field on  $E$  can be represented as  $A = A_{\bar{z}}d\bar{z} + A_z dz$  in the complex coordinate  $z$ . We take  $A$  to be anti-hermitian so that the field strength is given by  $F = dA + A \wedge A$ . For flat connections, the  $\bar{z}$ -component  $A_{\bar{z}}$  is expanded as

$$A_{\bar{z}} = -(A_z)^\dagger \xrightarrow{\text{flat connection}} \frac{-2\pi i}{(\tau - \tau^*)} \text{diag}(\phi_1, \dots, \phi_N), \tag{3.3}$$

where  $\phi_i$  are complex scalars which are independent of  $z$  and satisfy  $\sum_{i=1}^N \phi_i = 0$ . The constant prefactor  $-2\pi i/(\tau - \tau^*)$  is chosen for later convenience. Wilson lines in the A- and B-cycles are given by

$$\begin{aligned} U_A &= \exp \int_A (-A_{\bar{z}}d\bar{z} - A_z dz) = \text{diag} \left( \dots, \exp \left( 2\pi i \cdot \frac{\phi_i - \phi_i^*}{\tau - \tau^*} \right), \dots \right), \\ U_B &= \exp \int_B (-A_{\bar{z}}d\bar{z} - A_z dz) = \text{diag} \left( \dots, \exp \left( 2\pi i \cdot \frac{\tau^* \phi_i - \tau \phi_i^*}{\tau - \tau^*} \right), \dots \right). \end{aligned} \tag{3.4}$$





**Figure 2.** For the case  $N = 2$ , the moduli space of flat connections  $\mathcal{M}_{\text{flat}}$  on  $E = \mathbb{T}^2$  is  $\mathbb{T}^2/\mathbb{Z}_2$ , which as in this figure can be identified with  $\mathbb{C}\mathbb{P}^1$ . The four fixed points of  $\mathbb{T}^4/\mathbb{Z}_2$  are mapped into the four singular points on  $\mathbb{C}\mathbb{P}^1$ , represented by black dots.

The scalars  $\phi_i$  have periodicities

$$\phi_i \sim \phi_i + (\tau m_i - n_i), \quad m_i, n_i \in \mathbb{Z}, \quad \sum_i m_i = \sum_i n_i = 0, \quad (3.5)$$

and thus can be regarded as a point on  $E$  with the constraint  $\sum_i \phi_i = 0$ . Furthermore,  $\phi_i$ 's are permuted by the Weyl group of  $SU(N)$ , namely the  $N$ -th symmetric group  $\mathfrak{S}_N$ : an element of the Weyl group  $\sigma \in \mathfrak{S}_N$  acts as  $\phi_i \rightarrow \phi_{\sigma(i)}$ . Therefore, the moduli space of flat connections is given by

$$\mathcal{M}_{\text{flat}} = \mathbb{E}/\mathfrak{S}_N, \quad (3.6)$$

$$\mathbb{E} := \left\{ (\phi_1, \dots, \phi_N) \in E^N; \sum_i \phi_i = 0 \right\}. \quad (3.7)$$

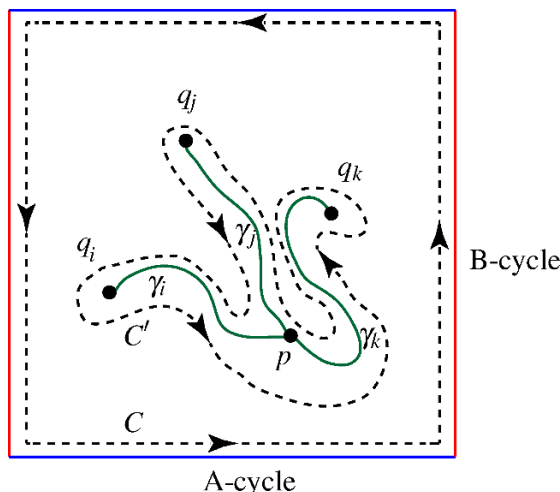
The case of  $N = 2$  is particularly simple, since then we have

$$\mathbb{E} := \{(\phi_1, -\phi_1) \in E^2\} = E, \quad \mathcal{M}_{\text{flat}} = \mathbb{T}^2/\mathbb{Z}_2. \quad (3.8)$$

One can then see as in figure 2 that this is indeed  $\mathbb{C}\mathbb{P}^1 = S^2$  as an algebraic variety. What has to be kept in mind is that the manifold has four singular points, where extra massless W-bosons appear.

The case of  $N > 2$  is more complicated, but there is a convenient way to describe such  $N$  points  $\phi_i$  on  $E$ . Consider a space of meromorphic functions  $F$  which are allowed to have  $N$ -th order pole at  $z = 0$ . Namely,  $F$  is allowed to behave as  $F(z) \sim z^{-N}$  ( $z \sim 0$ ). We denote such a space of meromorphic functions on  $E$  as  $H^0(E, \mathcal{O}(Np))$ , where  $p$  represents the point  $p = \{z = 0 \in E\}$ , and  $\mathcal{O}(Np)$  indicates that the functions are allowed to have  $N$ -th order pole at  $p$ .<sup>6</sup> When  $F \in H^0(E, \mathcal{O}(Np))$ , the Cauchy theorem for  $dF/F = \partial_z F/F dz$  on  $E$  implies that there must be  $N$ -points  $q_i$  ( $i = 1, \dots, N$ ) at which  $F(z)$  has a zero. This

<sup>6</sup>More precisely,  $\mathcal{O}(Np)$  is a line bundle on  $E$  associated to the divisor  $Np$ . See any textbook on algebraic geometry (e.g. [53]) for the meaning of those words “line bundle” and “divisor”. The reader who is not familiar with algebraic geometry can neglect them.



**Figure 3.** The path  $\gamma_i$  connects  $p$  and  $q_i$ . Since this is a two-torus, the blue (red) edges are identified, and are the A-cycle and the B-cycle, respectively. The contour  $C = ABA^{-1}B^{-1}$  can be deformed into the contour  $C'$ , which can be decomposed into small cycles surrounding  $\gamma_i$ 's.

is because by the Cauchy theorem, the sum of residues of  $dF/F$  must be zero and it has the residue  $+1$  at a simple zero of  $F$  and  $-1$  at a simple pole, with obvious generalization when zero or pole have multiplicity. If  $q_i = p$  for some  $i$ , that means that the degree of the pole at  $p = \{z = 0\}$  is reduced by the number of  $q_i$  for which  $q_i = p$ . If  $q_i = q_j$ , that means that the function  $F$  has double (or more generally multiple) zero at that point.

Then the claim is that the set of unordered points  $\{\phi_i\}_{1 \leq i \leq N}$  on the torus  $E$ , that is  $\mathcal{M}_{\text{flat}} = \mathbb{E}/\mathfrak{S}_N$ , is given by  $\mathbb{P}H^0(E, \mathcal{O}(Np))$ , where  $\mathbb{P}H^0(E, \mathcal{O}(Np))$  is the projective space associated to the vector space  $H^0(E, \mathcal{O}(Np))$ . The correspondence is given by mapping the  $N$  points  $q_i$  of zeros of a function  $F \in H^0(E, \mathcal{O}(Np))$  to the points on  $\mathcal{M}_{\text{flat}} = \mathbb{E}/\mathfrak{S}_N$ . Also, the vector space  $H^0(E, \mathcal{O}(Np))$  has complex dimension  $N$  and hence  $H^0(E, \mathcal{O}(Np)) \cong \mathbb{C}^N$ , so we get  $\mathcal{M}_{\text{flat}} \cong \mathbb{C}\mathbb{P}^{N-1}$ . This can be shown by well-known techniques in algebraic geometry (see e.g. [51]). Here we would like to give more or less elementary explanation.

Let  $\phi_i$  be the coordinates corresponding to  $q_i$  associated to the zero points of a function  $F$ . Then, first we want to show that they satisfy  $\sum_i \phi_i = 0$  so that they define a point on  $\mathbb{E}$  defined in (3.7). One way to see this is as follows. Corresponding to each  $q_i$ , take a path  $\gamma_i$  from  $p$  to  $q_i$  such that  $\gamma_i$  and  $\gamma_j$  do not intersect other than at the point  $p$ . By eliminating  $\gamma_i$  from  $E$ , the  $\log F$  becomes well defined which has branch cuts at  $\gamma_i$ . Consider the integration of  $\log F dz$  on a loop of the form  $ABA^{-1}B^{-1}$ , where  $A$  and  $B$  are the A-cycle and B-cycle of the torus  $E$ . Then we get a value of the form  $2\pi i(\mathbb{Z} + \tau\mathbb{Z})$ . This is because the contributions from  $A$  and  $A^{-1}$  add up to give  $2\pi i n \int_A dz$  where  $2\pi i n \in 2\pi i \mathbb{Z}$  comes from the difference of  $\log F$  before and after going around the loop  $B$ . Similarly, the contributions from  $B$  and  $B^{-1}$  add to give  $2\pi i m \int_B dz$ . On the hand, applying the Cauchy theorem to this integral, we get  $-\sum_i 2\pi i \int_{\gamma_i} dz$ . By using  $\int_{\gamma_i} dz = (\phi_i - 0)$ , we get the desired result  $(\sum_i \phi_i) \in (\mathbb{Z} + \tau\mathbb{Z})$ . See figure 3 for the situation.

From the results obtained above, we see that a meromorphic function  $F \in H^0(E, \mathcal{O}(Np))$  gives a data  $\{\phi_i\}/\mathfrak{S}_N$  needed for a flat connection. Suppose that two meromorphic func-

tions  $F$  and  $G$  have the same set of zero points  $\{\phi_i\}$ . Then, their ratio  $F/G$  does not have either a zero or pole. Such a holomorphic function must be a constant,  $F/G = \text{const}$ . This means the following. First, notice that the space of functions  $H^0(E, \mathcal{O}(Np))$  is a complex vector space. Let  $\mathbb{P}H^0(E, \mathcal{O}(Np))$  be the projective space associated to this vector space. Namely, we introduce equivalence relation  $F \sim cF$  for constants  $c \in \mathbb{C} - \{0\}$ , and divide the space  $H^0(E, \mathcal{O}(Np)) - \{0\}$  by this equivalence relation. Then, what we found above is that there is an injective map  $\mathbb{P}H^0(E, \mathcal{O}(Np)) \rightarrow \mathcal{M}_{\text{flat}}$ .

It turns out that the map above is also surjective. Indeed, given a set of points  $\{\phi_i\}$  satisfying  $\sum_i \phi_i = 0$ , we can explicitly construct a function  $F \in H^0(E, \mathcal{O}(Np))$  which has zeros at these points and an  $N$ -th order pole at  $p = \{z = 0\}$ :

$$F(z) = \prod_{m,n \in \mathbb{Z}} f(z + m + n\tau), \quad f(z) = \prod_{i=1}^N \frac{(z - \phi_i)}{z}. \quad (3.9)$$

The infinite product is convergent if  $\sum_i \phi_i = 0$ , because  $\log f(z) \rightarrow -z^{-1} \sum_i \phi_i + \mathcal{O}(z^{-2})$  for  $z \rightarrow \infty$ , and  $\sum_{(m,n)} 1/(z + m + n\tau)^k$  is convergent for  $k \geq 2$ .

This establish that the moduli space of flat connections is given by

$$\mathcal{M}_{\text{flat}} = \mathbb{P}H^0(E, \mathcal{O}(Np)) = \mathbb{C}\mathbb{P}^{N-1}, \quad (3.10)$$

where we have used the fact that  $H^0(E, \mathcal{O}(Np))$  is an  $N$ -dimensional vector space, as can be seen by counting the number of parameters to specify  $F \in H^0(E, \mathcal{O}(Np))$  (i.e.,  $\phi_i$  with  $\sum_i \phi_i = 0$  gives  $N - 1$  parameters and the overall scale gives one parameter).

We remark that the equivalence (3.10) should be interpreted as equivalence as algebraic varieties, and at the classical level, the metric of the moduli space  $\mathcal{M}_{\text{flat}}$  is not the Fubini-Study metric. However, this point does not affect our qualitative discussions in this paper.

### 3.2 Explicit map from $\mathcal{M}_{\text{flat}}$ to $\mathbb{C}\mathbb{P}^{N-1}$

The above discussion was rather abstract. We now make the map  $\mathcal{M}_{\text{flat}} = \mathbb{E}/\mathfrak{S}_N \rightarrow \mathbb{C}\mathbb{P}^{N-1}$  more explicit.

We denote the root lattice of  $\text{SU}(N)$  by  $\mathbb{L}$

$$\mathbb{L} = \left\{ \vec{\ell} = (\ell_1, \dots, \ell_N) \in \mathbb{Z}^N; \sum_i \ell_i = 0 \right\}. \quad (3.11)$$

The weight lattice is spanned by the fundamental weights

$$\vec{e}_k = \left( \overset{1}{1}, \dots, \overset{k}{1}, 0, \dots, 0 \right) - \frac{k}{N} (1, \dots, 1). \quad (3.12)$$

We define theta functions as

$$\theta_k(\vec{\phi}) := \sum_{\vec{\ell} \in \mathbb{L}} e^{\pi i \tau (\vec{\ell} + \vec{e}_k)^2 + 2\pi i (\vec{\ell} + \vec{e}_k) \cdot \vec{\phi}} \quad (k = 1, \dots, N), \quad (3.13)$$

where  $\vec{\phi} = (\phi_1, \dots, \phi_N)$  and the inner product between vectors is defined as  $\vec{\phi} \cdot \vec{\ell} = \sum_i \phi_i \ell_i$ . These theta functions are invariant under the Weyl symmetry  $\mathfrak{S}_N$  acting on  $\vec{\phi}$ , because each set

$$\vec{e}_k + \mathbb{L} = \{\vec{e}_k + \vec{\ell}; \vec{\ell} \in \mathbb{L}\} \tag{3.14}$$

is Weyl invariant, and the Weyl symmetry preserves the inner product. Furthermore, under the shift

$$\vec{\phi} \rightarrow \vec{\phi} + \tau \vec{m} - \vec{n} \quad (\vec{m}, \vec{n} \in \mathbb{L}), \tag{3.15}$$

they transform as

$$\theta_k(\vec{\phi} + \tau \vec{m} - \vec{n}) = e^{-\pi i \tau \vec{m}^2 - 2\pi i \vec{m} \cdot \vec{\phi}} \theta_k(\vec{\phi}). \tag{3.16}$$

Note that the factor  $e^{-\pi i \tau \vec{m}^2 - 2\pi i \vec{m} \cdot \vec{\phi}}$  is independent of  $k$ .

We denote points of  $\mathbb{C}\mathbb{P}^{N-1}$  by using homogeneous coordinates as  $[Z_1, \dots, Z_N]$ . Then, if we define

$$\varphi(\vec{\phi}) := [\theta_1(\vec{\phi}), \dots, \theta_N(\vec{\phi})], \tag{3.17}$$

then the above properties imply that this is a well-defined map from  $\mathcal{M}_{\text{flat}}$  to  $\mathbb{C}\mathbb{P}^{N-1}$ .<sup>7</sup> We claim that this is an isomorphism between  $\mathcal{M}_{\text{flat}}$  and  $\mathbb{C}\mathbb{P}^{N-1}$ .

In the above discussion, we have given  $\theta_k$  explicitly. In fact, there is an algebraic geometric reason for the existence of such a map. (The reader who is not interested in algebraic geometry can skip this paragraph.) Let  $\mathbb{E}$  be the space of flat connections before dividing by the Weyl group as defined in (3.7). In the previous subsection, we established that  $\mathbb{E}/\mathfrak{S}_N \cong \mathbb{C}\mathbb{P}^{N-1}$ . Thus there is a natural projection map

$$\pi : \mathbb{E} \rightarrow \mathbb{C}\mathbb{P}^{N-1}. \tag{3.18}$$

On the  $\mathbb{C}\mathbb{P}^{N-1}$ , there is a line bundle  $\mathcal{O}(1)$  (which is also written as  $\mathcal{O}(H)$  for a hyperplane  $H$  in  $\mathbb{C}\mathbb{P}^{N-1}$  and is called the hyperplane line bundle). It has the properties that its holomorphic sections are one-to-one correspondence with linear functions of the homogeneous coordinates  $Z_k$  ( $k = 1, \dots, N$ ) of the  $\mathbb{C}\mathbb{P}^{N-1}$ . With this identification, there are  $N$  linearly independent sections  $Z_k$  of  $\mathcal{O}(1)$ . Now, we can pull-back this line bundle by  $\pi$  to define a line bundle  $L = \pi^* \mathcal{O}(1)$  on  $\mathbb{E}$ . Then we also get  $N$  linearly independent sections  $\theta_k = \pi^* Z_k$  of the line bundle  $L$  by the pull-back. By using these sections, we can define a map

$$\tilde{\varphi} : \mathbb{E} \ni \vec{\phi} \mapsto [\pi^* Z_1(\vec{\phi}), \dots, \pi^* Z_N(\vec{\phi})] \in \mathbb{C}\mathbb{P}^{N-1}. \tag{3.19}$$

By construction, this map coincides with the original projection map  $\pi$ . On the other hand,  $\mathbb{E}$  is an Abelian variety because it is equivalent to  $E^{N-1}$  as is clear from (3.7). Also,  $L$  is a positive line bundle in the sense that its curvature can be positive definite by choosing an appropriate metric on  $L$ . There is a general classification and explicit construction of such line bundles and their sections on Abelian varieties. See the section on ‘‘Theta Functions’’ in [53]. These are precisely the theta functions  $\theta_k(\vec{\phi})$  given in (3.13). In fact, the nontrivial transformation (3.16) implies that  $\theta_k$  are not functions, but sections of a line bundle  $L$ .

<sup>7</sup>The functions  $\theta_k$  ( $k = 1, \dots, N$ ) are expected to be not simultaneously zero at any value of  $\vec{\phi}$ , since there are  $N$  functions of  $N - 1$  variables. For more mathematical justifications, see the paragraph below.

### 3.3 1-form $\mathbb{Z}_N$ symmetry in $\mathbb{C}\mathbb{P}^{N-1}$

By compactification of space-time on  $S^1$ , a 1-form symmetry splits to a 1-form symmetry and a 0-form symmetry. Here, “0-form” symmetry means a usual global symmetry. So, we get a usual global symmetry for each  $S^1$ .

In the context of reducing four dimensional Yang-Mills to two dimensional  $\mathbb{C}\mathbb{P}^{N-1}$ , we are interested in the two 0-form symmetries  $\mathbb{Z}_N^{(A)}$  and  $\mathbb{Z}_N^{(B)}$  associated with the torus  $E = \mathbb{T}^2 = S_A^1 \times S_B^1$ . Their action on the Wilson lines  $U_A, U_B$  along  $S_A^1, S_B^1$  is described as

$$(p, q) \in \mathbb{Z}_N^{(A)} \times \mathbb{Z}_N^{(B)} : (U_A, U_B) \mapsto \left( e^{-\frac{2\pi ip}{N}} U_A, e^{-\frac{2\pi iq}{N}} U_B \right), \quad (3.20)$$

where the minus signs are introduced just for later convenience. Notice that the two  $\mathbb{Z}_N$ 's,  $\mathbb{Z}_N^{(A)}$  and  $\mathbb{Z}_N^{(B)}$  commute with each other.

Let us translate the action of  $\mathbb{Z}_N$ -symmetries  $\mathbb{Z}_N^{(A)}$  and  $\mathbb{Z}_N^{(B)}$  in (3.20) into the language of  $\mathbb{C}\mathbb{P}^{N-1}$ -model.

By using the relation (3.4), we find the  $\mathbb{Z}_N$ -symmetries in (3.20) act on  $\phi_k$  as

$$(p, q) \in \mathbb{Z}_N^{(A)} \times \mathbb{Z}_N^{(B)} : \phi_k \mapsto \phi_k - \frac{p\tau - q}{N} + \tau\mathbb{Z} + \mathbb{Z}. \quad (3.21)$$

where  $\tau\mathbb{Z} + \mathbb{Z}$  means an arbitrary constant of the form  $\tau m_i - n_i$  such that  $\sum_i \phi_i = 0$  is preserved. By defining a vector

$$\vec{c}_j = (0, \dots, 0, \overset{j}{1}, 0, \dots, 0) - \frac{1}{N}(1, \dots, 1) = \vec{e}_j - \vec{e}_{j-1}. \quad (3.22)$$

the above action is written as

$$\vec{\phi} \mapsto \vec{\phi} + (p\tau - q)\vec{c}_j \quad (3.23)$$

for arbitrary  $j$ . The reason that  $j$  can be arbitrary is because  $\vec{c}_j - \vec{c}_{j'} \in \mathbb{L}$  for any  $j$  and  $j'$ .

Now we can study the action of them on the theta functions  $\theta_k$  defined in section 3.2. First, one can check that for  $(p, q) = (0, 1)$ , the change of the theta functions is given as

$$\theta_k(\vec{\phi} - \vec{c}_j) = e^{\frac{2\pi ik}{N}} \theta_k(\vec{\phi}). \quad (3.24)$$

Next, for  $(p, q) = (1, 0)$ , we get

$$\theta_k(\vec{\phi} + \tau\vec{c}_j) = \sum_{\vec{\ell} \in \mathbb{L}} e^{\pi i \tau (\vec{\ell} + \vec{e}_k + \vec{c}_j)^2 + 2\pi i (\vec{\ell} + \vec{e}_k + \vec{c}_j) \cdot \vec{\phi} - \pi i \tau \vec{c}_j^2 - 2\pi i \vec{c}_j \cdot \vec{\phi}}. \quad (3.25)$$

Now, note that  $\vec{e}_k + \vec{c}_{k+1} = \vec{e}_{k+1}$  and  $\vec{c}_{k+1} - \vec{c}_j \in \mathbb{L}$  implies

$$\vec{e}_k + \vec{c}_j + \mathbb{L} = \vec{e}_{k+1} + \mathbb{L}. \quad (3.26)$$

Therefore we get

$$\theta_k(\vec{\phi} + \tau\vec{c}_j) = e^{-\pi i \tau \vec{c}_j^2 - 2\pi i \vec{c}_j \cdot \vec{\phi}} \theta_{k+1}(\vec{\phi}). \quad (3.27)$$

Note that the overall factor  $e^{-\pi i \tau \vec{c}_j^2 - 2\pi i \vec{c}_j \cdot \vec{\phi}}$  is independent of  $k$ , and hence is irrelevant when  $\theta_k$  are mapped to homogeneous coordinates of  $\mathbb{CP}^{N-1}$ .

In summary, in terms of the homogeneous coordinates, the symmetries  $\mathbb{Z}_N^{(A)} \times \mathbb{Z}_N^{(B)}$  act as

$$1 \in \mathbb{Z}_N^{(A)} : [\dots, Z_k, \dots] \mapsto [\dots, Z_{k+1}, \dots], \tag{3.28}$$

$$1 \in \mathbb{Z}_N^{(B)} : [\dots, Z_k, \dots] \mapsto [\dots, e^{\frac{2\pi i k}{N}} Z_k, \dots], \tag{3.29}$$

or more generally

$$(p, q) \in \mathbb{Z}_N^{(A)} \times \mathbb{Z}_N^{(B)} : [\dots, Z_k, \dots] \mapsto [\dots, e^{\frac{2\pi i q k}{N}} Z_{k+p}, \dots]. \tag{3.30}$$

As expected,  $\mathbb{Z}_N^{(A)}$  and  $\mathbb{Z}_N^{(B)}$  commute with each other.

## 4 The $\mathbb{CP}^{N-1}$ instantons as Yang-Mills instantons

In this section we explain that the instantons of  $\mathbb{CP}^{N-1}$ -model can be identified with the Yang-Mills instantons. First, we argue that their  $\theta$  angles are identified,  $\theta_{\mathbb{CP}^{N-1}} = \theta_{\text{YM}}$ . Second, we argue the relation between one instanton solution of  $\mathbb{CP}^{N-1}$ -model and that of the Yang-Mills theory. Readers willing to accept the fact that [Yang-Mills instanton] = [ $\mathbb{CP}^{N-1}$  instanton] can skip this section. The relation between 4d instantons and 2d instantons are discussed in more mathematically precise way by Atiyah [52] motivated by the analogy between the Yang-Mills and the  $\mathbb{CP}^{N-1}$  model.

### 4.1 The topological theta term

We explain the equivalence of the topological terms between the  $\mathbb{CP}^{N-1}$  and Yang-Mills. Actually, we discuss more details of the actions. We follow the explanation in section 4 of [54] adapted to our case.

#### 4.1.1 4d Yang-Mills as 2d gauge theory with infinite gauge group

Let  $E$  be a Riemann surface which for our application is  $\mathbb{T}^2$ . We consider a 4d spacetime  $\mathbb{R}^2 \times E$  with coordinates  $(x^i, y^a)$ , where  $x^i$  ( $i = 0, 1$ ) are the coordinates of  $\mathbb{R}^2$  and  $y^a$  ( $a = 2, 3$ ) are the coordinates of  $E$ . They are related to the complex coordinate  $z$  on  $E$  as  $z = y^2 + iy^3$ . The metric on  $E$  is denoted as  $ds^2 = g_{ab} dy^a dy^b = 2g_{z\bar{z}} |dz|^2$ . The following discussion can be easily generalized even if we replace  $\mathbb{R}^2 \times E$  by more general  $X_2 \times \Sigma$  for 2d space-time  $X_2$  and a Riemann surface  $\Sigma$ .

The 4d Yang-Mills theory with a compact, simple and simply connected gauge group  $G$  on  $\mathbb{R}^2 \times E$  can be regarded as a 2d gauge theory on  $\mathbb{R}^2$  with gauge group  $\mathcal{G}$ , where  $\mathcal{G}$  is the group of all gauge transformations on  $E$ . Here we are regarding the space  $E$  as an internal manifold, and considering a gauge theory on the 2d space-time  $\mathbb{R}^2$ . Notice that this  $\mathcal{G}$  is an infinite dimensional group,

$$\mathcal{G} = \{g : E \rightarrow G\}, \tag{4.1}$$

with the obvious group structure. In the following, all the Lie algebra generators are taken to be anti-hermitian which is different from the standard physics convention.

The total gauge field  $A$  is split as  $A = a + b$ , where  $a = \sum_{i=0,1} a_i dx^i$  is the gauge field of  $\mathcal{G}$  and  $b = \sum_{a=2,3} b_a dy^a$  is now regarded as “matter fields” from the 2-dimensional point of view. Regarding  $b$  as matter fields may be intuitively understood if we perform Kaluza-Klein reduction, since in the 2d space-time  $\mathbb{R}^2$  they are just scalar fields. However, we can proceed in an abstract way without any Kaluza-Klein reduction.

A gauge transformation  $g(x, y)$  of  $G$  on  $\mathbb{R}^2 \times E$  can be regarded as a gauge transformation of  $\mathcal{G}$  on  $\mathbb{R}^2$ , and it acts as

$$a \rightarrow g^{-1} a g + g^{-1} d_x g, \quad b \rightarrow g^{-1} b g + g^{-1} d_y g, \quad (4.2)$$

where  $d_x$  and  $d_y$  are exterior derivatives on  $\mathbb{R}^2$  and  $E$ , respectively.

The matter field  $b$  takes values in the space of connections on  $E$  which we denote as  $\mathcal{A}$ . The space  $\mathcal{A}$  is an infinite dimensional Kähler manifold, once we fix a complex structure on  $E$ . The holomorphic coordinates on  $\mathcal{A}$  are given by  $b_{\bar{z}}$ , i.e., the anti-holomorphic component of the connection  $b = b_{\bar{z}} d\bar{z} + b_z dz$  on  $E$ . The Kähler form  $\omega$  is given by<sup>8</sup>

$$\begin{aligned} 2\pi\omega &= -\frac{1}{4\pi} \int_E \text{tr}(\delta b \wedge \delta b) \\ &= \frac{1}{2\pi} \int_E (dz \wedge d\bar{z}) \text{tr}(\delta b_{\bar{z}} \wedge \delta b_z), \end{aligned} \quad (4.3)$$

where  $\delta$  is the exterior derivative on  $\mathcal{A}$ , or more intuitively,  $\delta b$  means infinitesimal fluctuations of  $b$ . The trace is normalized in such a way that the usual instanton number is given by integration of  $\frac{1}{8\pi^2} \text{tr}(F \wedge F)$ . (For the case of  $G = \text{SU}(N)$ , the trace is in the defining  $N$ -dimensional representation.) Even if we forget about the complex structure of  $E$ , the  $\omega$  still defines a natural symplectic structure on  $\mathcal{A}$ . Under the gauge transformation, the  $\delta b$  transforms as  $\delta b \rightarrow g^{-1}(\delta b)g$ , and hence the gauge transformation preserves the Kähler form.

Because  $\mathcal{G}$  acts on  $\mathcal{A}$  preserving the Kähler structure, we can define the moment map of  $\mathcal{G}$  as follows. An element of the Lie algebra of  $\mathcal{G}$  is an infinitesimal gauge transformation  $\epsilon$  on  $E$ . It is described by a vector field on  $\mathcal{A}$  given by

$$V(\epsilon) = \int_E ([b, \epsilon] + d_y \epsilon) \frac{\partial}{\partial b}, \quad (4.4)$$

where  $[b, \epsilon] + d_y \epsilon$  is the infinitesimal gauge transformation of  $b$ . Let  $\iota_V$  be the contraction operator of a vector field  $V$  acting on differential forms. We get

$$\iota_{V(\epsilon)}(2\pi\omega) = -\frac{1}{2\pi} \int_E \text{tr}([(b, \epsilon] + d_y \epsilon) \wedge \delta b) = \frac{1}{2\pi} \int_E \text{tr}(\epsilon D_y \delta b), \quad (4.5)$$

where  $D_y = d_y + [b, \cdot]$  is the covariant exterior derivative. Then we can write

$$\iota_{V(\epsilon)}(2\pi\omega) = \delta\mu(\epsilon), \quad (4.6)$$

---

<sup>8</sup>We change the normalization of the Kähler form  $\omega$  from that of [54] by a factor of  $2\pi$ .

where  $\mu(\epsilon)$  is the moment map associated to  $\epsilon$ , and is given by

$$\mu(\epsilon) = \frac{1}{2\pi} \int_E \text{tr} \left( \epsilon f^b \right) , \tag{4.7}$$

$$f^b = d_y b + b \wedge b . \tag{4.8}$$

We define  $\mu$  by

$$\mu = \frac{\sqrt{g^{-1}}}{2\pi} f_{23}^b = -\frac{i g^{\bar{z}z}}{2\pi} f_{z\bar{z}}^b . \tag{4.9}$$

This  $\mu$  takes values in the Lie algebra of  $\mathcal{G}$ .

Now we can rewrite the 4d Yang-Mills action as a  $\mathcal{G}$  gauge theory on the 2d spacetime  $\mathbb{R}^2$ . First, let us introduce a notation for an inner product on the internal space. For elements of the Lie algebra of  $\mathcal{G}$ ,  $r$  and  $s$ , we define

$$\langle r, s \rangle = - \int_E \sqrt{g} d^2 y \text{tr}(rs) . \tag{4.10}$$

The minus sign comes from the fact that we take the Lie algebra to be anti-hermitian, and hence this inner product is positive definite. Now the kinetic term can be written as

$$- \int_{\mathbb{R}^2 \times E} \frac{1}{2g^2} \text{tr}(F_{\mu\nu} F_{\mu\nu}) = \int_{\mathbb{R}^2} L_{2\text{d,kin}} . \tag{4.11}$$

The Lagrangian  $L_{2\text{d,kin}}$  is given by

$$L_{2\text{d,kin}} = \frac{1}{g^2} \left[ \frac{1}{2} \langle f_{ij}^a, f_{ij}^a \rangle + 2 \langle D_i b_{\bar{z}}, g^{\bar{z}z} D_i b_z \rangle + (2\pi)^2 \langle \mu, \mu \rangle \right] , \tag{4.12}$$

where

$$D_i b_{\bar{z}} = \partial_i b_{\bar{z}} - \partial_{\bar{z}} a_i + [a_i, b_{\bar{z}}] , \quad f_{ij}^a = \partial_i a_j - \partial_j a_i + [a_i, a_j] . \tag{4.13}$$

Therefore, the 4d Yang-Mills is now rewritten as a 2d gauge theory with the gauge kinetic term  $\langle f_{ij}^a, f_{ij}^a \rangle$ , the matter kinetic term  $\langle D_i b_{\bar{z}}, g^{\bar{z}z} D_i b_z \rangle$  and the potential energy  $\langle \mu, \mu \rangle$ .

The topological term of 4d Yang-Mills is given as

$$\frac{i\theta}{8\pi^2} \int_{\mathbb{R}^2 \times E} \text{tr}(F \wedge F) = \int_{\mathbb{R}^2} L_{2\text{d,top}} , \tag{4.14}$$

where

$$L_{2\text{d,top}} = -\frac{i\theta}{4\pi^2} \left[ 2\pi \langle f^a, \mu \rangle - i \langle D_x b_{\bar{z}}, g^{\bar{z}z} D_x b_z \rangle \right] , \tag{4.15}$$

where  $f^a = \frac{1}{2} f_{ij}^a dx^i \wedge dx^j$  and  $D_x b_{\bar{z}} = D_i b_{\bar{z}} dx^i$ .



### 4.1.2 $\mathbb{CP}^{N-1}$ as Kähler quotient $\mathcal{A}/\mathcal{G}$

Having derived the 2d Lagrangian  $L_{2d} = L_{2d,\text{kin}} + L_{2d,\text{top}}$ , let us look at the potential minima of this Lagrangian. The potential energy is proportional to  $\langle \mu, \mu \rangle$ , and hence the potential minima at the classical level is given by  $2\pi\mu = f_{23}^b = 0$ . This is just the flat connections on  $E$ . If we divide the space of the potential minima by the gauge group  $\mathcal{G}$ , we get the  $\mathbb{CP}^{N-1}$  for the case  $G = \text{SU}(N)$  as discussed before. On the other hand, taking  $\mu = 0$  and dividing by the group  $\mathcal{G}$  is precisely the Kähler (or symplectic) quotient of the space  $\mathcal{A}$  by  $\mathcal{G}$  denoted as  $\mathcal{A}/\mathcal{G}$ . Therefore we get

$$\mathbb{CP}^{N-1} = \mathcal{A}/\mathcal{G}. \tag{4.16}$$

The Kähler form (4.3) descends to this space  $\mathcal{A}/\mathcal{G}$ .

Now let us look at the topological term in this subspace  $\mu = 0$ . We get

$$\begin{aligned} L_{2d,\text{top}} &\xrightarrow{\mu=0} \frac{i\theta}{2\pi} \left[ \frac{i}{2\pi} \langle D_x b_{\bar{z}}, g^{\bar{z}z} D_x b_z \rangle \right] \\ &= \frac{i\theta}{2\pi} \left[ \frac{1}{2\pi} \int (dz \wedge d\bar{z}) \text{tr}(D_x b_{\bar{z}} D_x b_z) \right]. \end{aligned} \tag{4.17}$$

However, this is precisely the pull-back of the Kähler form given by (4.3). Let  $\Phi$  be the map  $\Phi : \mathbb{R}^2 \ni x \mapsto b_{\bar{z}}(x) \in \mathcal{A}$ . Then, by the pull-back we get  $\Phi^* \delta b_{\bar{z}} = D_x b_{\bar{z}}$ , and hence

$$\left[ \frac{1}{2\pi} \int (dz \wedge d\bar{z}) \text{tr}(D_x b_{\bar{z}} D_x b_z) \right] = 2\pi \Phi^* \omega. \tag{4.18}$$

Therefore, we finally get

$$L_{2d,\text{top}} \xrightarrow{\mu=0} i\theta \Phi^* \omega. \tag{4.19}$$

The  $\omega$  is a Kähler form on  $\mathbb{CP}^{N-1} = \mathcal{A}/\mathcal{G}$ . Furthermore, it must be integrally quantized on  $\mathcal{A}/\mathcal{G}$  and can take the value 1 as is clear from the fact that  $\Phi^* \omega$  comes from the 4d topological term  $\frac{1}{8\pi^2} \int \text{tr}(F \wedge F)$ . Therefore  $\omega$  is the generator of  $H^2(\mathbb{CP}^{N-1}, \mathbb{Z})$ , and the instanton number of  $\mathbb{CP}^{N-1}$  corresponds to the instanton number of 4d Yang-Mills.

## 4.2 Moduli space of instantons

Next let us compare the moduli spaces of instantons of Yang-Mills and those of  $\mathbb{CP}^{N-1}$ -model. To simplify the problem, we compactify the 2d space-time  $\mathbb{R}^2$  to  $S^2$ .

For the Yang-Mills theory, the total 4d space-time is  $S^2 \times E$ . Then, the Atiyah-Hitchin-Singer theorem [55] states that the number of (virtual<sup>9</sup>) dimensions of the instanton moduli space for  $\text{SU}(N)$  with topological charge  $Q$  is given by

$$4NQ - (N^2 - 1) \frac{\chi + \sigma}{2}, \tag{4.20}$$

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<sup>9</sup>Virtual dimensions coincides with the actual dimensions for irreducible solutions in which the gauge group is completely broken by the solutions.

where  $\chi$  and  $\sigma$  are the Euler number and the signature of the 4-manifold.<sup>10</sup> For the 4-manifold  $S^2 \times E$ , we have  $\chi = \sigma = 0$ , and thus the dimensions is  $4NQ$ .

Let us next consider the instantons of the  $\mathbb{C}\mathbb{P}^{N-1}$ -model on  $S^2$ . If we regard  $S^2 = \mathbb{C}\mathbb{P}^1$ , then the instantons of  $\mathbb{C}\mathbb{P}^{N-1}$ -model is given by a holomorphic map  $\mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^{N-1}$ , and the instanton number is given by the winding number of this map. The standard explanation is as follows. Consider a general sigma model whose target space  $\mathcal{M}$  is Kähler. Let  $\xi^I$  be holomorphic coordinates of the sigma model, and let  $g_{I\bar{J}}$  be the Kähler metric such that the Kähler form  $\omega = ig_{I\bar{J}}d\xi^I \wedge d\bar{\xi}^{\bar{J}}$  is integrally quantized, i.e., the generator of  $H^2(\mathcal{M}, \mathbb{Z})$ . Also let  $x^i$  be the coordinates of the spacetime  $S^2 = \mathbb{C}\mathbb{P}^1$  with  $w = x^0 + ix^1$  its complex coordinate. Then we get

$$g_{I\bar{J}}(\partial_i \xi^I \pm \epsilon_{ij} \partial_j \xi^I)(\partial_i \bar{\xi}^{\bar{J}} \pm \epsilon_{ij} \partial_j \bar{\xi}^{\bar{J}}) \geq 0$$

$$\Rightarrow \int d^2x g_{I\bar{J}}(\partial_i \xi^I)(\partial_i \bar{\xi}^{\bar{J}}) \geq \left| \int d^2x g_{I\bar{J}} \epsilon_{ij}(\partial_i \xi^I)(\partial_j \bar{\xi}^{\bar{J}}) \right| = \left| \int \omega \right| = |Q|, \quad (4.21)$$

where  $Q$  is the instanton number defined in terms of the Kähler form  $\omega$ . The equality is saturated if and only if  $\partial_i \xi^I \pm \epsilon_{ij} \partial_j \xi^I = 0$ , which implies  $\partial_{\bar{w}} \xi^I = 0$  (instanton) or  $\partial_w \xi^I = 0$  (anti-instanton) in terms of  $w = x^0 + ix^1$ . The equation  $\partial_{\bar{w}} \xi^I = 0$  means that the map from the space-time  $\mathbb{C}\mathbb{P}^1$  to the target space  $\mathcal{M}$  is holomorphic.

Here we consider the case where the instanton number is integer. The case of fractional instantons is discussed in section 5.

Let  $[Z_k]$  ( $k = 1, \dots, N$ ) be the homogeneous coordinates of  $\mathbb{C}\mathbb{P}^{N-1}$ , and let  $w \in \mathbb{C} \cup \{\infty\} = \mathbb{C}\mathbb{P}^1$  be the usual coordinate of  $\mathbb{C}\mathbb{P}^1$ . Then, holomorphic maps are given by

$$Z_k = \sum_{\ell=0}^Q a_{k,\ell} w^\ell, \quad (4.22)$$

where  $a_{k,\ell}$  are constants. The fact that the degree of the polynomial  $Q$  corresponds to the instanton number is easy to see by considering simple cases. Let us consider the case  $N = 2$  and  $[Z_1, Z_2] = [1, w]$ . This map is the identity map from the spacetime  $\mathbb{C}\mathbb{P}^1$  to the target space  $\mathbb{C}\mathbb{P}^{N-1=1}$ . Therefore it gives the instanton number  $Q = 1$ . In the case  $[Z_1, Z_2] = [1, w^Q]$ , the spacetime  $\mathbb{C}\mathbb{P}^1$  wraps  $Q$  times around the target  $\mathbb{C}\mathbb{P}^{N-1}$ , so the instanton number is  $Q$ . The general case is seen by using the invariance of the topological charge under continuous deformation, and also by using the embedding  $\mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^{N-1}$  as  $[Z_1, Z_2] \rightarrow [Z_1, Z_2, 0, \dots, 0]$ .

There are  $N(Q + 1)$  complex parameters  $a_{k,\ell}$ . However, the overall scale is irrelevant because  $Z_k$  are homogeneous coordinates. So there are  $N(Q + 1) - 1$  complex parameters, or real  $2N(Q + 1) - 2$  parameters.

When  $Q = 1$ , the  $\mathbb{C}\mathbb{P}^{N-1}$  instantons have real  $4N - 2$  parameters. Thus there are two missing parameters compared with the dimension of the Yang-Mills instantons  $4N$

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<sup>10</sup>Small fluctuations of gauge fields  $\delta A$  are described as  $\delta A_{\alpha\dot{\beta}}^a$ , where  $\alpha$  and  $\dot{\beta}$  are spinor indices of Lorentz symmetry  $SO(4) \cong SU(2)_L \times SU(2)_R$ , and  $a$  is the index of the adjoint representation of  $SU(N)$ . By regarding it as an  $SU(2)_L$  spinor field  $\delta A_{\alpha\dot{\beta}}^a = (\lambda_\alpha)_\beta^a$  which takes values in the  $SU(N) \times SU(2)_R$  bundle, one can apply the usual Atiyah-Singer index theorem to get the Atiyah-Hitchin-Singer index theorem.

discussed above. These missing moduli may be understood as follows. On generic points of the moduli space  $\mathcal{M}_{\text{flat}} = \mathbb{CP}^{N-1}$ , the  $SU(N)$  gauge group is broken down to  $U(1)^{N-1}$ . However, along some locus on  $\mathcal{M}_{\text{flat}}$ , some non-abelian gauge symmetry is recovered. The  $\mathbb{CP}^{N-1}$ -model is an approximate description of the Yang-Mills theory away from the locus where non-abelian symmetries are enhanced. When  $\mathbb{CP}^{N-1}$  instantons hit such a locus, the approximation in terms of the flat connection (3.3) is not valid and the gauge field develops nontrivial profile along  $E$  [51]. Then the full Yang-Mills solution may not be invariant under the translations along the torus  $E$ . The moduli about translations along  $E$  are not visible in the low energy  $\mathbb{CP}^{N-1}$  description. Since  $E$  has 2 directions, there are 2 translation moduli. This provides the missing 2 moduli parameters.

When  $Q > 1$ , the number of the parameters of instantons of  $\mathbb{CP}^{N-1}$ -model are much smaller than those of the Yang-Mills theory. The missing moduli parameters can still be described in algebraic geometry by using spectral cover construction [51] because  $\mathbb{CP}^1 \times E$  can be regarded as a (trivial) elliptic manifold. We do not discuss it in this paper, however.

## 5 Fractional instantons, vacuum structure, and center symmetry

Having studied the relation between instantons in 4d and 2d theories, we now study fractional instantons. In particular we show that such instantons restore the center symmetry of the system.

Let  $(t, x)$  be the coordinates of  $\mathbb{R} \times S^1_C$  with  $x \sim x + 1$ . The metric may then be taken to be

$$ds^2 = L_C^2(dt^2 + dx^2) + L_E^2|dz|^2, \tag{5.1}$$

where  $L_C$  and  $L_E$  are some length parameters. The multiplication of the term  $dt^2$  by the same  $L_C^2$  as the one multiplying  $dx^2$  is introduced just for later convenience.

For the twisted compactification, we choose to use the symmetry  $\mathbb{Z}_N^{(B)}$  discussed in section 2.2. Suppose we first compactify the theory on  $E$  and reduce it to  $\mathbb{CP}^{N-1}$ . Then we learn from (3.29) that the twist by  $\mathbb{Z}_N^{(B)}$  on the homogeneous coordinates is given by

$$[\dots, Z_k(t, x + 1), \dots] = \left[ \dots, e^{\frac{2\pi ik}{N}} Z_k(t, x), \dots \right]. \tag{5.2}$$

This is precisely the boundary condition which played an important role in the analysis of the  $\mathbb{CP}^{N-1}$  model [23, 24, 34]. For this reason let us start our analysis in the  $\mathbb{CP}^{N-1}$ -model, and then subsequently discuss the four-dimensional theory.

### 5.1 Fractional instantons and the vacuum structure in $\mathbb{CP}^{N-1}$ -model

Here we discuss the vacuum structure of the  $\mathbb{CP}^{N-1}$ -model. First of all, let us discuss the candidates for the vacuum at the classical level in the presence of the twist (5.2). A minimal energy configuration should be constant on 2d space-time  $(t, x)$ , and also invariant under the above twist. This requires  $e^{2\pi ik/N} Z_k = c Z_k$  ( $k = 1, \dots, N$ ), where  $c$  is a constant independent of  $k$  which takes care of the fact that  $Z_k$  are homogeneous coordinates and

hence an overall constant is irrelevant. There are  $N$  solutions of these equations, which are given by

$$P_k = [0, \dots, 0, \overset{k}{1}, 0, \dots, 0] \in \mathbb{CP}^{N-1}. \tag{5.3}$$

These points are the fixed points of the twisting (5.2). These are the classical vacua.

Thanks to the twisting, the fields have energy gap. For example, near the point  $P_N$ , the fields are described by inhomogeneous coordinates  $U_k := Z_k/Z_N$  with the boundary condition  $U_k(t, x+1) = e^{\frac{2\pi ik}{N}} U_k(t, x)$ . This boundary condition forces  $U_k$  to have a nonzero momentum in the direction  $S_C^1$ . Therefore, the vacua have energy gap already at the classical level, so there are no IR divergences. This makes straightforward perturbation theory possible around these vacua.

There are transitions among these classical vacua  $P_k$  by fractional instantons and anti-instantons, as we now describe. First, let us identify  $\mathbb{R} \times S_C^1$  as  $S^2 = \mathbb{CP}^1$  by regarding the infinite future  $t = \infty$  and infinite past  $t = -\infty$  as the north pole and south pole of  $S^2 = \mathbb{CP}^1$ , respectively. More explicitly, we take the coordinate

$$w = e^{2\pi(t+ix)} \in \mathbb{C} + \{0\} = \mathbb{CP}^1. \tag{5.4}$$

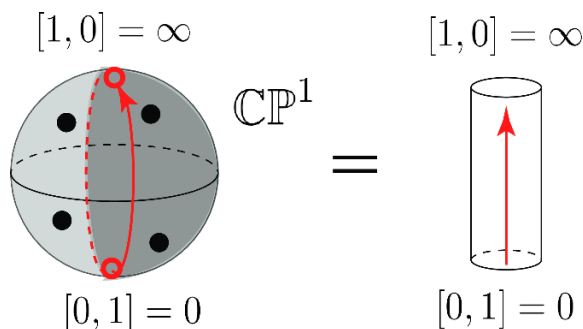
The point  $w = 0$  corresponds to the infinite past  $t = -\infty$  and the point  $w = \infty$  corresponds to the infinite future  $t = +\infty$ . Then, as stated in the previous section, instantons are given by holomorphic maps from  $\mathbb{CP}^1$  to  $\mathbb{CP}^{N-1}$ . However, we have to take into account the boundary condition (5.2) appropriately when we consider instanton solutions.

In the present case of twisted compactification, there exist fractional instantons (see [56–60] for early works). There are  $1/N$  instantons connecting  $P_k$  and  $P_{k+1}$  which we denote as  $I_{k \rightarrow k+1}$ . They are given by

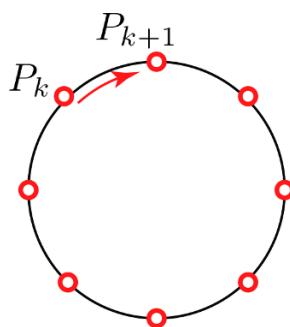
$$I_{k \rightarrow k+1} = (P_k \rightarrow P_{k+1}) : [\dots, Z_k(w), Z_{k+1}(w), \dots] = \left[ 0, \dots, 0, a, w^{\frac{1}{N}}, 0, \dots, 0 \right], \tag{5.5}$$

where  $a$  is a constant complex moduli parameter of the  $1/N$  instantons. Notice that the twisted boundary condition (5.2) is satisfied by these fractional instanton solutions. The loop  $x \rightarrow x+1$  corresponds to  $w \rightarrow e^{2\pi i} w$ , under which  $w^{1/N} \rightarrow e^{2\pi i/N} w^{1/N}$ . By using the fact that homogeneous coordinates  $Z_k$  are defined only up to overall multiplication by constants, one can check that (5.2) is satisfied. This really has  $1/N$  topological charge, because by going to the  $N$ -covering space of  $\mathbb{CP}^1$  as  $w' = w^{1/N}$ , the above solutions are  $[\dots, 0, a, w', 0, \dots]$  which is the standard one instanton of  $\mathbb{CP}^{N-1}$ . There are also  $-1/N$  anti-instantons  $\bar{I}_{k \rightarrow k+1}$  which are obtained by replacing  $w \rightarrow 1/\bar{w}$ . See figure 4 and 5 for these fractional instantons.

Note that the Atiyah-Hitchin-Singer theorem [55] predicts that there are  $4NQ = 4$  moduli fields for  $Q = 1/N$  instantons. Two of the four real parameters are provided by  $a$ . The other two may be provided by the moduli associated to translations along  $E$  as discussed earlier in the case of  $Q = 1$ . Actually, the  $\frac{1}{2\pi i} \log a$  also gives translation moduli



**Figure 4.** A fractional instanton for  $\mathbb{C}\mathbb{P}^1$ -model interpolates between two different vacua, 0 and  $\infty$  (the black dots represents singular points, as discussed in figure 2). In cylindrical coordinate (so that the geometry is  $\mathbb{R} \times S^1$ ) with fixed  $x$ , this moves along the  $\mathbb{R}$ -direction from past to future infinities, as we vary the parameter  $t$ . The full trajectory, with both  $t$  and  $x$  varied, is a hemisphere, and when the two such hemispheres are combined we obtain a full sphere. This is a manifestation of the fact that two fractional instantons combine into a single instanton.



**Figure 5.** The geometry is more complicated for  $N > 2$  case than the  $N = 2$  case in figure 4. However, the idea is the same: the classical moduli space  $\mathbb{C}\mathbb{P}^{N-1}$  is lifted by twisted boundary condition along  $S^1_C$  into  $N$  different points  $P_1, \dots, P_N$ , however tunneling between different these points, as given by fractional instantons, lifts the  $N$ -fold degeneracy, restoring the center symmetry dynamically.

along the direction  $\mathbb{R} \times S^1_C$ . Therefore, all the four moduli parameters of the Yang-Mills  $1/N$  instantons are the translations on  $\mathbb{R} \times \mathbb{T}^3$ .<sup>11</sup>

Now the true vacuum structure of the theory at the quantum level is described as follows. First of all, note that in the present case, the  $1/N$  instantons do not have size modulus because all the four moduli are associated to translations on space-time  $\mathbb{R} \times \mathbb{T}^3$ . The absence of the size modulus is related to the fact that there are no zero modes in the twisted compactification used in this paper. Then, at least if the length scale of the compactification is small, the dilute gas approximation of fractional instanton/anti-instanton is valid. This is the crucial difference from the instanton computation on  $\mathbb{R}^4$  discussed in the Introduction.

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<sup>11</sup>For some gauge groups  $G$  other than  $SU(N)$ , there exist fractional instantons on  $\mathbb{R} \times \mathbb{T}^3$  even without twisting. In those cases, the fact that the four moduli parameters are given by translations on  $\mathbb{R} \times \mathbb{T}^3$  can be seen by using string duality. See section 3.1 of [61] for these discussions.

Let  $H$  be the Hamiltonian of the system, and  $|P_k\rangle$  be the classical vacuum staying at the point  $P_k$ . Then the usual dilute gas computation of instantons would give the transition amplitudes

$$\langle P_{k+n}|H|P_k\rangle = C_n(g, e^{i\theta}, L_A, L_B, L_C, \mu) e^{\frac{in\theta}{N}}, \quad (5.6)$$

where  $C_n(g, e^{i\theta}, L_A, L_B, L_C, \mu)$  is a function of the coupling  $g$ , the exponential of the theta angle  $e^{i\theta}$ , the lengths  $L_{A,B,C}$  along  $S^1_{A,B,C}$  and the renormalization scale  $\mu$ . The  $k$  and  $n$  should be considered mod  $N$  by regarding  $C_{n+N} := C_n e^{-i\theta}$ . This  $C_n(g, e^{i\theta}, L_A, L_B, L_C, \mu)$  has perturbative as well as non-perturbative terms, and is expected to have the form of a resurgent function. The dependence of  $C_n(g, e^{i\theta}, L_A, L_B, L_C, \mu)$  on the theta angle  $\theta$  through  $e^{i\theta}$  is because one-instanton  $\prod_{k=1}^N I_{k \rightarrow k+1}$  is a transition from one vacuum to itself. In the dilute gas approximation, we roughly get

$$C_{n=\pm 1}(g, e^{i\theta}, L_A, L_B, L_C, \mu) \sim -M \exp\left(-\frac{8\pi^2}{Ng^2}\right), \quad (5.7)$$

where  $M > 0$  is given by a combination of the parameters  $(g, e^{i\theta}, L_A, L_B, L_C, \mu)$  with mass dimension one, while we neglect other  $C_n(g, e^{i\theta}, L_A, L_B, L_C, \mu)$  for  $|n| \geq 2$ . The minus sign was put to make the later results consistent with the general argument of Vafa and Witten [40], as was already mentioned around (1.3).

By diagonalizing the Hamiltonian, we get the energy eigenstates and eigenvalues as

$$H|e\rangle = E_e|e\rangle, \quad |e\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^N e^{\frac{2\pi i k e}{N}} |P_k\rangle, \quad (5.8)$$

where

$$E_e = \sum_{n=0}^{N-1} C_n(g, e^{i\theta}, L_A, L_B, L_C, \mu) e^{\frac{in(\theta-2\pi e)}{N}} \quad (5.9)$$

$$\sim C_0 - 2M \exp\left(-\frac{8\pi^2}{Ng^2}\right) \cos\left(\frac{\theta - 2\pi e}{N}\right). \quad (5.10)$$

There are  $N$  states, and the true vacuum is the one which minimizes this energy while the others are metastable.

Let us notice several important properties of the above vacuum structure. First, the energies  $E_e$  depend on  $e^{i\theta/N}$ , while the  $\theta$  is supposed to be a periodic variable  $\theta \sim \theta + 2\pi$ . They are consistent due to the existence of  $N$  (meta)stable vacua. For each  $e \in \mathbb{Z}_N$ , we have one vacuum. When the  $\theta$  is adiabatically changed to  $\theta + 2\pi$ , the vacuum  $|e\rangle$  goes to the vacuum  $|e-1\rangle$ , mapping one vacuum to another. This turns a metastable vacuum to a true vacuum, or a true vacuum into a metastable vacuum. This is a manifestation of the Witten effect [62], because our twisted boundary condition by the center symmetry is equivalent to including a unit 't Hooft magnetic flux  $\mathfrak{m} = 1$  on  $S^1_B \times S^1_C$  [63], and the  $e \in \mathbb{Z}_N$  can be identified with the 't Hooft electric flux, as will be seen in later discussions. In this way, the dependence on  $\theta$  as  $e^{i\theta/N}$  is consistent with the periodicity. See [46] for recent related discussion of this type of Witten effect.

Next, in the large  $N$  limit and with finite  $e$ , the energy is proportional to

$$E_e \propto (\theta - 2\pi e)^2 \quad (N \rightarrow \infty). \quad (5.11)$$

These properties are exactly as expected from the large  $N$  analysis of  $\mathbb{CP}^{N-1}$ -model and Yang-Mills theory, as mentioned already in section 1.2.

The results above are obtained in a small volume region where weak coupling analysis is reliable. However, the above vacuum structure such as  $E_e \propto \cos[(\theta - 2\pi e)/N]$  was also obtained in a different regime where the volume of the space-time is infinite while there is a light adjoint Majorana fermion [45]. The dependence of  $E_e$  on  $\theta$  can have important implications for cosmology of (non-QCD) axions such as string axions, and hence it would be very interesting to investigate it further.

## 5.2 4d situation

The discussion of the previous subsection was for the  $\mathbb{CP}^{N-1}$  model. Let us see what is happening in the 4d Yang-Mills theory. We introduce coordinates  $(x_A, x_B, x_C)$  related to the previously introduced coordinates as  $z = x_A + \tau x_B$  and  $x = x_C$  (where  $\tau = iL_B/L_A$ ), and call the corresponding directions as A, B and C cycles, respectively.

We would like to identify the points  $P_k$  given in (5.3) in terms of the 4d Wilson lines  $U_A, U_B$ , and  $U_C$ , where these Wilson lines are computed along the A, B and C cycles. In the following, it may be convenient to consider the covering space  $\mathbb{R}^3$  of  $S_A^1 \times S_B^1 \times S_C^1$  and regard the gauge fields as living on this  $\mathbb{R}^3$  with certain periodic properties. The Wilson lines  $U_{A,B,C}$  are defined by integration over  $0 \leq x_{A,B,C} \leq 1$  as

$$U_A(x_B, x_C) = \exp \left( - \int_{0 \leq x_A \leq 1, \text{ with fixed } x_B, x_C} A \right), \quad (5.12)$$

and similarly for  $U_B$  and  $U_C$ .

Implementation of the twist is a little more complicated in the Yang-Mills [63]. (See [64] for a detailed explanation of mathematical background behind the following discussion.) We introduce a gauge transformation  $h(x_B)$  which has the property that

$$h(x_B + 1) = e^{\frac{2\pi i}{N}} h(x_B), \quad h(x_B = 0) = 1. \quad (5.13)$$

Then, the gauge field is assumed to have the periodic boundary condition

$$\begin{aligned} A(x_B, x_C + 1) &= h^{-1}(x_B) A(x_B, x_C) h(x_B) + h^{-1}(x_B) dh(x_B), \\ A(x_B + 1, x_C) &= A(x_B, x_C), \end{aligned} \quad (5.14)$$

where we suppressed  $x_A$  because it is irrelevant for the discussion of the nontrivial part of the periodic boundary condition. This periodicity property means in particular that

$$\begin{aligned} U_B(x_C + 1) &= h(x_B = 1)^{-1} U_B(x_C) h(x_B = 0) \\ &= e^{-\frac{2\pi i}{N}} U_B(x_C). \end{aligned} \quad (5.15)$$

This corresponds to the twisted boundary condition  $Z_k(x_C + 1) = e^{2\pi i k/N} Z_k(x_C)$  of the  $\mathbb{CP}^{N-1}$  as can be seen from (3.20) and (3.29).



Another point which requires a care is the following. Under gauge transformation  $g$  on the space-time, the  $U_C$  transforms as  $g^{-1}(x_B, x_C = 1)U_C g(x_B, x_C = 0)$ . However, the  $g$  also satisfies the boundary condition  $g(x_B, x_C + 1) = h^{-1}(x_B)g(x_B, x_C)h(x_B)$  and hence the gauge transformation is  $U_C \rightarrow h^{-1}(x_B)g^{-1}(x_B, x_C = 0)h(x_B)U_C g(x_B, x_C = 0)$ . This requires us to modify the definition of  $U_C$  as

$$U'_C(x_B) = h(x_B)U_C(x_B) . \tag{5.16}$$

Then the gauge invariant Wilson loops can be defined as  $\text{tr} U'_C$ . This new  $U'_C$  has a boundary condition as

$$U'_C(x_B + 1) = e^{\frac{2\pi i}{N}} U'_C(x_B) . \tag{5.17}$$

The reason for this is as follows. As mentioned in section 2.2 and reviewed in appendix A, the symmetry operators of the 1-form symmetry  $C(Y)$  is defined on some codimension-2 surface  $Y$ . One way to implement the twisted boundary condition (which might look different from the implementation by (5.14) but is actually equivalent) is by inserting such an operator. In the present case, the  $Y$  extends in the direction  $\mathbb{R} \times S_A$  while it is localized on a point  $\text{pt} \in S_B \times S_C$ , so  $Y = \text{pt} \times \mathbb{R} \times S_A$ . Schematically we write this twist as  $C(\mathbb{R} \times S_A)$ . Then the directions  $S_B$  and  $S_C$  are on the same footing in this implementation of the 1-form twist. In particular, when we move  $U'_C$  from  $x_B$  to  $x_B + 1$ , this Wilson line is crossed across the operator  $C(\mathbb{R} \times S_A)$ . This crossing produces the phase factor  $e^{2\pi i/N}$  in the boundary condition of  $U'_C$ . Similarly  $U_B$  gets the phase  $U_B(x_C + 1) = e^{-\frac{2\pi i}{N}} U_B(x_C)$  by crossing  $C(\mathbb{R} \times S_A)$ . However, instead of  $C(\mathbb{R} \times S_A)$ , we just use the boundary conditions (5.14) in this subsection.<sup>12</sup>

Now let us restrict our attention to flat connections  $F = dA + A^2 = 0$ . In this case we have

$$U_C(x_B + 1)U_B(x_C)U_C^{-1}(x_B)U_B^{-1}(x_C + 1) \xrightarrow{\text{flat connection}} 1 . \tag{5.18}$$

By using  $U_C(x_B + 1) = U_C(x_B) = h^{-1}(x_B)U'_C(x_B)$  and setting  $U_B := U_B(0)$  and  $U'_C := U'_C(0)$ , we get

$$U'_C U_B = e^{-\frac{2\pi i}{N}} U_B U'_C . \tag{5.19}$$

More generally, if  $U'_C U_B = e^{-\frac{2\pi i m}{N}} U_B U'_C$ , then there is a 't Hooft magnetic flux  $m \in \mathbb{Z}_N$  on  $S_B^1 \times S_C^1$  [63, 64].

Up to conjugation (i.e., gauge transformation), there is a unique solution to the equation (5.19). This is because  $U'_C$  can be regarded as the “raising operator” of the eigenvalues of  $U_B$ . The solution up to conjugation is given by

$$(U_B)_{ij} = \delta_{i,j} \exp\left(2\pi i \frac{j - \frac{N+1}{2}}{N}\right), \quad (U'_C)_{ij} = \delta_{i,j+1} . \tag{5.20}$$

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<sup>12</sup>Mathematically speaking, any realization of the twist is described by a nontrivial topology of the  $SU(N)/\mathbb{Z}_N$  bundle over  $S_B^1 \times S_C^1$ . The (5.14) is one way to realize such a nontrivial bundle. Another way to realize it is to consider a small patch near  $\text{pt} \in S_B^1 \times S_C^1$ , and take a nontrivial transition function around this point by using an element of  $\pi_1(SU(N)/\mathbb{Z}_N)$ . These two are equivalent.



On the other hand,  $U_A = U_A(0)$  satisfies

$$U_A U_B = U_B U_A, \quad U_A U'_C = U'_C U_A, \quad (5.21)$$

because the boundary condition in the direction  $x_A$  is trivial (i.e., there is no twist in the  $S_A$  direction). Then, there are  $N$  solutions for  $U_A$  given by

$$U_A = e^{2\pi i \frac{K}{N}} \cdot 1, \quad (5.22)$$

where  $K \in \mathbb{Z}_N$ .

Notice that the gauge symmetry is completely broken by the Wilson lines  $U_B$  and  $U'_C$ , because the subset of  $SU(N)$  whose elements commute with both  $U_B$  and  $U'_C$  is discrete. This makes all fields massive and there is no infrared divergence, and we are away from the singular points of the moduli space of flat connections on the two-torus  $E$  because W-bosons are massive.

The above Wilson lines correspond to the coordinates of  $\mathbb{C}\mathbb{P}^{N-1}$  as follows. In terms of  $\vec{\phi}$  introduced in (3.4), the above  $(U_A, U_B)$  are given by

$$\vec{\phi} = \frac{\vec{\rho}}{N} - \tau K \vec{c}_j, \quad (5.23)$$

where  $\vec{c}_j$  was introduced in (3.22), and  $\vec{\rho}$  is the Weyl vector (i.e., half the sum of all positive roots) of  $SU(N)$  with respect to the root lattice  $\mathbb{L}$  (3.11):

$$\vec{\rho} = \left( \frac{N-1}{2}, \frac{N-3}{2}, \dots, -\frac{N-1}{2} \right). \quad (5.24)$$

Now let us map the above  $\vec{\phi}$  to  $\mathbb{C}\mathbb{P}^{N-1}$ . First, by a computation as in section 2.2 we get

$$\theta_k \left( \frac{\vec{\rho}}{N} - \tau K \vec{c}_j \right) = e^{-\pi i \tau K^2 \vec{c}_j^2 + 2\pi i K \vec{c}_j \cdot \frac{\vec{\rho}}{N}} \theta_{k-K} \left( \frac{\vec{\rho}}{N} \right). \quad (5.25)$$

Next, let  $W \in \mathfrak{S}_N$  be an element of the Weyl group such that it sends the  $i$ -th component to  $i+1$ -th component. Then, one can see that  $W(\vec{\rho}) = \vec{\rho} - N\vec{c}_1$ . Noticing the fact that the lattices  $\vec{e}_k + \mathbb{L}$  are invariant under the Weyl group, we get

$$\theta_k \left( \frac{\vec{\rho}}{N} \right) = \theta_k \left( W \left( \frac{\vec{\rho}}{N} \right) \right) = \theta_k \left( \frac{\vec{\rho}}{N} - \vec{c}_1 \right) = e^{\frac{2\pi i k}{N}} \theta_k \left( \frac{\vec{\rho}}{N} \right), \quad (5.26)$$

and hence  $\theta_k(\vec{\rho}/N) = 0$  for  $k \neq 0 \pmod N$ . Therefore, we conclude that

$$\theta_k \left( \frac{\vec{\rho}}{N} - \tau K \vec{c}_j \right) = 0 \quad \text{for } k \neq K. \quad (5.27)$$

By the identification  $Z_k = \theta_k$ , this is precisely the point  $P_K$  introduced in (5.3);

$$U_A = e^{\frac{2\pi i K}{N}} \longleftrightarrow P_K = [\dots, 0, \overset{K}{1}, 0, \dots]. \quad (5.28)$$

### 5.3 Dynamical restoration of the center symmetry

Now we can show that the center symmetry (i.e. dimensional reduction of 1-form symmetry) is unbroken. Because we are compactifying the theory on  $S_A^1 \times S_B^1 \times S_C^1$ , we get three center symmetries  $\mathbb{Z}_N^{(A)} \times \mathbb{Z}_N^{(B)} \times \mathbb{Z}_N^{(C)}$  from the dimensional reduction of 1-form symmetry, and they are associated to the gauge invariant Wilson loop operators  $\text{tr } U_A$ ,  $\text{tr } U_B$  and  $\text{tr } U'_C$ . For operators  $\text{tr } U_B$  and  $\text{tr } U'_C$ , we have

$$\text{tr } U_B = \text{tr } U'_C = 0 . \tag{5.29}$$

Therefore, the only nontrivial point is about the operator  $\text{tr } U_A$  which acts on  $|P_k\rangle$  as

$$\text{tr } U_A |P_k\rangle = N e^{\frac{2\pi i k}{N}} |P_k\rangle . \tag{5.30}$$

Thus, at each classical vacuum on  $P_k$ , the center symmetry is broken by the VEV (vacuum expectation value) of the Wilson loop  $\text{tr } U_A$ . However, we found that the true quantum vacua are linear combinations of  $|P_k\rangle$  given in (5.8). For them, we get

$$\langle \mathbf{e} | \text{tr } U_A | \mathbf{e} \rangle = \frac{1}{N} \sum_{k=1}^N N e^{\frac{2\pi i k}{N}} = 0 . \tag{5.31}$$

Therefore, the center symmetry is dynamically restored by  $1/N$  fractional instanton/anti-instanton effects! The fact that the symmetry is restored itself is not so surprising because all the spatial directions are compact  $\mathbb{T}^3 = S_A^1 \times S_B^1 \times S_C^1$  and a symmetry breaking does not usually happen in quantum mechanics. However, we stress that the “strong dynamics” properties of confinement such as the center symmetry restoration and the dependence of the vacuum energy on  $\theta$  are realized completely in weak coupling regime. Because of this, we conjecture that the vacua discussed in the previous subsection are adiabatically continued when we go from small to large volume limit.

Although the center symmetry is unbroken, the vacua have charges under  $\mathbb{Z}_N^{(A)}$ . The vacuum states go to itself under the action of the symmetry and hence all operators having nonzero charge under it have vanishing VEVs. But there can be a possible phase factor on the action of the symmetry operator on the vacua. To see this, let  $C_A$  be the generator of the center symmetry  $\mathbb{Z}^{(A)}$  acting on the Hilbert space.<sup>13</sup> From (3.28), it acts as  $C_A |P_k\rangle = |P_{k-1}\rangle$ .<sup>14</sup> Then we find that the basis  $|\mathbf{e}\rangle$  diagonalizes the electric one-form center symmetry  $C_A$  (hence the notation  $\mathbf{e}$  for “electric”):

$$C_A |\mathbf{e}\rangle = e^{\frac{2\pi i \mathbf{e}}{N}} |\mathbf{e}\rangle . \tag{5.32}$$

<sup>13</sup>In the description  $C(Y)$  in appendix A,  $C_A$  is defined by choosing  $Y$  which is extending in the direction  $S_B \times S_C$  while localized in the direction  $\mathbb{R} \times S_A$ . So schematically we can write  $C_A = C(S_B \times S_C)$ .

<sup>14</sup>In principle, we can have a nontrivial phase factors here as  $C_A |P_k\rangle = \eta_k |P_{k-1}\rangle$  for  $|\eta_k| = 1$ . By successive redefinitions of states  $|P_k\rangle$ , we can assume  $\eta_k = 1$  for  $k \neq 0 \pmod N$ . Furthermore, the fact that the symmetry is  $\mathbb{Z}_N$  implies that  $(C_A)^N = 1$ , and then we get  $\prod_{k=1}^N \eta_k = 1$ . From these facts, we can take all  $\eta_k$  to be 1. Once we fix the phase factors of the states  $|P_k\rangle$  in this way, there is no more freedom to change them except for the overall factor. Then the phase  $e^{i\theta/N}$  appearing in (5.6) is well-defined, and hence  $\theta$  has a physical meaning. However, there is a freedom to change the definition of  $C_A$  as  $C'_A = e^{2\pi i/N} C_A$  and  $|P'_k\rangle = e^{-2\pi i k/N} |P_k\rangle$ . They still satisfy  $C'_A |P'_k\rangle = |P'_{k-1}\rangle$ , but we get the change  $\theta \rightarrow \theta' = \theta + 2\pi$ . Of course, this is the usual periodicity of  $\theta$ .

Therefore, the different vacua has different charges under  $C_A$ . The  $\mathbf{e} \in \mathbb{Z}_N$  which determines the eigenvalues of  $C_A$  is the 't Hooft electric flux [63, 64].

We stress again that the center symmetry is still preserved, because  $C_A|\mathbf{e}\rangle \propto |\mathbf{e}\rangle$  and hence they define the same ray in the Hilbert space (i.e., the same physical state). So, the center symmetry is unbroken at each vacuum. By using (5.31), one can see that we can move between these vacua by the action of the operator  $\text{tr} U_A$ ,

$$\frac{1}{N} \text{tr} U_A |\mathbf{e}\rangle = |\mathbf{e} + 1\rangle . \tag{5.33}$$

This is consistent because  $U_A$  is charged under  $C_A$ .

## 6 Borel plane of QFTs and Linde's problem

Having constructed a setup where IR divergence is properly taken into account and the center symmetry ensured, we now have a setup where the resurgence program can be applicable, at least in principle. While the detailed analysis of such an analysis is left for future work, let us here point out an important subtlety: we demonstrate that there can exist singularities in Borel plane in QFT which depend on the shapes of manifolds. Such manifold-dependent singularities could in some cases dominate over renormalons, and hence dramatically change the singularity structure of the Borel plane of a compactified theory. This dependence is on the shapes of manifolds, so they survive even in the large volume limit. This might suggest that there is no possibilities for resurgence directly on  $\mathbb{R}^4$  for Yang-Mills theory, and we always have to start from a compactified geometry without IR problems.

### 6.1 Quantum mechanics

That we have manifold-dependent singularities in the Borel plane is a general phenomenon. To explain this, let us start with 1d quantum mechanics with target space  $\mathcal{M}$ . The Lagrangian is

$$L_{1d} = \frac{1}{2g_{\text{QM}}^2} h_{IJ} \partial_t \phi^I \partial_t \phi^J , \tag{6.1}$$

where  $h_{IJ}$  is the metric on the target space  $\mathcal{M}$ , and  $g_{\text{QM}}^2$  is the coupling constant (which is usually called  $\hbar$ ). The canonical momentum is

$$\pi_I = \frac{\partial L_{1d}}{\partial(\partial_t \phi^I)} = \frac{1}{g_{\text{QM}}^2} h_{IJ} \partial_t \phi^J \longrightarrow -i \nabla_i ,$$

where  $\nabla_I$  is the covariant derivative on the target space  $\mathcal{M}$  with respect to the metric  $h_{IJ}$ . The Hamiltonian is therefore given by

$$H = \frac{g_{\text{QM}}^2}{2} \Delta , \tag{6.2}$$

where  $\Delta = -\nabla^I \nabla_I$  is the Laplacian.

Let  $\lambda_\ell$  be the  $\ell$ -th eigenvalue of the Laplacian  $\Delta$ , and let  $n_\ell$  be the multiplicity of that eigenvalue. Then the partition function of the theory on  $S^1$  with circumference  $2\pi R$  is given by

$$Z = \text{tr} e^{-2\pi RH} = \sum_{\ell=0}^{\infty} n_\ell e^{-\pi R g_{\text{QM}}^2 \lambda_\ell} . \tag{6.3}$$

**Asymptotic expansion.** Now we restrict our attention to the case  $\mathcal{M} = \mathbb{CP}^1 = S^2$  which has the unit radius. We want to perform perturbative expansion of  $Z$  in terms of the coupling constant. In this case, we know that the eigenfunctions of the Laplacian  $\Delta$  are given by spherical harmonics and the eigenvalues and their degeneracies are given by

$$\lambda_\ell = \ell(\ell + 1) , \quad n_\ell = 2\ell + 1 , \tag{6.4}$$

where  $\ell = 0$  is the ground state and  $\ell \geq 1$  are excited states. Now we can expand the partition function  $Z$  by using the Euler-Maclaurin formula. If  $f(x)$  is an analytic function, we have asymptotic expansion given by

$$\sum_{k=0}^{K-1} f\left(k + \frac{1}{2}\right) = \int_0^K f(x) dx - \sum_{m=1}^{\infty} \frac{(1 - 2^{-2m+1}) B_{2m}}{(2m)!} [f^{(2m-1)}(0) - f^{(2m-1)}(K)] , \tag{6.5}$$

where  $f^{(2m-1)}$  means  $(2m-1)$ -th order differential of the function  $f$ . The  $B_{2m}$  are Bernoulli numbers and can be expressed as

$$B_{2m} = (-1)^{m+1} \frac{2(2m)!}{(2\pi)^{2m}} \zeta(2m) , \tag{6.6}$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} . \tag{6.7}$$

Note that  $\zeta(s) \rightarrow 1$  as  $s \rightarrow \infty$ . Thus these formulas give the asymptotic behavior of  $B_{2m}$ .

Now we can rewrite the partition function of  $\mathbb{CP}^1 = S^2$  quantum mechanics. We set

$$f(x) = x e^{-\pi R g_{\text{QM}}^2 x^2} . \tag{6.8}$$

Then we get

$$\begin{aligned} e^{-\frac{1}{4}\pi R g_{\text{QM}}^2} \cdot Z &= \sum_{\ell=0}^{\infty} 2 \left(\ell + \frac{1}{2}\right) e^{-\pi R g_{\text{QM}}^2 (\ell + \frac{1}{2})^2} \\ &= 2 \int_0^{\infty} f(x) dx - 4 \sum_{m=1}^{\infty} \frac{(1 - 2^{-2m+1})(-1)^{m+1} \zeta(2m)}{(2\pi)^{2m}} f^{(2m-1)}(0) . \end{aligned} \tag{6.9}$$

One can compute

$$\int_0^{\infty} f(x) dx = \frac{1}{2\pi R g_{\text{QM}}^2} , \tag{6.10}$$

and

$$f^{(2m-1)}(0) = (-1)^{m-1} \frac{(2m-1)!}{(m-1)!} (\pi R g_{\text{QM}}^2)^{m-1}. \quad (6.11)$$

Thus we get

$$e^{-\frac{1}{4}\pi R g_{\text{QM}}^2} \cdot Z = \frac{1}{\pi R g_{\text{QM}}^2} - 4 \sum_{m=1}^{\infty} \frac{(1-2^{-2m+1})\zeta(2m)}{(2\pi)^{2m}} \frac{(2m-1)!}{(m-1)!} (\pi R g_{\text{QM}}^2)^{m-1}. \quad (6.12)$$

Let us study the asymptotic behavior. The coefficient grows as

$$\frac{(1-2^{-2m+1})\zeta(2m)}{(2\pi)^{2m}} \frac{(2m-1)!}{(m-1)!} = C_m \frac{m!}{\pi^{2m}}, \quad (6.13)$$

where  $C_m \rightarrow 1/\sqrt{4\pi m}$ . Therefore, a crude asymptotic behavior of  $Z$  is given as

$$Z \sim \sum_m m! \left( \frac{R g_{\text{QM}}^2}{\pi} \right)^m.$$

This has a singularity on the positive real axis of the Borel plane with the associated ambiguity of order

$$\exp\left(-\frac{\pi}{R g_{\text{QM}}^2}\right). \quad (6.14)$$

Assuming that resurgence works, this implies that the path integral of this quantum mechanics has unstable saddle points with the classical action  $\pi/R g_{\text{QM}}^2$ . Interestingly, we can indeed find such a saddle point.

**Classical saddle points.** In the path integral with the Lagrangian (6.1), there are saddle points which are given by geodesics on the target space  $\mathcal{M}$ . Let us consider the simple case  $\mathcal{M} = \mathbb{C}\mathbb{P}^1 = S^2$ . Let  $(\theta, \phi)$  be the polar coordinates of  $\mathcal{M} = S^2$ . Then, classical solutions of equations of motion are geodesics wrapping great circles of  $S^2$ . For example, we have

$$(\theta, \phi) = \left( \frac{\pi}{2}, \frac{nt}{R} \right) \quad 0 \leq t \leq 2\pi R, \quad (6.15)$$

where  $n \in \mathbb{Z}$ . The action can be easily evaluated. The  $n = 0$  is the lowest stable solutions. For  $n = \pm 1$  it is given as

$$S_{\text{cl}} = \int_0^{2\pi R} L dt = \frac{\pi}{R g_{\text{QM}}^2}. \quad (6.16)$$

This is exactly as expected from the above analysis of the asymptotic expansion and the singularities of the Borel plane. Notice that this action explicitly depends on the parameter  $R$  which specify the manifold  $S^1$ .

**Twist.** We can also introduce a twist in the partition function above. Explicitly, we can impose the boundary condition such as

$$(\theta, \phi)_{(t+2\pi R)} = (\theta, \phi + \pi)_{(t)}. \tag{6.17}$$

Then the classical solutions (6.15) still exists, but now with the condition that  $n \in \frac{1}{2} + \mathbb{Z}$ . The stable lowest action solutions are  $\theta = 0, \pi$ . Therefore, the above solutions are unstable. The order of the classical action is the same up to numerical constants. The analysis of the asymptotic behavior of the perturbative expansion should also be straightforward, but we do not perform that explicitly.

## 6.2 2d sigma models

Having established that for 1d quantum mechanics the asymptotic behavior is reproduced from corresponding unstable saddle points, we can easily see that there are singularities on the Borel plane of 2d sigma models.

The Lagrangian for the 2d theory is

$$\mathcal{L}_{2d} = \frac{1}{2g_{2d}^2} h_{IJ} \partial_i \phi^I \partial_i \phi^J. \tag{6.18}$$

Now let us consider partition function of this theory on the spacetime  $\mathbb{T}^2 = S^1 \times S^1$ , where one  $S^1$  has radius  $R_1$  and the other  $S^1$  has radius  $R_2$ ; in this section we compactify the temporal direction of the 2d sigma models. If  $R_1 \ll R_2$ , we may first dimensionally reduce the theory to the quantum mechanics discussed above. The coupling constants are related as

$$g_{\text{QM}}^2 = \frac{g_{2d}^2}{2\pi R_1}. \tag{6.19}$$

The radiative corrections from the Kaluza-Klein modes are higher orders of  $g_{2d}^2$ . Then, in the dimensionally reduced theory, we can use the above results about quantum mechanics.

Let us consider the case  $\mathcal{M} = \mathbb{CP}^1 = S^2$ . The classical action obtained above is given by

$$S_{\text{cl}} = \frac{\pi}{R_2 g_{\text{QM}}^2} = \frac{R_1}{R_2} \frac{2\pi^2}{g_{2d}^2}. \tag{6.20}$$

From this, we conclude that the Borel plane contains a singularity which depends on the shape  $R_1/R_2$  of the manifold. Notice that when  $R_1/R_2 \ll 1$ , this pole is very close to the origin than renormalons and instanton-anti-instanton pairs.

In the large volume limit

$$\frac{R_1}{R_2} = \text{fixed}, \quad R_1 R_2 \rightarrow \infty, \tag{6.21}$$

we expect to recover the theory on flat space  $\mathbb{R}^2$ . In particular, we expect that it is sensitive only to the volume  $R_1 R_2$  of spacetime, and not to the shape  $R_1/R_2$  if the theory has a

mass gap.<sup>15</sup> Nevertheless, the above-mentioned singularity of the Borel plane with the corresponding action (6.20) still exists. This suggests that the structure of the Borel plane is more complicated than is usually thought in this IR regulated situation.

On the other hand, if we take

$$R_1 = \text{fixed}, \quad R_2 \rightarrow \infty, \quad (6.22)$$

then the partition function gives the thermal free energy with temperature  $T = (2\pi R_1)^{-1}$ . In this limit, the action (6.20) becomes extremely small and hence the singularity on the Borel plane merges into the origin. This is a manifestation of the Linde's problem in thermal free energy in the present case of 2d  $\mathbb{CP}^1$  sigma model.

However, we stress that if we can complete the resurgence program of QFT, we can still get a sensible answer. Let us demonstrate this point by using a toy function. Consider

$$-\frac{1}{R_1 R_2} \log Z_{\text{toy}} = R_1^{-2} \int_0^1 dt \frac{\exp\left(-\frac{R_1}{R_2} \frac{2\pi^2}{g_{2d}^2} t\right)}{\sqrt{1-t}}. \quad (6.23)$$

This has an asymptotic expansion given roughly as

$$-\frac{1}{R_1 R_2} \log Z_{\text{toy}} \sim R_1^{-2} \sum_m m! \left(\frac{R_2}{R_1} \frac{g_{2d}^2}{2\pi^2}\right)^m. \quad (6.24)$$

Naively from this perturbative expansion, it seems hopeless to get a sensible answer in the limit  $R_2 \rightarrow \infty$ . In the notation of the subsection 1.2, we identify  $\beta = T^{-1} \rightarrow 2\pi R_1$  and  $m_{\text{IR}} \rightarrow R_2^{-1}$ , so the IR cutoff  $m_{\text{IR}}$  goes to zero as  $R_2 \rightarrow \infty$ . However, the integral in (6.23) makes perfect sense in the limit  $R_2 \rightarrow \infty$  with a finite value for the toy thermal free energy. Of course, the actual partition functions of QFT's are more complicated than the above toy function. But the point here is that Linde's problem can in principle be overcome by resurgence.

**Twist.** Suppose that we introduce the twisted boundary condition on the  $S^1$  with radius  $R_2$ . If  $R_1 \ll R_2$ , the result is qualitatively similar to the one above; we have a singularity on the Borel plane which depends on  $R_1/R_2$ . However, in the region  $R_1 \gg R_2$ , the theory has a mass gap of order  $R_2^{-1}$  which is large, and hence there is no IR divergences even in the limit  $R_1 \rightarrow \infty$ . Thus we expect that there is no problem in resurgence in this limit. This is the situation studied in e.g., [23, 24, 31, 65–68] and also in this paper.

### 6.3 Speculation on 4d Yang-Mills

We would like to make speculative comments on the structure of the Borel plane and Linde's problem in 4d Yang-Mills. By a compactification on small  $\mathbb{T}^2$  and considering only flat connections, we have seen that the 4d  $SU(N)$  Yang-Mills is reduced to 2d  $\mathbb{CP}^{N-1}$  at a

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<sup>15</sup>Instead, if the theory is gapless in the large volume limit such as conformal field theories, it is natural that quantities such as partition functions depend on  $R_1/R_2$ .

qualitative level.<sup>16</sup> Then, we expect that the 4d Yang-Mills encounters all the phenomena discussed above for  $\mathbb{CP}^1$ .

For example, we can find a classical solution of the Yang-Mills equation on a torus. We consider the 4-manifold  $\mathbb{T}^4 = S_T^1 \times S_A^1 \times S_B^1 \times S_C^1$  with the circumference of each circle given by  $L_T, L_A, L_B$  and  $L_C$ . We take the metric as

$$ds^2 = L_T^2 dx_T^2 + L_A^2 dx_A^2 + L_B^2 dx_B^2 + L_C^2 dx_C^2, \quad (6.25)$$

with each  $x_{T,A,B,C}$  having periodicity  $x_{T,A,B,C} \sim x_{T,A,B,C} + 1$ . Without twist, we can find a classical solution of the Yang-Mills equation as

$$iA = iA_\mu dx^\mu = \text{diag}(a, -a, 0, \dots, 0), \quad (6.26)$$

where

$$a = 2\pi x_C dx_B. \quad (6.27)$$

This solution is included in the Cartan of  $SU(N)$ , and has a constant curvature

$$f = da = 2\pi dx_C \wedge dx_B. \quad (6.28)$$

Thus it satisfies the Yang-Mills equations. Notice that this configuration  $a = 2\pi x_C dx_B$  is similar to the classical solutions in  $\mathbb{CP}^{N-1}$  given above via the relations between Yang-Mills and  $\mathbb{CP}^{N-1}$  model given in (3.4) and (3.13).

With the twist on  $S_B^1 \times S_C^1$ , we still have solutions such as

$$iA = iA_\mu dx^\mu = \text{diag}(a, , 0, \dots, 0) - \frac{1}{N} \text{diag}(a, a, \dots, a) \quad (6.29)$$

with the above  $a$ .

These classical solutions have actions of order

$$S_{\text{cl}} \sim \frac{L_T L_A}{L_B L_C} \frac{8\pi^2}{g_{\text{YM}}^2}, \quad (6.30)$$

up to order 1 factor. These actions explicitly depend on the shape of the 4-manifold even in the large volume limit. We have not analyzed the asymptotic behavior of the perturbative expansion of the free energy, but the existence of these unstable solutions suggests that the structure of the Borel plane is more complicated than is usually thought.

If  $L_T L_A \ll L_B L_C$ , this classical action becomes arbitrary small. On the other hand, if  $L_T L_A \gg L_B L_C$ , it depends on whether there is the twist or not. Without the twist, we can have a similar solution  $a = 2\pi x_T dx_A$  with the action proportional to  $(L_B L_C)/(L_T L_A)$ . So the situation is parallel to the above case. With twist, such a solution is forbidden basically because the twist generates a mass gap determined by  $L_{B,C}$  and there is no IR

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<sup>16</sup>Quantitatively, the classical action of the 4d Yang-Mills gives a metric of  $\mathbb{CP}^{N-1}$  which is different from the standard Fubini-Study metric. Also, there are radiative corrections from W-bosons. It would be interesting to perform detailed computations on these points.



divergences in the limit  $L_{T,A} \rightarrow \infty$ . Thus there is no classical saddle points with arbitrarily small action.

The value of the action (6.30) is very large in the parameter region  $L_T \rightarrow \infty$  which we have studied in this paper, and hence the above unstable saddle point is unimportant for the resurgence analysis at zero temperature  $L_T \rightarrow \infty$ . By contrast, when we decompactify the spatial regions  $L_{A,B,C} \rightarrow \infty$  with fixed ratios and constant finite  $L_T$ , then the action becomes arbitrarily small. This is a manifestation of Linde’s problem of 4d Yang-Mills and this problem should be overcome by resurgence as discussed above.

## A One-form global symmetry

Let us quickly review the concept of the one-form global symmetry [48, 69]. The usual (i.e., 0-form) symmetries act on local operators, while 1-form symmetries act on line operators such as Wilson loop operators. More explicitly, they can be described as follows. The usual 0-form symmetry for a continuous symmetry can be described by picking up a codimension-1 submanifold  $Y$  inside a spacetime  $X$ . By integrating a current  $j = j_\mu dx^\mu$  on  $Y$  we get a charge as  $Q(Y) = \int_Y *j$ . For discrete symmetries, there is no current operator, but we can still consider charge operators  $Q(Y)$  defined on  $Y$ . Conservation of the charge is translated to the statement that this operator  $Q(Y)$  only depends on the topology of  $Y$  outside the locus where other operators are inserted. If we cross a local operator across  $Y$ , then that operator is transformed by  $Q(Y)$ . We can do similar things for  $p$ -form symmetries. Pick up a codimension- $(p + 1)$  submanifold  $Y$  and define a charge operator  $Q(Y)$ . Operators charged under them are extended in  $p$  dimensional manifold  $Z$ . We denote that operator as  $O(Z)$ . If the  $Z$  is crossed across  $Y$ , the operator under consideration is changed by the charge operator  $Q(Y)$ .

For example, if  $Z \cong \mathbb{R}^p$  is a flat subspace inside the total spacetime  $X = \mathbb{R}^d$ , and  $Y \cong S^{d-p-1}$  is a sphere surrounding  $Z$  where  $d = \dim X$ , then we get

$$Q(Y) O(Z) = q O(Z) , \tag{A.1}$$

where  $q$  is the charge of the operator  $O$  under the generator  $Q$  of the symmetry.

If we consider the case  $p = 1$ ,  $d = 4$  and  $O(Z) = \text{tr} U(Z)$  is the Wilson loop operator, there is a 1-form  $\mathbb{Z}_N$  symmetry  $C$  such that<sup>17</sup>

$$C(Y) \text{tr} U(Z) = e^{\frac{2\pi i L(Y,Z)}{N}} \text{tr} U(Z) , \tag{A.2}$$

where  $L(Y, Z)$  is the linking number of  $Y$  and  $Z$ . For example, in the case discussed above, namely  $Z \cong \mathbb{R}^p$  and  $Y \cong S^{d-p-1}$  is a sphere surrounding  $Z$ , we get  $L(Y, Z) = 1$ .

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<sup>17</sup> $C(Y)$  here is a Gukov-Witten surface operator [70, 71].

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