

Families of gauge conditions in BV formalism

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ABSTRACT: In BV formalism we can consider a Lagrangian submanifold as a gauge condition. Starting with the BV action functional we construct a closed form on the space of Lagrangian submanifolds. If the action functional is invariant with respect to some group H and Λ is an H -invariant family of Lagrangian submanifold then under certain conditions we construct a form on Λ that descends to a closed form on Λ/H . Integrating the latter form over a cycle in Λ/H we obtain numbers that can have interesting physical meaning. We show that one can get string amplitudes this way. Applying this construction to topological quantum field theories one obtains topological invariants.

KEYWORDS: BRST Quantization, Gauge Symmetry, Topological Strings

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1 Introduction

A physical theory can be represented by various equivalent action functionals. For example, in the case of degenerate action functionals we can impose different gauge conditions. In BRST-formalism infinitesimal Q -exact variation of action functional leads to equivalent action functional. In BV-formalism the role of choice of gauge condition is played by the choice of Lagrangian submanifold.

As an example one can consider topological quantum field theories of Witten type, where the action functional in BRST-formalism depends on metric, but the variation of this functional by an infinitesimal change of the metric (the energy-momentum tensor) is Q -exact.

The first impression is that it is sufficient to consider only one functional from a family of physically equivalent action functionals. As was noticed in [1] this is wrong. The consideration of a family of equivalent action functionals or family of gauge conditions labeled by points of (super) manifold Λ leads to a construction of a closed differential form

Ω on Λ (a closed pseudodifferential form if Λ is a supermanifold). If our action functionals are invariant with respect to some group H then the form Ω is H -invariant, but it does not necessarily descend to Λ/H . Under some conditions we construct a closed H -equivariant form Ω_H and show that this equivariant form is homologous to a form descending to Ω/H . This allows us to obtain interesting physical quantities integrating over cycles in Λ/H .

For example, we can start with topological quantum field theory on some manifold Σ . One can apply our results to the family of equivalent action functionals labeled by metrics on Σ . We obtain topological invariants of Σ this way; it would be interesting to calculate them and compare with known invariants.

This machinery can be applied to string amplitudes. The worldsheet of bosonic string can be considered as two-dimensional topological quantum field theory. Considering Λ as a space of metrics and H as a group generated by diffeomorphisms and Weyl transformations we get formulas for string amplitudes; for appropriate choice of Lagrangian submanifolds these formulas coincide with the standard ones. Similar constructions work for other types of strings.

Some remarks about terminology and notations. We are saying “manifold” instead of “supermanifold”, “group” instead of “supergroup”, etc. We understand an element of super Lie algebra as a linear combination $\sum \epsilon^A T_A$ where T_A are even or odd generators of \mathbb{Z}_2 -graded Lie algebra and ϵ^A are even or odd elements of some Grassmann algebra; hence in our understanding an element of super Lie algebra is always an even object (see [2, 3] for the definitions of supermanifold, super Lie algebra, etc. that we are using).

We work in BV-formalism assuming that the BV action functionals are defined on odd symplectic manifold M equipped with volume element (SP-manifold in terminology of [4, 5]). In this situation the odd Laplacian Δ is defined on the space of functions on M . It was noticed in [6] that in the absence of the volume element the odd Laplacian is defined on semidensities; this allows the reformulation of BV-formalism for any odd symplectic manifold. In appendix C we show how to prove our main results in this more general setting. Some basic formulas of BV-formalism are listed in appendix A.

The space of (smooth) functions on a supermanifold M is denoted $\text{Fun}(M)$. This space is \mathbb{Z}_2 -graded: $\text{Fun}(M) = \text{Fun}_{\bar{0}}(M) + \text{Fun}_{\bar{1}}(M)$. Functions on ΠTM (on the space of tangent bundle with reversed parity of fibers) are called pseudodifferential forms (PDF) on M . (Differential forms can be considered as polynomial functions on ΠTM .) Diff stands for the group of diffeomorphisms, Vect for its Lie algebra (the algebra of vector fields), Weyl for the group of Weyl transformations. As we have noticed an element of any super Lie algebra (and hence a vector field) is considered an even object.

We use the term “canonical transformation” for a transformation of (odd) symplectic manifold preserving the symplectic form (another word for this notion is “symplectomorphism”). On a simply connected manifold infinitesimal canonical transformations can be characterized as Hamiltonian vector fields. Notice that in our terminology the Hamiltonian on odd symplectic manifold is an odd function B ; the first order differential operator corresponding to the Hamiltonian vector field with the Hamiltonian B is expressed in terms of the odd Poisson bracket as an operator transforming a function G into $\{B, G\}$; this

operator is even (parity preserving). The condition $\Delta B = 0$ means that the Hamiltonian vector field is volume preserving (= divergence free).

2 Families of equivalent action functionals

Let us consider a functional S defined on an odd symplectic manifold M with volume element and satisfying the quantum master equation $\Delta e^{S_{\text{BV}}} = 0$. (Here Δ stands for the odd Laplacian [23, BV-formalism/OddLaplace.html].) Then the physical quantities corresponding to the BV action functional S_{BV} can be expressed as integrals $\int_L A e^{S_{\text{BV}}}$ where L is a Lagrangian submanifold of M and the integral is taken with respect to the volume element induced on this submanifold; A stands for quantum observable (i.e. $\Delta(Ae^{S_{\text{BV}}}) = 0$ or equivalently $\Delta A + \{A, S_{\text{BV}}\} = 0$). These integrals depend only on the homology class of the Lagrangian submanifold.

Let us consider now a family of physically equivalent BV-action functionals $S_\lambda, \lambda \in \Lambda$ obeying $\{S_\lambda, S_\lambda\} = 0, \Delta S_\lambda = 0$. We can consider S as a function on $\Lambda \times M$. We assume that Λ is simply connected; then S_λ being physically equivalent for different values of λ is equivalent to the existence of functions B_a such that:

$$\frac{\partial}{\partial \lambda^a} S_\lambda = \{B_a, S_\lambda\} \tag{2.1}$$

for some $B_a \in \text{Fun}_1(M), \Delta B_a = 0$ (one can describe B_a as Hamiltonians of infinitesimal volume preserving canonical transformations giving equivalence of functionals S_λ for infinitesimally close λ). The eq. (2.1) implies that $\left\{ \frac{\partial B_a}{\partial \lambda^b} - \frac{\partial B_b}{\partial \lambda^a} + \{B_a, B_b\}, S_\lambda \right\} = 0$. We will assume a stronger condition:

$$dB - \frac{1}{2}\{B, B\} = 0 \tag{2.2}$$

$$\text{where } B = d\lambda^a B_a \tag{2.3}$$

Then the following PDF on Λ is closed:

$$\Omega(\lambda, d\lambda) = \int_L \exp(S_\lambda + B) \tag{2.4}$$

Indeed using eqs. (A.13) and (A.14) we obtain

$$d\Omega(\lambda, d\lambda) = \int_L \left(\{B, S\} + \frac{1}{2}\{B, B\} \right) e^{S+B} = \int_L \Delta e^{S+B} = 0 \tag{2.5}$$

More generally, let us define:

$$\Omega\langle F \rangle(\lambda, d\lambda) = \int_L F e^{S+B} \tag{2.6}$$

$$\text{where } F \in \text{Fun}(\Lambda \times M) \text{ such that } dF = \{B, F\} \tag{2.7}$$

Then:

$$d\Omega\langle F \rangle = -\Omega\langle \Delta F + \{S, F\} \rangle \tag{2.8}$$

Eq. (2.6) follows from the following chain of equalities:

$$d\Omega = \int_L \left(\{B, F\} + \frac{1}{2}\{B, B\}F + \{B, S\}F \right) e^{S+B} = \tag{2.9}$$

$$= \int_L \Delta(Fe^{S+B}) - \int_L (\Delta F + \{S, F\})e^{S+B} \tag{2.10}$$

and $\int_L \Delta(\dots) = 0$.

Notice that eq. (2.1) does *not* define B_a unambiguously; there is a freedom to add to B_a a function $\{S, A_a\}$ where $\Delta A_a = 0$. One can use this freedom to obtain B_a satisfying (2.1) and (2.2). This is not always possible globally, but always possible locally (in small pieces of the parameter space Λ). To check this we consider a fiber bundle over Λ having as a fiber over a point $\lambda \in \Lambda$ the set of volume preserving canonical transformations transforming S_{λ_0} in S_λ . (Here λ_0 is a fixed point of Λ .) A continuous (even differentiable) section of this bundle not necessarily exists globally, but always exists locally. It exists globally, in particular, in the case when Λ is contractible. Differentiating the section U_λ we obtain infinitesimal canonical transformations $\hat{B}_a = \frac{\partial U}{\partial \lambda^a} U^{-1}$. Their Hamiltonians B_a obey (2.1) and (2.2). (This is not quite correct: the operators $\hat{B} = d\lambda^a \hat{B}_a$ obey $d\hat{B} - 1/2[\hat{B}, \hat{B}] = 0$, but their Hamiltonians B_a specified *via* $\hat{B}_a = \{B_a, -\}$ are defined only up to a λ -dependent constant and (2.2) is true only for an appropriate choice of these constants; see appendix C for details.)

3 Families of Lagrangian submanifolds in BV phase space

We will show that one can construct some interesting quantities (including string amplitudes) considering *families* of Lagrangian submanifolds instead of families of action functionals.

Let us fix a connected family Λ of simply connected Lagrangian submanifolds. In other words we assume that L depends on parameters $\lambda_1, \dots, \lambda_k, \dots$ (these parameters can be odd, but for simplicity we assume that they are even). Let G be the group of canonical transformations of M (transformations preserving the odd symplectic structure), and \mathfrak{g} its Lie algebra. Elements of \mathfrak{g} correspond to odd functions on M (Hamiltonians).

Tentative definition of the closed form Ω . We want to define a closed pseudo-differential form Ω on the space LAG of all simply-connected Lagrangian submanifolds:

$$\Omega \in \text{Fun}(\Pi T \text{ LAG}) \tag{3.1}$$

Roughly speaking, the value of Ω at a point $v \in \Pi T \text{ LAG}$ is computed as follows. Notice that v corresponds to a pair (L, σ) where $L \in \text{LAG}$ and $\sigma \in \text{Fun}(L)$ is an odd function on L describing the tangent vector.¹ The variation of L can be described by infinitesimal

¹As in Classical Mechanics, a function on Lagrangian manifold L specifies a tangent vector to LAG (an infinitesimal deformation of L). In our case the symplectic form is odd, hence the correspondence is parity reversing. These functions are called “infinitesimal gauge fermions”. We have assumed that L is simply-connected, in this case the map of functions to infinitesimal deformations is surjective and its kernel consists of constant functions.

canonical transformation; one can say that σ is a restriction to L of the Hamiltonian of this transformation. (Notice that the canonical transformation is not unique, but the restriction of its Hamiltonian to L is well defined up to a constant summand.) In other words, for any vector field v inducing a tangent vector to LAG at L we have:

$$d\sigma = -(\iota_v \omega)|_L. \tag{3.2}$$

The function σ depends on $v \in \Pi T_L \text{LAG}$ (on odd tangent vector to LAG at L) linearly, hence it can be considered as a one-form on LAG.

By definition:

$$\Omega(L, v) = \int_L e^{S_{\text{BV}} + \sigma} \tag{3.3}$$

More generally, for every function F on M we define:

$$\Omega\langle F \rangle(L, v) = \int_L F e^{S_{\text{BV}} + \sigma} \tag{3.4}$$

As a complication, the one-form σ is defined only up to a constant:

$$\sigma \mapsto \sigma + \text{const} \tag{3.5}$$

Therefore the definition of Ω by eq. (3.3) is strictly speaking ambiguous. We will prove that it is always possible to resolve this ambiguity in such a way, that the form Ω is closed. Moreover,

$$d\Omega\langle F \rangle = -\Omega\langle \Delta F + \{S, F\} \rangle \tag{3.6}$$

It is enough to prove this formula for restriction to any finite-dimensional submanifold $\Lambda \subset \text{LAG}$ (i.e. a family of Lagrangian submanifolds). Let us parameterize Λ by coordinates $\lambda^1, \dots, \lambda^n$. This means that we have a family of Lagrangian submanifolds $(L(\lambda))$.

Let us find a family of volume preserving canonical transformations $g(\lambda)$ such that:

$$L(\lambda) = g(\lambda)L_0 \tag{3.7}$$

(locally this is always possible). The introduction of such $g(\lambda)$ is essentially a trick. It does not participate in any way in the *definition* of Ω ; we will use it just to compute $d\Omega$. Using $g(\lambda)$ we can construct a family of physically equivalent action functionals S_λ obeying

$$\int_{L_0} e^{S_\lambda} = \int_{L_\lambda} e^S.$$

Here S_λ is obtained from S by means of the transformation g_λ . It is easy to check that the form Ω introduced in present section coincides with the form constructed in the section 2 for the family S_λ and denoted by the same symbol; hence it is closed. (The second summand in the definition of Ω in section 2 is a Hamiltonian \mathcal{H} of the infinitesimal canonical transformation governing the variation of S_λ . The Hamiltonian governing the variation of L_λ enters the definition of Ω in present section. These two Hamiltonians coincide up to a constant; resolving the ambiguity in the definition of second Hamiltonian in appropriate way we can say that the Hamiltonians coincide.)

If we know the precise definition of Ω we can give also a precise definition of $\Omega\langle F \rangle$. The formula (3.6) follows from (2.6).

A more formal proof of the results of this section is given in appendix C.

4 Gauge symmetries

Form Ω is not necessarily base with respect to gauge symmetries. We assume that the action functional S , the observable A , the volume element on M , and the family Λ are invariant under a subgroup $H \subset G$ (or Lie algebra $\mathfrak{h} \subset \mathfrak{g}$).² We denote by $\hat{\mathfrak{h}}$ the set of Hamiltonians of elements of \mathfrak{h} ; then the \mathfrak{h} -invariance of S , A and volume element means that for every $h \in \hat{\mathfrak{h}}$ we have $\{S, h\} = 0, \{A, h\} = 0$ and $\Delta h = 0$. (It is enough to impose a weaker requirement:

$$\Delta h + \{S_{\text{BV}}, h\} = 0, \tag{4.1}$$

see [7].) It follows from these assumptions that the form Ω is also H -invariant (or \mathfrak{h} -invariant). In general the form Ω is *not horizontal*, and therefore does not descend to Λ/H [23, omega/Descent.To.Double.Coset.html]. However, in some important cases, in particular in string theory, the form Ω does descend to Λ/H for appropriate choice of the family of Lagrangian submanifolds.

We will now construct a modified form Ω which is base.

Under the assumptions of previous section, let us make the following additional assumption [23, introduction/Proof.of.equivariance.html]. Suppose that there exists a map $\Phi : \hat{\mathfrak{h}} \rightarrow \text{Fun}(M)$ such that every Hamiltonian $h \in \hat{\mathfrak{h}}$ satisfies:

$$h = \{S_{\text{BV}}, \Phi(h)\} + \Delta\Phi(h) + \frac{1}{2}\{\Phi(h), \Phi(h)\} \tag{4.2}$$

(Notice that the Hamiltonian h is odd, but $\Phi(h)$ is even.) We will also require that Φ satisfies the following ‘‘equivariance’’ property. For any two elements $h \in \hat{\mathfrak{h}}$ and $\tilde{h} \in \hat{\mathfrak{h}}$:

$$\{h, \Phi(\tilde{h})\} = \Phi(\{h, \tilde{h}\}) \tag{4.3}$$

Let us suppose that the action of \mathfrak{h} on Λ comes from a free action of the corresponding Lie group H (this Lie group is not necessarily connected). Then we can construct [23, omega/Descent.To.Double.Coset.html#(part...Modified.P.D.F)] closed form Ω_H , which descends to Λ/H . (In other words this is a base form, i.e. H -invariant and H -horizontal form.)

Technically, we use the formalism of equivariant cohomology [23, equivariant-cohomology/index.html]. The conditions we impose on the map Φ allow us to prove that the form

$$\Omega_H^c(\lambda, d\lambda, h) = \int_{L_\lambda} e^{S_{\text{BV}} + \sigma + \Phi(h)} \tag{4.4}$$

represents a class of H -equivariant cohomology of Λ in the Cartan model [23, equivariant-cohomology/Equivariant.Cohomology.html]. (We consider here σ as a one-form on Λ .)

Recall that in this model an equivariant cohomology class is represented by a differential form depending on an element of \mathfrak{h} and belonging to the kernel of Cartan differential $d - \iota_h$ where $h \in \mathfrak{h}$. (The dependence of \mathfrak{h} should agree with the action of the group H .)

²Notice, that H is not necessarily the full group of automorphisms. In string worldsheet theory, typically H is the group of diffeomorphisms.

We modify the definition allowing pseudodifferential forms instead of differential forms. We do not impose the condition of polynomial dependence of \mathbf{h} .

The proof of the fact that eq. (4.4) is equivariantly closed uses (3.6) and the relation [23, omega/As_Intertwiner.html]

$$\iota_r \Omega \langle F \rangle = \Omega \langle RF \rangle \tag{4.5}$$

where $r \in \mathfrak{g}$ and R stands for the corresponding Hamiltonian. This formula immediately follows from:

$$\iota_r \sigma = R|_L \tag{4.6}$$

which is essentially the definition of σ .

In the case when H is a conventional group the Poisson bracket corresponds to usual commutator hence $\{h, h\} = 0$; combining this with eq. (4.3) we get:

$$\{h, \Phi(h)\} = 0 \tag{4.7}$$

(this also can be derived just from eqs. (4.1) and (4.2)).

From Cartan to base. If the action of H on Λ is free the H -equivariant cohomology is isomorphic to the cohomology of Λ/H . An explicit formula for a base form belonging to the same class of equivariant cohomology as $\Omega_{\mathfrak{h}}^{\mathbb{C}}$ can be written as follows ([23, equivariant-cohomology/Direct_Computation.html#(elem...Def.Underline.Alpha)]). We need to choose a connection θ on Λ (the cohomology class of the resulting base form will not depend on the choice of θ). Then we have to replace σ with the horizontal projection of σ , and substitute the curvature $f = d\theta - \frac{1}{2}\theta^2$ for h (see [8] for a review):

$$\Omega^{\text{base}} = \int_{L_\lambda} \exp \left[S_{\text{BV}} + (\sigma - \iota(\theta)\sigma) + \left(d\theta - \frac{1}{2}\theta^2 \right) \Phi \right] \tag{4.8}$$

The second term $\sigma - \iota(\theta)\sigma$ is the horizontal projection of θ . The third term $(d\theta - \frac{1}{2}\theta^2) \Phi$ should be understood as follows. Consider the curvature $d\theta - \frac{1}{2}\theta^2$ of the connection in the fiber bundle $\Lambda \rightarrow \Lambda/H$; this is an H -equivariant \mathfrak{h} -valued 2-form on Λ . Composing it with our map Φ we get a two-form with values in $\text{Fun}(M)$, which is denoted $(d\theta - \frac{1}{2}\theta^2) \Phi$ in eq. (4.8).

The considerations above are rigorous in finite-dimensional case, however, we will use them in infinite-dimensional case where they can be justified in the framework of perturbation theory. Notice in the case when the dimension is infinite one should impose some additional conditions. In particular, the quadratic part of the BV action functional restricted to the Lagrangian submanifold should be non-degenerate. This condition (non-degeneracy condition) is necessary to have well defined perturbation theory. It is not needed in finite-dimensional case when the integral has a definition independent of the perturbation theory and the integral of degenerate functional makes sense. The situation with the completeness condition is similar: it is necessary only in infinite-dimensional case. The conditions of irreducibility and completeness are discussed in more details in [16].

The odd Laplacian Δ is ill-defined in the infinite-dimensional case unless we are working in the framework of perturbation theory when we can apply the methods of [9] or [10].

However the equation $\Delta S = 0$ does make sense; it just means that the nilpotent vector field Q corresponding to the first order differential operator transforming a function f into $\{f, S\}$ is volume preserving. (There exist standard ways to check that an operator in infinite-dimensional space is volume preserving; for example a method based on the calculation of Seeley coefficients is explained in [11].) Replacing S by $\exp[\frac{S}{\hbar}]$ we can write the quantum master equation $\Delta e^{\frac{S}{\hbar}} = 0$ as $\{S, S\} + \hbar\Delta S = 0$; in infinite-dimensional case we assume that both summands vanish: $\{S, S\} = 0$ (classical classical master equation) and $\Delta S = 0$. Similarly, we assume that in (4.2) $\Delta\Phi = 0$. In infinite-dimensional case we require that a quantum observable A satisfies the equations $\Delta A = 0$ and $\{A, S\} = 0$.

5 From BRST to BV

Let us suppose that we have a functional $S(\psi)$ with an odd symmetry Q_{BRST} (BRST symmetry) that is nilpotent *off-shell* (i.e. nilpotent without using the equations of motion). Then we can construct an odd symplectic manifold adding antifields ψ^* and solution to the classical Master Equation given by the formula

$$S_{\text{BV}} = S(\psi) + (Q_{\text{BRST}}\psi^i)\psi_i^* \tag{5.1}$$

In the case when Q_{BRST} is volume preserving (divergence-free) S_{BV} obeys also quantum master equation $\Delta S_{\text{BV}} = 0$. This statement is rigorous in finite-dimensional situation; it remains true also in the infinite-dimensional case.

A special case of this construction comes from the “standard” BRST formalism. It works for gauge theories as Yang-Mills/QCD or Chern-Simons, and also for the bosonic string worldsheet theory [23, bosonic-string/index.html] and the RNS superstring [23, Heterotic-RNS/index.html].

One starts from the “classical action” $S_{\text{cl}}(\varphi)$, which is invariant with respect to group H , hence with respect to its Lie algebra \mathcal{H} with generators T_A (“gauge symmetry”). Then one introduces additional variables c^A (“the ghosts”) with the quantum numbers of the symmetry parameter, but opposite statistics.

The nilpotent symmetry Q is defined by the following formulas [23, BRST-formalism/Construction.html]:

$$Q_{\text{BRST}}\varphi^i = T_A^i c^A, \quad Q_{\text{BRST}}c^A = \frac{1}{2}f_{BC}^A c^B c^C \tag{5.2}$$

where f_{BC}^A are structure constants of the Lie algebra \mathcal{H} . To continue from BRST to BV [23, BRST-formalism/BV_from_BRST.html], we define an odd symplectic manifold adding to φ^i, c^A their antifields φ_i^*, c_A^* having opposite parity (geometrically this means that we consider cotangent bundle with reversed parity of fibers). Here φ^i is the collective notation for the “old fields”. In such a situation, a solution of the classical Master Equation (a special case of (5.1)) can be written in the form:

$$S_{\text{BV}} = S_{\text{cl}}(\varphi) + \frac{1}{2}f_{BC}^A c^B c^C c_A^* + T_A^i c^A \varphi_i^* = S_{\text{cl}}(\varphi) + (Q_{\text{BRST}}c^A)c_A^* + (Q_{\text{BRST}}\varphi^i)\varphi_i^* \tag{5.3}$$

Our goal will be to solve the eq. (4.2) for BV action functional (5.3). Notice that this action functional is invariant with respect to the action of the group H and its Lie algebra \mathcal{H} ; the hamiltonian of the element $\xi = \xi^A T_A \in \mathcal{H}$ has the form $h = T_A^i \xi^A \varphi_i^* + [\xi, c]^A c_A^*$.

There exists a solution mapping this Hamiltonian into $\Phi(h) = \xi^A c_A^*$; it satisfies the conditions $\{\Phi(h), \Phi(h)\} = \Delta\Phi(h) = 0$.

To check (4.2) it is sufficient to notice that

$$\{S_{\text{BV}}, \xi^A c_A^*\} = T_A^i \xi^A \varphi_i^* + [\xi, c]^A c_A^* \tag{5.4}$$

A solution of (4.2) should obey (4.3). To verify this condition we notice that $\{T_A^i \xi^A \varphi_i^* + [\xi, c]^A c_A^*, \tilde{\xi}^A c_A^*\} = f_{BC}^A \xi^B \tilde{\xi}^C c_A^* = [\xi, \tilde{\xi}]^A c_A^*$.

Physically meaningful quantities should be obtained from integrating $e^{S_{\text{BV}}}$ over a suitably chosen Lagrangian submanifold. However, a suitable choice of Lagrangian submanifold is not completely trivial. The most straightforward guess would be to put all antifields to zero, i.e. $\varphi^* = c^* = 0$. But this is a wrong choice, because the restriction of S_{BV} to such a Lagrangian submanifold would be just S_{cl} , a degenerate functional. The next guess would be to consider some *deformation* of $\varphi^* = c^* = 0$, such that the restriction of S_{BV} be nonzero. But this also does not quite work, for the following reason: we want to keep the ghost number symmetry. It turns out that in the class of Lagrangian submanifolds respecting the ghost number symmetry, the one given by $\varphi^* = c^* = 0$ is typically rigid. It has no deformations, because there are no fields with negative ghost number.

In fact there are several solutions to this problem. One solution is to consider a family [23, BRST-formalism/Family_of_Lagrangian_submanifolds.html] of Lagrangian submanifolds. Each individual member of a family may be not ghost-number-invariant, but we require that the family as a whole be closed under the action of the ghost number symmetry. Another solution is to add a field/antifield quartet, in other words non-minimal fields, with some non-minimal fields carrying negative ghost number. (In finite-dimensional case this gives the same answer [23, omega/As_integration_over_single_L.html] as considering a family.) Yet another approach is to start with a Lagrangian submanifold [23, BRST-formalism/Family_of_Lagrangian_submanifolds.html] where some antifields *and some fields* are zero. Schematically:

$$\varphi_1^* = \varphi_2 = c_1^* = c_2 = 0 \tag{5.5}$$

In the particular case of bosonic string worldsheet theory in section 6, where H is the group of diffeomorphisms, we will use this third method.

Comment about antifields. If ϕ is a scalar field, we will consider ϕ^* a density (i.e. a volume form, or an area form in the two-dimensional case). This is very natural:

- The odd symplectic form is given by the integral of the density $(-1)^{\bar{\phi}} \delta\phi \wedge \delta\phi^*$, i.e. $\omega = \int (-1)^{\bar{\phi}} \delta\phi \wedge \delta\phi^*$
- A local infinitesimal field redefinition $\phi \mapsto \phi + \varepsilon V(\phi)$ is generated by the odd Hamiltonian $\int V(\phi)\phi^*$ (in order for this integral to make sense, ϕ^* should be a density).

In the same sense, we actually think of the “variational derivative” $\frac{\delta}{\delta\phi}$ as a density; it is “generated by” ϕ^* in terms of odd Poisson bracket.

6 Topological quantum field theories. Bosonic strings

In BRST formalism a topological quantum field theory is defined by a family of action functionals depending on riemannian metric on some manifold X and satisfying the condition that the variation of the action functional by infinitesimal variation of the metric is BRST exact (topological quantum field theories of Witten type). In BV formalism we should have solutions to the master equation $\{S, S\} = 0$ depending on riemannian metric and obeying $dS = \{b, S\}$ where d is the de Rham differential on the space MET of all metrics and b is a 1-form on this space. (If V is a vector field on the space of metrics we can write $dS/dV = \{b(V), S\}$.) Alternatively we can assume that the solution to the master equation is fixed, but the Lagrangian submanifold depends on the choice of metric.

We can construct an n -form Ω_n on MET integrating $b(V_1) \dots b(V_n)e^S$ over some Lagrangian submanifold L in the space of fields. Summing the forms Ω_n we can get an inhomogeneous closed form Ω that can be obtained by integrating e^{S+b} over L . Under certain conditions (see section 2) one can prove that this form is closed and descends to the quotient space of MET with respect to the action of the group Diff of diffeomorphisms of X . We obtain a closed form on the quotient MET/Diff; integrating this form over a cycle we can get new invariants. In particular, applying these ideas to Chern-Simons theory one obtains invariants constructed by Kontsevich [12]; see [1] for detail. (Another construction of these invariants was given in [13].)

In the rest of this section we will outline applications of these ideas to string perturbation theory. The target of string theory can be regarded as two-dimensional topological quantum field theory; the above considerations can be applied to this TQFT. We will show that string amplitudes are particular cases of new invariants we have mentioned. Instead of formalism of families of equivalent action functionals we will use more flexible formalism of families of Lagrangian submanifolds.

Bosonic string. Master action in terms of world sheet metric. The construction outlined in section 5 works for both bosonic string [23, bosonic-string/index.html] and RNS superstring [23, Heterotic-RNS/index.html].

Let us consider bosonic string. For definiteness we are writing formulas for bosonic string in flat space. (To avoid anomalies we should assume that we work in the dimension 26.) We start with the action functional

$$S_{\text{mat}}[g, x] = \frac{1}{2} \int \sqrt{g} g^{\alpha\beta} \partial_\alpha x^m \partial_\beta x^m d^2\xi \tag{6.1}$$

We integrate here over a compact surface of genus h with metric $g_{\alpha\beta}$. We always assume that $h > 1$. The subindex mat stands for “matter”, although this action also involves the dynamical metric $g_{\alpha\beta}$. This functional is invariant with respect to diffeomorphisms and Weyl transformations $g'_{\alpha\beta} = e^\phi g_{\alpha\beta}$; hence we can construct a BV action functional introducing diffeomorphism ghosts c , Weyl ghosts ζ and antifields to $g_{\alpha\beta}$, x^m and ghosts.³

³BV formalism was previously applied to bosonic string in [14].

Following the general scheme outlined in section 5 we obtain:

$$S_{\text{BV}} = S_{\text{mat}}[g, x] + \int \left((\mathcal{L}_c g)_{\alpha\beta} g^{*\alpha\beta} + \zeta g_{\alpha\beta} g^{*\alpha\beta} + ((c^\alpha \partial_\alpha) x^m) x_m^* + \frac{1}{2} [c, c]^\alpha c_\alpha^* + (\mathcal{L}_c \zeta) \zeta^* \right) \quad (6.2)$$

Here \mathcal{L}_c is the Lie derivative along the vector field $c^\alpha \partial_\alpha$.

We now choose the Lagrangian submanifold in the following way:

$$g_{\alpha\beta} = g_{\alpha\beta}^{(0)}, \quad x^* = c^* = \zeta^* = 0 \quad (6.3)$$

where $g_{\alpha\beta}^{(0)}$ is a fixed metric (6.4)

The family (6.3) of Lagrangian submanifolds is closed under the action of diffeomorphisms. On Lagrangian submanifold (6.3) the action is quadratic and the form Ω is equal to:⁴

$$\Omega(g^{(0)}, \delta g^{(0)}) = \int [Dx Dg^* Dc D\zeta] \exp \left(S_{\text{BV}} + \int \delta g_{\alpha\beta}^{(0)} g^{*\alpha\beta} \right) \quad (6.5)$$

$$= \int [Dx Dg^* Dc D\zeta] \exp \left(S_{\text{mat}} + \int \left((\mathcal{L}_c g)_{\alpha\beta} g^{*\alpha\beta} + \zeta t + \delta g_{\alpha\beta}^{(0)} g^{*\alpha\beta} \right) \right) \quad (6.6)$$

$$= \int [Dx Db Dc] \exp \left(S_{\text{mat}} + \int \left((\nabla_\alpha c_\beta + \nabla_\beta c_\alpha) b^{\alpha\beta} + \delta g_{\alpha\beta}^{(0)} b^{\alpha\beta} \right) \right). \quad (6.7)$$

We introduced the notation $t = g_{\alpha\beta} g^{*\alpha\beta}$, $b^{\alpha\beta} =$ traceless part of $g^{*\alpha\beta}$, i.e. $g^{*\alpha\beta} = b^{\alpha\beta} + \frac{1}{2} t g^{\alpha\beta}$. In the transition to the last line we integrated over ζ and t .

Non-degeneracy. The exponential in (6.5) is non-degenerate. (The restriction of S_{BV} to the Lagrangian submanifold of eq. (6.3) is non-degenerate [23, BRST-formalism/Family_of_Lagrangian_submanifolds.html#%28part...Non-degeneracy%29] modulo a finite-dimensional space of zero modes of $b^{\alpha\beta}$. This finite-dimensional degeneracy is removed by the second term in the exponential of (6.5).)

Symmetries. The form Ω is invariant with respect to diffeomorphisms; moreover on the family (6.3) it is a base form, because for any worldsheet vector field ξ :

$$\iota_\xi \Omega = \int [Dx Dg^* Dc D\zeta] \left(\int d^2 z (\mathcal{L}_\xi g_{\alpha\beta}) g^{*\alpha\beta} \right) \exp \left(S_{\text{BV}} + \int \delta g_{\alpha\beta}^{(0)} g^{*\alpha\beta} \right) = \quad (6.8)$$

$$= \int [Dx Dg^* Dc D\zeta] \int d^2 z \xi^\alpha \frac{\partial}{\partial c^\alpha} \exp \left(S_{\text{BV}} + \int \delta g_{\alpha\beta}^{(0)} g^{*\alpha\beta} \right) = 0 \quad (6.9)$$

To check that the last line is zero we notice that the derivative with respect to c_α under the sign of two-dimensional integral can be replaced by variational derivative under the sign of infinite-dimensional integral.

Let us study the behavior of this form with respect to Weyl transformations $g'_{\alpha\beta} = e^\phi g_{\alpha\beta}$. The 0-th component Ω^0 of inhomogeneous form Ω can be regarded as a partition function of conformal field theory. The variation of partition function by infinitesimal Weyl

⁴We denote the de Rham differential on the infinite-dimensional space of metrics by δ instead of d .

transformation is governed by trace anomaly $\delta Z/\delta\phi = (-\frac{cR}{12} + const)Z$ where c stands for the central charge and R denotes the curvature of the worldsheet. In our case the central charge vanishes (we are working in critical dimension $d = 26$; in general the central charge is equal to $d - 26$). We see that Ω^0 does not change by Weyl transformations. The k -th component of the form Ω can be expressed in terms of correlation functions of the same conformal theory. The behavior of correlation functions by Weyl transformations is governed by conformal dimensions Δ_i of fields Ψ_i :

$$\langle \Psi'_1(\xi_1) \dots \Psi'_k(\xi_k) \rangle_{g'} = e^{-\sum \Delta_i \Psi_i(\xi_i)} \langle \Psi_1(\xi_1) \dots \Psi_k(\xi_k) \rangle_g \quad (6.10)$$

([22], formula (13,50)). To check the Weyl invariance of Ω we notice that the dimension of $b^{\alpha\beta}$ is 2 (it coincides with conformal dimension) and the dimension of $g_{\alpha\beta}$ is -2 .

We have proved that in critical theory Ω is Weyl invariant. Moreover, it descends not only to MET/Diff, but also to MET/Diff \times Weyl, that can be identified with the moduli space of complex structures on a compact surface of genus h . (A formal proof of the fact that Ω is a base form for the Weyl group repeats the proof of similar statement for Diff.) We can get the partition function of bosonic string integrating the form over this moduli space. (Notice that we are working with inhomogeneous forms, but the integration singles out one component of this form.)

We can solve eq. (4.2) using the general considerations [23, omega/Case_Of_Standard_BRST.html] of section 5. Namely, we should take a map sending a worldsheet vector field $\xi^\alpha(z, \bar{z})$ plus infinitesimal Weyl transformation $\varphi(z, \bar{z})$ to:

$$\Phi(\xi, \varphi) = \int \xi^\alpha c_\alpha^* + \varphi \zeta^* \quad (6.11)$$

Then the functional $\{S, \Phi(\xi, \varphi)\}$ can be considered as a Hamiltonian of infinitesimal transformation of fields corresponding to the vector field ξ and Weyl factor φ . This means eq. (6.11) defines a solution of eq. (4.2) for the Lie algebra of the group Diff \times Weyl acting on the space of fields. This allows us to construct an equivariant form

$$\Omega_{\text{Lie}(\text{Diff} \times \text{Weyl})}^c(\xi, \varphi) = \int_{g_L} \exp \left(S_{\text{BV}} + \sigma + \int \xi^\alpha c_\alpha^* + \varphi \zeta^* \right) \quad (6.12)$$

We can then construct the corresponding base form [23, omega/Base_Form.html] which descends to $\Lambda/(\text{Diff} \times \text{Weyl})$. On the standard family of Lagrangian submanifolds given by eq. (6.3) $c^* = \zeta^* = 0$. Therefore $\Omega_{\text{Lie}(\text{Diff} \times \text{Weyl})}^c(\xi, \varphi)$ becomes essentially Ω of eq. (6.5).

Singular metrics. Notice that in the action functional (6.1) we can allow slightly singular metrics. We say that the worldsheet metric on a surface of genus h is slightly singular if on some real curves one of the eigenvalues of the metric $g_{\alpha\beta}$ vanishes and another eigenvalue remains positive. More precisely we suppose that $g = \det g_{\alpha\beta}$ vanishes on a family of closed real curves and in the neighborhood of one of these curves it takes the form $\rho^2\sigma$ where $\rho = 0$ is the equation of the curve and σ is a positive function.⁵ It is easy to check

⁵The simplest example of this picture is a cylinder with coordinates (ρ, ϕ) and metric $ds^2 = d\rho^2 + \rho^2 d\phi^2$. Here $-a < \rho < a, 0 \leq \phi < 2\pi$.

that under these conditions the action functional (6.1) is finite if we make an additional assumption that x^m is constant on every closed curve where the metric is singular. The formulas for BV action (6.2) and Lagrangian submanifold (6.3) can be applied to slightly singular metrics. We obtain a family of Lagrangian manifolds labelled by these metrics. Factorizing the topological space Λ of slightly singular metrics with respect to diffeomorphisms and Weyl transformations we obtain the space $\Lambda/\text{Diff} \times \text{Weyl}$. Points of this space can be identified with complex curves having simplest singularities (nodes). (Every closed curve where the metric is singular should be contracted to a point; the metric specifies a complex structure in the complement to these points.) A part of this space that consists of stable curves (curves having only finite number of automorphisms) can be identified with Deligne-Mumford compactification of the moduli space of algebraic curves of genus h . This is a good topological space (an orbifold). The remaining part is a “bad” (non-separable) space, but it does not play any role (a heuristic explanation of this fact is the remark that its contribution to the partition function is suppressed by the infinite volume of the automorphism group). The form of eq. (6.5) descends to Deligne-Mumford space as a form having some singularities. To obtain physical quantities we should integrate the form over a cycle in Deligne-Mumford space; to obtain the partition function we should integrate over the fundamental cycle. (Of course, this is only a formal calculation—due to the tachyon in the spectrum of bosonic string the integral is divergent.)

Master equation in terms of complex structures. A worldsheet complex structure can be specified by a field of linear operators I acting on tangent spaces and obeying $I^2 = -1$. Another way to specify a complex structure is to fix a complex vector field e such that the complex conjugate vector field \bar{e} together with e specifies a basis of complexified tangent space. (To relate these descriptions we define e as the eigenvector of I having eigenvalue i .) Notice that e is only defined up to multiplication: $e \sim ue$, where u is a complex function on the worldsheet.

Due to Weyl invariance one can express the functional (6.1) in terms of complex structures. We obtain the following functional:

$$S_{\text{mat}}[I, x] = \int e^\alpha \partial_\alpha x \bar{e}^\beta \partial_\beta x d\mu \tag{6.13}$$

where the measure μ on the worldsheet is specified by the condition the vectors e, \bar{e} span a parallelogram of measure 1 in tangent space. The functional is invariant with respect to diffeomorphisms. We can now follow the standard procedure by first introducing the diffeomorphism ghosts [23, BRST-formalism/Construction.html] c (BRST formalism) and then adding antifields [23, BRST-formalism/BV_from_BRST.html]. The result is the Master Action of the following form:

$$S_{\text{BV}} = S_{\text{mat}}[I, x] + \int \left((\mathcal{L}_c I)_\alpha^\beta I^{*\alpha}_\beta + (\mathcal{L}_c x) x^* + \frac{1}{2} [c, c]^\alpha c^*_\alpha \right) \tag{6.14}$$

In the expression for the action we integrate over a worldsheet. In the h -loop contribution the worldsheet is a surface of genus h .

Notice that one can introduce a notion of slightly degenerate complex structure assuming that the vectors e and \bar{e} can be linearly dependent on a family of closed curves on a worldsheet. (In a neighborhood of such a curve we should have a relation $\bar{e} = \lambda e + \rho f + \dots$ where tangent vectors e and f are linearly independent, $\rho = 0$ is the equation of the curve and \dots are higher order terms with respect to ρ .)

7 String amplitudes

7.1 String amplitudes for critical string

To represent the string theory in BV form we have applied the general constructions of the section 5 to the action functional $S_{\text{mat}}[g, x]$. This functional depends on the metric $g_{\alpha\beta}$ on the worldsheet (on a compact surface of genus h) and a map $\mathbf{x}(\xi) = x^m(\xi)$ of this surface to \mathbb{R}^d . This functional is invariant with respect to diffeomorphisms and Weyl transformations. We applied the standard BRST construction in this setting and used (5.1) to get the BV action. To describe string amplitudes we should add marked points (punctures) (ξ_1, \dots, ξ_n) on the worldsheet to this picture. Following [14] we will consider ξ_i as dynamical variables on equal footing with the metric.

Using again the constructions of the section 5 we get the new BV action S'_{BV} with an extra term $c^\alpha(\xi_i)\xi_\alpha^{i*}$:

$$S'_{\text{BV}} = S_{\text{BV}} + c^\alpha(\xi_i)\xi_\alpha^{i*} \quad (7.1)$$

where S_{BV} is defined by (6.14). As was noticed in section 5 this functional obeys quantum master equation in the case when the volume is Q -invariant; this remark forces us to use the diffeomorphism invariant measure $\sqrt{g(\xi_1)}d^2\xi_1 \cdots \sqrt{g(\xi_n)}d^2\xi_n$ on the space of marked points.

Let us consider functionals $V_i(\xi_i)$ (vertices) which are invariant under diffeomorphisms. The typical examples of such vertices are tachyonic vertex $e^{i\mathbf{p}\mathbf{x}(\xi)}$ and graviton vertex $\epsilon_{kl}g^{\alpha\beta}\partial_\alpha x^k(\xi)\partial_\beta x^l(\xi)e^{i\mathbf{p}\mathbf{x}(\xi)}$. We can introduce a new action functional

$$S''_{\text{BV}} = S'_{\text{BV}} + \sum \epsilon^i V_i(\xi_i) \quad (7.2)$$

where ϵ_i are infinitesimally small.

To define string amplitudes it is convenient to work with BV-action functional that is obtained from (7.2) by means of “integrating out” Weyl ghosts.⁶ We obtain the new BV action \tilde{S}_{BV} given by the formula

$$e^{\tilde{S}_{\text{BV}}} = e^{S_{\text{mat}}[g, x] + \int \left((\mathcal{L}_c g)_{\alpha\beta} g^{*\alpha\beta} + ((c^\alpha \partial_\alpha) x^m) x_m^* + \frac{1}{2} [c, c]^\alpha c_\alpha^* - c^\alpha(\xi_i)\xi_\alpha^{i*} + \sum \epsilon^i V_i(\xi_i) \right)} \delta(g^{*\alpha\beta} g_{\alpha\beta})$$

Denoting the traceless part of $g^{*\alpha\beta}$ by $b^{\alpha\beta}$ we can represent this action functional in the form

$$\tilde{S}_{\text{BV}} = \hat{S}_{\text{BV}} + \sum \epsilon^i V_i(\xi_i) \quad (7.3)$$

⁶If a solution A of the equation $\Delta A = 0$ is defined on direct product of two odd symplectic manifolds \mathcal{Y}' and \mathcal{Y}'' we can obtain a solution of similar equation on \mathcal{Y}' integrating over Lagrangian submanifolds $L \in \mathcal{Y}''$. (See for example [10].) In our case we integrate over Lagrangian submanifold $\zeta^* = 0$ of manifold with coordinates ζ, ζ^* .

where

$$\hat{S}_{\text{BV}} = S_{\text{mat}}[g, x] + \int \left((\mathcal{L}_c g)_{\alpha\beta} b^{\alpha\beta} + ((c^\alpha \partial_\alpha) x^m) x_m^* + \frac{1}{2} [c, c]^\alpha c_\alpha^* - c^\alpha (\xi_i) \xi_\alpha^{i*} \right) \quad (7.4)$$

Now we can use the standard construction of the form Ω starting with the action functional \hat{S}_{BV} . However, we prefer to construct the form Ω starting with the functional \hat{S}_{BV} and including the factor $V_1 \dots V_n$ into defining integral. (The form coming from the second construction can be obtained from the first one by means of differentiation with respect to parameters.) We consider a family of Lagrangian submanifolds parameterized by $g_{\alpha\beta}^{(0)}, \xi_i^{(0)}$ taking

$$g_{\alpha\beta} = g_{\alpha\beta}^{(0)}, \quad \xi_i = \xi_i^{(0)}, \quad x^* = c^* = 0 \quad (7.5)$$

The form Ω , restricted to one of these Lagrangian submanifolds looks as follows:

$$\begin{aligned} \Omega(g_{\alpha\beta}^{(0)}, \xi_i^{(0)}, \delta g_{\alpha\beta}^{(0)}, d\xi_i^{(0)}) = \\ = \int [Dx Db D\xi^{*i} Dc] \sqrt{g(\xi_1^{(0)})} V_1(\xi_1^{(0)}) \cdots \sqrt{g(\xi_n^{(0)})} V_n(\xi_n^{(0)}) \times \end{aligned} \quad (7.6)$$

$$\begin{aligned} \times \exp \left(S_{\text{mat}} + \int (\nabla_\alpha c_\beta + \nabla_\beta c_\alpha) b^{\alpha\beta} + \right. \\ \left. + \int \delta g_{\alpha\beta}^{(0)} b^{\alpha\beta} + \xi^{*i} (c(\xi^i) - d\xi_i^{(0)}) \right) \end{aligned} \quad (7.7)$$

Using this formula we can get an expression of Ω in terms of correlation functions of conformal field theory. This allows us to analyze the behavior of Ω with respect to Weyl transformations. It is easy to see that in our case of critical string this form is Weyl invariant if conformal fields corresponding to vertices V_i have conformal dimension 2 (dimension $(1, 1)$ in the language of complex geometry). In this case the form descends to the moduli space $\mathcal{M}_{h,n}$ of compact complex curves of genus h with n marked points and to its Deligne-Mumford compactification $\bar{\mathcal{M}}_{h,n}$. Integrating over the fundamental cycle of $\bar{\mathcal{M}}_{h,n}$ we obtain the h -loop contribution to string amplitudes. To check this we notice that after integration over $d\xi^*$ (and omitting indices $^{(0)}$ for brevity) we get:

$$\int [Dx Db Dc] \Pi_j \left(\sqrt{g(\xi_j)} (-d\xi_j^1 + c^1(\xi_j)) (-d\xi_j^2 + c^2(\xi_j)) V_j(\xi_j) \right) \times \quad (7.8)$$

$$\times \exp \left(S_{\text{mat}} + \int (\nabla_\alpha c_\beta + \nabla_\beta c_\alpha) b^{\alpha\beta} + \delta g_{\alpha\beta} b^{\alpha\beta} \right) \quad (7.9)$$

This result is equivalent to the standard expression for the string amplitude [15]. To see this we notice that $\Pi_j (d\xi_j^1 + c^1(\xi_j)) (d\xi_j^2 + c^2(\xi_j))$ consist on 2^n summands; one of them gives the standard expression for string amplitudes with non-integrated vertices, another gives the standard expression with integrated vertices, and the rest correspond to the situation when some vertices are integrated and some are non-integrated. All these summands are equal, hence we obtain the standard answer up to a factor 2^n .

Another way to calculate the string amplitudes is to work with infinitesimal deformations of BV action functional. Such deformations can be identified with (classical or

quantum) observables. In string theory they can be considered as integrated vertices. Applying our approach to the deformation of BV action we obtain the standard expression of string amplitudes in terms of integrated vertices (see [16] for detail).

An important method of calculation of scattering amplitudes in string theory is based on the consideration of *off-shell string amplitudes*. This is the best method to calculate amplitudes when the mass gets quantum corrections. The off-shell amplitudes should be defined in such a way that the particle masses correspond to their poles (in momentum representation) and scattering amplitudes should be expressed in terms of residues in these poles.

To define off-shell string amplitudes for critical string one can consider surfaces with marked points and local coordinate systems in the neighborhoods of these points [17], [18]. This is equivalent to consideration of surfaces with boundary. The BV formalism on manifolds with boundary was analyzed in [10]. It should be possible to combine our approach [23, boundary/index.html] with BV-BFV formalism of [10]; these would lead to generalization of definitions given in [17, 18].

For non-critical strings very nice definition of off-shell amplitudes was suggested by A. Polyakov [19]; it works well in our setting. Polyakov considers maps $\mathbf{x}(\xi) = \mathbf{x}^m(\xi)$ of a surface with marked points ξ_1, \dots, ξ_k into \mathbb{R}^d and includes the factor

$$\prod_i \int \delta(\mathbf{x}_i - \mathbf{x}(\xi_i)) \sqrt{g(\xi_i)} d^2 \xi_i \tag{7.10}$$

in the functional integral that defines the partition function. Geometrically this means that we integrate over all surfaces in \mathbb{R}^d that contain the points $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^d$ (surfaces with pinned points $\{\mathbf{x}_i\}$ in Polyakov’s terminology). Doing the functional integral we obtain a function $G(\mathbf{x}_1, \dots, \mathbf{x}_k)$ that can be interpreted as off-shell amplitude in coordinate representation. The off-shell amplitude in the momentum representation $G(\mathbf{p}_1, \dots, \mathbf{p}_k)$ can be defined as Fourier transform of $G(\mathbf{x}_1, \dots, \mathbf{x}_k)$ or directly as a functional integral for partition function with insertion

$$\prod_j \int e^{i\mathbf{p}_j \mathbf{x}(\xi_j)} \sqrt{g(\xi_j)} d^2 \xi_j \tag{7.11}$$

Polyakov considers off-shell amplitudes only at tree level (genus zero surfaces), however they can be considered also in multi-loop case.

8 Pure spinor superstring

We hope that our ideas will lead to better understanding of pure spinor formalism in superstring theory and to simplified expressions for amplitudes in this formalism.

The worldsheet sigma-model of the pure spinor sigma-model has different versions which are quasiisomorphic to each other, as usual in the topological field theory. There is a “minimal version”, which (in case of Type II theory⁷) describes matter fields (x, θ_L, θ_R) and “ghost fields” λ_L, λ_R constrained to live on the pure spinor cone:

$$(\lambda_L \Gamma^m \lambda_L) = (\lambda_R \Gamma^m \lambda_R) = 0 \tag{8.1}$$

⁷For the heterotic string the right-moving variables are those of the heterotic RNS formalism.

The flat space sigma-model [24] requires introduction of the momenta p_+^L and p_-^R conjugate to θ_L and θ_R , and the fermionic part of the action is of the first order in derivatives:

$$\int d^2z (p_+^L \partial_- \theta_L + p_-^R \partial_+ \theta_R) \tag{8.2}$$

The action for pure spinors is, schematically:

$$\int d^2z (w_+^L \partial_- \lambda_L + w_-^R \partial_+ \lambda_R) \tag{8.3}$$

where the “conjugate momenta” w_+^L, w_-^R take values in the cotangent bundle of the pure spinor cone.⁸ The bosonic part of the action is the usual $\int d^2z \partial_+ x^m \partial_- x^m$.

The model is invariant under a fermionic nilpotent symmetry Q . Importantly, it splits (for Type II case) into the sum of left and right symmetries:

$$Q = Q_L + Q_R \tag{8.4}$$

such that the conserved currents corresponding to Q_L and Q_R are holomorphic and anti-holomorphic, respectively.

In the case of flat target space, it is easy to obtain the corresponding BV action functional: for every field Φ one should add its antifield Φ^* and a term in the action having the form $(Q\Phi) \cdot \Phi^*$. (This is a special case of general construction described in Sec 5; see (5.1).)

However, the solution of eq. (4.2) requires different methods. As a first step, let us restrict ourselves to the left sector.⁹ The explicit form of eq. (4.2) for the left sector of the pure spinor string is:

$$\{S_{BV}, a(\xi)\} + \frac{1}{2}\{a(\xi), a(\xi)\} = \mathcal{H}(\xi) \tag{8.5}$$

$$\begin{aligned} \text{where } \mathcal{H}(\xi) = & (\xi^z \partial_z x^m) x_m^* + (\xi^z \partial_z \theta) \theta^* + (\mathcal{L}_\xi p_+) p^{*+} + \\ & + (\xi^z \partial_z \lambda_L) \lambda_L^* + (\mathcal{L}_\xi w_+) w^{*+} \end{aligned} \tag{8.6}$$

— this has to be solved for the unknown $a(\xi)$; notice that $\mathcal{H}(\xi)$ is linear in ξ , but $a(\xi)$ does not have to be linear in ξ . (We have assumed that $\Delta a(\xi) = 0$; otherwise we should add an ill-defined term $\Delta a(\xi)$.) One solution can be obtained as follows. Since the worldsheet theory is conformal, a holomorphic vector field ξ^+ is symmetry; it is generated by $\xi^+ T_{++}$. It was shown by Berkovits that the energy-momentum tensor is BRST-trivial: $T_{++} = Q_L b_{++}$ (even off-shell) where b_{++} is a composite b -ghost. This means that one should expect that the worldsheet action can be included into topological conformal field theory. A rigorous proof of this statement is still unknown; the most convincing treatment of this problem was given in [20].¹⁰ Notice that $\xi^+ b_{++}$ is a holomorphic current and therefore also

⁸One way of describing the pure spinor system is to cover the cone with patches. On each patch, both λ_L and w^L take values in flat 11-dimensional space.

⁹Applying this to the “full” sigma-model, i.e. left plus right sector, is work in progress in collaboration with R. Lipinski Jusinskas.

¹⁰That paper contains also the calculation of superstring amplitudes in the framework of BV-formalism; some ideas of this calculation can be used in our approach.

corresponds to some symmetry. We can identify $a(\xi) = \Phi\langle\xi\rangle$, a BV Hamiltonian generating the infinitesimal action of that symmetry. Then the second term in (8.5) vanishes and this equation is satisfied. However we hope that there exist simpler solutions of eq. (8.5) with non-vanishing second term; we leave this question for future work. We believe, that applying the techniques described above one can not only justify the pure spinor formalism, but also simplify the formulas (hopefully we can avoid using the complicated and not very well defined “composite b -ghost”).

A Some useful formulas

BV phase space [23, BV-formalism/index.html] is an odd symplectic supermanifold M with a nondegenerate closed odd 2-form ω . For any $F \in \text{Fun}(M)$ we can define its Hamiltonian vector field. We will think of this vector field as a first order linear differential operator, acting on $\text{Fun}(M)$:

$$G \mapsto \{F, G\} = F \frac{\overleftarrow{\partial}}{\partial Z^A} \pi^{AB}(Z) \frac{\partial}{\partial Z^B} G \quad (\text{A.1})$$

and denote this operator $\{F, -\}$. (Here $\pi^{AB}(Z)$ is a matrix inverse to $\omega_{AB}(Z)$.) By definition:

$$dF = (-)^{\bar{F}+1} \iota_{\{F, -\}} \omega \quad (\text{A.2})$$

where ι is the operator of contraction, satisfying $[\iota_V, d] = \mathcal{L}_V$. This implies:

$$\{F, G\} = \iota_{\{F, -\}} dG = (-)^{\bar{G}+1} \iota_{\{F, -\}} \iota_{\{G, -\}} \omega \quad (\text{A.3})$$

In coordinates:

$$\omega = dZ^A dZ^B \omega_{AB} \quad (\text{A.4})$$

$$\omega_{AB} = (-1)^{(\bar{A}+1)(\bar{B}+1)} \omega_{BA} \quad (\text{A.5})$$

$$d = dZ^A \frac{\partial}{\partial Z^A} \quad (\text{A.6})$$

$$\iota_V = V^A \frac{\partial}{\partial dZ^A} \quad (\text{A.7})$$

$$\pi^{AB} = (-1)^{1+(\bar{A}+1)(\bar{B}+1)} \pi^{BA} \quad (\text{A.8})$$

Locally it is possible to choose the Darboux coordinates:

$$\{F, G\} = F \left(\frac{\overleftarrow{\partial}}{\partial \phi_A^*} \frac{\partial}{\partial \phi^A} - \frac{\overleftarrow{\partial}}{\partial \phi^A} \frac{\partial}{\partial \phi_A^*} \right) G \quad (\text{A.9})$$

$$\omega = (-1)^A d\phi^A d\phi_A^* \quad (\text{A.10})$$

If the manifold M is equipped with a volume element (with a density) we can define the odd Laplacian acting on functions by the formula

$$\Delta F = \text{div}\{F, -\} \quad (\text{A.11})$$

where div stands for the divergence of vector field with respect to the volume element.

The volume element should be chosen in such a way that $\Delta^2 = 0$. The relation between odd Laplace operator [23, BV-formalism/OddLaplace.html] and $\{-, -\}$ is:

$$\Delta(XY) = (\Delta X)Y + (-)^{\bar{X}} \Delta Y + (-)^{\bar{X}} \{X, Y\} \quad (\text{A.12})$$

$$\Delta e^\Phi = \left(\Delta \Phi + \frac{1}{2} \{ \Phi, \Phi \} \right) e^\Phi \quad (\text{A.13})$$

In Darboux coordinates Δ is:

$$\Delta = (-1)^{\bar{A}+1} \frac{\partial}{\partial \phi_A^*} \frac{\partial}{\partial \phi^A} \quad (\text{A.14})$$

One can prove that ΔX given by this formula does not depend on the choice of Darboux coordinates if X transforms as a semidensity (recall that semi-densities transform as square roots of densities = volume elements). Hence for any odd symplectic manifold one can define Δ on semi-densities (volume element is not necessary), see [6].

B Definition of Ω using marked points

Let LAG_+ denote the space of Lagrangian submanifolds with marked points. A point of LAG_+ is a pair (L, a) where $L \in \text{LAG}$ and $a \in L$. This defines the double fibration:

$$M \xleftarrow{p} \text{LAG}_+ \xrightarrow{\pi} \text{LAG} \quad (\text{B.1})$$

Given $v \in \Pi T_{(L,a)} \text{LAG}_+$, we can consider two projections $\pi_* v \in \Pi T_L \text{LAG}$ and $p_* v \in \Pi T_a M$. We will define Ω is a pseudo-differential form, i.e. a function of L, a, v . It will depend on v only through $\pi_* v$. We can characterize $\pi_* v$ as a section of $\Pi T M|_L$ modulo $\Pi T L$. We then define σ as follows:

$$\begin{aligned} \sigma &\in \text{Fun}(L) \\ d\sigma &= -(\iota_{\pi_* v} \omega)|_L \end{aligned} \quad (\text{B.2})$$

$$\sigma(a) = 0 \quad (\text{B.3})$$

This definition specifies σ as a linear function of v , i.e. as a one-form on LAG_+ . In order to make sense of $\iota_{\pi_* v} \omega$ we must think of $\pi_* v$ as a section of $\Pi T M$; the fact that it is only defined up to tangent to TL does not matter because L is isotropic. Eq. (B.3) eliminates the ambiguity, and we can now safely define a function Ω on $\Pi T \text{LAG}_+$ (a pseudodifferential form on LAG_+) as in eq. (3.3):

$$\Omega(L, a, v) = \int_L e^{S_{\text{BV}} + \sigma} \quad (\text{B.4})$$

More generally, for every function F on M we define:

$$\Omega\langle F \rangle(L, a, v) = \int_L F e^{S_{\text{BV}} + \sigma} \quad (\text{B.5})$$

We will now prove the following formula:

$$(d - p^* \omega) \Omega\langle F \rangle = -\Omega\langle \Delta F + \{S_{\text{BV}}, F\} \rangle \quad (\text{B.6})$$

Comment. As a straightforward generalization, we can consider a product of Ω with the pullback under p of any differential or pseudo-differential form ν on M . It satisfies: [23, omega/Descent_To_LAGs.html#(part...Upgrade_to_)]

$$d(p^*\nu \Omega\langle F \rangle) = (-)^{|\nu|+1} p^*\nu \Omega\langle \Delta F + \{S_{\text{BV}}, F\} \rangle + p^*(d\nu + \omega\nu) \Omega\langle F \rangle \quad (\text{B.7})$$

Notice the appearance of the nilpotent operator $d + \omega$ which was studied in [21].

Proof. We take a family of Lagrangian submanifolds with marked points $(L(\lambda), a(\lambda))$ and represent it in the form

$$L(\lambda) = g(\lambda)L_0 \quad (\text{B.8})$$

$$a(\lambda) = g(\lambda)a_0 \quad (\text{B.9})$$

where $g(\lambda)$ are volume preserving canonical transformations (locally this is always possible).

It is sufficient to analyze the restriction $\Omega\langle F \rangle(\lambda, d\lambda)$ of the form (B.5) to this family.

As in section 3 using the canonical transformations $g(\lambda)$ we can construct a family of action functionals S_λ and corresponding forms that will be denoted by $\tilde{\Omega}$ and $\tilde{\Omega}\langle F \rangle$. These forms do not coincide with the forms $\Omega\langle F \rangle(\lambda, d\lambda)$ constructed by means of family of Lagrangian submanifolds with marked points, but they are closely related. As we noticed in section 3 the second summand in the exponential in the formula defining $\tilde{\Omega}\langle F \rangle(\lambda, d\lambda)$ is the Hamiltonian of the infinitesimal canonical transformation governing the variation of S_λ . The second summand in the formula defining $\Omega\langle F \rangle(\lambda, d\lambda)$ is the Hamiltonian $\mathcal{H}(\lambda, d\lambda)$ of the infinitesimal canonical transformation¹¹ governing the variation of L_λ . They coincide up to a constant summand. This constant can be calculated from (B.2). We obtain [23, omega/Descent_To_LAGs.html#(part...Upgrade_to_)]

$$\Omega\langle F \rangle(\lambda, d\lambda) = C\tilde{\Omega}\langle F \rangle(\lambda, d\lambda) \quad (\text{B.10})$$

where $C = e^{-\mathcal{H}(\lambda, d\lambda)(g(\lambda)a_0)}$. (One can say that C is expressed in terms of the value of the Hamiltonian of the infinitesimal canonical transformation at the marked point.)

We have calculated already the differential of $\tilde{\Omega}\langle F \rangle(\lambda, d\lambda)$. But we also have to evaluate d_Λ of the prefactor C . Using eq. (A.3), appendix, and $p^*\omega = \frac{1}{2} \left(\iota \left(d\lambda^k \frac{\partial a^A}{\partial \lambda^k} \frac{\partial}{\partial a^A} \right) \right)^2 \omega$ we get:

$$d_\Lambda e^{-\mathcal{H}(\lambda, d\lambda)(ga_0)} = \quad (\text{B.11})$$

$$= e^{-\mathcal{H}(\lambda, d\lambda)(ga_0)} \left(-(d_\Lambda \mathcal{H}(\lambda, d\lambda))(ga_0) - \{\mathcal{H}(\lambda, d\lambda), \mathcal{H}(\lambda, d\lambda)\}(ga_0) \right) = \quad (\text{B.12})$$

$$= -\frac{1}{2} e^{-\mathcal{H}(\lambda, d\lambda)(ga_0)} \{\mathcal{H}(\lambda, d\lambda), \mathcal{H}(\lambda, d\lambda)\}(ga_0) = \quad (\text{B.13})$$

$$= \frac{1}{2} e^{-\mathcal{H}(\lambda, d\lambda)(ga_0)} \left((\iota_{\{\mathcal{H}(\lambda, d\lambda), \cdot\}})^2 \omega \right)(ga_0) = \quad (\text{B.14})$$

$$= \frac{1}{2} e^{-\mathcal{H}(\lambda, d\lambda)(ga_0)} \left(\iota_{\{\mathcal{H}(\lambda, d\lambda), a^A\}} \partial / \partial a^A \right)^2 \omega(a) |_{a=ga_0} = \quad (\text{B.15})$$

$$= \frac{1}{2} e^{-\mathcal{H}(\lambda, d\lambda)(ga_0)} \left(\iota_{d\lambda^k (\partial_k a^A)} \partial / \partial a^A \right)^2 \omega(a) |_{a=ga_0} = e^{-\mathcal{H}(\lambda, d\lambda)(ga_0)} p^*\omega \quad (\text{B.16})$$

¹¹It is related to the B_a used in section 2 as follows: $\mathcal{H}(\lambda, d\lambda)(g(\lambda)x) = \sum d\lambda^a B_a(\lambda, x)$.

This concludes the proof.

Given a “symplectic potential” α satisfying $d\alpha = \omega$ we can construct a closed form as follows:

$$\Omega_+ = (p^* e^{-\alpha}) \int_L e^\sigma \tag{B.17}$$

We will choose the following ansatz for the equivariantly closed analogue of Ω :

$$\Omega_+^c = (p^* \nu) \int_L e^{S+\sigma+\Phi(h)} \tag{B.18}$$

where ν is of the same formal type as a Cartan cochain:

$$\nu \in \text{Fun}((\Pi TM) \times \mathfrak{h}) \tag{B.19}$$

The expression defined in eq. (B.18) is a cocycle of the Cartan complex of equivariant cohomology of LAG_+ if in addition to (4.2) we have

$$(d + \omega - \iota_{\{h, \cdot\}} + h) \nu = 0 \tag{B.20}$$

Even though ν lives in the same space as cochains of the Cartan complex, the differential defined by eq. (B.20) is different. (The Cartan differential would be $d - \iota_{\{h, \cdot\}}$.)

Comment. In particular, *when* we can choose an H -invariant “symplectic potential” α such that $d\alpha = \omega$, eq. (B.20) has a simple solution:

$$\nu = e^\alpha \tag{B.21}$$

Proof of Ω_+^c being equivariantly closed. We have to prove that:

$$(d - \iota_{\{h, \cdot\}}) \Omega_+^c = 0 \tag{B.22}$$

where d is the de Rham differential on LAG_+ . The action of d is given by eq. (B.7). The action of $\iota_{\{h, \cdot\}}$ on σ is essentially as in eq. (4.6), but we have to remember to subtract the compensating constant to make sure that σ vanishes at the marked point; therefore:

$$\iota_{\{h, \cdot\}} \sigma = h - h(a) \tag{B.23}$$

The vanishing of $(d - \iota_{\{h, \cdot\}}) \Omega_+^c$ when eqs. (4.2) and (B.20) are satisfied follows from direct computation.

C Central extension of the group of canonical transformations

In this section we will give a precise definition of Ω using a well-defined closed PDF $\widehat{\Omega}$ on a central extension \widehat{G} of the group of canonical transformations.¹² This group is infinite-dimensional, however, in this section we will keep the notation d for the de Rham differential on the group and on the space of Lagrangian submanifolds LAG .

¹²The existence of a central extension of the group of canonical transformations (symplectomorphisms) of odd symplectic manifold M can be proven in the same way as for an even symplectic manifold. Namely, as in the even case one constructs a bundle with connection over M , the fiber of this bundle is a one-dimensional odd vector space. The group \widehat{G} can be defined as a group of transformations of the total space of the bundle that are compatible with the fibration (transform fibers into fibers), induce canonical transformation on the base and are compatible with connection.

C.1 Definition of $\widehat{\Omega}$

Let us consider the Lie superalgebra $\Pi\text{Fun}(M)$ with the commutator given by the odd Poisson bracket. It is a central extension of the Lie superalgebra of Hamiltonian vector fields which we denote \mathfrak{g} ; therefore we denote it $\widehat{\mathfrak{g}}$:

$$\widehat{\mathfrak{g}} = \Pi\text{Fun}(M) \quad (\text{C.1})$$

We consider the central extension of the group of canonical transformations \widehat{G} , whose Lie algebra is $\widehat{\mathfrak{g}}$.

As a variation on our theme, we will now construct a map from LAG to the space of closed PDFs on \widehat{G} , which we will call $\widehat{\Omega}$:

$$\widehat{\Omega} \in \text{Fun}(\text{LAG} \times \Pi T\widehat{G}) \quad (\text{C.2})$$

$$\widehat{\Omega}(L, \widehat{g}, d\widehat{g}) = \int_{gL} \exp(S_{\text{BV}} + d\widehat{g}\widehat{g}^{-1}) \quad (\text{C.3})$$

Here following [6] we consider $\exp(S_{\text{BV}})$ as a semidensity, $d\widehat{g}\widehat{g}^{-1}$ is the right-invariant form on \widehat{G} taking values in the Lie algebra (Maurer-Cartan form), and g stands for an element of G corresponding to $\widehat{g} \in \widehat{G}$. In eq. (C.3) we consider $d\widehat{g}\widehat{g}^{-1}$ as a function on M , using the fact that the Lie algebra of \widehat{G} is $\Pi\text{Fun}(M)$. This form satisfies the Maurer-Cartan equation:

$$d(d\widehat{g}\widehat{g}^{-1}) + \frac{1}{2}\{d\widehat{g}\widehat{g}^{-1}, d\widehat{g}\widehat{g}^{-1}\} = 0 \quad (\text{C.4})$$

This $\widehat{\Omega}$ is closed as a PDF on \widehat{G} , i.e.:

$$d\widehat{\Omega} = 0 \quad (\text{C.5})$$

$$\text{where } d = d\widehat{g} \frac{\partial}{\partial \widehat{g}} \quad (\text{C.6})$$

The proof of eq. (C.5) is a straightforward computation [23, omega/Definition.html] very similar to the computations in section 2.

We must stress that this $\widehat{\Omega}$ is well-defined (does not contain any ambiguities).

C.2 How to build a form on LAG starting from $\widehat{\Omega}$

Since G (and therefore \widehat{G}) acts on LAG, there is a natural projection:

$$\widehat{\pi} : \text{LAG} \times \Pi T\widehat{G} \rightarrow \Pi T\text{LAG} \quad (\text{C.7})$$

However, it is *not true* that $\widehat{\Omega}$ is constant along the fibers of $\widehat{\pi}$. Indeed, for a $\xi \in \text{Lie}(\text{St}(L_0))$, where $\text{St}(L_0)$ stands for the stable subgroup of $L_0 \in \text{LAG}$ in \widehat{G} one can check that the restriction of ξ on L_0 is a constant c . Using $\widehat{g}\widehat{\xi}\widehat{g}^{-1} = \widehat{\xi} \circ g^{-1}$ we get:

$$\widehat{\Omega}(L_0, \widehat{g}, d\widehat{g} + \widehat{g}\widehat{\xi}) = k\widehat{\Omega}(L_0, \widehat{g}, d\widehat{g}) \quad (\text{C.8})$$

where k is some number. Therefore $\widehat{\Omega}$ does not automatically provide a PDF on LAG.

We could impose some additional restrictions, such as ghost number symmetry,¹³ which would guarantee that $k = 0$.

Let us suppose now that a subset of LAG is represented in the form $g(\lambda)L_0$ where $g(\lambda) \in G, \lambda \in \Lambda$. Assume that we can find a “lift” $\widehat{g}(\lambda)$ of $g(\lambda)$ to \widehat{G} . Then we can define a closed form

$$\Omega(L, dL) = \widehat{\Omega}(L_0, \widehat{g}(\lambda), d(\widehat{g}(\lambda))) \quad (\text{C.9})$$

This coincides with the “tentative” definition of section 3, because the restriction of $d\widehat{g}\widehat{g}^{-1}$ to gL_0 gives σ . This is a general fact, true both in classical mechanics and in BV formalism. In classical mechanics it is essentially the Hamilton-Jacobi equation, which describes the evolution of a Lagrangian submanifold (specified by a generating function usually called S) under the Hamiltonian flow. It says that $\frac{\partial S}{\partial t}$ equals the restriction of H on L plus a constant (which can depend on t).

Notice that by the variation of $\widehat{g}(\lambda)$ the form $\Omega(L, dL)$ obviously remains in the same cohomology class.

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¹³We do not require that the semidensity be invariant under the ghost number symmetry; just that $d\widehat{g}\widehat{g}^{-1}$ have ghost number -1 .

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