

Cluster functions and scattering amplitudes for six and seven points

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ABSTRACT: Scattering amplitudes in planar super-Yang-Mills theory satisfy several basic physical and mathematical constraints, including physical constraints on their branch cut structure and various empirically discovered connections to the mathematics of cluster algebras. The power of the bootstrap program for amplitudes is inversely proportional to the size of the intersection between these physical and mathematical constraints: ideally we would like a list of constraints which determine scattering amplitudes uniquely. We explore this intersection quantitatively for two-loop six- and seven-point amplitudes by providing a complete taxonomy of the $\text{Gr}(4, 6)$ and $\text{Gr}(4, 7)$ cluster polylogarithm functions of [15] at weight 4.

KEYWORDS: Scattering Amplitudes, Supersymmetric Gauge Theory

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1 Introduction

Several recent papers following [1] have explored the connection between (multi-loop) scattering amplitudes in planar $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory and cluster algebras, a subject of great interest to mathematicians. This line of research has two closely related branches: (1) investigating purely mathematical questions having to do with the classification of functions with certain cluster algebraic properties, i.e. “how rare are special

functions of the type we see in SYM theory?”, and (2) exploiting these mathematical properties, together with physical input as needed, to carry out calculations of new, previously intractable amplitudes, i.e. “how far can we get by exploiting the special properties of cluster algebras?”.

The most basic aspect of the observed connection, supported by all evidence available to date, is that n -point scattering amplitudes in SYM theory have singularities only at points in $\text{Conf}_n(\mathbb{P}^3)$ (the space of massless n -point kinematics modulo dual conformal invariance) where some cluster coordinate of the associated $\text{Gr}(4, n)$ cluster algebra vanishes. More specifically, all known multi-loop amplitudes may be expressed as linear combinations of generalized polylogarithm functions written in the symbol alphabet consisting of such cluster coordinates. We expect this to be true to all loop order for all MHV and NMHV amplitudes.

Deeper connections to the underlying cluster algebra have been found for the two-loop MHV remainder functions $R_n^{(2)}$. The algebra of generalized polylogarithm functions modulo products admits a cobracket δ satisfying $\delta^2 = 0$, giving it the structure of a Lie coalgebra [2]. It has been observed that $\delta R_n^{(2)}$ has a very rigid connection to the Poisson structure on the kinematic domain $\text{Conf}_n(\mathbb{P}^3)$. Specifically, the $(2, 2)$ component of $\delta R_n^{(2)}$ can always be written as a linear combination of $\text{Li}_2(-x_i) \wedge \text{Li}_2(-x_j)$ for pairs of cluster coordinates having Poisson bracket $\{\log x_i, \log x_j\} = 0$, while the $(3, 1)$ component can always be written as a linear combination of $\text{Li}_3(-x_i) \wedge \log(x_j)$ for pairs having $\{\log x_i, \log x_j\} = \pm 1$. These mathematical properties are tightly constraining: it has been argued in [3] that, when combined with a few physical constraints, they uniquely determine the $(2, 2)$ component of $\delta R_n^{(2)}$ for all n .

It is an interesting open problem to determine whether (and, if so, precisely how) the structure of more general amplitudes may be dictated by the underlying Poisson structure on $\text{Conf}_n(\mathbb{P}^3)$. This is a difficult question to address because data on multi-loop amplitudes is very hard to come by — beyond the two-loop MHV amplitudes, explicit results for complete amplitudes at fixed loop order are available only for $n = 6$ [4–10] (in addition, the symbol of the two-loop $n = 7$ NMHV amplitude has been computed in [11], and that of the three-loop $n = 7$ MHV amplitude in [12]). With only a handful of results available it may be difficult to identify a pattern which might let one tease out the underlying structure. Moreover, accidental simplifications may occur at small n which can obscure the general structure. (For example, the $(2, 2)$ component of $\delta R_6^{(2)}$ is identically zero [13].) It is known that the $(3, 3)$ component of $\delta R_6^{(3)}$ is not expressible in terms of cluster \mathcal{X} -coordinates [14], but there could be some more deeply hidden structure in this amplitude.

The primary goal of this paper is to further explore the taxonomy of two-loop cluster functions, as defined in [15], for $n = 6, 7$. We are particularly interested in the interplay between various mathematically natural but physically obscure conditions that certain functions can satisfy (such as the tight cluster constraints satisfied by all two-loop MHV amplitudes, mentioned above) and physically natural constraints, such as the requirement that amplitudes can only have physical branch points on the principal sheet (the so-called “first-entry condition” [16]). In previous work including [3] it has been remarked that the mathematical and physical constraints on MHV amplitudes seem almost orthogonal. One

of our goals here is to explore this question quantitatively by fully classifying the dimensions of function spaces satisfying various properties.

We begin in section 2 with a lightning review to set some notation and terminology. In sections 3 and 4 respectively we exhaustively analyze the spaces of cluster functions on the $\text{Gr}(4, 6)$ and $\text{Gr}(4, 7)$ cluster algebras respectively of relevance to $n = 6, 7$ -point amplitudes in planar SYM theory.

2 Review and notation

A kinematic configuration of n massless on-shell particles, with a cyclic order (which comes naturally in gauge theories when one looks at planar scattering amplitudes), can be parameterized in terms of n momentum twistors [17], $Z_i \in \mathbb{P}^3$, $i = 1, \dots, n$. The dual conformal symmetry of planar n -point amplitudes in SYM theory further implies that they are functions not on $(\mathbb{P}^3)^n$ but on the smaller space $\text{Conf}_n(\mathbb{P}^3) \cong \text{Gr}(4, n)/(\mathbb{C}^*)^{n-1}$ [1].

Viewing each Z_i as a four-component vector of homogeneous coordinates, the Plücker coordinates are defined by $\langle ijkl \rangle \equiv \det(Z_i Z_j Z_k Z_l)$. Functions on $\text{Conf}_n(\mathbb{P}^3)$ may be written in terms of ratios of Plücker coordinates such as

$$\frac{\langle ijkl \rangle \langle abcd \rangle}{\langle ijcd \rangle \langle abkl \rangle}, \tag{2.1}$$

or more generally in terms of ratios of homogeneous polynomials in Plücker coordinates having total weight zero under rescaling any of the Z_i .

Such objects form the building blocks for the $\text{Gr}(4, n)$ Grassmannian cluster algebra [18, 19], which is the algebra generated by certain preferred sets of coordinates on $\text{Gr}(4, n)$. These coordinates come in two related varieties: the \mathcal{A} -coordinates, which consist of the Plücker coordinates and certain homogeneous polynomials in them, and the \mathcal{X} -coordinates [20], which consist of certain scale-invariant ratios of \mathcal{A} -coordinates.

In this paper we focus on the cases $n = 6, 7$, for which the corresponding cluster algebras have respectively 15, 49 \mathcal{A} -coordinates and 15, 385 \mathcal{X} -coordinates.¹ The reader may find these coordinates tabulated in [1]. Of course, the \mathcal{X} -coordinates are not algebraically independent since the dimension of $\text{Conf}_n(\mathbb{P}^3)$ is only $3(n - 5)$. A “cluster” is a particular choice of $3(n - 5)$ cluster \mathcal{X} -coordinates in terms of which all others may be determined by a simple set of rational transformations called mutations.

A still mysterious but apparently important role is played by the fact that $\text{Conf}_n(\mathbb{P}^3)$ admits a natural Poisson structure, which it inherits from the Grassmannian [18]. A characteristic feature of cluster coordinates is that within each cluster, the \mathcal{X} -coordinates are log-canonical with respect to this Poisson structure, i.e.

$$\{\log x_i, \log x_j\} = B_{ij}, \quad i, j = 1, \dots, 3(n - 5), \tag{2.2}$$

where B is an antisymmetric integer-valued matrix (which for $n = 6, 7$ only takes the values $0, \pm 1$).

¹In some applications it is sensible to count x and $1/x$ separately, in which case these numbers would be 30, 770.

We expect all six- and seven-point L -loop scattering amplitudes in planar SYM theory to be (generalized) polylogarithm functions of uniform transcendental weight $2L$ whose symbols may be written in terms of the $\text{Gr}(4, n)$ cluster coordinates. For the purpose of writing a symbol alphabet the relevant question is not how many coordinates are algebraically independent, but how many are multiplicatively independent — we say that a finite collection $\{y_1, \dots, y_m\}$ is multiplicatively independent if there is no collection of integers $\{n_1, \dots, n_m\}$ such that $\prod y_i^{n_i} = 1$, i.e. if the collection $\{\log y_1, \dots, \log y_m\}$ is linearly independent over \mathbb{Z} .

As mentioned above there are respectively 15 (385) cluster \mathcal{X} -coordinates x_i for $n = 6$ ($n = 7$), but the corresponding sets of $\log x_i$ only span spaces of dimension 9 (42). Choosing bases for these spaces provides a collection of 9 (42) multiplicatively independent ratios to serve as symbol alphabets for building cluster polylogarithm functions.

2.1 The $\text{Gr}(4,6)$ cluster algebra

For six-point amplitudes the relevant cluster algebra is $\text{Gr}(4, 6)$, which is isomorphic to the A_3 cluster algebra. Its 15 cluster \mathcal{A} -coordinates are just the Plücker coordinates $\langle ijkl \rangle$. This algebra has 15 \mathcal{X} -coordinates. In the notation of [15] these are named v_i, x_i^\pm for $i = 1, 2, 3$ and e_i for $i = 1, \dots, 6$.

The reader may find explicit formulas for these as ratios of Plücker coordinates in [15]. Since one of the goals of this paper is to make contact with the work of Dixon et al. we will instead provide this information via the connection to the variables u, v, w, y_u, y_v, y_w used in [4–10].

The three-dimensional kinematic configuration space $\text{Conf}_6(\mathbb{P}^3)$ may be parameterized in terms of the three coordinates

$$y_u = \frac{\langle 1236 \rangle \langle 1345 \rangle \langle 2456 \rangle}{\langle 1235 \rangle \langle 1246 \rangle \langle 3456 \rangle}, \quad y_v = \frac{\langle 1235 \rangle \langle 1456 \rangle \langle 2346 \rangle}{\langle 1234 \rangle \langle 1356 \rangle \langle 2456 \rangle}, \quad y_w = \frac{\langle 1246 \rangle \langle 1356 \rangle \langle 2345 \rangle}{\langle 1256 \rangle \langle 1345 \rangle \langle 2346 \rangle}. \tag{2.3}$$

Note that a cyclic rotation $Z_i \rightarrow Z_{i+1}$ maps

$$y_u \rightarrow 1/y_v, \quad y_v \rightarrow 1/y_w, \quad y_w \rightarrow 1/y_u, \tag{2.4}$$

while reflection $Z_i \rightarrow Z_{1-i}$ (all indices are understood to be cyclic modulo 6) takes

$$y_u \rightarrow y_v, \quad y_v \rightarrow y_u, \quad y_w \rightarrow y_w. \tag{2.5}$$

The spacetime parity operator acts on momentum twistors as²

$$Z_i \rightarrow W_i = *(Z_{i-1} \wedge Z_i \wedge Z_{i+1}), \tag{2.6}$$

which transforms the cross-ratios defined in (2.3) according to

$$y_u \rightarrow 1/y_u, \quad y_v \rightarrow 1/y_v, \quad y_w \rightarrow 1/y_w. \tag{2.7}$$

²The notation means that W_i spans the one-dimensional subspace orthogonal to the 3-plane spanned by Z_{i-1}, Z_i, Z_{i+1} in \mathbb{C}^4 .

It is a curious accident that for $n = 6$ spacetime parity reversal is equivalent on $\text{Conf}_n(\mathbb{P}^3)$ to an element (namely, shift-by-three) of the cyclic group.

Three other variables used by Dixon et al. may be defined in terms of these via

$$u = \frac{y_u(1-y_v)(1-y_w)}{(1-y_u y_v)(1-y_u y_w)}, \quad v = \frac{y_v(1-y_u)(1-y_w)}{(1-y_u y_v)(1-y_v y_w)}, \quad w = \frac{y_w(1-y_u)(1-y_v)}{(1-y_u y_w)(1-y_v y_w)}. \quad (2.8)$$

Central to our investigations is the Poisson structure on $\text{Conf}_6(\mathbb{P}^3)$, which may be expressed in terms of the y variables as

$$\{\log y_u, \log y_v\} = \{\log y_v, \log y_w\} = \{\log y_w, \log y_u\} = \frac{(1-y_u)(1-y_v)(1-y_w)}{1-y_u y_v y_w}. \quad (2.9)$$

It is invariant under the full cyclic group (and hence, it is parity symmetric) but antisymmetric under reflection.

In terms of these variables, the cluster \mathcal{X} -coordinates may be expressed as

$$\begin{aligned} v_1 &= \frac{1-v}{v}, & v_2 &= \frac{1-w}{w}, & v_3 &= \frac{1-u}{u}, \\ x_1^+ &= \frac{y_v(1-y_u y_w)}{1-y_v}, & x_2^+ &= \frac{y_w(1-y_u y_v)}{1-y_w}, & x_3^+ &= \frac{y_u(1-y_v y_w)}{1-y_u}, \\ x_1^- &= \frac{1-y_u y_w}{y_u y_w(1-y_v)}, & x_2^- &= \frac{1-y_u y_v}{y_u y_v(1-y_w)}, & x_3^- &= \frac{1-y_v y_w}{y_v y_w(1-y_u)}, \\ e_1 &= \frac{1-y_v}{y_v(1-y_u)}, & e_2 &= \frac{y_v(1-y_w)}{1-y_v}, & e_3 &= \frac{1-y_u}{y_u(1-y_w)}, \\ e_4 &= \frac{y_u(1-y_v)}{1-y_u}, & e_5 &= \frac{1-y_w}{y_w(1-y_v)}, & e_6 &= \frac{y_w(1-y_u)}{1-y_w}. \end{aligned} \quad (2.10)$$

Note that under a cyclic shift $Z_i \rightarrow Z_{i+1}$ we have

$$v_i \rightarrow v_{i+1}, \quad x_i^\pm \rightarrow x_{i+1}^\mp, \quad e_i \rightarrow e_{i+1}, \quad (2.11)$$

while under parity the v_i are invariant and

$$x_i^\pm \rightarrow x_i^\mp, \quad e_i \rightarrow e_{i+3}. \quad (2.12)$$

Of particular importance are pairs x_1, x_2 of distinct \mathcal{X} -coordinates with simple Poisson brackets. By “simple” we mean specifically that $\{\log x_1, \log x_2\}$ is either 0 or ± 1 . There are three pairs with Poisson bracket zero,

$$\{\log x_i^+, \log x_i^-\} = 0, \quad (2.13)$$

and 30 pairs with Poisson bracket +1,

$$\{\log e_i, \log e_{i+4}\} = \{\log x_{i+1}^\pm, \log v_i\} = \{\log v_{i+1}, \log x_i^\pm\} = \{\log x_{i+1}^\pm, \log e_i\} = 1 \quad (2.14)$$

together with their cyclic images, for $6+6+6+12 = 30$ pairs. The remaining 72 pairs have “complicated” Poisson brackets (specifically, non-integer-valued; see for example (2.9)).

2.2 The Gr(4,7) cluster algebra

For seven-point amplitudes the relevant cluster algebra is $\text{Gr}(4, 7)$, which is isomorphic to the E_6 algebra. The 49 cluster \mathcal{A} -coordinates consist of the 35 Plücker coordinates $\langle ijkl \rangle$ together with 14 homogeneous polynomials denoted by $\langle 1(23)(45)(67) \rangle$, $\langle 2(13)(45)(67) \rangle$ (and their cyclic images), where

$$\langle i(i-1, i+1)(j, j+1)(k, k+1) \rangle = \langle i-1 i j j+1 \rangle \langle i i+1 k k+1 \rangle - \langle i-1 i k k+1 \rangle \langle i i+1 j j+1 \rangle. \tag{2.15}$$

One can build from these 49 \mathcal{A} -coordinates a total of 385 cluster \mathcal{X} -coordinates (or 770 if we count their multiplicative inverses). These are tabulated on pages 40–41 of [1]. Out of $\frac{1}{2} \cdot 385 \cdot 384 = 73920$ pairs of \mathcal{X} -coordinates, 2520 have Poisson bracket ± 1 while 833 have Poisson bracket zero.

2.3 The cobracket and Bloch groups

We recall that the algebra \mathcal{A} of generalized polylogarithm functions admits a coproduct giving it the structure of a Hopf algebra [2]. When we work with the quotient space \mathcal{L} of polylogarithm functions modulo products of functions of lower weight, the coproduct descends onto the quotient space to a cobracket δ which satisfies $\delta^2 = 0$. We review here only the barest essentials, and refer the reader to [1, 15] for additional details.

The cobracket of a weight-4 function has two components,

$$\delta \mathcal{L}_4 \in (B_3 \otimes \mathbb{C}^*) \oplus (B_2 \wedge B_2), \tag{2.16}$$

where the Bloch group B_k is, for our purposes, the free abelian group generated by functions of the form $\{x\}_k \equiv -\text{Li}_k(-x)$, where Li_k is the classical polylogarithm function and x is a function on $\text{Conf}_n(\mathbb{P}^3)$ which is rational in Plücker coordinates.

The fact that $\delta^2 = 0$ and that δ has trivial cohomology means that if $a \in B_3 \otimes \mathbb{C}^*$ and $b \in B_2 \wedge B_2$, then there exists a function f whose cobracket components are $a \oplus b$ if and only if $\delta_{31}(a) + \delta_{22}(b) = 0$. As explained in [15], this condition can be used to explicitly enumerate cluster functions, at least on algebras of finite type. For such algebras $B_3 \otimes \mathbb{C}^*$ and $B_2 \wedge B_2$ are finite dimensional vector spaces on which δ acts linearly, so the space of cluster \mathcal{A} -functions is simply the kernel of δ .

At weight 4 a general polylogarithm can be expressed in terms of the classical functions Li_k if and only if its $B_2 \wedge B_2$ cobracket component vanishes. We will often be interested in counting the number of non-classical functions, since the classical ones (which correspond to solutions of $\delta_{31}(a) = 0$) are trivial to enumerate. To answer this question we compute the dimension of the subspace of $B_2 \wedge B_2$ such that the equation $\delta_{31}(a) + \delta_{22}(b) = 0$ is solvable for some $a \in B_3 \otimes \mathbb{C}^*$.

One final piece of terminology concerns the interplay between the Poisson structure on the Grassmannian cluster algebras and the cobracket of polylogarithm functions. We recall that two cluster \mathcal{X} -coordinates x, y have $\{\log x, \log y\} \in \mathbb{Z}$ only if there exists a cluster containing either x or $1/x$, and either y or $1/y$. As reviewed in [1], the combinatorics of mutations is encoded in a graph called the (generalized) Stasheff polytope associated to the

algebra. We therefore say that a function has “Stasheff local” $B_2 \wedge B_2$ if it can be expressed as a linear combination of terms of the form $\{x\}_2 \wedge \{y\}_2$ for pairs having integer Poisson bracket (for $\text{Gr}(4, 6)$ and $\text{Gr}(4, 7)$, this integer will always be in the set $\{-1, 0, +1\}$).

3 The cluster structure of hexagon functions at weight 4

3.1 Setup

In this section we consider cluster functions on the $A_3 \cong \text{Gr}(4, 6)$ cluster algebra. The term “cluster \mathcal{A} -function” introduced in [15] refers, in the present application, to an integrable symbol written in the 9-letter alphabet of cluster coordinates (specifically, this means any multiplicatively independent set of \mathcal{X} -coordinates; or equivalently, homogeneous ratios of \mathcal{A} -coordinates) on $\text{Gr}(4, 6)$.

Any linear combination of cluster \mathcal{A} -functions with the property that only the three variables u, v, w appear in the first-entry of the symbol, reflecting the physically allowed branch points for a scattering amplitude [16], is called a “physical function” or, following the terminology of [6], a “hexagon function”. These have been studied through high weight in the series of papers [4–10], but we restrict our analysis to weight 4 as our aim is to explore connections between the cobrackets and the cluster Poisson structure of these functions.

Let \mathcal{A}_k denote the vector space of all weight- k cluster \mathcal{A} -functions. Such functions are easy to count for any A_m type cluster algebra (see [21, 22]); for A_3 we have the generating function

$$f_{A_3}(t) = 1 + \sum_{k=1}^{\infty} t^k \dim(\mathcal{A}_k) = \frac{1}{1-2t} \frac{1}{1-3t} \frac{1}{1-4t}, \tag{3.1}$$

so that

$$\dim(\mathcal{A}_k) = 9, 55, 285, 1351, \dots \quad k = 1, 2, 3, 4, \dots \tag{3.2}$$

Let \mathcal{L}_k denote the quotient of \mathcal{A}_k by products of functions of lower weight. The number of such functions can be computed by taking the plethystic logarithm of the generating function $f_{A_3}(t)$ (see for example [23]), which gives

$$\dim(\mathcal{L}_k) = 9, 10, 30, 81, \dots \quad k = 1, 2, 3, 4, \dots \tag{3.3}$$

Finally we denote by \mathcal{B}_k the subspace of \mathcal{L}_k generated by the classical polylogarithms (we do not yet restrict their arguments to be cluster \mathcal{X} -coordinates). We have

$$\dim(\mathcal{B}_k) = 10, 30, 45, \dots \quad k = 2, 3, 4, \dots \tag{3.4}$$

For $k < 4$ the agreement with (3.3) reflects the fact that all such generalized polylogarithms can be expressed in terms of the classical functions; for higher k these numbers can be obtained by choosing a basis for \mathcal{L}_k and computing $\dim \ker \delta$ as described in the previous section.

3.2 The non-classical functions

Beginning at $k = 4$ we can distinguish between classical and non-classical functions. At weight $k = 4$, the “non-classicalness” of a function is completely characterized by its $B_2 \wedge B_2$ cobracket component (see for example [1]). Since B_2 has dimension 10 according to (3.4), $B_2 \wedge B_2$ evidently has dimension 45. However, a random element of this vector space is not guaranteed to be the $B_2 \wedge B_2$ cobracket component of any cluster \mathcal{A} -function — there is a nontrivial integrability constraint.

In fact, by comparing (3.4) to (3.3) we see that there are 81 functions in all, minus 45 classical functions, for a total of 36 non-classical functions. We conclude that in the 45-dimensional space $B_2 \wedge B_2$ spanned by objects of the form $\{x\}_2 \wedge \{y\}_2$, for cluster coordinates x and y , only the linear combinations lying in a particular 36-dimensional subspace correspond to cobracket components of actual cluster \mathcal{A} -functions.³ We will shortly characterize this 36-dimensional space completely.

Let us write PB_0 to denote the subspace of $B_2 \wedge B_2$ spanned by objects of the form $\{x\}_2 \wedge \{y\}_2$ for pairs having Poisson bracket $\{\log x, \log y\} = 0$. In what follows we will for example say that a function “lives in PB_0 ” if its $B_2 \wedge B_2$ cobracket component can be expressed in terms of such pairs. Similarly, let PB_1 be the subspace spanned by pairs having Poisson bracket 1, and let us also use the shorthand $PB_* = B_2 \wedge B_2$, meaning that the Poisson bracket can be anything. We found in (2.13) and (2.14) that there are respectively 3, 30 pairs with Poisson bracket 0, 1. It is simple to check that the corresponding elements are linearly independent in $B_2 \wedge B_2$, so we have that $\dim PB_0 = 3$ and $\dim PB_1 = 30$, while of course $\dim PB_* = \dim B_2 \wedge B_2 = 45$.

With this notation in hand let us now summarize our findings on the 36 non-classical cluster \mathcal{A} -functions at weight four, which we find fall into two broad groups:

(A) 6 of these functions are the “ A_2 cluster functions” introduced in [15]. There is one such function for each A_2 subalgebra of A_3 ; these subalgebras and the associated functions are represented visually in equation (4.3) of that paper. These six functions have additional “cluster structure”: their $B_3 \otimes \mathbb{C}^*$ cobracket components can be expressed entirely in terms of cluster \mathcal{X} -coordinates — this means that they are “cluster \mathcal{X} -functions” in the terminology of [15]. General elements of this six-dimensional space are not Stasheff local — their $B_2 \wedge B_2$ cobracket components are not expressible in terms of pairs of coordinates with Poisson bracket $0, \pm 1$. Only one particular linear combination of these 6 — the one called the A_3 function in [15]—has a nice $B_2 \wedge B_2$, in fact lying inside PB_0 . The $B_2 \wedge B_2$ cobracket component of this A_3 function is

$$\sum_{i=1}^3 \{x_i^+\}_2 \wedge \{x_i^-\}_2. \tag{3.5}$$

This quantity is parity-odd so it cannot possibly appear in the two-loop six-point MHV remainder function, which is parity-even. This “explains” why the hypothesis that two-

³Linear combinations which fall outside this 36-dimensional subspace are certainly integrable [24], but they integrate to functions with symbols involving letters which are not cluster coordinates, for example differences of \mathcal{X} -coordinates $x_i - x_j$, which does not in general factor into a product of cluster coordinates. Hence they are not cluster \mathcal{A} -functions.

loop MHV remainder functions must live in PB_0 , which we know to be true for all n [3], implies that the case $n = 6$ must be classical.

(B) The remaining 30 functions are sort of the opposite: no linear combination of these 30 has a $B_3 \otimes \mathbb{C}^*$ content which can be expressed entirely in terms of \mathcal{X} -coordinates, so none of them are cluster \mathcal{X} -functions. On the other hand, all of them are Stasheff local — they all have “nice” $B_2 \wedge B_2$, in fact they span exactly the 30-dimensional subspace $PB_1 \subset B_2 \wedge B_2$.

3.3 The physical (hexagon) functions

Dixon et al. find that there are precisely 15 functions at weight 4 (modulo products of functions of lower weight) satisfying the first-entry condition, which they call hexagon functions. Let us put aside 9 which are purely classical and focus on the two types of functions named Ω_2 and F_1 in [6].

(A) The function F_1 is parity-odd and comes in three cyclic permutations (i.e., $i \rightarrow i+2$ and $i \rightarrow i+4$). These functions are rather interesting; each of them has a $B_2 \wedge B_2$ coproduct component given by (3.5) plus additional terms which cannot be expressed in terms of pairs having simple Poisson bracket. Since (3.5) is invariant under $i \rightarrow i+2$, we can throw out these terms by taking the difference between any two pairs of the three permutations of F_1 . Indeed such linear combinations have appeared in the literature, as in (B.18) and (B.20) of [6] which define the function \tilde{V} by

$$8\tilde{V} = -F_1(u, v, w) + F_1(w, u, v) + \text{products of lower-weight functions.} \quad (3.6)$$

Hence only two of the three distinct cyclic permutations of \tilde{V} are linearly independent.

(B) Next we look at the parity-even function Ω_2 which also comes in three cyclic permutations. At the level of $B_2 \wedge B_2$, where we can ignore all terms involving only classical polylogarithms, the function Ω_2 is equivalent (modulo an overall multiplicative factor) to the function called V by Dixon et al.; see for example (7.1) through (7.3) of [4]. In that paper it was also observed that the three cyclic permutations of this function add up to a purely classical function, so the three different permutations of V span only a two-dimensional subset of $B_2 \wedge B_2$.

To summarize, we find that the subspace of $B_2 \wedge B_2$ spanned by physical (hexagon) functions has dimension 5. Two dimensions are spanned by the parity-even functions of type V , while three dimensions are spanned by the parity-odd functions of type F_1 . Although a generic vector in the three-dimensional parity-odd subspace has terms with “bad” Poisson brackets, there is something especially nice about the subspace spanned by the permutations of V and \tilde{V} together. To see this we exhibit here a formula for their cobracket components, which we find are most simply packaged in the formula

$$\delta_{|2,2}(V + \tilde{V}) = \frac{1}{2}\{v_2\}_2 \wedge \{x_1^-\}_2 - \frac{1}{2}\{v_1\}_2 \wedge \{x_3^-\}_2 - \frac{1}{2}\{x_1^+\}_2 \wedge \{v_3\}_2 + \frac{1}{2}\{x_2^+\}_2 \wedge \{v_1\}_2. \quad (3.7)$$

Since V, \tilde{V} have parity even and odd, respectively, $\delta_{|2,2}(V - \tilde{V})$ is given by the same formula but with $x^\pm \rightarrow x^\mp$. We now see that each term in (3.7) involves only the PB_1

pairs listed in (2.14)! Moreover, it is trivial to check directly from (3.7) and the cyclic transformations (2.11) that the six functions V, \tilde{V} altogether span only a four-dimensional subspace of PB_1 .

3.4 Summary

The results of this section can be summarized in the following classification of weight-4 cluster functions on $A_3 \cong \text{Gr}(4, 6)$:

There are a total of 81 irreducible weight-four cluster \mathcal{A} -functions

- ↳ 45 classical, 10 of which are physical
- ↳ 36 non-classical, 5 of which are physical
 - (three permutations of F_1 and two of Ω_2)
 - ↳ 30 PB_1 functions, 4 of which are physical
 - (two permutations each of V, \tilde{V})
 - ↳ 6 A_2 functions; these are all of the cluster \mathcal{X} -functions
 - ↳ 1 PB_0 function, the A_3 function
 - (as mentioned above, this is F_1 plus terms in PB_*)
 - ↳ 5 PB_* functions

Let us emphasize that these numbers count only irreducible functions, and that starting from the third line they moreover count functions modulo the classical function Li_4 (i.e., the numbers refer to dimensions of subspaces of $B_2 \wedge B_2$). When we say that a function is physical modulo additional terms, we mean that it is possible to choose the additional terms to render the function physical.

3.5 The two-loop hexagon MHV amplitude

Let us now comment on the relevance of these functions to the two-loop six-point MHV remainder function $R_6^{(2)}$, which was found to be expressible in terms of the classical polylogarithm functions Li_k in [13] (a fact that we “explained” below (3.5)). In fact, this amplitude is even more special because it is a cluster \mathcal{X} -function, which means that it can be expressed entirely in terms of the $\text{Li}_k(-x)$; the $\text{Li}_k(1+x)$ and $\text{Li}_k(1+1/x)$ functions, whose $B_3 \otimes \mathbb{C}^*$ cobracket components are not expressible in terms of cluster \mathcal{X} -coordinates, are not needed [1].

Above we tabulated our finding that (modulo products of lower-weight functions) there are only 10 physical and classical polylogarithms at weight four. In this space we now search for functions whose coproducts are expressible entirely in terms of the $\text{Li}_k(-x)$. We find that there is a unique linear combination that is invariant under the discrete symmetries (parity and dihedral invariance) that MHV amplitudes must possess. That linear combination is proportional to the two-loop MHV remainder function

$$R_6^{(2)\text{MHV}} = \sum_{i=1}^3 \left[\text{Li}_4(-x_i^+) + \text{Li}_4(-x_i^-) - \frac{1}{2} \text{Li}_4(-v_i) \right] + \text{products of lower-weight functions}, \tag{3.8}$$

in agreement with the known result [13]. (This argument, of course, does not fix the overall coefficient.) Of course, in this case it is very well known that the product terms are also completely fixed by simple considerations, but our focus in this paper is on the leading term.

3.6 The two-loop hexagon NMHV amplitude

The $n = 6$ NMHV two-loop ratio function is given by [4]

$$\mathcal{P}_{6,\text{NMHV}}^{(2)} = [23456][V(u, v, w) + \tilde{V}(y_u, y_v, y_w)] + \text{cyclic} \tag{3.9}$$

where [23456] is the R -invariant

$$[abcde] = \frac{\delta^4(\chi_a \langle bcde \rangle + \text{cyclic})}{\langle abcd \rangle \langle bcde \rangle \langle cdea \rangle \langle deab \rangle \langle eabc \rangle} \tag{3.10}$$

and V, \tilde{V} are the two generalized polylogarithm functions of uniform transcendental weight four reviewed in section 3.3 above. These two functions were computed explicitly in [4] (see also [21] for a different presentation of these functions). The $B_2 \wedge B_2$ component of the cobracket of this amplitude was computed in (3.7), where it was found to be expressible entirely in terms of pairs living in PB_1 .⁴

The NMHV ratio function provides us (at the level of $B_2 \wedge B_2$) with a total of four linearly independent non-classical functions of weight 4 (as reviewed above, each of V and \tilde{V} comes in three cyclic permutations, but the cyclic sum of each is separately zero inside $B_2 \wedge B_2$). We see from the summary in section 3.4 that precisely 5 functions of this type exist. Only four linear combinations of them, however, actually appear in the amplitude — these are precisely the four linear combinations which live in PB_1 ! The one additional non-classical weight-4 hexagon function which exists but does not appear in the amplitude, F_1 by itself, has terms with “bad” Poisson brackets (i.e., non-Stasheff local terms) in its $B_2 \wedge B_2$ content.

4 The cluster structure of heptagon functions at weight 4

4.1 Setup

In this section the term “cluster function” refers to an integrable symbol written in the 42-letter alphabet of cluster coordinates on $\text{Gr}(4, 7)$. Any linear combination of such symbols with the property that only the Plücker coordinates of the form $\langle i \ i+1 \ j \ j+1 \rangle$ appear in the first entry of the symbol, reflecting the physically allowed branch points for a scattering amplitude, is called (the symbol of) a “physical function” or a “heptagon function” following the terminology of [12] where they have been studied through weight six. The analysis here, where we aim to make finer statements about the connection to the Poisson bracket of the cluster algebra, is again restricted to weight 4, of relevance to two-loop amplitudes.

Let \mathcal{A}_k denote the vector space of all weight- k functions. In contrast to the A_m cluster algebras and the example shown in (3.1), we do not know of any generating function which

⁴This observation was first made by C. Vergu [14].

counts the number of cluster functions for the E_6 algebra. These may be tabulated through weight 3 by explicit enumeration, but at higher weight these numbers must be computed by analyzing the integrability constraint. This boils down to a linear algebra problem, since counting the number of cluster functions at weight k is the same as finding how many linear combinations of the 42^k weight- k symbols satisfy the integrability constraint. (This calculation can be rendered more manageable by imposing integrability at the level of the cobracket rather than at the level of the symbol.) We have carried this out at $k = 4$ to find that

$$\dim(\mathcal{A}_k) = 42, 1035, 19536, 312578, \dots \quad k = 1, 2, 3, 4, \dots \quad (4.1)$$

Let \mathcal{L}_k denote the quotient of \mathcal{A}_k by products of functions of lower weight. As in (3.3) taking the plethystic logarithm [23] gives

$$\dim(\mathcal{L}_k) = 42, 132, 748, 4193, \dots \quad k = 1, 2, 3, 4, \dots \quad (4.2)$$

Finally we denote by B_k the subspace of \mathcal{L}_k generated by the classical polylogarithms (we do not yet restrict their arguments to be cluster \mathcal{X} -coordinates). We have

$$\dim(B_k) = 132, 748, 1155, \dots \quad k = 2, 3, 4, \dots \quad (4.3)$$

As mentioned before, agreement of these numbers with (4.2) is guaranteed for $k < 4$, and we obtained the value 1155 for $k = 4$ by computing $\dim \ker \delta$ as described in section 2.

Before we turn to weight 4, a minor interesting comment about $k = 3$ is in order. It is simple to write down classical cluster functions of the form $\text{Li}_k(-x)$, $\text{Li}_k(1+x)$ and $\text{Li}_k(1+1/x)$ for any weight k , where x runs over the set of 385 \mathcal{X} -coordinates. For $k = 3$, this set of functions is overcomplete due to the identity

$$\text{Li}_3(-x) + \text{Li}_3(1+x) + \text{Li}_3(1+1/x) = 0 \quad \text{mod products of lower-weight functions.} \quad (4.4)$$

Among the 385 functions of type $\text{Li}_3(-x)$ there are exactly 22 additional linear relations. These were discovered in [1], where they were called D_4 identities since the simplest manifestation of this identity occurs for the D_4 algebra. Altogether then these identities account for the $3 \times 385 - 385 - 22 = 748$ linearly independent weight-3 cluster \mathcal{A} -functions tabulated in (4.2).

4.2 The non-classical functions

Let us now repeat the analysis done in the beginning of section 3.2 for the E_6 algebra. Since B_2 has dimension 132, $B_2 \wedge B_2$ has dimension 8646. We again use the notation PB_0 , PB_1 , and $PB_* = B_2 \wedge B_2$ to denote the subspaces spanned by elements of the form $\{x\}_2 \wedge \{y\}_2$ for pairs x, y having Poisson bracket 0, ± 1 , or “anything.” We find that PB_0 has dimension 455 and PB_1 has dimension 2520.

A quick glance at (4.2) and (4.3) reveals that there are $4193 - 1155 = 3038$ non-classical cluster functions at weight $k = 4$. We find that these fall into three groups:

- (A) First, there are the A_2 functions. We recall from (for example) [1] that E_6 has 1071 A_2 subalgebras, so one can construct 1071 A_2 functions according to the definition

given in [15], but only 448 of these are linearly independent inside $B_2 \wedge B_2$.⁵ These functions are moreover cluster \mathcal{X} -functions: their $B_3 \otimes \mathbb{C}^*$ cobracket components can be expressed entirely in terms of cluster \mathcal{X} -coordinates, but their $B_2 \wedge B_2$ content is, in general, not Stasheff local — not expressible in terms of pairs with Poisson bracket $0, \pm 1$.

There are no linear combinations of these 448 functions which live in PB_1 — these are covered in (B) just ahead — but we find that 195 linear combinations live in PB_0 . This 195-dimensional space is spanned by the set of A_3 functions associated to the various A_3 subalgebras of E_6 .

- (B) There are 2520 functions which span the 2520-dimensional subspace $PB_1 \subset B_2 \wedge B_2$. We found the same phenomenon in the six-point case discussed in the previous section. There we furthermore found that no linear combination of these PB_1 functions had a $B_3 \otimes \mathbb{C}^*$ component that could be expressed entirely in terms of \mathcal{X} -coordinates. We have not repeated this analysis for the 2520 seven-point functions; the computation seems formidable.
- (C) There are an additional $3038 - 448 - 2520 = 70$ functions which we can tabulate explicitly (at least at the level of their cobrackets), but seem to have no nice characterization.

4.3 The physical (heptagon) functions

It was found in [12] that there are precisely 1288 functions at weight 4 satisfying the first-entry condition, which are called physical, or heptagon functions. We have computed the $B_2 \wedge B_2$ cobracket of each of them, and found that there are only 126 non-zero linear combinations. This means that there are 1162 classical heptagon functions and 126 non-classical heptagon functions at weight 4. We have found that these 126 heptagon functions fall into three types:

- (A) A total of 105 of these functions live in PB_0 ; they come in 15 families related by cyclic permutations.
- (B) A total of 14 of these functions live in PB_1 ; they come in 2 families related by cyclic permutations.
- (C) There is one remaining family of 7 functions related by cyclic permutations. No linear combination of these is Stasheff local (i.e., lives within the union of PB_0 and PB_1).

⁵This result was first obtained in the undergraduate thesis of A. Scherlis.

4.4 Summary

The results of this section can be summarized in the following classification of weight-4 cluster functions on $E_6 \cong \text{Gr}(4, 7)$:

- There are a total of 4193 irreducible weight-four cluster \mathcal{A} -functions
 - ↳ 1155 classical, 770 of which are physical
 - ↳ 3038 non-classical, 126 of which are physical
 - ↳ 2520 PB_1 functions, 105 of which are physical
 - ↳ 448 A_2 functions; these are all of the cluster \mathcal{X} -functions
 - ↳ 195 PB_0 function, 14 of which are physical
 - ↳ 253 PB_* functions
 - ↳ 70 other PB_* functions

Again let us emphasize that these numbers count only irreducible functions, and that starting from the third line they moreover count functions modulo the classical function Li_4 (i.e., the numbers refer to dimensions of subspaces of $B_2 \wedge B_2$). When we say that a function is physical modulo additional terms, we mean that it is possible to choose the additional terms to render the function physical.

4.5 The two-loop heptagon MHV amplitude

The symbol of the two-loop seven-point MHV remainder function $R_7^{(2)}$ was computed in [25], and its cobracket was computed in [1], where it was observed to be a cluster \mathcal{X} -function living in PB_0 . An analytic formula for $R_7^{(2)}$ was obtained in [26] and checked against the earlier numerical results of [27].

If we start from the hypothesis that $R_7^{(2)}$ should be a cluster \mathcal{X} -function living in PB_0 , then we see from the above chart that there are only 14 physical functions with these properties. It was shown in [3] that only one linear combination of these has the dihedral symmetry required of the amplitude, is well-defined in the collinear limit, and satisfies the “last-entry” condition [25] required by supersymmetry.

In fact these constraints, while all true, are vastly stronger than necessary to pin down $R_7^{(2)}$: in [12] it was found that the symbol of $R_7^{(2)}$ is the unique weight-4 heptagon function (up to an overall multiplicative factor) which is well-defined in all $i+1 \parallel i$ collinear limits!

4.6 The two-loop heptagon NMHV amplitude

The symbol of the seven-point 2-loop NMHV ratio function $\mathcal{P}_{7,\text{NMHV}}^{(2)}$ was first computed in [11]. It may be expressed as a linear combination of the 21 seven-point NMHV R -invariants (of which 15 are linearly independent), with coefficients that have uniform transcendentality weight 4. Due to the linear relations between R -invariants there is some freedom in how to represent the amplitude (i.e., one can shift terms from one transcendental function to another by adding zero to the amplitude in various ways).

Despite this freedom, we find that it impossible to write the $B_2 \wedge B_2$ cobracket of this amplitude in a Stasheff local manner, i.e. in terms of $\{x\}_2 \wedge \{y\}_2$ for pairs x, y having

Poisson bracket $0, \pm 1$. The local terms having “good” Poisson brackets may be expressed (in one particular representation of the amplitude) as

$$\delta_{22} \mathcal{P}_{7, \text{NMHV}}^{(2)} |_{\text{“good”}} = (f_{12} R_{12} + f_{13} R_{13} + f_{14} R_{14}) + \text{cyclic}, \quad (4.5)$$

where the quantities f_{12} , f_{13} and f_{14} are presented explicitly in the appendix, and R_{ij} is the R -invariant whose arguments are 1234567 (in that order) but with i and j omitted — this is the same as the notation used in [4]. Meanwhile the “bad” terms are given by:

$$\delta_{22} \mathcal{P}_{7, \text{NMHV}}^{(2)} |_{\text{“bad”}} = (R_{25} - R_{26} + R_{37} - R_{47}) B_1 + \text{cyclic} \quad (4.6)$$

in terms of a single element $B_1 \in B_2 \wedge B_2$ (also given in the appendix) which is not expressible solely in terms of pairs having Poisson bracket zero or one.

In fact we can point our finger directly at the “offending” function corresponding to B_1 in the summary presented at the end of section 4.4. There we found that of the 126 non-classical weight-4 heptagon functions, 105 live in PB_1 while 14 live in PB_0 , leaving $127 - 105 - 14 = 7$ unaccounted for. These other seven functions have $B_2 \wedge B_2$ cobracket components given exactly by B_1 in its seven cyclic arrangements.

5 Conclusion

In this paper we have studied in detail the taxonomy of weight-4 cluster functions on the cluster algebras relevant for 6- and seven-point amplitudes in planar SYM theory. In particular we have counted the numbers of linearly independent functions satisfying various mathematical constraints on their cobrackets, and the physical “first-entry” constraint which specifies the locations where amplitudes are permitted to have branch points on the principal sheet. These results are summarized in sections 3.4 and 4.4.

For $n = 6$ the story is very simple: there is no non-classical weight-4 generalized polylogarithm function which is consistent with the discrete symmetries of the MHV amplitude and whose $B_2 \wedge B_2$ cobracket component is expressible in terms of pairs of cluster \mathcal{X} -coordinates having Poisson bracket 0. This “explains” why the two-loop six-point MHV remainder function “must be” expressible in terms of classical polylogarithms [13].

Meanwhile, there are precisely 4 linearly independent non-classical functions which satisfy the first-entry condition and are Stasheff local (they have $B_2 \wedge B_2$ cobracket components are expressible in terms of pairs of cluster \mathcal{X} -coordinates having Poisson bracket 1). These are precisely the (non-classical parts of the) 4 independent functions which appear in the two-loop six-point NMHV ratio function [4].

For $n = 7$, as has already been observed in [3, 12], the cobracket (indeed, the whole symbol) of the two-loop MHV amplitude is uniquely determined by a simple list of mathematical and physical constraints. However the story for the two-loop NMHV ratio function is a little more complicated. We find that the cobracket of this amplitude is not expressible in a Stasheff local manner (that means, in terms of pairs having Poisson bracket $0, \pm 1$). It would be very interesting to learn if there is some other question one may ask about the cluster structure of this amplitude, to which a more affirmative answer may be given. We

expect to be the case since it is known that there is a cluster structure at the level of the integrand (aspects of which have been explored in [28, 29]), of which some echo ought to remain for integrated amplitudes.

One of our results might be of more mathematical than physical interest. For both the A_3 and E_6 cluster algebras, we find that for any pair of \mathcal{X} -coordinates with Poisson bracket $\{\log x, \log y\} = 1$, there exists a weight-4 cluster \mathcal{A} -function (that is, an integrable symbol whose letters are drawn from the alphabet of cluster coordinates) whose $B_2 \wedge B_2$ cobracket component is $\{x\}_2 \wedge \{y\}_2$. It would be interesting to learn if there is a mathematical explanation for this fact, and whether it is valid for more general cluster algebras (in particular, for ones of infinite type). In contrast, pairs of \mathcal{X} -coordinates having Poisson bracket 0 are rarely integrable in this manner; the two-loop MHV amplitudes of planar SYM theory remarkably provide functions of this relatively rare type.

In the introduction we mentioned that in previous work including [3] it has been remarked that the mathematical and physical constraints on MHV amplitudes seem almost orthogonal. This is both good and bad. On the one hand it is good to discover a short list of simple criteria which uniquely, or almost uniquely, determine an amplitude of interest — this is the core goal of the S -matrix program. On the other hand it is bad when there is no known formalism which simultaneously manifests both types of constraints. We do not yet know of any way, besides explicit enumeration, to actually identify and write down functions satisfying both the physical and mathematical we expect amplitudes to possess. Explicit results for higher loop planar SYM amplitudes remain, at least for the moment, difficult needles to find.

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A Two-loop heptagon NMHV coproduct data

In the first three subsections we list the Stasheff local contributions to the $B_2 \wedge B_2$ cobracket component of the two-loop heptagon NMHV ratio function, in terms of the quantities f_{12} , f_{13} , and f_{14} appearing in (4.5). Specifically, these contain all terms of the form $\{x\}_2 \wedge \{y\}_2$ for pairs x, y having Poisson bracket $0, \pm 1$. The additional “bad” contributions to the cobracket are shown in (4.6) and given explicitly in the fourth subsection.

A.1 f_{13}

This function is cyclically invariant and lives entirely in PB_1 . We find

$$\begin{aligned}
 \delta_{22}f_{13} = & \frac{1}{7} \left(\left\{ \frac{\langle 1367 \rangle \langle 2347 \rangle}{\langle 1237 \rangle \langle 3467 \rangle} \right\}_2 \wedge \left\{ \frac{\langle 1367 \rangle \langle 2347 \rangle \langle 4567 \rangle}{\langle 1467 \rangle \langle 2367 \rangle \langle 3457 \rangle} \right\}_2 \right. \\
 & - \left\{ \frac{\langle 1247 \rangle \langle 1256 \rangle}{\langle 1245 \rangle \langle 1267 \rangle} \right\}_2 \wedge \left\{ \frac{\langle 1245 \rangle \langle 1567 \rangle}{\langle 1257 \rangle \langle 1456 \rangle} \right\}_2 \\
 & + \left\{ \frac{\langle 1256 \rangle \langle 2345 \rangle}{\langle 1235 \rangle \langle 2456 \rangle} \right\}_2 \wedge \left(\left\{ \frac{\langle 1236 \rangle \langle 1245 \rangle}{\langle 1234 \rangle \langle 1256 \rangle} \right\}_2 - \left\{ \frac{\langle 1235 \rangle \langle 1567 \rangle \langle 2456 \rangle}{\langle 1257 \rangle \langle 1456 \rangle \langle 2356 \rangle} \right\}_2 \right) \\
 & + \left\{ \frac{\langle 1247 \rangle \langle 1345 \rangle}{\langle 1234 \rangle \langle 1457 \rangle} \right\}_2 \wedge \left(\left\{ \frac{\langle 1345 \rangle \langle 1467 \rangle}{\langle 1347 \rangle \langle 1456 \rangle} \right\}_2 - \left\{ \frac{\langle 1245 \rangle \langle 1467 \rangle}{\langle 1247 \rangle \langle 1456 \rangle} \right\}_2 \right) \\
 & + \left(\left\{ \frac{\langle 1247 \rangle \langle 1345 \rangle \langle 1567 \rangle}{\langle 1257 \rangle \langle 1347 \rangle \langle 1456 \rangle} \right\}_2 - \left\{ \frac{\langle 1247 \rangle \langle 1256 \rangle \langle 1345 \rangle}{\langle 1234 \rangle \langle 1257 \rangle \langle 1456 \rangle} \right\}_2 \right) \\
 & \quad \wedge \left(\left\{ -\frac{\langle 1267 \rangle \langle 1345 \rangle}{\langle 1(27)(34)(56) \rangle} \right\}_2 + \left\{ -\frac{\langle 1237 \rangle \langle 1456 \rangle}{\langle 1(27)(34)(56) \rangle} \right\}_2 \right) \\
 & + \left(\left\{ \frac{\langle 1247 \rangle \langle 1256 \rangle \langle 1346 \rangle}{\langle 1234 \rangle \langle 1267 \rangle \langle 1456 \rangle} \right\}_2 - \left\{ \frac{\langle 1237 \rangle \langle 1345 \rangle \langle 1567 \rangle}{\langle 1257 \rangle \langle 1347 \rangle \langle 1356 \rangle} \right\}_2 \right) \wedge \left\{ -\frac{\langle 1234 \rangle \langle 1567 \rangle}{\langle 1(27)(34)(56) \rangle} \right\}_2 \Big) \\
 & + \text{cyclic.}
 \end{aligned}$$

A.2 f_{12}

If we first define the quantity X_1 by

$$\begin{aligned}
 X_1 = & \left\{ \frac{\langle 1367 \rangle \langle 2347 \rangle}{\langle 1237 \rangle \langle 3467 \rangle} \right\}_2 \wedge \left\{ \frac{\langle 1267 \rangle \langle 3467 \rangle}{\langle 1467 \rangle \langle 2367 \rangle} \right\}_2 + \left\{ \frac{\langle 1467 \rangle \langle 2347 \rangle}{\langle 1247 \rangle \langle 3467 \rangle} \right\}_2 \wedge \left\{ \frac{\langle 1347 \rangle \langle 4567 \rangle}{\langle 1467 \rangle \langle 3457 \rangle} \right\}_2 \\
 & - \left\{ \frac{\langle 1247 \rangle \langle 1345 \rangle}{\langle 1234 \rangle \langle 1457 \rangle} \right\}_2 \wedge \left\{ \frac{\langle 1245 \rangle \langle 3457 \rangle}{\langle 1457 \rangle \langle 2345 \rangle} \right\}_2 - \left\{ \frac{\langle 1457 \rangle \langle 2347 \rangle}{\langle 1247 \rangle \langle 3457 \rangle} \right\}_2 \wedge \left\{ \frac{\langle 1347 \rangle \langle 4567 \rangle}{\langle 1467 \rangle \langle 3457 \rangle} \right\}_2 \\
 & + \left\{ \frac{\langle 1256 \rangle \langle 2345 \rangle}{\langle 1235 \rangle \langle 2456 \rangle} \right\}_2 \wedge \left\{ \frac{\langle 1236 \rangle \langle 1245 \rangle \langle 2567 \rangle}{\langle 1235 \rangle \langle 1267 \rangle \langle 2456 \rangle} \right\}_2 - \left\{ \frac{\langle 1267 \rangle \langle 2356 \rangle}{\langle 1236 \rangle \langle 2567 \rangle} \right\}_2 \wedge \left\{ \frac{\langle 1236 \rangle \langle 2345 \rangle \langle 2567 \rangle}{\langle 1235 \rangle \langle 2367 \rangle \langle 2456 \rangle} \right\}_2 \\
 & + \left(\left\{ \frac{\langle 1234 \rangle \langle 1467 \rangle \langle 3457 \rangle}{\langle 1247 \rangle \langle 1345 \rangle \langle 3467 \rangle} \right\}_2 - \left\{ \frac{\langle 1245 \rangle \langle 1467 \rangle \langle 3457 \rangle}{\langle 1247 \rangle \langle 1345 \rangle \langle 4567 \rangle} \right\}_2 \right) \wedge \left\{ -\frac{\langle 1467 \rangle \langle 2345 \rangle}{\langle 4(12)(35)(67) \rangle} \right\}_2 \\
 & + \left\{ \frac{\langle 1467 \rangle \langle 2367 \rangle \langle 2457 \rangle}{\langle 1267 \rangle \langle 2347 \rangle \langle 4567 \rangle} \right\}_2 \wedge \left\{ -\frac{\langle 1237 \rangle \langle 4567 \rangle}{\langle 7(16)(23)(45) \rangle} \right\}_2 \\
 & - \left\{ \frac{\langle 1467 \rangle \langle 2367 \rangle \langle 3457 \rangle}{\langle 1367 \rangle \langle 2347 \rangle \langle 4567 \rangle} \right\}_2 \wedge \left\{ -\frac{\langle 1267 \rangle \langle 3457 \rangle}{\langle 7(16)(23)(45) \rangle} \right\}_2 \\
 & + 2 \left\{ \frac{\langle 1245 \rangle \langle 2467 \rangle \langle 3457 \rangle}{\langle 1247 \rangle \langle 2345 \rangle \langle 4567 \rangle} \right\}_2 \wedge \left\{ -\frac{\langle 1234 \rangle \langle 4567 \rangle}{\langle 4(12)(35)(67) \rangle} \right\}_2
 \end{aligned}$$

and X_2, \dots, X_7 by taking $i \rightarrow i + 1$, then we find

$$\begin{aligned}
 \delta_{22}f_{12} = & \frac{1}{7} (3, -4, 3, -4, 3, -4, 3) \cdot (X_1, X_2, X_3, X_4, X_5, X_6, X_7) \\
 & + \left\{ \frac{\langle 1237 \rangle \langle 1246 \rangle}{\langle 1234 \rangle \langle 1267 \rangle} \right\}_2 \wedge \left(\left\{ \frac{\langle 1246 \rangle \langle 1345 \rangle}{\langle 1234 \rangle \langle 1456 \rangle} \right\}_2 + \left\{ \frac{\langle 1234 \rangle \langle 1467 \rangle \langle 3456 \rangle}{\langle 1246 \rangle \langle 1345 \rangle \langle 3467 \rangle} \right\}_2 + \left\{ \frac{\langle 1467 \rangle \langle 3456 \rangle}{\langle 1346 \rangle \langle 4567 \rangle} \right\}_2 \right. \\
 & + \left. \left\{ \frac{\langle 1246 \rangle \langle 1345 \rangle \langle 4567 \rangle}{\langle 1245 \rangle \langle 1467 \rangle \langle 3456 \rangle} \right\}_2 \right) + \left\{ \frac{\langle 1457 \rangle \langle 3456 \rangle}{\langle 1345 \rangle \langle 4567 \rangle} \right\}_2 \wedge \left(\left\{ \frac{\langle 1234 \rangle \langle 1457 \rangle}{\langle 1247 \rangle \langle 1345 \rangle} \right\}_2 \right. \\
 & + \left. \left\{ \frac{\langle 1234 \rangle \langle 1267 \rangle \langle 1457 \rangle}{\langle 1237 \rangle \langle 1245 \rangle \langle 1467 \rangle} \right\}_2 + \left\{ \frac{\langle 1237 \rangle \langle 1467 \rangle}{\langle 1267 \rangle \langle 1347 \rangle} \right\}_2 + \left\{ \frac{\langle 1237 \rangle \langle 1345 \rangle \langle 1467 \rangle}{\langle 1234 \rangle \langle 1367 \rangle \langle 1457 \rangle} \right\}_2
 \end{aligned}$$

$$\begin{aligned} & \wedge \left\{ -\frac{\langle 1237 \rangle \langle 4567 \rangle}{\langle 7(16)(23)(45) \rangle} \right\}_2 + \left(\left\{ -\frac{\langle 1234 \rangle \langle 3567 \rangle}{\langle 3(17)(24)(56) \rangle} \right\}_2 - \frac{3}{7} \left\{ \frac{\langle 1347 \rangle \langle 1356 \rangle \langle 2346 \rangle}{\langle 1234 \rangle \langle 1367 \rangle \langle 3456 \rangle} \right\}_2 \right. \\ & \left. + \left\{ \frac{\langle 1367 \rangle \langle 2347 \rangle \langle 3456 \rangle}{\langle 1347 \rangle \langle 2346 \rangle \langle 3567 \rangle} \right\}_2 \right) \wedge \left\{ -\frac{\langle 1237 \rangle \langle 3456 \rangle}{\langle 3(17)(24)(56) \rangle} \right\}_2. \end{aligned}$$

A.3 f_{14}

This function lives entirely in PB_1 . If we first define the quantity

$$\begin{aligned} Y = & \left\{ \frac{\langle 2347 \rangle \langle 2356 \rangle}{\langle 2345 \rangle \langle 2367 \rangle} \right\}_2 \wedge \left\{ \frac{\langle 2346 \rangle \langle 3567 \rangle}{\langle 2367 \rangle \langle 3456 \rangle} \right\}_2 + \left\{ \frac{\langle 1367 \rangle \langle 2347 \rangle}{\langle 1237 \rangle \langle 3467 \rangle} \right\}_2 \wedge \left\{ \frac{\langle 2347 \rangle \langle 3567 \rangle}{\langle 2367 \rangle \langle 3457 \rangle} \right\}_2 \\ & + \left\{ \frac{\langle 1257 \rangle \langle 1456 \rangle}{\langle 1245 \rangle \langle 1567 \rangle} \right\}_2 \wedge \left\{ \frac{\langle 1257 \rangle \langle 1456 \rangle \langle 2345 \rangle}{\langle 1235 \rangle \langle 1457 \rangle \langle 2456 \rangle} \right\}_2 - \left\{ \frac{\langle 1367 \rangle \langle 1457 \rangle \langle 2347 \rangle}{\langle 1237 \rangle \langle 1467 \rangle \langle 3457 \rangle} \right\}_2 \wedge \left\{ \frac{\langle 1347 \rangle \langle 4567 \rangle}{\langle 1467 \rangle \langle 3457 \rangle} \right\}_2 \\ & - \left\{ \frac{\langle 1237 \rangle \langle 2356 \rangle}{\langle 1235 \rangle \langle 2367 \rangle} \right\}_2 \wedge \left\{ \frac{\langle 1236 \rangle \langle 2567 \rangle}{\langle 1267 \rangle \langle 2356 \rangle} \right\}_2 - \left\{ \frac{\langle 1256 \rangle \langle 2345 \rangle}{\langle 1235 \rangle \langle 2456 \rangle} \right\}_2 \wedge \left\{ \frac{\langle 1235 \rangle \langle 2567 \rangle}{\langle 1257 \rangle \langle 2356 \rangle} \right\}_2 \\ & + \left\{ \frac{\langle 1257 \rangle \langle 1456 \rangle \langle 2345 \rangle}{\langle 1235 \rangle \langle 1457 \rangle \langle 2456 \rangle} \right\}_2 \wedge \left\{ -\frac{\langle 1235 \rangle \langle 4567 \rangle}{\langle 5(17)(23)(46) \rangle} \right\}_2 \\ & - \left\{ \frac{\langle 1257 \rangle \langle 1456 \rangle \langle 2356 \rangle}{\langle 1235 \rangle \langle 1567 \rangle \langle 2456 \rangle} \right\}_2 \wedge \left\{ -\frac{\langle 1235 \rangle \langle 4567 \rangle}{\langle 5(17)(23)(46) \rangle} \right\}_2 \\ & + \left\{ \frac{\langle 1235 \rangle \langle 2367 \rangle \langle 2457 \rangle}{\langle 1237 \rangle \langle 2345 \rangle \langle 2567 \rangle} \right\}_2 \wedge \left\{ -\frac{\langle 1267 \rangle \langle 2345 \rangle}{\langle 2(13)(45)(67) \rangle} \right\}_2 \\ & - \left\{ \frac{\langle 1367 \rangle \langle 1457 \rangle \langle 2347 \rangle}{\langle 1237 \rangle \langle 1467 \rangle \langle 3457 \rangle} \right\}_2 \wedge \left\{ -\frac{\langle 1567 \rangle \langle 2347 \rangle}{\langle 7(16)(23)(45) \rangle} \right\}_2 \\ & + \left\{ \frac{\langle 1357 \rangle \langle 2347 \rangle \langle 2356 \rangle}{\langle 1237 \rangle \langle 2345 \rangle \langle 3567 \rangle} \right\}_2 \wedge \left\{ -\frac{\langle 1237 \rangle \langle 3456 \rangle}{\langle 3(17)(24)(56) \rangle} \right\}_2 \\ & - \left\{ \frac{\langle 1467 \rangle \langle 2367 \rangle \langle 3457 \rangle}{\langle 1367 \rangle \langle 2347 \rangle \langle 4567 \rangle} \right\}_2 \wedge \left\{ -\frac{\langle 1567 \rangle \langle 2347 \rangle}{\langle 7(16)(23)(45) \rangle} \right\}_2 \end{aligned}$$

then we find

$$\begin{aligned} \delta_{22} f_{14} = & \frac{2}{7} (Y + \text{cyclic}) - 2Y + \left(\left\{ \frac{\langle 1257 \rangle \langle 1456 \rangle}{\langle 1245 \rangle \langle 1567 \rangle} \right\}_2 - \left\{ \frac{\langle 1267 \rangle \langle 2356 \rangle}{\langle 1236 \rangle \langle 2567 \rangle} \right\}_2 - \left\{ \frac{\langle 1257 \rangle \langle 2456 \rangle}{\langle 1245 \rangle \langle 2567 \rangle} \right\}_2 \right. \\ & \left. + \left\{ \frac{\langle 1235 \rangle \langle 1267 \rangle \langle 2456 \rangle}{\langle 1236 \rangle \langle 1245 \rangle \langle 2567 \rangle} \right\}_2 \right) \wedge \left\{ \frac{\langle 1256 \rangle \langle 2345 \rangle}{\langle 1235 \rangle \langle 2456 \rangle} \right\}_2 + \left\{ \frac{\langle 1235 \rangle \langle 2367 \rangle \langle 2456 \rangle}{\langle 1236 \rangle \langle 2345 \rangle \langle 2567 \rangle} \right\}_2 \\ & \wedge \left(\left\{ \frac{\langle 1267 \rangle \langle 2356 \rangle}{\langle 1236 \rangle \langle 2567 \rangle} \right\}_2 - \left\{ -\frac{\langle 1267 \rangle \langle 2345 \rangle}{\langle 2(13)(45)(67) \rangle} \right\}_2 \right) + \left(\left\{ \frac{\langle 1235 \rangle \langle 2367 \rangle \langle 2456 \rangle}{\langle 1236 \rangle \langle 2345 \rangle \langle 2567 \rangle} \right\}_2 \right. \\ & \left. + \left\{ \frac{\langle 1236 \rangle \langle 1245 \rangle \langle 2567 \rangle}{\langle 1235 \rangle \langle 1267 \rangle \langle 2456 \rangle} \right\}_2 \right) \wedge \left\{ -\frac{\langle 1237 \rangle \langle 2456 \rangle}{\langle 2(13)(45)(67) \rangle} \right\}_2 - \left\{ \frac{\langle 1367 \rangle \langle 1457 \rangle \langle 2357 \rangle}{\langle 1237 \rangle \langle 1567 \rangle \langle 3457 \rangle} \right\}_2 \\ & \wedge \left\{ -\frac{\langle 1237 \rangle \langle 4567 \rangle}{\langle 7(16)(23)(45) \rangle} \right\}_2 - \left\{ \frac{\langle 1367 \rangle \langle 2347 \rangle \langle 3456 \rangle}{\langle 1347 \rangle \langle 2346 \rangle \langle 3567 \rangle} \right\}_2 \wedge \left\{ -\frac{\langle 1237 \rangle \langle 3456 \rangle}{\langle 3(17)(24)(56) \rangle} \right\}_2 \\ & + \left(\left\{ \frac{\langle 1367 \rangle \langle 2347 \rangle \langle 2356 \rangle}{\langle 1237 \rangle \langle 2346 \rangle \langle 3567 \rangle} \right\}_2 + \left\{ \frac{\langle 1347 \rangle \langle 2346 \rangle \langle 3567 \rangle}{\langle 1367 \rangle \langle 2347 \rangle \langle 3456 \rangle} \right\}_2 \right) \wedge \left\{ -\frac{\langle 1367 \rangle \langle 2345 \rangle}{\langle 3(17)(24)(56) \rangle} \right\}_2 \\ & - \left\{ \frac{\langle 2346 \rangle \langle 3567 \rangle}{\langle 2367 \rangle \langle 3456 \rangle} \right\}_2 \wedge \left\{ \frac{\langle 1237 \rangle \langle 2346 \rangle \langle 3567 \rangle}{\langle 1367 \rangle \langle 2347 \rangle \langle 2356 \rangle} \right\}_2 - \left(\left\{ \frac{\langle 1457 \rangle \langle 2456 \rangle}{\langle 1245 \rangle \langle 4567 \rangle} \right\}_2 \right. \\ & \left. + \left\{ \frac{\langle 1567 \rangle \langle 2456 \rangle}{\langle 1256 \rangle \langle 4567 \rangle} \right\}_2 \right) \wedge \left\{ \frac{\langle 1257 \rangle \langle 1456 \rangle}{\langle 1245 \rangle \langle 1567 \rangle} \right\}_2 - \left\{ \frac{\langle 1457 \rangle \langle 2357 \rangle \langle 2456 \rangle}{\langle 1257 \rangle \langle 2345 \rangle \langle 4567 \rangle} \right\}_2 \wedge \left\{ -\frac{\langle 1567 \rangle \langle 2345 \rangle}{\langle 5(17)(23)(46) \rangle} \right\}_2 \\ & - \left(\left\{ \frac{\langle 1567 \rangle \langle 2357 \rangle \langle 2456 \rangle}{\langle 1257 \rangle \langle 2356 \rangle \langle 4567 \rangle} \right\}_2 + \left\{ \frac{\langle 1567 \rangle \langle 2357 \rangle \langle 3456 \rangle}{\langle 1357 \rangle \langle 2356 \rangle \langle 4567 \rangle} \right\}_2 \right) \wedge \left\{ -\frac{\langle 1567 \rangle \langle 2345 \rangle}{\langle 5(17)(23)(46) \rangle} \right\}_2 \\ & + \left(\left\{ \frac{\langle 1347 \rangle \langle 1567 \rangle}{\langle 1367 \rangle \langle 1457 \rangle} \right\}_2 + \left\{ \frac{\langle 1567 \rangle \langle 3467 \rangle}{\langle 1367 \rangle \langle 4567 \rangle} \right\}_2 \right) \wedge \left\{ \frac{\langle 1467 \rangle \langle 3457 \rangle}{\langle 1347 \rangle \langle 4567 \rangle} \right\}_2 - \left(\left\{ \frac{\langle 1347 \rangle \langle 2346 \rangle \langle 3567 \rangle}{\langle 1367 \rangle \langle 2347 \rangle \langle 3456 \rangle} \right\}_2 \right. \end{aligned}$$

$$\begin{aligned}
& - \left\{ \frac{\langle 2346 \rangle \langle 3567 \rangle}{\langle 2367 \rangle \langle 3456 \rangle} \right\}_2 + \left\{ \frac{\langle 1347 \rangle \langle 3567 \rangle}{\langle 1367 \rangle \langle 3457 \rangle} \right\}_2 - \left\{ \frac{\langle 1347 \rangle \langle 4567 \rangle}{\langle 1467 \rangle \langle 3457 \rangle} \right\}_2 \Big) \wedge \left\{ \frac{\langle 1237 \rangle \langle 3467 \rangle}{\langle 1367 \rangle \langle 2347 \rangle} \right\}_2 \\
& + \left(\left\{ \frac{\langle 1237 \rangle \langle 1467 \rangle \langle 3457 \rangle}{\langle 1367 \rangle \langle 1457 \rangle \langle 2347 \rangle} \right\}_2 + \left\{ \frac{\langle 1567 \rangle \langle 2367 \rangle \langle 2457 \rangle}{\langle 1267 \rangle \langle 2357 \rangle \langle 4567 \rangle} \right\}_2 + \left\{ \frac{\langle 1567 \rangle \langle 2367 \rangle \langle 3457 \rangle}{\langle 1367 \rangle \langle 2357 \rangle \langle 4567 \rangle} \right\}_2 \right. \\
& \left. + \left\{ \frac{\langle 1367 \rangle \langle 2347 \rangle \langle 4567 \rangle}{\langle 1467 \rangle \langle 2367 \rangle \langle 3457 \rangle} \right\}_2 \right) \wedge \left\{ - \frac{\langle 1237 \rangle \langle 4567 \rangle}{\langle 7(16)(23)(45) \rangle} \right\}_2.
\end{aligned}$$

A.4 B_1

Here we display the non-Stasheff local contributions to the $B_2 \wedge B_2$ coproduct component of the two-loop seven-point NMHV ratio function (4.6). Exceptionally in this formula we make use of the cross-ratios a_{ij} defined in equation (2.1) of [12]. We find that

$$\begin{aligned}
B_1 = & (a_{12} \wedge a_{16}) \wedge (a_{12} \wedge a_{61}) + (a_{12} \wedge a_{16}) \wedge (a_{17} \wedge a_{61}) - (a_{12} \wedge a_{23}) \wedge (a_{12} \wedge a_{61}) \\
& - (a_{12} \wedge a_{23}) \wedge (a_{17} \wedge a_{61}) - (a_{12} \wedge a_{32}) \wedge (a_{12} \wedge a_{61}) - (a_{12} \wedge a_{32}) \wedge (a_{17} \wedge a_{61}) \\
& - (a_{12} \wedge a_{61}) \wedge (a_{13} \wedge a_{16}) + (a_{12} \wedge a_{61}) \wedge (a_{13} \wedge a_{23}) + (a_{12} \wedge a_{61}) \wedge (a_{13} \wedge a_{32}) \\
& - (a_{12} \wedge a_{61}) \wedge (a_{16} \wedge a_{23}) - (a_{12} \wedge a_{61}) \wedge (a_{16} \wedge a_{32}) + (a_{13} \wedge a_{16}) \wedge (a_{17} \wedge a_{61}) \\
& - (a_{13} \wedge a_{23}) \wedge (a_{17} \wedge a_{61}) - (a_{13} \wedge a_{32}) \wedge (a_{17} \wedge a_{61}) + (a_{16} \wedge a_{23}) \wedge (a_{17} \wedge a_{61}) \\
& + (a_{16} \wedge a_{32}) \wedge (a_{17} \wedge a_{61})
\end{aligned}$$

where we follow the slight abuse of notation explained in [15] of writing B_1 not explicitly as an element of $B_2 \wedge B_2$, but rather by writing the result of the iterated coproduct acting on B_1 according to $\{a\}_2 \wedge \{b\}_2 \mapsto (a \wedge (1+a)) \wedge (b \wedge (1+b))$ and then expanding all multiplicative terms out using the usual symbol rules. In other words, the above formula represents the symbol of the function B_1 antisymmetrized according to $a \otimes b \otimes c \otimes d \mapsto (a \wedge b) \wedge (c \wedge d)$.

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