BPS index and 4d $\mathcal{N} = 2$ superconformal field theories

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ABSTRACT: We study the BPS index for the four-dimensional rank-one $\mathcal{N} = 2$ superconformal field theories $H_0, H_1, H_2, E_6, E_7, E_8$. We consider compactifications of the E-string theory on T^2 in which these theories arise as low energy limits. Using this realization we clarify the general structure of the BPS index. The index is characterized by two exponents and a sequence of invariants. We determine the exponents and the first few invariants.

KEYWORDS: Field Theories in Higher Dimensions, Supersymmetric gauge theory, Topological Strings

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1 Introduction and summary

The moduli space of vacua of $\mathcal{N} = 2$ supersymmetric theories in four dimensions often contains singularities where a nontrivial superconformal field theory (SCFT) arises [1]. Nowadays many such 4d $\mathcal{N} = 2$ SCFTs are known, even in the rank-one case alone (see, e.g., [2]). A classic example is the sequence of SCFTs denoted by $H_0, H_1, H_2, D_4, E_6, E_7, E_8$ [3–6], which can be realized by a single D3-brane probing F-theory singularities with constant dilaton [7–12].

In the study of 4d $\mathcal{N} = 2$ supersymmetric theories the BPS index is of crucial importance as it captures the details of the exact quantum spectrum. For $U(n)$ gauge theories the BPS index can be expressed in a very concise, explicit form known as the Nekrasov partition function [13, 14].¹ It is natural to ask how the BPS index looks near the SCFT singularities on the moduli space of supersymmetric theory. The answer is not obvious, even in the case of a gauge theory whose Nekrasov partition function is explicitly known. This is because the Nekrasov partition function is given by the instanton expansion about the classical singularity of the moduli space, but the expansion is no longer valid at the SCFT singularities. Some sort of analytic continuation is required.

¹In this paper the term ‘Nekrasov partition function’ means the explicit expression (typically given as a sum over partitions), while the term ‘BPS index’ abstractly denotes the observable that can, in certain cases, be expressed as the Nekrasov partition function. For 5d gauge theories on $\mathbb{R}^4 \times S^1$ the BPS index is defined as a generalized supersymmetric index [13]. For 4d gauge theories it can be defined as the partition function given by the path integral in the Omega background [14]. For the SCFTs studied in this paper we do not know any explicit, direct definition of the BPS index. It can, however, be defined indirectly by means of theories which flow to the SCFTs and one can unambiguously compute it (at least as a series expansion).

In this paper we focus on the 4d rank-one $\mathcal{N} = 2$ SCFTs H_n, E_{8-n} ($n = 0, 1, 2$) and study the BPS index. (The index for the D_4 SCFT can be computed from the Nekrasov partition function by using the modular properties of the index [15].) These SCFTs are realized in many ways. For example, the H_n SCFTs can be studied by means of the original realization in $SU(2)$ super Yang-Mills [3]. Analysis of the BPS index along these lines was carried out [16]. In this paper we study the SCFTs by means of toroidal compactifications of the 6d E-string theory [17–23]. This approach has the following advantages.

Firstly, one can make full use of the known results of the well-studied BPS index of E-strings. It has several entirely different interpretations, providing us with complementary ways to compute it. In particular, the E-string theory has a world-sheet description, which enables us to compute the index as the generating function for the sequence of elliptic genera of multiple E-strings [23]. Secondly, in this approach one can study the above SCFTs (including the D_4 case [24]) in a unified manner. The E-string theory encompasses almost all 5d $\mathcal{N} = 1$ and 4d $\mathcal{N} = 2$ rank-one gauge theories [20, 22]. All the above SCFTs arise in the moduli space of this single theory.

We clarify the general structure of the BPS index. For the sake of simplicity we consider the unrefined case, where the chemical potentials ϵ_1, ϵ_2 for the Lorentz spins are fixed as $\epsilon_1 = -\epsilon_2 =: \hbar$. Our main results are summarized as follows. The BPS index for the type $\mathfrak{g} = E_{8-n}, H_n$ ($n = 0, 1, 2$) SCFTs takes the form

$$Z^{\mathfrak{g}}(\phi, \hbar) = \exp(\beta^{\mathfrak{g}} \hbar^2 \partial_{\phi}^2) \left[\phi^{-\gamma^{\mathfrak{g}}} \exp \sum_{k=1}^{\infty} c_k^{\mathfrak{g}} \left(\frac{\hbar}{\phi} \right)^{m^{\mathfrak{g}} k} \right], \quad (1.1)$$

where ϕ is the Higgs expectation value and

$$\beta^{E_{8-n}} = \beta^{H_n} = \frac{3^{(n-1)/2}}{4\pi}, \quad (1.2)$$

$$\gamma^{E_{8-n}} = \frac{1}{2} \left(\frac{12}{n+2} - 1 \right), \quad \gamma^{H_n} = \frac{1}{2} \left(\frac{12}{n+2} - 1 \right)^{-1}, \quad (1.3)$$

$$m^{E_{8-n}} = m^{H_n} = 4 + 2|n-1|. \quad (1.4)$$

The differential operator is introduced so that the ‘descendants’ that are determined by the modular anomaly equation are concealed from view. Put in this form the BPS index is characterized by two exponents $\beta^{\mathfrak{g}}, \gamma^{\mathfrak{g}}$ and a sequence of invariants $c_k^{\mathfrak{g}}$. In this paper we determine $\beta^{\mathfrak{g}}, \gamma^{\mathfrak{g}}$ (as above), invariants $c_k^{E_8}, c_k^{H_0}$ ($k = 1, 2, 3, 4$) and $c_1^{E_7}, c_1^{H_1}$ by using the known results of the BPS index for the E-string theory. The values of γ^{H_n} have been known [16] and our results are in agreement with them.

Our method can easily be generalized to the case with general ϵ_1, ϵ_2 . Another interesting generalization of this work is to turn on the chemical potentials for the other global symmetry charges. It would also be interesting to clarify how our results are related to the superconformal index for the SCFTs [25–30].

The rest of the paper is organized as follows. In section 2 we review the definition and the basic properties of the BPS index of E-strings. In section 3 we consider the setting of the E-string theory on $\mathbb{R}^4 \times T^2$ that realizes $E_8 \oplus H_0$ singularities on the moduli space. We

first study the index of E-strings in this setting and then take limits to obtain the index for the E_8 and H_0 SCFTs. Section 4 and section 5 are devoted to the $E_7 \oplus H_1$ case and the $E_6 \oplus H_2$ case respectively.

2 Review of BPS index of E-strings

The BPS index of E-strings is defined as the 5d BPS index [13] for the E-string theory on $\mathbb{R}^5 \times S^1$. It is given by a trace over the space of the BPS particles as follows:

$$Z(\phi, \tau, \mathbf{m}, \epsilon_1, \epsilon_2) := \text{Tr}(-1)^{2J_L+2J_R} y_L^{J_L} y_R^{J_R+J_I} p^n q^k e^{i\mathbf{\Lambda} \cdot \mathbf{m}}, \quad (2.1)$$

where

$$y_L := e^{i(\epsilon_1 - \epsilon_2)}, \quad y_R := e^{i(\epsilon_1 + \epsilon_2)}, \quad p := e^{-\phi}, \quad q := e^{2\pi i \tau}. \quad (2.2)$$

Here J_L, J_R, J_I and $\mathbf{\Lambda} = (\Lambda_1, \dots, \Lambda_8)$ are spins (or weights of the associated Lie algebras) of the little group $\text{SO}(4) = \text{SU}(2)_L \times \text{SU}(2)_R$, the R-symmetry group $\text{SU}(2)_I$ and the global symmetry group E_8 respectively. Nonnegative integers n, k are respectively the winding number and the momentum along S^1 . Z is a function in twelve variables. ϕ is the tension of the E-strings and τ is proportional to the inverse of the radius of S^1 . In the Seiberg-Witten description [4, 31] of the E-string theory on $\mathbb{R}^4 \times T^2$, ϕ is interpreted as the Higgs expectation value of the $\text{U}(1)$ vector multiplet and τ is the complex structure of T^2 . $\mathbf{m} = (m_1, \dots, m_8)$ and ϵ_1, ϵ_2 are respectively the Wilson line parameters or the chemical potentials for the global symmetries E_8 and $\text{SO}(4)$. Throughout this paper we consider the unrefined case $\epsilon_1 = -\epsilon_2 =: \hbar$.

The index Z is interpreted in several ways. One interpretation is associated with the expansion

$$Z = 1 + \sum_{n=1}^{\infty} p^n Z_n, \quad (2.3)$$

where Z_n is the elliptic genus of n E-strings. Z_n with any n can in principle be computed by using the localization technique [23]. Explicit forms of Z_n ($n \leq 4$) were obtained in [23]. Another useful interpretation is

$$Z = \exp \sum_{g=0}^{\infty} \hbar^{2g-2} F_g, \quad (2.4)$$

where F_g is the genus- g topological string amplitude for the local $\frac{1}{2}\text{K3}$. F_g ($g \leq 3$) with general $\mathbf{m}$ were computed in [32], which we will use mainly in this paper. A third interpretation relates Z with the partition function of a certain five-brane web system [33]. This picture enables us to compute Z as a power series expansion in q .

The index Z satisfies two important constraints. One is known as the modular anomaly equation [34]

$$\partial_{E_2} Z = \frac{1}{24} \hbar^2 \partial_{\phi} (\partial_{\phi} - 1) Z, \quad (2.5)$$

where ∂_{E_2} is the formal derivative in Eisenstein series $E_2(\tau)$. The other is the gap condition

$$\ln Z = \sum_{n=1}^{\infty} p^n \left(\frac{1}{n \left(2 \sin \frac{n\hbar}{2} \right)^2} + \mathcal{O}(q^n) \right). \quad (2.6)$$

This follows from the geometric structure of the local $\frac{1}{2}$ K3 [21].

3 $E_8 \oplus H_0$ case

3.1 Seiberg-Witten curve

Let us first consider the E-string theory on $\mathbb{R}^4 \times T^2$ without E_8 Wilson lines, i.e.

$$\mathbf{m} = \mathbf{0}. \quad (3.1)$$

The low energy effective theory is characterized by the following elliptic Seiberg-Witten curve [20, 22]

$$y^2 = 4x^3 - \frac{1}{12}E_4(\tau)u^4x - \frac{1}{216}E_6(\tau)u^6 + 4u^5. \quad (3.2)$$

Here u parametrizes the Coulomb branch of the moduli space of vacua. In this section we focus on the case where the complex structure of the torus is fixed to the special value

$$\tau = \tau_* := e^{2\pi i/3}. \quad (3.3)$$

This is a value for which Eisenstein series $E_4(\tau)$ vanishes

$$E_{4*} := E_4(\tau_*) = 0. \quad (3.4)$$

We let subscript $*$ denote that the quantity is evaluated at $\tau = \tau_*$. For example,

$$q_* = e^{2\pi i\tau_*} = -e^{-\pi\sqrt{3}}. \quad (3.5)$$

Other Eisenstein series take the following values (see, e.g., [35])

$$E_{2*} = \frac{2\sqrt{3}}{\pi}, \quad E_{6*} = 216 \frac{\Gamma(1/3)^{18}}{(2\pi)^{12}}. \quad (3.6)$$

The Seiberg-Witten curve (3.2) becomes

$$y^2 = 4x^3 - \frac{1}{216}E_{6*}u^6 + 4u^5 \quad (3.7)$$

$$= 4x^3 - \frac{\Gamma(1/3)^{18}}{(2\pi)^{12}}u^6 + 4u^5. \quad (3.8)$$

Because of the absence of the linear term in x , the j -invariant of this elliptic curve is identically 0. Therefore the complex structure of the curve takes the constant value τ_* over the moduli space. The discriminant of the curve has zeros at $u = 0$ and $u = 864/E_{6*}$, where the elliptic fibration has singular fibers of Kodaira type II^* and II respectively. This means that near the point at $u = 0$ the theory looks like the E_8 SCFT while near the point at $u = 864/E_{6*}$ the theory looks like the H_0 SCFT [20]. In the following, we will study how the BPS index looks near these SCFT points.

3.2 Mirror map

In this subsection we establish the mirror map which connects the global moduli space coordinate u with the scalar expectation value ϕ on local patches of the moduli space. We will see that the mirror map reduces to a very simple form in the present case with $\tau = \tau_*$. This is similar to what was observed in the $D_4 \oplus D_4$ case [24].

Since the mirror map described below is merely a special case of the general one [22], we will summarize the main points only. The reader is referred for the details to [32]. Note that the variable ϕ used in [32] should not be confused with ϕ in this paper. They are related to each other by

$$\phi_{\text{there}} = -\phi_{\text{here}} + \ln \left(-q \prod_{k=1}^{\infty} (1 - q^k)^{12} \right). \quad (3.9)$$

In our present case with $\mathbf{m} = \mathbf{0}$ and $\tau = \tau_*$, one of the periods (divided by 2π) of the Seiberg-Witten curve is written as

$$\omega = \frac{1}{u \left(1 - \frac{864}{E_{6*}u} \right)^{1/6}} = \frac{E_{6*} z^6 - 1}{864 z^5}. \quad (3.10)$$

Here z is a new coordinate of the moduli space defined by

$$z := \left(1 - \frac{864}{E_{6*}u} \right)^{-1/6}. \quad (3.11)$$

The singularities corresponding to the E_8 and H_0 SCFTs are mapped to $z = 0$ and $z = \infty$ respectively. The Higgs expectation value is given by

$$\begin{aligned} \phi &= \int \omega du \\ &= 6 \int \frac{dz}{1 - z^6} \end{aligned} \quad (3.12)$$

$$= \ln \frac{1+z}{1-z} + \frac{1}{2} \ln \frac{1+z+z^2}{1-z+z^2} + \sqrt{3} \arctan \left(\frac{\sqrt{3}z}{1-z^2} \right) + \phi_0^{E_8}. \quad (3.13)$$

Here $\phi_0^{E_8}$ is an integration constant. Near the E_8 singularity at $z = 0$, ϕ is expanded as

$$\phi - \phi_0^{E_8} = 6 \sum_{k=0}^{\infty} \frac{z^{6k+1}}{6k+1} = 6 \left(z + \frac{z^7}{7} + \frac{z^{13}}{13} + \cdots \right). \quad (3.14)$$

Near the H_0 singularity at $z = \infty$, on the other hand, ϕ is expanded as

$$\phi - \phi_0^{H_0} = 6 \sum_{k=0}^{\infty} \frac{1}{(6k+5)z^{6k+5}} = 6 \left(\frac{1}{5z^5} + \frac{1}{11z^{11}} + \frac{1}{17z^{17}} + \cdots \right). \quad (3.15)$$

It is possible to determine the values of the integration constants $\phi_0^{E_8}$ and $\phi_0^{H_0}$ with respect to the convention of [32], but for our purposes their values are not important.

3.3 6d amplitudes

Let us now consider the BPS index of E-strings in the present setting

$$Z^{E_8 \oplus H_0}(\phi, \hbar) := Z(\phi, \tau = \tau_*, \mathbf{m} = \mathbf{0}, \epsilon_1 = \hbar, \epsilon_2 = -\hbar). \quad (3.16)$$

We will study it mainly in the expansion of the form

$$\ln Z^{E_8 \oplus H_0} = \sum_{g=0}^{\infty} \hbar^{2g-2} F_g^{E_8 \oplus H_0}(\phi). \quad (3.17)$$

By using the modular anomaly equation (2.5), the gap condition (2.6), the known forms of F_g ($g \leq 3$) [32] and elliptic genera Z_n ($n \leq 4$) [23], we are able to determine $F_g^{E_8 \oplus H_0}$ for $g \leq 15$. The first few amplitudes are as follows:

$$F_0^{E_8 \oplus H_0} = 0, \quad (3.18)$$

$$\begin{aligned} F_1^{E_8 \oplus H_0} &= \frac{1}{2} \ln \omega + \frac{\phi}{2} - \frac{1}{2} \ln \left(-q_* \prod_{k=1}^{\infty} (1 - q_*^k)^{12} \right) \\ &= \frac{1}{2} \ln \frac{z^6 - 1}{z^5} + \frac{\phi}{2} + \frac{\sqrt{3}}{4} \pi - \ln 2 - \frac{3}{4} \ln 3, \end{aligned} \quad (3.19)$$

$$F_2^{E_8 \oplus H_0} = \left(\frac{35 - 10z^6 + 11z^{12}}{3456z^2} - \frac{1}{96} \right) E_{2*}, \quad (3.20)$$

$$F_3^{E_8 \oplus H_0} = \frac{5(1 - z^6)^2(14 + 23z^6 + 44z^{12})}{746496z^4} E_{2*}^2, \quad (3.21)$$

$$\begin{aligned} F_4^{E_8 \oplus H_0} &= \frac{(1 - z^6)^2}{z^6} \left[\frac{5(2485 + 3128z^6 + 3246z^{12} - 31000z^{18} + 92125z^{24})}{7739670528} E_{2*}^3 \right. \\ &\quad \left. + \frac{12625 - 34792z^6 + 632886z^{12} - 2352376z^{18} + 2208217z^{24}}{38698352640} E_{6*} \right]. \end{aligned} \quad (3.22)$$

The rest of the results are rather lengthy and thus we do not present them here. Instead, in what follows we clarify the structure of $F_g^{E_8 \oplus H_0}$ with general g .

First, $F_g^{E_8 \oplus H_0}$ ($g \geq 3$) takes the following form

$$F_g^{E_8 \oplus H_0} = \frac{(1 - z^6)^2}{z^{2g-2}} \sum_{k=0}^{\lfloor (g-1)/3 \rfloor} E_{2*}^{g-1-3k} E_{6*}^k \sum_{i=0}^{2g-4} c_{g,k,i} z^{6i}, \quad (3.23)$$

where $c_{g,k,i}$ are some numerical coefficients. Next, one can easily see that $F_g^{E_8 \oplus H_0}$ with $g = 3n + 2, 3n + 3$ ($n \in \mathbb{Z}_{\geq 0}$) are completely determined by the modular anomaly equation, given the data of $F_g^{E_8 \oplus H_0}$ ($g \leq 3n + 1$). Therefore all the essential data are provided by $F_g^{E_8 \oplus H_0}$ with $g = 3n + 1$ ($n \in \mathbb{Z}_{\geq 0}$). Furthermore, apart from the $n = 0$ case, they are written as

$$F_{3n+1}^{E_8 \oplus H_0} = \frac{(1 - z^6)^2}{z^{6n}} \sum_{k=0}^n E_{2*}^{3(n-k)} E_{6*}^k \sum_{i=0}^{6n-2} c_{3n+1,k,i} z^{6i} \quad (3.24)$$

with unknowns $c_{3n+1,k,i}$, but except for $k = n$ they are again determined by the modular anomaly equation, given the data of $F_{3m+1}^{E_8 \oplus H_0}$ ($m < n$). To sum up, the data of $F_1^{E_8 \oplus H_0}$ and $c_{3n+1,n,i}$ ($n \in \mathbb{Z}_{>0}$, $i = 0, \dots, 6n - 2$) completely determine the BPS index.

Before closing this subsection let us sketch out a proof of the general form (3.23). First, it is well-known that topological string amplitudes for Calabi-Yau threefolds are polynomials in a finite number of generators [36, 37]. The generators for the most general BPS index of E-strings were identified [32]. By reducing the results to the present case, one sees that $F_g^{E_8 \oplus H_0}$ ($g \geq 2$) are polynomials in $\partial_\phi^k \ln \omega$ ($k = 1, 2, 3$) and in E_{2*}, E_{6*} . Moreover, $F_g^{E_8 \oplus H_0}$ ($g \geq 3$) is of degree $2g - 2$ and weight $2g - 2$, where we assign degree k to $\partial_\phi^k \ln \omega$ and weight $2n$ to E_{2n*} . It then follows that $F_g^{E_8 \oplus H_0}$ ($g \geq 3$), as a function in z , is written as

$$F_g^{E_8 \oplus H_0} = \frac{f_{2g-2}(z^6)}{z^{2g-2}}, \quad (3.25)$$

with $f_{2g-2}(x)$ being a degree $2g - 2$ polynomial function. Next, let us show that $F_g^{E_8 \oplus H_0}$ ($g \geq 3$) contains the factor $(1 - z^6)^2$. We first note that the elliptic genus of a single E-string in the present case vanishes

$$Z_1^{E_8 \oplus H_0} = q_*^{1/2} \frac{E_{4*}}{\eta(\tau_*)^6 \vartheta_1(\hbar, \tau_*)^2} = 0. \quad (3.26)$$

In terms of $F_g^{E_8 \oplus H_0}$ this means that

$$F_g^{E_8 \oplus H_0} = \mathcal{O}(p^2) \quad (3.27)$$

in the power series expansion in $p = e^{-\phi}$. Since the variable z is expanded in p as

$$z = 1 + 2\sqrt{3}e^{-\pi\sqrt{3}/2}p + 30e^{-\pi\sqrt{3}}p^2 + \mathcal{O}(p^3), \quad (3.28)$$

$F_g^{E_8 \oplus H_0}$ ($g \geq 2$) has to contain the factor $(1 - z)^2$ in order to satisfy (3.27). This, combined with (3.25), means that $F_g^{E_8 \oplus H_0}$ ($g \geq 3$) has to contain the factor $(1 - z^6)^2$.

3.4 4d limits

We have so far studied 6d theory on $\mathbb{R}^4 \times T^2$. Let us now consider scaling limits in which the torus T^2 shrinks to zero size. To do this we first recover the length scale R , which is proportional to the radii of the T^2 . We then take the limit of $R \rightarrow 0$ while keeping the complex structure of the T^2 to be $\tau = \tau_*$. In the low energy Seiberg-Witten description, this procedure corresponds to zooming in on a point in the moduli space. In particular, by suitably zooming in on the singularity at $z = 0$ ($z = \infty$), one obtains the $E_8(H_0)$ SCFT.

Let us first consider the scaling limit in which only the local structure of the E_8 singularity at $z = 0$ contributes to the physics. This is achieved by first replacing the variables as

$$\phi - \phi_0^{E_8} \rightarrow R\phi, \quad z \rightarrow Rz, \quad \hbar \rightarrow R\hbar \quad (3.29)$$

and then taking the limit of $R \rightarrow 0$. In this limit a function in z is dominated by the leading order term of its expansion in z . The mirror map (3.14) is simplified as

$$\phi = 6z. \quad (3.30)$$

The BPS index for the 4d E_8 SCFT is obtained from that for the E-string theory as

$$Z^{E_8}(\phi, \hbar) = (\text{const.}) \lim_{R \rightarrow 0} Z^{E_8 \oplus H_0} \left(R\phi + \phi_0^{E_8}, R\hbar \right) \Big|_{z \sim 0}. \quad (3.31)$$

The normalization constant is fixed as follows. Z^{E_8} is expanded in $\hbar$ as

$$\ln Z^{E_8} = \sum_{g=0}^{\infty} F_g^{E_8} \hbar^{2g-2} \quad (3.32)$$

with

$$\begin{aligned} F_0^{E_8} &= 0, & F_1^{E_8} &= -\frac{5}{2} \ln \phi, & F_2^{E_8} &= \frac{35E_{2*}}{96} \frac{1}{\phi^2}, & F_3^{E_8} &= \frac{35E_{2*}^2}{288} \frac{1}{\phi^4}, \\ F_4^{E_8} &= \frac{25(497E_{2*}^3 + 101E_{6*})}{165888} \frac{1}{\phi^6}, & \dots \end{aligned} \quad (3.33)$$

We fix the normalization of Z^{E_8} so that $F_1^{E_8}$ takes the above simple form without constant term. Note also that the form of $F_0^{E_8}$ is not characteristic of the SCFT itself, but rather of how it is embedded in the bulk theory. The explicit forms of $F_g^{E_8}$ ($g \leq 15$) are immediately obtained from the results of the last subsection. Instead of listing them all, we will present the results in a very concise form, taking account of the fact that the BPS index satisfies the modular anomaly equation.

For Z that has the structure (2.3), one can show that the modular anomaly equation (2.5) is formally solved as

$$Z = \exp\left(\frac{\phi}{2} - \frac{E_2}{96} \hbar^2\right) \exp\left(\frac{E_2}{24} \hbar^2 \partial_\phi^2\right) \tilde{Z}, \quad (3.34)$$

where $\tilde{Z}$ is independent of E_2 . After we set $\tau = \tau_*$, E_2 becomes a numerical constant and the modular anomaly equation does not make sense, but the structure (3.34) remains intact in the index. Note that the first prefactor in (3.34) becomes trivial when we take the 4d limit.

By exploiting this fact, the results of F_g ($g \leq 15$) are packed into the following concise expression:

$$Z^{E_8} = \exp\left(\frac{E_{2*}}{24} \hbar^2 \partial_\phi^2\right) \left[\phi^{-5/2} \exp \sum_{k=1}^{\infty} c_k^{E_8} \left(\frac{\hbar}{\phi}\right)^{6k} \right], \quad (3.35)$$

with

$$\begin{aligned} c_1^{E_8} &= 2525 \left(\frac{E_{6*}}{2^{11} \cdot 3^4} \right), \\ c_2^{E_8} &= 1941020160 \left(\frac{E_{6*}}{2^{11} \cdot 3^4} \right)^2, \\ c_3^{E_8} &= 26440099120581792 \left(\frac{E_{6*}}{2^{11} \cdot 3^4} \right)^3, \\ c_4^{E_8} &= 2512057097259272539155456 \left(\frac{E_{6*}}{2^{11} \cdot 3^4} \right)^4. \end{aligned} \quad (3.36)$$

Similarly, one can compute Z^{H_0} by zooming in on the H_0 singularity at $z = \infty$. This is done by first replacing the variables as

$$\phi - \phi_0^{H_0} \rightarrow R\phi, \quad z \rightarrow R^{-1/5}z, \quad \hbar \rightarrow R\hbar \quad (3.37)$$

and then taking the limit of $R \rightarrow 0$. The mirror map (3.15) becomes

$$\phi = \frac{6}{5z^5}. \quad (3.38)$$

The BPS index for the H_0 SCFT is obtained as

$$Z^{H_0}(\phi, \hbar) = (\text{const.}) \lim_{R \rightarrow 0} Z^{E_8 \oplus H_0} \left(R\phi + \phi_0^{H_0}, R\hbar \right) \Big|_{z \sim \infty}. \quad (3.39)$$

The results are given as follows:

$$Z^{H_0} = \exp \left(\frac{E_{2*}}{24} \hbar^2 \partial_\phi^2 \right) \left[\phi^{-1/10} \exp \sum_{k=1}^{\infty} c_k^{H_0} \left(\frac{\hbar}{\phi} \right)^{6k} \right], \quad (3.40)$$

with

$$\begin{aligned} c_1^{H_0} &= 2208217 \left(\frac{E_{6*}}{2^{11} \cdot 3^4 \cdot 5^7} \right), \\ c_2^{H_0} &= 85679149172703360 \left(\frac{E_{6*}}{2^{11} \cdot 3^4 \cdot 5^7} \right)^2, \\ c_3^{H_0} &= 76522745976613844587093143840 \left(\frac{E_{6*}}{2^{11} \cdot 3^4 \cdot 5^7} \right)^3, \\ c_4^{H_0} &= 516442282343494619890254623170666086528000 \left(\frac{E_{6*}}{2^{11} \cdot 3^4 \cdot 5^7} \right)^4. \end{aligned} \quad (3.41)$$

4 $E_7 \oplus H_1$ case

In this section we consider the compactification of the E-string theory that realizes the E_7 and H_1 SCFTs and compute the BPS index. Let us first recall the $E_7 \oplus A_1$ reduction of the global symmetry E_8 of the E-string theory. This is achieved by setting the Wilson line parameters as [38]

$$\mathbf{m} = (0, 0, 0, 0, 0, 0, \pi, \pi). \quad (4.1)$$

In this case the general Seiberg-Witten curve [22, 32] reduces to [39]

$$y^2 = 4x^3 + (\vartheta_3^4 + \vartheta_4^4) u^2 x^2 + \left(\frac{\vartheta_3^4 \vartheta_4^4}{4} u - \frac{16}{\vartheta_3^2 \vartheta_4^2} \right) u^3 x. \quad (4.2)$$

Here $\vartheta_j \equiv \vartheta_j(0, \tau)$. The E_7 singularity has already realized in this setting. To obtain the H_1 singularity and make the complex structure of the curve constant, we set the complex structure of T^2 as

$$\tau = \tau_{\sharp} := \frac{1+i}{2}. \quad (4.3)$$

This value is connected with $\tau = i$ by an $\text{SL}(2, \mathbb{Z})$ transformation.² As in the last section, subscript $\sharp$ denotes that the quantity is evaluated at $\tau = \tau_\sharp$. We see that

$$q_\sharp = -e^{-\pi}, \quad (4.4)$$

$$E_{2\sharp} = \frac{6}{\pi}, \quad E_{4\sharp} = -12 \frac{\Gamma(1/4)^8}{(2\pi)^6}, \quad E_{6\sharp} = 0, \quad (4.5)$$

$$\vartheta_{3\sharp}^4 + \vartheta_{4\sharp}^4 = 0, \quad \vartheta_{3\sharp}^2 \vartheta_{4\sharp}^2 = 2 \frac{\Gamma(1/4)^4}{(2\pi)^3}. \quad (4.6)$$

The curve (4.2) then becomes

$$y^2 = 4x^3 + \left(\frac{\Gamma(1/4)^8}{(2\pi)^6} u - 8 \frac{(2\pi)^3}{\Gamma(1/4)^4} \right) u^3 x. \quad (4.7)$$

The mirror map is given as follows. The period is

$$\omega = \frac{1}{u \left(1 - \frac{64}{\vartheta_{3\sharp}^6 \vartheta_{4\sharp}^6 u} \right)^{1/4}} = \frac{\vartheta_{3\sharp}^6 \vartheta_{4\sharp}^6}{64} \frac{z^4 - 1}{z^3}, \quad (4.8)$$

where this time we have defined z as

$$z := \left(1 - \frac{64}{\vartheta_{3\sharp}^6 \vartheta_{4\sharp}^6 u} \right)^{-1/4}. \quad (4.9)$$

The Higgs expectation value is given by

$$\begin{aligned} \phi &= \int \omega du \\ &= 4 \int \frac{dz}{1 - z^4} \end{aligned} \quad (4.10)$$

$$= 2 \arctan z + 2 \operatorname{arctanh} z + \phi_0^{E_7}. \quad (4.11)$$

Near the E_7 singularity at $z = 0$, ϕ is expanded as

$$\phi - \phi_0^{E_7} = 4 \sum_{k=0}^{\infty} \frac{z^{4k+1}}{4k+1} = 4 \left(z + \frac{z^5}{5} + \frac{z^9}{9} + \cdots \right). \quad (4.12)$$

Near the H_1 singularity at $z = \infty$, on the other hand, ϕ is expanded as

$$\phi - \phi_0^{H_1} = 4 \sum_{k=0}^{\infty} \frac{1}{(4k+3)z^{4k+3}} = 4 \left(\frac{1}{3z^3} + \frac{1}{7z^7} + \frac{1}{11z^{11}} + \cdots \right). \quad (4.13)$$

²We could study the present case by adopting the value $\tau = i$. This gives rise to different values of $E_{2n}(\tau)$. In this case, however, the value of $\mathbf{m}$ which gives the Seiberg-Witten curve with E_7 and H_1 degenerations also changes. (It changes to the modular S-transform of (4.1) [38].) Accordingly the form of the curve (4.2) and the normalization of ϕ are modified, which compensates the changes of $E_{2n}(\tau)$ and eventually leads us to the same final results.

Using the explicit forms of F_g ($g \leq 3$) [32], one can immediately compute $F_g^{E_7 \oplus H_1}$ ($g \leq 3$). The results are

$$F_0^{E_7 \oplus H_1} = 0, \quad (4.14)$$

$$\begin{aligned} F_1^{E_7 \oplus H_1} &= \frac{1}{2} \ln \omega + \frac{1}{2} \phi - \frac{1}{2} \ln \left(-q_{\sharp} \prod_{k=1}^{\infty} (1 - q_{\sharp}^k)^{12} \right) \\ &= \frac{1}{2} \ln \frac{z^4 - 1}{z^3} + \frac{1}{2} \phi + \frac{\pi}{4} - \frac{3}{2} \ln 2, \end{aligned} \quad (4.15)$$

$$F_2^{E_7 \oplus H_1} = \left(\frac{15 - 6z^4 + 7z^8}{1536z^2} - \frac{1}{96} \right) E_{2\sharp}, \quad (4.16)$$

$$F_3^{E_7 \oplus H_1} = \frac{(1 - z^4)^2}{z^4} \left[\frac{15 + 22z^4 + 35z^8}{98304} E_{2\sharp}^2 + \frac{9 - 11z^4 + 56z^8}{221184} E_{4\sharp} \right]. \quad (4.17)$$

By taking limits as in the last section, one obtains the BPS index for the E_7 and H_1 SCFTs as

$$Z^{E_7} = \exp \left(\frac{E_{2\sharp}}{24} \hbar^2 \partial_{\phi}^2 \right) \left[\phi^{-3/2} \exp \sum_{k=1}^{\infty} c_k^{E_7} \left(\frac{\hbar}{\phi} \right)^{4k} \right], \quad (4.18)$$

$$Z^{H_1} = \exp \left(\frac{E_{2\sharp}}{24} \hbar^2 \partial_{\phi}^2 \right) \left[\phi^{-1/6} \exp \sum_{k=1}^{\infty} c_k^{H_1} \left(\frac{\hbar}{\phi} \right)^{4k} \right] \quad (4.19)$$

with

$$c_1^{E_7} = \frac{E_{4\sharp}}{2^5 \cdot 3}, \quad c_1^{H_1} = \frac{7E_{4\sharp}}{2^2 \cdot 3^7}. \quad (4.20)$$

To determine the invariants $c_k^{E_7}, c_k^{H_1}$ for $k \geq 2$, we need more data of F_g or Z_n . We expect that $c_k^{E_7}, c_k^{H_1}$ for general k are given by $E_{4\sharp}^k$ multiplied by some rational numbers, as in the case of the E_8 and H_0 SCFTs.

5 $E_6 \oplus H_2$ case

In this section we consider the compactification of the E-string theory that realizes the E_6 and H_2 SCFTs and compute the BPS index. Let us first recall the $E_6 \oplus A_2$ reduction of the global symmetry E_8 of the E-string theory. This is achieved by setting the Wilson line parameters as [38]

$$\mathbf{m} = \left(0, 0, 0, 0, 0, \frac{4\pi}{3}, \frac{4\pi}{3}, \frac{4\pi}{3} \right). \quad (5.1)$$

The general Seiberg-Witten curve [22, 32] reduces to [39]

$$y^2 = 4x^3 + 3\alpha_3^2 u^2 x^2 + \frac{2}{3} \alpha_3 \left(\beta_3 u - \frac{27}{\beta_3} \right) u^3 x + \frac{1}{27} \left(\beta_3 u - \frac{27}{\beta_3} \right)^2 u^4, \quad (5.2)$$

where

$$\begin{aligned} \alpha_3 &:= \sum_{(m,n) \in \mathbb{Z}^2} q^{m^2 + n^2 - mn} = \vartheta_3(0, 2\tau) \vartheta_3(0, 6\tau) + \vartheta_2(0, 2\tau) \vartheta_2(0, 6\tau), \\ \beta_3 &:= \frac{\eta(\tau)^9}{\eta(3\tau)^3}. \end{aligned} \quad (5.3)$$

The E_6 singularity has already realized in this setting. To obtain the H_2 singularity and make the complex structure of the curve constant, we set the complex structure of T^2 as

$$\tau_b = \frac{1}{2} + \frac{i}{2\sqrt{3}}. \quad (5.4)$$

This value is connected with $\tau = e^{2\pi i/3}$ by an $\text{SL}(2, \mathbb{Z})$ transformation. We let subscript b denote that the quantity is evaluated at $\tau = \tau_b$. We see that

$$q_b = -e^{-\pi/\sqrt{3}}, \quad (5.5)$$

$$E_{2b} = \frac{6\sqrt{3}}{\pi}, \quad E_{4b} = 0, \quad E_{6b} = -2^3 \cdot 3^6 \frac{\Gamma(1/3)^{18}}{(2\pi)^{12}}, \quad (5.6)$$

$$\alpha_{3b} = 0, \quad \beta_{3b} = 27 \frac{\Gamma(1/3)^9}{(2\pi)^6}. \quad (5.7)$$

The curve (5.2) then becomes

$$y^2 = 4x^3 + \frac{1}{27} \left(27 \frac{\Gamma(1/3)^9}{(2\pi)^6} u - \frac{(2\pi)^6}{\Gamma(1/3)^9} \right)^2 u^4. \quad (5.8)$$

The mirror map is given as follows. The period is

$$\omega = \frac{1}{u \left(1 - \frac{27}{\beta_{3b}^2} u \right)^{1/3}} = \frac{\beta_{3b}^2 z^3 - 1}{27 z^2}, \quad (5.9)$$

where this time we have defined z as

$$z := \left(1 - \frac{27}{\beta_{3b}^2} u \right)^{-1/3}. \quad (5.10)$$

The Higgs expectation value is given by

$$\begin{aligned} \phi &= \int \omega du \\ &= 3 \int \frac{dz}{1 - z^3} \end{aligned} \quad (5.11)$$

$$= -\ln(1 - z) + \frac{1}{2} \ln(1 + z + z^2) + \sqrt{3} \arctan \frac{1 + 2z}{\sqrt{3}} - \frac{\pi}{2\sqrt{3}} + \phi_0^{E_6}. \quad (5.12)$$

Near the E_6 singularity at $z = 0$, ϕ is expanded as

$$\phi - \phi_0^{E_6} = 3 \sum_{k=0}^{\infty} \frac{z^{3k+1}}{3k+1} = 3 \left(z + \frac{z^4}{4} + \frac{z^7}{7} + \cdots \right). \quad (5.13)$$

Near the H_2 singularity at $z = \infty$, on the other hand, ϕ is expanded as

$$\phi - \phi_0^{H_2} = 3 \sum_{k=0}^{\infty} \frac{1}{(3k+2)z^{3k+2}} = 3 \left(\frac{1}{2z^2} + \frac{1}{5z^5} + \frac{1}{8z^8} + \cdots \right). \quad (5.14)$$

Using the explicit forms of F_g ($g \leq 3$) [32], one can immediately compute $F_g^{E_6 \oplus H_2}$ ($g \leq 3$). The results are

$$F_0^{E_6 \oplus H_2} = 0, \quad (5.15)$$

$$\begin{aligned} F_1^{E_6 \oplus H_2} &= \frac{1}{2} \ln \omega + \frac{1}{2} \phi - \frac{1}{2} \ln \left(-q_b \prod_{k=1}^{\infty} (1 - q_b^k)^{12} \right) \\ &= \frac{1}{2} \ln \frac{z^3 - 1}{z^2} + \frac{1}{2} \phi + \frac{\pi}{4\sqrt{3}} - \frac{3}{4} \ln 3, \end{aligned} \quad (5.16)$$

$$F_2^{E_6 \oplus H_2} = \left(\frac{8 - 4z^3 + 5z^6}{864z^2} - \frac{1}{96} \right) E_{2b}, \quad (5.17)$$

$$F_3^{E_6 \oplus H_2} = \frac{(1 - z^3)^2 (20 + 26z^3 + 35z^6)}{93312z^4} E_{2b}^2. \quad (5.18)$$

The BPS index for the E_6 and H_2 SCFTs are

$$Z^{E_6} = \exp \left(\frac{E_{2b}}{24} \hbar^2 \partial_\phi^2 \right) \left[\phi^{-1} \exp \sum_{k=1}^{\infty} c_k^{E_6} \left(\frac{\hbar}{\phi} \right)^{6k} \right], \quad (5.19)$$

$$Z^{H_2} = \exp \left(\frac{E_{2b}}{24} \hbar^2 \partial_\phi^2 \right) \left[\phi^{-1/4} \exp \sum_{k=1}^{\infty} c_k^{H_2} \left(\frac{\hbar}{\phi} \right)^{6k} \right]. \quad (5.20)$$

In contrast to the $E_8 \oplus H_0$ case, the holomorphic anomaly equation, the gap condition and the data of elliptic genera Z_n ($n \leq 4$) are not enough to determine the invariants $c_k^{E_6}$, $c_k^{H_2}$. To determine the first invariants $c_1^{E_6}$, $c_1^{H_2}$, one needs slightly more data, e.g., the explicit form(s) of $F_4^{E_6 \oplus H_2}$ or $Z_n^{E_6 \oplus H_2}$ with $n = 5, 6$. We expect that $c_k^{E_6}$, $c_k^{H_2}$ for general k are given by E_{6b}^k multiplied by some rational numbers, as in the case of the E_8 and H_0 SCFTs.

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