Abelian duality at higher genus

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Abstract: In three dimensions, a free, periodic scalar field is related by duality to an abelian gauge field. Here I explore aspects of this duality when both theories are quantized on a Riemann surface of genus g. At higher genus, duality involves an identification of winding with momentum on the Jacobian variety of the Riemann surface. I also consider duality for monopole and loop operators on the surface and exhibit the operator algebra, a refinement of the Wilson-'t Hooft algebra.

Contents

1 Introduction

Very broadly, dualities in quantum field theory often involve an interchange between classical and quantum data. Perhaps the simplest and best-known example occurs for the theory of a free periodic[1] scalar field $\phi \sim \phi + 2\pi$ on a Riemann surface Σ, with sigma model action

$$\mathbf{I}(\phi) \,=\, \frac{R^2}{4\pi} \int_\Sigma d\phi \wedge \star d\phi \,=\, \frac{R^2}{4\pi} \int_\Sigma d^2x \, \sqrt{h} \, \partial_\mu \phi \, \partial^\mu \phi \,, \qquad \mu = 1, 2 \,. \qquad (1.1)$$

Here R is a parameter which determines the radius of the circle for maps $\phi : \Sigma \to S^1$, and $\star$ is the Hodge star associated to a given metric h on Σ.

When ϕ is quantized on the circle, meaning that we take $\Sigma = \mathbb{R} \times S^1$, one finds that the Hilbert space is graded by a pair of integers (p, w),

$$\mathscr{H}_{S^1} \,=\, \bigoplus_{(p,w)\in\mathbb{Z}\oplus\mathbb{Z}} \mathscr{H}_{S^1}^{p,w} \,. \qquad (1.2)$$

The integers p and w are naturally interpreted as charges for a combined $\mathrm{U}(1)_\ell \times \mathrm{U}(1)_r$ action on $\mathscr{H}_{S^1}$, under which each summand in (1.2) transforms with the specified weights. The integer p is associated to the global $\mathrm{U}(1)_\ell$ symmetry under which the value of ϕ shifts by a constant,

$$\mathrm{U}(1)_\ell : \quad \phi \longmapsto \phi + c \,, \qquad c \in \mathbb{R}/2\pi\mathbb{Z} \,. \qquad (1.3)$$

This transformation clearly preserves the classical action in (1.1). Concretely, p labels the states in $\mathscr{H}_{S^1}$ which arise from the quantization of the constant mode $\phi_0 \in S^1$ of the scalar field. These states correspond to a Fourier basis for $L^2(S^1; \mathbb{C})$ as below,

$$\Psi_p(\phi_0) \,=\, \mathrm{e}^{\,i\,p\,\phi_0} \,, \qquad p \in \mathbb{Z} \,, \qquad (1.4)$$

and p is the momentum conjugate to ϕ_0.

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[1] "Periodic" is perhaps better stated as "circle-valued".

The other charge w describes the winding-number of ϕ as a map from the circle to itself. Hence w labels the connected components of the configuration space

$$\mathcal{X} = \bigsqcup_{w \in \mathbb{Z}} \mathcal{X}_w, \qquad \mathcal{X} = \mathrm{Map}(S^1, S^1), \tag{1.5}$$

where

$$\mathcal{X}_w = \left\{ \phi : S^1 \to S^1 \,\big|\, \phi(x + 2\pi) = \phi(x) + 2\pi w \right\}. \tag{1.6}$$

Each component of the classical configuration space must be quantized separately, and those states which arise from $\mathcal{X}_w$ span the subspace of the Hilbert space at the grade w.

Though the Hilbert space on S^1 is bigraded by $(p, w) \in \mathbb{Z} \oplus \mathbb{Z}$, the individual gradings have very different physical origins. The momentum p appears only after quantization, so the grading by p is inherently quantum. Conversely, the grading by winding-number w can be understood in terms of the topology of the configuration space $\mathcal{X}$, so the grading by w is classical.

In a similar vein, the conserved currents on Σ associated to the $\mathrm{U}(1)_\ell \times \mathrm{U}(1)_r$ global symmetry are respectively

$$j_\ell = d\phi, \qquad j_r = \star d\phi. \tag{1.7}$$

The topological current j_r, whose charge is the winding-number w, trivially satisfies the conservation equation $d^\dagger j_r = 0$ (with $d^\dagger = -\star d\star$) for arbitrary configurations of the field ϕ on Σ. By contrast, $d^\dagger j_\ell = 0$ only when ϕ satisfies the classical equation of motion $\triangle \phi = d^\dagger d\phi = 0$. Thus conservation of j_ℓ is a feature of the dynamics — or lack thereof — in the abelian sigma model.

Because Σ has dimension two, the conserved currents j_ℓ and j_r are exchanged under the action by Poincaré-Hodge duality on the space of one-forms Ω^1_Σ. As familiar, the classical action by Poincaré duality extends to an action by T-duality [5–7, 12, 19] on the quantum field ϕ, under which the respective quantum and classical gradings by momentum and winding are exchanged, and the parameter R in (1.1) is inverted to $1/R$. See for instance Lecture 8 in [21] for further discussion of abelian duality on Σ.

The present paper is a continuation of [2], in which we examine global issues surrounding abelian duality in dimension three, on a Riemannian three-manifold M. In this case, duality now relates the periodic scalar field $\phi : M \to S^1$ to a $\mathrm{U}(1)$ gauge field A on M. Both quantum field theories are free, so both can be quantized on the product $M = \mathbb{R} \times \Sigma$ to produce respective Hilbert spaces $\mathscr{H}_\Sigma$ and $\mathscr{H}_\Sigma^\vee$. Duality is an equivalence of quantum field theories, so we expect an isomorphism[2]

$$\mathscr{H}_\Sigma \simeq \mathscr{H}_\Sigma^\vee. \tag{1.8}$$

Exploring how the identification in (1.8) works when Σ is a compact Riemann surface of genus g will be our main goal in this paper.

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[2] Our notation for the Maxwell Hilbert space $\mathscr{H}_\Sigma^\vee$ is *not* intended to suggest that it is naturally dual as a vector space to the scalar Hilbert space $\mathscr{H}_\Sigma$.

Just as T-duality acts in a non-trivial way on $\mathscr{H}_{S^1}$ by exchanging momentum and winding in (1.2), we will see that the dual identification $\mathscr{H}_\Sigma \simeq \mathscr{H}_\Sigma^\vee$ relies upon an analogous exchange of classical and quantum data for the scalar field and the gauge field on Σ. However, the Hilbert spaces $\mathscr{H}_\Sigma$ and $\mathscr{H}_\Sigma^\vee$ are now more interesting due to their dependence on the geometry of the Riemann surface Σ, and unraveling the isomorphism in (1.8) turns out to be a richer story than for quantization on S^1.

In genus zero, when $\Sigma = \mathbb{CP}^1$ and the Hilbert space has a physical interpretation via radial quantization on $\mathbb{R}^3$, nothing that we say will be new. As usual in the world of Riemann surfaces, though, genus zero is a rather degenerate case, and several important features only emerge at genus $g \geq 1$. From the perspective of the scalar field, these features are related to topological winding-modes on Σ, and from the perspective of the gauge field, they are related to the existence of a moduli space of non-trivial flat connections on Σ.

Given the current rudimentary understanding of duality, especially in dimension three, the existence of any tractable example is important. Both this work and [2] are motivated by questions about non-abelian duality for a certain topological version of the $\mathcal{N} = 8$ supersymmetric Yang-Mills theory in three dimensions, considered to a certain extent in §3.3 of [22]. From the latter perspective, the abelian analysis here provides a useful toy model in which everything can be understood directly and in detail.

The plan of the paper. In section 2, we construct the respective Hilbert spaces $\mathscr{H}_\Sigma$ and $\mathscr{H}_\Sigma^\vee$ associated to the periodic scalar field and the abelian gauge field on $M = \mathbb{R} \times \Sigma$. Because the quantum field theories are free, the quantization holds no mystery and can be carried out quite rigorously, if one wishes. Both Hilbert spaces depend on the detailed choice of the Riemannian metric on Σ. In either case, though, we identify a particularly simple, infinite-dimensional subspace of 'quasi-topological' states which depend only upon the overall volume and complex structure of Σ. These quasi-topological states are exchanged under duality, analogous to the exchange of momentum and winding states for quantization on S^1.

In section 3, we proceed to the consider the algebra satisfied by a natural set of operators (Wilson loops, vortex loops, and monopole operators in the language of Maxwell theory) which act on the Hilbert spaces constructed in section 2. For a free quantum field theory, there is only one possible operator algebra that can arise — namely, the Heisenberg algebra, in a suitable geometric realization. When $\Sigma = \mathbb{CP}^1$ there is not much to say, but in higher genus, the operator algebra has a non-trivial holomorphic dependence on Σ that seems not to have been previously noted. This algebra is a refinement of the celebrated Wilson-'t Hooft algebra [15]. See [9, 10] for a somewhat related appearance of the Heisenberg algebra in four-dimensional Maxwell theory, and [3, 4, 8, 16, 18] for some recent discussions of monopoles and vortices in the setting of $\mathcal{N} = 2$ supersymmetric gauge theory.

Along the way, we also consider in section 3.1 the dual identification of operators acting on $\mathscr{H}_\Sigma \simeq \mathscr{H}_\Sigma^\vee$. For the convenience of the reader, a complementary review of the path integral perspective on the order-disorder correspondence for our operators can be found in section 5.2 of [2].

2 Abelian duality on a Riemann surface

In this section, we quantize both the periodic scalar field and the abelian gauge field on a compact surface Σ of genus g, with Riemannian metric h. We then compare the results.

Some geometric preliminaries. Though the quantum field theories under consideration are free, they definitely depend upon the choice of the metric h on Σ. The most basic invariant of h is the total volume

$$\ell^2 = \int_\Sigma \mathrm{vol}_\Sigma\,, \qquad \mathrm{vol}_\Sigma = \star 1 \in \Omega_\Sigma^2\,, \tag{2.1}$$

where ℓ is the length associated to the chosen metric.

As in [2], the Hamiltonians for both the scalar field and the gauge field on Σ will depend upon a parameter e^2, identified with the electric coupling in the Maxwell theory on $M = \mathbb{R} \times \Sigma$. All our constructions will respect the classical scaling under which the metric h transforms by

$$h \longmapsto \Lambda^2 h\,, \qquad \Lambda \in \mathbb{R}_+\,, \tag{2.2}$$

along with

$$\ell \longmapsto \Lambda \ell\,, \tag{2.3}$$

and

$$e^2 \longmapsto \Lambda^{-1} e^2\,. \tag{2.4}$$

Hence ℓ and e^2 are redundant parameters, since either can be scaled to unity with an appropriate choice of Λ in (2.2). Nonetheless, we leave the dependence on both ℓ and e^2 explicit, so that the naive dimensional analysis holds.

At least when the genus of Σ is positive, a more refined invariant of the metric h is the induced complex structure on Σ. Concretely, specifying a complex structure on Σ amounts to specifying a Hodge decomposition for complex one-forms

$$\Omega_\Sigma^1 \otimes \mathbb{C} \simeq \Omega_\Sigma^{1,0} \oplus \Omega_\Sigma^{0,1}\,, \tag{2.5}$$

where $\Omega_\Sigma^{1,0}$ and $\Omega_\Sigma^{0,1}$ refer to complex one-forms of given holomorphic/anti-holomorphic type. With this decomposition, one can define a Dolbeault operator $\overline{\partial}$ by projection onto $\Omega_\Sigma^{0,1}$, from which one obtains a notion of holomorphy on Σ.

The Hodge star associated to the metric h satisfies $\star^2 = -1$ when acting on Ω_Σ^1. The eigenspaces of the Hodge star then provide the decomposition in (2.5), where by convention

$$\star = -i \ \text{ on } \ \Omega_\Sigma^{1,0}\,, \qquad \star = +i \ \text{ on } \ \Omega_\Sigma^{0,1}\,. \tag{2.6}$$

In this manner, the metric h determines a complex structure on Σ.

Finally, we will make great use of the de Rham Laplacians $\triangle_0$ and $\triangle_1$ acting on differential forms of degrees zero and one on Σ. As usual, both Laplacians are defined in terms of the L^2-adjoint $d^\dagger$ via $\triangle_{0,1} = d^\dagger d + d d^\dagger$. With this convention, the Laplacian is a positive operator. Because Σ is smooth and compact, the spectra of $\triangle_0$ and $\triangle_1$ are discrete, and the kernels of each Laplacian are identified with the cohomology groups

$$H^0(\Sigma; \mathbb{R}) = \mathbb{R}\,, \qquad H^1(\Sigma; \mathbb{R}) = \mathbb{R}^{2g}\,. \tag{2.7}$$

2.1 The quantum sigma model

As in section 2 of [2], we consider a periodic scalar field ϕ on $M = \mathbb{R} \times \Sigma$,

$$\phi : M \longrightarrow S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}\,, \tag{2.8}$$

where we interpret ϕ as an angular quantity, subject to the identification

$$\phi \sim \phi + 2\pi\,. \tag{2.9}$$

Unlike in [2], though, for the purpose of quantization we work in Lorentz signature $(-++)$ on $\mathbb{R} \times \Sigma$, with the product metric

$$ds_M^2 = -dt^2 + h_{z\bar{z}}\,dz \otimes d\bar{z}\,. \tag{2.10}$$

Here t is interpreted as the "time" along $\mathbb{R}$, and $(z, \bar{z})$ are local holomorphic/anti-holomorphic coordinates on Σ. Philosophically, quantization is more naturally carried out in Lorentz as opposed to Euclidean signature, since only in the former case does one expect to obtain a physically-sensible, unitary quantum field theory.

In Lorentz signature on $M = \mathbb{R} \times \Sigma$, the free sigma model action is given by

$$\mathbf{I}_0(\phi) = \frac{e^2}{4\pi} \int_{\mathbb{R} \times \Sigma} dt \left[(\partial_t \phi)^2 \, \mathrm{vol}_\Sigma - d\phi \wedge \star d\phi \right]. \tag{2.11}$$

Throughout, we follow the convention that the de Rham operator d and the Hodge star $\star \equiv \star_\Sigma$ refer to quantities on Σ, as opposed to M. As in (2.1), vol_Σ is the Riemannian volume form on Σ induced by the metric h. Finally, e^2 is a dimensionful parameter which will eventually be identified under duality with the electric coupling in Maxwell theory on M.

Under the scaling by Λ in (2.2) and (2.4), the volume form on Σ transforms by

$$\mathrm{vol}_\Sigma \longrightarrow \Lambda^2 \, \mathrm{vol}_\Sigma\,, \tag{2.12}$$

and $d\phi \wedge \star d\phi$ is invariant, a fact familiar in the context of two-dimensional conformal field theory. Hence $\mathbf{I}_0(\phi)$ will be invariant under (2.2) and (2.4) provided that the time t is also scaled by

$$t \longmapsto \Lambda t\,. \tag{2.13}$$

This scaling of the time coordinate is moreover necessary for a homogeneous scaling of the three-dimensional metric (2.10) on M.

As we observed in [2], the free sigma model action in (2.11) can be extended by topological terms

$$\mathbf{I}_1(\phi) = \frac{e^2}{2\pi} \int_{\mathbb{R} \times \Sigma} dt\, \alpha \wedge d\phi + \frac{\theta}{2\pi\ell^2} \int_{\mathbb{R} \times \Sigma} dt\, \partial_t \phi \cdot \mathrm{vol}_\Sigma\,. \tag{2.14}$$

Here α is a harmonic one-form on Σ,

$$\alpha \in \mathcal{H}^1(\Sigma)\,, \tag{2.15}$$

and θ is a real constant,

$$\theta \in \mathbb{R} \,. \tag{2.16}$$

Together, α and θ specify the components of the complex harmonic two-form γ that appears on the compact three-manifold M in [2]. The prefactor of $1/2\pi$ in (2.14) is just a convention, and the factors of e^2 and $1/\ell^2$ in the respective terms are dictated by invariance under the scaling in (2.2), (2.4), and (2.13). Note also that the two-form $\mathrm{vol}_\Sigma/\ell^2$ which enters the second term in (2.14) is properly normalized to serve as an integral generator for $H^2(\Sigma; \mathbb{Z})$.

We take the total sigma model action to be the sum

$$\mathbf{I}_{\mathrm{tot}}(\phi) \,=\, \mathbf{I}_0(\phi) \,+\, \mathbf{I}_1(\phi) \,, \tag{2.17}$$

or more explicitly,

$$\mathbf{I}_{\mathrm{tot}}(\phi) \,=\, \frac{e^2}{4\pi} \int_{\mathbb{R} \times \Sigma} dt \left[(\partial_t \phi)^2 \,\mathrm{vol}_\Sigma \,+\, 2\, \frac{\theta}{e^2 \ell^2} \, \partial_t \phi \,\mathrm{vol}_\Sigma \,-\, d\phi \wedge \star d\phi \,+\, 2\, \alpha \wedge d\phi \right]. \tag{2.18}$$

Because α is closed by assumption, $d\alpha = 0$, the topological terms in (2.18) do not alter the classical equation of motion

$$\partial_t^2 \phi \,+\, \triangle_0 \phi \,=\, 0 \,, \tag{2.19}$$

a version of the usual wave equation on $\mathbb{R} \times \Sigma$. As a small check on our signs, note that $\triangle_0 \geq 0$ is positive while $\partial_t^2 \leq 0$ is negative, so the equation of motion for ϕ does admit non-trivial, time-dependent solutions. Clearly, α in (2.18) serves to distinguish the various topological winding-sectors associated to the circle-valued map ϕ.

Like α, the constant θ multiplies a term in the action (2.14) which is a total derivative. Hence θ has no effect on the classical physics. However, θ does change the definition of the canonical momentum Π_ϕ conjugate to ϕ,

$$\Pi_\phi \,=\, \frac{e^2}{2\pi} \left(\partial_t \phi \,+\, \frac{\theta}{e^2 \ell^2} \right), \tag{2.20}$$

in terms of which we write the classical Hamiltonian

$$\mathbf{H} \,=\, \int_\Sigma \left[\frac{\pi}{e^2} \left(\Pi_\phi \,-\, \frac{\theta}{2\pi \ell^2} \right)^2 \mathrm{vol}_\Sigma \,+\, \frac{e^2}{4\pi} \, d\phi \wedge \star d\phi \,-\, \frac{e^2}{2\pi} \, \alpha \wedge d\phi \right]. \tag{2.21}$$

As will be clear, the quantum sigma model does depend upon θ non-trivially, and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ becomes an angular parameter closely analogous to the theta-angle of Yang-Mills theory in two and four dimensions.

In addition to the Hamiltonian $\mathbf{H}$, another important quantity is the conserved momentum $\mathbf{P}$ associated to the global U(1) symmetry under which the value of ϕ shifts by a constant, exactly as in (1.3). Constant shifts in ϕ manifestly preserve the sigma model action, with conserved current $j = e^2 \, d\phi/2\pi$ and charge

$$\mathbf{P} \,=\, \frac{e^2}{2\pi} \int_\Sigma \partial_t \phi \cdot \mathrm{vol}_\Sigma \,. \tag{2.22}$$

Because of the U(1) symmetry, the Hilbert space for ϕ will automatically carry an integral grading by the eigenvalue of $\mathbf{P}$ when we quantize.

Classical mode expansion. In principle, the Hilbert space for the periodic scalar field on the surface Σ is straightforward to describe, though the detailed spectrum of the Hamiltonian depends very much on the geometry of Σ.

Very briefly, just as for quantization on S^1, the quantization on Σ involves a countable number of topological sectors, corresponding to homotopy classes of the map $\phi : \Sigma \to S^1$. These homotopy classes are labelled by a winding-number ω which is valued in the cohomology lattice

$$\mathbb{L} = H^1(\Sigma; \mathbb{Z}) \simeq \mathbb{Z}^{2g}, \tag{2.23}$$

as discussed for instance in section 2.1 of [2]. Abusing notation slightly, I write the cohomology class associated to the circle-valued map ϕ as

$$\omega = \left[\frac{d\phi}{2\pi}\right] \in H^1(\Sigma; \mathbb{Z}). \tag{2.24}$$

Globally, the configuration space $\mathcal{X} = \mathrm{Map}(\Sigma, S^1)$ is a union of components

$$\mathcal{X} = \bigsqcup_{\omega \in \mathbb{L}} \mathcal{X}_\omega, \tag{2.25}$$

where

$$\mathcal{X}_\omega = \left\{ \phi : \Sigma \to S^1 \,\middle|\, \left[\frac{d\phi}{2\pi}\right] = \omega \right\}. \tag{2.26}$$

As standard in quantum field theory, each component $\mathcal{X}_\omega \subset \mathcal{X}$ must be quantized separately, leading to a topological grading by the cohomology lattice $\mathbb{L}$ on the full Hilbert space $\mathscr{H}_\Sigma$. This grading by $\omega \in \mathbb{L}$ is the obvious counterpart for quantization on Σ to the grading (1.2) by winding-number $w \in \mathbb{Z}$ for quantization on S^1.

Concretely, as the first step towards constructing the sigma model Hilbert space, we solve the classical equation of motion for ϕ in (2.19). For the moment, we assume ϕ to have trivial winding, so that the time-dependent field $\phi : \mathbb{R} \times \Sigma \to S^1$ can be equivalently considered as a map $\phi : \mathbb{R} \to \mathcal{X}_0$ to the identity component of $\mathcal{X}$.

The general solution of (2.19) then takes the form

$$\phi(t, z, \overline{z}) = \frac{\phi_0}{e^2 \ell} + \frac{2\pi t}{\ell} p_0 + \sqrt{\pi} \sum_{\lambda > 0} \frac{e^2}{\lambda} \psi_\lambda(z, \overline{z}) \left[a_\lambda \, e^{-i\lambda t} + a_\lambda^\dagger \, e^{i\lambda t} \right]. \tag{2.27}$$

Generalizing the standard solution to the wave equation on $\mathbb{R}^{1,2}$, this expression for ϕ is written in terms of an orthonormal basis $\{\psi_\lambda\}$ of eigenmodes for the scalar Laplacian $\triangle_0$ on Σ,

$$\triangle_0 \psi_\lambda = \lambda^2 \, \psi_\lambda, \qquad \psi_\lambda \in \Omega^0_\Sigma, \tag{2.28}$$

with the convention that $\lambda > 0$ is a positive real number. For simplicity, we assume that all non-vanishing eigenvalues of the scalar Laplacian are distinct, so that each eigenfunction $\psi_\lambda(z, \overline{z})$ is uniquely labelled by λ. Of course, the precise spectrum for the scalar Laplacian depends sensitively on the geometry of the surface Σ, but we will not require any detailed information about the spectrum here, other than that it is discrete.

To ensure invariance of the eigenmode expansion for ϕ under the scaling in (2.2) and (2.4), we employ the invariant (and coupling-dependent) normalization condition

$$||\psi_\lambda||^2 = e^4 \int_\Sigma \psi_\lambda^2 \, \text{vol}_\Sigma = 1 \, . \tag{2.29}$$

A similar coupling-dependent normalization condition is used in (2.46) of [2], for the same reason. Accordingly, the constant function with unit norm on Σ is

$$\psi_0 = \frac{1}{e^2 \ell} \, . \tag{2.30}$$

This constant function appears implicitly in (2.27) as the coefficient of the zero-mode ϕ_0. Because ϕ is an angular quantity with period 2π, the zero-mode ϕ_0 must have its own periodicity

$$\phi_0 \sim \phi_0 + 2\pi e^2 \ell \, . \tag{2.31}$$

Hence ϕ_0 effectively decompactifies in the large-volume limit $\ell \to \infty$ with e^2 fixed.

Otherwise, p_0, a_λ, and $a_\lambda^\dagger$ for $\lambda > 0$ in (2.27) are constants which specify the classical solution for ϕ. The constant $p_0 \in \mathbb{R}$ is real and determines the classical momentum via

$$\mathbf{P} = \frac{e^2}{2\pi} \int_\Sigma \partial_t \phi \cdot \text{vol}_\Sigma = e^2 \ell \, p_0 \, , \tag{2.32}$$

whereas $(a_\lambda, a_\lambda^\dagger)$ are a conjugate pair of complex numbers associated to the oscillating modes of ϕ. The various factors of e^2 and ℓ sprinkled about (2.27) are necessary for invariance under the scaling in (2.2) and (2.4). In this regard, I observe that the eigenvalues of the Laplacian $\triangle_0$ themselves scale with Λ as

$$\lambda \longmapsto \Lambda^{-1} \lambda \, . \tag{2.33}$$

When $\phi : \mathbb{R} \times \Sigma \to S^1$ has non-trivial winding, the classical mode expansion in (2.27) must be generalized only slightly. Exactly as in section 2.2 in [2], we consider a harmonic representative for the cohomology class $\omega \in H^1(\Sigma; \mathbb{Z})$. Mildly abusing notation, I re-use ω to refer to this representative in the space $\mathcal{H}^1(\Sigma)$ of harmonic one-forms on Σ. Associated to the harmonic one-form ω with integral periods on Σ is a fiducial harmonic map $\Phi_\omega : \Sigma \to S^1$ satisfying

$$\frac{d\Phi_\omega}{2\pi} = \omega \, , \qquad \omega \in \mathcal{H}^1(\Sigma) \, . \tag{2.34}$$

Since $d^\dagger \omega = 0$, we see that $\triangle_0 \Phi_\omega = d^\dagger d\Phi_\omega = 0$ automatically. By Hodge theory, the map Φ_ω is determined by ω up to a constant. To fix that constant, we select a basepoint $\sigma_0 \in \Sigma$, which will re-occur later in section 2.2, and we impose

$$\Phi_\omega(\sigma_0) = 0 \mod 2\pi \, . \tag{2.35}$$

Together, the conditions in (2.34) and (2.35) uniquely determine the fiducial map Φ_ω with given winding.

Because the winding-number is additive, the general solution to the equation of motion in (2.19) with winding-number $\omega \in \mathbb{L}$ can now be written as the sum of the topologically-trivial solution in (2.27) with the fiducial harmonic map $\Phi_\omega(z, \overline{z})$,

$$\phi(t, z, \overline{z}) = \Phi_\omega(z, \overline{z}) + \frac{\phi_0}{e^2 \ell} + \frac{2\pi t}{\ell} p_0 + \sqrt{\pi} \sum_{\lambda > 0} \frac{e^2}{\lambda} \psi_\lambda(z, \overline{z}) \left[a_\lambda \, e^{-i \lambda t} + a_\lambda^\dagger \, e^{i \lambda t} \right]. \quad (2.36)$$

In these terms, the coefficients ϕ_0, p_0, and $(a_\lambda, a_\lambda^\dagger)$ for all $\lambda > 0$ parametrize the classical phase space for maps $\phi : \mathbb{R} \to \mathcal{X}_\omega$.

Sigma model Hilbert space. To quantize, we promote both the scalar field ϕ and the momentum Π_ϕ in (2.20) to operators which obey the canonical commutation relations

$$\left[\phi(z), \Pi_\phi(w) \right] = i \, \delta_\Sigma(z, w), \qquad z, w \in \Sigma. \quad (2.37)$$

Here δ_Σ is a delta-function with support on the diagonal $\Delta \subset \Sigma \times \Sigma$. We will take either of two perspectives on (2.37).

From the first perspective, the commutator in (2.37) can be realized through the functional identification

$$\Pi_\phi(w) = -i \frac{\delta}{\delta \phi(w)}, \quad (2.38)$$

or equivalently via (2.20),

$$\frac{e^2}{2\pi} \partial_t \phi(w) = -i \frac{D}{D\phi(w)}. \quad (2.39)$$

Here $D/D\phi(w)$ is interpreted as a covariant functional derivative incorporating the shift by θ in the canonical momentum,

$$\frac{D}{D\phi(w)} = \frac{\delta}{\delta \phi(w)} - i \frac{\theta}{2\pi \ell^2}. \quad (2.40)$$

Because θ is just a constant,

$$\left[\frac{D}{D\phi(z)}, \frac{D}{D\phi(w)} \right] = 0, \qquad z \neq w. \quad (2.41)$$

As we will see quite explicitly, $D/D\phi$ thus describes a flat connection with non-trivial holonomy over the configuration space $\mathcal{X}$ of maps from Σ to S^1.

For the alternative perspective on the commutator in (2.37), we rewrite the delta-function δ_Σ in terms of the orthonormal eigenbasis $\{\psi_\lambda\}$ for the scalar Laplacian,

$$\delta_\Sigma(z, w) = \frac{1}{\ell^2} + \sum_{\lambda > 0} e^4 \, \psi_\lambda(z) \, \psi_\lambda(w), \qquad z, w \in \Sigma. \quad (2.42)$$

The first term on the right in (2.42) arises from the constant mode ψ_0, and the factor of e^4 in the sum over the higher eigenmodes is a result of the normalization condition in (2.29).

After substituting the mode expansions in (2.36) and (2.42) into the canonical commutation relation, we find that (ϕ_0, p_0) and $(a_\lambda, a_\lambda{}^\dagger)$ for $\lambda > 0$ satisfy the free-field Heisenberg algebra

$$[\phi_0,\, p_0] = i\,,$$
$$\left[a_\lambda,\, a_{\lambda'}{}^\dagger\right] = \frac{\lambda}{e^2}\,\delta_{\lambda\lambda'}\,, \tag{2.43}$$

with all other commutators vanishing identically. As usual, in the second line of (2.43) we introduce the Kronecker delta, defined by $\delta_{\lambda\lambda'} = 1$ if $\lambda = \lambda'$ and $\delta_{\lambda\lambda'} = 0$ otherwise.

These commutation relations hold in each winding-sector, independent of the class $\omega \in H^1(\Sigma; \mathbb{Z})$, so the quantization will also be independent of ω. As the counterpart to the topological decomposition of $\mathcal{X} = \mathrm{Map}(\Sigma, S^1)$ in (2.25), the total Hilbert space $\mathscr{H}_\Sigma$ for the periodic scalar field on Σ decomposes into the direct sum

$$\mathscr{H}_\Sigma = \bigoplus_{\omega \in \mathbb{L}} \mathscr{H}_\Sigma^\omega\,, \qquad\qquad \mathbb{L} = H^1(\Sigma; \mathbb{Z})\,, \tag{2.44}$$

where each subspace $\mathscr{H}_\Sigma^\omega$ is itself a tensor product (independent of ω)

$$\mathscr{H}_\Sigma^\omega = \mathsf{H}_0 \otimes \bigotimes_{\lambda > 0} \mathsf{H}_\lambda\,. \tag{2.45}$$

Of the two factors in the tensor product, H_λ is the less interesting. Up to an irrelevant choice of normalization, a_λ and $a_\lambda{}^\dagger$ in (2.43) satisfy the usual commutator algebra for a harmonic oscillator with frequency λ. Hence H_λ is the Fock space for that oscillator.

More interesting for us is the universal factor H_0. This factor arises from the quantization of the zero-modes (ϕ_0, p_0) of the scalar field and thus does not depend upon the spectral geometry of the surface Σ. Together ϕ_0 and p_0 simply describe the position and momentum of a free particle moving on a circle with radius $e^2\ell$. The corresponding phase space is the cotangent bundle T^*S^1 with the canonical symplectic structure, and at least when $\theta = 0$ in (2.14), the quantization is entirely standard. Directly,

$$\mathsf{H}_0 \simeq L^2(S^1; \mathbb{C})\,, \qquad\qquad [\,\theta = 0\,] \tag{2.46}$$

spanned by the Fourier wavefunctions

$$\Psi_m(\phi_0) = \exp\!\left(i\,\frac{m}{e^2\ell}\,\phi_0\right), \qquad\qquad m \in \mathbb{Z}\,. \tag{2.47}$$

As usual, the classical momentum p_0 becomes identified with the operator $-i\,\partial/\partial\phi_0$. Via the identification $\mathbf{P} = e^2\ell\,p_0$ in (2.32), each Fourier wavefunction in (2.47) is an eigenstate of the total momentum operator

$$\mathbf{P} = -i\,e^2\ell\,\frac{\partial}{\partial\phi_0}\,. \tag{2.48}$$

When the topological parameter θ is non-zero, the quantization of ϕ_0 and p_0 is modified. After we project (2.39) and (2.40) to the space of zero-modes, p_0 becomes identified with the θ-dependent operator

$$p_0 = -i\,\frac{D}{D\phi_0}\,, \qquad\qquad \frac{D}{D\phi_0} = \frac{\partial}{\partial\phi_0} - i\,\frac{\theta}{2\pi e^2\ell}\,. \tag{2.49}$$

Evidently, $D/D\phi_0$ in (2.49) is the covariant derivative for a unitary flat connection on a complex line-bundle $\mathcal{L}$ over the circle, with holonomy

$$\mathrm{Hol}_{S^1}(D/D\phi_0) = \exp(i\,\theta)\,. \tag{2.50}$$

As the natural generalization of (2.46), the zero-mode Hilbert space H_0 is the space of square-integrable sections of $\mathcal{L}$,

$$\mathsf{H}_0 = L^2(S^1; \mathcal{L})\,, \tag{2.51}$$

on which $\mathbf{P}$ now acts covariantly by

$$\mathbf{P} = -i\,e^2\ell\,\frac{D}{D\phi_0}\,. \tag{2.52}$$

For the Fourier wavefunction $\Psi_m(\phi_0)$ in (2.47), all this is just to say that

$$\mathbf{P}\cdot\Psi_m(\phi_0) = \left(m - \frac{\theta}{2\pi}\right)\cdot\Psi_m(\phi_0), \qquad m \in \mathbb{Z}, \tag{2.53}$$

as follows directly from (2.49). Hence the topological parameter θ induces a uniform shift on the eigenvalues of $\mathbf{P}$ away from integral values. Manifestly, the spectrum of $\mathbf{P}$ depends only on the value of θ modulo 2π.

Because the zero-mode Hilbert space H_0 is graded by $\mathbf{P}$, the full sigma model Hilbert space $\mathscr{H}_\Sigma$ is bigraded by the lattice $\mathbb{Z} \oplus \mathbb{L}$,

$$\mathscr{H}_\Sigma \simeq \bigoplus_{(m,\,\omega)\in\mathbb{Z}\oplus\mathbb{L}} \mathscr{H}_\Sigma^{m,\omega}\,, \tag{2.54}$$

in parallel to the bigrading by $\mathbb{Z} \oplus \mathbb{Z}$ for $\mathscr{H}_{S^1}$ in (1.2). As a convenient shorthand, I let $|m;\omega\rangle$ denote the Fourier wavefunction $\Psi_m(\phi_0)$, considered in the topological sector with winding-number ω and satisfying the vacuum condition

$$a_\lambda|m;\omega\rangle = 0\,, \qquad \lambda > 0\,. \tag{2.55}$$

All other Fock states in $\mathscr{H}_\Sigma$ are obtained by acting with the oscillator raising-operators $a_\lambda^\dagger$ on each Fock vacuum $|m;\omega\rangle$, so the summands in $\mathscr{H}_\Sigma$ above are more explicitly

$$\mathscr{H}_\Sigma^{m,\omega} = \mathbb{C}\cdot|m;\omega\rangle \otimes \bigotimes_{\lambda>0}\mathsf{H}_\lambda\,. \tag{2.56}$$

Philosophically, the grading by the eigenvalue m in (2.54) is a quantum grading (since we must quantize ϕ_0 to define it!), whereas the grading by the winding-number ω is classical, just as we saw in section 1 for quantization on S^1. But needless to say, because $\mathbb{Z} \neq \mathbb{L} \simeq \mathbb{Z}^{2g}$, duality on Σ cannot exchange the two gradings, as occurs for duality on S^1. Rather, the role of duality will be to exchange the quantum versus classical interpretations of each.

Finally, let us consider the action of the sigma model Hamiltonian $\mathbf{H}$ on the Hilbert space. After the identification in (2.39), the classical Hamiltonian becomes the operator

$$\mathbf{H} = \int_\Sigma \left[-\frac{\pi}{e^2}\,\frac{D^2}{D\phi^2}\,\mathrm{vol}_\Sigma \,+\, \frac{e^2}{4\pi}\,d\phi\wedge\star d\phi \,-\, \frac{e^2}{2\pi}\,\alpha\wedge d\phi\right]. \tag{2.57}$$

Upon substituting for the momentum $\mathbf{P}$ in (2.52),

$$\mathbf{H} \,=\, \int_\Sigma \left[\frac{\pi}{e^2} \left(\frac{\mathbf{P}^2}{\ell^4} + \cdots \right) \mathrm{vol}_\Sigma \,+\, \frac{e^2}{4\pi} \, d\phi \wedge \star d\phi \,-\, \frac{e^2}{2\pi} \, \alpha \wedge d\phi \right], \qquad (2.58)$$

where the ellipses indicate terms in $D^2/D\phi^2$ which involve the non-zero eigenmodes of ϕ and thus the Fock operators $(a_\lambda, a_\lambda^\dagger)$.

The spectrum of $\mathbf{H}$ depends upon the corresponding spectrum of eigenvalues $\{\lambda^2\}$ for the scalar Laplacian $\triangle_0$, which in turn depends upon the geometry of Σ. To simplify the situation, we consider the action of $\mathbf{H}$ only on the Fock vacua $|m; \omega\rangle$ in (2.55). From (2.53) and (2.58),

$$\mathbf{H} \, |m; \omega\rangle \,=\, e^2 \left[\frac{\pi}{(e^2\ell)^2} \left(m - \frac{\theta}{2\pi} \right)^2 \,+\, \pi \, (\omega, \omega) \,-\, \langle \alpha, \omega \rangle + \frac{E_0}{e^2\ell} \right] |m; \omega\rangle. \qquad (2.59)$$

Here (ω, ω) is the L^2-norm of the harmonic one-form appearing in (2.34),

$$(\omega, \omega) \,=\, \int_\Sigma \omega \wedge \star \omega, \qquad \omega \,\in\, \mathcal{H}^1(\Sigma), \qquad (2.60)$$

and $\langle \alpha, \omega \rangle$ denotes the intersection pairing

$$\langle \alpha, \omega \rangle \,=\, \int_\Sigma \alpha \wedge \omega, \qquad \alpha \,\in\, \mathcal{H}^1(\Sigma). \qquad (2.61)$$

As will be important later, note that (ω, ω) in (2.60) is a conformal invariant, for which only the complex structure on Σ matters, and of course $\langle \alpha, \omega \rangle$ is purely topological.

The energy in (2.59) also includes a constant term E_0, independent of e^2, m, and ω, which arises from the sum over the zero-point energies $\frac{1}{2}\lambda$ of each oscillating eigenmode of ϕ. The factor of $1/\ell$ which multiplies E_0 is fixed by the scaling in (2.33), and we have pulled out an overall factor of e^2 from $\mathbf{H}$ so that the quantity in brackets is scale-invariant (or dimensionless). Physically, E_0/ℓ is a Casimir energy on the compact surface Σ, and some method of regularization must be chosen to make sense of the divergent sum $E_0 \sim \sum_{\lambda > 0} \frac{1}{2}\lambda$ which naively defines E_0, eg. by normal-ordering or use of the zeta-function. For comparison under duality, the particular method used to define E_0 will not matter, so we simply assume that E_0 has been determined in some way from the non-zero eigenvalues of the scalar Laplacian $\triangle_0$ on Σ.

Finally, let us consider the dependence of the spectrum of $\mathbf{H}$ on the effective coupling $1/e^2\ell$. Though the abelian sigma model is a free quantum field theory, there remains a definite sense in which the spectrum simplifies in the weakly-coupled regime $1/e^2\ell \ll 1$. As apparent from (2.59), in this limit the quantum states with least energy in any given topological sector are precisely the Fock vacua $|m; \omega\rangle$, for arbitrary values of the Fourier momentum $m \in \mathbb{Z}$.

Conversely, when $1/e^2\ell$ is of order-one, we do not find a clean separation in energy between the Fock vacua $|m; \omega\rangle$ and oscillator states such as $a_\lambda^\dagger |0; \omega\rangle$ for suitable λ. Hence in the latter case, the low-lying energy spectrum of the quantum sigma model depends much more delicately on the geometry of Σ.

2.2 The quantum Maxwell theory

We now consider the quantization of Maxwell theory on $M = \mathbb{R} \times \Sigma$, with the same Lorentzian product metric already appearing in (2.10).

Classically, the Maxwell gauge field A is a connection on a fixed principal U(1)-bundle P over M,

$$
\begin{array}{c}
\mathrm{U}(1) \to P \\
\downarrow \\
M
\end{array}
\quad . \tag{2.62}
$$

When $M = \mathbb{R} \times \Sigma$, the restriction of P determines an associated complex line-bundle L over Σ, with Chern class

$$
c_1(L) = \left[\frac{F_A}{2\pi} \right] \in H^2(\Sigma; \mathbb{Z}). \tag{2.63}
$$

Here $F_A = dA$ is the curvature, and we specify the Chern class of L by a single integer

$$
m = \deg(L) \in \mathbb{Z}. \tag{2.64}
$$

The coincidence in notation between m in (2.47) and (2.64) is no accident.

The integer m suffices to fix the topological type of both P and L. However, for purpose of quantization, we will need to endow the line-bundle L with a holomorphic structure as well. Because Σ carries a complex structure associated to its Riemannian metric h as in (2.6), L can be given a holomorphic structure uniformly for all degrees as soon as we pick a basepoint $\sigma_0 \in \Sigma$. We set

$$
L = \mathcal{O}_\Sigma(m\,\sigma_0), \qquad \sigma_0 \in \Sigma. \tag{2.65}
$$

By definition, holomorphic sections of L can be identified with meromorphic functions on Σ which have a pole of maximum degree m at the point $\sigma_0 \in \Sigma$.

Note that the choice of basepoint is only relevant when Σ has genus $g \geq 1$, since the holomorphic structure on any line-bundle of degree m over $\mathbb{CP}^1$ is unique. The same remark also applies to our previous choice of basepoint for the sigma model: in genus zero, the only fiducial harmonic map Φ_ω is constant, so the condition in (2.35) does not actually depend upon the choice of σ_0.

Specialized to $M = \mathbb{R} \times \Sigma$, the free Maxwell action becomes

$$
\mathbf{I}_0(A) = \frac{1}{4\pi e^2} \int_{\mathbb{R} \times \Sigma} dt \left[E_A \wedge \star E_A - F_A \wedge \star F_A \right], \qquad E_A = \iota_{\partial/\partial t} F_A \in \Omega_\Sigma^1. \tag{2.66}
$$

Here we stick to the assumption that $\star \equiv \star_\Sigma$ is the Hodge operator on Σ, so we have separated the curvature into the electric component E_A, which transforms like a one-form on Σ, along with the magnetic component $F_A \equiv F_A|_\Sigma$, which transforms like a two-form on Σ. Explicitly in local coordinates,

$$
E_A = F_{A,tz}\, dz + F_{A,t\bar{z}}\, d\bar{z}. \tag{2.67}
$$

Invariance under the scaling in (2.2), (2.4), and (2.13) fixes the dependence of the Maxwell action on the electric coupling e^2, and the overall factor of $1/4\pi$ in (2.66) appears by convention.

As in section 2.1, topological terms can also be added to the Maxwell action, of the form

$$\mathbf{I}_1(A) \; = \; \frac{1}{2\pi} \int_{\mathbb{R}\times\Sigma} dt \, \upbeta \wedge E_A \; + \; \frac{\theta}{2\pi e^2 \ell^2} \int_{\mathbb{R}\times\Sigma} dt \, F_A \,. \tag{2.68}$$

Like $\upalpha$ in (2.15), $\upbeta$ is a real harmonic one-form,

$$\upbeta \; \in \; \mathcal{H}^1(\Sigma) \,, \tag{2.69}$$

and $\theta \in \mathbb{R}$ is a real parameter that will correspond under duality to the angle already appearing in (2.59). With some malice aforethought, the coefficient $1/e^2\ell^2$ in (2.68) has been chosen to achieve this identification, along with invariance under the scaling in (2.2), (2.4), and (2.13).

We then consider the total gauge theory action

$$\mathbf{I}_{\text{tot}}(A) \; = \; \mathbf{I}_0(A) \; + \; \mathbf{I}_1(A) \,, \tag{2.70}$$

or more explicitly,

$$\mathbf{I}_{\text{tot}}(A) \; = \; \frac{1}{4\pi e^2} \int_{\mathbb{R}\times\Sigma} dt \, \left[E_A \wedge \star E_A \; - \; F_A \wedge \star F_A \; - \; 2e^2 E_A \wedge \upbeta \; + \; 2\frac{\theta}{\ell^2} F_A \right] \,. \tag{2.71}$$

Previously for the periodic scalar field, the angular parameter θ served to modify the definition (2.20) of the canonical momentum Π_ϕ. This role is now taken by the harmonic one-form $\upbeta$, which appears in the canonical momentum

$$\Pi_A \; = \; \frac{1}{2\pi e^2} \star E_A \; - \; \frac{1}{2\pi} \upbeta \,. \tag{2.72}$$

In terms of Π_A, the classical Hamiltonian is[3]

$$\mathbf{H}^\vee \; = \; \int_\Sigma \left[\pi e^2 \left(\Pi_A + \frac{\upbeta}{2\pi} \right) \wedge \star \left(\Pi_A + \frac{\upbeta}{2\pi} \right) + \frac{1}{4\pi e^2} F_A \wedge \star F_A \; - \; \frac{\theta}{2\pi e^2 \ell^2} F_A \right] \,. \tag{2.73}$$

The degree m of the line-bundle L is measured by the net magnetic flux through the surface,

$$\int_\Sigma \frac{F_A}{2\pi} \; = \; m \,, \tag{2.74}$$

so θ in (2.73) now serves to distinguish the topological sectors labelled by m.

Because we have yet to fix a gauge, the classical Maxwell Hamiltonian $\mathbf{H}^\vee$ is degenerate along gauge orbits. As a remedy, we work throughout in Coulomb gauge,

$$A_t \; = \; 0 \,, \tag{2.75}$$

where A_t is the time-component of the gauge field on $M = \mathbb{R} \times \Sigma$. In Coulomb gauge, the equation of motion for A_t holds identically as the Gauss law constraint

$$d^\dagger E_A \; = \; 0 \,. \tag{2.76}$$

---

[3]The superscript on $\mathbf{H}^\vee$ in (2.73) serves to differentiate the Maxwell Hamiltonian notationally from the Hamiltonian $\mathbf{H}$ for the periodic scalar field in section 2.1.

(Because $d\beta = 0$, the topological terms do not modify the Gauss law on Σ.) On $\mathbb{R}^{1,2}$, Coulomb gauge does not respect Lorentz invariance, which is the main disadvantage of Coulomb gauge. For quantization on $M = \mathbb{R} \times \Sigma$, though, Lorentz invariance is neither here nor there, and the gauge condition in (2.75) is perfectly natural.

To fix the remaining time-independent gauge transformations on Σ, we impose the further harmonic condition

$$d^\dagger A = 0. \tag{2.77}$$

Harmonic gauge on Σ is particularly convenient from the geometric perspective. In this gauge, the Gauss constraint in (2.76) is automatically obeyed, and A satisfies the classical wave equation

$$\partial_t^2 A + \triangle_1 A = 0, \qquad A \in \Omega_\Sigma^1, \tag{2.78}$$

where $\triangle_1 = d^\dagger d + d d^\dagger$ is the de Rham Laplacian for one-forms on Σ. We considered precisely the same equation of motion in (2.19) for the periodic scalar field ϕ, so quantization of A in harmonic gauge will share many features with quantization of ϕ, and duality will be manifest.

Finally, if A is any time-independent connection on Σ, the equation of motion in (2.78) implies that the curvature is also harmonic,

$$d^\dagger F_A = 0, \qquad F_A \in \Omega_\Sigma^2. \tag{2.79}$$

Thus the classical vacua of Maxwell theory on Σ correspond to harmonic connections on the line-bundle L.

When Σ has genus $g \geq 1$, Maxwell theory on $M = \mathbb{R} \times \Sigma$ is invariant under a continuous $\mathrm{U}(1)^{2g}$ global symmetry, which does not occur at genus zero. To describe the action of the symmetry on the gauge field, we first select an integral harmonic basis $\{\mathfrak{e}_1, \ldots, \mathfrak{e}_{2g}\}$ for the cohomology lattice

$$\mathbb{L} \simeq \mathbb{Z}\mathfrak{e}_1 \oplus \cdots \oplus \mathbb{Z}\mathfrak{e}_{2g}, \qquad \mathbb{L} = H^1(\Sigma; \mathbb{Z}). \tag{2.80}$$

The group $\mathrm{U}(1)^{2g}$ then acts on the gauge field by shifts

$$\mathrm{U}(1)^{2g}: \quad A \longmapsto A + \sum_{j=1}^{2g} c_j \, \mathfrak{e}_j, \qquad c_j \in \mathbb{R}/2\pi\mathbb{Z}. \tag{2.81}$$

Such shifts for any constant c_j trivially preserve both the Maxwell action in (2.71) and the gauge conditions in (2.75) and (2.77). Note also that shifts by elements in the lattice $2\pi\mathbb{L}$ are induced by homotopically non-trivial, "large" gauge transformations

$$A^u = A + i \, u^{-1} \, du, \qquad u : \Sigma \to \mathrm{U}(1). \tag{2.82}$$

so these lattice elements act as the identity modulo gauge-equivalence. As a result, the parameters c_j in (2.81) are circle-valued, and the global symmetry group is compact.

The group $\mathrm{U}(1)^{2g}$ acts to shift the holonomies of the gauge field, so we can think of this global symmetry group more intrinsically as the Jacobian torus of Σ,

$$\mathscr{J}_\Sigma = H^1(\Sigma; \mathbb{R})/2\pi\mathbb{L} \simeq \mathrm{U}(1)^{2g}, \qquad \mathbb{L} = H^1(\Sigma; \mathbb{Z}). \tag{2.83}$$

The Jacobian $\mathscr{J}_\Sigma$, in its role as the moduli space of flat U(1)-connections on Σ, will be essential in analyzing abelian duality at higher genus.

The U(1)^{2g} global symmetry of Maxwell theory on Σ is the counterpart to the more obvious U(1) symmetry of the periodic scalar field in (1.3). Just as for the conserved momentum $\mathbf{P}$ in (2.22), the global symmetry of Maxwell theory leads to a set of $2g$ conserved charges

$$\mathbf{W_j} = \frac{1}{2\pi e^2} \int_\Sigma \mathfrak{e}_j \wedge \star E_A \,, \qquad\qquad \mathbf{j} = 1, \ldots, 2g \,,$$
$$= \int_\Sigma \mathfrak{e}_j \wedge \left(\Pi_A + \frac{\beta}{2\pi} \right) . \tag{2.84}$$

Conservation of $\mathbf{W_j}$ follows from the harmonic condition $d\mathfrak{e}_j = d^\dagger \mathfrak{e}_j = 0$, as well as the classical equation of motion in (2.78).

Because of the U(1)^{2g} global symmetry, upon quantization the Maxwell Hilbert space will automatically carry an integral grading by the eigenvalues of $\mathbf{W_j}$. Moreover, since $\star E_A$ is directly related (2.72) to the canonical momentum Π_A for the gauge field, the grading by $\mathbf{W_j}$ will again be interpreted physically as a quantum grading by total momentum.

Classical mode expansion. This background material out of the way, we now quantize the Maxwell gauge field on the Riemann surface Σ. As for the periodic scalar field, our main interest lies in a universal set of low-lying energy levels which are not sensitive to the detailed spectral geometry of Σ.

The quantization of A on Σ involves a countable number of topological sectors, labelled by the degree m of the line-bundle L. By analogy to the decomposition (2.25) of the configuration space for the scalar field, we write the configuration space for the Maxwell gauge field as a union of components

$$\mathcal{A} = \bigsqcup_{m \in \mathbb{Z}} \mathcal{A}_m \,, \tag{2.85}$$

where $\mathcal{A}_m$ is the affine space of unitary connections on the complex line-bundle L of degree m over Σ,

$$\mathcal{A}_m = \left\{ A \in \mathcal{A} \,\middle|\, \int_\Sigma F_A = 2\pi m \right\} . \tag{2.86}$$

Each connected component $\mathcal{A}_m \subset \mathcal{A}$ of the configuration space must be quantized separately, so the Maxwell Hilbert space $\mathscr{H}_\Sigma^\vee$ automatically carries an integral grading by the degree $m \in \mathbb{Z}$.

Having broken our quantization problem into countably-many pieces, we solve the classical equation of motion (2.78) for A in harmonic gauge. As a special case, we begin by considering only time-independent classical solutions, corresponding to connections with harmonic curvature on Σ.

When L has degree $m = 0$ and hence is topologically trivial, a harmonic connection on L is simply a flat connection, of the form

$$A = \sum_{j=1}^{2g} \varphi_0{}^j \, \mathfrak{e}_j \,, \qquad\qquad \varphi_0^j \in \mathbb{R}/2\pi\mathbb{Z} \,, \tag{2.87}$$

for the fixed harmonic basis $\{\mathfrak{e}_1, \cdots, \mathfrak{e}_{2g}\}$ of $H^1(\Sigma; \mathbb{Z})$. The expansion coefficients $\varphi_0{}^j$ for $j = 1, \ldots, 2g$ are thus angular coordinates on the Jacobian $\mathscr{J}_\Sigma$, which has already appeared in (2.83). Equivalently, $\left(\varphi_0{}^1, \ldots, \varphi_0{}^{2g}\right)$ characterize the holonomies of A around a generating set of closed one-cycles on Σ.

Because Σ carries a complex structure, each harmonic one-form $\mathfrak{e}_j$ in (2.87) can be decomposed according to its holomorphic/anti-holomorphic type via (2.6), in which case $\mathscr{J}_\Sigma$ itself inherits a complex structure. Intrinsically as a complex torus,

$$\begin{aligned} \mathscr{J}_\Sigma &= H^1_{\bar{\partial}}(\Sigma, \mathcal{O}_\Sigma)/2\pi\mathbb{L}, \qquad\qquad \mathbb{L} = H^1(\Sigma; \mathbb{Z}), \\ &\simeq \mathrm{Pic}_0(\Sigma), \end{aligned} \tag{2.88}$$

where $\mathrm{Pic}_0(\Sigma)$ denotes the group of isomorphism classes of holomorphic line-bundles of degree zero on Σ, with group multiplication given by the tensor product of line-bundles.

If L has degree $m \neq 0$, then a harmonic connection on L cannot be flat. Instead, the curvature $\widehat{F}_m$ of any harmonic connection on L is proportional to the Riemannian volume form on Σ,

$$\widehat{F}_m = \frac{2\pi m}{\ell^2}\,\mathrm{vol}_\Sigma, \tag{2.89}$$

where the proportionality constant in (2.89) is determined by the topological condition in (2.86). The formula in (2.89) is insufficient to fix a fiducial U(1)-connection $\widehat{A}_m$ with the given curvature, since $\widehat{A}_m$ may have non-trivial holonomies not detected by $\widehat{F}_m$. To fix $\widehat{A}_m$ uniquely, we use our auxiliary choice of basepoint $\sigma_0 \in \Sigma$ and the resulting holomorphic identification $L = \mathcal{O}_\Sigma(m\,\sigma_0)$. Precisely the same choice appeared in the quantization (2.35) of the periodic scalar field, for precisely the same reason.

Abstractly, the basepoint $\sigma_0 \in \Sigma$ provides an isomorphism between distinct components of the Picard group of all holomorphic line-bundles on Σ,

$$\mathrm{Pic}(\Sigma) = \bigsqcup_{m \in \mathbb{Z}} \mathrm{Pic}_m(\Sigma), \tag{2.90}$$

via the tensor product

$$\begin{aligned} \otimes \mathcal{O}_\Sigma(\sigma_0) : \mathrm{Pic}_m(\Sigma) &\xrightarrow{\simeq} \mathrm{Pic}_{m+1}(\Sigma), \\ \mathfrak{L} &\longmapsto \mathfrak{L} \otimes \mathcal{O}_\Sigma(\sigma_0). \end{aligned} \tag{2.91}$$

Here $\mathrm{Pic}_m(\Sigma)$ denotes the component of the Picard group consisting of degree m holomorphic line-bundles on Σ. Under the isomorphism in (2.91), all components of the Picard group are identified with the distinguished component $\mathrm{Pic}_0(\Sigma) \simeq \mathscr{J}_\Sigma$. Because we already have a fiducial connection in $\mathrm{Pic}_0(\Sigma)$, namely $\widehat{A}_0 = 0$ in (2.87), we just take $\widehat{A}_m$ to be the image of $\widehat{A}_0$ under the isomorphism. Equivalently from the differential perspective, $\widehat{A}_m$ is the unique harmonic, unitary connection compatible with the holomorphic structure on $\mathcal{O}_\Sigma(m\,\sigma_0)$. See for instance Ch. 4 of [11] for more about the existence and uniqueness of $\widehat{A}_m$.

Our choice for the fiducial harmonic connection $\widehat{A}_m$ is natural in the following sense. Trivially, $\mathcal{O}_\Sigma(m\,\sigma_0) = \mathcal{O}_\Sigma(\sigma_0)^{\otimes m}$. Thus $\widehat{A}_m$ for general m is related to the basic connection $\widehat{A}_1$ on $\mathcal{O}_\Sigma(\sigma_0)$ by

$$\widehat{A}_m = m\,\widehat{A}_1, \qquad\qquad m \in \mathbb{Z}. \tag{2.92}$$

This identity is clearly compatible with the formula for the harmonic curvature $\widehat{F}_m$ in (2.89). For the remainder of the paper, I simplify the notation by setting $\widehat{A} \equiv \widehat{A}_1$.

According to the preceding discussion, in each degree $m \in \mathbb{Z}$, the arbitrary time-independent solution to the classical equation of motion for A in (2.78) is given up to gauge-equivalence by a point on the Jacobian $\mathscr{J}_\Sigma \simeq \mathrm{U}(1)^{2g}$. To describe the more general, time-dependent solution in harmonic gauge, we perform an expansion of A in eigenmodes of the de Rham Laplacian $\triangle_1$,

$$
\begin{aligned}
A(t, z, \overline{z}) \,=\, m\,\widehat{A} + \sum_{j=1}^{2g} \varphi_0{}^j\, \mathfrak{e}_j + 2\pi e^2\, t \sum_{j,k=1}^{2g} p_{0,j}\, (\mathsf{Q}^{-1})^{jk}\, \mathfrak{e}_k \\
+ \sqrt{\pi} \sum_{\lambda > 0} \frac{e^2}{\lambda}\, \chi_\lambda(z, \overline{z}) \Big[a_\lambda\, \mathrm{e}^{-i\,\lambda\,t} + a_\lambda^\dagger\, \mathrm{e}^{i\,\lambda\,t} \Big].
\end{aligned}
\tag{2.93}
$$

The eigenmode expansion of A in (2.93) requires several comments.

First, $p_{0,j} \in \mathbb{R}$ for $j = 1, \ldots, 2g$ are the classical momenta conjugate to the angular coordinates $\varphi_0{}^j$ on the Jacobian. The coefficient of e^2 which multiplies $p_{0,j}$ is fixed by scaling, due to the explicit t-dependence. Also, Q is the positive-definite, symmetric matrix of L^2 inner-products[4]

$$
\mathsf{Q}_{jk} \,=\, (\mathfrak{e}_j, \mathfrak{e}_k) \,=\, \int_\Sigma \mathfrak{e}_j \wedge \star \mathfrak{e}_k\,, \qquad j, k = 1, \ldots, 2g\,.
\tag{2.94}
$$

The inverse matrix Q^{-1} satisfies $(\mathsf{Q}^{-1})^{ij}\, \mathsf{Q}_{jk} = \delta_k^i$. Since $E_A = \partial_t A$ in Coulomb gauge, the expansion of A ensures that $p_{0,j}$ is equal to the conserved charge $\mathbf{W_j}$ in (2.84),

$$
\mathbf{W_j} \,=\, p_{0,j}\,, \qquad j = 1, \ldots, 2g\,.
\tag{2.95}
$$

Proceeding to the second line of (2.93), $(a_\lambda, a_\lambda^\dagger)$ are a conjugate pair of complex parameters associated to the oscillating modes of A, and the coefficient $\sqrt{\pi}\, e^2/\lambda$ has been chosen to simplify the commutator algebra upon quantization. We also introduce an orthonormal basis of one-forms $\chi_\lambda \in \Omega_\Sigma^1$ which satisfy the joint conditions

$$
d^\dagger \chi_\lambda \,=\, 0\,, \qquad \chi_\lambda \in \Omega_\Sigma^1\,,
\tag{2.96}
$$

as well as

$$
\triangle_1 \chi_\lambda \,=\, \lambda^2\, \chi_\lambda\,, \qquad \lambda \in \mathbb{R}\,,
\tag{2.97}
$$

where by convention $\lambda > 0$ is positive. The first condition instantiates the harmonic gauge in (2.77), and the second condition states that χ_λ is an eigenform for the de Rham Laplacian $\triangle_1$ acting on one-forms on Σ.

The eigenforms χ_λ have a simple relation to the eigenfunctions $\psi_\lambda \in \Omega_\Sigma^0$ which appear in the corresponding mode expansion for the scalar field ϕ. For if ψ_λ is an eigenfunction of the scalar Laplacian,

$$
\triangle_0 \psi_\lambda \,=\, \lambda^2\, \psi_\lambda\,, \qquad \lambda > 0\,,
\tag{2.98}
$$

---

[4]The same notation for Q and Q^{-1} is used in [2], but in reference to similar quantities defined on a closed three-manifold M.

then we obtain a corresponding eigenform χ_λ for $\triangle_1$ by setting

$$\chi_\lambda = \frac{e^2}{\lambda} \star d\psi_\lambda \,. \tag{2.99}$$

Since $d^\dagger = -\star d\star$ and $\star^2 = -1$ on Ω^1_Σ, trivially $d^\dagger \chi_\lambda = 0$. Also,

$$\triangle_1 \chi_\lambda = d^\dagger d\chi_\lambda = -\star d\star d\left(\frac{e^2}{\lambda} \star d\psi_\lambda\right) = \frac{e^2}{\lambda} \star d\left(\triangle_0 \psi_\lambda\right) = \lambda^2 \chi_\lambda \,. \tag{2.100}$$

The relative coefficient e^2/λ in (2.99) just ensures that χ_λ has unit norm,

$$||\chi_\lambda||^2 = \int_\Sigma \chi_\lambda \wedge \star \chi_\lambda = \frac{e^4}{\lambda^2} \int_\Sigma \psi_\lambda \wedge \star \triangle_0 \psi_\lambda = 1 \,, \tag{2.101}$$

assuming that ψ_λ is normalized according to (2.29).

Conversely, given any eigenform χ_λ satisfying (2.96) and (2.97) with $\lambda > 0$, we obtain a normalized eigenfunction ψ_λ via

$$\psi_\lambda = -\frac{1}{e^2\lambda} \star d\chi_\lambda \,. \tag{2.102}$$

Following the same steps in (2.100), one can verify directly that ψ_λ in (2.102) is a normalized eigenfunction of the scalar Laplacian $\triangle_0$. The minus sign in (2.102) is a nicety which ensures that the map from ψ_λ to χ_λ and back is the identity.

Together, the relations in (2.99) and (2.102) constitute a Hodge isomorphism between the non-zero spectrum of $\triangle_0$ acting on Ω^0_Σ and the non-zero spectrum of $\triangle_1$ acting on the intersection $\Omega^1_\Sigma \cap \mathrm{Ker}(d^\dagger)$. Consequently, the oscillator frequencies $\lambda > 0$ which appear in the harmonic expansion (2.93) of the gauge field A are precisely the same frequencies which appear in the harmonic expansion (2.27) of the periodic scalar field ϕ. This equality is essential for abelian duality to hold on Σ.

Canonical commutation relations. Naively, to quantize the Maxwell theory, we promote both A and the canonical momentum Π_A to operator-valued one-forms which satisfy the canonical equal-time commutation relation

$$\Big[A(z),\,\Pi_A(w)\Big] = i\,\mathrm{vol}_\Sigma \cdot \delta_\Sigma(z,w)\,, \qquad\qquad z, w \,\in\, \Sigma\,. \tag{2.103}$$

In making sense of (2.103) geometrically, the Riemannian volume form on the right is to be interpreted as a section of the tensor product $\mathrm{vol}_\Sigma \in \Omega^1_\Sigma \otimes \Omega^1_\Sigma$, and $\delta_\Sigma(z,w)$ remains the delta-function with support along the diagonal $\Delta \subset \Sigma \times \Sigma$.

In actuality, the situation is more complicated, because the naive commutator in (2.103) is not compatible with the Coulomb gauge conditions

$$d^\dagger A = d^\dagger E_A = 0\,, \qquad\qquad A, E_A \,\in\, \Omega^1_\Sigma\,, \tag{2.104}$$

the latter of which implies

$$d\Pi_A = d\left(\frac{1}{2\pi e^2} \star E_A - \frac{1}{2\pi}\,\beta\right) = 0\,. \tag{2.105}$$

I let d_z and d_w denote the respective de Rham operators acting individually on the left and right factors in the product $\Omega^1_\Sigma \otimes \Omega^1_\Sigma$. Then the left-hand side of (2.103) is annihilated by $d_z{}^\dagger$ and d_w according to the gauge conditions in (2.104) and (2.105), but the right-hand side is not.

This situation is a familiar feature of Coulomb gauge, as is the remedy. Let $G(z,w)$ be the Green's function for the scalar Laplacian $\triangle_0$ on Σ, such that

$$\triangle_0 G(z,w) \,=\, \delta_\Sigma(z,w) \,-\, \frac{1}{\ell^2}\,. \qquad (2.106)$$

Because Σ is compact, we are careful to subtract the contribution from the constant mode in (2.106), so that both sides of the equation for $G(z,w)$ integrate to zero over Σ. Equivalently, $G(z,w)$ can be expanded in terms of the orthonormal eigenbasis $\{\psi_\lambda\}$ for Ω^0_Σ,

$$G(z,w) \,=\, e^4 \sum_{\lambda>0} \frac{\psi_\lambda(z)\,\psi_\lambda(w)}{\lambda^2}\,. \qquad (2.107)$$

For the ambitious reader who enjoys keeping track of factors of e^2, recall that e^2 appears in the normalization condition (2.29) for ψ_λ and thus enters the expansion for $G(z,w)$.

The scalar Green's function $G(z,w)$ can be used to correct the naive commutation relation in (2.103) so that the right-hand side is actually compatible with the Coulomb gauge constraints $d^\dagger A = d\Pi_A = 0$. To wit, the corrected commutator will be

$$\Big[A(z),\,\Pi_A(w)\Big] \,=\, i\,\mathrm{vol}_\Sigma \cdot \delta_\Sigma(z,w) \,-\, i\,(d_z \otimes \star d_w)G(z,w)\,. \qquad (2.108)$$

Here $(d_z \otimes \star d_w)G(z,w)$ is the section of $\Omega^1_\Sigma \otimes \Omega^1_\Sigma$ obtained from the action of the individual de Rham operators on $G(z,w)$. In terms of the spectral decomposition in (2.107),

$$(d_z \otimes \star d_w)G(z,w) \,=\, e^4 \sum_{\lambda>0} \frac{d\psi_\lambda(z) \otimes \star d\psi_\lambda(w)}{\lambda^2} \,\in\, \Omega^1_\Sigma \otimes \Omega^1_\Sigma\,. \qquad (2.109)$$

Most crucially, the right-hand side of the corrected commutator (2.108) does lie in the kernels of both $d_z{}^\dagger$ and d_w. This statement can be verified either by direct computation from the defining equation for $G(z,w)$ in (2.106) or, as will be more relevant here, by applying the spectral decomposition for $(d_z \otimes \star d_w)G(z,w)$ in (2.109). I take the latter approach.

In close analogy to (2.42), the term involving the delta-function in (2.108) can be presented as the sum

$$\mathrm{vol}_\Sigma \cdot \delta_\Sigma(z,w) \,=$$
$$\sum_{j,k=1}^{2g} (\mathsf{Q}^{-1})^{jk}\, \mathfrak{e}_j(z) \otimes \star \mathfrak{e}_k(w) \,+\, \sum_{\lambda>0} \Big[\chi_\lambda(z) \otimes \star \chi_\lambda(w) \,-\, \star \chi_\lambda(z) \otimes \chi_\lambda(w)\Big]\,. \qquad (2.110)$$

Very briefly, the three terms on the right in (2.110) reflect the three terms in the Hodge decomposition

$$\Omega^1_\Sigma \,\simeq\, \mathcal{H}^1(\Sigma) \oplus \mathrm{Im}\big(d^\dagger|_{\Omega^2_\Sigma}\big) \oplus \mathrm{Im}\big(d|_{\Omega^0_\Sigma}\big)\,. \qquad (2.111)$$

The first term in (2.110), involving the harmonic forms $\mathfrak{e}_j$ for $j = 1, \ldots, 2g$, describes the action of $\mathrm{vol}_\Sigma \cdot \delta_\Sigma(z, w)$ by wedge-product and convolution on $\mathcal{H}^1(\Sigma)$. Otherwise, the two sets of eigenforms $\{\chi_\lambda\}$ and $\{\star\chi_\lambda\}$ for $\lambda > 0$ span the respective images of $d^\dagger$ and d, as discussed previously in relation to (2.99). The latter two terms on the right in (2.110) then account for the action of $\mathrm{vol}_\Sigma \cdot \delta_\Sigma(z, w)$ on $\mathrm{Im}(d^\dagger|_{\Omega_\Sigma^2})$ and $\mathrm{Im}(d|_{\Omega_\Sigma^0})$. The relative minus sign in (2.110) can be checked directly by working in a local frame on Σ or understood as a consequence of anti-symmetry under the exchange of the factors in $\Omega_\Sigma^1 \otimes \Omega_\Sigma^1$.

Because $\mathfrak{e}_j$ is harmonic and $d^\dagger\chi_\lambda = 0$, the first two terms on the right of (2.110) are annihilated by $d_z{}^\dagger$ and d_w. On the other hand, for the third term in (2.110) the relation between χ_λ and ψ_λ in (2.99) implies

$$\sum_{\lambda > 0} \star\chi_\lambda(z) \otimes \chi_\lambda(w) \,=\, -e^4 \sum_{\lambda > 0} \frac{d\psi_\lambda(z) \otimes \star d\psi_\lambda(w)}{\lambda^2} \,. \qquad (2.112)$$

The minus sign appears since $\star^2 = -1$ on Ω_Σ^1. Thence from (2.109) and (2.110), the right-hand side of the corrected commutation relation is given by

$$\mathrm{vol}_\Sigma \cdot \delta_\Sigma(z, w) - (d_z \otimes \star d_w)G(z, w) = \sum_{j,k=1}^{2g} \left(\mathsf{Q}^{-1}\right)^{jk} \mathfrak{e}_j(z) \otimes \star\mathfrak{e}_k(w) + \sum_{\lambda > 0} \chi_\lambda(z) \otimes \star\chi_\lambda(w) \,. \qquad (2.113)$$

The expression in (2.113) is manifestly annihilated by $d_z{}^\dagger$ and d_w, so is compatible with Coulomb gauge.

Maxwell Hilbert space. With the corrected commutation relation in (2.108), the quantization of Maxwell theory on Σ is now straightforward. From the (slightly formal) functional perspective, the momentum Π_A becomes identified with the operator

$$\Pi_A(w) \,=\, -i \left[\mathrm{vol}_\Sigma \cdot \frac{\delta}{\delta A(w)} \,+\, \int_\Sigma d^2u \,(d_u \otimes \star d_w)G(u, w)\,\mathrm{vol}_\Sigma \cdot \frac{\delta}{\delta A(u)}\right]. \qquad (2.114)$$

Here the smeared term involving the Green's function $G(u, w)$ is the necessary price of working in Coulomb gauge. I do not wish to belabor the interpretation of the non-local, smeared term, as we will be primarily interested in situations for which it does not matter. However, let me say a word about the basic geometric meaning of (2.114), which may be somewhat opaque.

Because $\Pi_A \in \Omega_\Sigma^1$ transforms as a one-form, the right-hand side of (2.114) must also transform as a one-form on Σ. Dually to A, the derivative $\delta/\delta A \in T\Sigma$ transforms as a vector field on Σ. The notation $\mathrm{vol}_\Sigma \cdot \delta/\delta A$ indicates that this vector field is to be contracted with the volume form to produce a one-form on Σ, as expressed in local coordinates

$$\mathrm{vol}_\Sigma \cdot \frac{\delta}{\delta A} \,=\, \sum_{\mu,\nu=1}^{2} (\mathrm{vol}_\Sigma)_{\mu\nu} \, dx^\mu \, \frac{\delta}{\delta A_\nu} \,\in\, \Omega_\Sigma^1 \,. \qquad (2.115)$$

For the smeared term in (2.114), we take the wedge-product of $(d_u \otimes \star d_w)G(u, w)$ as a section of $\Omega_\Sigma^1 \otimes \Omega_\Sigma^1$ with the one-form $\mathrm{vol}_\Sigma \cdot \delta/\delta A$ to obtain a section of $\Omega_\Sigma^2 \otimes \Omega_\Sigma^1$. The first factor is integrated over Σ to produce yet another one-form.

The classical relation $\Pi_A = \star E_A / 2\pi e^2 - \beta / 2\pi$ implies that the electric field E_A acts as the covariant operator

$$\frac{1}{2\pi e^2} \star E_A(w) \, = \, -i \, \frac{D}{DA(w)} \, , \tag{2.116}$$

where

$$\frac{D}{DA(w)} \, = \, \mathrm{vol}_\Sigma \cdot \frac{\delta}{\delta A(w)} \, + \, \frac{i\,\beta}{2\pi} \, + \, \int_\Sigma d^2u \, (d_u \otimes \star d_w) G(u,w) \, \mathrm{vol}_\Sigma \cdot \frac{\delta}{\delta A(u)} \, . \tag{2.117}$$

This expression for $D/DA(w)$ should be compared to the corresponding expression for $D/D\phi(w)$ appearing in (2.40).

Smearing or no, since A itself does not appear on the right-hand side of (2.117), the functional derivative $D/DA(w)$ describes a flat connection on the subspace of the affine space $\mathcal{A}$ where $d^\dagger A = 0$,

$$\left[\frac{D}{DA(z)}, \, \frac{D}{DA(w)} \right] \, = \, 0 \, . \tag{2.118}$$

In precise analogy to the angle θ appearing in $D/D\phi(w)$, the harmonic one-form β in $D/DA(w)$ will describe the holonomy of a flat connection over the Jacobian $\mathscr{J}_\Sigma$, after we reduce to zero-modes.

As a more down-to-earth alternative to the functional calculus, quantization of A can be carried out in terms of the eigenmode expansion in (2.93), coupled with the spectral identity in (2.113). To realize the commutator in (2.108), the pairs $\left(\varphi_0{}^j, \, p_{0,j} \right)$ for $j = 1, \ldots, 2g$ and $\left(a_\lambda, a_\lambda{}^\dagger \right)$ for all $\lambda > 0$ are promoted to operators which obey the Heisenberg algebra

$$\begin{aligned}
\left[\varphi_0{}^j, \, p_{0,k} \right] &= i \, \delta_k{}^j \, , \qquad\qquad j, k = 1, \ldots, 2g \, , \\
\left[a_\lambda, \, a_{\lambda'}{}^\dagger \right] &= \frac{\lambda}{e^2} \, \delta_{\lambda\lambda'} \, ,
\end{aligned} \tag{2.119}$$

with all other commutators vanishing. This algebra is akin to that for the periodic scalar field ϕ in (2.43), but rather than quantizing a single periodic zero-mode, we quantize a set of $2g$ zero-modes which describe the motion of a free particle on the Jacobian $\mathscr{J}_\Sigma$ of Σ.

The free-field algebra holds in each topological sector labelled by $m = \deg(L)$, so the Maxwell Hilbert space $\mathscr{H}_\Sigma^\vee$ is the direct sum

$$\mathscr{H}_\Sigma^\vee \, = \, \bigoplus_{m \in \mathbb{Z}} \left(\mathscr{H}_\Sigma^\vee \right)^m \, , \tag{2.120}$$

where each summand is itself the tensor product

$$\left(\mathscr{H}_\Sigma^\vee \right)^m \, = \, \mathsf{H}_0^\vee \otimes \bigotimes_{\lambda > 0} \mathsf{H}_\lambda \, . \tag{2.121}$$

Exactly as in (2.45), H_λ is the oscillator Fock space acted upon by the pair $(a_\lambda, a_\lambda{}^\dagger)$. We have already noted that an identical spectrum of non-zero frequencies $\lambda > 0$ occurs for both the periodic scalar field and the U(1) gauge field on Σ. Thus the same tensor product of Fock spaces H_λ appears in both the scalar Hilbert space $\mathscr{H}_\Sigma$ and the Maxwell Hilbert space $\mathscr{H}_\Sigma^\vee$. At least for the excited oscillator states in the two Hilbert spaces, abelian duality is a trivial equivalence.

The remaining factor $\mathsf{H}_0^\vee$ arises from the quantization of the zero-modes for the gauge field. Classically, $(\varphi_0{}^j, p_{0,j})$ for $j = 1, \ldots, 2g$ are coordinates on the cotangent bundle $T^* \mathscr{J}_\Sigma$ with its canonical symplectic structure, so the standard quantization yields

$$\mathsf{H}_0^\vee \simeq L^2(\mathscr{J}_\Sigma; \mathbb{C}). \qquad [\, \beta = 0 \,] \qquad (2.122)$$

Because $\mathscr{J}_\Sigma \simeq \mathrm{U}(1)^{2g}$ is just a torus, the Hilbert space $\mathsf{H}_0^\vee$ is naturally spanned by the collection of Fourier wavefunctions

$$\Psi_\omega(\varphi_0) \,=\, \exp\!\left(i \sum_{j=1}^{2g} \varphi_0{}^j \int_\Sigma \mathfrak{e}_j \wedge \omega \right), \qquad \omega \in \mathbb{L}, \qquad (2.123)$$

each labelled by an element ω in the cohomology lattice $\mathbb{L} = H^1(\Sigma; \mathbb{Z})$. Again, the coincidence of notation with the winding-number in section 2.1 is no accident. Here though, integrality of ω is not due to topology per se, but rather to the requirement that the wavefunction in (2.123) be invariant under shifts $\varphi_0{}^j \mapsto \varphi_0{}^j + 2\pi$ for all $j = 1, \ldots, 2g$.

Physically, ω determines the conserved momentum carried by the state $\Psi_\omega(\varphi_0)$, where we apply the identification

$$\mathbf{W_j} \,=\, p_{0,j} \,=\, -i \frac{\partial}{\partial \varphi_0{}^j}. \qquad [\, \beta = 0 \,] \qquad (2.124)$$

Directly, the Fourier wavefunction in (2.123) is a momentum eigenstate,

$$\mathbf{W_j} \cdot \Psi_\omega \,=\, \langle \mathfrak{e}_j, \omega \rangle \cdot \Psi_\omega. \qquad (2.125)$$

As in (2.61), $\langle \cdot, \cdot \rangle$ is shorthand for the intersection pairing of one-forms on Σ.

When the topological parameter $\beta \in \mathcal{H}^1(\Sigma)$ is non-zero, the interpretation of the zero-mode momentum $p_{0,j}$ is modified via (2.116) and (2.117) to

$$p_{0,j} \,=\, -i \frac{D}{D\varphi_0{}^j}, \qquad \frac{D}{D\varphi_0{}^j} \,=\, \frac{\partial}{\partial \varphi_0{}^j} + i \frac{\langle \mathfrak{e}_j, \beta \rangle}{2\pi}. \qquad (2.126)$$

In precise analogy to the expression for $D/D\phi_0$ in (2.49), $D/D\varphi_0$ is the covariant derivative associated to a unitary flat connection on a complex line-bundle $\mathcal{L}$ over the Jacobian $\mathscr{J}_\Sigma$, and the harmonic one-form β determines the holonomies of this connection around each one-cycle on the Jacobian. Because $\mathscr{J}_\Sigma$ is the quotient $H^1(\Sigma; \mathbb{R})/2\pi\mathbb{L}$, each generating one-cycle $C_j \in H_1(\mathscr{J}_\Sigma; \mathbb{Z})$ can be identified with a corresponding lattice generator $\mathfrak{e}_j \in \mathbb{L}$, for which

$$\mathrm{Hol}_{C_j}(D/D\varphi_0) \,=\, \exp[-i \langle \mathfrak{e}_j, \beta \rangle]. \qquad (2.127)$$

When $\beta \neq 0$, the Hilbert space $\mathsf{H}_0^\vee$ generalizes to the space of square-integrable sections of the complex line-bundle $\mathcal{L}$,

$$\mathsf{H}_0^\vee \,\simeq\, L^2(\mathscr{J}_\Sigma; \mathcal{L}). \qquad (2.128)$$

As a result of the covariant identification in (2.126), $\mathbf{W_j} \equiv p_{0,j}$ then acts on the Fourier basis for $\mathsf{H}_0^\vee$ with the new eigenvalues

$$\mathbf{W_j} \cdot \Psi_\omega \,=\, \left\langle \mathfrak{e}_j, \omega + \frac{\beta}{2\pi} \right\rangle \cdot \Psi_\omega, \qquad \omega \in \mathbb{L}. \qquad (2.129)$$

Again in comparison to (2.53), the role of the harmonic one-form β is to shift the integral grading by the cohomology lattice $\mathbb{L}$ on the Maxwell Hilbert space $\mathscr{H}_\Sigma^\vee$.

Because the zero-mode Hilbert space $\mathsf{H}_0^\vee$ is graded by the eigenvalues of $\mathbf{W_j}$ for $j = 1, \ldots, 2g$, the full Maxwell Hilbert space $\mathscr{H}_\Sigma^\vee$ is bigraded by the lattice $\mathbb{L} \oplus \mathbb{Z}$,

$$\mathscr{H}_\Sigma^\vee \simeq \bigoplus_{(\omega,\, m) \in \mathbb{L} \oplus \mathbb{Z}} \left(\mathscr{H}_\Sigma^\vee\right)^{\omega, m}. \tag{2.130}$$

Following the notation in section 2.1, I let $|\omega; m\rangle$ denote the Fourier wavefunction $\Psi_\omega(\varphi_0)$, considered in the topological sector with magnetic flux $m = \deg(L)$, and satisfying the vacuum condition

$$a_\lambda |\omega; m\rangle = 0, \qquad \lambda > 0. \tag{2.131}$$

All other Fock states in $\mathscr{H}_\Sigma$ are obtained by acting with the oscillator raising-operators $a_\lambda^\dagger$ on the Fock vacuum $|\omega; m\rangle$, so more explicitly

$$\left(\mathscr{H}_\Sigma^\vee\right)^{\omega, m} = \mathbb{C} \cdot |\omega; m\rangle \otimes \bigotimes_{\lambda > 0} \mathsf{H}_\lambda. \tag{2.132}$$

Clearly $\left(\mathscr{H}_\Sigma^\vee\right)^{\omega, m}$ is isomorphic to the scalar field summand $\mathscr{H}_\Sigma^{m, \omega}$ in (2.56).

Finally, let us consider the action of the Maxwell Hamiltonian $\mathbf{H}^\vee$ on states in the Hilbert space $\mathscr{H}_\Sigma^\vee$. Under the identification (2.116) of the electric field $\star E_A$ with the covariant operator D/DA, the Hamiltonian becomes

$$\mathbf{H}^\vee = \int_\Sigma \left[-\pi e^2 \frac{D}{DA} \wedge \star \frac{D}{DA} + \frac{1}{4\pi e^2} F_A \wedge \star F_A - \frac{\theta}{2\pi e^2 \ell^2} F_A \right]. \tag{2.133}$$

In terms of the conserved momenta $\mathbf{W_j}$ in (2.84),

$$\mathbf{H}^\vee = \int_\Sigma \left[\pi e^2 \left(\mathsf{Q}^{-1}\right)^{jk} \mathbf{W_j} \mathbf{W_k} + \cdots + \frac{1}{4\pi e^2} F_A \wedge \star F_A - \frac{\theta}{2\pi e^2 \ell^2} F_A \right], \tag{2.134}$$

where the omitted terms involve the action of D/DA on the excited oscillator states in the Hilbert space.

The complete spectrum of the Maxwell Hamiltonian depends upon the set of eigenvalues $\{\lambda^2\}$ for the scalar Laplacian on Σ, exactly as for the periodic scalar field. Following the strategy in section 2.1, we ask instead the more limited question of how $\mathbf{H}^\vee$ acts on the Fock vacua $|\omega; m\rangle$ associated to the harmonic modes of the gauge field. Evidently from (2.89) and (2.129),

$$\mathbf{H}^\vee |\omega; m\rangle = e^2 \left[\pi \left(\omega + \frac{\beta}{2\pi}, \, \omega + \frac{\beta}{2\pi} \right) + \frac{\pi\, m^2}{(e^2 \ell)^2} - \frac{\theta\, m}{(e^2 \ell)^2} + \frac{E_0}{e^2 \ell} \right] |\omega; m\rangle, \tag{2.135}$$

where E_0/ℓ is again a Casimir energy on Σ. Because the zero-point energies $\frac{1}{2}\lambda$ of the oscillating modes are the same for both the periodic scalar field and the gauge field, the constant E_0 in (2.135) will agree with the corresponding constant in (2.59) so long as we use the same regularization method to define both (as we assume).

For Maxwell theory on Σ, the spectrum of $\mathbf{H}^\vee$ simplifies in the regime $e^2\ell \ll 1$ of weak electric coupling. Only then do the Fock vacua $|\omega; m\rangle$ for arbitrary Fourier momentum $\omega \in \mathbb{L}$ have parametrically smaller energy than the typical oscillator state such as $a_\lambda^\dagger|0; m\rangle$. Not surprisingly, this case is opposite to the strong-coupling regime $1/e^2\ell \ll 1$ in which the states of least-energy arise by quantizing the single zero-mode of the periodic scalar field ϕ.

2.3 Topological Hilbert space

Let us summarize our results so far.

We have obtained an explicit identification between the Hilbert spaces for the $U(1)$ gauge field A and the periodic scalar field ϕ on the surface Σ,

$$
\begin{aligned}
\mathscr{H}_\Sigma{}^\vee &\simeq \mathscr{H}_\Sigma \,, \\
&= \bigoplus_{(m,\omega)\in\mathbb{Z}\oplus\mathbb{L}} \left[\mathbb{C} \cdot |m; \omega\rangle \otimes \bigotimes_{\lambda>0} \mathsf{H}_\lambda \right].
\end{aligned}
\tag{2.136}
$$

The isomorphism for the oscillator Fock spaces H_λ for $\lambda > 0$ follows from classical Hodge theory after we pass to Coulomb gauge for A, so it is relatively uninteresting. The non-trivial content in (2.136) is the identification between the Fock vacua $|m; \omega\rangle$, which arise from the quantization of the harmonic modes of A and ϕ in each topological sector.

For the periodic scalar field, $m \in \mathbb{Z}$ is a quantum label which arises from Fourier modes on S^1, and the lattice vector $\omega \in \mathbb{L}$ is a classical label which measures the winding-number of the map $\phi : \Sigma \to S^1$. Conversely for the gauge field, the integer m is the classical label, corresponding to the degree of the line-bundle L, and the lattice vector ω is the quantum label, arising from Fourier modes on the Jacobian $\mathscr{J}_\Sigma$. Under the isomorphism in (2.136), the classical and quantum labels are swapped, characteristic of abelian duality in any dimension.

A dual role is also played by the topological parameters (θ, α) and (θ, β) which enter the respective Hamiltonians in (2.21) and (2.73). For the periodic scalar field, the angle θ is a quantum parameter which determines the holonomy of a flat, unitary connection on a complex line bundle over S^1 as in (2.50), and the harmonic one-form α is a classical parameter which weights each winding-sector. For the gauge field, θ is the classical parameter which weights the magnetic flux on Σ, and β is now the quantum parameter which determines the holonomy of a flat connection on a complex line-bundle over $\mathscr{J}_\Sigma$ as in (2.127).

Nonetheless, under the dual correspondence

$$
\alpha = \star\beta \,, \qquad \alpha, \beta \in \mathcal{H}^1(\Sigma) \,,
\tag{2.137}
$$

the Hamiltonians $\mathbf{H}$ in (2.59) and $\mathbf{H}^\vee$ in (2.135) act identically on the states $|m; \omega\rangle$ up to a constant shift δ,

$$
\mathbf{H}|m; \omega\rangle = \left(\mathbf{H}^\vee + \delta \right)|m; \omega\rangle \,, \qquad \delta = \frac{e^2}{4\pi} \left[\frac{\theta^2}{(e^2\ell)^2} - (\beta, \beta) \right] .
\tag{2.138}
$$

Let us introduce the quantum partition functions for the scalar and Maxwell theories,

$$Z_\Sigma(R) \,=\, \text{Tr}_{\mathscr{H}_\Sigma}\, e^{-R\,\mathbf{H}}\,, \qquad\qquad Z_\Sigma^\vee(R) \,=\, \text{Tr}_{\mathscr{H}_\Sigma}\, e^{-R\,\mathbf{H}^\vee}\,, \qquad\qquad (2.139)$$

both of which depend upon a real parameter $R \in \mathbb{R}$ which can be interpreted as the length of the circle in $M = S^1 \times \Sigma$.

The constant shift in (2.138) then implies the relation

$$Z_\Sigma^\vee(R) \,=\, Z_\Sigma(R) \cdot \exp\left[\frac{e^2 R}{4\pi}\left(\frac{\theta^2}{(e^2\ell)^2} \,-\, (\alpha,\alpha)\right)\right]. \qquad\qquad (2.140)$$

The same duality relation appears under a different guise in [2], where it arises from the non-trivial modular transformation of a theta-function $\Theta_M(\gamma)$ associated to any Riemannian three-manifold M. See section 4.1 of [2] for a complete discussion of the theta-function and section 5.1 of the same work for a path integral derivation of the relation in (2.140). Compare especially to equation (5.1) in [2].

Finally, as we have already mentioned, the spectrum of $\mathbf{H}$ dramatically simplifies in either the small-volume limit $e^2\ell \ll 1$ or the large-volume limit $e^2\ell \gg 1$. In both cases, the quantum states of minimal energy within each topological sector are the Fock vacua $|m;\omega\rangle$, for all pairs (m,ω) in the lattice $\mathbb{Z} \oplus \mathbb{L}$. Hence we can sensibly restrict attention to the subspace of the full Hilbert space spanned by these states,

$$\mathscr{H}_\Sigma^{\text{top}} \,=\, \bigoplus_{(m,\omega)\in\mathbb{Z}\oplus\mathbb{L}} \mathbb{C} \cdot |m;\omega\rangle \,\subset\, \mathscr{H}_\Sigma\,. \qquad\qquad (2.141)$$

Essential for the following, the description of $\mathscr{H}_\Sigma^{\text{top}}$ does not require detailed knowledge of the Riemannian metric on Σ. Instead, the action of operators such as $\mathbf{H}$ on $\mathscr{H}_\Sigma^{\text{top}}$ will only depend upon the complex structure and the overall volume of Σ. In that sense, $\mathscr{H}_\Sigma^{\text{top}}$ is a subspace of 'quasi-topological' states. Unlike the typical situation in topological quantum field theory, though, $\mathscr{H}_\Sigma^{\text{top}}$ has infinite dimension. As a result, the action of various operators on $\mathscr{H}_\Sigma^{\text{top}}$ can be quite interesting, a topic to which we turn next. Elsewhere, I will discuss some important related notions in the context of $\mathcal{N} = 2$ supersymmetric quantum field theory in three dimensions.

3 Operator algebra at higher genus

Given the explicit construction of the Hilbert space on Σ, we now discuss the action of several natural classes of operators on that Hilbert space.

As mentioned at the end of section 2.3, we simplify life by considering only the action on the quasi-topological subspace $\mathscr{H}_\Sigma^{\text{top}}$ spanned by the Fock vacua $|m;\omega\rangle$,

$$\mathscr{H}_\Sigma^{\text{top}} \,=\, \bigoplus_{(m,\,\omega)\in\mathbb{Z}\oplus\mathbb{L}} \mathbb{C} \cdot |m;\omega\rangle\,. \qquad\qquad (3.1)$$

The restriction to $\mathscr{H}_\Sigma^{\text{top}}$ is natural in either the regime $e^2\ell \ll 1$ or $e^2\ell \gg 1$, for which the Fock vacua describe states of minimal energy within each topological sector. Because the

abelian theories in question are non-interacting, we do not need to worry about the effects of high-energy states, which would otherwise be integrated-out in passing from the big Hilbert space $\mathscr{H}_\Sigma$ to the subspace $\mathscr{H}_\Sigma^{\text{top}}$.

Following section 5.2 in [2], we analyze three classes of operators on Σ. We first have the local vertex operator $\mathsf{V}_k(\sigma)$ which is inserted at a point $\sigma \in \Sigma$,

$$\mathsf{V}_k(\sigma) = \mathrm{e}^{ik\phi(\sigma)}, \qquad k \in \mathbb{Z}. \tag{3.2}$$

Periodicity of the scalar field $\phi \sim \phi + 2\pi$ dictates that k be an integer so that $\mathsf{V}_k(\sigma)$ is single-valued.

Next we have the Wilson loop operator $\mathsf{W}_n(C)$ associated to an oriented, smoothly embedded curve $C \subset \Sigma$,

$$\mathsf{W}_n(C) = \exp\left[i\,n\oint_C A\right], \qquad n \in \mathbb{Z}. \tag{3.3}$$

For generic choices of C, the charge n of the Wilson loop operator must be an integer to ensure gauge-invariance with respect to the compact gauge group U(1).

Perhaps less appreciated, when C is a homologically-trivial curve which bounds a two-cycle $D \subset \Sigma$, the Wilson loop operator can be defined for an arbitrary real charge via

$$\mathsf{W}_\nu(C) = \exp\left[i\,\nu\int_D F_A\right], \qquad \nu \in \mathbb{R}. \tag{3.4}$$

This expression for $\mathsf{W}_\nu(C)$ is manifestly gauge-invariant for all values of ν, and it reduces to (3.3) by Stokes' theorem for $C = \partial D$. Unlike the situation for $\mathsf{W}_\nu(C)$ in three dimensions [2], where the role of D is played by a Seifert surface with some homological ambiguity, here there is no ambiguity about D. Because D is a two-cycle on Σ, the choice of D is fixed entirely by the orientations of the pair (Σ, C).

Without delay, let me emphasize that $\mathsf{W}_\nu(C)$ will act non-trivially on $\mathscr{H}_\Sigma^{\text{top}}$ even when C is trivial in homology. Likewise, $\mathsf{W}_n(C)$ will depend upon the geometry of $C \subset \Sigma$, not just the homology class $[C] \in H_1(\Sigma)$.

By contrast, we do have a purely homological loop operator $\mathsf{L}_\alpha(C)$, given by

$$\mathsf{L}_\alpha(C) = \exp\left[\frac{i\,\alpha}{2\pi}\oint_C d\phi\right], \qquad \alpha \in \mathbb{R}/2\pi\mathbb{Z}. \tag{3.5}$$

Clearly $\mathsf{L}_\alpha(C)$ detects the classical winding-number of the map $\phi : \Sigma \to S^1$, for which only the homology class $[C] \in H_1(\Sigma)$ is relevant. Since the one-form $d\phi/2\pi$ always has integral periods, $\mathsf{L}_\alpha(C)$ also depends only upon the value of α modulo 2π.

The operators in (3.2), (3.3), and (3.5) are presented in order-form, as classical functionals of the scalar field ϕ or the gauge field A. As well-known and reviewed for instance in section 5.2 of [2], each of these operators admits a dual disorder description, in which the operator creates a classical singularity at the point σ or along the curve C, respectively.

Very briefly, the vertex operator $\mathsf{V}_k(\sigma)$ creates a local monopole singularity of magnetic charge k in the gauge field A, and the loop operator $\mathsf{L}_\alpha(C)$ creates a codimension-two

singularity in $M = \mathbb{R} \times \Sigma$ around which A has monodromy α. Note that this interpretation is consistent with the angular nature of α. For the periodic scalar field, the Wilson loop operator $\mathsf{W}_n(C)$ dually creates an additive monodromy $\phi \mapsto \phi + 2\pi n$ around any small path encircling C inside M. Note that integrality of n is required for the monodromy to make sense for general C.

Such classical geometric descriptions of the disorder operators suffice for the path integral analysis of duality in [2]. Our goal in section 3.1 is to provide an alternative, quantum description of these operators — in both order and disorder form — by their action on the Hilbert space $\mathscr{H}_\Sigma^{\text{top}}$. Using these results, we then exhibit directly in section 3.2 the combined algebra of vertex and loop operators on Σ.

3.1 Monopoles and loops on a Riemann surface

To discuss the action of $\mathsf{V}_k(\sigma)$, $\mathsf{W}_n(C)$, and $\mathsf{L}_\alpha(C)$ on the Hilbert space, we assume that each operator acts at time $t = 0$ on $M = \mathbb{R} \times \Sigma$. These operators do not generally preserve the topological subspace $\mathscr{H}_\Sigma^{\text{top}}$ inside the full Hilbert space $\mathscr{H}_\Sigma$. To obtain an action on $\mathscr{H}_\Sigma^{\text{top}}$ alone, we compose with the projection from $\mathscr{H}_\Sigma$ onto $\mathscr{H}_\Sigma^{\text{top}}$, which occurs naturally in either the geometric limits $e^2\ell \ll 1$ or $e^2\ell \gg 1$. This projection onto $\mathscr{H}_\Sigma^{\text{top}}$ will be implicit throughout. At the classical level, projection onto $\mathscr{H}_\Sigma^{\text{top}}$ amounts to the Hodge projection onto harmonic configurations for ϕ and A.

Monopole operators. We begin with the action of the vertex operator $\mathsf{V}_k(\sigma)$ in (3.2). Directly via the eigenmode expansion (2.36) for the periodic scalar field,

$$\mathsf{V}_k(\sigma)|m;\omega\rangle = \exp\left[i\,k\,\Phi_\omega(\sigma) + i\,\frac{k}{e^2\ell}\,\phi_0 + \cdots\right]|m;\omega\rangle, \qquad (m,\omega) \in \mathbb{Z} \oplus \mathbb{L}. \qquad (3.6)$$

Here $\Phi_\omega : \Sigma \to S^1$ is the fiducial harmonic map (2.34) with winding-number ω, and the ellipses indicate terms involving the Fock operators a_λ and $a_\lambda^\dagger$, whose action becomes irrelevant after the projection to $\mathscr{H}_\Sigma^{\text{top}}$.

According to the description of the Fourier wavefunction in (2.47), the Fock ground-state $|m;\omega\rangle$ can itself be written as

$$|m;\omega\rangle \equiv \Psi_m(\phi_0)|\omega\rangle = \exp\left(i\,\frac{m}{e^2\ell}\,\phi_0\right)|\omega\rangle. \qquad (3.7)$$

Evidently from (3.6), $\mathsf{V}_k(\sigma)$ shifts the Fourier mode number m to $m + k$,

$$\mathsf{V}_k(\sigma)|m;\omega\rangle = \exp\left[i\,k\,\Phi_\omega(\sigma)\right]|m+k;\omega\rangle, \qquad (3.8)$$

up to an additional phase which depends upon the value of Φ_ω at the point σ. When $\Sigma = \mathbb{CP}^1$ has genus zero, the harmonic map Φ_ω in (3.8) is constant and equal to zero modulo 2π by the defining condition in (2.35). Hence in this case, the action of $\mathsf{V}_k(\sigma)$ on the quasi-topological subspace $\mathscr{H}_\Sigma^{\text{top}}$ does not actually depend upon the position of the vertex operator on Σ.

In higher genus, the situation is more interesting.

To evaluate the phase in (3.8), we use the defining conditions for the fiducial map, namely

$$d\Phi_\omega = 2\pi\omega\,, \qquad\qquad \Phi_\omega(\sigma_0) = 0 \mod 2\pi\,, \qquad\qquad (3.9)$$

where $\omega \in H^1(\Sigma; \mathbb{Z})$ is harmonic and $\sigma_0 \in \Sigma$ is the basepoint used for quantization. By Stokes' theorem, the phase factor in (3.8) can be recast in the form

$$
\begin{aligned}
\exp\Big[i\,k\,\Phi_\omega(\sigma)\Big] &= \exp\Big[i\,k\big(\Phi_\omega(\sigma) - \Phi_\omega(\sigma_0)\big)\Big]\,, \\
&= \exp\Big[2\pi i\,k \int_\Gamma \omega\Big]\,,
\end{aligned}
\qquad (3.10)
$$

where Γ is any oriented path on Σ which connects the basepoint σ_0 to the point σ where the vertex operator is inserted,

$$\partial\Gamma = \sigma - \sigma_0\,. \qquad\qquad (3.11)$$

The homotopy class of Γ is not unique, as clear when $\sigma = \sigma_0$ and Γ is an arbitrary closed curve based at σ_0. However, as usual in the business, integrality of both k and ω ensures that the phase in (3.10) is independent of the choice of the integration contour Γ. For the remainder, we suppress the appearance of Γ and simply write the vertex operator phase as

$$\mathsf{V}_k(\sigma)|m;\omega\rangle = \exp\Big[2\pi i\,k \int_{\sigma_0}^{\sigma} \omega\Big]|m+k;\omega\rangle\,. \qquad (3.12)$$

Thus, even when we restrict to the low-energy subspace $\mathscr{H}_\Sigma^{\mathrm{top}} \subset \mathscr{H}_\Sigma$, the action of the vertex operator $\mathsf{V}_k(\sigma)$ is still sensitive to the location at which the operator is inserted.

How does (3.12) arise when we describe the quantum theory on Σ dually in terms of the Maxwell gauge field A? To answer this question, we recall that the Fock vacua $|m;\omega\rangle$ in $\mathscr{H}_\Sigma^{\mathrm{top}}$ correspond to wavefunctions for A on the disjoint union of tori

$$\mathrm{Pic}(\Sigma) = \bigsqcup_{m\in\mathbb{Z}} \mathrm{Pic}_m(\Sigma)\,, \qquad\qquad \mathrm{Pic}_m(\Sigma) \simeq \mathscr{J}_\Sigma\,, \qquad\qquad (3.13)$$

each isomorphic to the Jacobian of Σ. A natural guess is that the effective action of the vertex operator $\mathsf{V}_k(\sigma)$ is induced from the tensor product (or Hecke modification) with the degree-k holomorphic line-bundle $\mathcal{O}_\Sigma(k\,\sigma)$,

$$
\begin{aligned}
\otimes\mathcal{O}_\Sigma(k\,\sigma): \mathrm{Pic}_m(\Sigma) &\xrightarrow{\ \simeq\ } \mathrm{Pic}_{m+k}(\Sigma)\,, \\
\mathfrak{L} &\longmapsto \mathfrak{L}\otimes\mathcal{O}_\Sigma(k\,\sigma)\,,
\end{aligned}
\qquad (3.14)
$$

as already appears in (2.91). In the gauge theory approach to geometric Langlands, this statement has been explained in §9.1 of [17], though we must make a few minor modifications to treat the non-topological (but free) theory here.

According to its definition as a monopole operator, reviewed in section 5.2 of [2], the vertex operator $\mathsf{V}_k(\sigma)$ acts topologically to increase the degree of the U(1)-bundle over Σ by k units, in accord with the shift $m \mapsto m + k$ in both (3.12) and (3.14). However, for a complete characterization of the phase in (3.12), we must also consider how the action of

$V_k(\sigma)$ on the state $|m;\omega\rangle$ depends upon the point $\sigma \in \Sigma$ and the Fourier mode $\omega \in \mathbb{L}$ for the gauge field wavefunction on $\mathscr{J}_\Sigma$.

To investigate the latter dependence, let us consider the composite operator

$$\mathsf{O}_k(\sigma,\sigma_0) = \mathsf{V}_k(\sigma) \circ \mathsf{V}_{-k}(\sigma_0), \qquad \sigma \neq \sigma_0, \qquad (3.15)$$

where σ is distinct from the basepoint σ_0. Because $\mathsf{V}_k(\sigma)$ and $\mathsf{V}_{-k}(\sigma_0)$ carry opposite monopole charges, $\mathsf{O}_k(\sigma,\sigma_0)$ does not alter the topology of the line-bundle over Σ. Nonetheless, $\mathsf{O}_k(\sigma,\sigma_0)$ may still act non-trivially on the state $|m;\omega\rangle$, at least in genus $g \geq 1$.

Since we work with zero-modes, let A be an arbitrary harmonic connection on a line-bundle of degree m over Σ, of the form

$$A = m\,\widehat{A} + \sum_{j=1}^{2g} \varphi_0{}^j\, \mathfrak{e}_j, \qquad \varphi_0{}^j \in \mathbb{R}/2\pi\mathbb{Z}. \qquad (3.16)$$

Here we have truncated the more general eigenform expansion for A in (2.93), and we recall that $\widehat{A}$ is the fiducial harmonic connection associated to the holomorphic line-bundle $\mathcal{O}_\Sigma(\sigma_0)$. As a harmonic connection, A determines a point in the Picard component $\mathrm{Pic}_m(\Sigma)$ of degree m, and we may interpret the Fock groundstate

$$|m;\omega\rangle \equiv \Psi_\omega(A)|m\rangle \qquad (3.17)$$

in terms of the wavefunction

$$\Psi_\omega(A) = \exp\left[i \int_\Sigma \left(A - m\,\widehat{A}\right) \wedge \omega\right], \qquad (3.18)$$

exactly as in (2.123). After we subtract $m\,\widehat{A}$ in the argument of the exponential, $\Psi_\omega(A)$ does not actually depend upon the degree m.

On this wavefunction, the composite operator $\mathsf{O}_k(\sigma,\sigma_0)$ acts via a modification of A induced from (3.14),

$$\mathsf{O}_k(\sigma,\sigma_0) \cdot \Psi_\omega(A) = \Psi_\omega(\widetilde{A}), \qquad (3.19)$$

where

$$\widetilde{A} = A + 2\pi k\,\delta_\Gamma, \qquad \delta_\Gamma \in \Omega_\Sigma^1. \qquad (3.20)$$

In this expression, δ_Γ is a one-form on Σ with delta-function support which represents the Poincaré dual of an oriented path Γ running from σ_0 to σ. Equivalently, for any smooth one-form η, the wedge product with δ_Γ satisfies

$$\int_\Sigma \delta_\Gamma \wedge \eta = \int_\Gamma \eta, \qquad \eta \in \Omega_\Sigma^1. \qquad (3.21)$$

Because the path Γ is open, the one-form δ_Γ is not closed but rather obeys

$$d\delta_\Gamma = \delta_\sigma - \delta_{\sigma_0}, \qquad \delta_\sigma, \delta_{\sigma_0} \in \Omega_\Sigma^2, \qquad (3.22)$$

where δ_σ and δ_{σ_0} are two-forms on Σ with delta-function support at the points $\sigma, \sigma_0 \in \Sigma$. As a consequence of (3.22), the curvature of the modified connection $\widetilde{A}$ in (3.20) is singular

at the locations where the vertex operators are inserted. Physically, these curvature singularities signal the creation of a monopole/anti-monopole pair of magnetic charge k on Σ.

Strictly speaking, the singular connection $\widetilde{A}$ in (3.20) is not harmonic, and so to interpret the dual action of $\mathsf{O}_k(\sigma, \sigma_0)$ on the zero-mode wavefunction, we should project the singular connection $\widetilde{A}$ onto the harmonic subspace of Ω_Σ^1. Thankfully, this projection is accomplished automatically for us when we evaluate

$$
\begin{aligned}
\Psi_\omega(\widetilde{A}) &= \exp\left[i \int_\Sigma \left(A + 2\pi k\,\delta_\Gamma - m\,\widehat{A}\right)\wedge\omega\right], \\
&= \exp\left(2\pi i k \int_\Sigma \delta_\Gamma\wedge\omega\right)\cdot\Psi_\omega(A), \\
&= \exp\left(2\pi i\,k \int_\Gamma \omega\right)\cdot\Psi_\omega(A),
\end{aligned}
\tag{3.23}
$$

since the one-form ω is harmonic by assumption. In passing from the second to the third line of (3.23), we use the defining property of δ_Γ in (3.21).

According to (3.19) and (3.23),

$$
\mathsf{O}_k(\sigma, \sigma_0)|m;\omega\rangle = \left[\mathsf{V}_k(\sigma) \circ \mathsf{V}_{-k}(\sigma_0)\right]|m;\omega\rangle = \exp\left(2\pi i\,k \int_{\sigma_0}^{\sigma} \omega\right)|m;\omega\rangle,
\tag{3.24}
$$

and again integrality of both k and ω ensures that the phase factor depends only upon the endpoints of the path Γ, where the monopoles are inserted. Clearly from its definition (3.14) via the tensor product, the monopole operator of charge k is the same as the k-th power of the unit monopole operator,

$$
\mathsf{V}_k(\sigma) = \mathsf{V}_1(\sigma)^k,
\tag{3.25}
$$

so we can rewrite the identity in (3.24) as

$$
\mathsf{V}_k(\sigma)|m;\omega\rangle = \exp\left(2\pi i\,k \int_{\sigma_0}^{\sigma} \omega\right)\mathsf{V}_k(\sigma_0)|m;\omega\rangle.
\tag{3.26}
$$

Hence we have determined the action of the monopole operator $\mathsf{V}_k(\sigma)$ for arbitrary points $\sigma \in \Sigma$ in terms of the action of the monopole operator $\mathsf{V}_k(\sigma_0)$ inserted at the basepoint σ_0.

We are left to discuss the action of the based monopole $\mathsf{V}_k(\sigma_0)$ on $|m;\omega\rangle$. The first claim is that $\mathsf{V}_k(\sigma_0)$ does not alter the quantum label ω,

$$
\langle m + k;\omega'|\mathsf{V}_k(\sigma_0)|m;\omega\rangle = 0, \qquad\qquad \omega \neq \omega'.
\tag{3.27}
$$

This statement follows by symmetry, since ω is interpreted as the charge under the group $\mathrm{U}(1)^{2g}$ which acts by translations on the Jacobian $\mathscr{J}_\Sigma$. For the harmonic connection A in (3.16), these translations are just shifts in the angular coordinates $\varphi_0{}^j$. Because the Hecke modification in (3.14) commutes with the action of $\mathrm{U}(1)^{2g}$, the monopole operator is uncharged under $\mathrm{U}(1)^{2g}$ and hence preserves ω.

Otherwise, from the gauge theory perspective we have left some ambiguity in the normalization of the basis state $|m;\omega\rangle$ for fixed ω as the degree m ranges over $\mathbb{Z}$. We fix this ambiguity up to an overall constant by declaring

$$
|m;\omega\rangle \equiv \left[\mathsf{V}_1(\sigma_0)\right]^m|0;\omega\rangle,
\tag{3.28}
$$

so that

$$\mathsf{V}_k(\sigma_0)|m;\omega\rangle = |m+k;\omega\rangle. \qquad (3.29)$$

Together, (3.26) and (3.29) imply the formula in (3.12), which we deduced from the more direct description of $\mathsf{V}_k(\sigma)$ as a vertex operator for the periodic scalar field ϕ.

Vortex loops. Just as we consider the action by the local vertex operator $\mathsf{V}_k(\sigma)$, we can also consider the action on $\mathscr{H}_\Sigma^{\mathrm{top}}$ by the respective loop operators $\mathsf{L}_\alpha(C)$ and $\mathsf{W}_n(C)$, where C is a closed curve in Σ. If C is not a spacelike curve in Σ but a timelike curve in M of the form $C = \mathbb{R} \times \{\sigma\}$ for some point $\sigma \in \Sigma$, then $\mathsf{L}_\alpha(C)$ and $\mathsf{W}_n(C)$ do not act on the Hilbert space $\mathscr{H}_\Sigma^{\mathrm{top}}$ but lead rather to the construction of new Hilbert spaces associated to the punctured surface $\Sigma^o = \Sigma - \{\sigma\}$. The analysis of such line operators is similar philosophically to the analysis of fibrewise Wilson loop operators in [1], so I omit the timelike case here.

Like the vertex operator $\mathsf{V}_k(\sigma)$, the loop operator $\mathsf{L}_\alpha(C)$ admits an elementary description in terms of the periodic scalar field ϕ,

$$\mathsf{L}_\alpha(C) = \exp\left(\frac{i\,\alpha}{2\pi} \oint_C d\phi\right), \qquad \alpha \in \mathbb{R}/2\pi\mathbb{Z}. \qquad (3.30)$$

On each state $|m;\omega\rangle$ in $\mathscr{H}_\Sigma^{\mathrm{top}}$, the operator $\mathsf{L}_\alpha(C)$ simply measures the winding-number $\omega \in H^1(\Sigma;\mathbb{Z})$,

$$\begin{aligned}
\mathsf{L}_\alpha(C)|m;\omega\rangle &= \exp\left(\frac{i\,\alpha}{2\pi} \oint_C d\phi\right)|m;\omega\rangle, \\
&= \exp\left(i\,\alpha \oint_C \omega\right)|m;\omega\rangle,
\end{aligned} \qquad (3.31)$$

where we recall that $[d\phi] = 2\pi\omega$ for the state $|m;\omega\rangle$. In particular, $\mathsf{L}_\alpha(C)$ respects the global U(1) symmetry by shifts $\phi \mapsto \phi + c$ for constant c and hence preserves the mode number m of the state $|m;\omega\rangle$. We also note that the phase in (3.31) depends only on the homology class of C in $H_1(\Sigma)$, and the integrality of ω ensures that the phase depends only on the value of the parameter α modulo 2π.

Again, our main goal is to understand how the formula in (3.31) arises dually in terms of the U(1) gauge field A. As reviewed in section 5.2 of [2], the loop operator $\mathsf{L}_\alpha(C)$ acts on A as a disorder operator which creates a curvature singularity along C of the form

$$F_A = -\alpha\,\delta_C, \qquad \alpha \in \mathbb{R}/2\pi\mathbb{Z}, \qquad (3.32)$$

where δ_C is a two-form with delta-function support that represents the Poincaré dual of $C \subset M$. Equivalently, near C the gauge field behaves as

$$A = -\frac{\alpha}{2\pi}\,d\vartheta + \cdots, \qquad (3.33)$$

where ϑ is an angular coordinate on the plane transverse to C, located at the origin. Globally, A has non-trivial monodromy about any small curve linking C in M, and $\mathsf{L}_\alpha(C)$ is the reduction to three dimensions of the basic Gukov-Witten [14] surface operator in four dimensions.

Unlike the monopole singularity, the singularity in (3.32) does not change the degree m of the line-bundle L over Σ. As will be useful later, let me give an elementary argument for this statement.

We consider an arbitrary configuration for the gauge field A on $M = \mathbb{R} \times \Sigma$ with the prescribed singularity in (3.32) at time $t = 0$, and smooth otherwise. We will measure the change in the degree m as the time t runs from $-\infty$ to $+\infty$ on M. By way of notation, $\Sigma_\pm \subset M$ will denote the copies of Σ at the times $t = \pm\infty$. Then the change Δm in the degree from $t = -\infty$ to $t = +\infty$ is computed by

$$
\begin{aligned}
\Delta m &= \int_{\Sigma_+} F_A - \int_{\Sigma_-} F_A = \int_M dF_A\,,\\
&= -\alpha \int_M d\delta_C = 0\,.
\end{aligned}
\tag{3.34}
$$

In the first line of (3.34) we apply Stokes' theorem, and in the second line we apply the Bianchi identity $dF_A = 0$ on the locus where A is smooth. Finally, because C is a closed curve on Σ, the Poincaré dual current δ_C is also closed, $d\delta_C = 0$. (The same computation would show that $\Delta m \neq 0$ for the monopole operator, which acts as a localized source for dF_A.) So as observed following (3.31), $\mathsf{L}_\alpha(C)$ must preserve the magnetic label m on the states $|m;\omega\rangle$ in $\mathscr{H}_\Sigma^{\mathrm{top}}$.

On the other hand, $\mathsf{L}_\alpha(C)$ does change the holonomies of A on Σ, from which the phase factor in (3.31) will be induced. To setup the computation, we consider the gauge theory on $M = \mathbb{R} \times \Sigma$, and we suppose that the line operator $\mathsf{L}_\alpha(C)$ is inserted on $\Sigma_0 \equiv \{0\} \times \Sigma$ in M. We fix an initial flat connection A_- on Σ_-. The operator $\mathsf{L}_\alpha(C)$ acts as a sudden perturbation to create the singularity in (3.32) at $t = 0$, after which we project A back onto the subspace of harmonic (ie. flat) connections. We then let A_+ be the final connection on Σ_+ which is obtained by subsequent time-evolution.[5] We wish to compare the holonomies of A_+ to those of A_-. See figure 1 for a sketch of M as a cylinder over the uniformization of Σ, where for concreteness we have drawn Σ as a Riemann surface of genus two.

For any closed, oriented curve γ on Σ, we evaluate

$$
\begin{aligned}
\Delta_\gamma A &= \oint_\gamma A_+ - \oint_\gamma A_- \qquad \mathrm{mod}\ 2\pi\,,\\
&= \oint_\gamma A_+ + \oint_{-\gamma} A_- \qquad \mathrm{mod}\ 2\pi\,,
\end{aligned}
\tag{3.35}
$$

where in the second line we reverse the orientation of γ when integrating A_-. As apparent from figure 1, we can use Stokes' theorem to evaluate the difference in (3.35) as an integral over the cylindrical surface $S = \mathbb{R} \times \gamma \subset M$,
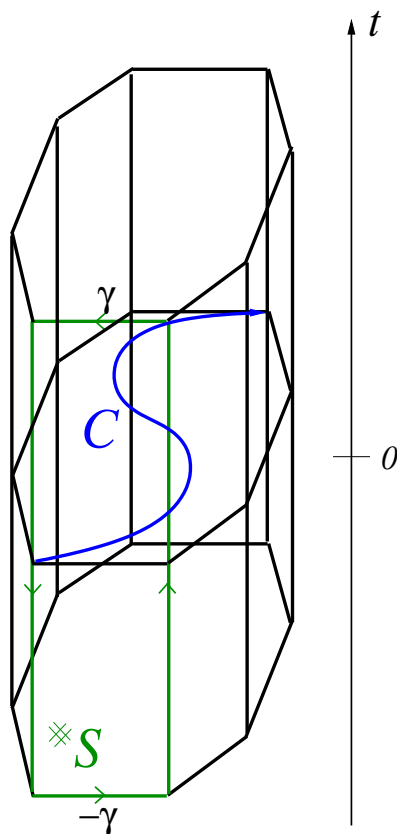
$$
\Delta_\gamma A = \int_{\partial S} A = \int_S F_A \qquad \mathrm{mod}\ 2\pi\,,
\tag{3.36}
$$

where S is oriented as the rectangle in the figure with boundary

$$
\partial S = \{+\infty\} \times \gamma - \{-\infty\} \times \gamma\,.
\tag{3.37}
$$

---

[5]Because the abelian gauge theory is free, the usual disorder path integral over arbitrary bulk configurations for A can be replaced by classical time-evolution.

Figure 1. M as a cylinder over the uniformization of Σ.

In passing from (3.36) to (3.37), we note that the vertical edges of S in figure 1 are identified after a reversal of orientation, so they make no contribution to the boundary integral over ∂S in (3.36).

We are left to evaluate the integral of the curvature F_A over S in (3.36). By assumption, F_A vanishes everywhere on S except for the explicit curvature singularity (3.32) created at $t = 0$ along C. Consequently,

$$\Delta_\gamma A = -\alpha \int_S \delta_C = \alpha \oint_\gamma [C]^\vee \qquad \mod 2\pi \,, \tag{3.38}$$

where $[C]^\vee \in H^1(\Sigma; \mathbb{Z})$ is a harmonic representative for the dual of the homology class of C. The flip of sign in the second equality arises from due care with orientations.

Because the curve γ is arbitrary, $\mathsf{L}_\alpha(C)$ must act classically by the shift

$$A_+ = A_- + \alpha \, [C]^\vee \quad \mod 2\pi\mathbb{L} \,. \tag{3.39}$$

We recall from (3.18) that the wavefunction Ψ_ω evaluated on a harmonic connection A of degree m is given by

$$\Psi_\omega(A) = \exp\Big[i \int_\Sigma \big(A - m\,\widehat{A}\big) \wedge \omega\Big] \,, \tag{3.40}$$

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Under the shift (3.39) induced by $\mathsf{L}_\alpha(C)$, we see that Ψ_ω transforms by

$$
\begin{aligned}
\mathsf{L}_\alpha(C)[\Psi_\omega(A)] &= \Psi_\omega\big(A + \alpha\,[C]^\vee\big)\,, \\
&= \exp\!\left[i\,\alpha \int_\Sigma [C]^\vee \wedge \omega\right] \cdot \Psi_\omega(A)\,, \\
&= \exp\!\left[i\,\alpha \oint_C \omega\right] \cdot \Psi_\omega(A)\,.
\end{aligned}
\tag{3.41}
$$

exactly as in (3.31).

I briefly mention two other ways to understand the formula in (3.41) from gauge theory.

So far we have introduced two disorder operators for the gauge field A, namely, the monopole operator $\mathsf{V}_k(\sigma)$ and the loop operator $\mathsf{L}_\alpha(C)$. These two operators are not unrelated. Let us again consider a monopole/anti-monopole pair $\mathsf{V}_k(\sigma) \circ \mathsf{V}_{-k}(\sigma_0)$ on Σ. If $\sigma = \sigma_0$, this composite operator is the identity, but for $\sigma \neq \sigma_0$, the operator acts on states in $\mathscr{H}_\Sigma^{\mathrm{top}}$ with the non-trivial phase

$$
\mathsf{O}_k(\sigma,\sigma_0)|m;\omega\rangle = \Big[\mathsf{V}_k(\sigma) \circ \mathsf{V}_{-k}(\sigma_0)\Big]|m;\omega\rangle = \exp\!\left(2\pi i\,k \int_{\sigma_0}^\sigma \omega\right)|m;\omega\rangle\,.
\tag{3.42}
$$

Using $\mathsf{O}_k(\sigma,\sigma_0)$, we can try to make a new operator á la Verlinde [20] from the monodromy action by $\mathsf{O}_k(\sigma,\sigma_0)$ on $\mathscr{H}_\Sigma^{\mathrm{top}}$ as the point σ is moved adiabatically around a closed curve $C \subset \Sigma$ based at σ_0. When k is integral, the induced phase in (3.42) is trivial, and the Verlinde operator acts as the identity. However, when $k \neq 0 \bmod \mathbb{Z}$ is allowed to be fractional, the Verlinde operator constructed from $\mathsf{O}_k(\sigma,\sigma_0)$ is non-trivial and acts by precisely the phase in (3.41), provided we set

$$
\alpha = 2\pi k \in \mathbb{R}/2\pi\mathbb{Z}\,.
\tag{3.43}
$$

So the line operator $\mathsf{L}_\alpha(C)$ can be interpreted as the Verlinde operator associated to transport of a monopole/anti-monopole pair with non-integral magnetic charge. See [14] for a somewhat different explanation of the same effect in four-dimensional gauge theory. This relation can also be understood directly from the order-type expressions for $\mathsf{V}_k(\sigma) \circ \mathsf{V}_{-k}(\sigma_0)$ and $\mathsf{L}_\alpha(C)$ in terms of the periodic scalar field ϕ.

Alternatively, the action by the line operator $\mathsf{L}_\alpha(C)$ can be understood using the Lagrangian formalism, the focus of [2]. For simplicity in the following discussion, we assume that the topological parameters θ and β from section 2.2 are both set to zero. In the Lagrangian formalism, the inner-product of states $\langle m;\omega'|\mathsf{L}_\alpha(C)|m;\omega\rangle$ for some $\omega,\omega' \in \mathbb{L}$ is computed by the path integral

$$
\langle m;\omega'|\mathsf{L}_\alpha(C)|m;\omega\rangle =
$$
$$
\frac{1}{\mathrm{Vol}(\mathcal{G})} \int_{\mathrm{Pic}_m(\Sigma)\times\mathcal{A}_m\times\mathrm{Pic}_m(\Sigma)} \mathcal{D}A_+\,\mathcal{D}A\,\mathcal{D}A_-\;\overline{\Psi}_{\omega'}(A_+) \exp\!\left[\frac{i}{4\pi e^2}\int_M \mathcal{F}_A \wedge \star \mathcal{F}_A\right] \Psi_\omega(A_-)\,,
\tag{3.44}
$$

with modified curvature

$$
\mathcal{F}_A = F_A + \alpha\,\delta_C\,.
\tag{3.45}
$$

Here $A_\pm$ denote the boundary values for the gauge field at $t = \pm\infty$, and the path integral ranges over the affine space $\mathcal{A}_m$ of connections on the U(1)-bundle with degree m on $M = \mathbb{R} \times \Sigma$. We also integrate over the boundary values of A with weights given by the wavefunctions Ψ_ω and $\Psi_{\omega'}$ as in (3.18). Finally, the term proportional to δ_C in (3.45) enforces the condition that F_A have the singular behavior in (3.32). For a more thorough discussion of the latter remark, see section 5.2 in [2].

The modified action in (3.44) can be expanded in terms of F_A as

$$\int_M \mathcal{F}_A \wedge \star \mathcal{F}_A \;=\; \int_M F_A \wedge \star F_A \;+\; 2\alpha \int_M \delta_C \wedge \star F_A \;+\; c_0 \,. \tag{3.46}$$

Here c_0 is a formally divergent constant arising from the norm-square of δ_C, which we shall ignore. By comparison of (3.46) to the standard Maxwell action, $\mathsf{L}_\alpha(C)$ can be identified semi-classically with the operator

$$\mathsf{L}_\alpha(C) \;=\; \exp\!\left[\frac{i\,\alpha}{2\pi e^2} \int_M \delta_C \wedge \star F_A\right] \;=\; \exp\!\left[\frac{i\,\alpha}{2\pi e^2} \oint_C \star_\Sigma E_A\right], \tag{3.47}$$

where I note in the second equality that only the electric component of F_A contributes to the integral over the spacelike curve $C \subset \Sigma$, and $\star_\Sigma$ indicates the two-dimensional Hodge operator on Σ. Upon quantization as in (2.116),[6]

$$\frac{1}{2\pi e^2} \star_\Sigma E_A(w) \;=\; -i \operatorname{vol}_\Sigma \cdot \frac{\delta}{\delta A(w)} \,, \tag{3.48}$$

so the loop operator becomes

$$\mathsf{L}_\alpha(C) \;=\; \exp\!\left[\alpha \oint_C \operatorname{vol}_\Sigma \cdot \frac{\delta}{\delta A}\right]. \tag{3.49}$$

Manifestly, $\mathsf{L}_\alpha(C)$ acts upon any wavefunction $\Psi_\omega(A)$ by the shift $A \mapsto A + \alpha\,[C]^\vee$ appearing in the first line of (3.41).

Although we began with a disorder characterization of the loop operator in gauge theory, the classical description for $\mathsf{L}_\alpha(C)$ in (3.47) amounts to an order expression for the same operator. A quantum operator may admit distinct classical descriptions, so there is no contradiction here. See for instance section 4.1 of [1] for an analogous disorder presentation of the usual Wilson loop operator in Chern-Simons gauge theory.

Wilson loops. We are left to consider the action on $\mathscr{H}_\Sigma^{\mathrm{top}}$ of the Wilson loop operator $\mathsf{W}_n(C)$. In terms of the gauge field, the Wilson loop operator acts simply by multiplication in the topological sector labelled by the degree m,

$$\begin{aligned}
\mathsf{W}_n(C) \cdot \Psi_\omega(A) &= \exp\!\left(i\,n \oint_C A\right) \cdot \exp\!\left[i \int_\Sigma (A - m\,\widehat{A}) \wedge \omega\right], \\
&= \exp\!\left(i\,n \int_\Sigma [C]^\vee \wedge A\right) \cdot \exp\!\left[i \int_\Sigma (A - m\,\widehat{A}) \wedge \omega\right], \\
&= \exp\!\left(i\,m\,n \oint_C \widehat{A}\right) \cdot \Psi_{\omega - n\,[C]^\vee}(A),
\end{aligned} \tag{3.50}$$

---

[6]The Coulomb-gauge smearing term in (2.116) can be ignored when we restrict to the topological subspace $\mathscr{H}_\Sigma^{\mathrm{top}}$.

where again $[C]^\vee \in H^1(\Sigma; \mathbb{Z})$ is the Poincaré dual of the curve $C \subset \Sigma$. In passing to the second line of (3.50), we recall that A is a harmonic connection with expansion (3.16) for wavefunctions in $\mathscr{H}_\Sigma^{\mathrm{top}}$. As a result,

$$\mathsf{W}_n(C)|m;\omega\rangle \,=\, \exp\left(i\,m\,n \oint_C \widehat{A}\right)|m;\omega - n\,[C]^\vee\rangle\,. \tag{3.51}$$

Clearly, integrality of n is necessary whenever the homology class $[C] \neq 0$ is non-trivial, else the Wilson loop operator does not act in a well-defined way on $\mathscr{H}_\Sigma^{\mathrm{top}}$.

The phase factor in (3.51) depends upon the fiducial harmonic connection $\widehat{A}$ on the degree-one line-bundle $L = \mathcal{O}_\Sigma(\sigma_0)$ over Σ. Because $\widehat{A}$ is not flat, this phase is not invariant under deformations of the curve C. For instance, even when $C = \partial D$ is trivial in homology, the Wilson loop operator still acts non-trivially on the state $|m;\omega\rangle$,

$$\begin{aligned}
\mathsf{W}_\nu(C)|m;\omega\rangle &\,=\, \exp\left(i\,m\,\nu \int_D \widehat{F}_A\right)|m;\omega\rangle\,, \\
&\,=\, \exp\left[2\pi i\,m\,\nu\left(\frac{\mathrm{vol}_\Sigma(D)}{\ell^2}\right)\right]|m;\omega\rangle\,, \qquad\quad C \,=\, \partial D\,.
\end{aligned} \tag{3.52}$$

In passing to the second line, we use the formula for $\widehat{F}_m$ in (2.89), and we let $\mathrm{vol}_\Sigma(D)$ be the volume of D in the given metric on Σ. If ν is integral, the phase in (3.52) does not depend on whether D or $D' = \Sigma - D$ is chosen to bound C. Otherwise, for arbitrary real values $\nu \in \mathbb{R}$, the orientation of C uniquely fixes the bounding two-cycle D with compatible orientation, so that the action of $\mathsf{W}_\nu(C)$ is well-defined.[7]

As usual, we now wish to understand the results in (3.51) and (3.52) dually in terms of the periodic scalar field ϕ. Like the previous disorder description for the loop operator $\mathsf{L}_\alpha(C)$, the Wilson loop operator $\mathsf{W}_n(C)$ will act on ϕ by creating a singularity along C such that ϕ winds by $2\pi n$ when traversing any small circle which links $C \subset M = \mathbb{R} \times \Sigma$.

The effective shift of $\omega \in H^1(\Sigma;\mathbb{Z})$ in (3.51) can be understood dually in close correspondence to the shift (3.39) induced by the vortex loop operator $\mathsf{L}_\alpha(C)$ on the gauge field A. Classically, ω is interpreted as the winding-number of ϕ, with $\omega = [d\phi/2\pi]$. So long as ϕ is a smooth map to the circle, then $d[d\phi] = 0$. However, in the background of the Wilson loop operator $\mathsf{W}_n(C)$, we replace the smooth map ϕ by a section $\widetilde{\phi}$ of a non-trivial S^1-bundle over the complement $M^o = M - C$, such that

$$d[d\widetilde{\phi}] \,=\, d(d\phi + B) \,=\, F_B \,=\, 2\pi n\,\delta_C\,. \tag{3.53}$$

For a path integral justification of the statement above, I refer the interested reader to the end of section 5.2 in [2].

By exactly the same computation as in (3.35), (3.36), and (3.38), we evaluate the change due to the insertion of $\mathsf{W}_n(C)$ in the winding-number of ϕ around an arbitrary

---

[7]As observed in section 5.2 of [2], the analogous definition of $\mathsf{W}_\nu(C)$ for null-homologous curves C in a three-manifold M generally does depend upon an extra discrete choice of a relative class in $H_2(M, C)$ for the bounding Seifert surface.

closed curve $\gamma \subset \Sigma$ as

$$
\begin{aligned}
\Delta_\gamma \omega &= \oint_\gamma \frac{d\phi_+}{2\pi} - \oint_\gamma \frac{d\phi_-}{2\pi}\,, \\
&= \int_S d\left[\frac{d\widetilde{\phi}}{2\pi}\right], \qquad\qquad S = \mathbb{R} \times \gamma\,, \\
&= n \int_S \delta_C = -n \oint_\gamma [C]^\vee\,.
\end{aligned}
\tag{3.54}
$$

reproducing the shift in (3.51).

The non-topological, m-dependent phase in (3.51) is slightly more subtle. For simplicity, we will reproduce this phase only in the special case that $C = \partial D$ is homologically-trivial, as assumed in (3.52). Then

$$
\mathsf{W}_\nu(C) = \exp\left[i\,\nu \int_D F_A\right], \qquad\qquad C = \partial D\,.
\tag{3.55}
$$

As the ur-statement of abelian duality, discussed in the Introduction to [2], we have the correspondence

$$
F_A = e^2 \star d\phi\,.
\tag{3.56}
$$

Hence the classical description of the Wilson loop operator in terms of ϕ must be

$$
\mathsf{W}_\nu(C) = \exp\left[i\,\nu\,e^2 \int_D \mathrm{vol}_\Sigma \cdot \partial_t \phi\right].
\tag{3.57}
$$

Upon quantization, we apply the functional identification in (2.40) with $\theta = 0$ to rewrite $\mathsf{W}_\nu(C)$ as the operator

$$
\mathsf{W}_\nu(C) = \exp\left[2\pi\nu \int_D \mathrm{vol}_\Sigma \cdot \frac{\delta}{\delta\phi}\right].
\tag{3.58}
$$

Hence $\mathsf{W}_\nu(C)$ acts upon any wavefunction $\Psi_m(\phi)$ by the shift $\phi \mapsto \phi + 2\pi\nu\,\mathrm{vol}_\Sigma(D)/\ell^2$.

According to our previous results in (2.36) and (2.47), the Fourier wavefunction $\Psi_m(\phi)$ which describes the Fock state $|m;\omega\rangle$ is given explicitly by

$$
\Psi_m(\phi) = \exp\left[i\,\frac{m}{\ell^2} \int_\Sigma \mathrm{vol}_\Sigma \cdot (\phi - \Phi_\omega)\right].
\tag{3.59}
$$

Immediately, the action by the operator in (3.58) on this wavefunction produces the geometric phase in the second line of (3.52).

To summarize, the Wilson loop $\mathsf{W}_n(C)$ and the vortex loop $\mathsf{L}_\alpha(C)$ play dual roles. When expressed in terms of the gauge field, $\mathsf{W}_n(C)$ acts classically by multiplication on any state $|m;\omega\rangle$. But when expressed in terms of the scalar field, $\mathsf{W}_n(C)$ acts quantum-mechanically as the differential (or shift) operator in (3.58). Conversely, the vortex loop $\mathsf{L}_\alpha(C)$ acts classically by multiplication when written in terms of ϕ, but quantum-mechanically as the differential (or shift) operator in (3.49) when written in terms of A.

3.2 Wilson-'t Hooft commutation relations

Finally, let us examine the commutation relations between the operators $\mathsf{V}_k(\sigma)$, $\mathsf{L}_\alpha(C)$, and $\mathsf{W}_n(C)$, all acting on the topological Hilbert space $\mathscr{H}_\Sigma^{\mathrm{top}}$. The idea of examining these commutators goes back to 't Hooft, and we will find a holomorphic refinement of the classic results in [15].

Collecting our previous formulas in (3.12), (3.31), and (3.51) we explicitly present the action of the operators on the Fock vacua $|m;\omega\rangle$ as

$$\mathsf{V}_k(\sigma)|m;\omega\rangle = \exp\left(2\pi i\, k \int_{\sigma_0}^{\sigma} \omega\right)|m+k;\omega\rangle\,,$$

$$\mathsf{L}_\alpha(C)|m;\omega\rangle = \exp\left(i\,\alpha \oint_C \omega\right)|m;\omega\rangle\,, \tag{3.60}$$

$$\mathsf{W}_n(C)\big|m;\omega\rangle = \exp\left(i\,m\,n \oint_C \widehat{A}\right)\big|m;\omega - n\,[C]^\vee\rangle\,.$$

Clearly for all pairs $\sigma, \sigma' \in \Sigma$ and $C, C' \subset \Sigma$,

$$\Big[\mathsf{V}_k(\sigma),\, \mathsf{V}_{k'}(\sigma')\Big] = \Big[\mathsf{L}_\alpha(C),\, \mathsf{L}_{\alpha'}(C')\Big] = \Big[\mathsf{W}_n(C),\, \mathsf{W}_{n'}(C')\Big] = 0\,. \tag{3.61}$$

Also,

$$\Big[\mathsf{V}_k(\sigma),\, \mathsf{L}_\alpha(C)\Big] = 0\,, \tag{3.62}$$

as follows directly from the elementary, order-type description of both the vertex and the homological loop operators in terms of the periodic scalar field ϕ.

On the other hand, the loop operators $\mathsf{L}_\alpha(C)$ and $\mathsf{W}_n(C)$ do not commute. Instead, the composition satisfies

$$\mathsf{L}_\alpha(C) \circ \mathsf{W}_n(C') = \exp\big[-i\,\alpha\,n\,(C \cdot C')\big]\, \mathsf{W}_n(C') \circ \mathsf{L}_\alpha(C)\,, \tag{3.63}$$

where

$$C \cdot C' = \oint_C [C']^\vee \in \mathbb{Z}\,. \tag{3.64}$$

Equivalently, $C \cdot C'$ is the topological intersection number of the curves $C, C' \subset \Sigma$. Because n and $C \cdot C'$ are integers, the phase in (3.63) only depends upon the value of α modulo 2π, consistent with its angular nature. When C' is trivial in homology, the charge n of the Wilson loop can be replaced by an arbitrary real parameter ν. In this special case, the phase in (3.63) remains well-defined, since $C \cdot C' = 0$.

Of course, the commutator in (3.63) appears in direct analogy to the celebrated commutation relation for Wilson and 't Hooft operators in four-dimensional abelian gauge theory, for which the corresponding phase is proportional to the linking number of the curves C and C' in $\mathbb{R}^3$. See §10.2 of [21] for a review of this story in four dimensions.

Though more or less obvious, the non-trivial commutation relation in (3.63) has an interesting consequence, because it implies that the monopole operator $\mathsf{V}_k(\sigma)$ and the Wilson loop operator $\mathsf{W}_n(C)$ similarly fail to commute. As one can check directly,

$$\mathsf{V}_k(\sigma) \circ \mathsf{W}_n(C) = \exp\left(-2\pi i\,k\,n \int_{\sigma_0}^{\sigma} [C]^\vee\right) \exp\left(-i\,k\,n \oint_C \widehat{A}\right) \mathsf{W}_n(C) \circ \mathsf{V}_k(\sigma)\,. \tag{3.65}$$

To make sense of (3.65), we must work with a definite, harmonic representative for the cohomology class $[C]^\vee$ which is Poincaré dual to $[C]$. Otherwise, absent a definite representative, the value of the line integral from σ_0 to σ in (3.65) would be ambiguous.

One potentially unsettling feature of the commutation relation in (3.65) is that the phase on the right-hand side appears to depend upon the auxiliary choices of the basepoint $\sigma_0 \in \Sigma$ and the harmonic connection $\widehat{A}$. These choices enter the definition of the states $|m;\omega\rangle$, but they do not enter the intrinsic definitions of the operators $\mathsf{V}_k(\sigma)$ and $\mathsf{W}_n(C)$ themselves and hence should not enter the commutator.[8]

Actually, the situation is slightly better than it first appears, since the fiducial connection $\widehat{A}$ is itself determined by the choice of σ_0. We recall that $\widehat{A}$ is defined as the unique harmonic connection compatible with the holomorphic structure on $\mathcal{O}_\Sigma(\sigma_0)$. As we now demonstrate, the explicit dependence on σ_0 in the first phase factor of (3.65) exactly cancels against the implicit dependence of $\widehat{A} \equiv \widehat{A}_{\sigma_0}$ on σ_0 in the second phase factor.

We begin by introducing another harmonic connection $\widehat{A}_\sigma$, associated to the degree-one holomorphic line-bundle $\mathcal{O}_\Sigma(\sigma)$. As harmonic connections, both $\widehat{A}_\sigma$ and $\widehat{A}_{\sigma_0}$ have the same curvature, proportional to the Riemannian volume form on Σ, so the difference $\widehat{A}_\sigma - \widehat{A}_{\sigma_0}$ is a closed one-form. For the first phase factor in (3.65), the classical Abel-Jacobi theory now provides the very beautiful reciprocity relation

$$\exp\left(-2\pi i\, k\, n \int_{\sigma_0}^{\sigma} [C]^\vee\right) \,=\, \exp\left[-i\, k\, n \oint_C \left(\widehat{A}_\sigma - \widehat{A}_{\sigma_0}\right)\right]. \tag{3.66}$$

See chapter 2.2 of [13] for a textbook reference on such reciprocity laws.

Substituting (3.66) into (3.65), we obtain a completely intrinsic reformulation of the commutation relation between the monopole operator and the Wilson loop,

$$\mathsf{V}_k(\sigma) \circ \mathsf{W}_n(C) \,=\, \exp\left(-i\, k\, n \oint_C \widehat{A}_\sigma\right) \mathsf{W}_n(C) \circ \mathsf{V}_k(\sigma)\,, \tag{3.67}$$

with no dependence on the arbitrary choice of the basepoint σ_0. We emphasize that the commutation relation in (3.67) *does* depend on the particular curve $C \subset \Sigma$, not merely the homology class $[C]$, because $\widehat{A}_\sigma$ is not flat. Moreover, the commutator depends holomorphically on the point at which the monopole operator is inserted, through the dependence of $\widehat{A}_\sigma$ on σ.

The commutation relation in (3.67) can be understood directly in terms of either the gauge field A or the periodic scalar field ϕ. Via the Hecke modification in (3.14), the monopole operator $\mathsf{V}_k(\sigma)$ induces the shift

$$A \longmapsto A + k\, \widehat{A}_\sigma\,. \tag{3.68}$$

On the other hand, the Wilson loop operator $\mathsf{W}_n(C)$ acts multiplicatively as in (3.50), from which (3.67) follows. Alternatively in terms of ϕ, the vertex operator $\mathsf{V}_k(\sigma)$ acts multiplicatively, and the Wilson loop operator $\mathsf{W}_\nu(C)$ for homologically-trivial $C = \partial D$ acts as the shift operator in (3.58), from which (3.67) again follows.

---

[8]I thank Marcus Benna for emphasizing this question to me.

To gain a final bit of additional insight into the meaning of the Wilson-'t Hooft commutation relation, let us consider the commutation relation of $\mathsf{W}_n(C')$ with the composite operator $\mathsf{O}_k(\sigma, \sigma_0) \equiv \mathsf{V}_k(\sigma) \circ \mathsf{V}_{-k}(\sigma_0)$ which has vanishing monopole charge. As an immediate consequence of (3.65),

$$\mathsf{O}_k(\sigma, \sigma_0) \circ \mathsf{W}_n(C') = \exp\left(-2\pi i\, k\, n \int_{\sigma_0}^{\sigma} \left[C'\right]^\vee\right) \mathsf{W}_n(C') \circ \mathsf{O}_k(\sigma, \sigma_0). \qquad (3.69)$$

But we have already identified $\mathsf{L}_\alpha(C)$ as the Verlinde operator derived from $\mathsf{O}_k(\sigma, \sigma_0)$, describing the creation and subsequent transport of a monopole/anti-monopole pair with fractional magnetic charge $\alpha = 2\pi k$. As σ is transported adiabatically around a curve C based at σ_0, the commutation relation in (3.69) reproduces the topological commutation relation of loop operators in (3.63).

Acknowledgments

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