

Exact results for vortex loop operators in 3d supersymmetric theories

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ABSTRACT: Three dimensional field theories admit disorder line operators, dubbed vortex loop operators. They are defined by the path integral in the presence of prescribed singularities along the defect line. We study half-BPS vortex loop operators for $\mathcal{N} = 2$ supersymmetric theories on S^3 , its deformation S_b^3 and $S^1 \times S^2$. We construct BPS vortex loops defined by the path integral with a fixed gauge or flavor holonomy for infinitesimal curves linking the loop. It is also possible to include a singular profile for matter fields. For vortex loops defined by holonomy, we perform supersymmetric localization by calculating the fluctuation modes, or alternatively by applying the index theory for transversally elliptic operators. We clarify how the latter method works in situations without fixed points of relevant isometries. Abelian mirror symmetry transforms Wilson and vortex loops in a specific way. In particular an ordinary Wilson loop transforms into a vortex loop for a flavor symmetry. Our localization results confirm the predictions of abelian mirror symmetry.

KEYWORDS: Supersymmetric gauge theory, Wilson, 't Hooft and Polyakov loops, Duality in Gauge Field Theories

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1 Introduction

In this paper we initiate the study of the exact expectation value of supersymmetric vortex loop operators in $\mathcal{N} = 2$ gauge theories in three dimensions.¹ In the case of $\mathcal{N} = 6$ supersymmetric Chern-Simons-matter theories [1] (known as ABJM theory) such operators were defined in [2] and evaluated at strong coupling. In this paper we will define them in more general theories. For abelian gauge groups, we perform the exact localization calculation of their expectation value, reducing the infinite dimensional path integral to a finite dimensional integral.

The basic definition of a vortex loop operator is that a gauge field has a singularity along a curve in space. Stated differently, it is the result of quantizing the theory in a background with a non-trivial singular connection. We start this paper by considering in great detail theories on a round \mathbb{S}^3 and later generalize to the case of the squashed sphere \mathbb{S}_b^3 and to the index calculation on $\mathbb{S}^2 \times \mathbb{S}^1$. In all these examples it is possible to introduce such singularities and with the appropriate choice of curve and boundary terms they preserve half of the supersymmetries.

¹Preliminary versions of these results were presented by T.O. at the “Autumn Symposium on String/M Theory”, KIAS, Seoul September 17-21 2012, at Kyoto University and at Rikkyo University and by F.P. at the “II Workshop on Geometric Correspondences of Gauge Theories,” SISSA-Trieste Italy, September 17-21 2012.

The allowed vortex loop operators depend intimately on the choice of gauge and global symmetries, matter content and the action. Particularly, the non-trivial connection may be dynamical and for gauge symmetry, or non-dynamical and for a global symmetry. In the next section we classify the possible types of 1/2 BPS loop operators in $\mathcal{N} = 2$ supersymmetric theories on \mathbb{S}^3 . Once we fix the path to be a large circle, a vortex loop operator is specified by some singularities of the gauge field, parameterized by a real diagonal matrix H (or for abelian theories a number η) and by singularities in the matter fields, encoded by a complex vector B (or number β).²

When B is completely generic, it serves effectively as a Higgs vacuum near the locus of the singularity. This can be made more precise by considering the gauge theory on $\mathbb{H}^2 \times \mathbb{S}^1$, where the singular classical solution becomes essentially a constant Higgs VEV on hyperbolic space \mathbb{H}^2 (with a holonomy along the \mathbb{S}^1). Localization reduces the partition function of supersymmetric theories on \mathbb{S}^3 to a finite dimensional integral over constant matrices, parameterizing the Coulomb branch. Even though the Higgs mechanism breaks the gauge symmetry only at the singularity, since the remaining fields are constant, they are effected by this local breaking and are frozen at the origin of the Coulomb branch. This can therefore be considered as localization on the Higgs branch, rather than the usual Coulomb branch.

We shall not perform the localization calculation of the operators with singularities for the matter fields in this paper and restrict ourselves to the case with $B = 0$.

Going back to the vortex loops without a singularity for the matter fields, these are studied in the following sections and supersymmetric localization is used to evaluate their expectation values. The calculation is similar to that in [3–8] and the final result is a very simple modification of the resulting matrix model.

In order to perform the localization calculation one should choose a localizing action, which for \mathbb{S}^3 is the usual supersymmetric Yang-Mills action and the dimensional reduction of the 4d chiral action to 3d. Both are known to be exact under some of the supercharges which preserve the vortex loops, though it also requires keeping track of boundary terms in the action near the singularity. Supersymmetric localization should allow therefore to compute the exact vacuum expectation value of the vortex loop operators by modifying the action with this exact term. With a diverging prefactor the calculation reduces to evaluating the classical action and one-loop quantum corrections around it.

We proceed to study the one-loop determinant by doing the spectral analysis in the background of the vortex loop operator. It breaks the supersymmetry of the vacuum, so the supersymmetry multiplets are short (similar to those on the deformed \mathbb{S}_b^3 [7]). It also effectively imposes modified non-periodic boundary conditions on the fields. We therefore classify non-periodic spherical harmonics on \mathbb{S}^3 , which is mainly done in appendices D and E and discussed in section 3. For small vorticity we expect that the spectrum does not change much (except for possible new almost zero modes). Indeed the spectrum is continuous with the vorticity parameter.

The result of the calculation is the usual finite dimensional matrix integral with an imaginary shift of the Coulomb brach parameters. The same shift appears in the one-loop

²There is an obstruction to having a singularity for the matter fields in the cases of \mathbb{S}_b^3 .

determinant and in the classical action. When the vortex is in a gauge connection, these parameters are integrated over and by contour deformation the result is trivial (up to a simple overall factor). When the vortex is defined for a background global symmetry the result is non-trivial.

As the vorticity grows larger, some modes which were perfectly regular turn singular, and worse, non-normalizable. We will assume that the expectation value of the vortex loop is analytic in vorticity. We thus propose a prescription for which modes to include in the spectrum and how to perform the integration over the Coulomb-branch parameters which guarantees this behavior for the gauge vortex loops and gives a prediction for vortex loops of global symmetries.

We also compute the partition function and the expectation value of loop operators on the deformed sphere \mathbb{S}_b^3 as well as $\mathbb{S}^1 \times \mathbb{S}^2$. In both cases the effect of the vortex loop operator is similar to that on the round \mathbb{S}^3 : vorticity in the gauge connection has no effect and vorticity for a flavor symmetry leads to shifts in physical parameters. To compute the one-loop determinants on these geometries, we apply the Atiyah-Singer index theory for transversally elliptic operators by generalizing the method used in [3]. In particular we manage to apply the Atiyah-Singer index theory despite the absence of fixed points for relevant isometries on these manifolds.³

We will also provide an intuitive explanation for how abelian mirror symmetry acts on vortex and Wilson loop operators, using the BF coupling between dynamical and non-dynamical gauge fields.

The vortex loop operators share some similarities to 't Hooft loop operators in four dimensions, whose exact expectation value in $\mathcal{N} = 2$ supersymmetric gauge theories was recently calculated in [10, 11]. They are both disorder line operators. They are also related to surface operators in 4d, see [12, 13] being co-dimension two defects.⁴ Like the surface operators, the vortex loop operators may involve a singularity for the matter fields as well as a non-trivial holonomy. We hope this work would be useful for an exact calculation of the expectation value of a BPS spherical surface operator in 4d.

Vortex loop operators with quantized vorticities are the same as Dirac strings, they may start and end on monopole operators. The ones we consider, though, permeate all of space (or a closed curve) instead of starting at a monopole. While in the presence of a monopole a cycle wrapping the string can be deformed and contracted to zero in a regular way on the other side of the monopole (so a Wilson loop around this cycle has to have trivial VEV), in the absence of the monopole, when considering an infinite or closed vortex loop operator, the holonomy does not have to be trivial and the vorticity may be non-integer.

It is important to distinguish between the vortex loop operators and dynamical vortices, like those of Nielsen-Olesen or Abrikosov, or those in supersymmetric theories studied in [14]. These vortices are dynamical objects, solutions to the vacuum equations of motion, while the vortex loop operators are external probes of the theory. If it were not for special

³The paper [9] also uses the fixed-point formula in such a situation, based on a similar logic.

⁴When a 3d theory lives on the boundary of a 4d spacetime, a bulk surface operator [12] ending on the boundary along a loop induces a vortex loop in the 3d theory.

boundary terms, the action of the vortex loop operators would diverge. But there is a relation, as a singular limit of the smooth solitonic vortices does reproduce the semiclassical vortex loop operator. The relation between the two is analogous to that between an 't Hooft-Polyakov monopole and an 't Hooft loop.

As this manuscript was being finalized the paper of Kapustin, Willett and Yaakov [15] appeared.⁵ That paper shares the same topic as ours and has a great deal of overlap to our discussion of vortex loop operators on the round \mathbb{S}^3 .

Note added: in the replacement on the arXiv, we elaborated on the index theory calculation of the one-loop determinants on \mathbb{S}_b^3 and $\mathbb{S}^1 \times \mathbb{S}^2$. We also made several corrections in the computation of the vortex loop expectation values. For the analysis of gauge vortex loops, we made use of the $SL(2, \mathbb{Z})$ action in the presence of loop operators considered in [15].

2 Half-BPS loop operators

Loop operators are non-local gauge invariant operators that are supported on a closed one-dimensional line. In three dimensional gauge theories there are two types of loop operators: Wilson loops, that are order type operators, and vortex loops that are disorder type operators. In the following, we provide a definition of the latter in a generic Euclidean theory on \mathbb{S}^3 with $\mathcal{N} \geq 2$ supersymmetry. Most of this is carried over to the cases of \mathbb{S}_b^3 and $\mathbb{S}^2 \times \mathbb{S}^1$ discussed in sections 4 and 5. It is assumed that the field content of the theory includes at least an $\mathcal{N} = 2$ vector multiplet, that is a gauge field A_μ , two spinors λ and $\bar{\lambda}$, and two auxiliary real scalars D and σ . This multiplet may be gauged or associated to a global symmetry. The matter vortices require of course matter fields, the dimensional reduction of a chiral multiplet in 4d with scalar ϕ , spinor ψ and auxiliary field F and anti-chiral multiplet with $\bar{\phi}$, $\bar{\psi}$ and \bar{F} . The parameterizations of the round \mathbb{S}^3 are described in appendix A and a few aspects of supersymmetry on \mathbb{S}^3 are collected in appendix B.

2.1 Half-BPS Wilson loop

Before focusing on vortex loop operators let us recall the construction of the half-BPS Wilson loops in $\mathcal{N} = 2$ supersymmetric theories in 3d [16]. We will then study all the singular field configurations preserving the same supercharges.

The ansatz for a supersymmetric Wilson loop operator is given by

$$W_R = \frac{1}{\dim R} \text{Tr}_R \mathcal{P} \left[\exp \left(\oint d\tau (iA_\mu \dot{x}^\mu + \sigma |\dot{x}|) \right) \right] \tag{2.1}$$

where $x^\mu(\tau)$ parameterizes the curve on which the Wilson loop is defined, \mathcal{P} denotes path-ordering and R is a representation of the gauge group. Applying the supersymmetry variations (B.5) to this operator it results [4]

$$\delta W_R \propto -\frac{1}{2} \bar{\epsilon} (\gamma_\mu \dot{x}^\mu - |\dot{x}|) \lambda + \frac{1}{2} \bar{\lambda} (\gamma_\mu \dot{x}^\mu - |\dot{x}|) \epsilon \tag{2.2}$$

⁵The mapping under Abelian mirror symmetry of loop operators was also obtained by Benjamin Assel, Ricardo Couso Santamaria and Jaume Gomis. We thank them for discussions.

and this is zero if

$$\bar{\epsilon}(\gamma_{\mu}\dot{x}^{\mu} - |\dot{x}|) = 0, \quad (\gamma_{\mu}\dot{x}^{\mu} - |\dot{x}|)\epsilon = 0, \quad (2.3)$$

or equivalently, using $\epsilon\bar{\epsilon} = \bar{\epsilon}\epsilon$ and $\epsilon\gamma^{\mu}\bar{\epsilon} = -\bar{\epsilon}\gamma^{\mu}\epsilon$ for fermionic SUSY parameters

$$(\gamma_{\mu}\dot{x}^{\mu} + |\dot{x}|)\bar{\epsilon} = 0, \quad (\gamma_{\mu}\dot{x}^{\mu} - |\dot{x}|)\epsilon = 0. \quad (2.4)$$

From these equations, it follows that a Wilson operator defined on a loop such that

$$\dot{x}^{\mu} = R e_3^{\mu} \quad (2.5)$$

preserves the supersymmetry generated by ϵ and $\bar{\epsilon}$ that satisfy

$$(\gamma_3 - 1)\epsilon = 0, \quad (\gamma_3 + 1)\bar{\epsilon} = 0. \quad (2.6)$$

We consider the Hopf fibration metric (A.5) with the left invariant vielbein (A.6), since in this vielbein basis the Killing spinors ϵ and $\bar{\epsilon}$ are constant. Given the expression for the inverse left invariant vielbein (A.7), the condition (2.5) implies that the Wilson loop is extended along a curve parameterized in the Hopf metric (A.5) as

$$\theta = \text{const}, \quad \phi = \text{const}, \quad \psi = 2\tau, \quad 0 \leq \tau \leq 2\pi. \quad (2.7)$$

Or in terms of the complex coordinate (u, v) in (A.1) as

$$u = u_0 e^{i\tau}, \quad v = v_0 e^{i\tau}, \quad 0 \leq \tau \leq 2\pi, \quad (2.8)$$

with arbitrary $|u_0|^2 + |v_0|^2 = R^2$. We will concentrate on the case of the loop at $u_0 = 0$ which is $\theta = 0$ in the Hopf coordinates. The submanifold described by $\theta = 0$ is codimension-2, since the metric (A.5) reduces to

$$ds^2 = \frac{R^2}{4}(d\psi + d\phi)^2 \quad (2.9)$$

and therefore, at $\theta = 0$ the loop is extended along $\psi + \phi$. In the torus fibration coordinates (A.8), the loop is extended along φ_2 .

Considering the commutators of the supersymmetries generated by ϵ and $\bar{\epsilon}$ that satisfy (2.6), one obtains the expression (B.6) where

$$v^{\theta} = 0, \quad v^{\phi} = 0, \quad v^{\psi} = \frac{2}{R}\bar{\epsilon}\gamma^3\epsilon, \quad (2.10)$$

that implies that the commutator of the supersymmetry includes a translation along the ψ angle that is a symmetry of the Wilson loop.

2.2 Half-BPS vortex loop operator: vector multiplet

Our purpose is not to study Wilson loops, which are electric order operators, but rather vortex loop operators, which are magnetic disorder operators. That amounts to considering the theory where certain fields have a singularity along a curve on \mathbb{S}^3 , which we take to be the same curve $\theta = 0$ as the aforementioned Wilson loops. We restrict to singularities

which preserve the supercharges with parameters ϵ and $\bar{\epsilon}$ satisfying the conditions (2.6) as above and use the localization scheme of [3, 4] to evaluate their expectation values.

We first examine which field configurations are invariant under the supercharge generated by $(\epsilon, \bar{\epsilon})$, which will restrict the allowed form of the singularities. Imposing reality of all the fields, then from the SUSY variation $\delta_\epsilon \lambda = 0$ in (B.5) we obtain

$$-\frac{1}{2}\epsilon_{\rho\mu\nu}F^{\mu\nu} + D_\rho\sigma = 0, \quad D + \frac{\sigma}{R} = 0, \quad (2.11)$$

and from $\delta_{\bar{\epsilon}}\bar{\lambda} = 0$

$$\frac{1}{2}\epsilon_{\rho\mu\nu}F^{\mu\nu} + D_\rho\sigma = 0, \quad D + \frac{\sigma}{R} = 0. \quad (2.12)$$

A field configuration that is invariant under the full set of supersymmetry preserved by the half-BPS Wilson loop (2.6), satisfies $\delta_\epsilon\lambda = \delta_{\bar{\epsilon}}\bar{\lambda} = 0$. Combining (2.11) and (2.12) we obtain

$$F_{\mu\nu} = 0, \quad D_\mu\sigma = 0, \quad D = -\frac{\sigma}{R}. \quad (2.13)$$

These are the same as the solutions to the localizing equations considered in [4, 5] and therefore we will be able to use the same localizing action.

In studying the \mathbb{S}^3 partition function the only classical solution of the supersymmetric Yang-Mills and Chern-Simons actions satisfying these conditions and the equations of motion are $A_\mu = 0$, $\sigma = 0$ and $D = 0$. BPS configurations include also a constant matrix $\sigma = \sigma_0$ and $D = -\sigma_0/R$. In studying the vortex loop operators we allow in addition singularities for the gauge field at $\theta = 0$. It is easiest to write the solution in the torus fibration coordinates (A.8) where the vortex is at $\vartheta = 0$ and is extended along the φ_2 circle. The curves along φ_1 at fixed ϑ are linked to the vortex, and therefore may have a nontrivial holonomy. We choose a gauge where

$$A_{\varphi_1}^{(0)} = H = \begin{pmatrix} \eta_1 \otimes 1_{N_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \eta_M \otimes 1_{N_M} \end{pmatrix}. \quad (2.14)$$

The allowed choices of H depend on the details of the gauge theory including the matter content. The requirement is that all observables are single valued when winding around the vortex loop $\varphi_1 \rightarrow \varphi_1 + 2\pi$. This includes the action, which should be well defined (up to integer shifts by 2π , as usual for Chern-Simons theory) and any gauge invariant local operator. This is automatically satisfied if all the fields of the theory are single valued, which happens if the eigenvalues η_i of H are all integers.

Local observables are gauge invariant under any gauge transformation and therefore will not be affected when rotating around the vortex loop. The partition function of Chern-Simons is not invariant under large gauge transformations, which leads to the usual quantization of the Chern-Simons level k . The presence of a vortex loop operator further restricts k such that $kH/2$ is a weight vector of a unitary representation of the gauge group [17]. For a fixed k this is a quantization condition on H .

In the Hopf coordinates (A.5), the vortex is associated to a constant gauge field $A_\mu^{(0)}$ given by

$$A_\theta^{(0)} = 0, \quad A_\phi^{(0)} = -\frac{1}{2}H, \quad A_\psi^{(0)} = \frac{1}{2}H. \quad (2.15)$$

Away from the singularity, this constant vector field configuration satisfies $F_{\mu\nu} = 0$. Indeed, in the presence of the vortex, the topology of the \mathbb{S}^3 is modified to $\mathbb{S}^1 \times \text{disk}$ and it is hence possible to have non-trivial flat connections.

An important point to notice is that thus far the background of the vortex does not seem to break any supersymmetry, as a flat connection is a solution to both the BPS and anti-BPS equations. One has to examine the singularity at $\theta \rightarrow 0$ to see that it indeed breaks half the supercharges. This is done in appendix C, where we write down the boundary terms for the Yang-Mills, Chern-Simons and Fayet-Iliopoulos actions and verify that the vortex loop operator breaks half the supersymmetries.

Given the singular behavior (2.14) at $\theta = 0$, the most general solution to the BPS equations (2.13) has this exact value for the gauge field as in the classical solution and in addition we can turn on $\sigma = \sigma_0$ and $D = -\frac{\sigma_0}{R}$ where σ_0 is covariantly constant, i.e.,

$$D_\mu^{(0)}\sigma_0 = 0, \quad (2.16)$$

and the covariant derivative $D_\mu^{(0)}$ is defined using the constant connection $A_\mu^{(0)}$ (2.15).

If we label by $(\sigma_0)^i_j$ one of the components of σ_0 in the i, j block, then (2.16) gives

$$\begin{aligned} \partial_\phi(\sigma_0)^i_j - \frac{i}{2}(\eta_i - \eta_j)(\sigma_0)^i_j &= 0, \\ \partial_\psi(\sigma_0)^i_j + \frac{i}{2}(\eta_i - \eta_j)(\sigma_0)^i_j &= 0. \end{aligned} \quad (2.17)$$

For $\eta_i \not\equiv \eta_j \pmod{1}$ the only regular periodic solution to these equations is $(\sigma_0)^i_j = 0$. For generic η_i, η_j the only nontrivial solutions are therefore for $i = j$, and this component $(\sigma_0)^i_j$ can be an arbitrary constant.

If the vortex loop operator is defined for a gauged vector multiplet (rather than a background vector field), we should define the integration measure. Since for generic H the allowed values of σ_0 are automatically diagonal, there is no extra Vandermonde determinant.

If there are degeneracies, and the singularity preserves a non-trivial Levi group $U(N_1) \times \dots \times U(N_M)$, the resulting Vandermonde determinant involves only the eigenvalues within the different blocks along the diagonal

$$\prod_{m=1}^M \prod_{i < j=1}^{N_m} [(\sigma_0)_{m,i} - (\sigma_0)_{m,j}]^2, \quad (2.18)$$

where we labeled $(\sigma_0)_{m,i}$ the i^{th} element on the diagonal of the m^{th} block of σ_0 .

Note that the symmetry is enlarged (and the resulting Vandermonde) also for values of η_i differing by integers, as can be seen by the periodic non-trivial solutions of (2.17). Furthermore, if some η_i form a representation of \mathbb{Z}_n for some $n \leq N$, so $\eta_j \equiv j/n \pmod{1}$ for $j = 1, \dots, n$, then an S_n subgroup is preserved allowing for twisted solutions which are

periodic only up to S_n transformations. Such solutions are important in ABJM theory [2] and exist also for surface operators in $\mathcal{N} = 4$ SYM in four dimensions in [18]. This mimics the construction of “long strings” in M(atr)ix theory [19, 20].

2.3 Half-BPS vortex loop operator: matter multiplet

We turn now to the matter sector, whose supersymmetry transformations are written in appendix B.3. We shall find non-trivial profiles for the scalar field which are invariant under the same supercharges as the Wilson loops and the vortex loop operators from the vector multiplet.

2.3.1 Abelian theory

Let us start with an abelian theory. Assuming the supercharges satisfy the half-BPS conditions in (2.6), the vanishing of the variation of ψ and $\bar{\psi}$ in (B.13) give the equations

$$\begin{aligned} ie_3^\mu D_\mu \phi + i\sigma\phi - \frac{\Delta}{R}\phi = 0, & \quad (e_1^\mu + ie_2^\mu)D_\mu \phi = 0, & \quad F = 0, \\ ie_3^\mu D_\mu \bar{\phi} - i\sigma\bar{\phi} + \frac{\Delta}{R}\bar{\phi} = 0, & \quad (e_1^\mu - ie_2^\mu)D_\mu \bar{\phi} = 0, & \quad \bar{F} = 0. \end{aligned} \tag{2.19}$$

where e_a^μ are the inverse vielbeins in (A.7). This expression applies for a massless field charged under a single gauge group. For a field charged under two groups there would be the appropriate modification to the connection D_μ and likewise σ would be replaced by the difference of $\sigma^{(i)}$ of the two vector multiplets. As usual a mass term is like a σ field for a non-dynamical vector field.

In terms of the Hopf coordinates (A.5) the equations for ϕ are⁶

$$D_\psi \phi = -\frac{i}{2}(\Delta - iR\sigma)\phi, \quad (\sin\theta D_\theta - iD_\phi + i\cos\theta D_\psi)\phi = 0. \tag{2.20}$$

We saw already that the supersymmetry conditions of the vector multiplet restrict $\sigma = \sigma_0$ a constant. For real $\sigma_0 \neq 0$ the first equation does not have periodic solutions other than $\phi = 0$. For $\sigma_0 = 0$ (or in the case with more than one gauge multiple or a mass term, the vanishing of their sum) there are extra solutions of the form

$$\phi(\theta, \phi, \psi) = e^{-i\frac{\Delta+\eta}{2}\psi}\phi(\theta, \phi), \tag{2.21}$$

with $\phi(\theta, \phi)$ satisfying

$$\left(\sin\theta \partial_\theta - i\partial_\phi - \frac{\eta}{2} + \frac{\Delta}{2}\cos\theta\right)\phi(\theta, \phi) = 0, \tag{2.22}$$

where η is the gauge vorticity (2.14) for an abelian theory, i.e., $H = \eta$. With the ansatz $\phi(\theta, \phi) = e^{in\phi}\phi_n(\theta)$ we get the solution

$$\phi_n(\theta) = \frac{\beta_n}{R^\Delta} \sin^{-\frac{\Delta-\eta}{2}-n}\frac{\theta}{2} \cos^{-\frac{\Delta+\eta}{2}+n}\frac{\theta}{2}. \tag{2.23}$$

⁶Hopefully there will be no confusion between the field ϕ and coordinate ϕ .

The values of β_n are determined by specifying the singularity of the field. This ansatz allows for singularities and zeros at $\theta = 0$ and $\theta = \pi$, but by taking linear combinations of these functions one can get singularities at any point on the base parameterized by (θ, ϕ) .

In terms of the torus coordinates (A.8) the solution is

$$\phi(\vartheta, \varphi_1, \varphi_2) = \frac{\beta_n}{R^\Delta} (\sin \vartheta e^{i\varphi_1})^{-\frac{\Delta-\eta}{2}-n} (\cos \vartheta e^{i\varphi_2})^{-\frac{\Delta+\eta}{2}+n} e^{-i\eta\varphi_1}. \quad (2.24)$$

Requiring periodicity in the φ_2 direction enforces $n - (\Delta + \eta)/2$ to be an integer. Furthermore, if we want singularities only at $\vartheta = 0$, then this integer cannot be negative. The simplest and least singular case is when it is zero, which gives

$$\phi(\vartheta, \varphi_1, \varphi_2) = \frac{\beta e^{-i\eta\varphi_1}}{(R \sin \vartheta e^{i\varphi_1})^\Delta}. \quad (2.25)$$

The behavior of the scalar field near the singularity is determined by its dimension Δ (and in addition the holonomy η). For a scalar of canonical dimension $\Delta = 1/2$ this is

$$\phi(\vartheta, \varphi_1, \varphi_2) = \frac{\beta e^{-i\eta\varphi_1}}{\sqrt{R \sin \vartheta e^{i\varphi_1}}}. \quad (2.26)$$

The field ϕ is complex, but the parameter β can, without loss of generality, be taken real. Its phase is unphysical as it is modified by taking $\varphi_1 \rightarrow \varphi_1 + 2\pi$ and can be changed by a gauge transformation with a constant gauge parameter.

One can also formulate the vortices in flat space (and on $\mathbb{H}_2 \times \mathbb{S}^1$), as was done in [2]. The flat space vortex arises in the large R limit after replacing $R \sin \vartheta \rightarrow r$, $\cos \vartheta \rightarrow 1$ and $R\varphi_2 \rightarrow x_3$. After rescaling we get the solution

$$\phi(r, \varphi_1, x_3) = \beta_n (r e^{i\varphi_1})^{-n-\Delta/2} (r e^{-i\varphi_1})^{\eta/2}. \quad (2.27)$$

In the special case of $n = \frac{\Delta+\eta}{2}$ we get from (2.25)

$$\phi(r, \varphi_1, x_3) = \frac{\beta e^{-i\eta\varphi_1}}{(r e^{i\varphi_1})^\Delta}. \quad (2.28)$$

This indeed matches with the vortex in ABJM theory [2] once we set $\Delta = 1/2$ and $\eta = 0$ (in [2] there was a gauge vortex, but it was in the diagonal sum of the two gauge groups which the matter fields are not charged under).

So far we discussed only the field ϕ . The same analysis applies also for the field $\bar{\phi}$, once we require the invariance under the $\bar{\epsilon}$ variation.

2.3.2 Non-abelian theory

Turning to the general non-abelian theory, the matter fields are in some representation R of the gauge group. We denote the generators of the algebra in the R representation as (X_α^R, K_i^R) , where K_i^R 's span the Cartan subset. The normalization of the generators is such that $\text{Tr}(X_\alpha^R, X_\beta^R) = \delta_{\alpha+\beta,0}$ and $(X_\alpha^R)^\dagger = X_{-\alpha}^R$.

The weights of R are denoted as ρ and the associated state is $|\rho\rangle$ such that for the Cartan generators $K_i^R|\rho\rangle = \rho_i|\rho\rangle$. The scalar field of the chiral multiplet ϕ is expressed as

$$\phi = \sum_{\rho} \phi^{\rho} |\rho\rangle \quad (2.29)$$

and likewise the other members of the multiplet. For the anti-chiral scalar we take bra states

$$\bar{\phi} = \sum_{\rho} \bar{\phi}^{\rho} \langle\rho| \quad (2.30)$$

where $\langle\rho|\rho'\rangle = \delta_{\rho,\rho'}$. The fields of the vector multiplet, which are in the adjoint representation, appear in the chiral Lagrangian and the supersymmetry transformations accompanied by the generators of the algebra in the representation R , so for example

$$\sigma \rightarrow \sigma^i K_i^R + \sigma^{\alpha} X_{\alpha}^R. \quad (2.31)$$

Then the first equation in (2.19) becomes

$$\begin{aligned} ie_3^{\mu} D_{\mu} \phi + i\sigma\phi - \frac{\Delta}{R}\phi &= \sum_{\rho} \left(ie_3^{\mu} (\nabla_{\mu} \phi^{\rho} + iA_{\mu}^i \rho_i \phi^{\rho}) + i\sigma^i \rho_i \phi^{\rho} - \frac{\Delta}{R} \phi^{\rho} \right) |\rho\rangle \\ &+ \sum_{\rho,\rho'} \left(ie_3^{\mu} A_{\mu}^{\alpha} (X_{\alpha}^R)_{\rho'}^{\rho} \phi^{\rho'} + i\sigma^{\alpha} (X_{\alpha}^R)_{\rho'}^{\rho} \phi^{\rho'} \right) |\rho'\rangle = 0. \end{aligned} \quad (2.32)$$

For $\sigma = 0$ and A_{μ} as in (2.14), the second line of this equation vanishes and we find $\dim(R)$ copies of the scalar equations in (2.19). The solution is then as in (2.25)

$$\phi(\vartheta, \varphi_1, \varphi_2) = \sum_{\rho} \frac{\beta^{\rho} e^{-i\rho(H)\varphi_1}}{(R \sin \vartheta e^{i\varphi_1})^{\Delta}} |\rho\rangle, \quad (2.33)$$

which can also be written as

$$\phi(\vartheta, \varphi_1, \varphi_2) = \frac{e^{-iH\varphi_1}}{(R \sin \vartheta e^{i\varphi_1})^{\Delta}} B, \quad B = \sum_{\rho} \beta^{\rho} |\rho\rangle. \quad (2.34)$$

The simplest solutions to the BPS equations are when either $\sigma = 0$, which allows for arbitrary β^{ρ} , or when $\phi = 0$ which allows for arbitrary constant $\sigma = \sigma_0$. More generally one can turn on just some components of σ and then there will be a restriction on which β^{ρ} may be nonzero. Viewed the other way, choosing non-generic β^{ρ} will leave some residual symmetry and the components of σ in the directions of the generators of this preserved symmetry will not be frozen to zero and will have to be integrated over after localization. For example, if R is the fundamental representation of $U(N)$ and $\beta^i = 0$ for $i = 1, \dots, n$, then there will be a residual $U(n)$ symmetry and after diagonalization, n elements of σ_0 to integrate over. If n of the β^i 's are equal to each-other but non zero, the symmetry will be $SU(n)$, and so on.

This analysis is nothing different from the usual breaking of gauge symmetry by the Higgs mechanism, only that here the scalar fields get a non-trivial profile, instead of a constant VEV. As mentioned in the introduction, this profile becomes constant upon conformal transformation to $\mathbb{H}_2 \times \mathbb{S}^1$.

2.4 Vortex and Wilson loop operators for flavor symmetries

Before plunging into the localization calculations we want to explore the relation between Wilson loop operators and vortex loop operators in different $\mathcal{N} = 2$ theories. A useful generalization of the loop operators discussed above is to consider operators defined with respect to global rather than gauge groups.

Given a global symmetry that commutes with SUSY (hence not an R -symmetry), it is natural to couple its current j^μ to a background, non-dynamical abelian gauge field \mathbf{A}_μ through the coupling in the action⁷

$$\int \mathbf{A}_\mu j^\mu, \tag{2.35}$$

which can be supersymmetrized. This procedure is sometimes called “gauging”, but we reserve the term for the case when the gauge field is dynamical. The vortex loop operator for this global symmetry is defined by letting the non-dynamical gauge field have the singularity (2.14)

$$\mathbf{A} \sim \eta d\varphi_1, \tag{2.36}$$

where φ_1 is the angular variable in the locally defined polar coordinates on the plane orthogonal to and centered at the loop. Whether this definition gives a BPS vortex loop depends on the space-time geometry since we need a globally defined supersymmetric profile of \mathbf{A}_μ with the singularity (2.36). In the case of \mathbb{S}^3 discussed so far (and \mathbb{S}_b^3 and $\mathbb{S}^1 \times \mathbb{S}^2$ studied later) this is indeed the case.

Consider a vortex loop for the topological symmetry $U(1)_J$ generated by the current $J^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho} F_{\nu\rho}$, the Hodge dual of the field strength of a dynamical abelian gauge field. This vortex loop is simply a rewriting of the usual Wilson loop. The singularity (2.36) means that the non-dynamical field strength $d\mathbf{A}$ has a delta function, and the coupling (2.35), which is precisely the BF coupling (B.18), becomes

$$\int \mathbf{A} \wedge F = \int d\mathbf{A} \wedge A = \eta \oint A + \int d\underline{\mathbf{A}} \wedge A, \tag{2.37}$$

where A is the dynamical gauge field and the underline indicates the smooth part of the field. Thus the singularity induces a Wilson loop (of charge η). This argument can be supersymmetrized. See section 3.

Though somewhat trivial, it is also natural to define Wilson loop operators associated with a global symmetry as the insertion of the function

$$e^{\oint(i\mathbf{A}+\dots)}, \tag{2.38}$$

where for supersymmetry one needs an appropriate curve for integration and certain terms in the ellipses. Since \mathbf{A} is non-dynamical, this term factors out of the path integral and is given by its background value.

In the case of the topological symmetry $U(1)_J$, the associated Wilson loop is in fact a gauge vortex loop, defined by the singularity (2.14) in a dynamical gauge field A . This

⁷If the global symmetry is non-abelian, we can use its Cartan subalgebra to define the flavor vortex loop.

can be seen through a manipulation similar to (2.37):

$$\int \mathcal{D}\underline{A} \dots e^{i \int A \wedge d\underline{A}} e^{-S[A, \dots]} \dots = \int \mathcal{D}A \dots e^{-i\eta \oint A} e^{i \int A \wedge dA} e^{-S[A, \dots]} \dots \quad (2.39)$$

On the left side A contains a singular part and \underline{A} is the smooth fluctuation, while on the right side A is regarded as a smooth gauge field by a change of the integration contour.⁸

This analysis applies to all vortex loop operators for dynamical gauge fields, and we can therefore conclude that they have a trivial expectation value, apart for a possible simple multiplicative factor. This will be verified in the rest of the paper by explicit localization calculations.

To prevent the impression that the following is an exercise in futility, we should point out that not all vortex loop operators are trivial. We saw above that vortex loop operators for the topological symmetry are the same as Wilson loops, which are not trivial. That still is not so exciting, as we can use the standard definition of the Wilson loop and do not require to define it via the vortex loop. If there is a global symmetry under which some of the chiral fields are charged (i.e., a flavor symmetry) then the flavor vortex loop for that group will not be trivial nor trivially related to a Wilson loop operator.⁹ Indeed as we explain in section 6, under abelian mirror symmetry flavor and topological symmetry are exchanged, so the flavor vortex loop operator gets mapped to the gauge Wilson loop operator.

This statement may seem surprising, since we are accustomed to continuous holonomies and discrete electric charges. It is therefore important to analyze which values of charges are allowed for the BPS loop operators. The answer seems to depend on the topology of the space.

As discussed after (2.14), in the case of S^3 the holonomies can be continuous, which is true also for the squashed sphere S_b^3 discussed in section 4. The situation on $S^2 \times S^1$ discussed in section 5 is slightly different. In that case introducing a non-integer vortex at the north pole of S^2 (wrapping the S^1) would automatically induce also a singularity at the south pole. The total vorticity will cancel, unless we introduce a nontrivial transition function at the equator, in which case the total vorticity is integral. The conclusion is therefore that each vortex can have a continuous parameter, but the total vorticity has to be an integer.

Normally Wilson loops are defined only for integer electric charges, which is due to the fact that the gauge group (in the abelian case) is $U(1)$ rather than \mathbb{R} . But on S^3 , which is simply connected there is no obstruction of using \mathbb{R} , with continuous electric charges, as the gauge group. The mirror of the flavor vortex would be such a Wilson loop. On $S^2 \times S^1$ there is a non-contractable cycle and large gauge transformations can wind around it leading to a quantization condition. Again, we can locally break the abelian Wilson loop into two which are not integer, say one at the north and one at the south poles of S^2 , but the total charge is quantized, which matches the mirror picture of the vortex loops.

⁸This manipulation becomes more natural when loop operators are smeared as in [15].

⁹Likewise, it is not clear whether the matter vortex loop operators of section 2.3 are trivial or not.

3 Localization on \mathbb{S}^3 and harmonic analysis

In this section we describe the localization of $\mathcal{N} = 2$ theories on the round \mathbb{S}^3 in the presence of a vortex operator defined in (2.14). We use the conventions of [21].

The gauge vortex loop operators are given by a choice of a real diagonal matrix H (2.14) breaking the gauge symmetry near the singularity to a subgroup. In the most general case, where all eigenvalues of H are distinct, the gauge symmetry is broken to the Cartan subalgebra. For degenerate H there will be larger residual gauge symmetry.

In the proceeding we will study the partition function of generic supersymmetric theories in the presence of a gauge vortex loop operator. The calculation is done using localization techniques.

3.1 Classical factor

The localization calculation reduces the path integral on \mathbb{S}^3 to a finite dimensional integral over BPS configurations. This is achieved (see appendix D) by adding Q -exact terms to the action, whose bulk part is proportional to the SYM and/or the Chiral actions. The modified action determines the localization locus and the 1-loop determinant about it. This locus turn out to be given by $\delta\lambda = 0$, which are just the BPS equations (2.13). We thus have the classical vortex configuration and in addition should integrate off-shell over the covariantly constant σ_0 matrix.

The original action does not necessarily vanish on the BPS configurations. We calculate this contribution first.

In the gauge sector there may be a supersymmetric Chern-Simons term with level k . The action on \mathbb{S}^3 is

$$\mathcal{S}_{\text{SCS}} = \frac{k}{4\pi} \int d^3x \sqrt{g} \text{Tr} \left[\epsilon^{\mu\nu\rho} \left(A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho \right) - \bar{\lambda} \lambda + 2D\sigma \right]. \quad (3.1)$$

Including the boundary term (C.11), which is required for supersymmetry and gauge invariance, and evaluated on the BPS vortex configuration we find

$$\begin{aligned} i\mathcal{S}_{\text{SCS}}^{\text{BPS}} + i\mathcal{S}_{\text{B}}^{\text{BPS}} &= \frac{ik}{4\pi} \int d^3x \sqrt{g} \text{Tr} \left[-2\frac{\sigma_0^2}{R} \right] + kR \int d\varphi_2 \text{Tr}[H\sigma_0] \\ &= -\pi ik \text{Tr} \left[(R\sigma_0 + iH)^2 + H^2 \right]. \end{aligned} \quad (3.2)$$

It is also possible to include the supersymmetric Yang-Mills action on \mathbb{S}^3

$$\begin{aligned} \mathcal{S}_{\text{SYM}} &= \frac{1}{g_{\text{YM}}^2} \int d^3x \sqrt{g} \text{Tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \sigma D^\mu \sigma + \frac{1}{2} \left(D + \frac{\sigma}{R} \right)^2 \right. \\ &\quad \left. + \frac{i}{2} \bar{\lambda} \gamma^\mu D_\mu \lambda + \frac{i}{2} \bar{\lambda} [\sigma, \lambda] - \frac{1}{4R} \bar{\lambda} \lambda \right]. \end{aligned} \quad (3.3)$$

Both the bulk and boundary terms (C.5) of the SYM action vanish on the BPS vortex configurations.

Lastly, another possible supersymmetric term for an abelian vector multiplet is the Fayet-Iliopoulos action

$$\mathcal{S}_{\text{FI}} = -\frac{i\zeta}{2\pi R} \int d^3x \sqrt{g} \left(D - \frac{\sigma}{R} \right). \quad (3.4)$$

Together with the boundary term (C.23) we find

$$\mathcal{S}_{\text{FI}}^{\text{BPS}} + \mathcal{S}_{\text{B}}^{\text{BPS}} = 2\pi i\zeta(R\sigma_0 + i\eta). \quad (3.5)$$

The matter fields are described by a chiral multiplet in a generic representation R of the gauge group, possibly reducible. The supersymmetric action for a chiral multiplet with fields with arbitrary dimension Δ is given by

$$\begin{aligned} \mathcal{L}_{\text{chiral}} = & D_\mu \bar{\phi} D^\mu \phi + \bar{\phi} \sigma^2 \phi + \frac{i(2\Delta - 1)}{R} \bar{\phi} \sigma \phi + \frac{\Delta(2 - \Delta)}{R^2} \bar{\phi} \phi + i\bar{\phi} D \phi + \bar{F} F \\ & - i\bar{\psi} \gamma^\mu D_\mu \psi + i\bar{\psi} \sigma \psi - \frac{2\Delta - 1}{2R} \bar{\psi} \psi + i\bar{\psi} \lambda \phi - i\bar{\phi} \bar{\lambda} \psi. \end{aligned} \quad (3.6)$$

We perform the localization calculation here only for the vortex loops with vanishing ϕ , so this term in the action vanishes on these BPS configurations.

In addition to the vortex we may have a Wilson loop which links it and does not break any further supersymmetry. From (2.1) we see that we will get a term

$$W^{\text{cl.}} = \frac{1}{\dim R} \text{Tr}_R [\exp(2\pi(R\sigma_0 + iH))]. \quad (3.7)$$

Examining the classical pieces in the different actions as well as in the Wilson loop, we see that the inclusion of the vortex loop amounts to the simple replacement $R\sigma_0 \rightarrow R\sigma_0 + iH$. The only exception is in the case of Chern-Simons, which has an extra $-\pi i k \text{Tr}(H^2)$, which is a simple constant multiplicative factor.

3.2 Fluctuation determinant

The localizing action on \mathbb{S}^3 is written in appendix D and is the sum of the SYM action and in the presence of matter fields also the chiral action, both multiplied by an arbitrary constant t . For large t the path integral reduces to the saddle points of the action and the one-loop determinant about it.

In appendix D the resulting kinetic operators are written down and diagonalized. For the vector multiplet they are

$$\nabla_\mu^{(0)} = \nabla_\mu + i\alpha(A_\mu^{(0)}), \quad (3.8)$$

which is the usual covariant derivative in the presence of the background gauge field. The background field can be removed by a singular gauge transformation (D.8), which makes all the fluctuation fields $\tilde{\Phi}$ non-periodic. Rather, they satisfy

$$\begin{aligned} \tilde{\Phi}^\alpha(\vartheta, \varphi_1 + 2\pi, \varphi_2) &= e^{2\pi i\alpha(H)} \tilde{\Phi}^\alpha(\vartheta, \varphi_1, \varphi_2), \\ \tilde{\Phi}^\alpha(\vartheta, \varphi_1, \varphi_2 + 2\pi) &= \tilde{\Phi}^\alpha(\vartheta, \varphi_1, \varphi_2). \end{aligned} \quad (3.9)$$

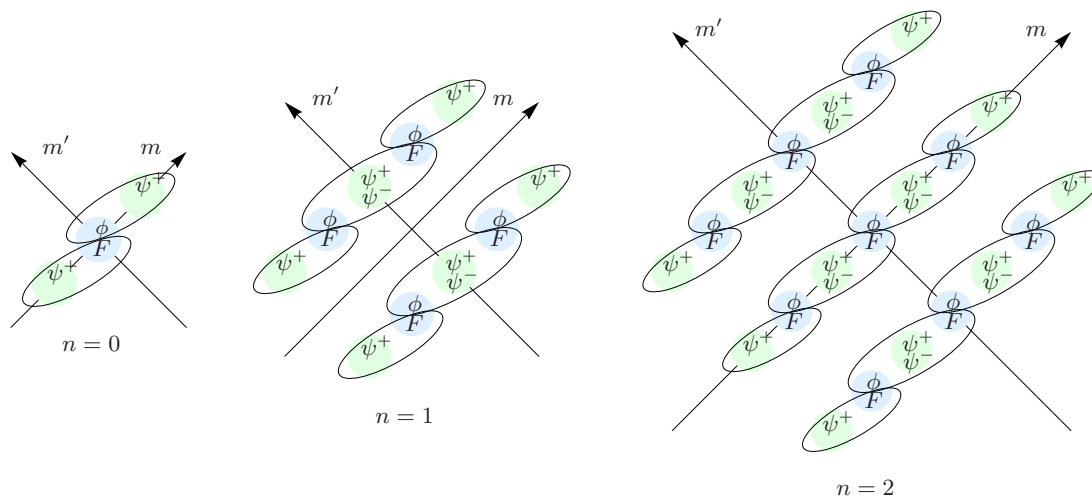


Figure 1. Fluctuation modes of a chiral multiplet. The lattices represent states with principle quantum numbers $n = 0, 1, 2$ and the allowed values of m and m' . In the presence of the vortex loop these multiplets are broken to smaller ones encapsulated by the ovals. Only the short representations (with two modes) contribute to the determinant.

The spherical harmonics with non-standard periodicity conditions are studied in appendix E and give for the vector multiplet the product representation of the determinant as (D.14)

$$Z_{1\text{-loop}}^{\text{vector}}(\sigma_0) = \prod_{\alpha > 0} \prod_n^{\infty} (n^2 + \alpha(R\sigma_0 + iH)^2). \quad (3.10)$$

For $\alpha(H) = 0$ the product over n starts at $n = 1$, but for $\alpha(H) \neq 0$ we expect there to be extra fermionic (almost-)zero modes and the product starts at $n = 0$. In that case we find after regularizing the infinite product (E.52)

$$Z_{1\text{-loop}}^{\text{vector}}(\sigma_0) = \prod_{\alpha > 0} \frac{1}{\pi^2} \sinh^2(\pi\alpha(R\sigma_0 + iH)). \quad (3.11)$$

For $\alpha(H) = 0$ this is multiplied by an extra factor of $1/\alpha(R\sigma_0 + iH)^2$. This extra factor exactly cancels the Vandermonde determinant, which as discussed at the end of section 2.2, appears only in the case of degenerate H .

A similar analysis for the chiral multiplet in appendix D.2 leads to (D.28)

$$Z_{1\text{-loop}}^{\text{chiral}}(\sigma_0) = \prod_{n=1}^{\infty} \prod_{\rho} \left(\frac{n+1-\Delta+i\rho(R\sigma_0+iH)}{n-1+\Delta-i\rho(R\sigma_0+iH)} \right)^n = \prod_{\rho} s_{b=1}(i-i\Delta-\rho(R\sigma_0+iH)), \quad (3.12)$$

where ρ are the weights of the representation of the matter fields and $s_b(x)$ is the double sine function.

3.3 Spectral analysis

We have now found that the different ingredients making up the matrix model representation of the S^3 partition function of an $\mathcal{N} = 2$ supersymmetric theory in 3d are modified.

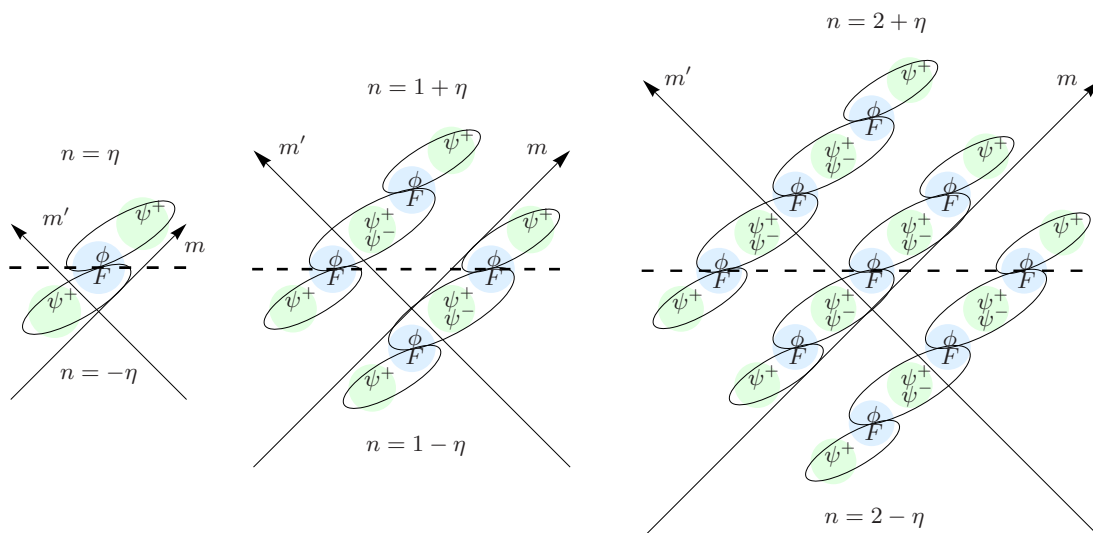


Figure 2. After introducing $\eta = 1/2$ the entire spectrum in figure 1 is shifted by $m \rightarrow m + \eta/2$ and $m' \rightarrow m' + \eta/2$. For the multiplets under the dashed line the principle quantum number is shifted $n \rightarrow n - \eta$ and above the dashed line $n \rightarrow n + \eta$, which effects the determinant.

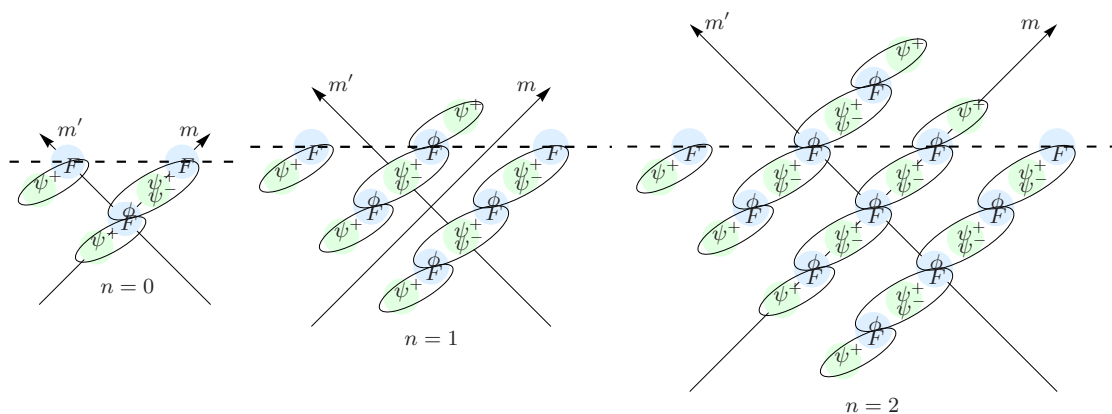


Figure 3. For $\eta = 1$ the spectrum is shifted by a full integer. Here are the new states with principle quantum numbers 0, 1 and 2. The states above the dashed line come from the original multiplet with $n - \eta$ and those below from $n + \eta$. Compared to the spectrum in figure 1, with the same value of n , there are the same number of ϕ modes, but an extra $\{\psi^+, F\}$ short multiplet, and one $\{\phi, \psi^+\}$ short multiplet gets enlarged by an extra ψ^- and F mode.

The contributions of the CS and FI actions evaluated on the BPS configurations are given by (3.2), (3.5) respectively. The one-loop determinants for the vector and chiral multiplets are in (3.11) and (3.12). Rather surprisingly, the change to all of them can be accounted for by an imaginary shift $\sigma_0 \rightarrow \sigma_0 + iH/R$.¹⁰ This is also true for the expectation value of a Wilson loop in the presence of the vortex (3.7).

Since σ_0 is integrated over, the deformation of the contour of integration will not change the answer as long as no singularities are crossed. The conclusion, as predicted in

¹⁰The one exception is the CS term which has an extra $-\pi ik \text{Tr}(H^2)$ term in the action, which gives an overall multiplicative factor to the partition function.

section 2.4, is therefore that at least for abelian theories *the vorticity H does not effect the partition function.*¹¹

There are some subtleties in this statement due to the fact that the double sine function arising in the one-loop determinant does have poles at integer values of H . In this section we discuss these subtleties and their origin and propose a prescription to resolve them.

After the imaginary shift of σ_0 , the determinant for a chiral field is given by (D.28). For $\eta = 1$ this is

$$s_{b=1}(i - i\Delta - (R\sigma_0 + i\eta)) = \frac{1}{\pi} \sin(\pi(i - i\Delta - R\sigma_0)) s_{b=1}(i - i\Delta - R\sigma_0). \quad (3.13)$$

The situation for the vector multiplet is simpler, since the adjoint representation is self conjugate. Under an integer imaginary shift $\sinh(\pi(R\sigma_0 + in)) = (-1)^n \sinh(\pi R\sigma_0)$ (3.11). The other change is that the denominator $1/(\pi(R\sigma_0 + in))$ which usually cancels the regular Vandermonde factor is no longer there, due to the extra goldstino zero modes.

But the transformation of the chiral multiplet (3.13) is a nontrivial transformation, meaning the spectrum of fluctuations really changes even for integer η . This is illustrated in figures 1–3. Starting with a supermultiplet of the $\text{OSp}(2|2, 2)$ supersymmetry group on \mathbb{S}^3 , the vortex breaks the symmetry down to $\text{SU}(1|1, 1)$, and the original symmetry is not restored at integer η , leading to a different multiplet structure. The breaking of the supersymmetry multiplet of fluctuation modes is analyzed in appendix E.4.

Exactly half of the states with principle quantum number n get deformed to states with quantum number $n - \eta$ and half to $n + \eta$. For small $\eta > 0$ there are $(n + 1)(n + 2)/2$ modes of the scalar field ϕ with $n + \eta$ and $n(n + 1)/2$ with $n - \eta$. For $\eta = 1$ there are $(n + 1)(n + 2)/2 + n(n + 1)/2 = (n + 1)^2$ states with principle quantum number n , which is the same as the number for $\eta = 0$. The number of fermi fields does change, with one extra fermion of either chirality. To keep the SUSY structure consistent, there are also two extra modes of the auxiliary field F .

In terms of the multiplets of the smaller group, for $\eta = 1$ there is one extra long multiplet and one less short multiplet with ϕ and ψ^+ and one more short multiplet with ψ^+ and F . The extra four modes are eigenstates of the Laplacian and Dirac operator with the relevant eigenvalues, which were not there for $\eta = 0$, so these are modes which do not belong to the $\text{OSp}(2|2, 2)$ representation. They are in fact singular modes, which normally are not included in the spectral analysis. They are part of larger nonunitary representations of this group, which are not part of the unitary subrepresentation. The explicit analysis of the spectral flow means that these states should be counted and they lead to the factor in (3.13).

Of course if the theory has only self conjugate representations, or all representations are paired up with their conjugates, then there are extra cancellations and for integer η one finds only at most a sign factor. This is the case for theories with $\mathcal{N} = 4$ SUSY. But as stated, for $\mathcal{N} = 2$ SUSY, the effect of spectral flow is very nontrivial.

¹¹In pure topological Chern-Simons theory a vortex loop was defined in [17]. It was argued there that it is equal to a Wilson loop observable. In an abelian Chern-Simons theory, both Wilson and vortex loops are almost trivial, acting on the partition function as multiplication by a phase.

The conclusion of the above discussion would seem to imply that the partition function in the presence of the vortex loop, while constant for $0 \leq \eta < 1$, jumps for integer η . There seem to be two possible prescriptions. The first is to use the values of the 1-loop determinant that we have found, but keep the integration contour such that it does not cross the poles. The second possibility is to not do the spectral flow as discussed above, but as the singular modes show up in the spectrum, replace them with other modes which were singular before and now become regular. In this way we restore the original spectrum for integer η .

While the second possibility seems more appealing physically, an analysis of flavor vortex loop operators, where σ_0 is not integrated over seems to prefer the first interpretation. This allows for them to be dual to regular gauge Wilson loops under abelian mirror symmetry.

4 Localization on \mathbb{S}_b^3 by index theory

In this section, we will compute by localization the expectation value of the gauge and flavor vortex loops on a deformation of the three-sphere, commonly denoted as \mathbb{S}_b^3 . As in previous sections, we will consider an arbitrary $\mathcal{N} = 2$ gauge theory with a chiral multiplet in representation R of the gauge group. We will first explain how to use the equivariant index theory to compute the one-loop determinant that appears in the partition function. Then we will apply the technique to compute the expectation value of the vortex loop. We provide many technical details in appendix F.

4.1 Partition function

This geometry \mathbb{S}_b^3 , also known as the ellipsoid, is defined by the metric

$$ds^2 = R^2 (f(\vartheta)^2 d\vartheta^2 + b^2 \sin^2 \vartheta d\varphi_1^2 + b^{-2} \cos^2 \vartheta d\varphi_2^2), \tag{4.1}$$

where

$$f(\vartheta) \equiv (b^{-2} \sin^2 \vartheta + b^2 \cos^2 \vartheta)^{1/2}. \tag{4.2}$$

To describe spinors, we will use the orthonormal frame given by¹²

$$e^{\hat{1}} = Rb^{-1} \cos \vartheta d\varphi_2, \quad e^{\hat{2}} = -Rb \sin \vartheta d\varphi_1, \quad e^{\hat{3}} = R f(\vartheta) d\vartheta. \tag{4.3}$$

For localization we will use the supercharge $Q = \delta_\epsilon + \delta_{\bar{\epsilon}}$ generated by the two spinors

$$\epsilon \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{i}{2}(\varphi_1 + \varphi_2 + \vartheta)} \\ e^{\frac{i}{2}(\varphi_1 + \varphi_2 - \vartheta)} \end{pmatrix}, \quad \bar{\epsilon} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{\frac{i}{2}(-\varphi_1 - \varphi_2 + \vartheta)} \\ e^{\frac{i}{2}(-\varphi_1 - \varphi_2 - \vartheta)} \end{pmatrix}. \tag{4.4}$$

As shown in [7], these spinors satisfy a variant of the Killing spinor equations, ensuring that the algebra generated by supersymmetry transformations of fields on \mathbb{S}_b^3 closes. In particular, the supercharge Q squares to a sum of bosonic symmetries

$$Q^2 = i\mathcal{L}_v + i\sigma - v^\mu A_\mu + \frac{1}{2R}(b + b^{-1})\mathcal{R}, \tag{4.5}$$

¹²Comparison with [7] is simple with this definition of $e^{\hat{\mu}}$: $(\vartheta, \varphi_1, \varphi_2)_{\text{here}} = (\theta, -\chi, \varphi)_{\text{there}}$. Also we have the following change of conventions: $(A_\mu, C, \bar{\epsilon}, \bar{\lambda}, \bar{\psi})_{\text{here}} = -(A_\mu, C, \bar{\epsilon}, \bar{\lambda}, \bar{\psi})_{\text{there}}$, $(\epsilon, \bar{\epsilon})_{\text{here}} = (-\bar{\epsilon}, \epsilon)_{\text{there}}$, $v_{\text{here}}^\mu = -v_{\text{there}}^\mu$, $(F, \bar{F})_{\text{here}} = (-F, -\bar{F})_{\text{there}}$.

where \mathcal{L}_v is the Lie derivative¹³ along the vector field

$$v \equiv \bar{\epsilon} \gamma^{\hat{\mu}} \epsilon e_{\hat{\mu}} = R^{-1} \left(b^{-1} \frac{\partial}{\partial \varphi_1} + b \frac{\partial}{\partial \varphi_2} \right). \quad (4.6)$$

We write $\mathcal{R} = \mathcal{R}_0 - \Delta \mathcal{F}$, where \mathcal{R}_0 generates the canonical R-symmetry, and \mathcal{F} is the generator of flavor symmetry $U(1)_F$. In our convention the lowest component ϕ has eigenvalues $\mathcal{R}_0 = 0$ and $\mathcal{F} = 1$.

In section 3 and appendix D, the one-loop determinant in the presence of a vortex loop on the round sphere $S_{b=1}^3$ was computed by expanding the fields in spherical harmonics. With the deformation $b \neq 1$ turned on, the analysis of harmonics is possible but more complicated [7], especially when the vortex loop is inserted. We thus work in an alternative approach based on the equivariant index theory [22]. First we will reproduce the known one-loop determinant that appears in the partition function on S_b^3 .

The equivariant index theory was used in [3, 10, 23] to compute one-loop determinants in other geometries. In this approach, one deforms the Lagrangian by $tQ \cdot V$ for some fermionic functional V . We choose

$$V = V_{\text{vec}} + V_{\text{chi}}, \quad (4.7)$$

where

$$V_{\text{vec}} = (Q\lambda)^\dagger \lambda + (Q\bar{\lambda})^\dagger \bar{\lambda}, \quad V_{\text{chi}} = (Q\psi)^\dagger \psi + (Q\bar{\psi})^\dagger \bar{\psi}. \quad (4.8)$$

Let H and K be two copies of $U(1)$ generated respectively by

$$-i(\partial_{\varphi_2} + \partial_{\varphi_1}) - \mathcal{R}_0 \quad \text{and} \quad -i(\partial_{\varphi_2} - \partial_{\varphi_1}). \quad (4.9)$$

The bosonic generator Q^2 in (4.6) specifies the action of the complexification $G_{\mathbb{C}}$ of the group G , which we define as¹⁴

$$G \equiv H \times K \times (\text{maximal torus of gauge group}) \times U(1)_F. \quad (4.10)$$

One then computes the equivariant index

$$\text{ind}_g D_{10} = \text{Tr}_{\text{Ker} D_{10}}(g) - \text{Tr}_{\text{Coker} D_{10}}(g) \quad (4.11)$$

of a differential operator D_{10} that appears in V as a function of $g \in G$. The precise definition of D_{10} is given in appendix F. We also show there that the path integral localizes to the configurations

$$A_\mu = 0, \quad \sigma = \text{constant}, \quad D = -\sigma/R, \quad \phi = F = 0. \quad (4.12)$$

The one-loop determinant is obtained from the index by the rule

$$\text{ind}_g D_{10}|_{g=\exp(cQ^2)} = \sum_j c_j e^{i w_j} \rightarrow Z_{1\text{-loop}} = \prod_j w_j^{-c_j/2}. \quad (4.13)$$

¹³Here the Lie derivative \mathcal{L}_v acts as $(\mathcal{L}_v + iv^\nu A_\nu) \cdot A_\mu = v^\nu F_{\nu\mu}$, $\mathcal{L}_v w_\mu = v^\nu \nabla_\nu w_\mu + [\nabla_\mu v^\nu] w_\nu$ etc., and in particular includes a Lorentz rotation. We also note that $\mathcal{L}_v + iv^\mu A_\mu$ is a gauge covariant Lie derivative.

¹⁴See also (F.13).

Here c_j is a sign ± 1 , iw_j is the eigenvalue of cQ^2 for mode j , and Q^2 is evaluated at the saddle point (4.12). The constant c affects only the overall normalization, and will be set to a convenient value.

In the set-ups of [3, 23], D_{10} was transversally elliptic and the index $\text{ind}_g D_{10}$ received contributions from the fixed points of the vector field in Q^2 . In our case of \mathbb{S}_b^3 , there is no fixed point with respect to the single $U(1)$ action generated by the vector field v . How can we compute the equivariant index in such a situation? We first rewrite the gauge field v in terms of the Hopf fibration coordinates $\phi = \varphi_2 - \varphi_1$, $\psi = \varphi_2 + \varphi_1$ (See (A.5)):

$$v = R^{-1}(b + b^{-1})\partial_\psi + R^{-1}(b - b^{-1})\partial_\phi. \tag{4.14}$$

The vector fields $2\partial_\psi$ and $2\partial_\phi$ respectively generate the action of H and K above. In particular, H rotates the Hopf fibers, and thus acts on \mathbb{S}_b^3 freely. With respect to the $H \times K$ -action, D_{10} fails to be elliptic but it is transversally elliptic. When part of the group action is free, the index of a transversally elliptic operator can be expressed in terms of the index of a transversally elliptic operator on the quotient space (\mathbb{S}^2 in our case) [22]. This is reviewed in appendix F. By the fixed point formula then, the index receives contributions from the fixed points of the other $U(1)$ action generated by ∂_ϕ . In terms of the original three-dimensional geometry, these fixed points correspond to the circle fibers at the north and the south poles of the \mathbb{S}^2 ($\theta = 2\vartheta$ equal to 0 and π respectively).

Let us set $Q = b + b^{-1}$, $\hat{\sigma} \equiv R\sigma$. We now compute the index $\text{ind}_g D_{10}$ with $g = e^{cQ^2}$, $c = -iR$. For the chiral multiplet, we can write $\text{ind}_g D_{10}^{\text{chi}} = \text{ind}_g D_{10,\mathbb{C}}^{\text{chi}} + \text{ind}_{g^{-1}} D_{10,\mathbb{C}}^{\text{chi}}$ as we show in appendix F. The reduction of $D_{10,\mathbb{C}}^{\text{chi}}$ to \mathbb{S}^2 near the north pole $\theta = 0$ is a twisted Dolbeault operator D_z . The local complex coordinate is given by $z \sim \theta e^{-i\phi}$.¹⁵ The equivariant index for the untwisted Dolbeault operator is $(1 - q^{-1})^{-1}$, where $q \in U(1)$ is the weight for the $U(1)$ action $z \mapsto tz$. We identify q with $e^{i(b-b^{-1})}$. As the contribution to $\text{ind}_g D_{10,\mathbb{C}}^{\text{chi}}$ from $\theta = 0$ we obtain

$$\sum_{n \in \mathbb{Z}} e^{inb} e^{\frac{1}{2}i\Delta Q} \frac{1}{1 - e^{-i(b-b^{-1})}} \sum_w e^{w \cdot \hat{\sigma}}, \tag{4.15}$$

where the sum is over the weights in the representation R . Similarly, the fixed point $\theta = \pi$ on \mathbb{S}^2 , where we identify $z \sim (\pi - \theta)e^{i\phi}$ and q with $e^{-i(b-b^{-1})}$, contributes

$$\sum_{n \in \mathbb{Z}} e^{inb^{-1}} e^{\frac{1}{2}i\Delta Q} \frac{1}{1 - e^{i(b-b^{-1})}} \sum_w e^{w \cdot \hat{\sigma}}. \tag{4.16}$$

As we explain in appendix F, the index theory instructs us to expand (4.15) as¹⁶

$$- \sum_{n \in \mathbb{Z}} e^{inb} e^{\frac{1}{2}i\Delta Q} \sum_{k=1}^{\infty} e^{ik(b-b^{-1})} \sum_w e^{w \cdot \hat{\sigma}}, \tag{4.17}$$

¹⁵The one-form in (A.5) can be written as $d\psi + \cos\theta d\phi = d(\psi \pm \phi) - (\pm 1 - \cos\theta)d\phi$. Thus at the north (south) pole $\theta = 0$ (π) the base is parameterized by (θ, ϕ) and the fiber by $\psi + \phi = 2\varphi_2$ ($\psi - \phi = 2\varphi_1$).

¹⁶There are two allowed choices as explained below (F.29).

and (4.16) as

$$\sum_{n \in \mathbb{Z}} e^{inb^{-1}} e^{\frac{1}{2}i\Delta Q} \sum_{k=0}^{\infty} e^{ik(b-b^{-1})} \sum_w e^{w \cdot \hat{\sigma}}. \quad (4.18)$$

By using the shift invariance $\sum_{n \in \mathbb{Z}} e^{inb^{\pm}} e^{\pm ikb^{\pm 1}} = \sum_{n \in \mathbb{Z}} e^{inb^{\pm}}$ and then splitting $\sum_{n \in \mathbb{Z}}$ into $\sum_{n \geq 0} + \sum_{n < 0}$ in (4.17) and (4.18), we obtain the total contribution

$$\sum_{n=0}^{\infty} e^{inb^{-1}} e^{\frac{1}{2}i\Delta Q} \sum_{k=0}^{\infty} e^{ikb} \sum_w e^{w \cdot \hat{\sigma}} - e^{-iQ} \sum_{k=0}^{\infty} e^{-ikb} e^{\frac{1}{2}i\Delta Q} \sum_{n=0}^{\infty} e^{-inb^{-1}} \sum_w e^{w \cdot \hat{\sigma}}. \quad (4.19)$$

The rule (4.13) applied to $\text{ind}_g D_{10}^{\text{chi}} = \text{ind}_g D_{10, \mathbb{C}}^{\text{chi}} + \text{ind}_{g^{-1}} D_{10, \mathbb{C}}^{\text{chi}}$ gives, up to an overall sign,

$$Z_{1\text{-loop}}^{\text{chi}} = \prod_{w \in R} \prod_{m, n \geq 0} \frac{mb + nb^{-1} + \frac{Q}{2} + iw \cdot \hat{\sigma} + \frac{Q}{2}(1 - \Delta)}{mb + nb^{-1} + \frac{Q}{2} - iw \cdot \hat{\sigma} - \frac{Q}{2}(1 - \Delta)} = \prod_{w \in R} s_b\left(\frac{iQ}{2}(1 - \Delta) - w \cdot \hat{\sigma}\right). \quad (4.20)$$

This is the well-known one-loop determinant for the chiral field [7]. If we had kept the constant c arbitrary, it would have canceled between the numerator and the denominator.

For a vector multiplet, the relevant differential operator is the differential in the de Rham complex twisted by the adjoint bundle (with a degree shifted by one). Since the de Rham and Dolbeault complexes are related by complexification ($\Omega_{\mathbb{C}}^0 = \Omega^{0,0}$, $\Omega_{\mathbb{C}}^1 = \Omega^{1,0} \oplus \Omega^{0,1}$, $\Omega_{\mathbb{C}}^2 = \Omega^{1,1}$), the index of the untwisted de Rham complex $D_{\text{dR}} : \Omega^0 \rightarrow \Omega^1 \rightarrow \Omega^2$ on \mathbb{C} is given as

$$\text{ind } D_{\text{dR}} = (1 - t^{-1}) \text{ind } \bar{\partial} = 1. \quad (4.21)$$

Let α denote the roots of the gauge group. Then the north pole $\theta = 0$ contributes

$$- \sum_{n \in \mathbb{Z}} e^{inb} \sum_{\alpha} e^{\alpha \cdot \hat{\sigma}} \quad (4.22)$$

to $\text{ind } D_{10}$ for the vector multiplet, and the south pole $\theta = \pi$ contributes

$$- \sum_{n \in \mathbb{Z}} e^{inb^{-1}} \sum_{\alpha} e^{\alpha \cdot \hat{\sigma}}. \quad (4.23)$$

The resulting one-loop determinant is

$$Z_{1\text{-loop}}^{\text{vec}} = \prod_{\alpha > 0} \sinh(\pi b \alpha \cdot \hat{\sigma}) \sinh(\pi b^{-1} \alpha \cdot \hat{\sigma}). \quad (4.24)$$

The product is over the positive roots. This also agrees with the results in the literature [7].

4.2 Vortex loop expectation values

On \mathbb{S}_b^3 with $b \neq 1$, BPS loop operators can only be supported along the circle fibers at $\theta = 0, \pi$. Let us for now focus on the gauge vortex loop with vorticity H along the fiber at $\theta \equiv 2\vartheta = 0$. This is characterized by the Q -invariant background configuration¹⁷

$$\begin{aligned} F_{\mu\nu} &= \frac{H}{b^2 R^2} \delta(1 - \cos \vartheta) \epsilon_{\mu\nu\rho} v^\rho, \quad \sigma = \text{const.}, \\ D &= -\frac{\sigma}{R f(\vartheta)} - i \frac{H}{b^2 R^2} \delta(1 - \cos \vartheta). \end{aligned} \quad (4.25)$$

¹⁷The delta functions should be understood to be $\delta(1 - \cos(\vartheta - \vartheta_0))$ with small $\vartheta_0 > 0$.

We claim that it has the effect of shifting the contributions from both the poles as $\hat{\sigma} \rightarrow \hat{\sigma} + ib^{-1}H$.

For the contribution from $\theta = \pi$, the gauge parameter in (4.5) becomes $\sigma + i(Rb)^{-1}H$ simply because the gauge field is turned on. This induces the shift in $\hat{\sigma}$.

The effect on the contribution from $\theta = 0$ is more subtle. In section 3, we constructed explicitly the eigenmodes of the kinetic operators on the round sphere. We saw that when the vortex loop is inserted and the eigenvalue η of H is turned on, generically certain modes that are singular must be allowed to fluctuate, contributing to the one-loop determinant. The analysis there was global and specific to the round metric, but the local behaviors of the allowed singular modes must be intrinsic to the vortex loop operator. Thus in the current approach to the one-loop determinant based on the index theory, we should compute the local contributions to the index by taking into account the local modes that are singular. This means that we should sum the U(1) weights for $z^{k+\eta}$ ($k = 0, 1, \dots$) instead of z^k if we work in the gauge where A_{φ_1} is zero.¹⁸ Then (4.15) receives an extra overall factor $e^{ib^{-1}\eta}$, which is equivalent to shifting $\hat{\sigma} \rightarrow \hat{\sigma} + ib^{-1}H$.

Thus the total effect of the vortex loop on the one-loop determinant is the shift $\hat{\sigma} \rightarrow \hat{\sigma} + ib^{-1}H$. This generalizes the results (3.11) and (D.28) for $\mathbb{S}_{b=1}^3$ to \mathbb{S}_b^3 .

We also need to evaluate the Chern-Simons term in the presence of a vortex loop on \mathbb{S}_b^3 . For $b = 1$, this was done in (3.2) using the boundary term (C.11). We specialize to the abelian case and set $H = \eta$. From (4.25) we find that $\mathcal{S}_{\text{SCS}} = -\pi k(\hat{\sigma} + ib^{-1}\eta)^2$.¹⁹ The effect of the vortex loop on the Chern-Simons action is again the shift $\hat{\sigma} \rightarrow \hat{\sigma} + ib^{-1}\eta$.

Repeating the same arguments above for a vortex loop with vorticity η at $\vartheta = \pi$, we find that the effect is the shift $\hat{\sigma} \rightarrow \hat{\sigma} - ib\eta$.

Let us assume that the gauge group is U(1) and consider the BF coupling (B.18) that appears as $e^{-\mathcal{S}_{\text{BF}}}$. It may be evaluated via the relation (B.19) between the BF and Chern-Simons terms. If we use the full gauge multiplet configuration in (4.12), we find that \mathcal{S}_{BF} equals $2\pi i(\hat{\sigma} + ib^{-1}\eta)\zeta$. If this were included in the path integral, then all the contributions inside the $\hat{\sigma}$ -integral would receive a uniform shift, so that after integration the vortex loop does not affect the partition function at all. From the point of view of the $\text{SL}(2, \mathbb{Z})$ action [24] on superconformal theories, however, it is more natural to not include terms proportional to η in the BF coupling, as follows from the discussion in [15].²⁰ See also (2.39). Thus $\mathcal{S}_{\text{BF}} = 2\pi i\hat{\sigma}\zeta$. Since all other contributions uniformly receive the shifts (5.25), the only effect of the vortex loop is to multiply the partition function by $\exp(-2\pi b^{-1}\eta\zeta)$. For a gauge vortex loop that has a singularity in a non-abelian gauge field, we cannot rule out the existence of non-perturbative corrections.

On the other hand, if the singularity is in a non-dynamical gauge field coupled to an

¹⁸If we work in the gauge where $A_{\varphi_1} = H$, the relevant modes are $z^k|z|^\eta$ and the H -dependence comes from the term in Q^2 that involves A_{φ_1} explicitly.

¹⁹If we add a constant boundary term $\propto \text{Tr } H^2$ in section 3.1, the results here and there agree.

²⁰See the discussion around (2.25) of [15]. They define loop operators as an action on the partition function that depends on a background gauge field coupled to a chosen global symmetry. The gauge vortex loop corresponds to SD_ω in their notation. Since $(D_\omega Z)[A] = Z[A + A_\omega]$ has A , not $A + A_\omega$, as the argument, the S -action yields the BF coupling between a new background field and A , not $A + A_\omega$.

(abelian or non-abelian) flavor symmetry, σ is replaced by a real mass and is not integrated over. Then the shift has a non-trivial effect.

Let us summarize the results of localization calculation for vortex loops placed at $\vartheta = \theta/2 = 0$ and $\pi/2$. The partition function $Z_{\mathbb{S}_b^3}(\zeta, \dots)$ of an abelian gauge theory is a function of the FI parameter ζ , and the effect of a gauge vortex loop V_η^{gauge} is the multiplication by an overall factor:

$$\langle V_\eta^{\text{gauge}} \rangle_{\mathbb{S}_b^3} = \begin{cases} e^{-2\pi b^{-1}\eta\hat{\zeta}} Z_{\mathbb{S}_b^3}(\hat{\zeta}) & \text{at } \vartheta = 0, \\ e^{2\pi b\eta\hat{\zeta}} Z_{\mathbb{S}_b^3}(\hat{\zeta}) & \text{at } \vartheta = \pi/2. \end{cases} \quad (4.26)$$

If the theory has a flavor symmetry, the partition function $Z_{\mathbb{S}_b^3}(\hat{m} = Rm, \dots)$ depends on the real mass parameters $m = \text{diag}(m_1, \dots)$. The expectation value of a flavor vortex loop V_H^{flavor} is the partition function whose argument \hat{m} is shifted in the imaginary direction:

$$\langle V_H^{\text{flavor}} \rangle_{\mathbb{S}_b^3} = \begin{cases} Z_{\mathbb{S}_b^3}(\hat{m} + ib^{-1}H) & \text{at } \vartheta = 0, \\ Z_{\mathbb{S}_b^3}(\hat{m} - ibH) & \text{at } \vartheta = \pi/2. \end{cases} \quad (4.27)$$

Later we will consider mirror symmetry. For reference, we quote results for the gauge and flavor Wilson loop expectation values [7]:

$$\langle W_\eta^{\text{gauge}} = e^{\eta \oint (A \pm i\sigma ds)} \rangle_{\mathbb{S}_b^3} = \begin{cases} Z_{\mathbb{S}_b^3}(\zeta + ib^{-1}\eta) & \text{at } \vartheta = 0, \\ Z_{\mathbb{S}_b^3}(\zeta - ib\eta) & \text{at } \vartheta = \pi/2, \end{cases} \quad (4.28)$$

$$\langle W_\eta^{\text{flavor}} \rangle_{\mathbb{S}_b^3} = \begin{cases} e^{2\pi b^{-1}\eta\hat{m}} Z_{\mathbb{S}_b^3}(\hat{m}) & \text{at } \vartheta = 0, \\ e^{-2\pi b\eta\hat{m}} Z_{\mathbb{S}_b^3}(\hat{m}) & \text{at } \vartheta = \pi/2, \end{cases} \quad (4.29)$$

5 Localization on $\mathbb{S}^1 \times \mathbb{S}^2$ by the index theory

In this section, we compute the expectation value of a vortex loop operator on the geometry $\mathbb{S}^1 \times \mathbb{S}^2$, or equivalently the (generalized) superconformal index in the presence of a vortex loop operator. For the purpose of explaining the computation, it is enough to consider the ordinary index of a general $\mathcal{N} = 2$ gauge theory, with a chiral multiplet of general R-charge $\mathcal{R} = -\Delta$ [8] in representation R of the gauge group. This simplifies the notation, and we will indicate only at the end the results for the generalized index, which incorporates the background magnetic flux on \mathbb{S}^2 for flavor symmetries [25]. As in the previous section where we studied the ellipsoid \mathbb{S}_b^3 , we compute the one-loop determinant using the equivariant index. We begin by explaining how to compute the superconformal index without a vortex loop in our approach.

5.1 Partition function

The geometry is defined by the metric

$$ds^2 = d\tau^2 + d\theta^2 + \sin^2\theta d\varphi^2, \quad (5.1)$$

with periodicity $\tau \sim \tau + \beta$. The vielbein are

$$e^1 = d\tau, \quad e^2 = d\theta, \quad e^3 = \sin\theta d\varphi. \quad (5.2)$$

Let us consider the supercharge generated by the following two conformal Killing spinors²¹

$$\varepsilon = \frac{1}{\sqrt{2}} e^{-\tau/2} \begin{pmatrix} -e^{\frac{i}{2}(\theta-\varphi)} \\ e^{\frac{i}{2}(-\theta-\varphi)} \end{pmatrix}, \quad \bar{\varepsilon} = \frac{1}{\sqrt{2}} e^{\tau/2} \begin{pmatrix} e^{\frac{i}{2}(-\theta+\varphi)} \\ e^{\frac{i}{2}(\theta+\varphi)} \end{pmatrix}. \quad (5.3)$$

Note, however, that these are not periodic in τ . To understand the origin of non-periodicity, let us look at the definition of the index

$$Z_{\mathbb{S}^1 \times \mathbb{S}^2} = \text{Tr}(-1)^F e^{-\beta_1(\mathcal{H}-\mathcal{R}-j_3)} e^{-\beta_2(\mathcal{H}+j_3)}. \quad (5.4)$$

Here $\mathcal{H} = -\partial_\tau$ is the Hamiltonian, \mathcal{R} is the R-symmetry generator, and j_3 is a generator of the isometry group $\text{SU}(2)$ that acts as $-i\partial_\varphi$ on neutral scalars. The index \mathcal{I} should be independent of β_1 because $\mathcal{H} - \mathcal{R} - j_3 = i\{\delta_\varepsilon, \delta_{\bar{\varepsilon}}\}$. Formally, the operators in the trace require the fields to satisfy the quasi-periodic boundary conditions

$$(\text{fields})_{\tau+\beta} = e^{\beta_1(-\mathcal{R}-j_3)+\beta_2j_3} (\text{fields})_\tau, \quad (5.5)$$

where $\beta = \beta_1 + \beta_2$. By assigning the R-charges $+1$ to ε and -1 to $\bar{\varepsilon}$, we see that ε and $\bar{\varepsilon}$ precisely satisfy the boundary conditions (5.5). Note, however, that the group action on the right hand side involves a rotation by an imaginary angle. The way to make sense of this is to rewrite everything including the Lagrangian and the SUSY transformations in terms of the redefined periodic fields

$$(\text{fields})_{\text{new}} := e^{-(\tau/\beta)(\beta_1(-\mathcal{R}-j_3)+\beta_2j_3)} (\text{fields}) \quad (5.6)$$

that are periodic in τ [26]. This is equivalent to replacing everywhere the time derivative $\partial_\tau = -\mathcal{H}$ by²²

$$\partial_\tau + \beta^{-1}[(-\mathcal{R} - j_3)\beta_1 + j_3\beta_2]. \quad (5.7)$$

In the new formulation, the spinors $(\epsilon, \bar{\epsilon}) := (\varepsilon_{\text{new}}, \bar{\varepsilon}_{\text{new}})$ that generate supersymmetry become τ -independent:

$$\epsilon = \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{\frac{i}{2}(\theta-\varphi)} \\ e^{\frac{i}{2}(-\theta-\varphi)} \end{pmatrix}, \quad \bar{\epsilon} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{i}{2}(-\theta+\varphi)} \\ e^{\frac{i}{2}(\theta+\varphi)} \end{pmatrix}. \quad (5.8)$$

In the localization approach [26] to the computation of the index (5.4), we deform the action by $tQ \cdot V$ for some fermionic functional V . Our choice is again (4.7). In the limit $t \rightarrow +\infty$, the path integral localizes to configurations

$$\begin{aligned} A_\tau &= -\frac{a}{\beta}, & A_\varphi^\pm &= \frac{m}{2}(\pm 1 - \cos\theta), & \sigma &= -\frac{m}{2}, & \text{where } a &= \text{const.}, m = \text{const.}, \\ \phi &= 0, & F &= 0. \end{aligned} \quad (5.9)$$

²¹There are four independent conformal Killing spinors on $\mathbb{R} \times \mathbb{S}^2$. If ϵ_0 is an arbitrary constant spinor, they are given by $e^{\mp\frac{1}{2}\tau} e^{\pm\frac{i}{2}\theta\gamma_\varphi} e^{\frac{i}{2}\varphi\gamma_\tau} \epsilon_0$.

²²Another way to understand the shift is that, after the redefinition that makes fields periodic, the shift in the derivatives cancels the twist in the trace and the index becomes $\text{Tr}(-1)^F e^{\beta\partial_\tau}$.

The flux m takes values in the Cartan subalgebra of the Lie group, and is further required to satisfy

$$\alpha(m), \rho(m) \in \mathbb{Z} \quad (5.10)$$

for any root α and any weight ρ in representation R . The expressions A_φ^\pm are valid in the standard two patches $U^+ = \{\theta \neq \pi\}$ and $U^- = \{\theta \neq 0\}$ of \mathbb{S}^2 .

The supercharge $Q = \delta_\epsilon + \delta_{\bar{\epsilon}}$ squares to

$$Q^2 = i\mathcal{L}_v + i(iv^\mu A_\mu + \sigma\bar{\epsilon}\epsilon) + i\mathcal{R} + i\beta^{-1}[(-\mathcal{R} - j_3)\beta_1 + j_3\beta_2], \quad (5.11)$$

where

$$v \equiv (\bar{\epsilon}\gamma^\mu\epsilon)\partial_\mu = \partial_\tau - i\partial_\varphi \quad (5.12)$$

and $\bar{\epsilon}\epsilon = -\cos\theta$. In order to simplify the expression (5.11) further, we need to take into account the saddle point configurations (5.9) and the representation of the SU(2) in the monopole background. On a scalar field with electric charge +1 in the background of monopole charge $\rho(m)$, the angular momentum operator j_3 acts as [27].²³

$$j_3 = -i(\partial_\varphi + i\rho(A_\varphi^\pm)) + \frac{\rho(m)}{2}\cos\theta = -i\partial_\varphi \pm \frac{\rho(m)}{2}. \quad (5.13)$$

The expression (5.11) can be rewritten as

$$Q^2 = i\mathcal{L}_{\partial_\tau} + \frac{a}{\beta} + i\frac{\beta_2}{\beta}(2j_3 + \mathcal{R}_0 - \Delta\mathcal{F}) \quad (5.14)$$

at the saddle point (5.9),²⁴ which is a linear combination of the generators of G defined again by (4.10), which acts on the coordinates $(h, t) \in H \times K$ by $(e^{2\pi i\tau/\beta}, e^{i\varphi}) \mapsto (h \cdot e^{2\pi i\tau/\beta}, t \cdot e^{i\varphi})$.

As in the case of \mathbb{S}_b^3 in the previous section, we would like to compute the equivariant index for the relevant differential operator D_{10} that appears in the fermionic functional V . Some details are given in appendix G. Note that if we choose $c = i\beta$, the index is independent of β_1 as required by the definition (5.4). The operator D_{10} is transversally elliptic with respect to the vector field ∂_τ that generates the free U(1) action on $\mathbb{S}^1 \times \mathbb{S}^2$ (this time in a trivial way as a translation along the circle), and thus reduces to a transversally elliptic operator on \mathbb{S}^2 .

Let $\rho \in R$ denote the weights in representation R of the gauge group. We show in appendix G that the equivariant index for the chiral multiplet in representation R is

$$\text{ind}_g(D_{10}^{\text{chi}})|_{g=e^{iQ^2}} = \sum_{n \in \mathbb{Z}} \sum_{r=0}^{\infty} \sum_{\rho \in R} \left(e^{iw(n,r,\rho)} + e^{-iw(n,r,\rho)} - e^{i\tilde{w}(n,r,\rho)} - e^{-i\tilde{w}(n,r,\rho)} \right) \quad (5.15)$$

with

$$\begin{aligned} iw(n, r, \rho) &\propto (2r - \rho(m) + \Delta)\beta_2 - 2\pi in + i\rho(a), \\ i\tilde{w}(n, r, \rho) &\propto -(2r - \rho(m) + 2 - \Delta)\beta_2 + 2\pi in + i\rho(a). \end{aligned} \quad (5.16)$$

²³The other generators act as $j_+ = e^{i\varphi}(\partial_\theta + i\cot\theta(\partial_\varphi + i\rho(A_\varphi^\pm))) + \frac{\rho(m)}{2}e^{i\varphi}\sin\theta$ and $j_- = e^{-i\varphi}(-\partial_\theta + i\cot\theta(\partial_\varphi + i\rho(A_\varphi^\pm))) + \frac{\rho(m)}{2}e^{-i\varphi}\sin\theta$.

²⁴The right hand side is the precise expression of \hat{Q}^2 for any field configuration. See appendix G.

The one-loop determinant that follows from the rule (4.13) is

$$\begin{aligned}
 Z_{1\text{-loop}}^{\text{chi}} &= \prod_{\rho \in R} \prod_{r=0}^{\infty} \prod_{n \in \mathbb{Z}} \frac{-(r - \frac{\rho(m)}{2} + 1 - \Delta/2)\beta_2 + \frac{i}{2}\rho(a) + \pi i n}{-(r - \frac{\rho(m)}{2} + \Delta/2)\beta_2 - \frac{i}{2}\rho(a) + \pi i n} \\
 &= \prod_{\rho \in R} \prod_{r=0}^{\infty} \frac{\sinh \left[-(r - \frac{\rho(m)}{2} + 1 - \Delta/2)\beta_2 + \frac{i}{2}\rho(a) \right]}{\sinh \left[-(r - \frac{\rho(m)}{2} + \Delta/2)\beta_2 - \frac{i}{2}\rho(a) \right]}.
 \end{aligned} \tag{5.17}$$

This can be rewritten as follows:

$$Z_{1\text{-loop}}^{\text{chi}} = \prod_{\rho \in R} \left(\prod_{r=0}^{\infty} \frac{e^{+(r - \frac{\rho(m)}{2} + 1 - \Delta/2)\beta_2 - \frac{i}{2}\rho(a)}}{e^{+(r - \frac{\rho(m)}{2} + \Delta/2)\beta_2 + \frac{i}{2}\rho(a)}} \times \prod_{r=0}^{\infty} \frac{1 - q^{r - \frac{\rho(m)}{2} + 1 - \Delta/2} e^{i\rho(a)}}{1 - q^{r - \frac{\rho(m)}{2} + \Delta/2} e^{-i\rho(a)}} \right), \tag{5.18}$$

where $q = e^{-2\beta_2}$. After regularizing the infinite product,²⁵ the one-loop determinant is given by

$$Z_{1\text{-loop}}^{\text{chi}} = \prod_{\rho \in R} \left(q^{-\frac{\rho(m)}{4}(1-\Delta)} e^{-\frac{i}{2}\rho(m)\rho(a)} \prod_{r=0}^{\infty} \frac{1 - q^{r - \frac{\rho(m)}{2} + 1 - \Delta/2} e^{i\rho(a)}}{1 - q^{r - \frac{\rho(m)}{2} + \Delta/2} e^{-i\rho(a)}} \right). \tag{5.20}$$

For the vector multiplet, the north and the south poles of \mathbb{S}^2 contribute identical amounts to the equivariant index, and in appendix G we compute the equivariant index for the vector multiplet. By applying the rule (4.13), the corresponding one-loop determinant is

$$\begin{aligned}
 Z_{1\text{-loop}}^{\text{vec}} &= \prod_{\alpha \in \text{adj}} \prod_{n \in \mathbb{Z}} \left(\frac{i}{2}\alpha(a) + \frac{1}{2}\alpha(m)\beta_2 + \pi i n \right)^{1/2} \left(\frac{i}{2}\alpha(a) - \frac{1}{2}\alpha(m)\beta_2 + \pi i n \right)^{1/2} \\
 &\sim \prod_{\alpha > 0} \left[2 \sinh \left(\frac{i}{2}\alpha(a) + \frac{1}{2}\alpha(m)\beta_2 \right) \right] \left[2 \sinh \left(\frac{i}{2}\alpha(a) - \frac{1}{2}\alpha(m)\beta_2 \right) \right] \\
 &= \prod_{\alpha \in \text{adj}} q^{-|\alpha(m)|/4} \left(1 - e^{-i\alpha(a)} q^{|\alpha(m)|/2} \right).
 \end{aligned} \tag{5.21}$$

Both (5.20)²⁶ and (5.21) agree with the results in the literature.

²⁵Following [26] (cf. [8]), we regularize the logarithm of the first factor as

$$\begin{aligned}
 &\sum_{r=0}^{\infty} \left[+ \left(r - \frac{\rho(m)}{2} + 1 - \Delta/2 \right) \beta_2 - \frac{i}{2}\rho(a) \right] - \sum_{r=0}^{\infty} \left[+ \left(r - \frac{\rho(m)}{2} + \Delta/2 \right) \beta_2 + \frac{i}{2}\rho(a) \right] \\
 &= \lim_{x, y \rightarrow 1} \left(\beta_2 \frac{\partial}{\partial x} + \frac{i}{2}\rho(a) \frac{\partial}{\partial y} \right) \left(+ \frac{x^{-\frac{\rho(m)}{2} + 1 - \Delta/2} y^{-1}}{1-x} - \frac{x^{-\frac{\rho(m)}{2} + \Delta/2} y}{1-x} \right) \\
 &= + \frac{\rho(m)}{2} ((1-\Delta)\beta_2 - i\rho(a)) - i\rho(a) \lim_{x \rightarrow 1} \left(\frac{1}{1-x} - 1 + \mathcal{O}(1-x) \right).
 \end{aligned} \tag{5.19}$$

By dropping the m -independent terms, we renormalize this to $+\frac{\rho(m)}{2}((1-\Delta)\beta_2 - i\rho(a))$ by taking the BF coupling as counter terms.

²⁶The corresponding formulas in [8] and [25] involve the absolute values $|\rho(m)|$. As explained in eq. (3.3) of [28], one can rewrite such an expression and eliminate $|\rho(m)|$ in favor of $-\rho(m)$.

5.2 Vortex loop expectation values

Let us now insert vortex loop operators with vorticities H^+ and H^- at the north and south poles of \mathbb{S}^2 , respectively. In the presence of background magnetic flux, the two loop operators are defined by the Q -invariant configurations

$$F_{\mu\nu} = [H^+\delta(1 - \cos\theta) + H^-\delta(-1 - \cos\theta)] \epsilon_{\mu\nu\rho} v^\rho + \frac{m}{2} \epsilon_{\tau\mu\nu}, \quad (5.22)$$

$$D = iH^+\delta(1 - \cos\theta) - iH^-\delta(-1 - \cos\theta), \quad \sigma = -\frac{m}{2}. \quad (5.23)$$

The gauge field is given as $A_\tau = -a/\beta$, $A_\theta = 0$,

$$A_\varphi^\pm = \frac{m}{2}(\pm 1 - \cos\theta) \pm H^\pm. \quad (5.24)$$

In order for the gauge fields on the two patches to be glued by a well-defined transition function, we need that $m + H^+ + H^-$ is a GNO charge that satisfies the Dirac quantization conditions $\alpha(m + H^+ + H^-), \rho(m + H^+ + H^-) \in \mathbb{Z}$.

It is clearest to restrict to the case $\beta_1 = \beta_2 = \beta/2$. In this case j_3 does not enter the field redefinition (5.6), after which

$$iQ^2 = -\partial_\tau + i\mathcal{L}_{\partial_\varphi} \mp \frac{m + H^+ + H^-}{2} + \frac{i}{\beta} \left(a + \frac{i\beta}{2}(H^+ - H^-) - \frac{i\Delta}{2}\beta \right) - \frac{1}{2}\mathcal{R}_0,$$

where the combination $-i\mathcal{L}_\varphi \pm \frac{m+H^++H^-}{2}$ is precisely j_3 . The effect on $Z_{1\text{-loop}}^{\text{chi}}$ of the vortex loop is the shifts

$$m \rightarrow m + H^+ + H^-, \quad a \rightarrow a - \frac{i}{2}(\log q)(H^+ - H^-). \quad (5.25)$$

We now specialize to the $U(1)$ gauge group and set $H^\pm = \eta^\pm$. We need the on-shell value of a supersymmetric Chern-Simons action in the vortex loop background. It enters as $e^{i\mathcal{S}_{\text{SCS}}}$ in the path integral. To evaluate it, we introduce a connection $A' = rA_\tau^{\text{new}}d\tau + A_\theta^{\text{new}}d\theta + A_\varphi^{\text{new}}d\varphi$ extended to the disk $\mathbb{D}^2 = \{re^{2\pi i\tau/\beta} | r \leq 1\}$ times \mathbb{S}^2 and put [26, 29, 30]

$$\mathcal{S}_{\text{SCS}}^{\text{boson}} = \frac{k}{4\pi} \left(\int_{\mathbb{D}^2 \times \mathbb{S}^2} dA' \wedge dA' + \int_{\mathbb{S}^1 \times \mathbb{S}^2} 2D\sigma \cdot \text{vol} \right). \quad (5.26)$$

After some calculations we find²⁷

$$\mathcal{S}_{\text{SCS}}^{\text{boson}} = k\beta \left(-\frac{a}{\beta} - \frac{i}{2}(\eta^+ - \eta^-) \right) (m + \eta^+ + \eta^-). \quad (5.27)$$

Let us consider the BF coupling (B.18) that appears as $e^{-\mathcal{S}_{\text{BF}}}$. It may be evaluated via the relation (B.19) between the BF and Chern-Simons terms. If we use the full gauge multiplet configuration in (5.23) and (5.24), we find that \mathcal{S}_{BF} equals

$$\left(ia + \frac{\Delta}{2}\beta \right) (m + \eta^+ + \eta^-) + \left(ia - \frac{\beta}{2}(\eta^+ - \eta^-) \right) \mathbf{m}. \quad (5.28)$$

²⁷For the flavor Chern-Simons action, we get $\mathcal{S}_{\text{SCS}}^{\text{boson}} = k\beta \left(-\frac{a}{\beta} + \frac{i}{2}\Delta - \frac{i}{2}(\eta^+ - \eta^-) \right) (m + \eta^+ + \eta^-)$, where bold fonts are used for quantities related to the flavor symmetry. Dependence on Δ arises because after the field redefinition (5.6) $\text{Im } \mathbf{A}_\tau$ contains $\frac{i}{2}\Delta$ as in the 4d case [31].

If this were included in the path integral, since the gauge vortex loops shift the parameters $m \in \mathbb{Z}$ and a that are summed or integrated over, it would not affect the partition function at all. From the point of view of the $SL(2, \mathbb{Z})$ action [24] on superconformal theories, however, it is more natural to not include terms proportional to η^\pm in the BF coupling, as follows from a discussion in [15]. See also (2.39). Thus we should drop η^\pm from (5.28). Since all other contributions uniformly receive the shifts (5.25), the only effect of the vortex loop is the multiplication by an overall factor:

$$\langle V_{\eta^+}^{\text{gauge}}(\text{north})V_{\eta^-}^{\text{gauge}}(\text{south}) \rangle_{\mathbb{S}^1 \times \mathbb{S}^2} = e^{+i(\eta^+ + \eta^-)(a - \frac{i}{2}\Delta)} q^{+(\eta^+ - \eta^-)m/2} Z_{\mathbb{S}^1 \times \mathbb{S}^2}(\mathbf{m}, e^{i\mathbf{a}}). \tag{5.29}$$

Here $\mathbf{a} = -\int_{\mathbb{S}^1} \mathbf{A}$ is the background holonomy along \mathbb{S}^1 , and $\mathbf{m} = (2\pi)^{-1} \int_{\mathbb{S}^2} \mathbf{F}$ is the background flux through \mathbb{S}^2 . For a non-Abelian gauge group, it is a possibility that there are non-perturbative contributions.

Next we consider an $\mathcal{N} = 2$ theory with a flavor symmetry. The generalized index is again a function of $\mathbf{a} = -\int_{\mathbb{S}^1} \mathbf{A}$ and $\mathbf{m} = (2\pi)^{-1} \int_{\mathbb{S}^2} \mathbf{F}$, though this time \mathbf{A} is coupled to the flavor symmetry. From the discussion above, we see that for two flavor vortex loops at the north and south poles, the correlator is given by the shifts in the partition function, i.e., the generalized index:

$$\langle V_{H^+}^{\text{flavor}}(\text{north})V_{H^-}^{\text{flavor}}(\text{south}) \rangle_{\mathbb{S}^1 \times \mathbb{S}^2} = Z_{\mathbb{S}^1 \times \mathbb{S}^2}(\mathbf{m} + H^+ + H^-, e^{i\mathbf{a}} q^{\frac{1}{2}(H^+ - H^-)}). \tag{5.30}$$

5.3 Wilson loop expectation values

We can also compute the expectation values of Wilson loops.²⁸ Consider an $\mathcal{N} = 2$ theory with at least one $U(1)$ gauge group. Such a theory possesses a global symmetry $U(1)_J$ generated by the conserved current $J^\mu = \epsilon^{\mu\nu\rho} \partial_\nu A_\rho$. The generalized index of the theory involves a sum over the gauge fluxes m , and depends on the flux \mathbf{m} of the background gauge field \mathbf{A}_μ for $U(1)_J$, whose coupling is given by the BF term (B.18). The partition function on $\mathbb{S}^1 \times \mathbb{S}^2$ then takes the form

$$Z_{\mathbb{S}^1 \times \mathbb{S}^2} = \sum_{m \in \mathbb{Z}} (q^c e^{-i\mathbf{a}})^m \oint \frac{da}{2\pi} e^{-ima} f(m, a, q, \dots). \tag{5.31}$$

Here c parameterizes the contribution of the gauge flux to the R-charge [28], \mathbf{a} is the chemical potential for $U(1)_J$, and f is some function. We now insert a Wilson loop of charge η^+ at the north pole, and another of charge η^- at the south pole. For $\beta_1 = \beta_2$, we find that the value of the product of the Wilson loops in the saddle point configuration is

$$e^{-i(\eta^+ + \eta^-)a} q^{-(\eta^+ - \eta^-)m/2}, \tag{5.32}$$

which is to be inserted inside the sum and the integral. Thus the effect of the Wilson loops is the shifts

$$\mathbf{m} \rightarrow \mathbf{m} + \eta^+ + \eta^-, \quad \mathbf{a} \rightarrow \mathbf{a} - \frac{i}{2}(\log q)(\eta^+ - \eta^-). \tag{5.33}$$

²⁸We thank J. Gomis for discussions on this calculation.

	Theory A	Theory B
Global symmetry	$U(1)_J$	Flavor symmetry
Vortex loop for global symmetry	Gauge Wilson loop	Flavor vortex loop
Wilson loop for global symmetry	Gauge vortex loop	Flavor Wilson loop

Table 1. Abelian mirror symmetry action on global symmetries and loop operators.

Thus the correlation function of the two ordinary Wilson loops is given by the partition function (generalized index) whose arguments are shifted:

$$\langle W_{\eta^+}^{\text{gauge}}(\text{north})W_{\eta^-}^{\text{gauge}}(\text{south}) \rangle_{\mathbb{S}^1 \times \mathbb{S}^2} = Z_{\mathbb{S}^1 \times \mathbb{S}^2}(\mathbf{m} + \eta^+ + \eta^-, e^{i\mathbf{a}} q^{\frac{1}{2}(\eta^+ - \eta^-)}). \quad (5.34)$$

In an $\mathcal{N} = 2$ theory with flavor symmetry, the correlator of two flavor Wilson loops with charges η^+ and η^- , inserted at the north and south poles respectively, is given by

$$\langle W_{\eta^+}^{\text{flavor}}(\text{north})W_{\eta^-}^{\text{flavor}}(\text{south}) \rangle_{\mathbb{S}^1 \times \mathbb{S}^2} = e^{-i(\eta^+ + \eta^-)(\mathbf{a} - \frac{i}{2}\Delta\beta)} q^{-(\eta^+ - \eta^-)\mathbf{m}/2} Z_{\mathbb{S}^1 \times \mathbb{S}^2}(\mathbf{m}, e^{i\mathbf{a}}). \quad (5.35)$$

6 Abelian mirror symmetry

We have employed the supersymmetric localization method to obtain exact quantitative results for the expectation values and correlators of vortex loop operators. Let us now discuss more qualitative and conceptual points regarding loop operators in three dimensional supersymmetric theories.

Any duality maps global symmetries of one theory to those of the other. In particular abelian mirror symmetry [32–36] by definition maps a topological symmetry $U(1)_J$ in one theory to a flavor symmetry in the dual theory. It was explained in section 2.4 that the vortex loop for $U(1)_J$ is the gauge Wilson loop, and that the Wilson loop for $U(1)_J$ is the gauge vortex loop. Thus the transformations of loop operators under abelian mirror symmetry follow from those of global symmetries. We can summarize the abelian mirror symmetry action on loop operators in $\mathcal{N} = 2$ theories.

Let us illustrate the mapping of global symmetries and loop operators in a well-known $\mathcal{N} = 2$ mirror pair [37]. As Theory A (SQED), we consider the $U(1)$ gauge theory with two chirals $(\Phi, \tilde{\Phi})$ of charges $(1, -1)$. This theory has a flavor symmetry $U(1)_{\text{axial}}$ for which the fields have charges $(1, 1)$, as well as a topological symmetry $U(1)_J$. As Theory B (XYZ model), we consider a theory of three chiral superfields (X, Y, Z) , interacting through the superpotential $W = XYZ$. The superpotential is invariant under two symmetries $U(1)_1$ and $U(1)_2$, whose charges are given by $(2, -1, -1)$ and $(0, 1, -1)$ respectively. It is known that $U(1)_{\text{axial}}$ is identified with $U(1)_1$, and $U(1)_J$ with $U(1)_2$. We summarize the symmetries and the loop operator spectra in table 2.

SQED				
	Φ	$\tilde{\Phi}$	Loop operator	
$U(1)_{\text{gauge}}$	1	-1	$\left\{ \begin{array}{l} \text{Flavor vortex} \\ \text{Flavor Wilson} \\ \text{Gauge Wilson} \\ \text{Gauge vortex} \end{array} \right.$	
$U(1)_{\text{axial}}$	1	1		
$U(1)_J$				
XYZ model				
	X	Y	Z	Loop operator
$U(1)_1$	2	-1	-1	$\left\{ \begin{array}{l} \text{Flavor vortex} \\ \text{Flavor Wilson} \end{array} \right.$
$U(1)_2$	0	1	-1	

Table 2. Mirror symmetry for the two-flavor SQED and the XYZ model.

Our localization results for loop operators provide a quantitative test of the mirror symmetry predictions. On $S^1 \times S^2$, the correlation function (5.34) of two gauge Wilson loops is identical to the correlation function (5.30) of two flavor vortex loops, confirming the correspondence on the middle row in table 1. Similarly, the equality between (5.29) and (5.35) verifies the mirror symmetry action on the bottom row of table 1. The same checks can also be made using the results for Wilson and vortex loop operators on S_b^3 .

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A Metric and vielbein on \mathbb{S}^3

The three dimensional sphere \mathbb{S}^3 with radius R can be represented by a pair of complex coordinates $(u, v) \in \mathbb{C}^2$ by the equation

$$u\bar{u} + v\bar{v} = R^2. \quad (\text{A.1})$$

The manifold is invariant under $\text{SO}(4) \cong \text{SU}(2)_L \times \text{SU}(2)_R$ symmetry. The generators of the two $\text{SU}(2)$ factors are denoted as L_1^L, L_2^L, L_3^L and L_1^R, L_2^R, L_3^R and they satisfy the following commutation relations

$$[L_a^L, L_b^L] = i\varepsilon_{abc}L_c^L, \quad [L_a^R, L_b^R] = i\varepsilon_{abc}L_c^R, \quad [L_a^L, L_b^R] = 0. \quad (\text{A.2})$$

We define raising operators $L_+^L = L_1^L + iL_2^L$, $L_+^R = L_1^R + iL_2^R$ and lowering operators $L_-^L = L_1^L - iL_2^L$, $L_-^R = L_1^R - iL_2^R$. The representation of the generators in the (u, v) coordinates is given by

$$L_-^L = \bar{u}\partial_v - \bar{v}\partial_u, \quad L_+^L = -u\partial_{\bar{v}} + v\partial_{\bar{u}}, \quad L_3^L = \frac{1}{2}(u\partial_u + v\partial_v - \bar{u}\partial_{\bar{u}} - \bar{v}\partial_{\bar{v}}), \quad (\text{A.3})$$

and

$$L_-^R = \bar{u}\partial_{\bar{v}} - v\partial_u, \quad L_+^R = -u\partial_v + \bar{v}\partial_{\bar{u}}, \quad L_3^R = \frac{1}{2}(u\partial_u - v\partial_v - \bar{u}\partial_{\bar{u}} + \bar{v}\partial_{\bar{v}}). \quad (\text{A.4})$$

In the main text we use two different parameterization of the \mathbb{S}^3 , the Hopf fibration and the torus fibration.

A.1 Hopf fibration

The Hopf fibration of \mathbb{S}^3 is given by the parameterization $u = R \sin \frac{\theta}{2} e^{i(\psi-\phi)/2}$ and $v = R \cos \frac{\theta}{2} e^{i(\psi+\phi)/2}$ where $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 4\pi$. The metric in the Hopf fibration is given by

$$\begin{aligned} ds^2 = g_{\mu\nu}dx^\mu dx^\nu &= \frac{R^2}{4}(d\theta^2 + \sin^2\theta d\phi^2 + (d\psi + \cos\theta d\phi)^2) \\ &= \frac{R^2}{4}(d\theta^2 + d\phi^2 + d\psi^2 + 2\cos\theta d\phi d\psi). \end{aligned} \quad (\text{A.5})$$

This metric can be derived considering that $\mathbb{S}^3 = \text{SU}(2)$, as shown also in appendix A of [21]. The left invariant vielbein basis is

$$\begin{aligned} e^1 &= \frac{R}{2}(\cos\psi d\theta + \sin\psi \sin\theta d\phi), \\ e^2 &= \frac{R}{2}(\sin\psi d\theta - \cos\psi \sin\theta d\phi), \\ e^3 &= \frac{R}{2}(\cos\theta d\phi + d\psi), \end{aligned} \quad (\text{A.6})$$

and the inverse vielbein defined as $e_a^\mu = e^b_\nu g^{\nu\mu} \delta_{ab}$ is given by

$$e_a^\mu = \frac{2}{R} \begin{pmatrix} \cos\psi & \frac{\sin\psi}{\sin\theta} & -\cot\theta \sin\psi \\ \sin\psi & -\frac{\cos\psi}{\sin\theta} & \cot\theta \cos\psi \\ 0 & 0 & 1 \end{pmatrix}. \quad (\text{A.7})$$

A.2 Torus fibration

The torus fibration is obtained parameterizing u and v as $u = R \sin \vartheta e^{i\varphi_1}$ and $v = R \cos \vartheta e^{i\varphi_2}$, where $0 \leq \vartheta \leq \pi/2$ and $0 \leq \varphi_1, \varphi_2 \leq 2\pi$. Torus fibration and Hopf fibration parameters are related by $\theta = 2\vartheta$, $\phi = \varphi_2 - \varphi_1$ and $\psi = \varphi_2 + \varphi_1$. The metric is given by

$$ds^2 = R^2(d\vartheta^2 + \sin^2 \vartheta d\varphi_1^2 + \cos^2 \vartheta d\varphi_2^2), \quad (\text{A.8})$$

and a natural frame is

$$e^1 = R d\vartheta, \quad e^2 = R \sin \vartheta d\varphi_1, \quad e^3 = R \cos \vartheta d\varphi_2. \quad (\text{A.9})$$

The vortex loop operator is located at $\vartheta = 0$ and extended along φ_2 . For this field configuration, the holonomy is constant when computed along a curve linked to the vortex loop, therefore the monodromy along the φ_1 circle will be independent from ϑ .

In the frame described above, the solution of the Killing spinor equation on \mathbb{S}^3 is given by

$$\epsilon = e^{\frac{i}{2}\vartheta\gamma_1} e^{\frac{i}{2}(\varphi_1+\varphi_2)\gamma_3} \epsilon_0. \quad (\text{A.10})$$

B SUSY on 3D Euclidean manifolds

B.1 Conventions

We follow the conventions as in [21]. The curved space gamma matrices γ_μ are defined as $\gamma_\mu = \gamma_a e^a{}_\mu$ where γ_a are Pauli matrices and $e^a{}_\mu$ is a vielbein. It follows

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \quad (\text{B.1})$$

where $g_{\mu\nu}$ is the spacetime metric. Some useful relations for Pauli matrices are

$$\gamma_{ab} = \frac{1}{2}[\gamma_a, \gamma_b] = i\varepsilon_{abc}\gamma_c \quad \text{with} \quad \varepsilon_{123} = \varepsilon^{123} = 1, \quad (\text{B.2})$$

$$\gamma_1\gamma_2\gamma_3 = i.$$

The spinors ψ and $\bar{\psi}$ are independent and have the same index structure, i.e., ψ^α and $\bar{\psi}^\alpha$. Spinor indices are omitted in the main text and contracted as

$$\bar{\psi}\psi = \bar{\psi}^\alpha C_{\alpha\beta}\psi^\beta, \quad \bar{\psi}\gamma^\mu\psi = \bar{\psi}^\alpha C_{\alpha\beta}(\gamma^\mu)^\beta{}_\gamma\psi^\gamma. \quad (\text{B.3})$$

We take $C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Given that $C_{\alpha\beta}$ is antisymmetric and $(C\gamma^\mu)_{\alpha\beta}$ is symmetric, considering Grassmann-odd spinors it follows

$$\bar{\psi}\psi = \psi\bar{\psi}, \quad \bar{\psi}\gamma^\mu\psi = -\psi\gamma^\mu\bar{\psi}, \quad (\gamma^\mu\bar{\psi})\psi = -\bar{\psi}\gamma^\mu\psi. \quad (\text{B.4})$$

B.2 Vector multiplet

The field content of Euclidean $\mathcal{N} = 2$ vector multiplet is given by the gauge field A_μ , two complex Dirac spinors λ and $\bar{\lambda}$ and two auxiliary real scalar fields D and σ . The supersymmetry variations are parameterized by two independent complex spinors ϵ and $\bar{\epsilon}$ and they are given by [5, 7, 21]

$$\begin{aligned}
 \delta A_\mu &= \frac{i}{2}(\bar{\epsilon}\gamma_\mu\lambda - \bar{\lambda}\gamma_\mu\epsilon), \\
 \delta\sigma &= \frac{1}{2}(\bar{\epsilon}\lambda - \bar{\lambda}\epsilon), \\
 \delta\lambda &= -\frac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} - D\epsilon + i\gamma^\mu\epsilon D_\mu\sigma + \frac{2i}{3}\sigma\gamma^\mu D_\mu\epsilon, \\
 \delta\bar{\lambda} &= -\frac{1}{2}\gamma^{\mu\nu}\bar{\epsilon}F_{\mu\nu} + D\bar{\epsilon} - i\gamma^\mu\bar{\epsilon}D_\mu\sigma - \frac{2i}{3}\sigma\gamma^\mu D_\mu\bar{\epsilon}, \\
 \delta D &= -\frac{i}{2}\bar{\epsilon}\gamma^\mu D_\mu\lambda - \frac{i}{2}D_\mu\bar{\lambda}\gamma^\mu\epsilon + \frac{i}{2}[\bar{\epsilon}\lambda, \sigma] + \frac{i}{2}[\bar{\lambda}\epsilon, \sigma] - \frac{i}{6}(D_\mu\bar{\epsilon}\gamma^\mu\lambda + \bar{\lambda}\gamma^\mu D_\mu\epsilon).
 \end{aligned} \tag{B.5}$$

D_μ is the covariant derivative with respect spacetime and gauge connection. For the \mathbb{S}_b^3 metric, D_μ is covariant also with respect to an R-symmetry gauge field²⁹ $V = -\frac{1}{2}\left(1 - \frac{b}{f}\right)d\varphi_1 - \frac{1}{2}\left(1 - \frac{1}{bf}\right)d\varphi_2$ [7]. Denoting as δ_ϵ and $\delta_{\bar{\epsilon}}$ the supersymmetry generated by ϵ and $\bar{\epsilon}$, it results $[\delta_\epsilon, \delta_{\epsilon'}] = [\delta_{\bar{\epsilon}}, \delta_{\bar{\epsilon}'}] = 0$ and [5, 7, 21]

$$\begin{aligned}
 [\delta_\epsilon, \delta_{\bar{\epsilon}}]A_\mu &= iv^\nu\partial_\nu A_\mu + i\partial_\mu v^\nu A_\nu - D_\mu\Lambda, \\
 [\delta_\epsilon, \delta_{\bar{\epsilon}}]\sigma &= iv^\mu\partial_\mu\sigma + i[\Lambda, \sigma] + \rho\sigma, \\
 [\delta_\epsilon, \delta_{\bar{\epsilon}}]\lambda &= iv^\mu\partial_\mu\lambda + \frac{i}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\lambda + i[\Lambda, \lambda] + \frac{3}{2}\rho\lambda + \alpha\lambda, \\
 [\delta_\epsilon, \delta_{\bar{\epsilon}}]\bar{\lambda} &= iv^\mu\partial_\mu\bar{\lambda} + \frac{i}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\bar{\lambda} + i[\Lambda, \bar{\lambda}] + \frac{3}{2}\rho\bar{\lambda} - \alpha\bar{\lambda}, \\
 [\delta_\epsilon, \delta_{\bar{\epsilon}}]D &= iv^\mu\partial_\mu D + i[\Lambda, D] + 2\rho D + \mathcal{W},
 \end{aligned} \tag{B.6}$$

where

$$\mathcal{W} = \frac{1}{3}\sigma(\bar{\epsilon}\gamma^\mu\gamma^\nu D_\mu D_\nu\epsilon - \epsilon\gamma^\mu\gamma^\nu D_\mu D_\nu\bar{\epsilon}). \tag{B.7}$$

Therefore, for all the fields except the scalar D , the commutator is a sum of a translation by v^μ , a rotation by $\Theta^{\mu\nu}$, a R-symmetry rotation by α , a gauge transformation by Λ and a dilation by ρ . The explicit expression of the symmetry generators is

$$\begin{aligned}
 v^\mu &= \bar{\epsilon}\gamma^\mu\epsilon, \\
 \Theta^{\mu\nu} &= D^{[\mu}v^{\nu]} + v^\lambda\omega_\lambda^{\mu\nu}, \\
 \Lambda &= v^\mu iA_\mu + \sigma\bar{\epsilon}\epsilon, \\
 \rho &= \frac{i}{3}(\bar{\epsilon}\gamma^\mu D_\mu\epsilon + D_\mu\bar{\epsilon}\gamma^\mu\epsilon), \\
 \alpha &= \frac{i}{3}(D_\mu\bar{\epsilon}\gamma^\mu\epsilon - \bar{\epsilon}\gamma^\mu D_\mu\epsilon) + v^\mu V_\mu,
 \end{aligned} \tag{B.8}$$

²⁹For a chiral scalar of R-charge $-\Delta$, we have $D_\mu\phi = (\nabla_\mu + iA_\mu - i\Delta V_\mu)\phi$.

where $\omega_\lambda^{\mu\nu}$ is the spin connection. The supersymmetry parameters satisfy the Killing spinor equations

$$D_\mu \epsilon = \gamma_\mu \tilde{\epsilon}, \quad D_\mu \bar{\epsilon} = \gamma_\mu \tilde{\bar{\epsilon}}. \quad (\text{B.9})$$

The explicit expression for the spinors $\tilde{\epsilon}$ and $\tilde{\bar{\epsilon}}$ for \mathbb{S}^3 is [4]

$$\tilde{\epsilon} = \frac{i}{2R} \epsilon, \quad \tilde{\bar{\epsilon}} = \frac{i}{2R} \bar{\epsilon}. \quad (\text{B.10})$$

For \mathbb{S}_b^3 is [7]

$$\tilde{\epsilon} = \frac{i}{2Rf(\vartheta)} \epsilon, \quad \tilde{\bar{\epsilon}} = \frac{i}{2Rf(\vartheta)} \bar{\epsilon} \quad (\text{B.11})$$

where $f(\vartheta)$ is defined in the main text and for $\mathbb{S}^1 \times \mathbb{S}^2$ [8]

$$\tilde{\epsilon} = -\frac{1}{2} \gamma_{\hat{r}} \epsilon, \quad \tilde{\bar{\epsilon}} = \frac{1}{2} \gamma_{\hat{r}} \bar{\epsilon}. \quad (\text{B.12})$$

With these supersymmetry generators, it follows that for all the spaces that we consider, it results $\mathcal{W} = 0$, where \mathcal{W} is defined in (B.7). This implies that the supersymmetry closes off-shell on all the fields. It also results $\rho = 0$ for all the spaces, that implies that the commutator $[\delta_\epsilon, \delta_{\bar{\epsilon}}]$ does not include a dilation.

B.3 Chiral multiplet

The field content of the chiral multiplet is given by two complex scalars ϕ and F and spinors ψ and $\bar{\psi}$ with two complex components. These fields are in a generic representation of the gauge group. The supersymmetry variations are given by [5, 7, 21]

$$\begin{aligned} \delta\phi &= \bar{\epsilon}\psi, \\ \delta\bar{\phi} &= \epsilon\bar{\psi}, \\ \delta\psi &= i\gamma^\mu \epsilon D_\mu \phi + i\epsilon\sigma\phi + \frac{2\Delta i}{3} \gamma^\mu D_\mu \epsilon \phi + \bar{\epsilon}F, \\ \delta\bar{\psi} &= i\gamma^\mu \bar{\epsilon} D_\mu \bar{\phi} + i\bar{\phi}\sigma\bar{\epsilon} + \frac{2\Delta i}{3} \bar{\phi} \gamma^\mu D_\mu \bar{\epsilon} + \bar{F}\epsilon, \\ \delta F &= \epsilon(i\gamma^\mu D_\mu \psi - i\sigma\psi - i\lambda\phi) + \frac{i}{3}(2\Delta - 1)D_\mu \epsilon \gamma^\mu \psi, \\ \delta \bar{F} &= \bar{\epsilon}(i\gamma^\mu D_\mu \bar{\psi} - i\bar{\psi}\sigma + i\bar{\phi}\bar{\lambda}) + \frac{i}{3}(2\Delta - 1)D_\mu \bar{\epsilon} \gamma^\mu \bar{\psi} \end{aligned} \quad (\text{B.13})$$

and the commutators give the following off-shell result [5, 7, 21]

$$\begin{aligned} [\delta_\epsilon, \delta_{\bar{\epsilon}}]\phi &= iv^\mu \partial_\mu \phi + i\Lambda\phi + \Delta\rho\phi - \Delta\alpha\phi, \\ [\delta_\epsilon, \delta_{\bar{\epsilon}}]\bar{\phi} &= iv^\mu \partial_\mu \bar{\phi} - i\bar{\phi}\Lambda + \Delta\rho\bar{\phi} + \Delta\alpha\bar{\phi}, \\ [\delta_\epsilon, \delta_{\bar{\epsilon}}]\psi &= iv^\mu \partial_\mu \psi + \frac{i}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\psi + i\Lambda\psi + \left(\Delta + \frac{1}{2}\right)\rho\psi + (1 - \Delta)\alpha\psi, \\ [\delta_\epsilon, \delta_{\bar{\epsilon}}]\bar{\psi} &= iv^\mu \partial_\mu \bar{\psi} + \frac{i}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\bar{\psi} - i\bar{\psi}\Lambda + \left(\Delta + \frac{1}{2}\right)\rho\bar{\psi} + (\Delta - 1)\alpha\bar{\psi}, \\ [\delta_\epsilon, \delta_{\bar{\epsilon}}]F &= iv^\mu \partial_\mu F + i\Lambda F + (\Delta + 1)\rho F + (2 - \Delta)\alpha F, \\ [\delta_\epsilon, \delta_{\bar{\epsilon}}]\bar{F} &= iv^\mu \partial_\mu \bar{F} - i\bar{F}\Lambda + (\Delta + 1)\rho\bar{F} + (\Delta - 2)\alpha\bar{F}, \end{aligned} \quad (\text{B.14})$$

that is for generic supersymmetry parameters, as for the vector multiplet, the commutator $[\delta_\epsilon, \delta_{\epsilon'}]$ is a sum of a translation, a rotation, a R -symmetry rotation, a gauge transformation and a dilation. The commutator of two generic unbarred supersymmetries is different from zero for the scalar F [5, 7, 21]

$$[\delta_\epsilon, \delta_{\epsilon'}]F = \epsilon\gamma^{\mu\nu}\epsilon'(2D_\mu D_\nu\phi + iF_{\mu\nu}\phi) + \frac{2\Delta}{3}\phi(\epsilon\gamma^\mu\gamma^\nu D_\mu D_\nu\epsilon' - \epsilon'\gamma^\mu\gamma^\nu D_\mu D_\nu\epsilon). \quad (\text{B.15})$$

However, considering the supersymmetry generators that we described in the previous section, for all the spaces it results $[\delta_\epsilon, \delta_{\epsilon'}]F = 0$. A similar results holds also for the commutators of two barred supersymmetries on the field \bar{F} . We can therefore conclude that, for the spaces considered in the main text, also for the chiral multiplet two unbarred supersymmetries and two barred supersymmetries commute. The commutator $[\delta_\epsilon, \delta_{\bar{\epsilon}}]$ is a sum of a translation, a rotation, a R -symmetry rotation and a gauge transformation.

B.4 Background gauge multiplet

In three-dimensional theories, one often considers a background non-dynamical gauge field \mathbf{A}_μ that couples to a global symmetry. In $\mathcal{N} = 2$ theories, one introduces the SUSY partners $(\boldsymbol{\sigma}, \mathbf{D}, \dots)$, on which the path integral depends. The gauginos are set to zero, and in order to preserve supersymmetry, we require that their variation vanish. Suppose that on \mathbb{S}_b^3 we have vortex loop of charge η at $\vartheta = 0$. Supersymmetry requires that

$$\mathbf{A} = \eta d\varphi_1, \quad \mathbf{D} = -\frac{\boldsymbol{\sigma}}{Rf}, \quad \boldsymbol{\sigma} = \text{constant}. \quad (\text{B.16})$$

On $\mathbb{S}^1 \times \mathbb{S}^2$, when we have a (anti-)vortex loop of charge η^+ (η^-) at the north (south) pole, the configuration preserving SUSY is given by

$$\mathbf{A}^\pm = -\frac{a}{\beta}d\tau + \frac{m}{2}(\pm 1 - \cos\theta)d\varphi \pm \eta^\pm d\varphi, \quad \boldsymbol{\sigma} = -\frac{m}{2}, \quad a = \text{constant}, \quad m \in \mathbb{Z} \quad (\text{B.17})$$

on the two patches $U^+ = \{\theta \neq \pi\}$ and $U^- = \{\theta \neq 0\}$.

The supersymmetric BF coupling between the background and dynamical gauge multiplets is given by the insertion of $e^{-\mathcal{S}_{\text{BF}}}$ in the path integral where

$$\mathcal{S}_{\text{BF}} = -\frac{i}{2\pi} \int \mathbf{A} \wedge dA - \frac{i}{2\pi} \int d^3x \sqrt{g} (\mathbf{D}\boldsymbol{\sigma} + \boldsymbol{\sigma}D). \quad (\text{B.18})$$

The invariance of the BF term under Q follows from that of the CS term because

$$\mathcal{S}_{\text{BF}}(\mathbf{A}, \dots; A, \dots) = -i [\mathcal{S}_{\text{SCS}}(\mathbf{A} + A, \dots) - \mathcal{S}_{\text{SCS}}(\mathbf{A}, \dots) - \mathcal{S}_{\text{SCS}}(A, \dots)]_{k=1}. \quad (\text{B.19})$$

On \mathbb{S}_b^3 , the scalar $\boldsymbol{\sigma} = \zeta/R$ is nothing but the FI parameter, and the second term in (B.18) is the standard FI term \mathcal{S}_{FI} that enters the path integral as

$$e^{-\mathcal{S}_{\text{FI}}}, \quad \mathcal{S}_{\text{FI}} = -\frac{i\zeta}{2\pi R} \int d^3x \sqrt{g} \left(D - \frac{\boldsymbol{\sigma}}{Rf} \right). \quad (\text{B.20})$$

C Boundary terms on the round sphere \mathbb{S}^3

In the presence of the vortex loop operator one needs to keep track of delta function contributions at the singularity, or alternatively of boundary terms arising from an excised tubular region of the loop operator. There are three main reasons why these terms are important:

1. Without these terms the localizing actions are not Q -exact.
2. Without these terms the vortex loop operators would seem not to break any supersymmetry at all, while the boundary terms ensure they preserve only one half.
3. The boundary terms may contribute to the value of the action evaluated at the saddle points of the localizing action.

In this appendix we study the boundary terms for the different pieces of the $\mathcal{N} = 2$ actions in three dimensions, focusing for simplicity on the case of the round \mathbb{S}^3 .

For the round sphere \mathbb{S}^3 , the supersymmetry variations for the vector multiplet spinors (B.5) simplify to

$$\begin{aligned}\delta\lambda &= -\frac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} + i\gamma^\mu\epsilon D_\mu\sigma - \left(D + \frac{\sigma}{R}\right)\epsilon, \\ \delta\bar{\lambda} &= -\frac{1}{2}\gamma^{\mu\nu}\bar{\epsilon}F_{\mu\nu} - i\gamma^\mu\bar{\epsilon}D_\mu\sigma + \left(D + \frac{\sigma}{R}\right)\bar{\epsilon},\end{aligned}\tag{C.1}$$

and the variations for the spinors and the auxiliary scalars in the chiral multiplet (B.13) are

$$\begin{aligned}\delta\psi &= i\gamma^\mu\epsilon D_\mu\phi + i\epsilon\sigma\phi - \frac{\Delta}{R}\epsilon\phi + \bar{\epsilon}F, \\ \delta\bar{\psi} &= i\gamma^\mu\bar{\epsilon}D_\mu\bar{\phi} + i\bar{\phi}\sigma\bar{\epsilon} - \frac{\Delta}{R}\bar{\epsilon}\bar{\phi} + \bar{F}\epsilon, \\ \delta F &= \epsilon(i\gamma^\mu D_\mu\psi - i\sigma\psi - i\lambda\phi) + \frac{1}{2R}(2\Delta - 1)\epsilon\psi, \\ \delta\bar{F} &= \bar{\epsilon}(i\gamma^\mu D_\mu\bar{\psi} - i\bar{\psi}\sigma + i\bar{\phi}\bar{\lambda}) + \frac{1}{2R}(2\Delta - 1)\bar{\epsilon}\bar{\psi}.\end{aligned}\tag{C.2}$$

C.1 Boundary terms for supersymmetric Yang-Mills action

When using the supersymmetric Yang-Mills action (3.3) as a localizing term it should be written as a total superderivative [5]. Keeping track of total derivative terms it is possible to show that for generic Killing spinors ϵ and $\bar{\epsilon}$

$$\begin{aligned}\delta_{\bar{\epsilon}}\delta_{\epsilon}\text{Tr}\left(\frac{1}{2}\bar{\lambda}\lambda - 2D\sigma\right) &= \bar{\epsilon}\epsilon\mathcal{L}_{\text{SYM}} + D_\mu\text{Tr}\left(i\bar{\epsilon}\gamma_\nu\epsilon F^{\nu\mu}\sigma - \bar{\epsilon}\epsilon\sigma D^\mu\sigma - \bar{\epsilon}\gamma^{\nu\mu}\epsilon\sigma D_\nu\sigma\right. \\ &\quad \left.+ i\bar{\epsilon}\gamma^\mu\epsilon\left(D + \frac{\sigma}{R}\right)\sigma + \frac{i}{2}\bar{\lambda}\gamma^\mu\epsilon(\bar{\epsilon}\lambda)\right).\end{aligned}\tag{C.3}$$

Considering a boundary at small $\vartheta = \vartheta_0$ in the coordinate system (A.8), we obtain

$$\begin{aligned} & \int d^3x \sqrt{g} \delta_{\bar{\epsilon}} \delta_{\epsilon} \text{Tr} \left(\frac{1}{2} \bar{\lambda} \lambda - 2D\sigma \right) \\ &= \bar{\epsilon} \epsilon g_{\text{YM}}^2 \mathcal{S}_{\text{SYM}} - R^3 \int d\varphi_1 d\varphi_2 \cos \vartheta_0 \sin \vartheta_0 \text{Tr} \left(i \bar{\epsilon} \gamma_{\nu} \epsilon F^{\nu\vartheta} \sigma - \bar{\epsilon} \epsilon \sigma D^{\vartheta} \sigma \right. \\ & \quad \left. - \bar{\epsilon} \gamma^{\nu\vartheta} \epsilon \sigma D_{\nu} \sigma + i \bar{\epsilon} \gamma^{\vartheta} \epsilon \left(D + \frac{\sigma}{R} \right) \sigma + \frac{i}{2} \bar{\lambda} \gamma^{\vartheta} \epsilon (\bar{\epsilon} \lambda) \right). \end{aligned} \quad (\text{C.4})$$

With ϵ and $\bar{\epsilon}$ satisfying (2.6), the previous expression simplifies to

$$\begin{aligned} & \int d^3x \sqrt{g} \delta_{\bar{\epsilon}} \delta_{\epsilon} \text{Tr} \left(\frac{1}{2} \bar{\lambda} \lambda - 2D\sigma \right) \\ &= g_{\text{YM}}^2 \mathcal{S}_{\text{SYM}} + \int d\varphi_1 d\varphi_2 \cos \vartheta_0 \sin \vartheta_0 \text{Tr} \left(\frac{i}{\cos \vartheta_0} \sigma F_{\vartheta\varphi_2} + \frac{R}{2} D_{\vartheta} \sigma^2 + \frac{iR^2}{4} \bar{\lambda} (\gamma_1 - i\gamma_2) \lambda \right). \end{aligned} \quad (\text{C.5})$$

Being a supersymmetry variation automatically implies that the sum of the bulk and boundary actions are invariant under supersymmetry. This can also be verified directly, as the supersymmetry variations of the SYM Lagrangian (3.3) are total derivatives

$$\delta_{\epsilon} \mathcal{S}_{\text{SYM}} = \frac{1}{g_{\text{YM}}^2} \int d^3x \sqrt{g} D_{\mu} \text{Tr} \left[-\frac{i}{2} \bar{\lambda} \gamma_{\nu} \epsilon F^{\mu\nu} - \frac{1}{2} \bar{\lambda} \epsilon D^{\mu} \sigma - \frac{i}{2} \bar{\lambda} \gamma^{\mu} \epsilon \left(D + \frac{\sigma}{R} \right) \right]. \quad (\text{C.6})$$

Putting a cutoff at small ϑ_0 gives the boundary term

$$\begin{aligned} \delta_{\epsilon} \mathcal{S}_{\text{SYM}} = \frac{1}{g_{\text{YM}}^2} \int d\varphi_1 d\varphi_2 \text{Tr} \left[\frac{i}{2} \cos \vartheta_0 \bar{\lambda} \gamma_2 \epsilon F_{\vartheta\varphi_1} + \frac{i}{2} \sin \vartheta_0 \bar{\lambda} \gamma_3 \epsilon F_{\vartheta\varphi_2} \right. \\ \left. + \frac{1}{2} \cos \vartheta_0 \sin \vartheta_0 \left(R \bar{\lambda} \epsilon D_{\vartheta} \sigma + i R^2 \bar{\lambda} \gamma_1 \epsilon \left(D + \frac{\sigma}{R} \right) \right) \right], \end{aligned} \quad (\text{C.7})$$

likewise

$$\begin{aligned} \delta_{\bar{\epsilon}} \mathcal{S}_{\text{SYM}} = \frac{1}{g_{\text{YM}}^2} \int d^3x \sqrt{g} D_{\mu} \text{Tr} \left[-\frac{1}{4} \varepsilon^{\mu\nu\rho} \bar{\epsilon} \lambda F_{\nu\rho} + \frac{R}{2} \bar{\epsilon} \gamma^{\mu\nu} \lambda D_{\nu} \sigma \right] \\ = \frac{1}{g_{\text{YM}}^2} \int d\varphi_1 d\varphi_2 \left[\frac{1}{2} \bar{\epsilon} \lambda F_{\varphi_1\varphi_2} - \frac{1}{2} \cos \vartheta_0 \bar{\epsilon} \gamma_{12} \lambda D_{\varphi_1} \sigma - \frac{1}{2} \sin \vartheta_0 \bar{\epsilon} \gamma_{13} \lambda D_{\varphi_3} \sigma \right]. \end{aligned} \quad (\text{C.8})$$

By adding the boundary term in (C.5) and using the projection equations (2.6) it is possible to check that $\delta_{\epsilon} (S_{\text{SYM}} + S_{\text{SYM}}^B) = \delta_{\bar{\epsilon}} (S_{\text{SYM}} + S_{\text{SYM}}^B) = 0$. Since this statement is true only assuming the projection equations, this also confirms that the vortex loop operators break half of the supersymmetry at the singularity.

Note also that the boundary action vanishes on the BPS solutions (2.14) with arbitrary σ_0 .

C.2 Boundary terms for Chern-Simons action

The supersymmetric CS action (3.1) is not a total superderivative, still it is possible to add boundary terms such that its variation vanishes. The supersymmetry variation of this

term is of course also a total derivative, which reads

$$\begin{aligned} \delta_\epsilon \mathcal{S}_{\text{SCS}} &= \frac{ik}{4\pi} \int d^3x \sqrt{g} \text{Tr} \left[\frac{1}{2} \epsilon^{\mu\nu\rho} \partial_\mu (A_\nu \bar{\lambda} \gamma_\rho \epsilon) - \partial_\mu (\sigma \bar{\lambda} \gamma^\mu \epsilon) \right] \\ &= \frac{ik}{4\pi} \int d^2x \sqrt{g} \text{Tr} \left[-\frac{1}{2} \epsilon^{n\mu\nu} A_\mu \bar{\lambda} \gamma_\nu \epsilon + \sigma \bar{\lambda} \gamma^n \epsilon \right], \end{aligned} \quad (\text{C.9})$$

where the integral on the second line is on the boundary and γ^n is in the normal direction.

Let us use the metric (A.8) and frame (A.9) with a boundary at small ϑ_0 , so we get explicitly

$$\begin{aligned} \delta_\epsilon \mathcal{S}_{\text{SCS}} &= \frac{ikR}{4\pi} \int d\varphi_1 d\varphi_2 \cos \vartheta_0 \sin \vartheta_0 \times \\ &\quad \times \text{Tr} \left[-\frac{1}{2 \sin \vartheta_0} A_{\varphi_1} \bar{\lambda} \gamma_3 \epsilon + \frac{1}{2 \cos \vartheta_0} A_{\varphi_2} \bar{\lambda} \gamma_2 \epsilon + R \sigma \bar{\lambda} \gamma_1 \epsilon \right]. \end{aligned} \quad (\text{C.10})$$

Let us consider the boundary term

$$\mathcal{S}_{\text{SCS}}^B = \frac{k}{4\pi} \int d\varphi_1 d\varphi_2 \text{Tr} [A_{\varphi_1} (A_{\varphi_2} - 2iR\sigma)]. \quad (\text{C.11})$$

Given that near the singularity $\gamma_3 \epsilon = \epsilon$ (2.6) it follows

$$\delta_\epsilon (\mathcal{S}_{\text{SCS}} + \mathcal{S}_{\text{SCS}}^B) = 0. \quad (\text{C.12})$$

Likewise for the $\bar{\epsilon}$ variation

$$\begin{aligned} \delta_{\bar{\epsilon}} \mathcal{S}_{\text{SCS}} &= \frac{ik}{4\pi} \int d^3x \sqrt{g} \text{Tr} \left[-\frac{1}{2} \epsilon^{\mu\nu\rho} \partial_\mu (A_\nu \bar{\epsilon} \gamma_\rho \lambda) - \partial_\mu (\bar{\epsilon} \gamma^\mu \lambda \sigma) \right] \\ &= \frac{ik}{4\pi} \int d^2x \sqrt{g} \text{Tr} \left[\frac{1}{2} \epsilon^{n\mu\nu} A_\mu \bar{\epsilon} \gamma_\nu \lambda + \bar{\epsilon} \gamma^n \lambda \sigma \right], \end{aligned} \quad (\text{C.13})$$

and given that near the singularity $\gamma_3 \bar{\epsilon} = -\bar{\epsilon}$, it results

$$\delta_{\bar{\epsilon}} (\mathcal{S}_{\text{SCS}} + \mathcal{S}_{\text{SCS}}^B) = 0. \quad (\text{C.14})$$

Let's now consider gauge transformations. Given an element g of the gauge group, the gauge vector A_μ transforms as

$$A_\mu \rightarrow A_\mu^g = g A_\mu g^{-1} - i g \partial_\mu g^{-1} \quad (\text{C.15})$$

and the remaining fields in the vector multiplet transform in the adjoint representation, for instance

$$\sigma \rightarrow \sigma^g = g \sigma g^{-1}. \quad (\text{C.16})$$

The super Chern-Simons Lagrangian

$$\mathcal{L}_{\text{SCS}} = \text{Tr} \left[\epsilon^{\mu\nu\rho} \left(A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho \right) - \bar{\lambda} \lambda + 2D\sigma \right] \quad (\text{C.17})$$

transforms as

$$\mathcal{L}_{\text{SCS}} \rightarrow \mathcal{L}_{\text{SCS}}^g = \mathcal{L}_{\text{SCS}} + i\varepsilon^{\mu\nu\rho} \partial_\mu \text{Tr} [\partial_\nu g^{-1} g A_\rho] - \frac{1}{3} \varepsilon^{\mu\nu\rho} \text{Tr} [g^{-1} \partial_\mu g g^{-1} \partial_\nu g g^{-1} \partial_\rho g] . \quad (\text{C.18})$$

Using this result, it follows that the sum of the bulk and boundary actions transforms to

$$(\mathcal{S}_{\text{SCS}} + \mathcal{S}_{\text{SCS}}^B)^g = (\mathcal{S}_{\text{SCS}} + \mathcal{S}_{\text{SCS}}^B) - \frac{k}{2\pi} \int d\varphi_1 d\varphi_2 \text{Tr} [\partial_{\varphi_1} g^{-1} g (iA_{\varphi_2} + \cos \theta_0 R \sigma)] + T , \quad (\text{C.19})$$

where T is given by

$$T = -\frac{k}{12\pi} \int_{\vartheta \geq \vartheta_0} d^3x \sqrt{g} \varepsilon^{\mu\nu\rho} \text{Tr} [g^{-1} \partial_\mu g g^{-1} \partial_\nu g g^{-1} \partial_\rho g] - \frac{k}{4\pi} \int_{\vartheta = \vartheta_0} d\varphi_1 d\varphi_2 \text{Tr} [g^{-1} \partial_{\varphi_1} g g^{-1} \partial_{\varphi_2} g] . \quad (\text{C.20})$$

In the limit $\vartheta_0 \rightarrow 0$, $g(\vartheta = \vartheta_0)$ is independent of φ_1 , and we recover the usual gauge transformation of the Chern-Simons action.

C.3 Boundary terms for Fayet-Iliopoulos term

Under a generic supersymmetry variation the Fayet-Iliopoulos action (3.4) transforms as

$$\delta \mathcal{S}_{\text{FI}} = -\frac{\zeta}{4\pi R} \int d^3x \sqrt{g} D_\mu (\bar{\lambda} \gamma^\mu \epsilon + \bar{\epsilon} \gamma^\mu \lambda) , \quad (\text{C.21})$$

and placing a boundary at small $\theta = \theta_0$, it results

$$\delta \mathcal{S}_{\text{FI}} = \frac{\zeta R^2}{4\pi} \int d\varphi_1 d\varphi_2 \cos \vartheta_0 \sin \vartheta_0 (\bar{\lambda} \gamma^\theta \epsilon + \bar{\epsilon} \gamma^\theta \lambda) . \quad (\text{C.22})$$

If we add the boundary term

$$\mathcal{S}_{\text{FI}}^B = -\frac{\zeta}{2\pi} \int d\varphi_1 d\varphi_2 \cos \vartheta_0 A_{\varphi_1} , \quad (\text{C.23})$$

and consider the supersymmetry generators that satisfy the condition (2.6) we find

$$\delta(\mathcal{S}_{\text{FI}} + \mathcal{S}_{\text{FI}}^B) = 0 . \quad (\text{C.24})$$

C.4 Boundary terms for chiral action

Like the SYM action, the chiral action (3.6) is a total superderivative [5] and can be used as a localizing term. In the presence of a boundary or a singularity we need to consider boundary terms.

For generic Killing spinors ϵ and $\bar{\epsilon}$ the double variation is

$$\begin{aligned} \delta_\epsilon \delta_{\bar{\epsilon}} \left(\bar{\psi} \psi - 2i \bar{\phi} \sigma \phi + \frac{2(\Delta - 1)}{R} \bar{\phi} \phi \right) \\ = \bar{\epsilon} \epsilon \mathcal{L}_{\text{chiral}} + D_\mu \left(\bar{\epsilon} \gamma^{\mu\nu} \epsilon \bar{\phi} D_\nu \phi - \bar{\epsilon} \gamma^\mu \epsilon \bar{\phi} \sigma \phi + \frac{i(2 - \Delta)}{R} \bar{\epsilon} \gamma^\mu \epsilon \bar{\phi} \phi + i \bar{\epsilon} \gamma^\mu \bar{\psi} (\epsilon \psi) \right) . \end{aligned} \quad (\text{C.25})$$

Placing a boundary at small $\vartheta = \vartheta_0$ in the coordinate system (A.8), we obtain

$$\int d^3x \sqrt{g} \delta_{\bar{\epsilon}} \delta_{\epsilon} \left(\bar{\psi} \psi - 2i \bar{\phi} \sigma \phi + \frac{2(\Delta - 1)}{R} \bar{\phi} \phi \right) = \bar{\epsilon} \epsilon \mathcal{S}_{\text{chiral}} - R^3 \int d\varphi_1 d\varphi_2 \cos \vartheta_0 \sin \vartheta_0 \left(\bar{\epsilon} \gamma^{\vartheta\nu} \epsilon \bar{\phi} D_{\nu} \phi - \bar{\epsilon} \gamma^{\vartheta} \epsilon \bar{\phi} \sigma \phi + \frac{i(2-\Delta)}{R} \bar{\epsilon} \gamma^{\vartheta} \epsilon \bar{\phi} \phi + i \bar{\epsilon} \gamma^{\vartheta} \bar{\psi} (\epsilon \psi) \right). \tag{C.26}$$

With ϵ and $\bar{\epsilon}$ satisfying (2.6), the previous expression simplifies to

$$\int d^3x \sqrt{g} \delta_{\bar{\epsilon}} \delta_{\epsilon} \left(\bar{\psi} \psi - 2i \bar{\phi} \sigma \phi + \frac{2(\Delta - 1)}{R} \bar{\phi} \phi \right) = \mathcal{S}_{\text{chiral}} - R \int d\varphi_1 d\varphi_2 \cos \vartheta_0 \left(i \bar{\phi} D_{\varphi_1} \phi + \frac{iR}{2} \sin \vartheta_0 \bar{\psi} (\gamma_1 + i\gamma_2) \psi \right). \tag{C.27}$$

Clearly the variation of the sum of bulk and boundary actions will vanish for all the supercharges parameterized by ϵ and $\bar{\epsilon}$ satisfying (2.6). This can be verified by an explicit calculation as done for the SYM action above.

D Kinetic operators on \mathbb{S}^3

In this appendix we study the kinetic operators arising from expanding the localizing actions on \mathbb{S}^3 and calculate their spectra.

D.1 Vector multiplet

In order to localize the vector multiplet we add to the Lagrangian a total superderivative $t \mathcal{S}_{\text{loc}} = t \delta_{\epsilon} \delta_{\bar{\epsilon}} \int (\bar{\lambda} \lambda - 4D\sigma)$, which is proportional to the Yang-Mills action (3.3) [5, 21] and the boundary term (C.5).

To compute the one-loop contribution we consider fluctuations around the BPS configuration, i.e.

$$\sigma = \sigma_0 + \frac{\sigma'}{\sqrt{t}}, \quad D = -\frac{\sigma_0}{R} + \frac{D'}{\sqrt{t}}, \quad A_{\mu} = A_{\mu}^{(0)} + \frac{A'_{\mu}}{\sqrt{t}}, \quad \lambda = \frac{\lambda'}{\sqrt{t}}, \tag{D.1}$$

and expand the Yang-Mills action up to quadratic order in the fluctuations. Considering the background gauge $D^{(0)\mu} A'_{\mu} = 0$, where $D^{(0)}$ is defined using the connection $A_{\mu}^{(0)}$

$$D_{\mu}^{(0)} = \nabla_{\mu} + i[A_{\mu}^{(0)}, \], \tag{D.2}$$

we obtain

$$t \mathcal{L}_{\text{SYM}} = \frac{1}{2} \text{Tr} \left[-A'^{\mu} D^{(0)\nu} D_{\nu}^{(0)} A'_{\mu} - [\sigma_0, A'_{\mu}]^2 - \sigma' D^{(0)\mu} D_{\mu}^{(0)} \sigma' + \left(D' + \frac{\sigma'}{R} \right)^2 + i \bar{\lambda}' \gamma^{\mu} D_{\mu}^{(0)} \lambda' + i \bar{\lambda}' [\sigma_0, \lambda'] - \frac{1}{2R} \bar{\lambda}' \lambda' \right] \tag{D.3}$$

and this quadratic action is invariant under the supersymmetry variations

$$\begin{aligned}
\delta A'_\mu &= \frac{i}{2}(\bar{\epsilon}\gamma_\mu\lambda' - \bar{\lambda}'\gamma_\mu\epsilon), \\
\delta\sigma' &= \frac{1}{2}(\bar{\epsilon}\lambda' - \bar{\lambda}'\epsilon), \\
\delta\lambda' &= -\gamma^{\mu\nu}\epsilon D_\mu^{(0)}A'_\nu + i\gamma^\mu\epsilon(D_\mu^{(0)}\sigma' + i[A'_\mu, \sigma_0]) - \left(D' + \frac{\sigma'}{R}\right)\epsilon, \\
\delta\bar{\lambda}' &= -\gamma^{\mu\nu}\bar{\epsilon}D_\mu^{(0)}A'_\nu - i\gamma^\mu\bar{\epsilon}(D_\mu^{(0)}\sigma' + i[A'_\mu, \sigma_0]) + \left(D' + \frac{\sigma'}{R}\right)\bar{\epsilon}, \\
\delta D' &= -\frac{i}{2}\bar{\epsilon}\gamma^\mu D_\mu^{(0)}\lambda' - \frac{i}{2}D_\mu^{(0)}\bar{\lambda}'\gamma^\mu\epsilon + \frac{i}{2}[\bar{\epsilon}\lambda', \sigma_0] + \frac{i}{2}[\bar{\lambda}'\epsilon, \sigma_0] - \frac{1}{4R}\bar{\epsilon}\lambda' + \frac{1}{4R}\bar{\lambda}'\epsilon,
\end{aligned} \tag{D.4}$$

where ϵ and $\bar{\epsilon}$ are the supersymmetry preserved by the vortex. Using the Cartan decomposition of the gauge group, a generic fluctuation field Φ' is written as

$$\Phi' = \Phi^\alpha X_\alpha + \Phi^i K_i. \tag{D.5}$$

In the following we ignore the contribution of the Cartan components Φ^i , since their actions do not depend on σ_0 . The action $t\mathcal{L}_{\text{SYM}}$ is therefore written as

$$\begin{aligned}
\frac{1}{2} \int d^3x \sqrt{g} \sum_\alpha \left[g^{\mu\nu} A_\mu^{-\alpha} \left(-\nabla_\mu^{(0)} \nabla^{\mu(0)} + \alpha(\sigma_0)^2 \right) A_\nu^\alpha + \bar{\lambda}^{-\alpha} \left(i\gamma^\mu \nabla_\mu^{(0)} + i\alpha(\sigma_0) - \frac{1}{2R} \right) \lambda^\alpha \right. \\
\left. - \sigma^{-\alpha} \nabla^{(0)\mu} \nabla_\mu^{(0)} \sigma^\alpha + \left(D^{-\alpha} + \frac{\sigma^{-\alpha}}{R} \right) \left(D^\alpha + \frac{\sigma^\alpha}{R} \right) \right]
\end{aligned} \tag{D.6}$$

where we defined the operator

$$\nabla_\mu^{(0)} = \nabla_\mu + i\alpha(A_\mu^{(0)}) \tag{D.7}$$

and used the fact that $A_\mu^{(0)}$ is in the Cartan of the gauge group. The supersymmetry transformation for any Φ^α can be easily obtained projecting on the X_α generator the expressions in (D.4). It results that the effect of the vortex on the localizing action corresponds to replacing ∇_μ with $\nabla_\mu^{(0)}$. It is therefore convenient to redefine the generic field Φ^α as

$$\Phi^\alpha = e^{-i\alpha(A_\mu^{(0)})x^\mu} \tilde{\Phi}^\alpha \tag{D.8}$$

so that $\nabla_\mu^{(0)}\Phi^\alpha = e^{-i\alpha(A_\mu^{(0)})x^\mu} \nabla_\mu \tilde{\Phi}^\alpha$. Since Φ^α should be a periodic function, it follows that $\tilde{\Phi}^\alpha$ satisfies

$$\begin{aligned}
\tilde{\Phi}^\alpha(\vartheta, \varphi_1 + 2\pi, \varphi_2) &= e^{2\pi i\alpha(A_{\varphi_1}^{(0)})} \tilde{\Phi}^\alpha(\vartheta, \varphi_1, \varphi_2) = e^{2\pi i\alpha(H)} \tilde{\Phi}^\alpha(\vartheta, \varphi_1, \varphi_2), \\
\tilde{\Phi}^\alpha(\vartheta, \varphi_1, \varphi_2 + 2\pi) &= \tilde{\Phi}^\alpha(\vartheta, \varphi_1, \varphi_2),
\end{aligned} \tag{D.9}$$

or in terms of u and v (A.1)

$$\begin{aligned}
\tilde{\Phi}^\alpha(e^{2\pi i}u, v) &= e^{2\pi i\alpha(H)} \tilde{\Phi}^\alpha(u, v), \\
\tilde{\Phi}^\alpha(u, e^{2\pi i}v) &= \tilde{\Phi}^\alpha(u, v).
\end{aligned} \tag{D.10}$$

Considering the redefinition (D.8), the action (D.6) is written in terms of $\tilde{\Phi}^\alpha$ fields as

$$\frac{1}{2} \int d^3x \sqrt{g} \sum_\alpha \left[g^{\mu\nu} \tilde{A}_\mu^{-\alpha} (-\nabla_\mu \nabla^\mu + \alpha(\sigma_0)^2) \tilde{A}_\nu^\alpha + \tilde{\lambda}^{-\alpha} \left(i\gamma^\mu \nabla_\mu + i\alpha(\sigma_0) - \frac{1}{2R} \right) \tilde{\lambda}^\alpha - \tilde{\sigma}^{-\alpha} \nabla^\mu \nabla_\mu \tilde{\sigma}^\alpha + \left(\tilde{D}^{-\alpha} + \frac{\tilde{\sigma}^{-\alpha}}{R} \right) \left(\tilde{D}^\alpha + \frac{\tilde{\sigma}^\alpha}{R} \right) \right] \quad (\text{D.11})$$

and the supersymmetry transformations are given by

$$\begin{aligned} \delta \tilde{A}_\mu^\alpha &= \frac{i}{2} (\bar{\epsilon} \gamma_\mu \tilde{\lambda}^\alpha - \tilde{\lambda}^\alpha \gamma_\mu \epsilon), \\ \delta \tilde{\sigma}^\alpha &= \frac{1}{2} (\bar{\epsilon} \tilde{\lambda}^\alpha - \tilde{\lambda}^\alpha \epsilon), \\ \delta \tilde{\lambda}^\alpha &= -\gamma^{\mu\nu} \epsilon \nabla_\mu \tilde{A}_\nu^\alpha + i\gamma^\mu \epsilon (\nabla_\mu \tilde{\sigma}^\alpha - i\alpha(\sigma_0) \tilde{A}_\mu^\alpha) - \left(\tilde{D}^\alpha + \frac{\tilde{\sigma}^\alpha}{R} \right) \epsilon, \\ \delta \tilde{\lambda}^\alpha &= -\gamma^{\mu\nu} \bar{\epsilon} \nabla_\mu \tilde{A}_\nu^\alpha - i\gamma^\mu \bar{\epsilon} (\nabla_\mu \tilde{\sigma}^\alpha - i\alpha(\sigma_0) \tilde{A}_\mu^\alpha) + \left(\tilde{D}^\alpha + \frac{\tilde{\sigma}^\alpha}{R} \right) \bar{\epsilon}, \\ \delta \tilde{D}^\alpha &= -\frac{i}{2} \bar{\epsilon} \gamma^\mu \nabla_\mu \tilde{\lambda}^\alpha + \frac{i}{2} \epsilon \gamma^\mu \nabla_\mu \tilde{\lambda}^\alpha - \left(\frac{i}{2} \alpha(\sigma_0) + \frac{1}{4R} \right) \bar{\epsilon} \tilde{\lambda}^\alpha + \left(-\frac{i}{2} \alpha(\sigma_0) + \frac{1}{4R} \right) \tilde{\lambda}^\alpha \epsilon. \end{aligned} \quad (\text{D.12})$$

The one-loop contribution of the vector multiplet that depends on σ_0 therefore is given by

$$Z_{1\text{-loop}}^{\text{vector}}(\sigma_0) = \prod_\alpha \frac{\det_\alpha \left(i\gamma^\mu \nabla_\mu + i\alpha(\sigma_0) - \frac{1}{2R} \right)}{\det_\alpha \left(-\nabla_\mu \nabla^\mu + \alpha(\sigma_0)^2 \right)^{1/2}}, \quad (\text{D.13})$$

where $\det_\alpha(\mathcal{O})$ is the determinant of the operator \mathcal{O} , evaluated in a space of fields $\tilde{\Phi}^\alpha$ that satisfy the boundary condition (D.10).

In appendix E we have constructed a basis of harmonics satisfying these boundary conditions. There are many such harmonics, most of which are non-normalizable and should not be included in the spectrum. We discuss the states in detail in appendix E and try to make an educated guess which modes should be included in the spectrum. The result is

$$Z_{1\text{-loop}}^{\text{vector}}(\sigma_0) = \prod_{\alpha>0} \prod_n^\infty (n^2 + \alpha(R\sigma_0 + iH)^2). \quad (\text{D.14})$$

D.2 Chiral multiplet

We now focus on the matter sector of the theory.

Since $\mathcal{L}_{\text{chiral}}$ (3.6) is a total superderivative [5, 21], multiplying the matter action by an arbitrary parameter t , the result of the path integral remain unchanged. Therefore, the contribution of the matter sector is given by the quadratic fluctuation with respect to the classical configuration that minimize (3.6), i.e., $\phi = \bar{\phi} = F = \bar{F} = \psi = \bar{\psi} = 0$.

As for the vector multiplet, we consider fluctuation with respect to the localizing configuration

$$\phi = \frac{1}{\sqrt{t}} \phi', \quad F = \frac{1}{\sqrt{t}} F', \quad \psi = \frac{1}{\sqrt{t}} \psi', \quad \bar{\psi} = \frac{1}{\sqrt{t}} \bar{\psi}', \quad (\text{D.15})$$

and in the large t limit,

$$\begin{aligned}
 t\mathcal{L}_{\text{chiral}} = & D_\mu^{(0)} \bar{\phi}' D^{\mu(0)} \phi' + \bar{\phi}' \sigma_0^2 \phi' + \frac{i2(\Delta-1)}{R} \bar{\phi}' \sigma_0 \phi' + \frac{\Delta(2-\Delta)}{R^2} \bar{\phi}' \phi' + \bar{F}' F' \\
 & - i\bar{\psi}' \gamma^\mu D_\mu^{(0)} \psi' + i\bar{\psi}' \sigma_0 \psi' - \frac{2\Delta-1}{2R} \bar{\psi}' \psi'
 \end{aligned} \tag{D.16}$$

where we defined

$$D_\mu^{(0)} = \nabla_\mu + iA_\mu^{(0)i} K_i^R, \tag{D.17}$$

where K_i^R are the elements of the Cartan subalgebra written in the R representation. Also the constant field σ_0 is expressed in the R representation of the Cartan subalgebra, i.e., $\sigma_0 = \sigma_0^i K_i^R$. The Lagrangian (D.16) is invariant under the following supersymmetry transformations

$$\begin{aligned}
 \delta\phi' &= \bar{\epsilon}\psi', \\
 \delta\bar{\phi}' &= \epsilon\bar{\psi}', \\
 \delta\psi' &= i\gamma^\mu \epsilon D_\mu^{(0)} \phi' + i\epsilon\sigma_0 \phi' - \frac{\Delta}{R} \epsilon\phi' + \bar{\epsilon}F', \\
 \delta\bar{\psi}' &= i\gamma^\mu \bar{\epsilon} D_\mu^{(0)} \bar{\phi}' + i\bar{\phi}' \sigma_0 \bar{\epsilon} - \frac{\Delta}{R} \bar{\epsilon}\bar{\phi}' + \bar{F}' \epsilon, \\
 \delta F' &= \epsilon(i\gamma^\mu D_\mu^{(0)} \psi' - i\sigma_0 \psi') + \frac{1}{2R} (2\Delta-1) \epsilon\psi', \\
 \delta \bar{F}' &= \bar{\epsilon}(i\gamma^\mu D_\mu^{(0)} \bar{\psi}' - i\bar{\psi}' \sigma_0) + \frac{1}{2R} (2\Delta-1) \bar{\epsilon}\bar{\psi}'.
 \end{aligned} \tag{D.18}$$

Expanding all the fields in the weights ρ of the representation R (see section 2.3.2) the Lagrangian becomes

$$\begin{aligned}
 t\mathcal{L}_{\text{chiral}} = & \sum_\rho \left[\bar{\phi}^\rho \left(\nabla_\mu^{(0)} \nabla^{\mu(0)} + \left(\rho(\sigma_0) + i\frac{\Delta-1}{R} \right)^2 + \frac{1}{R^2} \right) \phi^\rho + \bar{F}^\rho F^\rho \right. \\
 & \left. + \bar{\psi}^\rho \left(-i\gamma^\mu \nabla_\mu^{(0)} + i\rho(\sigma_0) - \frac{2\Delta-1}{2R} \right) \psi^\rho \right]
 \end{aligned} \tag{D.19}$$

where we defined

$$\nabla_\mu^{(0)} = \nabla_\mu + i\rho(A_\mu^{(0)}). \tag{D.20}$$

Like for the vector multiplet, we redefine the fields as

$$\Phi^\rho = e^{-i\rho(A_\mu^{(0)})x^\mu} \tilde{\Phi}^\rho, \tag{D.21}$$

so that $\nabla_\mu^{(0)} \Phi^\rho = e^{-i\rho(A_\mu^{(0)})x^\mu} \nabla_\mu \tilde{\Phi}^\rho$ and $\tilde{\Phi}^\rho$ satisfy

$$\tilde{\Phi}^\rho(e^{2\pi i} u, v) = e^{2\pi i\rho(H)} \tilde{\Phi}^\rho(u, v), \quad \tilde{\Phi}^\rho(u, e^{2\pi i} v) = \tilde{\Phi}^\rho(u, v). \tag{D.22}$$

The fields $\tilde{\Phi}^\rho$ can be thought as the complex conjugate of Φ^ρ fields, although in the Euclidean formulation of the theory $\tilde{\Phi}^\rho$ and Φ^ρ are independent. It is however natural to redefine $\tilde{\Phi}^\rho$ as

$$\bar{\Phi}^\rho = e^{i\rho(A_\mu^{(0)})x^\mu} \tilde{\tilde{\Phi}}^\rho \tag{D.23}$$

so that

$$\tilde{\tilde{\Phi}}^\rho(e^{2\pi i}u, v) = e^{-2\pi i\rho(H)}\tilde{\tilde{\Phi}}^\rho(u, v), \quad \tilde{\tilde{\Phi}}^\rho(u, e^{2\pi i}v) = \tilde{\tilde{\Phi}}^\rho(u, v). \quad (\text{D.24})$$

In terms of $\tilde{\Phi}^\rho$ and $\tilde{\tilde{\Phi}}^\rho$ fields, the Lagrangian reads

$$t\mathcal{L}_{\text{chiral}} = \sum_\rho \left[\tilde{\tilde{\Phi}}^\rho \left(-\nabla_\mu \nabla^\mu + \left(\rho(\sigma_0) + i\frac{\Delta-1}{R} \right)^2 + \frac{1}{R^2} \right) \tilde{\Phi}^\rho + \tilde{F}^\rho \tilde{F}^\rho \right. \\ \left. + \tilde{\psi}^\rho \left(-i\gamma^\mu \nabla_\mu + i\rho(\sigma_0) - \frac{2\Delta-1}{2R} \right) \tilde{\psi}^\rho \right] \quad (\text{D.25})$$

and it is invariant under the following supersymmetry transformations

$$\begin{aligned} \delta\tilde{\Phi}^\rho &= \bar{\epsilon}\tilde{\psi}^\rho, \\ \delta\tilde{\tilde{\Phi}}^\rho &= \epsilon\tilde{\psi}^\rho, \\ \delta\tilde{\psi}^\rho &= i\gamma^\mu \epsilon \nabla_\mu \tilde{\Phi}^\rho + i\epsilon\sigma_0\tilde{\Phi}^\rho - \frac{\Delta}{R}\epsilon\tilde{\Phi}^\rho + \bar{\epsilon}\tilde{F}^\rho, \\ \delta\tilde{\tilde{\psi}}^\rho &= i\gamma^\mu \bar{\epsilon} \nabla_\mu \tilde{\tilde{\Phi}}^\rho + i\tilde{\tilde{\Phi}}^\rho\sigma_0\bar{\epsilon} - \frac{\Delta}{R}\bar{\epsilon}\tilde{\tilde{\Phi}}^\rho + \tilde{F}^\rho\epsilon, \\ \delta\tilde{F}^\rho &= \epsilon(i\gamma^\mu \nabla_\mu \tilde{\psi}^\rho - i\sigma_0\tilde{\psi}^\rho) + \frac{1}{2R}(2\Delta-1)\epsilon\tilde{\psi}^\rho, \\ \delta\tilde{\tilde{F}}^\rho &= \bar{\epsilon}(i\gamma^\mu \nabla_\mu \tilde{\tilde{\psi}}^\rho - i\tilde{\tilde{\psi}}^\rho\sigma_0) + \frac{1}{2R}(2\Delta-1)\bar{\epsilon}\tilde{\tilde{\psi}}^\rho. \end{aligned} \quad (\text{D.26})$$

The one-loop contribution of the chiral multiplet that depend on σ_0 is given by

$$Z_{1\text{-loop}}^{\text{chiral}}(\sigma_0) = \prod_\rho \frac{\det_\rho \left(-i\gamma^\mu \nabla_\mu + i\rho(\sigma_0) - \frac{2\Delta-1}{2R} \right)}{\det_\rho \left(-\nabla_\mu \nabla^\mu + \left(\rho(\sigma_0) + i\frac{\Delta-1}{R} \right)^2 + \frac{1}{R^2} \right)}, \quad (\text{D.27})$$

where $\det_\rho(\mathcal{O})$ is computed on the space of fields that satisfy the boundary conditions (D.22).

As in the case of the vector multiplet we have to consider spherical harmonics with modified periodicity, discussed in appendix E. Our result is

$$Z_{1\text{-loop}}^{\text{chiral}}(\sigma_0) = \prod_{n=1}^{\infty} \prod_\rho \left(\frac{n+1-\Delta+i\rho(R\sigma_0+iH)}{n-1+\Delta-i\rho(R\sigma_0+iH)} \right)^n = \prod_\rho s_{b=1}(i-i\Delta-\rho(R\sigma_0+iH)), \quad (\text{D.28})$$

where $s_b(x)$ is the double sine function. There are many subtleties in this expression which are discussed in section 3.3.

E Spherical harmonics with non-standard periodicity

As explained in appendix D we are interested in computing the spectrum of operators using a basis that does not satisfy the standard periodicity condition. In particular, we consider eigenfunctions $\Phi(\theta, \phi, \psi)$ that satisfy

$$\Phi(e^{2\pi i}u, v) = e^{2\pi i\alpha(H)}\Phi(u, v) = e^{2\pi i\eta}\Phi(u, v). \quad (\text{E.1})$$

Here $\alpha(H) = \eta$ is the value of H for one of the weights, appropriate for a field in the adjoint representation. For other representations it is replaced by $\rho(H)$, which to avoid clutter we will also denote by η .

E.1 Scalar harmonics

Let us recall the construction of the usual scalar harmonics on \mathbb{S}^3 . The scalar Laplacian $-\nabla^2$ can be expressed in terms of the $SU(2)_L$ or $SU(2)_R$ angular momentum operators (A.3), (A.4) as

$$-\nabla^2 = \frac{4}{R^2}(L^L)^2 = \frac{4}{R^2}(L^R)^2. \quad (\text{E.2})$$

The spherical harmonics are classified by representations of $SU(2)_L \times SU(2)_R$ which obviously should have the same quadratic Casimir. The states $S(n, m, m')$ are labeled by three integers n, m, m' , such that $j = n/2 \geq |m|, |m'|$. n is the principal quantum number and m and m' are eigenvalues of the operators L_3^L and L_3^R .

These spherical harmonics can be written in terms of homogenous polynomials of degree n in the four coordinates u, \bar{u}, v and \bar{v} . We can construct a highest weight state³⁰

$$S(n, n/2, n/2) \propto u^n, \quad (\text{E.3})$$

which is annihilated by L_+^L and L_+^R . The full multiplet with $(n+1)^2$ states can be constructed by acting with L_-^L and L_-^R

$$S(n, m, m') \propto (L_-^L)^{n/2-m} (L_-^R)^{n/2-m'} u^n. \quad (\text{E.4})$$

The lowest weight state is reached by acting n times with both L_-^L and L_-^R and it has the form

$$S(n, -n/2, -n/2) \propto \bar{u}^n. \quad (\text{E.5})$$

We would like now to generalize this to functions which satisfy the periodicity condition (E.1). We require that the functions vanish at $u = 0$ and are regular at $v = 0$. m and m' are shifted by $\eta/2$ and a natural highest weight state is

$$S_H(n + \eta, \frac{n + \eta}{2}, \frac{n + \eta}{2}) \propto u^{n+\eta}. \quad (\text{E.6})$$

This function has the desired periodicity conditions, for $\eta > -n$ it vanishes at $u = 0$ and is regular at $v = 0$. Acting with the Laplacian on it gives

$$-\nabla^2 S_H \left(n + \eta, \frac{n + \eta}{2}, \frac{n + \eta}{2} \right) = \frac{(n + \eta)(n + \eta + 2)}{R^2} S_H \left(n + \eta, \frac{n + \eta}{2}, \frac{n + \eta}{2} \right). \quad (\text{E.7})$$

We can create other states solving the same equation by acting on this state with any number of L_-^L and L_-^R . For non-integer η they form a non-unitary representation of $SO(4)$ which is infinite dimensional, so it is not clear how many of the states in this representation we should include. We are not required to include the full representation, since the loop operator breaks the $SO(4)$ symmetry. Note that both L_-^L and L_-^R include a ∂_u derivative,

³⁰The normalization is not important for our purposes, so we will ignore it.

so acting with a total of k lowering operators will give a term proportional to $u^{n-k+\eta}$. For $k > n + \eta$ this mode is singular at $u = 0$. We rely on the analysis of supersymmetry multiplets in appendix E.4 to determine which states should be included.

In addition to the states which are descendants of the highest weight state there are more regular states that we can construct by starting with the modified lowest weight state

$$S_L \left(n - \eta, \frac{-n + \eta}{2}, \frac{-n + \eta}{2} \right) \propto \bar{u}^{n-\eta}. \tag{E.8}$$

Note that it has the same periodicity $e^{2\pi i \eta}$ under $u \rightarrow e^{2\pi i} u$, but a different Casimir

$$-\nabla^2 S_L \left(n - \eta, \frac{-n + \eta}{2}, \frac{-n + \eta}{2} \right) = \frac{(n - \eta)(n - \eta + 2)}{R^2} S_L \left(n - \eta, \frac{-n + \eta}{2}, \frac{-n + \eta}{2} \right). \tag{E.9}$$

Acting on this state up to k times with either L_+^L or L_+^R will generate $(k + 1)(k + 2)/2$ states. If $k < n - \eta$ then these states are regular at $u = 0$.

The space of scalar harmonics with non-trivial periodicity is equipped with a scalar product defined as for standard scalar harmonics

$$\langle S(n_1, m_1, m'_1), S(n_2, m_2, m'_2) \rangle = \int d\Omega \bar{S}(n_1, m_1, m'_1) S(n_2, m_2, m'_2) \tag{E.10}$$

where $d\Omega$ is the volume element on the \mathbb{S}^3 ,³¹ and \bar{S} is the complex conjugate of S . It results

$$\begin{aligned} \langle S_H(n_1, m_1, m'_1), S_H(n_2, m_2, m'_2) \rangle &\propto \delta_{n_1, n_2} \delta_{m_1, m_2} \delta_{m'_1, m'_2} \\ \langle S_L(n_1, m_1, m'_1), S_L(n_2, m_2, m'_2) \rangle &\propto \delta_{n_1, n_2} \delta_{m_1, m_2} \delta_{m'_1, m'_2} \\ \langle S_H(n_1, m_1, m'_1), S_L(n_2, m_2, m'_2) \rangle &= 0 \end{aligned} \tag{E.11}$$

where all the functions have the same deformation parameter η .

E.2 Vector harmonics

There are two sets of divergenceless vector harmonics on \mathbb{S}^3 [38, 39]: $\mathbf{V}^+(n, m, m')$ that form a representation $(\frac{n+1}{2}, \frac{n-1}{2})$ of the symmetry group $SU(2)_L \times SU(2)_R$, and $\mathbf{V}^-(n, m, m')$ that form a representation $(\frac{n-1}{2}, \frac{n+1}{2})$.³² The two sets satisfy

$$-\nabla_\mu \nabla^\mu \mathbf{V}^\pm(n, m, m') = (n + 1)^2 \mathbf{V}^\pm(n, m, m'), \tag{E.12}$$

and are related to each other by the parity operator P as

$$P \mathbf{V}^+(n, m, m') = (-1)^{n+1} \mathbf{V}^-(n, m', m), \tag{E.13}$$

where the action of the parity on the complex variables u, v is given by

$$P u = -u, \quad P v = -\bar{v}. \tag{E.14}$$

³¹For instance, in the torus fibration coordinates it is given by $d\Omega = dx^3 \sqrt{g} = R^3 \sin \vartheta \cos \vartheta d\vartheta d\varphi_1 d\varphi_2$.

³²We use bold characters for vectors in the four dimensional embedding space.

Complex conjugation acts as

$$\bar{\mathbf{V}}^\pm(n, m, m') = (-1)^{m+m'+1} \mathbf{V}^\pm(n, -m, -m'). \quad (\text{E.15})$$

In each of the $\mathbf{V}^+(n, m, m')$ and $\mathbf{V}^-(n, m, m')$ multiplet there are $n(n+2)$ states. To explicitly write the states it is convenient to consider the scalar product of the vector harmonics with an auxiliary vector $\mathbf{r}' = (u', v')$ defined in the embedding space \mathbb{C}^2 .³³ The highest weight state $\mathbf{V}^+(n, \frac{n+1}{2}, \frac{n-1}{2})$ is given (ignoring normalizations) by

$$\mathbf{r}' \cdot \mathbf{V}^+\left(n, \frac{n+1}{2}, \frac{n-1}{2}\right) \propto (-u)^{n-1}(vu' - uv') \quad (\text{E.16})$$

and the other $\mathbf{r}' \cdot \mathbf{V}^+(n, m, m')$ in the multiplet are obtained applying the annihilation operators $L_-^L + L_-'^L$ and $L_-^R + L_-'^R$, where $L_-'^L$ and $L_-'^R$ are generators acting on the auxiliary variables u' and v' . \mathbf{V}^- are gotten by acting with the parity operator P .

We would like to construct vector harmonics that satisfy the periodicity condition (E.1). As with the scalars, we take the (unnormalized) modified highest weight states

$$\begin{aligned} \mathbf{r}' \cdot \mathbf{V}_H^+\left(n + \eta, \frac{n+1+\eta}{2}, \frac{n-1+\eta}{2}\right) &\propto u^{n-1+\eta}(vu' - uv'), \\ \mathbf{r}' \cdot \mathbf{V}_H^-\left(n + \eta, \frac{n-1+\eta}{2}, \frac{n+1+\eta}{2}\right) &\propto u^{n-1+\eta}(\bar{v}u' - u\bar{v}'). \end{aligned} \quad (\text{E.17})$$

We can act on these states with lowering operators to create other states all of which satisfy

$$-\nabla_\mu \nabla^\mu \mathbf{V}^\pm(n + \eta, m, m') = (n + \eta + 1)^2 \mathbf{V}^\pm(n + \eta, m, m'). \quad (\text{E.18})$$

Let us examine their behavior at $u \rightarrow 0$. The lowest power of u in the descendants are

$$\begin{aligned} \mathbf{r}' \cdot \mathbf{V}_H^+\left(n + \eta, \frac{n+1+\eta}{2} - l, \frac{n-1+\eta}{2} - r\right) &\sim u' \bar{v}^l v^{r+1} (-\partial_u)^{l+r} u^{n-1+\eta} + \dots \\ \mathbf{r}' \cdot \mathbf{V}_H^-\left(n + \eta, \frac{n-1+\eta}{2} - \tilde{l}, \frac{n+1+\eta}{2} - \tilde{r}\right) &\sim u' \bar{v}^{\tilde{l}+1} v^{\tilde{r}} (-\partial_u)^{\tilde{l}+\tilde{r}} u^{n-1+\eta} + \dots \end{aligned} \quad (\text{E.19})$$

We find that for $0 < \eta < 1$ there are $\frac{n(n+1)}{2}$ modes of each of \mathbf{V}_H^\pm which are regular as $u \rightarrow 0$. The same statement holds true when considering singularities of the field strength rather than the gauge field.

In a similar fashion to before we can also start with the lowest weight states

$$\begin{aligned} \mathbf{r}' \cdot \mathbf{V}_L^+\left(n - \eta, \frac{-n-1+\eta}{2}, \frac{-n+1+\eta}{2}\right) &\propto \bar{u}^{n-1-\eta}(\bar{v}\bar{u}' - \bar{u}\bar{v}'), \\ \mathbf{r}' \cdot \mathbf{V}_L^-\left(n - \eta, \frac{-n+1+\eta}{2}, \frac{-n-1+\eta}{2}\right) &\propto \bar{u}^{n-1-\eta}(v\bar{u}' - \bar{u}v'), \end{aligned} \quad (\text{E.20})$$

and act on them with raising operators, giving eigenstate of the vector Laplacian

$$-\nabla_\mu \nabla^\mu \mathbf{V}^\pm(n - \eta, m, m') = (n - \eta + 1)^2 \mathbf{V}^\pm(n - \eta, m, m'). \quad (\text{E.21})$$

³³The scalar product for four dimensional vectors is defined as $A \cdot B = A_1 B_1 + A_2 B_2 + A_3 B_3 + A_4 B_4 = A_u B^u + A_{\bar{u}} B^{\bar{u}} + A_v B^v + A_{\bar{v}} B^{\bar{v}} = 2A_u B_{\bar{u}} + 2A_{\bar{u}} B_u + 2A_v B_{\bar{v}} + 2A_{\bar{v}} B_v$.

Their leading behavior at $u \rightarrow 0$ is

$$\begin{aligned} \mathbf{r}' \cdot \mathbf{V}_L^+ \left(n - \eta, \frac{-n - 1 + \eta}{2} + l, \frac{-n + 1 + \eta}{2} + r \right) &\sim \bar{u}' v^l \bar{v}^{r+1} (\partial_{\bar{u}})^{l+r} (\bar{u})^{n-1-\eta} + \dots \\ \mathbf{r}' \cdot \mathbf{V}_L^- \left(n - \eta, \frac{-n + 1 + \eta}{2} + \tilde{l}, \frac{-n - 1 + \eta}{2} + \tilde{r} \right) &\sim \bar{u}' v^{\tilde{l}+1} \bar{v}^{\tilde{r}} (\partial_{\bar{u}})^{\tilde{l}+\tilde{r}} (\bar{u})^{n-1-\eta} + \dots \end{aligned} \quad (\text{E.22})$$

E.3 Spinor harmonics

We now study the spectrum of the Dirac operator

$$-i\cancel{\nabla} = -i\gamma^\mu \nabla_\mu \quad (\text{E.23})$$

where the covariant derivative for spinors in the left-invariant frame is given by

$$\nabla_\mu = \partial_\mu + \frac{i}{2R} \gamma_\mu. \quad (\text{E.24})$$

In terms of \mathbf{L}^L , the left-invariant angular momentum generators (A.3) and the spin operator $S^a = \frac{1}{2}\gamma^a$ this is

$$-i\cancel{\nabla} = \frac{1}{R} \left(4\mathbf{L}^L \cdot \mathbf{S} + \frac{3}{2} \right). \quad (\text{E.25})$$

Considering $\mathbf{J} = \mathbf{L}^L + \mathbf{S}$, we have

$$-i\cancel{\nabla} = \frac{1}{R} \left(2(\mathbf{J}^2 - (\mathbf{L}^L)^2 - \mathbf{S}^2) + \frac{3}{2} \right). \quad (\text{E.26})$$

Given that \mathbf{S} has spin $s = 1/2$, then for \mathbf{L} with spin $l = n/2$, the spin of \mathbf{J} is $j = (n \pm 1)/2$ and we label the eigenstate as $\chi^\pm(n, m, m')$. It follows that for $\chi^+(n, m, m')$ states $n \geq 0$, and for $\chi^-(n, m, m')$ states $n \geq 1$. The eigenvalues are $\pm(j + 1)/R$, or explicitly

$$\begin{aligned} -i\cancel{\nabla} \chi^+(n, m, m') &= \frac{1}{R} (n + 3/2) \chi^+(n, m, m'), \\ -i\cancel{\nabla} \chi^-(n, m, m') &= -\frac{1}{R} (n + 1/2) \chi^-(n, m, m'). \end{aligned} \quad (\text{E.27})$$

The multiplicity for $j = (n + 1)/2$ is $(n + 2)(n + 1)$ and for $j = (n - 1)/2$ is $n(n + 1)$.

The highest and lowest states are given in [38] (in slightly different notations)

$$\begin{aligned} j = \frac{n+1}{2} : \quad \chi^+ \left(n, \frac{n+1}{2}, \frac{n}{2} \right) &= \begin{pmatrix} u^n \\ 0 \end{pmatrix} & \chi^+ \left(n, -\frac{n+1}{2}, -\frac{n}{2} \right) &= \begin{pmatrix} 0 \\ \bar{u}^n \end{pmatrix} \\ j = \frac{n-1}{2} : \quad \chi^- \left(n, \frac{n-1}{2}, \frac{n}{2} \right) &= \begin{pmatrix} \bar{v} u^{n-1} \\ u^n \end{pmatrix} & \chi^- \left(n, -\frac{n-1}{2}, -\frac{n}{2} \right) &= \begin{pmatrix} \bar{u}^n \\ v \bar{u}^{n-1} \end{pmatrix} \end{aligned} \quad (\text{E.28})$$

As with the other harmonics, we can deform them for $\eta \neq 0$ as

$$\begin{aligned}
 j = \frac{n+1}{2} : \quad & \chi_H^+ \left(n + \eta, \frac{n+1+\eta}{2}, \frac{n+\eta}{2} \right) = \begin{pmatrix} u^{n+\eta} \\ 0 \end{pmatrix} \\
 & \chi_L^+ \left(n - \eta, \frac{-n-1+\eta}{2}, \frac{-n+\eta}{2} \right) = \begin{pmatrix} 0 \\ \bar{u}^{n-\eta} \end{pmatrix} \\
 j = \frac{n-1}{2} : \quad & \chi_H^- \left(n + \eta, \frac{n-1+\eta}{2}, \frac{n+\eta}{2} \right) = \begin{pmatrix} \bar{v}u^{n-1+\eta} \\ u^{n+\eta} \end{pmatrix} \\
 & \chi_L^- \left(n - \eta, \frac{-n+1+\eta}{2}, \frac{-n+\eta}{2} \right) = \begin{pmatrix} \bar{u}^{n-\eta} \\ v\bar{u}^{n-1-\eta} \end{pmatrix}
 \end{aligned} \tag{E.29}$$

More modes are obtained by applying the annihilators $J_- = L_-^L + S_-$ and L_-^R to highest weight states or the creators $J_+ = L_+^L + S_+$ and L_+^R to the lowest weight states. All these modes are eigenstate of the Dirac operator (E.26) with eigenvalues

$$\begin{aligned}
 -i\nabla\chi_H^\pm(n+\eta, m, m') &= \frac{1}{R}(\pm(n+\eta+1)+1/2)\chi_H^\pm(n+\eta, m, m'), \\
 -i\nabla\chi_L^\pm(n-\eta, m, m') &= \frac{1}{R}(\pm(n-\eta+1)+1/2)\chi_L^\pm(n-\eta, m, m').
 \end{aligned} \tag{E.30}$$

As before, these states will become singular when acting with too many creation/annihilation operators. Some of these states have to be included to complete the supersymmetry multiplets analyzed now.

E.4 Supersymmetry multiplets

In order to determine which fluctuation modes one should include in the calculation of the determinant it is helpful to consider the multiplets they form under the supercharges preserved by the vortex loop operator. This is analyzed here and the final expressions for the determinants determined.

E.4.1 Chiral multiplet

The fluctuations of bosons and fermions can be expressed in terms of fields that satisfy non trivial boundary conditions, as discussed in previous sections. These fields are related by supersymmetry transformations (D.12), (D.26), that when written in terms of symmetry generators $D_\mu = \frac{2i}{R}L_\mu^L$ (A.3) and $\gamma_\mu = 2S_\mu$ are

$$\begin{aligned}
 \delta\tilde{\phi}^\rho &= \bar{\epsilon}\tilde{\psi}^\rho, \\
 \delta\tilde{\psi}^\rho &= -\frac{4}{R}L_3^L\tilde{\phi}^\rho S_3\epsilon - \frac{2}{R}L_+^L\tilde{\phi}^\rho S_-\epsilon - \frac{2}{R}L_-^L\tilde{\phi}^\rho S_+\epsilon + \left(i\sigma_0\tilde{\phi}^\rho - \frac{\Delta}{R}\tilde{\phi}^\rho \right)\epsilon + \tilde{F}^\rho\bar{\epsilon}, \\
 \delta\tilde{F}^\rho &= -\frac{4}{R}\epsilon(\mathbf{L}^L \cdot \mathbf{S})\tilde{\psi}^\rho - i\sigma_0\epsilon\tilde{\psi}^\rho + \frac{1}{R}(\Delta-2)\epsilon\tilde{\psi}^\rho, \\
 \delta\tilde{\bar{\phi}}^\rho &= \epsilon\tilde{\bar{\psi}}^\rho, \\
 \delta\tilde{\bar{\psi}}^\rho &= -\frac{4}{R}L_3^L\tilde{\bar{\phi}}^\rho S_3\bar{\epsilon} - \frac{2}{R}L_+^L\tilde{\bar{\phi}}^\rho S_-\bar{\epsilon} - \frac{2}{R}L_-^L\tilde{\bar{\phi}}^\rho S_+\bar{\epsilon} + \left(i\tilde{\bar{\phi}}^\rho\sigma_0 - \frac{\Delta}{R}\tilde{\bar{\phi}}^\rho \right)\bar{\epsilon} + \tilde{F}^\rho\epsilon, \\
 \delta\tilde{\bar{F}}^\rho &= -\frac{4}{R}\bar{\epsilon}(\mathbf{L}^L \cdot \mathbf{S})\tilde{\bar{\psi}}^\rho - i\bar{\epsilon}\tilde{\bar{\psi}}^\rho\sigma_0 + \frac{1}{R}(\Delta-2)\bar{\epsilon}\tilde{\bar{\psi}}^\rho.
 \end{aligned} \tag{E.31}$$

For the supersymmetry preserved by the loop operators (2.6), the parameter ϵ has spin $+1/2$ and $\bar{\epsilon}$ spin $-1/2$, therefore $\gamma_+\epsilon = \gamma_-\bar{\epsilon} = 0$. They can be written as $\epsilon = \begin{pmatrix} \epsilon_1 \\ 0 \end{pmatrix}$ and $\bar{\epsilon} = \begin{pmatrix} 0 \\ \epsilon_2 \end{pmatrix}$ and a few terms of (E.31) drops out.

Focusing on specific components ρ of the fluctuation fields with $\rho(H) = \eta$, they can be expanded in the harmonic bases as

$$\begin{aligned}
 \tilde{\phi} &= \sum_{n,m,m'} \phi_{n+\eta,m,m'}^H S_H(n+\eta, m, m') + \sum_{n,m,m'} \phi_{n-\eta,m,m'}^L S_L(n-\eta, m, m'), \\
 \tilde{\psi} &= \sum_{n,m,m'} \psi_{n+\eta,m,m'}^{H+} \chi_H^+(n+\eta, m, m') + \sum_{n,m,m'} \psi_{n+\eta,m,m'}^{H-} \chi_H^-(n+\eta, m, m'), \\
 &\quad + \sum_{n,m,m'} \psi_{n-\eta,m,m'}^{L+} \chi_L^+(n-\eta, m, m') + \sum_{n,m,m'} \psi_{n-\eta,m,m'}^{L-} \chi_L^-(n-\eta, m, m'), \\
 \tilde{F} &= \sum_{n,m,m'} F_{n+\eta,m,m'}^H S_H(n+\eta, m, m') + \sum_{n,m,m'} F_{n-\eta,m,m'}^L S_L(n-\eta, m, m').
 \end{aligned} \tag{E.32}$$

The expansions of $\tilde{\phi}$, $\tilde{\psi}$ and \tilde{F} are similar and from orthogonality of the states and the eigenvalues calculated in the previous sections (E.7), (E.30) one sees that the action (D.25) is³⁴

$$\begin{aligned}
 \mathcal{S}_{\text{chiral}} &= \sum_{n,m,m'} \left(\frac{(n+\eta)(n+\eta+2) + (R\rho(\sigma_0) + i(\Delta-1))^2 + 1}{R^2} \bar{\phi}_{n+\eta,-m,-m'}^L \phi_{n+\eta,m,m'}^H \right. \\
 &\quad \left. + \frac{(n-\eta)(n-\eta+2) + (R\rho(\sigma_0) + i(\Delta-1))^2 + 1}{R^2} \bar{\phi}_{n-\eta,-m,-m'}^H \phi_{n-\eta,m,m'}^L \right) \\
 &+ \sum_{n,m,m',\pm} \left(\frac{(\pm(n+\eta+1) + iR\rho(\sigma_0) - \Delta + 1)}{R} \bar{\psi}_{n+\eta,-m,-m'}^{L\pm} \psi_{n+\eta,m,m'}^{H\pm} \right. \\
 &\quad \left. + \frac{(\pm(n-\eta+1) + iR\rho(\sigma_0) - \Delta + 1)}{R} \bar{\psi}_{n-\eta,-m,-m'}^{H\pm} \psi_{n-\eta,m,m'}^{L\pm} \right) \\
 &+ \sum_{n,m,m'} (\bar{F}_{n+\eta,-m,-m'}^L F_{n+\eta,m,m'}^H + \bar{F}_{n-\eta,-m,-m'}^H F_{n-\eta,m,m'}^L).
 \end{aligned} \tag{E.33}$$

Note that $\bar{\phi}$ is in the conjugate representation to $\tilde{\phi}$, so the allowed values of η , which are the eigenvalues of the weights ρ have the opposite signs. This matches with the fact that the shift of n in the states arising from the highest and lowest weight states have the opposite signs.

To see the supermultiplet structure we can plug the expansion (E.32) into (E.31). If we project the variation $\delta\tilde{\phi}^\rho$ into eigenstates of the total angular momentum \mathbf{L}^2 , L_3^L and L_3^R and find that

$$\begin{aligned}
 \delta\phi_{n+\eta,m,m'}^H &\sim \psi_{n+\eta,m+1/2,m'}^{H+} + \psi_{n+\eta,m+1/2,m'}^{H-}, \\
 \delta\phi_{n-\eta,m,m'}^L &\sim \psi_{n-\eta,m+1/2,m'}^{L+} + \psi_{n-\eta,m+1/2,m'}^{L-}.
 \end{aligned} \tag{E.34}$$

In the last expression we ignored numerical factors and assumed the states on the right hand side exist.

³⁴This result follows from the relation $\bar{S}_{H,L}(n+\eta, m, m') = S_{L,H}(n+\eta, -m, -m')$ and similar relations for the other type of harmonics, and the scalar products discussed in the previous sections.

Likewise, when projecting $\delta\tilde{\psi}$ on eigenstates of $\mathbf{L} \cdot \mathbf{S}$, J_3^L and L_3^R we find

$$\delta\psi_{n+\eta,m+1/2,m'}^{H\pm} \sim \phi_{n+\eta,m,m'}^H + F_{n+\eta,m+1,m'}^H \quad (\text{E.35})$$

(and likewise for $\psi^{L\pm}$). The variation of the modes of F give back the same modes ψ as above.

We therefore conclude that the states

$$\{\phi_{n+\eta,m,m'}^H, \psi_{n+\eta,m+1/2,m'}^{H+}, \psi_{n+\eta,m+1/2,m'}^{H-}, F_{n+\eta,m+1,m'}^H\} \quad (\text{E.36})$$

are multiplets of the unbroken supersymmetry and likewise

$$\{\phi_{n-\eta,m,m'}^L, \psi_{n-\eta,m+1/2,m'}^{L+}, \psi_{n-\eta,m+1/2,m'}^{L-}, F_{n-\eta,m+1,m'}^L\}. \quad (\text{E.37})$$

For each such multiplet there is another multiplet of the barred fields, which couple to them in the action (E.33). The contribution of each multiplet in (E.36) to the determinant is

$$\frac{(n+\eta+2+iR\rho(\sigma_0)-\Delta)(-n-\eta+iR\rho(\sigma_0)-\Delta)}{(n+\eta)(n+\eta+2)+(R\rho(\sigma_0)+i(\Delta-1))^2+1} = -1. \quad (\text{E.38})$$

So to minus signs, which we will not try to keep track of, the determinant is trivial.

The only exception to this statement is when the full multiplet does not exist, rather it gets shortened, in which case the determinant is nontrivial.

The largest value of m for which the state $\phi_{n+\eta,m,m'}^H$ exists is $m = \frac{n+\eta}{2}$. In these cases the multiplets get shortened, as the states $\psi_{n+\eta,\frac{n+1+\eta}{2},m'}^{H-}$ and $F_{n+\eta,1+\frac{n+\eta}{2},m'}^H$ do not exist. Likewise there is a state $\psi_{n-\eta,-\frac{n+1-\eta}{2}}^{L+}$ but no modes ϕ^L and ψ^{L-} with the relevant quantum numbers, only F^L . The shortened multiplets are therefore associated to $m = \frac{n+\eta}{2}$ and $m = -\frac{n-\eta}{2} - 1$ and are respectively given by

$$\begin{aligned} & \left\{ \phi_{n+\eta,\frac{n+\eta}{2},m'}^H, \psi_{n+\eta,\frac{n+1+\eta}{2},m'}^{H+} \right\}, \\ & \left\{ \psi_{n-\eta,-\frac{n+1-\eta}{2},m'}^{L+}, F_{n-\eta,-\frac{n-\eta}{2},m'}^L \right\}. \end{aligned} \quad (\text{E.39})$$

Of course a similar statement applies to $\bar{\phi}$, $\bar{\psi}^\pm$ and \bar{F} .

Each of the multiplets on the first line of (E.39) contributes to the determinant a factor of

$$\frac{(n+\eta+2+iR\rho(\sigma_0)-\Delta)}{(n+\eta)(n+\eta+2)+(R\rho(\sigma_0)+i(\Delta-1))^2+1} = \frac{1}{n+\eta-iR\rho(\sigma_0)+\Delta}, \quad (\text{E.40})$$

and each multiplet on the second line

$$n-\eta+2+iR\rho(\sigma_0)-\Delta. \quad (\text{E.41})$$

For $\eta = 0$ there are $n+1$ copies of each of these multiplets, which we expect to not change when turning on $\eta \neq 0$. The only question is how many states get a shift $n \rightarrow n+\eta$ and how many $n \rightarrow n-\eta$, which is answered by the supersymmetry analysis above and the

assumption of minimal singularities. We finally find that the determinant for the full chiral multiplet including the full representation R is

$$\begin{aligned}
 Z_{1\text{-loop}}^{\text{chiral}}(\sigma_0) &= \prod_{\rho} \prod_{n=0}^{\infty} \left(\frac{n+2-\Delta+i\rho(R\sigma_0+iH)}{n+\Delta-i\rho(R\sigma_0+iH)} \right)^{n+1} \\
 &= \prod_{\rho} \prod_{n=1}^{\infty} \left(\frac{n+1-\Delta+i\rho(R\sigma_0+iH)}{n-1+\Delta-i\rho(R\sigma_0+iH)} \right)^n \\
 &= \prod_{\rho} s_{b=1}(i-i\Delta-\rho(R\sigma_0+iH))
 \end{aligned} \tag{E.42}$$

where $s_b(x)$ is the double sine function.

E.4.2 Vector multiplet

We can repeat the same analysis for the vector multiplet. We expand the fluctuation fields as³⁵

$$\begin{aligned}
 A'_{\mu} &= \sum_{n,m,m'} \left(A_{n+\eta,m,m'}^{H+} V_{H\mu}^+ + A_{n+\eta,m,m'}^{H-} V_{H\mu}^- + A_{n-\eta,m,m'}^{L+} V_{L\mu}^+ + A_{n-\eta,m,m'}^{L-} V_{L\mu}^- \right), \\
 \lambda' &= \sum_{n,m,m'} \left(\lambda_{n+\eta,m,m'}^{H+} \chi_H^+ + \lambda_{n+\eta,m,m'}^{H-} \chi_H^- + \lambda_{n-\eta,m,m'}^{L+} \chi_L^+ + \lambda_{n-\eta,m,m'}^{L-} \chi_L^- \right), \\
 \bar{\lambda}' &= \sum_{n,m,m'} \left(\bar{\lambda}_{n+\eta,m,m'}^{H+} \chi_H^+ + \bar{\lambda}_{n+\eta,m,m'}^{H-} \chi_H^- + \bar{\lambda}_{n-\eta,m,m'}^{L+} \chi_L^+ + \bar{\lambda}_{n-\eta,m,m'}^{L-} \chi_L^- \right), \\
 \sigma' &= \sum_{n,m,m'} \left(\sigma_{n+\eta,m,m'}^H S_H + \sigma_{n-\eta,m,m'}^L S_L \right), \\
 D' &= \sum_{n,m,m'} \left(D_{n+\eta,m,m'}^H S_H + D_{n-\eta,m,m'}^L S_L \right), \\
 c &= \sum_{n,m,m'} \left(c_{n+\eta,m,m'}^H S_H + c_{n-\eta,m,m'}^L S_L \right),
 \end{aligned} \tag{E.43}$$

and we included the ghost field c , which thus far has been ignored, but should be included in a full analysis of the theory.

Expanding the supersymmetry transformations (D.12) as

$$\begin{aligned}
 \delta \tilde{A}_{\mu}^{\alpha} &= i e_{\mu}^a (\bar{\epsilon} S_a \tilde{\lambda}^{\alpha} - \tilde{\lambda}^{\alpha} S_a \epsilon), \\
 \delta \tilde{\sigma}^{\alpha} &= \frac{1}{2} (\bar{\epsilon} \tilde{\lambda}^{\alpha} - \tilde{\lambda}^{\alpha} \epsilon), \\
 \delta \tilde{\lambda}^{\alpha} &= \frac{4}{R} \epsilon^{abc} e_a^{\nu} S_b \epsilon L_c^L \tilde{A}_{\nu}^{\alpha} - \frac{4}{R} S^a \epsilon L_a^L \tilde{\sigma}^{\alpha} + 2\alpha(\sigma_0) S^a \epsilon e_a^{\mu} \tilde{A}_{\mu}^{\alpha} - \left(\tilde{D}^{\alpha} + \frac{\tilde{\sigma}^{\alpha}}{R} \right) \epsilon, \\
 \delta \tilde{\bar{\lambda}}^{\alpha} &= \frac{4}{R} \epsilon^{abc} e_a^{\nu} S_b \bar{\epsilon} L_c^L \tilde{A}_{\nu}^{\alpha} + \frac{4}{R} S^a \bar{\epsilon} L_a^L \tilde{\sigma}^{\alpha} - 2\alpha(\sigma_0) S^a \bar{\epsilon} e_a^{\mu} \tilde{A}_{\mu}^{\alpha} + \left(\tilde{D}^{\alpha} + \frac{\tilde{\sigma}^{\alpha}}{R} \right) \bar{\epsilon}, \\
 \delta \tilde{D}^{\alpha} &= \frac{1}{2R} \bar{\epsilon} (4\mathbf{L}^L \cdot \mathbf{S} + \frac{3}{2}) \tilde{\lambda}^{\alpha} - \frac{1}{2R} \epsilon (4\mathbf{L}^L \cdot \mathbf{S} + \frac{3}{2}) \tilde{\bar{\lambda}}^{\alpha} - \left(\frac{i}{2} \alpha(\sigma_0) + \frac{1}{4R} \right) \bar{\epsilon} \tilde{\lambda}^{\alpha}, \\
 &\quad + \left(-\frac{i}{2} \alpha(\sigma_0) + \frac{1}{4R} \right) \tilde{\bar{\lambda}}^{\alpha} \epsilon,
 \end{aligned} \tag{E.44}$$

³⁵For brevity we omit the indices of the harmonic functions which match the modes they multiply.

one finds that the modes that belong to the same multiplets have quantum numbers

$$\left\{ A_{n+1+\eta, m, m'}^{H+}, A_{n-1+\eta, m, m'}^{H-}, \sigma_{n+\eta, m, m'}^H, D_{n+\eta, m, m'}^H, c_{n+\eta, m, m'}^H, \right. \\ \left. \lambda_{n+\eta, m+1/2, m'}^{H+}, \lambda_{n+\eta, m+1/2, m'}^{H-}, \bar{\lambda}_{n+\eta, m-1/2, m'}^{H+}, \bar{\lambda}_{n+\eta, m-1/2, m'}^{H-} \right\} \quad (\text{E.45})$$

and likewise for those arising from the lowest weight states. Though we did not include the ghosts in the explicit SUSY transformations, it is clear that they should appear in the off-shell multiplets as above. Note that for the vectors the principle quantum number n is shifted by ± 1 .

As with the chiral multiplet, a full multiplet contributes nothing to the determinant of the associated kinetic operators, and the only multiplets which contribute are shortened ones.³⁶

The shortest multiplets for the H modes are obtained with $m = \frac{n+\eta}{2} + 1$ and for the L modes with $m = -\frac{n-\eta}{2} - 1$. They are of the form

$$\left\{ A_{n+1+\eta, 1+\frac{n+\eta}{2}, m'}^{H+}, \bar{\lambda}_{n+\eta, \frac{n+1+\eta}{2}, m'}^{H+} \right\}, \\ \left\{ A_{n+1-\eta, -1-\frac{n-\eta}{2}, m'}^{L+}, \lambda_{n-\eta, -\frac{n+1-\eta}{2}, m'}^{L+} \right\}. \quad (\text{E.46})$$

The action (D.11) couples a mode arising from the highest weight state and a mode from the lowest weight state with opposite weights $\pm\alpha$, hence with opposite signs of η . Thus a pair of short multiplets as above is coupled by (cf., (E.18), (E.30))

$$\left((n+2+\alpha(H))^2 + R^2\alpha(\sigma_0)^2 \right) A_{n+1+\eta, 1+\frac{n+\eta}{2}, m'}^{H+} A_{n+1+\eta, -1-\frac{n-\eta}{2}, m'}^{L+} \\ - (n+2+i\alpha(R\sigma_0 - iH)) \bar{\lambda}_{n+\eta, \frac{n+1+\eta}{2}, m'}^{H+} \lambda_{n+\eta, -\frac{n+1-\eta}{2}, m'}^{L+}. \quad (\text{E.47})$$

The contribution of such pair of multiplets to the determinant is therefore

$$-\frac{1}{n+2-i\alpha(R\sigma_0 + iH)}, \quad (\text{E.48})$$

Other longer, but still short multiplets are

$$\left\{ A_{n+1+\eta, \frac{n+\eta}{2}, m'}^{H+}, \sigma_{n+\eta, \frac{n+\eta}{2}, m'}^H, D_{n+\eta, \frac{n+\eta}{2}, m'}^H, c_{n+\eta, \frac{n+\eta}{2}, m'}^H, \right. \\ \left. \lambda_{n+\eta, \frac{n+1+\eta}{2}, m'}^{H+}, \bar{\lambda}_{n+\eta, \frac{n-1+\eta}{2}, m'}^{H+}, \bar{\lambda}_{n+\eta, \frac{n-1+\eta}{2}, m'}^{H-} \right\} \\ \left\{ A_{n+1-\eta, -\frac{n-\eta}{2}, m'}^{L+}, \sigma_{n-\eta, -\frac{n-\eta}{2}, m'}^L, D_{n-\eta, -\frac{n-\eta}{2}, m'}^L, c_{n-\eta, -\frac{n-\eta}{2}, m'}^L, \right. \\ \left. \lambda_{n-\eta, -\frac{n-1-\eta}{2}, m'}^{L+}, \lambda_{n-\eta, -\frac{n-1-\eta}{2}, m'}^{L-}, \bar{\lambda}_{n-\eta, -\frac{n+1-\eta}{2}, m'}^{L+} \right\}. \quad (\text{E.49})$$

These together give (ignoring overall signs)

$$(n - i\alpha(R\sigma_0 + iH)). \quad (\text{E.50})$$

³⁶Note that the ghost c cancels the contribution of the scalar σ .

As before we assume that the number of short multiplets is the same as for $\eta = 0$, which gives $n + 1$. Multiplying (E.48) and (E.50) each with multiplicity $n + 1$ gives

$$\begin{aligned}
 Z_{1\text{-loop}}^{\text{vector}}(\sigma_0) &= \prod_{\alpha} \prod_{n=0}^{\infty} \left(\frac{n - i\alpha(R\sigma_0 + iH)}{n + 2 - i\alpha(R\sigma_0 + iH)} \right)^{n+1} \\
 &= \prod_{\alpha>0} \prod_{n=0}^{\infty} \left(\frac{n^2 + \alpha(R\sigma_0 + iH)^2}{(n + 2)^2 + \alpha(R\sigma_0 + iH)^2} \right)^{n+1} \\
 &= \prod_{\alpha>0} \left(\alpha(R\sigma_0 + iH)^2 \prod_{n=1}^{\infty} (n^2 + \alpha(R\sigma_0 + iH)^2)^2 \right).
 \end{aligned}
 \tag{E.51}$$

The first term on the last line arises from the numerator of the $n = 0$ case in the line above, all the rest comes from the numerator of the $n = 1$ case and the difference arising from shifting the index $n + 2 \rightarrow n$ in the denominator.

It should be pointed out that for $\eta = 0$ the product in the numerator starts at $n = 1$, not $n = 0$. The relevant multiplet is (E.49) with $n = 0$, where the state $\bar{\lambda}_{0, -\frac{1}{2}, 0}^{H-}$ does not exist, without which the determinant of this multiplet is one. For $0 < \eta < 1$ this mode is singular, but we think it should still be included to complete the multiplet, and because other modes that are equally singular are also included. Physically this mode should be thought of as an almost goldstino mode due to the broken supersymmetry induced by the vortex, which exists only in the vortex background.

Regularizing the infinite product in (E.51) we obtain

$$\begin{aligned}
 Z_{1\text{-loop}}^{\text{vector}}(\sigma_0) &= \prod_{\alpha>0} \alpha(R\sigma_0 + iH)^2 \frac{\sinh^2(\pi\alpha(R\sigma_0 + iH))}{(\pi\alpha(R\sigma_0 + iH))^2} \\
 &= \prod_{\alpha>0} \frac{1}{\pi^2} \sinh^2(\pi\alpha(R\sigma_0 + iH)).
 \end{aligned}
 \tag{E.52}$$

For $\alpha(H) = 0$ the product should start at $n = 1$ and the denominator is instead $\pi^2 \rightarrow \pi^2\alpha(\sigma_0)^2$.

F Index theory calculations for \mathbb{S}_b^3

In this appendix, we provide some details on the localization computation by the index theory for \mathbb{S}_b^3 . To avoid cluttering equations we will suppress the mass parameters that are associated with flavor symmetries. They can be easily restored by the replacement $\sigma \rightarrow \sigma + \text{mass}$.

F.1 Vanishing theorem

Let us show that in the limit $t \rightarrow +\infty$, the path integral weighted by $e^{-tQ \cdot V}$ with V given in (4.7) and (4.8) localizes to the configurations (4.12). We will also show that in the presence of a vortex loop, the fields acquire, on top of the smooth configurations (4.12), the appropriate singular parts that characterize the operator. Given our choice of V , this will be done by solving the equations $Q \cdot \text{fermion} = 0$.

For a vector multiplet, noting that our SUSY parameters (4.4) satisfy $\epsilon = C^{-1}\bar{\epsilon}^*$, let us compute³⁷

$$\begin{aligned} 0 = C^{-1}(Q\bar{\lambda})^* &= C^{-1}\left(-\frac{1}{2}\gamma^{\mu\nu}\bar{\epsilon}F_{\mu\nu} + D\bar{\epsilon} - i\gamma^\mu\bar{\epsilon}D_\mu\sigma + \frac{1}{Rf}\sigma\bar{\epsilon}\right)^* \\ &= -\frac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} + D^\dagger\epsilon - i\gamma^\mu\epsilon D_\mu\sigma + \frac{1}{Rf}\sigma\epsilon. \end{aligned} \quad (\text{F.1})$$

and compare it with

$$0 = \delta\lambda = -\frac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} - D\epsilon + i\gamma^\mu\epsilon D_\mu\sigma - \frac{1}{Rf}\sigma\epsilon. \quad (\text{F.2})$$

We find that

$$-\frac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} - i(\text{Im } D)\epsilon = 0, \quad (\text{Re } D)\epsilon - i\gamma^\mu\epsilon D_\mu\sigma + \frac{1}{Rf}\sigma\epsilon = 0. \quad (\text{F.3})$$

In the absence of a vortex loop, we take D to be real (hermitian), and hence obtain $F_{\mu\nu} = 0$ from the first equation. If we have vortex loops at $\vartheta = 0$ or $\pi/2$,³⁸ then $\text{Im } D$ develops delta function singularities there so that the first equation in (F.3) is satisfied.

Contracting the second equation with ϵ^\dagger , we deduce from the real part that $\text{Re } D + \frac{1}{Rf}\sigma = 0$. Then we have $0 = \|\gamma^\mu\epsilon D_\mu\sigma\|^2 = \|D_\mu\sigma\|^2$, thus $D_\mu\sigma = 0$.

For a chiral multiplet we compute

$$C^{-1}(Q\bar{\psi})^* = i\gamma^\mu\epsilon D_\mu\phi - i\sigma\phi\epsilon - \frac{\Delta}{Rf}\phi\epsilon - F\bar{\epsilon}. \quad (\text{F.4})$$

Comparing this with

$$0 = Q\psi = i\gamma^\mu\epsilon D_\mu\phi + i\epsilon\sigma\phi - \frac{\Delta}{Rf}\epsilon\phi + \bar{\epsilon}F, \quad (\text{F.5})$$

we find that $\sigma\phi = F = 0$, and we are left with

$$0 = i\gamma^\mu\epsilon D_\mu\phi - \frac{\Delta}{Rf}\epsilon\phi. \quad (\text{F.6})$$

Let us substitute the explicit expression for ϵ given in (4.4), and take linear combinations of the two components in (F.6). From one combination we get $\frac{ib}{R}D_{\varphi_2}\phi + \frac{i}{Rb}D_{\varphi_1}\phi - \frac{\Delta}{Rf}\phi = 0$, or, by taking gauge and R-symmetry backgrounds into account, obtain

$$[ib\partial_{\varphi_2} + ib^{-1}\partial_{\varphi_1} - (\Delta + bH_2 + b^{-1}H_1)]\phi = 0. \quad (\text{F.7})$$

A non-zero solution to this is the matter vortex configuration discussed in section 2.3. It is singular for generic values of Δ and does not contribute to the path integral unless we choose to insert the corresponding disorder operators. We will not include the contributions from the configuration in this paper.

³⁷The symbol $*$ acts as complex conjugation on Grassmann-even and -odd numbers, and as hermitian conjugation on fields: $\phi^* = \bar{\phi}, \bar{\phi}^* = \phi, \sigma^* = \sigma, D^* = D^\dagger$.

³⁸In the following, we will set $A_\mu = 0$ without a vortex loop, and $A_\vartheta = 0, A_{\varphi_1} = H_1, A_{\varphi_2} = H_2$ with vortex loops, where H_1 and H_2 are the vorticities of the operators at $\vartheta = 0$ and $\vartheta = \pi/2$.

F.2 Gauge fixing

In order to perform the one-loop calculation, we need to fix the gauge for the field configurations around the chosen saddle point. As usual, we introduce ghosts (c, \bar{c}) and a bosonic auxiliary field B and require that by the BRST charge Q_B acts as³⁹

$$\begin{aligned} Q_B \cdot c &= -\frac{i}{2}[c, c], \\ Q_B \cdot \bar{c} &= B, \\ Q_B \cdot B &= 0, \end{aligned} \tag{F.8}$$

on (c, \bar{c}, B) , and as⁴⁰

$$Q_B \cdot (\text{field}) = -\mathcal{G}(c) \cdot (\text{field}), \tag{F.9}$$

on the original fields. It is also standard to define the functional

$$V_{\text{gh}} = \int d^3x \sqrt{h} \text{Tr} \left(\bar{c} (G(\tilde{A}) + \frac{\xi}{2} B) \right) \tag{F.10}$$

with a choice of gauge fixing term G .⁴¹ Let us indicate by (0) objects defined at the saddle point, and by tilde the difference of the dynamical field from its background value. For example $\tilde{A} = A - H_1 d\varphi_1 - H_2 d\varphi_2$, $\tilde{\sigma} = \sigma - \sigma^{(0)}$.⁴² The standard choice of $G(\tilde{A})$ is $D_{(0)}^\mu \tilde{A}_\mu$, but we will make a slightly different choice below. In the familiar background field gauge, we would gauge-fix by adding a gauge-fixing Lagrangian $Q_B \cdot V_{\text{gh}}$. For localization, we need to modify $Q_B \cdot V_{\text{gh}}$ so that it is compatible with the supercharge Q . We do this by defining the Q transformations of (c, \bar{c}, B) as

$$\begin{aligned} Q \cdot c &= \tilde{\sigma} \bar{\epsilon} \epsilon + iv^\mu \tilde{A}_\mu, & Q \cdot \bar{c} &= 0, \\ Q \cdot B &= iv^\mu D_\mu^{(0)} \bar{c} + i[\sigma^{(0)}, \bar{c}]. \end{aligned} \tag{F.11}$$

One can check that on all fields including (c, \bar{c}, B) , $\hat{Q} \equiv Q + Q_B$ acts as the bosonic symmetry

$$\hat{Q}^2 = i\mathcal{L}_v + i\sigma^{(0)} \bar{\epsilon} \epsilon - v^\mu A_\mu^{(0)} + \frac{1}{2R}(b + b^{-1})\mathcal{R}, \tag{F.12}$$

which is the same as (4.5) except that the fields take values at the saddle point. With gauge-fixing, the localization procedure involves adding to the action the term $t\hat{Q} \cdot \hat{V}$ instead of $tQ \cdot V$, where $\hat{V} = V + V_{\text{gh}}$.

As explained in section 4.1, $e^{c\hat{Q}^2}$ represents the action on the fields of the group G , which is the product of H , K , the maximal torus of the gauge group, and the flavor $U(1)$. For $c = -iR$, the corresponding group elements are parameterized as⁴³

$$g = (h, t, e^{ia}, f) \in G \quad \longrightarrow \quad (e^{-\frac{i}{2}(b+b^{-1})}, e^{-\frac{i}{2}(b-b^{-1})}, e^{\hat{\sigma}}, e^{\frac{i}{2}\Delta(b+b^{-1})}) \in G_{\mathbb{C}}. \tag{F.13}$$

³⁹Also note that $[c, c] \equiv c_\alpha c_\beta [T^\alpha, T^\beta]$ if $c = c_\alpha T^\alpha$.

⁴⁰Here $\mathcal{G}(c) \cdot$ is the gauge transformation with parameter c . For example $Q_B A_\mu = D_\mu c$, $Q_B \lambda = -i\{c, \lambda\}$.

⁴¹As in the standard R_ξ gauge, ξ is an arbitrary parameter that does not affect the result of path integral.

⁴²In the text the saddle point value $\sigma^{(0)}$ is simply denoted as σ , and is only distinguished by the context.

⁴³This is obtained by identifying $(g \cdot \Phi)(x) \equiv g \cdot \Phi(g^{-1} \cdot x)$ with $e^{c\hat{Q}^2} \Phi(x)$, where Φ is an arbitrary field. The action of $H \times K$ on coordinates is defined as $(h, t) \cdot (e^{i\psi/2}, e^{i\phi/2}) = (he^{i\psi/2}, te^{i\phi/2})$.

F.3 Cohomological organization of fields

From now on we will set R to one. As in [3], we want to organize fields in the cohomological form. We begin with a vector multiplet on S_b^3 . Let us define⁴⁴

$$\Lambda_\mu \equiv \bar{\epsilon}\gamma_\mu\lambda + \epsilon\gamma_\mu\bar{\lambda}, \quad \Lambda \equiv \bar{\epsilon}\lambda + \epsilon\bar{\lambda}. \quad (\text{F.14})$$

On the space of fields, we define bosonic and fermionic coordinates (X_0, X_1) as⁴⁵

$$X_0 = (X_0^{\text{vec}}; X_0^{\text{chi}}) \equiv (\tilde{A}_\mu; \phi, \bar{\phi}), \quad X_1 = (X_1^{\text{vec}}; X_1^{\text{chi}}) \equiv (\Lambda, c, \bar{c}; \epsilon\psi, \bar{\epsilon}\bar{\psi}). \quad (\text{F.15})$$

The field σ is a dynamical equivariant parameter. The remaining fields are interpreted as the differentials $\hat{Q}X_0$ and $\hat{Q}X_1$. In the following, we pick a saddle point and expand the action up to the quadratic order. We can write the quadratic part $\hat{V}^{(2)}$ of \hat{V} in the form

$$\hat{V}^{(2)} = \begin{pmatrix} \hat{Q}X_0 & X_1 \end{pmatrix} \mathcal{D} \begin{pmatrix} X_0 \\ \hat{Q}X_1 \end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix}. \quad (\text{F.16})$$

Then we have $\hat{Q} \cdot \hat{V}^{(2)} = X_{\text{bos}}K_{\text{bos}}X_{\text{bos}} + X_{\text{ferm}}K_{\text{ferm}}X_{\text{ferm}}$, where

$$K_{\text{bos}} = \begin{pmatrix} -\hat{Q}^2 & \\ & 1 \end{pmatrix} \mathcal{D} + \mathcal{D}^T \begin{pmatrix} \hat{Q}^2 & \\ & 1 \end{pmatrix} \quad \text{and} \quad K_{\text{ferm}} = -\mathcal{D} \begin{pmatrix} 1 & \\ & \hat{Q}^2 \end{pmatrix} + \begin{pmatrix} 1 & \\ & -\hat{Q}^2 \end{pmatrix} \mathcal{D}^T \quad (\text{F.17})$$

can be viewed as infinite dimensional real matrices that are symmetric and anti-symmetric, respectively. The invariance of \hat{V} under \hat{Q}^2 implies that \mathcal{D} commutes with \hat{Q}^2 . Then

$$\begin{pmatrix} 1 & \\ & -\hat{Q}^2 \end{pmatrix} K_{\text{bos}} = K_{\text{ferm}} \begin{pmatrix} \hat{Q}^2 & \\ & 1 \end{pmatrix}.$$

The one-loop determinant is thus given, up to a sign, by

$$Z_{1\text{-loop}} = \left(\frac{\det K_{\text{ferm}}}{\det K_{\text{bos}}} \right)^{1/2} = \left(\frac{\det_{\text{Coker}D_{10}} \hat{Q}^2}{\det_{\text{Ker}D_{10}} \hat{Q}^2} \right)^{1/2} \quad (\text{F.18})$$

and is related to the equivariant index (4.11) by the rule (4.13).

The fermionic functional $\hat{V} = \hat{V}_{\text{vec}} + \hat{V}_{\text{chi}}$ is given by⁴⁶

$$\begin{aligned} \hat{V}_{\text{vec}} &= (Q\lambda)^\dagger\lambda + (Q\bar{\lambda})^\dagger\bar{\lambda} + V_{\text{gh}} \\ &= -F_{\nu\rho}v^\nu\Lambda^\rho - \frac{i}{2}v^\mu\epsilon_{\mu\nu\rho}F^{\nu\rho}\Lambda - iD_\mu\sigma\Lambda^\mu - D\Lambda - (\sigma/f)v^\mu\Lambda_\mu + \bar{c}\left(G(\tilde{A}) + \frac{\xi}{2}B\right) \end{aligned} \quad (\text{F.19})$$

⁴⁴Throughout this section, the symbol Λ denotes a component of the gaugino, and should not be confused with a gauge parameter in section B.2.

⁴⁵For the spinors (4.4) and vector (4.6), we have $\bar{\epsilon}\epsilon = 1$, $v_\mu\bar{\epsilon}\gamma^\mu = \bar{\epsilon}$, $v_\mu\epsilon\gamma^\mu = -\epsilon$, $v^\mu v_\mu = 1$.

⁴⁶It is useful to define $\epsilon^0 := \begin{pmatrix} \epsilon \\ \bar{\epsilon} \end{pmatrix}$ and $\epsilon^\mu := \begin{pmatrix} \gamma^\mu \\ \gamma^\mu \end{pmatrix} \epsilon_0$, which satisfy $\epsilon_m^\dagger \epsilon^n = 2\delta_m^n$ and $\epsilon^m \epsilon_m^\dagger = 2\mathbf{1}_{4 \times 4}$ for $m, n = 0, \dots, 3$. We then have $\Lambda_\mu = \epsilon_\mu^\dagger \begin{pmatrix} \lambda \\ -\bar{\lambda} \end{pmatrix}$ and $\Lambda = \epsilon_0^\dagger \begin{pmatrix} \lambda \\ -\bar{\lambda} \end{pmatrix}$. We also use the identities $\bar{\epsilon}\gamma^\mu\lambda = \frac{1}{2}\Lambda^\mu - \frac{i}{2}\epsilon^\mu_{\nu\rho}v^\nu\Lambda^\rho + \frac{1}{2}v^\mu\Lambda$, $\epsilon\gamma^\mu\bar{\lambda} = \frac{1}{2}\Lambda^\mu + \frac{i}{2}\epsilon^\mu_{\nu\rho}v^\nu\Lambda^\rho - \frac{1}{2}v^\mu\Lambda$, $\bar{\epsilon}\lambda = \frac{1}{2}v^\mu\Lambda_\mu + \frac{1}{2}\Lambda$, and $\epsilon\bar{\lambda} = -\frac{1}{2}v^\mu\Lambda_\mu + \frac{1}{2}\Lambda$.

and

$$\begin{aligned}
 \hat{V}_{\text{chi}} &= (Q\psi)^\dagger\psi + (Q\bar{\psi})^\dagger\bar{\psi} \\
 &= -iD_\mu\bar{\phi}(\bar{\epsilon}\gamma^\mu\psi)_\perp - iD_\mu\bar{\phi}v^\mu\bar{\epsilon}\psi - i\bar{\phi}(\sigma+m)\bar{\epsilon}\psi - \frac{\Delta}{f}\bar{\phi}\bar{\epsilon}\psi - F\epsilon\psi \\
 &\quad + i(\epsilon\gamma^\mu\bar{\psi})_\perp D_\mu\phi + i\epsilon\bar{\psi}v^\mu D_\mu\phi + i(\sigma+m)\phi\epsilon\bar{\psi} + \frac{\Delta}{f}\phi\epsilon\bar{\psi} + F\bar{\epsilon}\bar{\psi}.
 \end{aligned} \tag{F.20}$$

The symbol \perp indicates the projection orthogonal to v^μ . We have $\bar{\epsilon}\gamma^\mu\psi = (\bar{\epsilon}\psi)v^\mu + (\bar{\epsilon}\gamma^\mu\psi)_\perp$.

F.4 Differential operator D_{10}

Given a differential operator \mathcal{D} on space X , its symbol $\sigma(\mathcal{D}; x, p)$ is a function of $x \in X$ and $p = p_\mu dx^\mu \in T_x^*X$, defined by replacing ∂_μ by ip_μ everywhere in \mathcal{D} , and collecting the terms that have the highest order in p . The operator \mathcal{D} is called elliptic if its symbol is invertible for any $x \in X$ and $p \neq 0$. When a compact Lie group G acts on X , there is a weaker notion of ellipticity. We say that \mathcal{D} is transversally elliptic if its symbol is invertible, at each $x \in X$, for all non-zero cotangent vectors $p \neq 0$ that are orthogonal to the G directions [22].⁴⁷ When \mathcal{D} is transversally elliptic with respect to the G -action, the equivariant index is well-defined as a distribution on G . In this section we will compute the differential operator D_{10} that appears in the fermionic functional $\hat{V}^{(2)}$, and show that it is transversally elliptic with respect to group G defined in (4.10).

Let us now study the differential operator D_{10} more closely. We begin with the vector multiplet. It is convenient to split the fluctuation $\tilde{A}_\mu \equiv A_\mu - A_\mu^{(0)}$ into the components parallel and orthogonal to the vector field v_μ : $\tilde{A}_\mu = a_\mu + v_\mu b$, $v^\mu a_\mu = 0$. The bosonic and fermionic coordinates (X_0, X_1) on the space of fields are defined in (F.15). A technical complication is that the ghost c appears with a derivative in

$$\Lambda_\mu = -2i\hat{Q}\tilde{A}_\mu + 2iD_\mu c. \tag{F.21}$$

Although $\hat{V}_{\text{vec}}^{(2)}$ in terms of the original fields involves only terms with a single derivative, several terms end up with two derivatives when we express $\hat{V}_{\text{vec}}^{(2)}$ in terms of X_0 , etc. Indeed, showing only the integrand, we find up to total derivatives⁴⁸

$$\begin{aligned}
 X_1^{\text{vec}} D_{10}^{\text{vec}} X_0^{\text{vec}} &= -2i[D_\nu(a_\mu + v_\mu b)](v^\nu D^\mu - v^\mu D^\nu)c \\
 &\quad - i\epsilon^{\mu\nu\rho}[D_\mu(a_\nu + v_\nu b)]v_\rho\Lambda + \bar{c}G(a + vb) \\
 &= \left(2ic \bar{c} \Lambda\right) \begin{pmatrix} (D^\mu v^\nu - D^\nu v^\mu)D_\nu v_\mu & (D^\mu v^\nu - D^\nu v^\mu)D_\nu \\ G(x, \partial)^\mu v_\mu & G(x, \partial)^\mu \\ -i\epsilon^{\mu\nu\rho}(D_\mu v_\nu)v_\rho & i\epsilon^{\mu\nu\rho}v_\rho D_\nu \end{pmatrix} \begin{pmatrix} b \\ a_\mu \end{pmatrix}. \tag{F.22}
 \end{aligned}$$

The operator is effectively a square matrix because a_μ has two independent components. Since the symbol is defined using the terms with most derivatives, superficially D_{10}^{vec} is

⁴⁷We say that p is orthogonal to the G directions if $p_\mu V^\mu = 0$ for any $V^\mu \partial_\mu$ whose flow is an action of G .

⁴⁸The square bracket $[\]$ indicates that the derivatives act only on the functions inside. If not in a square bracket, derivatives are understood to act on all the factors on the right.

neither elliptic nor transversally elliptic. We can, however, make a field redefinition so that D_{10}^{vec} is block diagonal, with one block being second order and the rest first order.⁴⁹

Let us introduce

$$\mathcal{D}_\mu = \frac{Rf}{2} \epsilon_{\mu\nu\rho} v^\nu D^\rho \quad (\text{F.23})$$

and take the gauge fixing term to be⁵⁰

$$G(\tilde{A}) = G^\mu \tilde{A}_\mu = (D^\mu + (D^\nu \mathcal{D}_\nu - v^\nu D_\nu) v^\mu) \tilde{A}_\mu. \quad (\text{F.24})$$

After some calculation, we find that the differential operator D_{10} for the vector multiplet can be block diagonalized:

$$\begin{aligned} & X_1^{\text{vec}} D_{10}^{\text{vec}} X_0^{\text{vec}} \\ &= \begin{pmatrix} 2ic & \bar{c} - 2iv^\nu D_\nu c & \Lambda \end{pmatrix} \begin{pmatrix} 2D^{[\mu} v^{\nu]} D_\nu v_\mu - D_\nu v^\nu G^\mu v_\mu & 0 \\ 0 & D^\mu \\ 0 & i\epsilon^{\mu\nu\rho} v_\rho D_\nu \end{pmatrix} \begin{pmatrix} b \\ a_\mu + \mathcal{D}_\mu b \end{pmatrix}. \end{aligned} \quad (\text{F.25})$$

Thus at the quadratic order in fluctuations, $b = v^\mu \tilde{A}_\mu$ and c decouple from the other combinations of fields in (F.25). The corresponding differential operator appears on the upper left corner of the matrix. Its symbol has determinant $p^2 - (p \cdot v)^2$, which vanishes for p parallel to v , but is non-zero for any non-zero p satisfying $p \cdot v = 0$. Since the vector field v given in (4.14) is a linear combination of the vector fields generating H and K , the operator for (b, c) is transversally elliptic, and the equivariant index is well-defined as a distribution on G . The equivariant index is however trivial because the differential operator maps the space of scalars to itself, and its kernel and cokernel are identical.

The differential operator in the lower right block of the matrix in (F.25) is first order, and its symbol has determinant $i(p^2 - (v \cdot p)^2)$. Thus the operator is again not elliptic, but is transversally elliptic.

For the chiral multiplet, we obtain from (F.20)

$$\begin{aligned} X_1^{\text{chi}} D_{10}^{\text{chi}} X_0^{\text{chi}} &= i(\epsilon\gamma^\mu \bar{\psi})_\perp D_\mu \phi + c.c. \\ &= i(\bar{\epsilon}\bar{\psi})(\epsilon\gamma^\mu \epsilon) D_\mu \phi + c.c. \end{aligned} \quad (\text{F.26})$$

where complex conjugation $*$ acts formally as $\psi^* = -C\bar{\psi}$. The vector field that appears in (F.26) can be decomposed as $\epsilon\gamma^\mu \epsilon = e^{i(\varphi_1 + \varphi_2)}(w^\mu + iu^\mu)$, where w^μ and u^μ are both real.⁵¹ It can be checked that (u, v, w) form an orthonormal basis of the tangent space. The symbol σ of the differential operator $i\epsilon\gamma^\mu \epsilon D_\mu$ then satisfies $|\sigma|^2 = (w \cdot p)^2 + (u \cdot p)^2 = p^2 - (v \cdot p)^2$, so the operator is not elliptic but is transversally elliptic. At the north and the south poles ($\vartheta = 0$ and $\pi/2$) of the base \mathbb{S}^2 , the operator acts as the Dolbeault operator in the directions orthogonal to v .

⁴⁹The same issue arises in localization in four dimensions. The authors of [3] and [40], working with momenta rather than derivatives, showed that highest order terms can be block diagonalized. Here we are pedantic and demonstrate that the whole differential operator can be block diagonalized.

⁵⁰See [40] for a similar choice of the gauge fixing term G in the four-dimensional case.

⁵¹Explicitly, $w = (Rf)^{-1} \partial_\vartheta$, $u = (Rb)^{-1} \cot \vartheta \partial_{\varphi_1} - R^{-1} b \tan \vartheta \partial_{\varphi_2}$.

F.5 Computation of $\text{ind } D_{10}$

Let $T_G^*X|_p$ be the space of cotangent vectors at $p \in X$ conormal to the G -orbit. Let T_G^*X be the collection of $T_G^*X|_p$ for all $p \in X$, and let $\pi : T_G^*X \rightarrow X$ be the projection. The symbol of a G -equivariant (pseudo)differential operator that maps a section of E_0 to a section of E_1 then defines a map $\pi^*E_0 \rightarrow \pi^*E_1$. The operator and its symbol are by definition transversally elliptic if the map is invertible on T_G^*X away from the zero section. Such a symbol defines a class in $K_G(T_G^*X)$, and the index depends only on this class [22].

We wish to compute the index of D_{10} . The index is determined by the homotopy class of the symbol of D_{10} . The key tool, when a factor H in $G = H \times K$ acts freely on X , is the following theorem (Corollary 3.3 of [22]).

Let us label the irreducible representations of H by R , and let E_R^* be the dual of the vector bundle E_R over X/H induced from R .⁵² For a symbol $\sigma_0 : \pi^*E_0 \rightarrow \pi^*E_1$ on X/H transversally elliptic with respect to K , the pull-back $p^*\sigma_0$ is a transversally elliptic symbol with respect to $H \times K$, where $p : X \rightarrow X/H$ is the projection. The index of $p^*\sigma_0$ is given by

$$\text{ind}_{(h,k)}(p^*\sigma_0) = \sum_R \text{ind}_k(\sigma_0 \otimes E_R^*) \cdot \chi_R(h), \quad (h, k) \in H \times K, \quad (\text{F.27})$$

where R labels irreducible representations of H , E_R^* is the dual of the vector bundle E_R induced from R , and χ_R is the character of R .

Let us apply this theorem to $X = \mathbb{S}_b^3$. Since we only need to know the K theory class of the symbol and H acts freely, let us set p_ψ to zero. Then $p^2 - (p \cdot v)^2$ cannot vanish unless $p = 0$. Thus the symbol of D_{10} reduces to an elliptic symbol σ_0 on $X/H = \mathbb{S}^2$, and we have $\sigma(D_{10}) = p^*\sigma_0$. For the vector multiplet, we obtain the de Rham complex on \mathbb{S}^2 with a degree shifted by one. The equivariant index of D_{10}^{vec} , obtained from (F.27) and the Atiyah-Bott formula, is

$$\text{ind}_g(D_{10}^{\text{vec}}) = - \sum_{n \in \mathbb{Z}} (t^n + t^{-n}) \chi_{\text{adj}}(e^{ia}) h^n, \quad (\text{F.28})$$

where $h \in H = \text{U}(1)$, $t \in K = \text{U}(1)$, and χ_{adj} is the character in the adjoint representation of the gauge group. The identification (F.13) leads to (4.22) and (4.23). For the chiral multiplet $\text{ind}_g D_{10}^{\text{chi}} = \text{ind}_g D_{10, \mathbb{C}}^{\text{chi}} + \text{ind}_{g^{-1}} D_{10, \mathbb{C}}^{\text{chi}}$, where $D_{10, \mathbb{C}}^{\text{chi}}$ reduces to the Dolbeault complex on \mathbb{S}^2 , whose index is

$$\sum_{n \in \mathbb{Z}} \left(\frac{t^n}{1 - t^{-2}} + \frac{t^{-n}}{1 - t^2} \right) \chi_R(e^{ia}) h^n f \quad (\text{F.29})$$

with χ_R the character of the matter representation R of the gauge group. Substitution (F.13) gives (4.15) and (4.16). Since the reduced symbol is elliptic, its index, the bracket in (F.29), is a polynomial. Thus there is no ambiguity in the index as long as we expand (4.15) and (4.16), we need to expand both in t or t^{-1} , so that the sum for fixed n is a finite polynomial.

⁵²The induced bundle E_R is the quotient of $X \times R$ by the H -action.

G Index theory calculations for $\mathbb{S}^1 \times \mathbb{S}^2$

In this appendix we repeat the steps in appendix F for $\mathbb{S}^1 \times \mathbb{S}^2$.

First let us determine the saddle point configurations that contribute in the localization calculation. For the vector multiplet, we again compare $Q\lambda = 0$ with a complex conjugate of $(Q\bar{\lambda})^* = 0$. Defining $B_\mu \equiv \frac{1}{2}\epsilon_\mu{}^{\nu\rho}F_{\nu\rho}$, we obtain $Q\lambda = \mathcal{E}_1 + \mathcal{E}_2$ and $\gamma_\tau C^{-1}(Q\bar{\lambda})^* = \mathcal{E}_1 - \mathcal{E}_2$, where

$$\begin{aligned}\mathcal{E}_1 &\equiv -i\gamma_\tau\epsilon(\text{Re}B_\tau) + (\text{Im}B^j)\gamma_j\epsilon - i(\text{Im}D)\epsilon + i\gamma^j\epsilon D_j\sigma - i\sigma\gamma_\tau\epsilon, \\ \mathcal{E}_2 &\equiv (\text{Im}B_\tau)\gamma^\tau\epsilon - i\gamma^j\epsilon(\text{Re}B_j) - (\text{Re}D)\epsilon + i\gamma^\tau\epsilon D_\tau\sigma.\end{aligned}\tag{G.1}$$

Here we used the properties $C^{-1}\epsilon^* = \gamma_\tau\bar{\epsilon}$, $C^{-1}\bar{\epsilon}^* = \gamma_\tau\epsilon$, $D_\mu\epsilon = -\frac{1}{2}\gamma_\mu\gamma_\tau\epsilon$, and $D_\mu\bar{\epsilon} = \frac{1}{2}\gamma_\mu\gamma_\tau\bar{\epsilon}$. Thus $\mathcal{E}_{1,2}$ must vanish separately. Let us set $\text{Im}D = \text{Im}B_\mu = 0$ here; turning on the imaginary parts corresponds to inserting vortex loops. Contract $\mathcal{E}_2 = 0$ with ϵ^\dagger . The real part of the equation implies that $\text{Re}D = 0$. Let us also consider contracting $\mathcal{E}_2 = 0$ with $\bar{\epsilon}^T C$. Since $\bar{\epsilon}\gamma^\mu\epsilon = (1, 0, -i)$ with $\mu = \tau, \theta, \varphi$, the real part implies that $B_\varphi = 0$, and the imaginary part implies that $D_\tau\sigma = 0$. It then follows that $B_\theta = 0$. Next contract $\mathcal{E}_1 = 0$ with $\bar{\epsilon}^T C$. We find that $D_\varphi\sigma = 0$, and that

$$B_\tau + \sigma = 0.\tag{G.2}$$

We also obtain $D_\theta\sigma = 0$. Thus we have $B_\tau + \sigma = 0$, where B_τ is the smooth part of B_τ . Diagonalizing σ , we find that $B_\tau = \frac{m}{2}$, $\sigma = -\frac{m}{2}$, where m is a quantized GNO charge. For the chiral multiplet, the same reasoning applied to $Q\psi = 0$ and $(Q\psi)^* = 0$ leads to

$$i\gamma^\tau\epsilon D_\tau\phi + F\bar{\epsilon} = 0, \quad i\sum_{j=\theta,\varphi}\gamma^j\epsilon D_j\phi + i\sigma\phi\epsilon - i\Delta\phi\gamma_\tau\epsilon = 0.\tag{G.3}$$

The first equation contracted with $\bar{\epsilon}^T C$ implies that $D_\tau\phi = F = 0$. Using the explicit expression (5.8), the remaining equations can be solved:

$$\phi = \beta \left(e^{\mp i\varphi} \tan \frac{\theta}{2} \right)^{m/2} (e^{-i\varphi} \sin \theta)^{-\Delta} e^{\mp \frac{i}{2} H^\pm \varphi},\tag{G.4}$$

where the upper and lower signs are respectively for the two patches U^\pm on \mathbb{S}^2 , and β is a constant. Again the configuration represents matter vortex loops, and we do not include the contributions from such a configuration in this paper. Thus the path integral localizes to the field configurations (5.9) in the absence of a vortex loop, and to the same configurations on top of singular backgrounds in the presence of vortex loops.

The BRST transformations and the localization action are the same as in the \mathbb{S}_b^3 case. In particular, the expressions (F.8)–(F.11) are valid. The square of \hat{Q}^2 is given precisely by the right hand side of (5.14). Thus for $c = i\beta$, the group element $g = e^{c\hat{Q}^2}$ is parameterized as⁵³

$$g = (h, t, e^{ia}, f) \in G \quad \longrightarrow \quad (e^{2\pi i}, e^{2\beta_2}, e^{ia}, e^{\beta_2\Delta}) \in \hat{G}_C,\tag{G.5}$$

where \hat{G}_C denotes the complexified universal covering of G . (So $e^{2\pi i}$ is non-trivial.)

⁵³The parameterization can be found as in footnote 43. The group action on coordinates is given by $(h, t) \cdot (e^{2\pi i\tau/\beta}, e^{i\varphi}) = (he^{2\pi i\tau/\beta}, te^{i\varphi})$.

Let us introduce $\varepsilon_0 := \begin{pmatrix} \varepsilon \\ -\bar{\varepsilon} \end{pmatrix}$, $\varepsilon_1 := \begin{pmatrix} \gamma_{\tilde{\tau}}\varepsilon \\ -\gamma_{\tilde{\tau}}\bar{\varepsilon} \end{pmatrix}$ and $\varepsilon_j := \begin{pmatrix} \gamma_j\varepsilon \\ \gamma_j\bar{\varepsilon} \end{pmatrix}$ with $j = 2, 3$, which satisfy $\varepsilon_m^\dagger \varepsilon^n = 2\delta_m^n$ and $\varepsilon^m \varepsilon_m^\dagger = \mathbf{21}_{4 \times 4}$ for $m, n = 0, \dots, 3$. We then define

$$\Lambda_m = \varepsilon_m^\dagger \begin{pmatrix} \gamma_{\tilde{\tau}}\lambda \\ \gamma_{\tilde{\tau}}\bar{\lambda} \end{pmatrix}$$

and take

$$X_0 = (X_0^{\text{vec}}; X_0^{\text{chi}}) \equiv (\tilde{A}_j, \tilde{\sigma}; \bar{\phi}, \phi), \quad X_1 = (X_1^{\text{vec}}; X_1^{\text{chi}}) \equiv (\Lambda_1, c, \bar{c}; \varepsilon\gamma_{\tilde{\tau}}\psi, \bar{\varepsilon}\gamma_{\tilde{\tau}}\bar{\psi}) \quad (\text{G.6})$$

as superspace coordinates. Gauge-fixing can be achieved with the localization action $\hat{V} = \hat{V}_{\text{vec}} + \hat{V}_{\text{chi}}$ given as

$$\hat{V}_{\text{vec}} \equiv (Q\lambda)^\dagger \lambda + (Q\bar{\lambda})^\dagger \bar{\lambda} + V_{\text{gh}} \quad (\text{G.7})$$

and

$$\begin{aligned} \hat{V}_{\text{chi}} &= (Q\psi)^\dagger \psi + (Q\bar{\psi})^\dagger \bar{\psi} \\ &= -i(\partial_\tau \bar{\phi} - i\bar{\phi}A_\tau^\dagger)\bar{\varepsilon}\psi + \varepsilon_\tau^{ij} D_i \bar{\phi} \bar{\varepsilon} \gamma_j \psi - i\sigma \bar{\phi} \bar{\varepsilon} \gamma_{\tilde{\tau}} \psi + i\Delta \bar{\phi} \bar{\varepsilon} \psi + \bar{F} \varepsilon \gamma_{\tilde{\tau}} \psi \\ &\quad - i(\varepsilon \bar{\psi})(\partial_\tau + iA_\tau^\dagger)\phi + \varepsilon_\tau^{ij} D_i \phi \varepsilon \gamma_j \bar{\psi} - i\phi \sigma \varepsilon \gamma_{\tilde{\tau}} \bar{\psi} - i\Delta \phi \varepsilon \bar{\psi} + F \bar{\varepsilon} \gamma_{\tilde{\tau}} \bar{\psi}. \end{aligned} \quad (\text{G.8})$$

Note that a vortex loop introduces a non-hermitian part to A_τ .

The functional \hat{V}_{vec} contains only first order differential operators acting on $(\lambda, \bar{\lambda}, c, \bar{c})$. When we express \hat{V}_{vec} in terms of $(X_{0,1}^{\text{vec}}, \hat{Q}X_{0,1}^{\text{vec}})$ to read off D_{10} , we find second order differentials because $D_\mu^{(0)}c$ appear in $\hat{Q}\tilde{A}_j$ and $\hat{Q}\tilde{\sigma}$. Thus the symbol determined by the highest order terms is, strictly speaking, degenerate everywhere. Instead of separating the first and the second order parts by block-diagonalizing D_{10} as in the \mathbb{S}_b^3 case, for $\mathbb{S}^1 \times \mathbb{S}^2$ we take an alternative approach, which we believe is more general. Namely, in order to compute the one-loop determinant around the saddle point (5.9), we consider the Gaussian functional integration of $e^{-t\hat{Q} \cdot V_u}$ with

$$V_u \equiv (1-u)\hat{V}_{\text{vec}}^{(2)} + uV' + \hat{V}_{\text{chi}}^{(2)}. \quad (\text{G.9})$$

Setting to zero the deformation parameter u gives back the original gauge-fixed action. If the bosonic part of $\hat{Q} \cdot V'$ is positive definite in directions transverse to the space of saddle point configurations, the path integral is independent of u , and can be evaluated at $u = 1$. As V' , we take

$$V' = (\hat{Q} \cdot X_1^{\text{vec}})^\dagger X_1^{\text{vec}} + (\hat{Q}^2 \cdot X_1^{\text{vec}})^\dagger \hat{Q} X_1^{\text{vec}} + V_{\text{gh}}. \quad (\text{G.10})$$

This looks almost the same as $\hat{V}_{\text{vec}}^{(2)}$, but D'_{10} , defined by replacing $\hat{V}^{(2)}$ in (F.16) with V' , has only first order differentials. Showing only the terms relevant for D'_{10} , we have

$$V' = (\hat{Q}c)^\dagger c + (\hat{Q}\Lambda_1)^\dagger \Lambda_1 + \dots + \bar{c}([\sigma^{(0)}, \tilde{\sigma}] - iD_{(0)}^j \tilde{A}_j) + \dots \quad (\text{G.11})$$

Recall that the embedding of \mathbb{S}^2 in \mathbb{R}^3 implies that $T^*\mathbb{S}^2$ and a trivial real line bundle add up to a trivial rank three bundle. Thus the combination $(\tilde{\sigma}, \tilde{A}_j)$, $j = \theta, \varphi$, can be expanded in three real scalars, and a convenient orthonormal basis is provided by supersymmetry:

$$(\tilde{\sigma}, \tilde{A}_j) = (\bar{\varepsilon}\varepsilon, i\bar{\varepsilon}\gamma_j\varepsilon)S + (i\varepsilon\gamma_\tau\varepsilon, \varepsilon\gamma_j\gamma_\tau\varepsilon)T + c.c., \quad (\text{G.12})$$

where the first term is real and the third is the conjugate of the second. The first section in the basis appears for $Q \cdot c$ in (F.11). Since only the term $iv^\tau \tilde{A}_\tau$ is imaginary,

$$(\hat{Q} \cdot c)^\dagger \sim 2(\bar{\epsilon}\epsilon\tilde{\sigma} + iv^j \tilde{A}_j) \tag{G.13}$$

up to \hat{Q} -exact and higher order terms. Let us set $w^j = \epsilon\gamma^j\gamma_\tau\epsilon$.⁵⁴ We claim that D'_{10} takes the form

$$X_1^{\text{vec}} D'_{10} X_0^{\text{vec}} = \begin{pmatrix} 2c & -i\bar{c} - 4\Lambda_1 & i\bar{c} - 4\Lambda_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ * & w^j D_j + \dots & \dots \\ * & \dots & \bar{w}^j D_j + \dots \end{pmatrix} \begin{pmatrix} S \\ T \\ \bar{T} \end{pmatrix}. \tag{G.14}$$

The ellipses do not involve differentials while $*$'s do. The first row in the matrix easily follows from orthonormality, while the rest needs some work. By rearranging the rows we can block-diagonalize D'_{10} to decouple S and c . The symbol of the remaining part of D'_{10} is proportional to $|w^j p_j|^2 = p_\theta^2 + \cot^2 \theta p_\varphi^2$. The symbol is invertible for non-zero momentum $(p_\mu)_{\mu=\tau,\theta,\varphi}$ transverse to ∂_τ and ∂_φ , and is transversally elliptic with respect to $H \times K$.

For the purpose of counting zero-modes and determining the index as a distribution, we complexify the complex and treat T and \bar{T} as independent complex scalars. For simplicity we suppress τ -dependence. The index is unchanged if we modify the operator, without changing the leading symbol, to⁵⁵

$$\begin{pmatrix} w^j D_j + se^{-i\varphi} \sin \theta & 0 \\ 0 & \bar{w}^j D_j + ise^{i\varphi} \sin \theta \end{pmatrix}. \tag{G.15}$$

In the limit $s \rightarrow +\infty$ we get the zero-modes of (G.15) localized near the north pole $\theta = 0$

$$(T, \bar{T}) \sim (e^{-ir\varphi} \sin^r \theta, 0) e^{-2s \sin^2 \frac{\theta}{2}}, \quad r = 0, 1, 2, \dots,$$

and those localized near the south pole $\theta = \pi$

$$(T, \bar{T}) \sim (0, e^{ir\varphi} \sin^r \theta) e^{-2s \cos^2 \frac{\theta}{2}}, \quad r = 0, 1, 2, \dots$$

The zero-modes of the dual operator also get localized. We have

$$(e^{-ir\varphi} \sin^r \theta, 0) e^{-2s \sin^2 \frac{\theta}{2}}, \quad r = 0, 1, 2, \dots \tag{G.16}$$

localized at $\theta = 0$, and

$$(0, e^{ir\varphi} \sin^r \theta) e^{-2s \cos^2 \frac{\theta}{2}}, \quad r = 0, 1, 2, \dots \tag{G.17}$$

localized at $\theta = \pi$. To the index, $e^{in\varphi}$ in each zero mode contributes t^{-n} , where $t \in K$. We also take into account the φ dependence of the basis sections in (G.12) as well as the flux

⁵⁴Explicitly, $w^\theta = ie^{-i\varphi}$, $w^\varphi = e^{-i\varphi} \cot \theta$.

⁵⁵This was explained using K-theory in [3], and the argument with explicit zero-mode solutions is due to [41]. Both considered the four-dimensional $\mathcal{N} = 2$ theory. See also [42] and [22].

contribution to j_3 in (5.13). We apply the reduction formula (F.27) to obtain

$$\begin{aligned} \text{ind } D_{10}^{\text{vec}} &= \sum_{n \in \mathbb{Z}} \sum_{r=0}^{\infty} \sum_{\alpha \in \text{adj}} h^n \left(t^{-\alpha(m)/2} (t^{r+1} - t^r) + t^{\alpha(m)/2} (t^{-r-1} - t^{-r}) \right) e^{i\alpha(a)} \\ &= - \sum_{n \in \mathbb{Z}} \sum_{\alpha \in \text{adj}} h^n \left(t^{-\alpha(m)/2} + t^{\alpha(m)/2} \right) e^{i\alpha(a)}. \end{aligned} \tag{G.18}$$

Since $\epsilon\gamma_{\hat{r}}\epsilon$ in (G.12) vanishes at the poles, the resulting local contributions coincide with those of the complexified de Rham complex with suitable twisting.

The chiral multiplet is simpler. We obtain from (G.8)⁵⁶

$$\begin{aligned} X_1^{\text{chi}} D_{10}^{\text{chi}} X_0^{\text{chi}} &= i\bar{w}^j D_j \bar{\phi}(\psi\gamma_{\hat{r}}\epsilon) + iw^j D_j \phi(\bar{\epsilon}\gamma_{\hat{r}}\bar{\psi}) \\ &\quad + i\frac{m}{2} \bar{\phi}(\bar{\epsilon}\gamma_{\hat{r}}\bar{\epsilon})(\psi\gamma_{\hat{r}}\epsilon) + i\phi\frac{m}{2}(\epsilon\gamma_{\hat{r}}\epsilon)(\bar{\epsilon}\gamma_{\hat{r}}\bar{\psi}) \end{aligned} \tag{G.19}$$

The operator D_{10}^{chi} is thus the “realification” of $w^j D_j + \dots$, where the ellipses contain no differentials. The symbol has determinant proportional to $p_{\theta}^2 + p_{\varphi}^2 \cot^2 \theta$, and is $H \times K$ -transversally elliptic. By deforming the operator to

$$w^j D_j + ise^{-i\varphi} \sin \theta + \dots \tag{G.20}$$

and taking $s \rightarrow +\infty$, we find zero-modes localized near $\theta = 0$

$$\phi \sim e^{-ir\varphi} \sin^r \theta e^{-2s \sin^2 \frac{\theta}{2}}, \quad r = 0, 1, 2, \dots,$$

and the zero-modes of the dual operator, localized near $\theta = \pi$, approximately given by

$$\epsilon\gamma_{\hat{r}}\psi \sim e^{ir\varphi} \sin^r \theta e^{-2s \cos^2 \frac{\theta}{2}}, \quad r = 0, 1, 2, \dots$$

Taking into account the R-charges and the gauge group action, the index is $\text{ind}_g D_{10}^{\text{chi}} = \text{ind}_g D_{10,\mathbb{C}}^{\text{chi}} + \text{ind}_{g^{-1}} D_{10,\mathbb{C}}^{\text{chi}}$, where $g = (h, t, e^{ia}, f)$ and

$$\text{ind}_g D_{10,\mathbb{C}}^{\text{chi}} = \sum_{n \in \mathbb{Z}} \sum_{r=0}^{\infty} \sum_{\rho \in R} h^n \left(t^{r-\frac{1}{2}\rho(m)} - t^{-r-1+\frac{1}{2}\rho(m)} \right) e^{i\rho(a)} f. \tag{G.21}$$

Substitution (G.5) gives (5.15).

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⁵⁶We use $\bar{\epsilon}\gamma_{\mu}\psi = (\bar{\epsilon}\gamma_{\mu}\gamma_{\hat{r}}\epsilon)(\bar{\epsilon}\psi) + (\bar{\epsilon}\gamma_{\mu}\bar{\epsilon})(\psi\gamma_{\hat{r}}\epsilon)$, $\epsilon\gamma_{\mu}\bar{\psi} = (\epsilon\gamma_{\mu}\epsilon)(\bar{\epsilon}\gamma_{\hat{r}}\bar{\psi}) + (\epsilon\gamma_{\mu}\gamma_{\hat{r}}\bar{\epsilon})(\bar{\epsilon}\bar{\psi})$ and $\bar{\epsilon}\gamma_{\hat{r}}\epsilon = 1$.

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