Published for SISSA by Springer

Received: April 2, 2014
Accepted: June 23, 2014
Published: July 23, 2014

# Unitarity cuts of integrals with doubled propagators

**Mads Søgaard**[a,b] **and Yang Zhang**[a]

[a] *Niels Bohr International Academy and Discovery Center, Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100 Copenhagen, Denmark*

[b] *Institut de Physique Théorique, CEA-Saclay, F–91191 Gif-sur-Yvette cedex, France*

*E-mail:* madss@nbi.dk, zhang@nbi.dk

Abstract: We extend the notion of generalized unitarity cuts to accommodate loop integrals with higher powers of propagators. Such integrals frequently arise in for example integration-by-parts identities, Schwinger parametrizations and Mellin-Barnes representations. The method is applied to reduction of integrals with doubled and tripled propagators and direct extract of integral coefficients at one and two loops. Our algorithm is based on degenerate multivariate residues and computational algebraic geometry.

Keywords: Scattering Amplitudes, Supersymmetric gauge theory

ArXiv ePrint: 1403.2463



doi:10.1007/JHEP07(2014)112

JHEP07(2014)112

## Contents



JHEP07(2014)112

# 1 Introduction

Perturbative scattering amplitudes of elementary particles in quantum field theories such as Quantum Chromodyanmics (QCD) are traditionally calculated by means of Feynman diagrams and rules. The Feynman approach is very intuitive, but suffers from a severe proliferation of terms and diagrams for increasing number of external particles and order in perturbation theory. Moreover, the simplicity of the underlying theory is by no means reflected by the results. Recent years have seen enourmous progress in quantitative determination of amplitudes at the one-loop level and beyond, catalyzed by the demand of precise theoretical predictions by the Large Hadron Collider (LHC) programme at CERN.

Modern computations of scattering amplitudes take advantage of the properties of analyticity and unitarity. Analyticity allow for amplitudes to be reconstructed directly from their analytic structure, while unitarity implies that residues at their singularities factorize onto products of simpler amplitudes. Advances along these lines include among others the unitarity method [1, 2] by Bern, Dixon, Dunbar and Kosower (see e.g. refs. [3–24]) and the Britto-Cachazo-Feng-Witten recursion relations [25, 26] for tree amplitudes. In the last couple of years, new developments in automation of two and three loop amplitudes calculations in arbitrary gauge theories at the level of an integral basis [27–34] and also at the integrand [35–44] have been reported. These papers generalize the works at one-loop of Britto, Cachazo, Feng [5], Forde [6] and Ossola, Papadopoulos and Pittau [14]. We also refer to [45–52] for more related literature.

In this paper, we define generalized unitarity cuts of integrals that otherwise appear incompatible with the usual cut prescription. Naively, applying a unitarity cut to an integral with higher powers of propagators, the immediate result is singular. Nevertheless, amplitude representations that contain integrals with doubled propagators can lead to significant simplifications as argued in [53]. Moreover, integrals with repeated propagators naturally appear in many actual calculations. We explain that such cuts can be treated as degenerate multivariate residues using computational algebraic geometry. Our algorithm makes it possible to use more general integral bases for loop amplitudes. In particular, we provide examples of two-loop integral bases, whose elements contain purely scalars and yet are adaptable for unitarity purposes. What is more, the algorithm can be used to derive IBP relations for integrals with repeated propagators.

The integrand reduction of two-loop diagrams with doubled propagators has been achieved in [44], via the synthetic polynomial division method. However, the full integral reduction for integrals with doubled propagators has not been considered from the unitarity viewpoint.

### 1.1 Generalized Feynman integrals

We define the generalized dimensionally regularized $n$-loop Feynman integral with arbitrary integer powers (also called indices) $(\sigma_1, \ldots, \sigma_p)$ of $p$ propagators by

$$I(\sigma_1, \ldots, \sigma_p) = \int \frac{d^D \ell_1}{(2\pi)^D} \cdots \int \frac{d^D \ell_n}{(2\pi)^D} \prod_{k=1}^{p} \frac{1}{f_k^{\sigma_k}(\{\ell_i\})} , \tag{1.1}$$

where the $f_k$'s are linear polynomials with respect to inner products of the $n$ loop momenta $\{\ell_i\}$ and $m$ external momenta $\{k_i\}$. The canonical integral is recovered when all indices are set to unity. Generally speaking, the integral will have a nontrivial numerator function and is in that case referred to as a tensor integral. We can always bring the numerator into the form of additional propagators raised to negative powers.

In a typical multiloop amplitude calculation, a huge number of Feynman integrals with different indices appear. A subset of the integrals can easily be reduced algebraically, e.g. by means of Gram matrix determinants. At first glance, the remaining integrals may seem irreducible and independent, but they are in fact related by *integration-by-parts identities* (IBP) [61]. Discarding the boundary term in $D$-dimensional integration, total derivatives vanish upon integration,

$$\int \frac{d^D \ell_1}{(2\pi)^D} \cdots \int \frac{d^D \ell_n}{(2\pi)^D} \frac{\partial}{\partial \ell_a^\mu} \left( k_b^\mu \prod_{k=1}^{p} \frac{1}{f_k^{\sigma_k}(\{\ell_i\})} \right) = 0 , \tag{1.2}$$

$$\int \frac{d^D \ell_1}{(2\pi)^D} \cdots \int \frac{d^D \ell_n}{(2\pi)^D} \frac{\partial}{\partial \ell_a^\mu} \left( \ell_b^\mu \prod_{k=1}^{p} \frac{1}{f_k^{\sigma_k}(\{\ell_i\})} \right) = 0 \,, \tag{1.3}$$

which can be recast as linear relations among integrals with shifted exponents,

$$\sum_i \mu_i I(\sigma_1 + \rho_{i,1}, \ldots, \sigma_n + \rho_{i,n}) = 0 \tag{1.4}$$

for $\rho_{i,j} \in \{-1, 0, 1\}$. The virtue of IBP relations is that within a given topology, a few integrals may be chosen as masters in the sense that all other integrals can be expressed in a basis of them. The importance is not to be underestimated. For example, the four-point massless planar triple box has several hundred renormalizable integrals which are reduced onto a linear combination of just three integrals with at most rank 1.

IBP relations can be generated by public computer codes such as `FIRE` [62] and `Reduze` [63]. In practice, the production of IBP relations is quite time consuming and requires considerable amount of memory.

### 1.2 Direct extraction of integral coefficients

We consider schematically an $n$-loop amplitude contribution which is denoted $\mathcal{A}^{(L)}$. After reduction onto a basis of master integrals, the amplitude can be written

$$\mathcal{A}^{(L)} = \sum_{k \in \text{Basis}} c_k I_k + \text{rational terms} \tag{1.5}$$

where the $c_k$'s are rational functions of external invariants. We refer to eq. (1.5) as the *master equation*. For example, at one loop the integral basis is very simple and contains only boxes, triangles, bubbles and rational terms. The integral itself is calculated once and for all and therefore the problem of computing the amplitude reduces to determining the coefficients.

The trick is to probe the loop integrand by applying generalized unitarity cuts on either side of the master equation. Originally, unitarity cuts were realized by replacing a set of propagators by Dirac delta functions restricting them to their mass shell. The framework of maximal unitarity at two loops initiated by Kosower and Larsen [28] naturally deals with amplitude contributions whose factorization properties are accessible only away from the real slices of Minkowski space, for example hepta-cuts of double boxes. Multidimensional complex contour integrals that compute multivariate residues provide the desired genelization of the localization property,

$$\int dz_1 \cdots \int dz_n h(\{z_i\}) \prod_{j=1}^{n} \delta(z_j - \xi_j) \equiv \frac{1}{(2\pi i)^n} \oint_{\Gamma_\epsilon(\xi)} dz_1 \wedge \cdots \wedge dz_n \frac{h(\{z_i\})}{\prod_{j=1}^{n}(z_j - \xi_j)} \tag{1.6}$$

for a given $\xi \in \mathbb{C}^n$. Here, $\Gamma_\epsilon$ is a torus of real dimension $n$ around the pole of the integrand at $z = \xi$. A generalized unitarity cut, even a maximal cut which puts as many propagators on-shell as possible, is typically shared among several basis integrals, hence intermediate algebra is in principle required. Instead one seeks to construct linear combinations of residues that in a certain sense are orthogonal to each other and thus project a single basis integral. In this way, the integral coefficient is expressed in terms of residues of products of tree amplitudes that arise when the loop amplitude factorizes. These combinations are subject to the consistency requirement that parity-odd integrands and total derivatives continue to vanish upon integration [28]. The tree-level data is easily manipulated within the spinor-helicity formalism by means of for instance superspace techniques [55, 56].

Direct extraction of master integral coefficients in maximal unitarity has been demonstrated for two-loop double boxes with up to four external massive or massless legs [28–31], for the nonplanar double box [32] and the three-loop triple box [33]. In these calculations, only basis integrals with single propagators were considered.

## 2 Multivariate residues

To extract the integral coefficients, we need to calculate *multivariate residues*. In many cases, these residues can simply be evaluated by Cauchy's theorem in higher dimensions and the Jacobian determinant. However, in some cases, like the unitarity cut of the triple box topology [33] or the unitarity cut of integral with doubled propagators, the residues are *degenerate* and have to be evaluated by algebraic geometry. In this section, we briefly review the concept and calculation of multivariate residues. Standard mathematical references include [64–66].

Consider a differential form $\omega$ in $n$ complex variables $z \equiv (z_1, \ldots, z_n)$,

$$\omega = \frac{h(z) dz_1 \wedge \cdots \wedge dz_n}{f_1(z) \cdots f_n(z)} , \tag{2.1}$$

where the numerator $h(z)$ and the denominators $f_1(z), \ldots, f_n(z)$ are holomorphic functions. If at a point $\xi$, $f_1(\xi) = \cdots = f_n(\xi) = 0$, then the residue of $\omega$ at $\xi$ regarding the divisors $\{f_1, \ldots, f_n\}$ is defined to be,

$$\mathrm{Res}_{\{f_1,\ldots,f_n\},\xi}(\omega) \equiv \left(\frac{1}{2\pi i}\right)^n \oint_\Gamma \frac{h(z) dz_1 \wedge \cdots \wedge dz_n}{f_1(z) \cdots f_n(z)} . \tag{2.2}$$

Here the contour $\Gamma$ is a real $n$-cycle $\Gamma = \{z : |f_i(z)| = \epsilon_i\}$ around $\xi$ and the orientation is specified by $d(\arg f_1) \wedge \cdots \wedge d(\arg f_n)$.

In two cases, a multivariate residue can be calculated straightforwardly,

- A residue is *non-degenerate*, if the Jacobian at $\xi$ is nonzero, i.e.,

  $$J(\xi) = \det(\partial f_i / \partial z_j)|_\xi \neq 0 . \tag{2.3}$$

  In this case, by the multi-dimensional verion of Cauchy's theorem, the value of residue is simply [64],

  $$\mathrm{Res}_{\{f_1,\ldots,f_n\},\xi}(\omega) = \frac{h(\xi)}{J(\xi)} . \tag{2.4}$$

- A residue is *factorizable* if each $f_i$ is a univariate polynomial, namely, $f_i(z) = f_i(z_i)$. In this case, the $n$-dimensional contour in is factorized to the product of $n$ univariate contours,

  $$\mathrm{Res}_{\{f_1,\ldots,f_n\},\xi}(\omega) = \left(\frac{1}{2\pi i}\right)^n \oint_{|f_1(z_1)|=\epsilon_1} \frac{dz_1}{f_1(z_1)} \cdots \oint_{|f_n(z_n)|=\epsilon_n} \frac{dz_n}{f_n(z_n)} h(z) . \tag{2.5}$$



JHEP07(2014)112

Then, we can evaluate this residue by applying the univariate residue formula $n$ times.

However, in general, a residue is neither non-degenerate nor factorizable. For example, consider a Feynman integrand with doubled (or higher-power) propagators,

$$\frac{1}{f_1^{\sigma_1} f_2 \cdots f_n}\,, \quad \sigma_1 > 1\,. \tag{2.6}$$

At a point $\xi$ where $f_1(\xi) = \cdots = f_k(\xi) = 0$, the Jacobian matrix is degenerate since

$$\left.\frac{\partial f_1^{\sigma_1}}{\partial z_i}\right|_{\xi} = \sigma_1 f_1^{\sigma_1 - 1} \left.\frac{\partial f_1}{\partial z_i}\right|_{\xi} = 0\,. \tag{2.7}$$

In general, this type of residues is not factorizable. To evaluate them, we need the *transformation law* [64] in algebraic geometry.

**Theorem 1 (Transformation law)** *Let $\{f_1, \ldots, f_n\}$ and $\{u_1, \ldots, u_n\}$ be two sets of holomorphic functions and $u_i = a_{ij} f_j$, where $a_{ij}$ are holomorphic functions. Assume that for each set, the common zeros are discrete points. Let $A$ be the matrix of the $a_{ij}$'s, then*

$$\mathrm{Res}_{\{f_1,\ldots,f_n\},\xi}\left(\frac{h(z)dz_1 \wedge \cdots \wedge dz_n}{f_1(z)\cdots f_n(z)}\right) = \mathrm{Res}_{\{u_1,\ldots,u_n\},\xi}\left(\frac{h(z)dz_1 \wedge \cdots \wedge dz_n}{u_1(z)\cdots u_n(z)} \det A\right). \tag{2.8}$$

This theorem holds for both non-degenerate and degenerate residues.

For Feynman integrals, the denominators are all polynomials. In this case, we can use the transformation law to convert a degenerate residue to a factorizable residue, via Gröbner basis. The algorithm involves the following steps [33]:

1. Calculate the Gröbner basis $\{g_1, \ldots, g_k\}$ of $\{f_1, \ldots, f_n\}$ in the *DegreeLexicographic* order and record the converting matrix $r_{ij}$, such that $g_i = r_{ij} f_j$.

2. For each $1 \le i \le n$, calculate the Gröbner basis of $\{f_1, \ldots, f_n\}$ in the *Lexicographic* order of $z_{i+1} \succ \cdots \succ z_n \succ z_1 \succ \cdots z_i$. Pick the univariate polynomial in $z_i$ from this Gröbner basis and name it as $u_i$.

3. For each $u_i$, divide it towards $\{g_1, \ldots, g_k\}$ so $u_i = s_{ij} g_j$.

4. The transformation matrix is $a_{ij} = s_{ik} r_{kj}$. By the transformation law, the degenerate residue is converted to a factorizable residue with the matrix $a_{ij}$.

Finally, we have a comment on the residues from the maximal cut of integrals with doubled (or multiple) propagators. For the residue from a general integrand,

$$\frac{N}{f_1^{\sigma_1} \cdots f_n^{\sigma_n}}\,, \tag{2.9}$$

to use (2.2), for each $f_i$, we have to collect all powers of $f_i$ as one denominator. Otherwise, the common zeros of denominators are not discrete points, so the residue is not well defined. Hence there is no ambiguity of defining denominators for the residue computation.

In our paper, we calculate the residues from the maximal cut of integrals with doubled and tripled propagators. The degenerate residues are evaluated by our Mathematica package `MathematicaM2`,[1] which calls Macaulay2 [67] to finish the computation of Gröbner bases. We demonstrate the multivariate residue computation explicitly by the one-loop box integral with double propagators. Then we show this method is general by two-loop examples. Related ideas were previously proposed to reduce one-loop integrals with generic powers of propagators [54].

## 3 Example: one-loop box

Consider a one-loop box integral with four massless legs $k_1, \ldots, k_4$,

$$I_4(\sigma_1, \ldots, \sigma_4)[N] \equiv \int_{\mathbb{R}^D} \frac{d^D\ell}{(2\pi)^D} \prod_{i=1}^{4} \frac{N}{f_i^{\sigma_i}(\ell)} \,, \tag{3.1}$$

where the denominators are,

$$f_1 = \ell^2 \,, \quad f_2 = (\ell - k_1)^2 \,, \quad f_3 = (\ell - K_{12})^2 \,, \quad f_4 = (\ell + k_4)^2 \,. \tag{3.2}$$

We suppress the Feynman $i\epsilon$-prescription as it is irrelevant for our purposes and assume for simplicity that all external momenta are massless and outgoing. External momenta are summed using the shorthand notation $K_{ij} = k_i + k_j + \cdots$.

In the following discussion, we set $D = 4$. We fix a basis of the four-dimensional space time $\{k_1, k_2, k_4, \omega\}$ where the spurious vector $\omega$ can be represented as

$$\omega \equiv \frac{1}{2s_{12}} \left( \langle 2|3|1] \langle 1|\gamma^\mu|2] - \langle 1|3|2] \langle 2|\gamma^\mu|1] \right) \,, \tag{3.3}$$

such that $\omega$ is orthogonal to the subspace spanned by the momentum vectors. The list of irreducible scalar products (ISP) can then be chosen as

$$\mathbb{ISP} = \{\ell \cdot \omega\} \,. \tag{3.4}$$

The loop momentum $\ell$ can be parameterized as

$$\ell^\mu = \alpha_1 k_1^\mu + \alpha_2 k_2^\mu + \frac{s_{12}\alpha_3}{2\langle 14\rangle[42]} \langle 1^-|\gamma^\mu|2^-\rangle + \frac{s_{12}\alpha_4}{2\langle 24\rangle[41]} \langle 2^-|\gamma^\mu|1^-\rangle \,, \tag{3.5}$$

and the Jacobian for this parametrization is

$$J = \det_{\mu,i} \frac{\partial \ell^\mu}{\partial \alpha_i} = -\frac{i s_{12}^2}{4\chi(\chi+1)} \,. \tag{3.6}$$

The cut equations $f_1(\ell) = \cdots = f_4(\ell) = 0$ have two solutions,

$$(\alpha_1, \ldots \alpha_4) = (1, 0, 0, -\chi) \equiv \xi_1 \,, \quad (\alpha_1, \ldots \alpha_4) = (1, 0, -\chi, 0) \equiv \xi_2 \,. \tag{3.7}$$

[1] The package can be downloaded from https://bitbucket.org/yzhphy/mathematicam2.



JHEP07(2014)112

There is only one master integral for one-loop box, the scalar integral,

$$I_4(\sigma_1, \dots \sigma_4)[N] = C_1 I_4(1,1,1,1)[1]_{\xi_1} + \cdots \tag{3.8}$$

where $\cdots$ stands for integrals with fewer than four propagators.

Localizing the contour around $\xi_1$ and $\xi_2$, it is clear that the Jacobian of $f$'s in $\alpha$'s is nonzero. So by Cauchy's theorem in higher dimensions (2.4), the residues are

$$I_4(1,1,1,1)[1]_{\xi_1} = -i\frac{1}{4s_{12}^2\chi}, \quad I_4(1,1,1,1)[1]_{\xi_2} = i\frac{1}{4s_{12}^2\chi} . \tag{3.9}$$

Together with the spurious integral condition $I_4(1,1,1,1)[\ell\cdot\omega]_{\xi_1} = 0$, we have the expression for the integral coefficient for the integral $I_4(\sigma_1, \dots \sigma_4)[N]$,

$$C_1 = 2is_{12}^2\chi(I_4(\sigma_1, \dots \sigma_4)[N]_{\xi_1} - I_4(\sigma_1, \dots \sigma_4)[N]_{\xi_2}) . \tag{3.10}$$

Now we consider integrals with doubled propagators, for example, $I_4(1,1,1,2)[1]$. The Jacobian of $\{f_1, f_2, f_3, f_4^2\}$ in $\alpha$'s is zero at both $\xi_1$ and $\xi_2$, so direct computation does not work. We can use the transformation law (2.8) to convert the denominators to a calculable form,

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} \equiv \begin{pmatrix} \alpha_1 - 1 \\ \alpha_2 \\ -\alpha_3(\alpha_3+\chi)^2 \\ -\alpha_4(\alpha_4+\chi)^2 \end{pmatrix} = M \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4^2 \end{pmatrix} , \tag{3.11}$$

where $M$ is $4 \times 4$ matrix and all entries are polynomials in $\alpha$'s, and

$$\det M = -\frac{\chi(1+\chi)(\alpha_3-\alpha_4)(\alpha_3+\alpha_4+2\chi)}{s_{12}^5} . \tag{3.12}$$

Hence, around either $\xi_1$ or $\xi_2$,

$$\oint \frac{d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \wedge d\alpha_4}{f_1 f_2 f_3 f_4^2} = \oint \frac{d\alpha_1}{\alpha_1 - 1} \oint \frac{d\alpha_2}{\alpha_2} \oint \frac{d\alpha_3}{\alpha_3(\alpha_3+\chi)^2} \oint \frac{\det M d\alpha_4}{\alpha_4(\alpha_4+\chi)^2} . \tag{3.13}$$

So the degenerate residue can be calculated by applying the univariate Cauchy's theorem four times. The explicit form of $M$ is found by our package `MathematicaM2`. The residues for $I_4(1,1,1,2)[1]$ are

$$I_4(1,1,1,2)[1]_{\xi_1} = -i\frac{1}{4s_{12}^3\chi^2} , \quad I_4(1,1,1,2)[1]_{\xi_2} = i\frac{1}{4s_{12}^3\chi^2} . \tag{3.14}$$

So we have

$$I_4(1,1,1,2)[1] = \frac{1}{s_{12}\chi} I_4(1,1,1,1)[1] + \cdots . \tag{3.15}$$

Similarly, using the same method, we find that,

$$I_4(2,1,1,1)[1] = \frac{1}{s_{12}} I_4(1,1,1,1)[1] + \cdots , \tag{3.16}$$

$$I_4(2,1,1,2)[1] = \frac{2}{s_{12}^2\chi} I_4(1,1,1,1)[1] + \cdots , \tag{3.17}$$

$$I_4(3,1,1,1)[1] = \frac{1}{s_{12}^2} I_4(1,1,1,1)[1] + \cdots . \tag{3.18}$$



JHEP07(2014)112

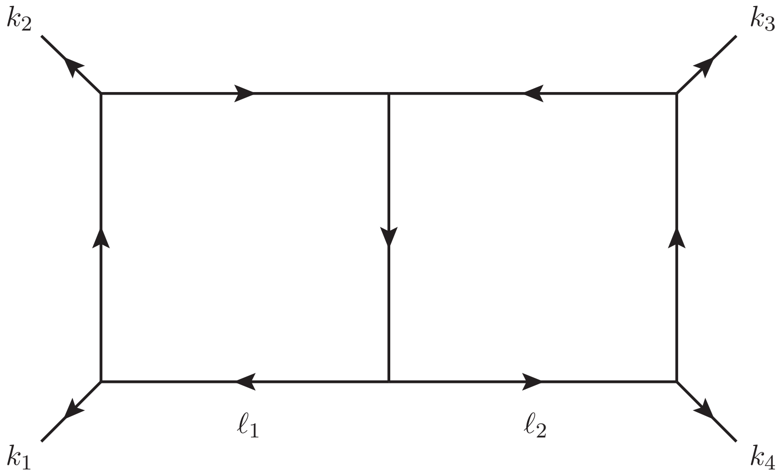


**Figure 1**. The massless four-point planar double box.

These results are consistent with the IBP relations in the $D = 4$ limit. For instance, from `FIRE` [62],

$$I_4(1,1,1,2)[1] = \frac{1+2\epsilon}{s_{12}\chi} I_4(1,1,1,1)[1] + \cdots \,, \tag{3.19}$$

$$I_4(2,1,1,1)[1] = \frac{1+2\epsilon}{s_{12}} I_4(1,1,1,1)[1] + \cdots \,, \tag{3.20}$$

$$I_4(2,1,1,2)[1] = \frac{2(1+\epsilon)(1+2\epsilon)}{s_{12}^2\chi} I_4(1,1,1,1)[1] + \cdots \,, \tag{3.21}$$

$$I_4(3,1,1,1)[1] = \frac{(1+\epsilon)(1+2\epsilon)}{s_{12}^2} I_4(1,1,1,1)[1] + \cdots \,. \tag{3.22}$$

## 4 Example: planar double box

We now proceed to two-loop integrals. The generalized dimensionally regularized two-loop planar double box scalar integral (figure 1) with arbitrary powers of propagators reads

$$P_{2,2}^{**}(\sigma_1,\ldots,\sigma_7) \equiv \int_{\mathbb{R}^D} \frac{d^D\ell_1}{(2\pi)^D} \int_{\mathbb{R}^D} \frac{d^D\ell_2}{(2\pi)^D} \prod_{i=1}^{7} \frac{1}{f_i^{\sigma_i}(\ell_1,\ell_2)} \,, \tag{4.1}$$

where the seven inverse propagators $\{f_i\}$ are given by

$$\begin{aligned} &f_1 = \ell_1^2 \,, \quad f_2 = (\ell_1 - k_1)^2 \,, \quad f_3 = (\ell_1 - K_{12})^2 \,, \\ &f_4 = \ell_2^2 \,, \quad f_5 = (\ell_2 - k_4)^2 \,, \quad f_6 = (\ell_2 - K_{34})^2 \,, \quad f_7 = (\ell_1 + \ell_2)^2 \,. \end{aligned} \tag{4.2}$$

Closed form expressions for planar double integrals can be found in refs. [57, 58].

As in the one-loop example, we choose $\{k_1, k_2, k_4, \omega\}$ as basis of the four-dimensional space time where again the spurious vector $\omega$ can be represented as

$$\omega \equiv \frac{1}{2s_{12}} \left( \langle 2|3|1] \langle 1|\gamma^\mu|2] - \langle 1|3|2] \langle 2|\gamma^\mu|1] \right) \,, \tag{4.3}$$

The list of irreducible scalar products (ISP) can then be chosen as [35]

$$\mathbb{ISP} = \{\ell_1 \cdot k_4,\ \ell_2 \cdot k_1,\ \ell_1 \cdot \omega,\ \ell_2 \cdot \omega\}\ , \tag{4.4}$$

and the integrand basis contains 16 spurious and 16 nonspurious elements. Whence the nine-propagator double box topology is defined by

$$P_{2,2}^{**}(\sigma_1, \ldots, \sigma_9) \equiv \int_{\mathbb{R}^D} \frac{d^D\ell_1}{(2\pi)^D} \int_{\mathbb{R}^D} \frac{d^D\ell_2}{(2\pi)^D} \prod_{i=1}^{9} \frac{1}{f_i^{\sigma_i}(\ell_1, \ell_2)}\ , \tag{4.5}$$

where $f_8 = \ell_1 \cdot k_4$ and $f_9 = \ell_2 \cdot k_1$ are the nonspurious ISPs. Then we have

$$P_{2,2}^{**}[(\ell_1 \cdot k_4)^n (\ell_1 \cdot k_2)^m] = P_{2,2}^{**}(1, \ldots, 1, -n, -m) \tag{4.6}$$

in the notation of refs. [27, 28].

### 4.1 Parametrization of hepta-cut solutions

In order to expose the singularity structure of the loop integrand, we adopt a particularly convenient loop momentum parametrization of previous works, see for instance ref. [28],

$$\ell_1^\mu = \alpha_1 k_1^\mu + \alpha_2 k_2^\mu + \frac{s_{12}\alpha_3}{2\langle 14\rangle[42]}\langle 1^-|\gamma^\mu|2^-\rangle + \frac{s_{12}\alpha_4}{2\langle 24\rangle[41]}\langle 2^-|\gamma^\mu|1^-\rangle\ , \tag{4.7}$$

$$\ell_2^\mu = \beta_1 k_3^\mu + \beta_2 k_4^\mu + \frac{s_{12}\beta_3}{2\langle 31\rangle[14]}\langle 3^-|\gamma^\mu|4^-\rangle + \frac{s_{12}\beta_4}{2\langle 41\rangle[13]}\langle 4^-|\gamma^\mu|3^-\rangle\ . \tag{4.8}$$

It is elementary to show that the Jacobians to compensate for the change of variables from loop momenta to parameter space are

$$J_\alpha = \det_{\mu,i} \frac{\partial \ell_1^\mu}{\partial \alpha_i} = -\frac{is_{12}^2}{4\chi(\chi+1)}\ , \quad J_\beta = \det_{\mu,i} \frac{\partial \ell_2^\mu}{\partial \beta_i} = -\frac{is_{12}^2}{4\chi(\chi+1)}\ , \tag{4.9}$$

where $\chi$ is a frequently used ratio of Mandelstam invariants,

$$\chi = \frac{s_{14}}{s_{12}}\ . \tag{4.10}$$

The zero locus of the ideal generated by the polynomials $f_i$ defines a reducible elliptic curve associated with a sixtuply pinched torus whose components are Riemann spheres,

$$\mathcal{S} = \big\{(\ell_1, \ell_2) \in (\mathbb{C}^4)^{\otimes 2} \mid f_i(\ell_1, \ell_2) = 0\big\} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_6\ . \tag{4.11}$$

The solutions can be summarized as follows,

$$\mathcal{S}_1 : (\alpha_3, \alpha_4, \beta_3, \beta_4) = (-\chi, 0, z, 0)\ , \qquad \mathcal{S}_2 : (\alpha_3, \alpha_4, \beta_3, \beta_4) = (z, 0, -\chi, 0)\ , \tag{4.12}$$

$$\mathcal{S}_3 : (\alpha_3, \alpha_4, \beta_3, \beta_4) = (0, -\chi, 0, z)\ , \qquad \mathcal{S}_4 : (\alpha_3, \alpha_4, \beta_3, \beta_4) = (0, z, 0, -\chi)\ , \tag{4.13}$$

$$\mathcal{S}_5 : (\alpha_3, \alpha_4, \beta_3, \beta_4) = (0, z, \tau(z), 0)\ , \qquad \mathcal{S}_6 : (\alpha_3, \alpha_4, \beta_3, \beta_4) = (z, 0, 0, \tau(z))\ , \tag{4.14}$$

with $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (1, 0, 0, 1)$ uniformly across all branches. Also, $\tau$ is defined by

$$\tau(z) \equiv -(\chi+1)\frac{z+\chi}{z+\chi+1}\ . \tag{4.15}$$



JHEP07(2014)112

## 4.2 Residues of the loop integrand

We follow the strategy of refs. [28, 29] and quickly rederive the hepta-cut of the massless planar double box. In the standard situation where all propagators are single, the residue of the scalar integrand is nondegenerate and is easy to calculate as the determinant of a Jacobian matrix as explained previously. For all six branches, the hepta-cut integral is [28]

$$P_{2,2}^{**}(1,\ldots,1,0,0)_{\mathcal{S}_i} = -\frac{1}{16s_{12}^3}\oint \frac{dz}{z(z+\chi)}\,. \tag{4.16}$$

It remains to choose an integral basis. The IBP identities generated with `FIRE` [62] grant that all double box integrals can be reduced onto two master integrals, such that a general integral (and the amplitude contribution itself) can be written

$$P_{2,2}^{**}(\sigma_1,\ldots,\sigma_9) = C_1 P_{2,2}^{**}(1,\ldots,1,1,0,0) + C_2 P_{2,2}^{**}(1,\ldots,1,-1,0) + \cdots \tag{4.17}$$

where hidden terms have less than seven propagators and therefore vanish on the maximal cut. Evidently, the integral basis consists of a scalar double box and a rank 1 tensor integral with single propagators.

We will focus on integrals that have at least one $\sigma_i > 1$, e.g.

$$P_{2,2}^{**}(2,1,\ldots,1,0,0) = C_1 P_{2,2}^{**}(1,\ldots,1,0,0) + C_2 P_{2,2}^{**}(1,\ldots,1,-1,0) + \cdots \tag{4.18}$$

$$P_{2,2}^{**}(1,\ldots,1,2,0,0) = C_1' P_{2,2}^{**}(1,\ldots,1,0,0) + C_2' P_{2,2}^{**}(1,\ldots,1,-1,0) + \cdots \tag{4.19}$$

for rational coefficients in external invariants and space-time dimension. The residue at the simultaneous zero of the inverse propagators in such an integral is clearly degenerate by the preceding discussion. Therefore we apply the algorithm to transform the residue to a factorized form and obtain the analog of eq. (4.16)

$$P_{2,2}^{**}(2,1,\ldots,1,0,0)_{\mathcal{S}_{1,3}} = -\frac{1}{16s_{12}^4}\oint \frac{dz}{z(z+\chi)}\,, \tag{4.20}$$

$$P_{2,2}^{**}(2,1,\ldots,1,0,0)_{\mathcal{S}_{2,4,5,6}} = -\frac{1}{16s_{12}^4}\oint \frac{dz}{z(z+\chi)^2}\,, \tag{4.21}$$

$$P_{2,2}^{**}(1,\ldots,1,2,0,0)_{\mathcal{S}_i} = -\frac{1}{16s_{12}^4}\oint \frac{dz}{z(z+\chi)^2}\,. \tag{4.22}$$

The integrand thus has two residues at each branch (we eliminate residues at infinity by the Global Residue Theorem) and since solution $\mathcal{S}_5$ and $\mathcal{S}_6$ give rise to additional poles in numerator insertions, we expect a total of fourteen independent residues. However, as explained in refs. [29, 43] and depicted in figure 2, the Jacobian poles are located at the nodal points of the elliptic curve defined by the hepta-cut. We truncate to a linearly independent set of residues for $\mathcal{S}_1 \cup \cdots \cup S_6$ and choose contours for the post hepta-cut degree of freedom that encircle eight global,

$$(\mathcal{G}_1,\ldots,\mathcal{G}_8) = (\mathcal{G}_{1\cap 2},\, \mathcal{G}_{2\cap 5},\, \mathcal{G}_{5\cap 3}\,, \mathcal{G}_{3\cap 4},\, \mathcal{G}_{4\cap 6},\, \mathcal{G}_{6\cap 1},\, \mathcal{G}_{5,\infty_R},\, \mathcal{G}_{6,\infty_R})\,, \tag{4.23}$$

with the corresponding weights,

$$\Omega = (\omega_{1\cap 2},\, \omega_{2\cap 5},\, \omega_{5\cap 3}\,, \omega_{3\cap 4},\, \omega_{4\cap 6},\, \omega_{6\cap 1},\, \omega_{5,\infty_R},\, \omega_{6,\infty_R})\,. \tag{4.24}$$

By convention, a residue with weight $\omega_{i\cap j}$ is evaluated on the $i$'th branch.

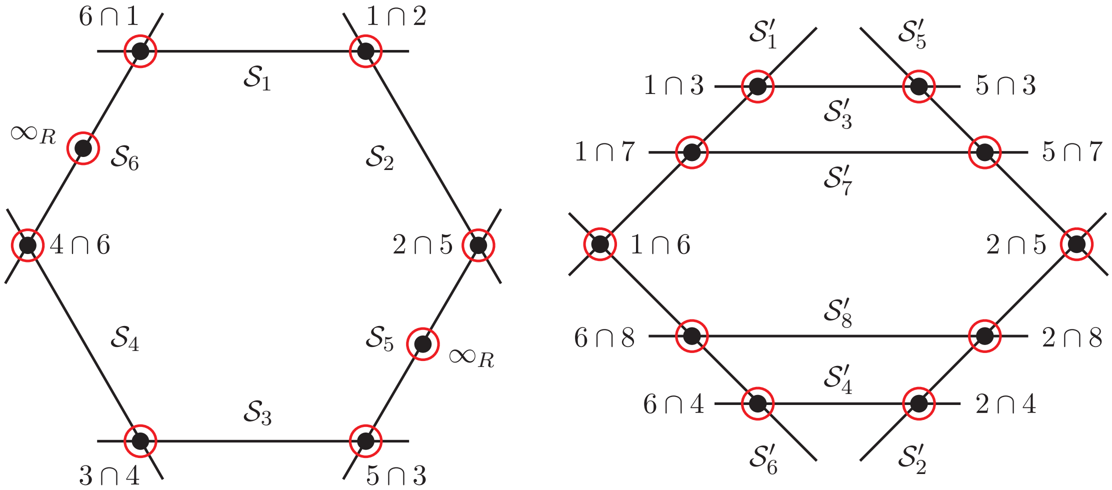


**Figure 2**. Global structure of the hepta-cut of the two-loop planar (left) and nonplanar (right) double box with purely massless kinematics and four external legs. The straight lines should be interpreted as genus-0 Riemann surfaces. Each branch may have an additional residue at $z = \infty$ which is eliminated here.

The associated residues of the integrand in the two master integrals in eq. (4.17) in the order displayed above then read

$$R_1 = \frac{1}{16\chi s_{12}^3}(1,-1,1,1,-1,1,0,0)\ , \quad R_2 = \frac{1}{32 s_{12}^2}(0,-1,0,0,-1,0,0,0)\ . \tag{4.25}$$

### 4.3 Master integral projectors

The hepta-cut contours are subject to consistency requirements in order to ensure that certain integral relations are preserved after pushing the real slice integrals into $\mathbb{C}^8$. It is well known that the integrand can be parametrized in terms of four irreducible products,

$$N = \sum_{a_1,\ldots,a_4} c_{a_1,\ldots,a_4}(\ell_1\cdot\omega)^{a_1}(\ell_2\cdot\omega)^{a_2}(\ell_1\cdot k_4)^{a_3}(\ell_2\cdot k_1)^{a_4}\ , \tag{4.26}$$

whose powers can be derived by renormalizability conditions and then multivariate polynomial division using the Gröbner basis method. The latter has been automated in the program BasisDet [38]. Both the spurious and nonspurious part of the basis contains sixteen elements. At the level of integrated expressions, the amplitude is expressed as a linear combination of just two masters. Accordingly, we demand that all integral identities on which the reduction relies are respected. The full list of IBP identities is available in ref. [32]. We arrange all parity-odd and IBP constraints as a $32\times 8$ matrix $M$ that acts on the residue weights $\Omega$. The two corresponding submatrices have rank 4 and 2 respectively. We then define two master contours by

$$\mathcal{M}_1\cdot(R_1,R_2) = (1,0)\ , \quad \mathcal{M}_2\cdot(R_1,R_2) = (0,1)\ . \tag{4.27}$$

Here, $\mathcal{M}_1$ and $\mathcal{M}_2$ are particular choices of the winding numbers (4.24), such that only the contribution from one of the basis integrals is picked up and normalized. The full $34\times 8$

residue matrix has rank 8 in either case and therefore the master contours are unique. The solutions take the very simple form

$$\mathcal{M}_1 = 4\chi s_{12}^3(1,0,1,1,0,1,1,1)\ , \quad \mathcal{M}_2 = -8s_{12}^2(1,2,1,1,2,1,3,3)\ . \tag{4.28}$$

Now we are ready to apply the master integral projectors in the context of double box integrals with doubled propagators. The residues of the integrands of the integrals on the left hand side of eq. (4.18) are

$$\begin{aligned}
\mathrm{Res}_{\{\mathcal{G}_i\}} P_{2,2}^{**}(2,1,\ldots,1,0,0) &= \frac{1}{16\chi s_{12}^4}(1,-1,1,1,-1,1,0,0)\ , \\
\mathrm{Res}_{\{\mathcal{G}_i\}} P_{2,2}^{**}(1,\ldots,1,2,0,0) &= \frac{1}{16\chi^2 s_{12}^4}(1,-1,1,1,-1,1,0,0)\ .
\end{aligned} \tag{4.29}$$

Therefore, applying the projectors yields the reduction identities

$$P_{2,2}^{**}(2,1,\ldots,1,0,0) = +\frac{1}{s_{12}}P_{2,2}^{**}(1,\ldots,1,0,0) + \cdots\ , \tag{4.30}$$

$$P_{2,2}^{**}(1,\ldots,1,2,0,0) = +\frac{1}{\chi s_{12}}P_{2,2}^{**}(1,\ldots,1,0,0) + \cdots\ . \tag{4.31}$$

It has been verified that our results are consistent with the four-dimensional limit of the following IBP relations in $D = 4-2\epsilon$,

$$P_{2,2}^{**}(2,1,\ldots,1,0,0) = \frac{1+2\epsilon}{s_{12}}P_{2,2}^{**}(1,\ldots,1,0,0) + \cdots \tag{4.32}$$

$$P_{2,2}^{**}(1,\ldots,1,2,0,0) = \frac{1+2\epsilon}{1+\epsilon}\left(\frac{1+3\epsilon}{\chi s_{12}}P_{2,2}^{**}(1,\ldots,1,0,0) + \frac{4\epsilon}{\chi s_{12}^2}P_{2,2}^{**}(1,\ldots,1,-1,0)\right) + \cdots \tag{4.33}$$

Indeed, as $\epsilon \to 0$ the tensor integral drops out and the integral with a doubled propagator and the canonical scalar integrals equate up the factors written above.

Any other powers of propagators may be treated similarly. A complete list of heptacuts of planar doubled boxes with a doubled propagator is given in appendix A.

## 5 Example: nonplanar double box

We define the four-point two-loop nonplanar double box integral (see figure 3) in dimensional regularization by

$$X_{1,1,2}^{**}(\sigma_1,\ldots,\sigma_7) \equiv \int_{\mathbb{R}^D}\frac{d^D\ell_1}{(2\pi)^D}\int_{\mathbb{R}^D}\frac{d^D\ell_2}{(2\pi)^D}\prod_{i=1}^{7}\frac{1}{\tilde{f}_i^{\sigma_i}(\ell_1,\ell_2)}\ , \tag{5.1}$$

and adopt the convention for propagators and momentum flow of ref [35],

$$\begin{aligned}
&\tilde{f}_1 = \ell_1^2\ , \quad \tilde{f}_2 = (\ell_1+k_1)^2\ , \quad \tilde{f}_3 = (\ell_2+k_4)^2\ , \\
&\tilde{f}_4 = \ell_2^2\ , \quad \tilde{f}_5 = (\ell_1-k_3)^2\ , \quad \tilde{f}_6 = (\ell_1+\ell_2-k_3)^2\ , \quad \tilde{f}_7 = (\ell_1+\ell_2-K_{23})^2\ .
\end{aligned} \tag{5.2}$$

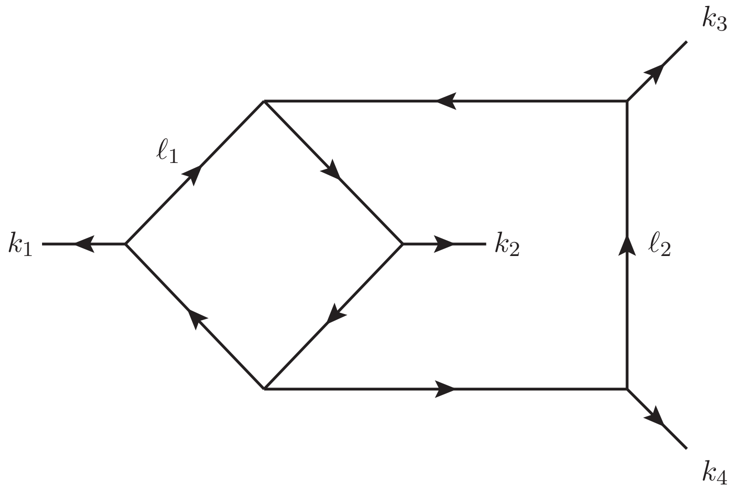


**Figure 3**. The nonplanar double box topology with four external particles.

All external and internal momenta are by assumption massless. We will consider four-dimensional unitarity cuts and therefore only reconstruct the master integral coefficients to leading order in the dimensional regulator. The Feynman integral itself was calculated in refs. [59, 60].

The set of vectors $\{k_1, k_2, k_3, \omega\}$ where $\omega$ is the spurious direction forms a basis of four-dimensional momentum space. There are again four irreducible scalar products,

$$\mathbb{ISP} = \{\ell_1 \cdot k_3,\, \ell_2 \cdot k_2,\, \ell_1 \cdot \omega,\, \ell_2 \cdot \omega\}\;, \tag{5.3}$$

and the minimal representation of the integrand consists of 19 spurious and 19 nonspurious monomials. Accordingly, we define the nine-propagator version of the two-loop crossed box integral by

$$X^{**}_{1,1,2}(\sigma_1, \ldots, \sigma_9) \equiv \int_{\mathbb{R}^D} \frac{d^D\ell_1}{(2\pi)^D} \int_{\mathbb{R}^D} \frac{d^D\ell_2}{(2\pi)^D} \prod_{i=1}^{9} \frac{1}{\tilde{f}_i^{\sigma_i}(\ell_1, \ell_2)}\;, \tag{5.4}$$

for $\tilde{f}_8 = \ell_1 \cdot k_3$ and $\tilde{f}_9 = \ell_1 \cdot k_2$.

### 5.1 Parametrization of hepta-cut solutions

In the nonplanar case, the zero locus of the ideal generated by inverse propagators defines a reducible genus-3 algebraic curve. The global structure of the hepta-cut for any configuration of external legs and masses was previously uncovered by computational algebraic geometry [43]. In the purely massless limit, the zero locus decomposes into a union of eight components which are exactly the inequivalent hepta-cut solutions,

$$\tilde{\mathcal{S}} = \big\{(\ell_1, \ell_2) \in (\mathbb{C}^4)^{\otimes 2} \mid \tilde{f}_i(\ell_1, \ell_2) = 0\big\} = \tilde{\mathcal{S}}_1 \cup \cdots \cup \tilde{\mathcal{S}}_8\;. \tag{5.5}$$

The spinorial loop momentum parametrization (4.7) applies equally well to the nonplanar double box. However, we choose a slightly different normalization in $\ell_2$ to adjust the flow

| | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\beta_4$ |
|---|---|---|---|---|---|---|---|---|
| $\mathcal{S}_1$ | $\chi - z$ | 0 | $\chi(z-\chi-1)$ | 0 | 0 | 0 | $z$ | 0 |
| $\mathcal{S}_2$ | $\chi - z$ | 0 | 0 | $\chi(z-\chi-1)$ | 0 | 0 | 0 | $z$ |
| $\mathcal{S}_3$ | 0 | 0 | $z$ | 0 | 0 | 0 | $\chi$ | 0 |
| $\mathcal{S}_4$ | 0 | 0 | 0 | $z$ | 0 | 0 | 0 | $\chi$ |
| $\mathcal{S}_5$ | $\chi - z$ | 0 | 0 | $(\chi+1)(z-\chi)$ | 0 | 0 | $z$ | 0 |
| $\mathcal{S}_6$ | $\chi - z$ | 0 | $(\chi+1)(z-\chi)$ | 0 | 0 | 0 | 0 | $z$ |
| $\mathcal{S}_7$ | $-1$ | 0 | 0 | $z$ | 0 | 0 | $1+\chi$ | 0 |
| $\mathcal{S}_8$ | $-1$ | 0 | $z$ | 0 | 0 | 0 | 0 | $1+\chi$ |

**Table 1**. Local hepta-cut solutions of the massless four-point nonplanar double box.

direction in comparison with refs. [32, 35],

$$\ell_1^\mu = \alpha_1 k_1^\mu + \alpha_2 k_2^\mu + \frac{s_{12}\alpha_3}{2\langle 14\rangle[42]}\langle 1^-|\gamma^\mu|2^-\rangle + \frac{s_{12}\alpha_4}{2\langle 24\rangle[41]}\langle 2^-|\gamma^\mu|1^-\rangle\,, \tag{5.6}$$

$$\ell_2^\mu = \beta_1 k_3^\mu + \beta_2 k_4^\mu + \frac{s_{12}\beta_3}{2\langle 32\rangle[24]}\langle 3^-|\gamma^\mu|4^-\rangle + \frac{s_{12}\beta_4}{2\langle 42\rangle[23]}\langle 4^-|\gamma^\mu|3^-\rangle\,. \tag{5.7}$$

The hepta-cut equations were solved using this parametrization in refs. [32, 35] and the resulting eight solutions are quoted here in table 1.

### 5.2 Residues of the loop integrand

Once the seven inverse propagators have been expanded in the loop momentum parametrization, it is straightforward to derive the hepta-cuts of the nonplanar double box scalar integral with single propagators as nondegenerate multivariate residues [32],

$$X^{**}_{1,1,2}(1,\ldots,1,0,0)_{\mathcal{S}_{3,4}} = -\frac{1}{16s_{12}^3}\oint\frac{dz}{z(z+\chi)}\,, \tag{5.8}$$

$$X^{**}_{1,1,2}(1,\ldots,1,0,0)_{\mathcal{S}_{7,8}} = -\frac{1}{16s_{12}^3}\oint\frac{dz}{z(z-\chi-1)}\,, \tag{5.9}$$

$$X^{**}_{1,1,2}(1,\ldots,1,0,0)_{\mathcal{S}_{1,2,5,6}} = -\frac{1}{16s_{12}^3}\oint\frac{dz}{z(z-\chi)(z-\chi-1)}\,. \tag{5.10}$$

For the topology and kinematical configuration in consideration, there are two master integrals, for instance the scalar integral and a rank 1 tensor. Therefore a general integral can be written

$$X^{**}_{1,1,2}(\sigma_1,\ldots,\sigma_9) = C_1 X^{**}_{1,1,2}(1,\ldots,1,0,0) + C_2 X^{**}_{1,1,2}(1,\ldots,1,-1,0) + \cdots\,. \tag{5.11}$$

We will consider integrals with doubled and also tripled propagators,

$$X^{**}_{1,1,2}(2,1,\ldots,1,0,0) = C_1 X^{**}_{1,1,2}(1,\ldots,1,0,0) + C_2 X^{**}_{1,1,2}(1,\ldots,1,-1,0) + \cdots\,, \tag{5.12}$$

$$X^{**}_{1,1,2}(1,\ldots,1,3,0,0) = C_1' X^{**}_{1,1,2}(1,\ldots,1,0,0) + C_2' X^{**}_{1,1,2}(1,\ldots,1,-1,0) + \cdots\,, \tag{5.13}$$



JHEP07(2014)112

and reconstruct the coefficients in strictly four dimensions. Evaluating the hepta-cuts of the displayed integrals by means of degenerate multivariate residues yields

$$X^{**}_{1,1,2}(2,1,\ldots,1,0,0)_{\mathcal{S}_{3,4}} = +\frac{1}{16s_{12}^4}\oint dz\frac{1+(1+\chi)z}{z^2(z+\chi)^2}\,, \tag{5.14}$$

$$X^{**}_{1,1,2}(2,1,\ldots,1,0,0)_{\mathcal{S}_{7,8}} = +\frac{1}{16s_{12}^4}\oint dz\frac{1+\chi}{z^2(z-\chi-1)}\,, \tag{5.15}$$

$$X^{**}_{1,1,2}(2,1,\ldots,1,0,0)_{\mathcal{S}_{1,2,5,6}} = +\frac{1}{16s_{12}^4}\oint dz\frac{1}{z(z-\chi)^2(z-\chi-1)}\,, \tag{5.16}$$

$$X^{**}_{1,1,2}(1,\ldots,1,3,0,0)_{\mathcal{S}_{3,4}} = -\frac{1}{16s_{12}^5}\oint dz\frac{h(z)}{z(z+\chi)^5}\,, \tag{5.17}$$

$$X^{**}_{1,1,2}(1,\ldots,1,3,0,0)_{\mathcal{S}_{7,8}} = -\frac{1}{16s_{12}^5}\oint dz\frac{(1+\chi)^2}{z(z-\chi-1)^3}\,, \tag{5.18}$$

$$X^{**}_{1,1,2}(1,\ldots,1,3,0,0)_{\mathcal{S}_{1,2,5,6}} = -\frac{1}{16s_{12}^5}\oint dz\frac{1}{z(z-\chi)^3(z-\chi-1)}\,, \tag{5.19}$$

where the numerator function $h(z)$ is defined by

$$h(z) = \chi^4 - \chi^3(4z+1) + \chi^2(z(z+1)+1) + 2\chi z(z+1) + z^2\,. \tag{5.20}$$

It is easiest to compare results with refs. [32] if we eliminate all residues at infinity by the Global Residue Theorem and thus only encircle poles at the nodal points of the algebraic curve defined by the hepta-cut. In this case, the parametrization is holomorphic and there are no additional poles in tensor integrals. Referring to figure 2, the contour weights are

$$\Omega = (\omega_{1\cap 6},\, \omega_{1\cap 3},\, \omega_{1\cap 7}\,, \omega_{2\cap 5},\, \omega_{2\cap 4},\, \omega_{2\cap 8},\, \omega_{5\cap 3},\, \omega_{5\cap 7},\, \omega_{6\cap 4},\, \omega_{6\cap 8})\,. \tag{5.21}$$

The residues computed by the master integrals for contours in this ordering read

$$R_1 = \frac{1}{16\chi(1+\chi)s_{12}^3}(-1, 1+\chi, -\chi, -1, 1+\chi, -\chi, 1+\chi, -\chi, 1+\chi, -\chi)\,, \tag{5.22}$$

$$R_2 = \frac{1}{32s_{12}^2}(0,1,-1,0,1,-1,0,0,0,0)\,. \tag{5.23}$$

In advance of what follows, we also need to collect the residues of the integrals in question with doubled and tripled propagators,

$$\mathrm{Res}_{\{\mathcal{G}_i\}}X^{**}_{1,1,2}(2,1,\ldots,1,0,0) = \frac{1}{16(1+\chi)\chi^2 s_{12}^4}(-1, 1-\chi^2, \chi^2, -1, 1-\chi^2, \chi^2, 1-\chi^2, \chi^2, 1-\chi^2, \chi^2)\,, \tag{5.24}$$

$$\mathrm{Res}_{\{\mathcal{G}_i\}}X^{**}_{1,1,2}(1,\ldots,1,3,0,0) = \frac{1}{16(1+\chi)\chi^3 s_{12}^5}(-1, 1+\chi^3, -\chi^3, -1, 1+\chi^3, -\chi^3, 1+\chi^3, -\chi^3, 1+\chi^3, -\chi^3)\,. \tag{5.25}$$

### 5.3 Master integral projectors

By integrand-level reduction using `BasisDet` [38] we find the general form of the nonplanar double box numerator, parametrized by the four irreducible scalar products,

$$N = \sum_{a_1,\ldots,a_4} c_{a_1,\ldots,a_4} (\ell_1 \cdot \omega)^{a_1} (\ell_2 \cdot \omega)^{a_2} (\ell_1 \cdot k_4)^{a_3} (\ell_2 \cdot k_1)^{a_4} \,. \tag{5.26}$$

The basis consists of 19 spurious and 19 nonspurious terms. Insisting that the reduction onto the two master integrals is respected by the unitarity procedure yields a $38 \times 10$ matrix $M$ whose submatrices corresponding to the parity-odd and parity-even parts are rank 5 and 3 respectively. Notice that all IBP relations used in this calculation can be obtained from ref. [32]. The full residue matrix obtained by adding either of the master integral projectors,

$$\mathcal{M}_1 \cdot (R_1, R_2) = (1, 0)\,, \quad \mathcal{M}_2 \cdot (R_1, R_2) = (0, 1)\,, \tag{5.27}$$

to $M$ has rank 10, which guarantees that the master contours are unique. In detail, the projectors are characterized by the 10-tuples

$$\mathcal{M}_1 = 2\chi(1+\chi)s_{12}^3(-2,1,1,-2,1,1,1,1,1,1)\,, \tag{5.28}$$

$$\begin{aligned}\mathcal{M}_2 = 4s_{12}^2(&2(1+2\chi), 1-2\chi, -3-2\chi, 2(1+2\chi), 1-2\chi,\\ &-3-2\chi, 1-2\chi, -3-2\chi, 1-2\chi, -3-2\chi)\,.\end{aligned} \tag{5.29}$$

We can now take advantage of the projectors to extract the coefficients in eq. (5.24),

$$X^{**}_{1,1,2}(2,1,\ldots,1,0,0) = +\frac{1}{\chi s_{12}} X^{**}_{1,1,2}(1,\ldots,1,0,0) - \frac{4}{\chi s_{12}^2} X^{**}_{1,1,2}(1,\ldots,1,-1,0) + \cdots \tag{5.30}$$

and similarly for the integral with a tripled propagator,

$$X^{**}_{1,1,2}(1\ldots,1,3,0,0) = +\frac{1}{\chi^2 s_{12}^2} X^{**}_{1,1,2}(1,\ldots,1,0,0) - \frac{4(1-\chi)}{\chi^2 s_{12}^3} X^{**}_{1,1,2}(1,\ldots,1,-1,0) + \cdots \tag{5.31}$$

The validity of our predictions for the coefficients has been tested against IBP relations generated by `FIRE` [62]. Taking the $D = 4$ limit of the following identities,

$$\begin{aligned}X^{**}_{1,1,2}(2,1,\ldots,1,0,0) = &+\frac{(1+2\epsilon)(1+(3+2\chi)\epsilon)}{(1+\epsilon)\chi s_{12}} X^{**}_{1,1,2}(1,\ldots,1,0,0)\\ &-\frac{4(1+2\epsilon)(1+4\epsilon)}{(1+\epsilon)\chi s_{12}^2} X^{**}_{1,1,2}(1,\ldots,1,-1,0) + \cdots\end{aligned} \tag{5.32}$$

$$\begin{aligned}X^{**}_{1,1,2}&(1,\ldots,1,3,0,0) =\\ &+\frac{(1+2\epsilon)(2+(9(1+\epsilon)+2\chi(1+2\epsilon-2(1+\epsilon)\chi))\epsilon)}{(2+\epsilon)\chi^2 s_{12}^2} X^{**}_{1,1,2}(1,\ldots,1,0,0)\\ &-\frac{4(1+2\epsilon)(1+4\epsilon)(2-2\chi(1+\epsilon)+3\epsilon)}{(2+\epsilon)\chi^2 s_{12}^3} X^{**}_{1,1,2}(1,\ldots,1,-1,0) + \cdots\end{aligned} \tag{5.33}$$

shows that the results are consistent.

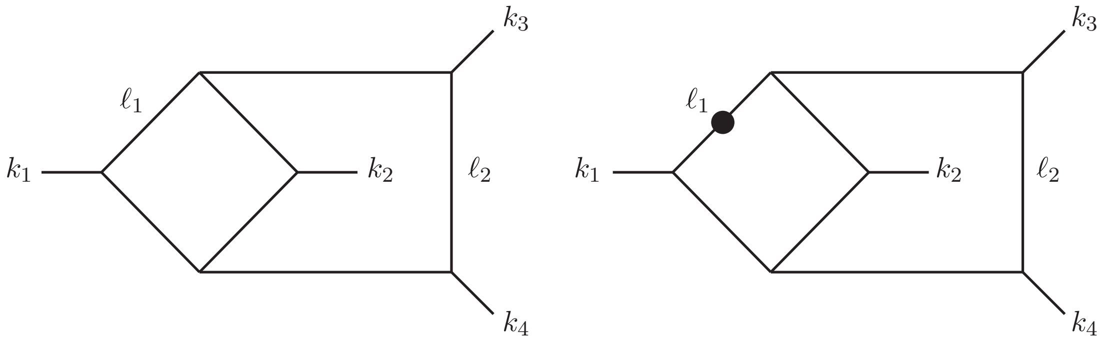


**Figure 4**. We use an integral basis for the massless four-point nonplanar double box that contains no tensor numerators. Instead we have (left) a scalar integral with single propagators and (right) a scalar integral with a doubled progator in the subbox.

We also examined hepta-cuts of all other four-point nonplanar double box integrals that have a doubled propagator. The results are similar to those presented here. Refer to appendix B for a complete list.

### 5.4 A scalar integral basis

The nonplanar double box amplitude contribution with four massless external legs has already been worked out in detail for an integral basis with a scalar integral and a rank 1 tensor [32, 35]. However, the reduction identities of the preceding subsection suggest an equivalent integral basis in which the tensor intergal is eliminated. We thus project the amplitude onto two scalar integrals. Our master equation reads (see also figure 4)

$$\mathcal{A}_4^{(2)} = C_1 X^{**}_{1,1,2}(1,\ldots,1,0,0) + C_2 X^{**}_{1,1,2}(2,1,\ldots,1,0,0) + \cdots \tag{5.34}$$

and we thus seek to determine $C_1$ and $C_2$.

It is not hard to show that the master integral projectors in this basis are

$$\begin{aligned}\mathcal{M}_1 = s_{12}^3(&-2(-1+2\chi^2),\, 1+2\chi^2,\, -3+2\chi^2,\, -2(-1+2\chi^2),\\ &1+2\chi^2,\, -3+2\chi^2,\, 1+2\chi^2,\, -3+2\chi^2,\, 1+2\chi^2,\, -3+2\chi^2)\,, \end{aligned}\tag{5.35}$$

$$\begin{aligned}\mathcal{M}_2 = \chi s_{12}^4(&-2(1+2\chi),\, -1+2\chi,\, 3+2\chi,\, -2(1+2\chi),\\ &-1+2\chi,\, 3+2\chi,\, -1+2\chi,\, 3+2\chi,\, -1+2\chi,\, 3+2\chi)\,, \end{aligned}\tag{5.36}$$

and the master integral coefficients in eq. (5.34) can then be written very compactly

$$C_i = -\frac{1}{16 s_{12}^3}\oint_{\mathcal{M}_i} \frac{dz}{z(z-\chi)(z-\chi-1)} \sum_{\substack{\text{helicities}\\ \text{particles}}} \prod_{k=1}^{6} A_{(k)}^{\text{tree}}(z)\,. \tag{5.37}$$

The coefficients produced here agree in $D = 4$ with those worked out in refs. [32, 35] as can be verified using the IBP identity (5.32).

## 6 Discussion

In this paper, we naturally generalized the maximal unitarity method to integrals with doubled propagators and provided a simple way of reducing integrals with doubled (or even higher-order) propagators onto a master integral basis. The residues of an integral with doubled propagators are degenerate, which cannot be directly calculated by Cauchy's theorem but can be evaluated by computational algebraic geometry methods (Gröbner basis). Then from the projector information, we obtain the master integral coefficients. This method has been successfully tested on several one-loop and two-loop examples.

Since the contour projector can be found by using IBPs without doubled propagators, our method implies that the complete set of IBPs (involving integrals with or without doubled propagators) can be derived from the set of IBPs without doubled propagators. Our method can also be used for converting between different integral bases.

So far, the maximal unitarity method for two-loop and higher-loop has been tested only for diagrams in the $D = 4$ limit. Therefore our paper only obtained the finite part of the reduction of integrals with doubled propagators, but not the $O(\epsilon)$ contribution.

On the other hand, in all our examples, the reduction coefficients of integral with doubled propagators are finite, i.e., without poles in $\epsilon$. It is not accidental: consider an integrand without doubled propagators $N/(f_1 \cdots f_k)$ in $n$ variables. If (1) its cut solution is $n - k$ dimensional, (2) the cut can be parameterized by a set of variables $z$, (3) the integration variables can be choosen to be $x$ and $z$, then we evaluate the multivariate residue regarding the ideal $I(z) = \langle f_1(x; z), \ldots, f_k(x; z)\rangle$ in $x$, parameterized by $z$,

$$\oint dz \oint dx \, \frac{N}{f_1 \ldots f_k} = \oint dz \, \frac{h(z)}{g(z)} \,. \tag{6.1}$$

For integrals with doubled propagators, a similar calculation regarding the ideal $\tilde{I}(z) = \langle f_1(x; z)^2, \ldots, f_k(x; z)\rangle$ gives

$$\oint dz \oint dx \, \frac{N}{f_1^2 \ldots f_k} = \oint dz \, \frac{\tilde{h}(z)}{\tilde{g}(z)} \,. \tag{6.2}$$

Note that $g(z_0) = 0$ if and only if the ideal $I(z_0)$ is not zero-dimensional. Since the ideals $I(z_0)$ and $\tilde{I}(z_0)$ have the same zero locus, $g(z_0) = 0$ if and only if $\tilde{g}(z_0) = 0$. Thus, $\tilde{g}$ and $g$ have the same zeros in $z$, just different multiplicities. Therefore the integral with doubled propagators does not generate new poles in $z$, and the residues are still finite. Hence the reduction coefficients are finite in the $D = 4$ limit.

There are several promising future directions. We expect that the maximal unitarity method (including integrals with doubled propagators) can be generalized to $D$-dimensional cases by a contour integral in the extra dimension and analytic continuation in $D$. Moreover, the reduction algorithm for integrals with higher powers of propagators should apply seamlessly to massive external legs.

For the computational aspect, the multivariate residue calculation can be sped up by using the relation between multivariate residues and the Bezoutian matrix [66]. Then we do not need to find the Gröbner basis in the lexicographic order.



JHEP07(2014)112

## Acknowledgments

We are grateful to Emil Bjerrum-Bohr, Simon Caron-Huot, Poul Henrik Damgaard, Tristan Dennen, Rijun Huang and David Kosower for useful discussions. It is a pleasure to thank Simon Badger and Hjalte Frellesvig for comments on the manuscript in draft stage. Both authors acknowledge the organizers and participants of the workshop *The Geometry and Physics of Scattering Amplitudes* at Simons Center for Geometry and Physics, Stony Brook University. We also express gratitude to Institut de Physique Théorique, CEA Saclay, and in particular David Kosower for hospitality during this project. The work of YZ is supported by Danish Council for Independent Research (FNU) grant 11-107241.

## A Planar double box hepta-cuts

For the sake of completeness, we include all hepta-cuts of four-point planar double box scalar integrals with a doubled propagator. The ordering of the propagators follows that of the main text.

$$P^{**}_{2,2}(2,1,\ldots,1,0,0)_{\mathcal{S}_{1,3}} = -\frac{1}{16s_{12}^4}\oint\frac{dz}{z(z+\chi)} \tag{A.1}$$

$$P^{**}_{2,2}(2,1,\ldots,1,0,0)_{\mathcal{S}_{2,4,5,6}} = -\frac{\chi}{16s_{12}^4}\oint\frac{dz}{z(z+\chi)^2} \tag{A.2}$$

$$P^{**}_{2,2}(1,2,1,\ldots,1,0,0)_{\mathcal{S}_{1,3}} = -\frac{1}{16s_{12}^4}\oint\frac{dz}{z(z+\chi)^2} \tag{A.3}$$

$$P^{**}_{2,2}(1,2,1,\ldots,1,0,0)_{\mathcal{S}_{2,4,5,6}} = +\frac{1}{16s_{12}^4}\oint dz\frac{\chi(1+\chi)+z(1+2\chi)}{z^2(z+\chi)^2} \tag{A.4}$$

$$P^{**}_{2,2}(1,1,2,1,\ldots,1,0,0)_{\mathcal{S}_{1,3}} = -\frac{1}{16s_{12}^4}\oint\frac{dz}{z(z+\chi)} \tag{A.5}$$

$$P^{**}_{2,2}(1,1,2,1,\ldots,1,0,0)_{\mathcal{S}_{2,4,5,6}} = -\frac{\chi}{16s_{12}^4}\oint\frac{dz}{z(z+\chi)^2} \tag{A.6}$$

$$P^{**}_{2,2}(1,1,1,2,1,1,1,0,0)_{\mathcal{S}_{1,3}} = -\frac{\chi}{16s_{12}^4}\oint\frac{dz}{z(z+\chi)^2} \tag{A.7}$$

$$P^{**}_{2,2}(1,1,1,2,1,1,1,0,0)_{\mathcal{S}_{2,4,5,6}} = -\frac{1}{16s_{12}^4}\oint\frac{dz}{z(z+\chi)} \tag{A.8}$$

$$P^{**}_{2,2}(1,\ldots,1,2,1,1,0,0)_{\mathcal{S}_{1,3}} = +\frac{1}{16s_{12}^4}\oint dz\frac{\chi(1+\chi)+z(1+2\chi)}{z^2(z+\chi)^2} \tag{A.9}$$

$$P^{**}_{2,2}(1,\ldots,1,2,1,1,0,0)_{\mathcal{S}_{2,4,5,6}} = -\frac{1}{16s_{12}^4}\oint\frac{dz}{z(z+\chi)^2} \tag{A.10}$$

$$P^{**}_{2,2}(1,\ldots,1,2,1,0,0)_{\mathcal{S}_{1,3}} = -\frac{\chi}{16s_{12}^4}\oint\frac{dz}{z(z+\chi)^2}\,, \tag{A.11}$$

$$P^{**}_{2,2}(1,\ldots,1,2,1,0,0)_{\mathcal{S}_{2,4,5,6}} = -\frac{1}{16s_{12}^4}\oint\frac{dz}{z(z+\chi)} \tag{A.12}$$

$$P^{**}_{2,2}(1,\ldots,1,2,0,0)_{\mathcal{S}_{i}} = -\frac{1}{16s_{12}^4}\oint\frac{dz}{z(z+\chi)^2} \tag{A.13}$$



JHEP07(2014)112

## B Nonplanar double box hepta-cuts

We also provide explicit forms of the hepta-cuts of all four-point nonplanar double box integrals with a single doubled propagator and a scalar numerator. The overall signs are determined by consistency of relations among residues.

$$X^{**}_{1,1,2}(2,1,\ldots,1,0,0)_{\mathcal{S}_{3,4}} = +\frac{1}{16s_{12}^4}\oint dz \frac{\chi+(1+\chi)z}{z^2(z+\chi)^2} \tag{B.1}$$

$$X^{**}_{1,1,2}(2,1,\ldots,1,0,0)_{\mathcal{S}'_{7,8}} = +\frac{1}{16s_{12}^4}\oint dz \frac{1+\chi}{z^2(z-\chi-1)} \tag{B.2}$$

$$X^{**}_{1,1,2}(2,1,\ldots,1,0,0)_{\mathcal{S}'_{1,2,5,6}} = +\frac{1}{16s_{12}^4}\oint dz \frac{1}{z(z-\chi)^2(z-\chi-1)} \tag{B.3}$$

$$X^{**}_{1,1,2}(1,2,1\ldots,1,0,0)_{\mathcal{S}'_{3,4}} = -\frac{1}{16s_{12}^4}\oint dz \frac{\chi}{z^2(z+\chi)}\ , \tag{B.4}$$

$$X^{**}_{1,1,2}(1,2,1,\ldots,1,0,0)_{\mathcal{S}'_{7,8}} = -\frac{1}{16s_{12}^4}\oint dz \frac{1+\chi(1+z)}{z^2(z-\chi-1)^2} \tag{B.5}$$

$$X^{**}_{1,1,2}(1,2,1,\ldots,1,0,0)_{\mathcal{S}'_{1,2,5,6}} = -\frac{1}{16s_{12}^4}\oint dz \frac{1}{z(z-\chi)(z-\chi-1)^2} \tag{B.6}$$

$$X^{**}_{1,1,2}(1,1,2,1,\ldots,1,0,0)_{\mathcal{S}'_{3,4}} = -\frac{1}{16s_{12}^4}\oint dz \frac{2\chi+z}{z(z+\chi)^2} \tag{B.7}$$

$$X^{**}_{1,1,2}(1,1,2,1,\ldots,1,0,0)_{\mathcal{S}'_{7,8}} = +\frac{1}{16s_{12}^4}\oint dz \frac{2(1+\chi)-z}{z(z-\chi-1)^2} \tag{B.8}$$

$$X^{**}_{1,1,2}(1,1,2,1,\ldots,1,0,0)_{\mathcal{S}'_{1,2,5,6}} = -\frac{1}{16s_{12}^4}\oint dz \frac{2\chi(1+\chi-z)-z}{z(z-\chi)^2(z-\chi-1)^2} \tag{B.9}$$

$$X^{**}_{1,1,2}(1,1,1,2,1,1,1,0,0)_{\mathcal{S}'_{3,4}} = -\frac{1}{16s_{12}^4}\oint dz \frac{1}{z(z+\chi)^2} \tag{B.10}$$

$$X^{**}_{1,1,2}(1,1,1,2,1,1,1,0,0)_{\mathcal{S}'_{7,8}} = -\frac{1}{16s_{12}^4}\oint dz \frac{1}{z(z-\chi-1)^2} \tag{B.11}$$

$$X^{**}_{1,1,2}(1,1,1,2,1,1,1,0,0)_{\mathcal{S}'_{1,2,5,6}} = -\frac{1}{16s_{12}^4}\oint dz \frac{(1+2(\chi-z))(\chi(1+\chi)-(1+2\chi)z)}{z^2(z-\chi)^2(z-\chi-1)^2} \tag{B.12}$$

$$X^{**}_{1,1,2}(1,\ldots,1,2,1,1,0,0)_{\mathcal{S}'_{3,4}} = -\frac{1}{16s_{12}^4}\oint dz \frac{2\chi+z}{z(z+\chi)^2} \tag{B.13}$$

$$X^{**}_{1,1,2}(1,\ldots,1,2,1,1,0,0)_{\mathcal{S}'_{7,8}} = +\frac{1}{16s_{12}^4}\oint dz \frac{2(1+\chi)-z}{z(z-\chi-1)^2} \tag{B.14}$$

$$X^{**}_{1,1,2}(1,\ldots,1,2,1,1,0,0)_{\mathcal{S}'_{1,2,5,6}} = +\frac{1}{16s_{12}^4}\oint dz \frac{2\chi(z-\chi-1)+z}{z(z-\chi)^2(z-\chi-1)^2} \tag{B.15}$$

$$X^{**}_{1,1,2}(1,\ldots,1,2,1,0,0)_{\mathcal{S}'_{3,4}} = -\frac{1}{16s_{12}^4}\oint dz \frac{2\chi+z}{z(z+\chi)^2} \tag{B.16}$$

$$X^{**}_{1,1,2}(1,\ldots,1,2,1,0,0)_{\mathcal{S}'_{7,8}} = +\frac{1}{16s_{12}^4}\oint dz \frac{2(1+\chi)-z}{z(z-\chi-1)^2} \tag{B.17}$$

$$X^{**}_{1,1,2}(1,\ldots,1,2,1,0,0)_{\mathcal{S}'_{1,2,5,6}} = +\frac{1}{16s_{12}^4}\oint dz \frac{2\chi(z-\chi-1)+z}{z(z-\chi)^2(z-\chi-1)^2} \tag{B.18}$$

JHEP07(2014)112

$$X^{**}_{1,1,2}(1,\ldots,1,2,0,0)_{\mathcal{S}'_{3,4}} = -\frac{1}{16s_{12}^4}\oint dz \frac{\chi(1-\chi)+(1+\chi)z}{z(z+\chi)^3} \tag{B.19}$$

$$X^{**}_{1,1,2}(1,\ldots,1,2,0,0)_{\mathcal{S}'_{7,8}} = -\frac{1}{16s_{12}^4}\oint dz \frac{1+\chi}{z(z-\chi-1)^2}\,, \tag{B.20}$$

$$X^{**}_{1,1,2}(1,\ldots,1,2,0,0)_{\mathcal{S}'_{1,2,5,6}} = +\frac{1}{16s_{12}^4}\oint dz \frac{1}{z(z-\chi)^2(z-\chi-1)} \tag{B.21}$$



## References

[1] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *Fusing gauge theory tree amplitudes into loop amplitudes*, *Nucl. Phys.* **B 435** (1995) 59 [hep-ph/9409265] [INSPIRE].

[2] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One loop n point gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys.* **B 425** (1994) 217 [hep-ph/9403226] [INSPIRE].

[3] Z. Bern and A.G. Morgan, *Massive loop amplitudes from unitarity*, *Nucl. Phys.* **B 467** (1996) 479 [hep-ph/9511336] [INSPIRE].

[4] Z. Bern, L.J. Dixon and D.A. Kosower, *One loop amplitudes for $e^+e^-$ to four partons*, *Nucl. Phys.* **B 513** (1998) 3 [hep-ph/9708239] [INSPIRE].

[5] R. Britto, F. Cachazo and B. Feng, *Generalized unitarity and one-loop amplitudes in $N=4$ super-Yang-Mills*, *Nucl. Phys.* **B 725** (2005) 275 [hep-th/0412103] [INSPIRE].

[6] D. Forde, *Direct extraction of one-loop integral coefficients*, *Phys. Rev.* **D 75** (2007) 125019 [arXiv:0704.1835] [INSPIRE].

[7] R. Britto, F. Cachazo and B. Feng, *Computing one-loop amplitudes from the holomorphic anomaly of unitarity cuts*, *Phys. Rev.* **D 71** (2005) 025012 [hep-th/0410179] [INSPIRE].

[8] Z. Bern, N.E.J. Bjerrum-Bohr, D.C. Dunbar and H. Ita, *Recursive calculation of one-loop QCD integral coefficients*, *JHEP* **11** (2005) 027 [hep-ph/0507019] [INSPIRE].

[9] S.J. Bidder, N.E.J. Bjerrum-Bohr, D.C. Dunbar and W.B. Perkins, *One-loop gluon scattering amplitudes in theories with $N<4$ supersymmetries*, *Phys. Lett.* **B 612** (2005) 75 [hep-th/0502028] [INSPIRE].

[10] R. Britto, E. Buchbinder, F. Cachazo and B. Feng, *One-loop amplitudes of gluons in SQCD*, *Phys. Rev.* **D 72** (2005) 065012 [hep-ph/0503132] [INSPIRE].

[11] R. Britto, B. Feng and P. Mastrolia, *The Cut-constructible part of QCD amplitudes*, *Phys. Rev.* **D 73** (2006) 105004 [hep-ph/0602178] [INSPIRE].

[12] P. Mastrolia, *On Triple-cut of scattering amplitudes*, *Phys. Lett.* **B 644** (2007) 272 [hep-th/0611091] [INSPIRE].

[13] A. Brandhuber, S. McNamara, B.J. Spence and G. Travaglini, *Loop amplitudes in pure Yang-Mills from generalised unitarity*, *JHEP* **10** (2005) 011 [hep-th/0506068] [INSPIRE].



JHEP07(2014)112

[14] G. Ossola, C.G. Papadopoulos and R. Pittau, *Reducing full one-loop amplitudes to scalar integrals at the integrand level*, *Nucl. Phys.* **B 763** (2007) 147 [hep-ph/0609007] [INSPIRE].

[15] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, *Unitarity cuts and Reduction to master integrals in d dimensions for one-loop amplitudes*, *JHEP* **03** (2007) 111 [hep-ph/0612277] [INSPIRE].

[16] S.D. Badger, *Direct Extraction Of One Loop Rational Terms*, *JHEP* **01** (2009) 049 [arXiv:0806.4600] [INSPIRE].

[17] W.T. Giele, Z. Kunszt and K. Melnikov, *Full one-loop amplitudes from tree amplitudes*, *JHEP* **04** (2008) 049 [arXiv:0801.2237] [INSPIRE].

[18] R. Britto and B. Feng, *Unitarity cuts with massive propagators and algebraic expressions for coefficients*, *Phys. Rev.* **D 75** (2007) 105006 [hep-ph/0612089] [INSPIRE].

[19] R. Britto and B. Feng, *Integral coefficients for one-loop amplitudes*, *JHEP* **02** (2008) 095 [arXiv:0711.4284] [INSPIRE].

[20] Z. Bern, J.J. Carrasco, T. Dennen, Y.-t. Huang and H. Ita, *Generalized Unitarity and Six-Dimensional Helicity*, *Phys. Rev.* **D 83** (2011) 085022 [arXiv:1010.0494] [INSPIRE].

[21] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, *D-dimensional unitarity cut method*, *Phys. Lett.* **B 645** (2007) 213 [hep-ph/0609191] [INSPIRE].

[22] Z. Bern, J.S. Rozowsky and B. Yan, *Two loop four gluon amplitudes in $N = 4$ super Yang-Mills*, *Phys. Lett.* **B 401** (1997) 273 [hep-ph/9702424] [INSPIRE].

[23] Z. Bern, L.J. Dixon and D.A. Kosower, *A Two loop four gluon helicity amplitude in QCD*, *JHEP* **01** (2000) 027 [hep-ph/0001001] [INSPIRE].

[24] Z. Bern, A. De Freitas and L.J. Dixon, *Two loop helicity amplitudes for gluon-gluon scattering in QCD and supersymmetric Yang-Mills theory*, *JHEP* **03** (2002) 018 [hep-ph/0201161] [INSPIRE].

[25] R. Britto, F. Cachazo and B. Feng, *New recursion relations for tree amplitudes of gluons*, *Nucl. Phys.* **B 715** (2005) 499 [hep-th/0412308] [INSPIRE].

[26] R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, *Phys. Rev. Lett.* **94** (2005) 181602 [hep-th/0501052] [INSPIRE].

[27] J. Gluza, K. Kajda and D.A. Kosower, *Towards a Basis for Planar Two-Loop Integrals*, *Phys. Rev.* **D 83** (2011) 045012 [arXiv:1009.0472] [INSPIRE].

[28] D.A. Kosower and K.J. Larsen, *Maximal Unitarity at Two Loops*, *Phys. Rev.* **D 85** (2012) 045017 [arXiv:1108.1180] [INSPIRE].

[29] S. Caron-Huot and K.J. Larsen, *Uniqueness of two-loop master contours*, *JHEP* **10** (2012) 026 [arXiv:1205.0801] [INSPIRE].

[30] H. Johansson, D.A. Kosower and K.J. Larsen, *Two-Loop Maximal Unitarity with External Masses*, *Phys. Rev.* **D 87** (2013) 025030 [arXiv:1208.1754] [INSPIRE].

[31] H. Johansson, D.A. Kosower and K.J. Larsen, *Maximal Unitarity for the Four-Mass Double Box*, *Phys. Rev.* **D 89** (2014) 125010 [arXiv:1308.4632] [INSPIRE].

[32] M. Søgaard, *Global Residues and Two-Loop Hepta-Cuts*, *JHEP* **09** (2013) 116 [arXiv:1306.1496] [INSPIRE].


JHEP07(2014)112

[33] M. Søgaard and Y. Zhang, *Multivariate Residues and Maximal Unitarity*, *JHEP* **12** (2013) 008 [arXiv:1310.6006] [INSPIRE].

[34] K.J. Larsen, *Global Poles of the Two-Loop Six-Point $N = 4$ SYM integrand*, *Phys. Rev.* **D 86** (2012) 085032 [arXiv:1205.0297] [INSPIRE].

[35] S. Badger, H. Frellesvig and Y. Zhang, *Hepta-Cuts of Two-Loop Scattering Amplitudes*, *JHEP* **04** (2012) 055 [arXiv:1202.2019] [INSPIRE].

[36] P. Mastrolia and G. Ossola, *On the Integrand-Reduction Method for Two-Loop Scattering Amplitudes*, *JHEP* **11** (2011) 014 [arXiv:1107.6041] [INSPIRE].

[37] S. Badger, H. Frellesvig and Y. Zhang, *An Integrand Reconstruction Method for Three-Loop Amplitudes*, *JHEP* **08** (2012) 065 [arXiv:1207.2976] [INSPIRE].

[38] Y. Zhang, *Integrand-Level Reduction of Loop Amplitudes by Computational Algebraic Geometry Methods*, *JHEP* **09** (2012) 042 [arXiv:1205.5707] [INSPIRE].

[39] S. Badger, H. Frellesvig and Y. Zhang, *A Two-Loop Five-Gluon Helicity Amplitude in QCD*, *JHEP* **12** (2013) 045 [arXiv:1310.1051] [INSPIRE].

[40] B. Feng and R. Huang, *The classification of two-loop integrand basis in pure four-dimension*, *JHEP* **02** (2013) 117 [arXiv:1209.3747] [INSPIRE].

[41] P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, *Scattering Amplitudes from Multivariate Polynomial Division*, *Phys. Lett.* **B 718** (2012) 173 [arXiv:1205.7087] [INSPIRE].

[42] P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, *Integrand-Reduction for Two-Loop Scattering Amplitudes through Multivariate Polynomial Division*, *Phys. Rev.* **D 87** (2013) 085026 [arXiv:1209.4319] [INSPIRE].

[43] R. Huang and Y. Zhang, *On Genera of Curves from High-loop Generalized Unitarity Cuts*, *JHEP* **04** (2013) 080 [arXiv:1302.1023] [INSPIRE].

[44] P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, *Multiloop Integrand Reduction for Dimensionally Regulated Amplitudes*, *Phys. Lett.* **B 727** (2013) 532 [arXiv:1307.5832] [INSPIRE].

[45] B. Feng, J. Zhen, R. Huang and K. Zhou, *Integral Reduction by Unitarity Method for Two-loop Amplitudes: A Case Study*, arXiv:1401.6766 [INSPIRE].

[46] E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, *Two loop QCD corrections to gluon-gluon scattering*, *Nucl. Phys.* **B 605** (2001) 467 [hep-ph/0102201] [INSPIRE].

[47] C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, *Two-loop QCD corrections to the scattering of massless distinct quarks*, *Nucl. Phys.* **B 601** (2001) 318 [hep-ph/0010212] [INSPIRE].

[48] C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, *Two loop QCD corrections to massless identical quark scattering*, *Nucl. Phys.* **B 601** (2001) 341 [hep-ph/0011094] [INSPIRE].

[49] C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, *Two loop QCD corrections to massless quark gluon scattering*, *Nucl. Phys.* **B 605** (2001) 486 [hep-ph/0101304] [INSPIRE].

[50] E.I. Buchbinder and F. Cachazo, *Two-loop amplitudes of gluons and octa-cuts in $N = 4$ super Yang-Mills*, *JHEP* **11** (2005) 036 [hep-th/0506126] [INSPIRE].

[51] F. Cachazo, *Sharpening The Leading Singularity*, arXiv:0803.1988 [INSPIRE].

[52] R.M. Schabinger, *A New Algorithm For The Generation Of Unitarity-Compatible Integration By Parts Relations*, *JHEP* **01** (2012) 077 [arXiv:1111.4220] [INSPIRE].

[53] J.M. Henn, *Multiloop integrals in dimensional regularization made simple*, *Phys. Rev. Lett.* **110** (2013) 251601 [arXiv:1304.1806] [INSPIRE].

[54] J.-H. Zhang, *Multidimensional Residues for Feynman Integrals with Generic Power of Propagators*, arXiv:1112.4136 [INSPIRE].

[55] Z. Bern, J.J.M. Carrasco, H. Ita, H. Johansson and R. Roiban, *On the Structure of Supersymmetric Sums in Multi-Loop Unitarity Cuts*, *Phys. Rev.* **D 80** (2009) 065029 [arXiv:0903.5348] [INSPIRE].

[56] M. Sogaard, *Supersums for all supersymmetric amplitudes*, *Phys. Rev.* **D 84** (2011) 065011 [arXiv:1106.3785] [INSPIRE].

[57] V.A. Smirnov, *Analytical result for dimensionally regularized massless on shell double box*, *Phys. Lett.* **B 460** (1999) 397 [hep-ph/9905323] [INSPIRE].

[58] V.A. Smirnov and O.L. Veretin, *Analytical results for dimensionally regularized massless on-shell double boxes with arbitrary indices and numerators*, *Nucl. Phys.* **B 566** (2000) 469 [hep-ph/9907385] [INSPIRE].

[59] J.B. Tausk, *Nonplanar massless two loop Feynman diagrams with four on-shell legs*, *Phys. Lett.* **B 469** (1999) 225 [hep-ph/9909506] [INSPIRE].

[60] C. Anastasiou, T. Gehrmann, C. Oleari, E. Remiddi and J.B. Tausk, *The Tensor reduction and master integrals of the two loop massless crossed box with lightlike legs*, *Nucl. Phys.* **B 580** (2000) 577 [hep-ph/0003261] [INSPIRE].

[61] K.G. Chetyrkin and F.V. Tkachov, *Integration by Parts: The Algorithm to Calculate $\beta$-functions in 4 Loops*, *Nucl. Phys.* **B 192** (1981) 159 [INSPIRE].

[62] A.V. Smirnov and V.A. Smirnov, *FIRE4, LiteRed and accompanying tools to solve integration by parts relations*, *Comput. Phys. Commun.* **184** (2013) 2820 [arXiv:1302.5885] [INSPIRE].

[63] A. von Manteuffel and C. Studerus, *Reduze 2 - Distributed Feynman Integral Reduction*, arXiv:1201.4330 [INSPIRE].

[64] P. Griffiths, J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience [John Wiley & Sons], New York (1978).

[65] R. Hartshorne, *Algebraic Geometry*, *Graduate Texts in Mathematics, No. 52*, Springer-Verlag, New York (1977).

[66] E. Cattani and A. Dickenstein. *Introduction to residues and resultants: Solving polynomial equations*, Springer Berlin Heidelberg (2005).

[67] D.R. Grayson and M.E. Stillman, *Macaulay2, a software system for research in algebraic geometry*. Available at http://www.math.uiuc.edu/Macaulay2.

JHEP07(2014)112