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# Trirefringence and the M5-brane

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Abstract: The Hamiltonian formulation for nonlinear chiral 2-form electrodynamics in six-dimensional Minkowski spacetime is used to show that small-amplitude plane-wave perturbations of a generic uniform constant 'magnetic' background exhibit trirefringence: all three independent wave-polarisations have distinct dispersion relations. While two coincide for Lorentz invariant theories, all three coincide uniquely for the chiral 2-form theory on the worldvolume of the M5-brane of M-theory. We argue that this is because, in this M-theory context, the waves propagate in a planar M5-M2-M2 bound-state preserving 16 supersymmetries. We also show how our results imply analogous results for nonlinear electrodynamics in a Minkowski spacetime of five and four dimensions.

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# 1 Introduction

The dispersion relation for an electromagnetic wave in an optically anisotropic medium is typically polarisation dependent; this is the phenomenon of birefringence. In theories of nonlinear electrodynamics (NLED), such as those arising as effective field theories for QED or SQED, a constant uniform electromagnetic background can be interpreted as an optical medium for small-amplitude plane-wave disturbances, which typically exhibit birefringence [1]. However, the Born-Infeld (BI) theory [2] is an exception to the rule [3]; in fact, Boillat [4] and Plebanski [5] have shown (in the closely related context of shock waves) that BI is the unique NLED with a weak-field limit for which there is no birefringence. There are others without a weak-field limit but, in contrast to BI, they are not electromagnetic-duality invariant [6].

Implicit in the above summary of NLED birefringence results of relevance here is the choice of a four-dimensional (4D) Minkowski spacetime. In higher dimensions there are more independent polarisations. In the 5D case, for example, the gauge vector field has three independent polarisations and one could expect to find three distinct dispersion relations for small-amplitude plane waves in a generic constant uniform electromagnetic background; i.e. *trirefringence*. As far as we are aware, this possibility has not been investigated, presumably because there is no obvious physical motivation but it is also less interesting from a purely theoretical perspective since one cannot expect to find any conformal limits of such theories. However, 5D NLEDs can be obtained by dimensional reduction from nonlinear theories of 6D chiral 2-form electrodynamics, without truncation since the chiral restriction ensures that there are still only three independent polarisations [7]. In this new context there are both weak-field and strong-field limits to interacting conformal chiral 2-form theories [8].

We have also shown in [8], extending observations of Perry and Schwarz [9], that there is a one-to-one correspondence, assuming Lorentz-invariance, between 6D chiral 2-form theories and 4D NLEDs that are electromagnetic duality invariant. This correspondence suggests that the 6D partner to the 4D BI theory will have special trirefringence properties. In fact, we find that it is the unique chiral 2-form electrodynamics theory for which all three dispersion relations coincide; i.e. the unique "zero-trirefringence" theory. This is an apparently stronger uniqueness result than could have been expected from the "zero-birefringence" property of the 4D BI theory because more conditions are needed to ensure coincidence of three dispersion relations than are needed for two. However, Lorentz invariance is not manifest in the Hamiltonian formulation used here and, as we shall show, 6D Lorentz invariance requires, by itself, a coincidence of two of the three dispersion relations.

Previous investigations into bifrefringence in the 4D NLED context have all started with a manifestly Lorentz invariant Lagrangian function of the electric and magnetic fields. The analogous starting point for 6D chiral 2-form electrodynamics is not immediately available because the (nonlinear) chirality condition on the 3-form field strength already implies the field equations. This difficulty can be circumvented by the inclusion of additional fields, in various ways but never without the need for some other non-manifest symmetry that imposes constraints on interactions (see [8] and references therein). As we briefly review below, chirality is trivially incorporated in the Hamiltonian formulation, which also applies to any theory with the same phase-space as the free-field theory, irrespective of whether it is Lorentz invariant.

Our motivation for the investigation leading to these results comes from the importance of BI and its 6D chiral 2-form partner in String/M-theory. The worldvolume action for the D3-brane of IIB superstring theory is (for suitable boundary conditions) the $\mathcal{N}=4$ supersymmetrization of the 4D BI theory [10]. The worldvolume action for the M5-brane of M-theory [11, 12] can be similarly interpreted as the $(2,0)$-supersymmetrization of the 6D chiral 2-form electrodynamics partner to the 4D BI theory [9]. In this context the 4D/6D pairing is a reflection of the String/M-theory dualities that relate the D3-brane with the M5-brane [13, 14]. We leave to the end of this article a discussion of implications of our results in this domain.

## 2 Hamiltonian field equations

In the Hamiltonian formulation of 6D chiral 2-form electrodynamics, the only independent field is a 2-form gauge potential $A$ on the Euclidean 5-space. For time-space coordinates $\{t, x^i; i=1,\ldots,5\}$, the phase-space Lagrangian density takes the form [7, 8]

$$\mathcal{L} = \frac{1}{2}\dot{A}\cdot B - \mathcal{H}(B)\,, \tag{2.1}$$

where $\dot{A} = \partial_t A$ and $B$ is the 'magnetic' 2-form field, with components

$$B^{ij} = (\boldsymbol{\nabla}\times A)^{ij} := \frac{1}{2}\varepsilon^{ijklm}\partial_k A_{lm}\,, \tag{2.2}$$

and $C\cdot C' = \frac{1}{2}C^{ij}C'_{ij}$ for any two 5-space 2-forms $(C, C')$. The Hamiltonian density $\mathcal{H}$ and the 5-vector field-momentum density $\mathbf{p}$, with components

$$p_i := (B\times B)_i = \frac{1}{8}\varepsilon_{ijklm}B^{jk}B^{lm}\,, \tag{2.3}$$



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are the Noether charge densities associated to time and space translation invariance; we follow here the notation and conventions of [8]:

$$\dot{B} = \boldsymbol{\nabla} \times H\,, \qquad H := \partial\mathcal{H}/\partial B\,. \tag{2.4}$$

A basis for rotationally invariant functions of $B$ is

$$s = \frac{1}{2}|B|^2 = \frac{1}{4}B^{ij}B_{ij}\,, \qquad p \equiv |\mathbf{p}| = |B \times B|\,, \tag{2.5}$$

but it is convenient to impose rotational invariance by requiring $\mathcal{H}$ to be a function of $(s, p^2)$, in which case

$$H = \mathcal{H}_s B + 2\mathcal{H}_{p^2}\left(\mathbf{p} \times B\right), \tag{2.6}$$

where, here and below, subscripts $s$ and $p^2$ denote partial derivatives with respect to these independent variables.

Lorentz boost invariance remains non-manifest but it is a symmetry iff $B \times B = H \times H$ (assuming unit speed of light) [8], and this is equivalent to

$$\mathcal{I} := \mathcal{H}_s^2 + 4s\mathcal{H}_s\mathcal{H}_{p^2} + 4p^2\mathcal{H}_{p^2}^2 = 1\,. \tag{2.7}$$

The choice $\mathcal{H} = s$ yields the free field theory.

### 2.1 Expansion about a constant background

For any choice of $\mathcal{H}$, the field equations are solved by $B = \bar{B}$, where $\bar{B}$ is both uniform and constant. We may expand the field equation for $B$ about this 'background' solution, which can then be viewed as a stationary homogeneous 'optical' fluid medium of energy density $\bar{\mathcal{H}} = \mathcal{H}(\bar{B})$ and momentum density $\bar{\mathbf{p}}$. We may consider perturbations about this background by setting $A = \bar{A} + \mathrm{a}$, where $\boldsymbol{\nabla} \times \bar{A} = \bar{B}$ (note that $\bar{A}$ cannot be uniform for non-zero $\bar{B}$). This implies that

$$B = \bar{B} + \mathrm{b}\,, \qquad \mathrm{b} = \boldsymbol{\nabla} \times \mathrm{a}\,, \tag{2.8}$$

and hence that $\partial_i b^{ij} \equiv 0$. Expanding the field equation (2.4) to first order in $b$ we find that

$$\dot{b} = \boldsymbol{\nabla} \times h(b)\,, \tag{2.9}$$

where $h(b)$ is a two-form depending linearly on $b$:

$$\begin{aligned} h_{ij} = Q\, b_{ij} &+ 2\bar{\mathcal{H}}_{p^2}\left(\bar{B}b\bar{B} + \bar{B}^2 b + b\bar{B}^2\right)_{ij} \\ &+ \left[(Y + 4\bar{s}X)(\bar{B}\cdot b) + 2X(\bar{B}^3\cdot b)\right]\bar{B}_{ij} \\ &+ 2\left[X(\bar{B}\cdot b) + 2\bar{\mathcal{H}}_{p^2p^2}(\bar{B}^3\cdot b)\right](\bar{B}^3)_{ij} \end{aligned} \tag{2.10}$$

for coefficient functions

$$\begin{aligned} Q &= \bar{\mathcal{H}}_s + 4\bar{s}\bar{\mathcal{H}}_{p^2}\,, \\ X &= \bar{\mathcal{H}}_{sp^2} + 4\bar{s}\,\bar{\mathcal{H}}_{p^2p^2}\,, \\ Y &= 4\bar{\mathcal{H}}_{p^2} + \bar{\mathcal{H}}_{ss} + 4\bar{s}\,\bar{\mathcal{H}}_{sp^2}\,. \end{aligned} \tag{2.11}$$

In what follows we shall omit the bars on the background values of $(s, p)$, and on $\mathcal{H}$ and its derivatives; it should be clear from the context when we are considering a constant uniform background and when we are considering generic field configurations. However, we will retain the $\bar{B}$ notation for the background value of $B$.

### 2.2 Plane waves on the background

For plane-wave solutions of (2.9) with angular frequency $\omega$ and wave 5-vector $\mathbf{k}$, the amplitudes $\mathrm{b}_{ij}$ satisfy

$$-\omega\mathrm{b} = \mathbf{k}\times h(\mathrm{b})\,, \qquad k_i\mathrm{b}^{ij} = 0\,, \tag{2.12}$$

but the second of these equations is implied by the first unless $\omega = 0$. The first equation can be written as a $10\times 10$ matrix equation of the form $M(\omega,k)\underline{\mathrm{b}} = 0$, where the components of $\underline{\mathrm{b}}$ are the ten independent components of the 2-form b. Each non-zero solution corresponds to a zero of $\det M(\omega,k)$, a 10th-order polynomial in $\omega$.

Using the $O(5)$ rotation/reflection symmetry, we may choose the 5-space axes such that the only non-zero components of $\bar{B}$ are

$$\bar{B}^{12} = -\bar{B}^{21} = \mathrm{B}_1\,, \qquad \bar{B}^{34} = -\bar{B}^{43} = \mathrm{B}_2\,, \tag{2.13}$$

for constants $\mathrm{B}_1 \geq \mathrm{B}_2 \geq 0$, so that $\mathrm{B}_1\mathrm{B}_2 = p$. This canonical form for $\bar{B}$ preserves an $\mathrm{SO}(2)\times\mathrm{SO}(2)$ subgroup of $O(5)$, which we may use to set

$$k_2 = k_4 = 0\,. \tag{2.14}$$

The matrix $M(\omega,k)$ is now block diagonal if we choose the first four components of $\underline{\mathrm{b}}$ to be $(\mathrm{b}_{24},\mathrm{b}_{13},\mathrm{b}_{15},\mathrm{b}_{35})$, so its determinant must factorise: $\det M = \Delta_4\Delta_6$. One finds that

$$\Delta_4 = \omega^2 P_2(\omega)\,, \qquad \Delta_6 = \omega^2 P_4(\omega)\,, \tag{2.15}$$

for polynomials $P_2$ and $P_4$ of, respectively, second and fourth order in $\omega$. The four linearly independent solutions with $\omega = 0$ are eliminated by the four conditions $k_i\mathrm{b}^{ij} = 0$, so plane-wave solutions correspond to zeros of either $P_2$ or $P_4$.

A calculation yields

$$P_2 = (\omega + 2k_5 p\,\mathcal{H}_{p^2})^2 - \chi \tag{2.16}$$

where

$$\chi = \mathcal{H}_s^2 k_5^2 + \mathcal{H}_s\left(Q_1 k_1^2 + Q_2 k_3^2\right)\,, \tag{2.17}$$

with

$$Q_\alpha = \mathcal{H}_s + 2\mathrm{B}_\alpha^2\mathcal{H}_{p^2} \qquad (\alpha = 1,2). \tag{2.18}$$

The dispersion relation for one polarisation is therefore $P_2 = 0$, with $P_2$ given by (2.16). As expected, it reduces to $\omega^2 = |\mathbf{k}|^2$ in the free-field limit. More generally, it depends on the direction of the wave-vector because of the term in (2.16) with the factor of $k_5 p$ (i.e. $\mathbf{k}\cdot\mathbf{p}$).

The remaining two dispersion relations must be obtained from the condition $P_4 = 0$. A calculation yields

$$P_4 = \left\{\left[\omega + k_5 p\left(2\mathcal{H}_{p^2} + \Lambda\right)\right]^2 - \chi'\right\} P_2 + \Upsilon k_1^2 k_3^2\,, \tag{2.19}$$

where

$$\Lambda = 2\mathcal{H}_{p^2} + \mathcal{H}_{ss} + 4s\mathcal{H}_{sp^2} + 4p^2\mathcal{H}_{p^2p^2} \tag{2.20}$$



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and

$$\chi' = \Xi_1\Xi_2 k_5^2 + \Xi_1 Q_2 k_1^2 + \Xi_2 Q_1 k_3^2\,, \tag{2.21}$$

for the additional coefficient functions

$$\begin{aligned}\Xi_\alpha = {} & \mathcal{H}_s + 2s\mathcal{H}_{ss} + 4p^2\mathcal{H}_{sp^2} \\ & + \mathrm{B}_\alpha^2(\Lambda - 4s\mathcal{H}_{sp^2} - 2\mathcal{H}_{ss})\,,\end{aligned} \tag{2.22}$$

and finally

$$\Upsilon = N_1N_2 - Q_1Q_2\, p^2\Lambda^2 \tag{2.23}$$

with

$$\begin{aligned}N_1 &= Q_2\Xi_1 - \mathcal{H}_s Q_1\,,\\ N_2 &= Q_1\Xi_2 - \mathcal{H}_s Q_2\,.\end{aligned} \tag{2.24}$$

### 2.3 Zero trirefringence conditions

The conditions required for all three dispersion relations to coincide is $P_4 = P_2^2$. From (2.19) we see that $P_2$ is a factor of $P_4$ for generic $\mathbf{k}$ only if $\Upsilon = 0$. The other factor is also $P_2$ only if both $\Lambda = 0$ and $\chi' = \chi$ for all $\mathbf{k}$, which requires only that $N_1 = N_2 = 0$ since these two relations imply $\Xi_1\Xi_2 = \mathcal{H}_s^2$. Moreover, the three relations

$$N_1 = N_2 = \Lambda = 0 \tag{2.25}$$

imply $\Upsilon = 0$, so these three relations are the necessary and sufficient conditions for coincidence of all three dispersion relations. They may be simplified by the observation that

$$N_1 + N_2 = 2\left(s\mathcal{H}_s + 2p^2\mathcal{H}_{p^2}\right)\Lambda - 8\left(s^2 - p^2\right)\Lambda_1, \tag{2.26}$$

$$N_1 - N_2 = 2\sqrt{s^2-p^2}\left(4s\Lambda_1 + 8p^2\Lambda_2 - \mathcal{H}_s\Lambda\right), \tag{2.27}$$

where

$$\begin{aligned}\Lambda_1 &:= \mathcal{H}_s\mathcal{H}_{sp^2} - \mathcal{H}_{p^2}\mathcal{H}_{ss}\\ \Lambda_2 &:= \mathcal{H}_s\mathcal{H}_{p^2p^2} - \mathcal{H}_{p^2}\mathcal{H}_{sp^2}\,.\end{aligned} \tag{2.28}$$

This shows that equations (2.25) are jointly equivalent to the following three "zero trirefringence" conditions:

$$\Lambda_1 = 0\,, \qquad \Lambda_2 = 0\,, \qquad \Lambda = 0\,. \tag{2.29}$$

The first two of these equations are trivially solved if $\mathcal{H}_{p^2} = 0$, but then the third requires $\mathcal{H}$ to be a linear function of $s$. Excluding this free field case, we may assume that $\mathcal{H}_{p^2} \neq 0$ and then define a new function $T(s,p^2)$ by the relation

$$\mathcal{H}_s = 2T\mathcal{H}_{p^2}\,. \tag{2.30}$$

Using this in the equations $\Lambda_1 = \Lambda_2 = 0$ we find that

$$T_s\mathcal{H}_{p^2}^2 = T_{p^2}\mathcal{H}_{p^2}^2 = 0\,, \tag{2.31}$$



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from which we conclude that $T$ is a constant. Using this fact to simplify the $\Lambda = 0$ condition, we then find the following simple second-order ODE for $\mathcal{H}$ as a function of $p^2$:

$$\mathcal{H}_{p^2} + 2\left(T^2 + 2Ts + p^2\right)\mathcal{H}_{p^2p^2} = 0\,. \tag{2.32}$$

The general solution has two integration constants. One is fixed by requiring positive energy and a choice of energy scale. The other is then fixed by requiring zero vacuum energy. The result is

$$\mathcal{H} = \mathcal{H}_{\rm M5} := \sqrt{T^2 + 2Ts + p^2} - T\,. \tag{2.33}$$

This is the Hamiltonian density for the chiral 2-form theory on the M5-brane [15, 16]; for brevity we shall call it the 'M5' theory. It is actually a family of theories labelled by the constant $T$ (the M5-brane tension) which has dimensions of energy density, and the free-field theory is included as the $T \to \infty$ limit. As already mentioned this 'M5' theory is the 6D partner to the 4D BI theory. It would be of interest to see whether there is a generalisation to 6D chiral 2-form dynamics of the recent characterisation of zero-birefrigence NLEDs as those with a Lagrangian satisfying a particular "$T\bar{T}$-like flow equation" [17].

To summarise: within the class of chiral 2-form electrodynamics invariant under rotations and time-space translations, and with the same phase space as the standard free-field theory, only the one-parameter 'M5' family exhibits "zero-trirefringence". For this exceptional family, the one dispersion relation for the three independent wave-polarizations is found from setting $\mathcal{H} = \mathcal{H}_{\rm M5}$ in (2.16) and then setting the resulting expression for $P_2$ to zero; this yields

$$\left[\omega + \frac{\mathbf{k}\cdot\mathbf{p}}{T_{\rm eff}}\right]^2 = \frac{T^2|\mathbf{k}|^2 - T|\bar{B}\mathbf{k}|^2}{T_{\rm eff}^2}\,, \tag{2.34}$$

where

$$T_{\rm eff} = \sqrt{T^2 + 2T\bar{s} + \bar{p}^2}\,. \tag{2.35}$$

Here we revert to the bar notation for background fields as a reminder that $T_{\rm eff}$ (which will play a role later) is constant. In the $T \to \infty$ (weak-field) limit this dispersion relation reduces to $\omega^2 = |\mathbf{k}|^2$, as expected.

We may also take the $T \to 0$ (strong-field) limit for which $\mathcal{H} = p$. This defines an interacting *conformal* 6D chiral 2-form electrodynamics theory [16, 18]; its 4D partner is Bialynicki-Birula electrodynamics [1, 19]. All constant uniform background solutions now have $p \neq 0$, and (2.34) reduces to the linear dispersion relation $\omega + \mathbf{k}\cdot\mathbf{n} = 0$, where $\mathbf{n} = \mathbf{p}/p$. In this case $b$ is a Fourier component of the first term in an expansion about $B = \bar{B}$ of an exact solution of the full field equations of the form $B = B_\perp(t - \mathbf{x}\cdot\mathbf{n}, \mathbf{x}_\perp)$, where $n_i B_\perp^{ij} = 0$ and $\mathbf{n}\cdot\mathbf{x}_\perp = 0$ for fixed direction $\mathbf{n}$.

### 2.4 Lorentz invariance

Surprisingly, the above results were obtained without the use of the Lorentz invariance condition (2.7), which is actually a consequence of the zero-trirefringence conditions (2.29). We shall now show that the Lorentz invariance condition by itself restricts trirefringence to birefringence; i.e. it implies that two of the three independent dispersion relations coincide. We begin with the observation that

$$\begin{aligned} \mathcal{I}_s &= 2(\mathcal{H}_s\Lambda - 2s\Lambda_1 - 4p^2\Lambda_2)\,, \\ \mathcal{I}_{p^2} &= 2(\mathcal{H}_{p^2}\Lambda + \Lambda_1 + 2s\Lambda_2)\,. \end{aligned} \tag{2.36}$$

Next, we observe that (2.27) may be rewritten as

$$N_1 - N_2 = 2\sqrt{s^2 - p^2}\,(\mathcal{H}_s\Lambda - \mathcal{I}_s)\ . \tag{2.37}$$

Now, using (2.37) together with (2.26) and (2.36) we obtain

$$\begin{aligned} N_1N_2 &\equiv \frac{1}{4}[(N_1+N_2)^2 - (N_1+N_2)^2] \\ &= 8(s^2-p^2)p^2\left(\Lambda_2\mathcal{I}_s - \Lambda_1\mathcal{I}_{p^2}\right) + \mathcal{I}p^2\Lambda^2\,. \end{aligned} \tag{2.38}$$

Substituting this expression into (2.23), and using the identity

$$Q_1Q_2 \equiv \mathcal{I}\,, \tag{2.39}$$

where $\mathcal{I}$ is the expression defined in (2.7), we deduce that

$$\Upsilon = 8(s^2-p^2)p^2\left(\Lambda_2\mathcal{I}_s - \Lambda_1\mathcal{I}_{p^2}\right)\ . \tag{2.40}$$

This result shows that $\Upsilon = 0$ for any $\mathcal{H}$ such that $\mathcal{I} = 1$, i.e. any Lorentz invariant theory. It then follows from (2.19) that Lorentz invariance implies $P_4 = P_2'P_2$, where $P_2'(\omega)$ is another quadratic polynomial in $\omega$. Thus, two of the three independent polarisations have coincident dispersion relations for generic ($P_2' \neq P_2$) Lorentz invariant theories while $P_2' = P_2$ uniquely for the 'M5' case.

## 3 Relation to M-theory

It is natural to wonder whether there is some M-theory explanation for the zero-trirefringence property of the 'M5' chiral 2-form theory. In the context of the M5-brane worldvolume dynamics, the Minkowski vacuum for the 'M5' theory is a planar static M5-brane, and perturbations about it are propagated by the free-field equations of a (2,0)-supersymmetric 6D field theory; its on-shell supermultiplet includes the three polarisation modes of the 'M5' chiral 2-form electrodynamics and five others, one for each of the five scalars representing transverse fluctuations of the planar M5-brane in an 11-dimensional space-time [20]. It might appear that some of the 16 supersymmetries of this (2,0)-supermultiplet must be broken when constant uniform background fields are introduced on the M5-brane, but this is not necessarily the case, as we now explain.

It was shown in [21] that a static planar M5-brane with constant uniform 3-form field strength is $\frac{1}{2}$-supersymmetric; i.e. it preserves 16 of the 32 supersymmetries of the M-theory 11D Minkowski vacuum, *independently of the strength of the 'background' 3-form field.* If the skew eigenvalues $\{\mathrm{B}_\alpha; \alpha = 1, 2\}$ of the background 'magnetic' 2-form are identified as "dissolved" M2-branes with charges

$$\zeta_\alpha = \sqrt{T}\,\mathrm{B}_\alpha \tag{3.1}$$

then the $\frac{1}{2}$ supersymmetry is also implied by the supertranslation algebra associated to the M5-brane worldvolume dynamics provided that [21, 22]

$$P_5 = \zeta_1\zeta_2/T\,, \tag{3.2}$$

which is the background field-momentum. The 'effective' M5-brane tension (i.e. total energy density) of these bound states is

$$P^0 = \sqrt{T^2 + \zeta_1^2 + \zeta_2^2 + P_5^2}\,. \tag{3.3}$$

The construction in [22] of 11D supergravity solutions sourced by these M5-M2-M2 "bound states" (generalizing the simpler M5-M2 $\frac{1}{2}$-supersymmetric solution of [23]) confirmed their $\frac{1}{2}$ supersymmetry. They are related by String/M dualities to the D2-D0-F1 "supertube" bound states of IIA superstring theory [24, 25] but with a *planar* D2-brane for which the generic $\frac{1}{4}$ supersymmetry is enhanced to $\frac{1}{2}$ supersymmetry [26].

Using (3.1) and (3.2), and reverting to the bar notation for background fields, we may rewrite (3.3) as

$$P^0 = \sqrt{T^2 + 2T\bar{s} + \bar{p}^2} = T_{\text{eff}}\,, \tag{3.4}$$

which is the expression of (2.35), now interpreted as the effective M5-brane tension for the M5-brane plus $\bar{B}$ background, which we can view as a new worldvolume 'vacuum' preserving all 16 supersymmetries. We should then re-normalize the M5-brane vacuum energy to be zero when $B = \bar{B}$. This means that we should replace $\mathcal{H}_{M5}$ by

$$\mathcal{H}'_{M5} = \sqrt{T^2 + 2Ts + p^2} - T_{\text{eff}}\,. \tag{3.5}$$

Now, in the expansion about the $B = \bar{B}$ background, the energy is zero when $b = 0$. We thus expect the field equations (2.9) to be part of a larger set of equations for disturbances of a planar M5-M2-M2 bound state configuration preserving 16 supersymmetries. This leads us to conjecture that the zero-trirefringence property of the 'M5' theory is a consequence of its unique status as a consistent truncation of the maximally-supersymmetric 6D field theory found from expansion of the full M5-brane dynamics about a novel 1/2-supersymmetric vacuum.

## 4 Implications for 5D and 4D NLED

We conclude with a brief explanation of how our results for 6D chiral 2-form electrodynamics imply analogous results for 5D and 4D NLEDs by means of dimensional reduction. As we used symmetries preserved by the 6D $B = \bar{B}$ background solution to choose the wave-vector of perturbations to have zero $k_2$ and $k_4$ components, we will first take all fields to be independent of $x^2$, to get 5D results, and then of both $x^2$ and $x^4$, to get 4D results. In the former case the 5-space 2-form $A = \frac{1}{2}dx^i \wedge dx^j A_{ij}$ can be written as

$$\frac{1}{2}dx^a \wedge dx^b \mathbb{A}_{ab} + dx^2 \wedge dx^a V_a\,, \quad (a,b = 1,3,4,5). \tag{4.1}$$

Correspondingly,

$$\begin{aligned} B^{ab} &= \frac{1}{2}\varepsilon^{abcd} F_{cd}\,, \quad (F_{ab} = 2\partial_{[a}V_{b]}) \\ B^{a2} &= \frac{1}{2}\varepsilon^{abcd}\partial_b \mathbb{A}_{cd} =: D^a\,. \end{aligned} \tag{4.2}$$

The gauge-invariant 4-space fields are therefore the two-form field strength $F_{ab}$ and a divergence-free 4-vector field $D^a$ that can be 'promoted' to an unconstrained 4-vector field by introducing a Lagrange multiplier field $V_0$ to impose the constraint $\partial_a D^a = 0$. One then finds (ignoring total derivative terms) that

$$\frac{1}{2}\dot{A}\cdot B = E_a\, D^a \qquad (E_a = \partial_a V_0 - \dot{V}_a)\,. \tag{4.3}$$

This is the 'symplectic' term in the phase-space Lagrangian density (2.1) reduced to a 5D NLED. Its Hamiltonian is a function of $(s,p^2)$ but now

$$s = \frac{1}{2}\left[|D|^2 + |F|^2\right]\,, \tag{4.4}$$

and, since the components of $\mathbf{p}$ in (2.3) are now

$$p_2 = \frac{1}{8}\varepsilon^{abcd}F_{ab}F_{cd}\,, \qquad p_a = F_{ab}D^b\,, \tag{4.5}$$

we also have

$$p^2 = \det F + |FD|^2\,. \tag{4.6}$$

For the special class of 5D NLEDs with Hamiltonian densities that are functions only of $(s,p^2)$, our 6D results imply that the unique zero-trirefringence family has a Hamiltonian density that is formally the same as $\mathcal{H}_{M5}$ of (2.33), but this is now equivalent to

$$\sqrt{T^2 \det\left(\mathbb{I}_4 + F/\sqrt{T}\right) + T|D|^2 + |FD|^2} - T\,. \tag{4.7}$$

This is the 5D BI Hamiltonian density, as can be seen from previous results on the Hamiltonian dynamics of the bosonic worldvolume fields for Dp-branes [27, 28].

We may now further dimensionally reduce by taking all fields to be independent of $x^4$. In this case $V_4$ becomes a scalar field, with canonical conjugate $D^4$, and if we truncate by setting to zero this conjugate pair then we are left with the 3-vector $\mathbf{D}$ (conjugate to $\mathbf{V}$) and the 3-space restriction of $F$, which is the Hodge dual of the magnetic 3-vector field $\mathbf{B}$. The phase space Lagrangian density is now that of a 4D NLED with Hamiltonian density $\mathcal{H}$ that is again a function of $(s,p^2)$, but now

$$s = \tfrac{1}{2}\left(|\mathbf{D}|^2 + |\mathbf{B}|^2\right)\,, \qquad p^2 = |\mathbf{D}\times\mathbf{B}|^2\,, \tag{4.8}$$



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which implies that $\mathcal{H}$ is electromagnetic duality invariant. Our 6D trirefringence results now imply that all zero-birefringence 4D NLEDs in this class are members of the family for which $\mathcal{H}(s,p^2)$ is formally the same as $\mathcal{H}_{M5}$ of (2.33), but this now defines the 4D BI theory. In other words, the 4D BI theory is the unique duality invariant zero-birefringence NLED, in agreement with [6] and with an earlier conclusion of [29] based on (what appear to us to be) slightly different premises. The main novelty here is that we find it in Hamiltonian form, and by dimensional reduction of the 'M5' theory of 6D chiral 2-form electrodynamics.

From the 6D perspective, the truncation described above (following dimensional reduction) amounts to setting to zero the components $(B_{24}, B_{13}, B_{15}, B_{35})$ of $B$, only two of which are independent because of the identity $\partial_i B^{ij} \equiv 0$. This truncation also removes the two linearly independent combinations of the $(\mathrm{b}_{24}, \mathrm{b}_{13}, \mathrm{b}_{15}, \mathrm{b}_{35})$ perturbations of $B$ about the $\bar{B}$ background. In 6D these were the amplitudes describing the phase space for the single polarisation mode with dispersion relation $P_2 = 0$ (as required for consistency of the truncation). In 4D they are the amplitudes for the 'extra' scalar field in the 4D $\mathcal{N}=4$ Maxwell supermultiplet relative to the 6D (2,0) antisymmetric tensor supermultiplet.

In the String/M-theory context, the 'dissolved' pair of orthogonal M2-branes (representing the $\bar{B}$ background on a static planar M5-brane) becomes a 'dissolved' orthogonal F1-D1 pair of IIB strings, i.e. a constant uniform background of orthogonal $(\mathbf{D}, \mathbf{B})$ fields, which generate the momentum $\mathbf{D} \times \mathbf{B}$ needed to preserve all 16 supersymmetries of a static planar D3-brane in the absence of the electromagnetic background fields.

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