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1/3 BPS loops and defect CFTs in ABJM theory

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Abstract: We address a longstanding question of whether ABJM theory has Wilson loop operators preserving eight supercharges (so 1/3 BPS). We present such Wilson loops made of a large supermatrix combining two 1/2 BPS Wilson loops. We study the spectrum of operator insertions into them including the displacement operator and several others and study their correlation functions. Another natural construction arising in this context are Wilson loops with alternating superconnections. This amounts to including "defect changing operators" along the loop, similar to a discrete cusp. This insertion is topological and preserves two supercharges. We study the multiplet of this operator and how it can be used to introduce further operators. We also construct the defect conformal manifold arising from marginal defect operators.

Keywords: Chern-Simons Theories, Global Symmetries, Supersymmetric Gauge Theory, Wilson, 't Hooft and Polyakov loops

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1 Introduction

ABJM theory [1] has a rich spectrum of line operators including the 1/2 BPS loop [2] and 1/6 BPS loops. The latter may be bosonic [3–5] with only a single gauge field or include fermi fields like the 1/2 BPS ones [6–9]. There are also Wilson loops preserving fewer supercharges [10–15], though they are not conformal. Finally there are vortex loops [16] that are 1/2 BPS or 1/3 BPS (though there should also be less supersymmetric versions).

A natural question that many experts have tried to address, is whether there are also 1/3 BPS Wilson loops in this theory. Given that a vortex loop exists, there is an appropriate superalgebra. Indeed, the $\mathfrak{osp}(6|4)$ superalgebra of ABJM is broken to $\mathfrak{su}(1,1|3)$ by the 1/2 BPS line and to $\mathfrak{su}(1,1|1) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ by the 1/6 BPS loops (for the bosonic loop, the $\mathfrak{u}(1)$ is enhanced to another $\mathfrak{su}(2)$). The 1/3 BPS algebra is $\mathfrak{su}(1,1|2) \oplus \mathfrak{u}(1)$. This latter algebra (up to a $\mathfrak{u}(1)$ factor) is also the symmetry of the 1/2 BPS loops of $\mathcal{N}=4$ Chern-Simons theories, which is a hint for our construction.

The 1/2 BPS Wilson loops of $\mathcal{N}=4$ theories [17, 18] have a degeneracy of pairs of loops preserving the same eight supercharges. Choosing then eight of the twelve supercharges of a 1/2 BPS loop W_1^+ of ABJM that generate an $\mathfrak{su}(1,1|2)$ subalgebra, there should be another Wilson loop, W_4^- preserving the same supercharges. This second Wilson loop is also 1/2 BPS, but the linear combination $W_{1/3} = n_1 W_1^+ + n_4 W_4^-$ is 1/3 BPS. Explicit expressions for W_1^+ and W_4^- are presented in appendix A: (A.7), (A.8).

Defining an operator as a linear combination of other ones may not seem fundamental, and one may raise the objection that they should each be studied independently. One way to see that this is not the case is that this linear combination arises naturally when considering Wilson loops based on superconnections larger than that of the 1/2 BPS loop. Such larger constructions with repeated entries from the same gauge field were proposed in [11–13] and give rise also to operators that cannot be expressed in the block-diagonal form of $n_1 W_1^+ + n_4 W_4^-$. While the 1/3 BPS loop itself can be written this way, it can be deformed into non-diagonal loops, so operator insertions into $W_{1/3}$ cannot be factorised as insertions into W_1^+ and those in W_4^-.

Having realised this 1/3 BPS line, we turn to studying its properties and in particular the defect CFT for operator insertions along it [19–23]. Part of this analysis relies on the explicit realisation of $W_{1/3}$ presented here and part is based on representation theory of the superconformal group, so is valid for any 1/3 BPS line operator including the vortex loop of [16] or any further line operators that may be found in the future.

The displacement and tilt operators are insertions that arise from broken translation and R-symmetry, respectively. As reviewed in appendix B, the two point functions of these operators are related to bremsstrahlung functions [24, 25]. ABJM theory has a rich spectrum of such functions [26–29], and the case of the 1/3 BPS Wilson loop is even richer.

Any 1/3 BPS line breaks the conformal group $\mathfrak{so}(4,1) \to \mathfrak{so}(2,1) \oplus \mathfrak{so}(2)$, just as the 1/2 BPS or any other conformal line operator, so has two displacement operators from the broken translations. The $\mathfrak{su}(4)$ R-symmetry is broken to $\mathfrak{su}(2) \oplus \mathfrak{u}(1)^2$ with five different pairs of tilt operators: a conjugate pair denoted $\mathbb{O}$ and $\bar{\mathbb{O}}$ for the broken generators between the two $\mathfrak{u}(1)$s and the others relating each of them to $\mathfrak{su}(2)$ (they are denoted as $\mathbb{O}^a$, $\bar{\mathbb{O}}_a$, $\mathbb{O}_a$, $\bar{\mathbb{O}}^a$ with $a \in \{2,3\}$). These are really different operators with different normalisations, i.e. there are three bremsstrahlung functions related to R-symmetry breaking in addition to the one for a real cusp.

Following [29, 30], we use in section 4 Ward identities and the explict form of the tilt operators to find relations between the bremsstrahlung functions. $\mathbb{O}$ is in the same multiplet with the displacement $\mathbb{D}$, so the associated bremsstrahlung functions are clearly related. For the other tilt operators, their sum is equal to that of $\mathbb{O}$.

In the case when the bremsstrahlung functions for $\mathbb{O}^a$ and $\mathbb{O}_a$ are equal (for $n_1 = n_4$), they are half of that of $\mathbb{O}$ or $\mathbb{D}$. A similar relation exists between the bremsstrahlung functions for the tilt and displacement of the bosonic loop, but here we find a very simple setting of the same phenomenon and a far easier proof of it.

Another natural object that arises in this context is a permutation operator, which we denote by σ, that replaces the connection of W_1^+ with that of W_4^-. This is the most clear manifestation of the nontrivial interplay between the two Wilson loops. This operator preserves two supercharges and has vanishing conformal dimension, so is topological. This enriches the spectrum of protected operators on $W_{1/3}$, but can also be considered as an operator on the 1/2 BPS W_1^+. We study some of its properties, but leave most of them for future work.

Going back to the five pairs of tilt operators, they are exactly marginal operators on $W_{1/3}$ and in section 5 we study the deformation of the loop by them. We follow [31] to calculate the geometry of the resulting defect conformal manifold and precisely match it with the quotient $\mathrm{SU}(4)/S(\mathrm{U}(2) \times \mathrm{U}(1) \times \mathrm{U}(1))$.

Some background information and many technical details are relegated to the appendices.

2 Realising 1/3 BPS Wilson loops

We construct here 1/3 BPS Wilson loops in ABJM theory and then recall some subtle features of general loops in this theory that play an important role for these new loops.

The simplest 1/2 BPS loop W_1^+ is formed out of $\mathcal{L}_1^+$ (A.7)

$$W_1^+ = \operatorname{Tr} \mathcal{P} \exp \int_{-\infty}^{\infty} i\mathcal{L}_1^+ \, dx \,. \tag{2.1}$$

For $\mathrm{SU}(N_1)\times\mathrm{SU}(N_2)$ ABJ(M) theory, the superconnection $\mathcal{L}_1^+$ is an $\mathrm{SU}(N_1|N_2)$ matrix. Here we take it to be a straight line in the x^3 direction (which we denote as x). For the circular loop there is subtlety of taking the trace or supertrace [2, 9], but here we are taking the infinite line, so really it is an open line. We write trace, and if adapting to a circle, one should include a twist operator (2.3) or following the convensions of [9], use a supertrace.

Another 1/2 BPS Wilson line is W_4^-, made out of the superconnection $\mathcal{L}_4^-$ (A.8). Each of W_1^+ and W_4^- preserves twelve supercharges, and when they are along the same line, they have eight supercharges in common.

To define the 1/3 BPS loop, we take a bigger structure combining both superconnections

$$W_{1/3}=\operatorname{Tr}\mathcal{P}\exp i\int_{-\infty}^{\infty}\operatorname{diag}(\underbrace{\mathcal{L}_1^+,\cdots,\mathcal{L}_1^+}_{n_1},\underbrace{\mathcal{L}_4^-,\cdots,\mathcal{L}_4^-}_{n_4})\,dx\,. \tag{2.2}$$

To avoid confusion, each $\mathcal{L}_i^\pm$ is an $(N_1|N_2)$ supermatrix, so this is not a $(2n_1N_1|2n_4N_2)$ supermatrix, but rather $((n_1+n_4)N_1|(n_1+n_4)N_2)$. With this diagonal structure, this loop can also be written as $n_1W_1^+ + n_4W_4^-$.

This loop on its own is 1/3 BPS, resolving this long-standing question. One may wonder whether there are other 1/3 BPS loops with non-diagonal structure, as there are many constructions of BPS Wilson loops that do not respect it. We made an extensive and systematic search, based on the techniques of [12, 13] and all the 1/3 BPS loops we found could be diagonalised to (2.2).

This construction of Wilson loops out of supermatrices has an $S(\mathrm{GL}(n_1+n_4)\times\mathrm{GL}(n_1+n_4))$ global symmetry, which was pointed out in [9, 11]. If we reorder the superconnection in a way that all $A_x^{(1)}$ are at the top left and $A_x^{(2)}$ at the bottom right, this group acts by independently rotating the n_1+n_4 copies of $A_x^{(1)}$ and of $A_x^{(2)}$, see [11] for details. This global symmetry is important in the analysis of the space of BPS Wilson loops, as the Wilson loop is a trace, so we should really identify operators related by conjugation. It is also this action that allows us to diagonalise all other 1/3 BPS loops we found to the same form as (2.2).

In general, this action is not a local symmetry. The simplest manifestation of that is in the case of a single $\mathcal{L}_1^+$, where the group is simply $\mathrm{GL}(1)=\mathbb{C}^*$ and it acts on the 2×2 structure within $\mathcal{L}_1^+$ (A.7) as conjugation by elements like

$$T=\begin{pmatrix} I_{N_1} & 0\\ 0 & -I_{N_2}\end{pmatrix}. \tag{2.3}$$

This has the effect of changing the signs $\alpha\to-\alpha$, $\bar\alpha\to-\bar\alpha$. The local action of this operator was studied in [32], where it was found to be a nontrivial operator in the defect CFT of the 1/2 BPS line. Though it has vanishing classical dimension, it is not BPS, so its dimension receives quantum corrections.

We focus instead on a diagonal $\mathrm{SL}(n_1+n_4)$ subgroup which acts simultaneously on $A_x^{(1)}$ and $A_x^{(2)}$, not modifying $\mathcal{L}_1^+$ and $\mathcal{L}_4^-$. This is the obvious group acting by conjugation on the n_1+n_4 matrix of superconnections in (2.2).

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2.1 Permutation operators

Of the diagonal $\mathrm{SL}(n_1+n_4)$ action on the matrix in (2.2), an $S(\mathrm{GL}(n_1)\times\mathrm{GL}(n_4))$ subgroups is in fact a local symmetry, as the superconnection is proportional to the identity in those blocks. Other group elements change the form of the connection and are nontrivial operations on the Wilson line and can be viewed as operators in a 1d defect CFT.

To keep the gauge fields on the diagonal, the group elements we employ are permutations, changing the order of the entries. Explicitly for the case of $n_1 = n_4 = 1$, there is a single non-trivial permutation

$$\sigma = \begin{pmatrix} 0 & I_{N_1+N_2} \\ I_{N_1+N_2} & 0 \end{pmatrix}, \qquad \sigma \begin{pmatrix} \mathcal{L}_1^+ & 0 \\ 0 & \mathcal{L}_4^- \end{pmatrix} \sigma = \begin{pmatrix} \mathcal{L}_4^- & 0 \\ 0 & \mathcal{L}_1^+. \end{pmatrix} \tag{2.4}$$

As both $\mathcal{L}_1^+$ and $\mathcal{L}_4^-$ have the same gauge-group structure, there is no obstruction to doing this, and it is particularly nice since W_1^+ and W_4^- share eight supercharges. Furthermore, as we show in appendix E.1, this combination preserves half the supercharges shared by the two lines.

Another natural operator is $\tau = \mathrm{diag}(I, -I)$, satisfying $\tau\sigma\tau = -\sigma$. This is different from T of (2.3), which acts within a single block of these matrices, so on a single $\mathcal{L}_i^\pm$. In this setting there are then two basic T-like operators, $\mathrm{diag}(T, 1)$, $\mathrm{diag}(1, T) = \sigma\,\mathrm{diag}(T, 1)\sigma$, and one can also multiply them with σ and τ.

Unlike T, the permutation σ and $\tau\sigma$ are protected local operators. As their conformal dimension vanishes, they are topological and correlation functions of any other operators do not depend on the exact position where the permutation happens, as long as it does not cross any of the other operators.

σ, T and τ are "line changing operators", similar to boundary changing operators in 2d CFTs.[1] The most studied such operators in Wilson lines are cusps, where the direction of the line changes, or there is a change in some internal parameters [26, 34–37]. Indeed in both $\mathcal{N} = 4$ SYM in 4d and in ABJM theory those were studied extensively and some cusps were shown to be BPS [26, 37, 38]. T is similar to a non-BPS cusp and σ to the BPS cusp. But unlike the usual cusps, both T and σ are discrete operations, so one cannot study them in a small or large angle expansion.

2.2 1/2 BPS loop with alternating superconnections

The operation of replacing part of a line with another connection arises naturally from the permutation symmetry above, but does not require large supermatrices. To see that consider

$$\sigma^+ = \frac{1}{2}(1+\tau)\sigma = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \qquad \sigma^- = \frac{1}{2}(1-\tau)\sigma = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}. \tag{2.5}$$

Clearly $(\sigma^+)^2 = (\sigma^-)^2 = 0$, so we should avoid that. On the other hand, $\sigma^+\sigma^-$ and $\sigma^-\sigma^+$ are projectors on the top or bottom parts of $\mathrm{diag}(\mathcal{L}_1^+, \mathcal{L}_4^-)$. Inserting this into the 1/3 BPS

[1] It is also natural to relate σ to permutation branes, see e.g. [33].

line we find

$$W_{1/3}[\sigma^+\sigma^-(0)] = \operatorname{Tr}\mathcal{P}e^{i\int_{-\infty}^{0}\begin{pmatrix}\mathcal{L}_1^+(x) & 0\\ 0 & \mathcal{L}_4^-(x)\end{pmatrix}dx}\begin{pmatrix}I & 0\\ 0 & 0\end{pmatrix}\mathcal{P}e^{i\int_{0}^{\infty}\begin{pmatrix}\mathcal{L}_1^+(x) & 0\\ 0 & \mathcal{L}_4^-(x)\end{pmatrix}dx} = W_1^+\,. \quad (2.6)$$

So this reduces the 1/3 BPS loop to the 1/2 BPS one. Inserting $\sigma^-\sigma^+$ reproduces W_4^-.

As stated, both σ and τ are protected topological operators, and hence also $\sigma^\pm$. We can therefore separate the two insertions and in particular move σ^- to $x \to \infty$, leaving us with the operator

$$W_{1/3}[\sigma^+(0)] = \operatorname{Tr}\mathcal{P}\left[\exp\int_{-\infty}^{0} i\mathcal{L}_1^+(x)\,dx\,\exp\int_0^\infty i\mathcal{L}_4^-(x)\,dx\right] = W_1^+[\sigma]. \qquad (2.7)$$

This loop starts with a single superconnection $\mathcal{L}_1^+$ and switches at $x=0$ to $\mathcal{L}_4^-$. In the last expression, we employed the notation $W_1^+[\sigma]$, where σ is now an insertion in the 1/2 BPS loop that changes the connection. We can then denote $\sigma^- \simeq \bar{\sigma}$, where it is assumed that there is first a σ insertion. On it's own $\bar{\sigma}$ is a good insertion in W_4^-.

Unlike $W_{1/3}$ (2.2), the line W_1^+ (2.6) is 1/2 BPS, but if we insert σ into it as in (2.7), it is natural to analyse it in the same context as the 1/3 BPS line. In the following we study both objects: the true 1/3 BPS line constructed from a larger superconection and the 1/2 BPS line with the topological operator σ in its spectrum and splitting the 1/2 BPS supermultiplets to 1/3 BPS ones.

A subtlety when writing expressions like (2.7) in terms of ABJM fields, is that one should treat the σ's as a book-keeping device indicating where the connection and spectrum of insertions changes. One is not meant to implement a substitution rule like $\sigma\bar{\psi}_+^1\bar{\sigma} = \bar{\psi}_-^4$. Such notations may also be possible, but they are not what we use here.

Not all line operators are Wilson lines, for example the vortex loops of [16], and in those cases we may not be able to write the expression on the right hand side of (2.7) explicitly. For that reason, we try to rely as much as possible on algebra, rather than on the explicit realisation of $W_{1/3}$ and the operator insertions.

3 Displacement multiplets of 1/3 BPS line operators

Among all operator insertions into the Wilson loop, the displacement operator and its superpartners are special, as they arise from broken global symmetries. The conservation equation for translation, supersymmetry and R-transformations are violated by the Wilson lines.

We study here how the displacement multiplets of 1/2 BPS line defects constructed in [23, 29, 39] split into 1/3 BPS multiplets. Most of the analysis is based on the breaking of global symmetries, so valid for any 1/3 BPS line operator.

3.1 First 1/2 BPS line

The 1/2 BPS defect along the $x = x_3$ axis preserves the rigid 1d conformal group, rotation around the line, and an $SU(3)\times U(1)$ R-symmetry, rotating $I, J = 2, 3, 4$ indicated by i, j. In addition, it preserves the supercharges Q_+^{12}, Q_+^{13}, Q_+^{14}, Q_-^{23}, Q_-^{34}, Q_-^{24} and the corresponding S's. A realisation of such an operator is W_1^+ with the superconnection in (A.7).

For all lines along x, the broken translation generators are the components $T^{\mu 1}$ and $T^{\mu 2}$ of the stress tensor. The other symmetries broken by this particular line include[2] the six components of the supercurrents $S_-^{\mu 1i}$ and $S_+^{\mu ij}$ and the 6 components of the R current ${J^\mu}_1{}^i$ and ${J^\mu}_i{}^1$. The conservation equations for the currents are then

$$\begin{aligned}
\partial_\mu T^{\mu\nu}(x) W_1^+ &= \delta(x_1)\delta(x_2)\delta_n^\nu\, W_1^+[\mathbb{D}^n(x)]\,, \qquad n\in\{2,3\}\,,\\
\partial_\mu S_\alpha^{\mu IJ}(x) W_1^+ &= \delta(x_1)\delta(x_2)\, W_1^+[\delta_1^I\delta_\alpha^-\bar{\mathbb{\Lambda}}^J(x) + \epsilon^{1IJK}\delta_\alpha^+\mathbb{\Lambda}_K(x)]\,,\\
\partial_\mu {J^\mu}_I{}^J(x) W_1^+ &= \delta(x_1)\delta(x_2)\, W_1^+[\delta_I^1\mathbb{O}^J(x) + \delta_1^J\bar{\mathbb{O}}_I(x)]\,.
\end{aligned} \tag{3.1}$$

Together, the operators on the right hand side form most of the displacement multiplet [23, 39] including the displacement itself $\mathbb{D} = \mathbb{D}^1 - i\mathbb{D}^2$, a fermionic operator $\mathbb{\Lambda}_i$ and the tilt $\mathbb{O}^i$. In fact there is one element missing, $\mathbb{F}$, which is fermionic and the lowest weight state in the multiplet. There are eight further operators in the complex conjugate multiplet.

The action of the preserved supersymmetries on the multiplet are [23, 29]

$$\{Q_+^{1i},\mathbb{F}\} = \mathbb{O}^i\,,\qquad [Q_+^{1i},\mathbb{O}^j] = \epsilon^{ijk}\mathbb{\Lambda}_k\,,\qquad \{Q_+^{1i},\mathbb{\Lambda}_j\} = -2\delta_j^i\mathbb{D}\,,\qquad [Q_+^{1i},\mathbb{D}] = 0\,. \tag{3.2}$$

From the Jacobi identities for the superalgebra one also finds $[Q_-^{ij},\mathbb{F}] = 0$ and

$$[Q_-^{ij},\mathbb{O}^k] = -2i\epsilon^{ijk}\mathcal{D}_x\mathbb{F}\,,\qquad \{Q_-^{ij},\mathbb{\Lambda}_k\} = 2i\delta_k^i\mathcal{D}_x\mathbb{O}^j - 2i\delta_k^j\mathcal{D}_x\mathbb{O}^i\,,\qquad [Q_-^{ij},\mathbb{D}] = i\epsilon^{ijk}\partial_x\mathbb{\Lambda}_k\,. \tag{3.3}$$

$\mathcal{D}_x$ is an appropriate covariant derivative along the line operator.

Explicit expressions for the operators in terms of the fields of ABJM theory are presented in [23] and also in appendix E. These operators can also be identified with fluctuations of the sigma-model describing an AdS_2 string in $AdS_4\times\mathbb{CP}^3$ [40].

3.2 Second 1/2 BPS line

The second line we consider also preserves the conformal group along x, rotation around the line, and an SU(3) $\times$ U(1) R-symmetry, rotating $I, J = 1, 2, 3$, now indicated as $\hat\imath, \hat\jmath$. It preserves the supercharges Q_+^{12}, Q_+^{13}, Q_+^{23}, Q_-^{34}, Q_-^{24}, Q_-^{14} and the corresponding S's. A realisation of such an operator is W_4^- with the superconnection in (A.8).

We can write the action of the symmetries broken by W_4^- on that loop as

$$\begin{aligned}
\partial_\mu T^{\mu\nu}(x) W_4^- &= \delta(x_1)\delta(x_2)\delta_n^\nu\, W_4^-[\mathbb{d}^n(x)]\,, \qquad n\in\{2,3\}\,,\\
\partial_\mu S_\alpha^{\mu IJ}(x) W_4^- &= \delta(x_1)\delta(x_2)\, W_4^-[\epsilon^{IJK4}\delta_\alpha^-\bar{\mathbb{\Lambda}}_I(x) + \delta_4^J\delta_\alpha^+\mathbb{\Lambda}^K(x)]\,,\\
\partial_\mu {J^\mu}_I{}^J(x) W_4^- &= \delta(x_1)\delta(x_2)\, W_4^-[\delta_4^J\mathbb{O}_I(x) + \delta_I^4\bar{\mathbb{O}}^J(x)]\,.
\end{aligned} \tag{3.4}$$

These operators fit into the displacement multiplet (and its conjugate) for the appropriate $\mathfrak{su}(1,1|3)$ superalgebra. Compared to the previous case we need to exchange $1 \leftrightarrow 4$, though one should also take into account that the spinors change chirality as do some signs in the matrix ${M^I}_J$.

[2] Broken rotations, special conformal transformations and superconformal generators all vanish at the origin, so do not give further operaotrs.

The analogue of (3.2) is now

$$\{Q_+^{\hat\imath\hat\jmath}, \reflectbox{$\mathbb{F}$}\} = \epsilon^{\hat\imath\hat\jmath\hat k}\reflectbox{$\mathbb{O}$}_{\hat k}\,, \qquad [Q_+^{\hat\imath\hat\jmath}, \reflectbox{$\mathbb{O}$}_{\hat k}] = \delta_{\hat k}^{\hat\imath}\reflectbox{$\mathbb{\Lambda}$}^{\hat\jmath} - \delta_{\hat k}^{\hat\jmath}\reflectbox{$\mathbb{\Lambda}$}^{\hat\imath}\,, \qquad \{Q_+^{\hat\imath\hat\jmath}, \reflectbox{$\mathbb{\Lambda}$}^{\hat k}\} = -2\epsilon^{\hat\imath\hat\jmath\hat k}\reflectbox{$\mathbb{D}$}\,, \qquad [Q_+^{\hat\imath\hat\jmath}, \reflectbox{$\mathbb{D}$}] = 0\,, \tag{3.5}$$

and the analogue of (3.3) is

$$[Q_-^{\hat\imath 4}, \reflectbox{$\mathbb{O}$}_{\hat\jmath}] = -2i\delta_{\hat\jmath}^{\hat\imath}\mathcal{D}_x\reflectbox{$\mathbb{F}$}\,, \qquad \{Q_-^{\hat\imath 4}, \reflectbox{$\mathbb{\Lambda}$}^{\hat\jmath}\} = 2i\epsilon^{\hat\imath\hat\jmath\hat k}\mathcal{D}_x\reflectbox{$\mathbb{O}$}_{\hat k}\,, \qquad [Q_-^{\hat\imath 4}, \reflectbox{$\mathbb{D}$}] = i\mathcal{D}_x\reflectbox{$\mathbb{\Lambda}$}^{\hat\imath}\,. \tag{3.6}$$

3.3 Decompsition into 1/3 BPS multiplets

The 1/2 BPS displacement multiplets are in the representation $\mathcal{B}^{1/2}_{3/2,0,0}$ of their respective $\mathfrak{su}(1,1|3)$ in the notations of [39] and $\mathbf{L\bar{A}_1}$ with primary $[\frac{3}{2}]^{0,0}_{1/2}$ in the notation of [41]. This representation splits into two representations of $\mathfrak{su}(1,1|2)$ denoted as $\mathbf{L\bar{A}_1}$ with primaries $[\frac{1}{2}]^0_{1/2}$ and $[1]^0_1$ in the notations of [41].

A simple way to see this in practice is to match the symmetries broken by both W_1^+ and W_4^- or only one of them. The symmetries broken by W_1^+ and preserved by W_4^- are

$$Q_+^{23}\,, \quad Q_-^{14}\,, \quad J_1{}^a\,, \quad J_a{}^1\,, \qquad a = 2,3\,. \tag{3.7}$$

Those give rise to the operators

$$Q_+^{23} \rightsquigarrow \mathbb{\Lambda}_4\,, \quad J_1{}^2 \rightsquigarrow \mathbb{O}^2\,, \quad J_1{}^3 \rightsquigarrow \mathbb{O}^3\,, \quad \mathbb{F}\,, \tag{3.8}$$

and their complex conjugates. We include $\mathbb{F}$ to complete the multiplet and in the following often omit the subscript 4 from the singlet $\mathbb{\Lambda}$. We call this the tilt multiplet.

Likewise the symmetries broken by W_4^- and not by W_1^+ are

$$Q_+^{14}\,, \quad Q_-^{23}\,, \quad J_4{}^a\,, \quad J_a{}^4\,, \qquad a = 2,3\,. \tag{3.9}$$

Those give rise to the operators

$$Q_+^{14} \rightsquigarrow \reflectbox{$\mathbb{\Lambda}$}^4\,, \quad J_2{}^4 \rightsquigarrow \reflectbox{$\mathbb{O}$}_2\,, \quad J_3{}^4 \rightsquigarrow \reflectbox{$\mathbb{O}$}_3\,, \quad \reflectbox{$\mathbb{F}$}\,, \tag{3.10}$$

and the complex conjugate multiplet. We name this the tlit multiplet, to distinguish from the tilt.

The symmetries broken by both W_1^+ and W_4^- are

$$P_1\,, \quad P_2\,, \quad Q_-^{12}\,, \quad Q_-^{13}\,, \quad Q_+^{24}\,, \quad Q_+^{34}\,, \quad J_1{}^4\,, \quad J_4{}^1\,. \tag{3.11}$$

In the case of W_1^+ they correspond to

$$P_1 - iP_2 \rightsquigarrow \mathbb{D}\,, \quad Q_+^{24} \rightsquigarrow -\mathbb{\Lambda}_3\,, \quad Q_+^{34} \rightsquigarrow \mathbb{\Lambda}_2\,, \quad J_1{}^4 \rightsquigarrow \mathbb{O}^4\,, \tag{3.12}$$

and in the case of W_4^-

$$P_1 - iP_2 \rightsquigarrow \reflectbox{$\mathbb{D}$}\,, \quad Q_+^{24} \rightsquigarrow \reflectbox{$\mathbb{\Lambda}$}^2\,, \quad Q_+^{34} \rightsquigarrow \reflectbox{$\mathbb{\Lambda}$}^3\,, \quad J_1{}^4 \rightsquigarrow \reflectbox{$\mathbb{O}$}_1\,. \tag{3.13}$$

3.4 Multiplets of $W_{1/3}$

In the case of a 1/3 BPS line operator, based on symmetry breaking alone, we should have the combination of terms in (3.1) and (3.4)

$$
\begin{aligned}
\partial_\mu T^{\mu\nu}(x) W_{1/3} &= \delta(x_1)\delta(x_2)\delta^\nu_n\, W_{1/3}[\mathbb{D}^n(x)]\,, \qquad n \in \{2,3\}\,,\\
\partial_\mu S^{\mu IJ}_\alpha(x) W_{1/3} &= \delta(x_1)\delta(x_2)\, W_{1/3}\big[\epsilon^{1aI4}\delta^J_4\delta^+_\alpha \mathbb{A}_a(x) + \delta^I_1\delta^J_a\delta^-_\alpha \bar{\mathbb{A}}^a(x)\\
&\quad + \delta^I_2\delta^J_3\,(\delta^+_\alpha \mathbb{\Lambda}(x) + \delta^-_\alpha \bar{\mathbb{\Lambda}}(x)) + \delta^I_1\delta^J_4\,(\delta^+_\alpha \mathbb{\Lambda}(x) + \delta^-_\alpha \bar{\mathbb{\Lambda}}(x))\big]\,, \qquad (3.14)\\
\partial_\mu J^\mu{}_I{}^J(x) W_{1/3} &= \delta(x_1)\delta(x_2)\, W_{1/3}\big[\delta^1_I\delta^J_4 \mathbb{O}(x) + \delta^4_I\delta^J_1 \bar{\mathbb{O}}(x) + \delta^1_I\delta^J_a \mathbb{O}^a(x)\\
&\quad + \delta^a_I\delta^J_1 \bar{\mathbb{O}}_a(x) + \delta^a_I\delta^J_4 \mathbb{O}_a(x) + \delta^4_I\delta^J_a \bar{\mathbb{O}}^a(x)\big]\,.
\end{aligned}
$$

For $W_{1/3}$ in (2.2), where the connection is a larger supermatrix, the operators on the right hand side are now matrices. In the $n_1 = n_4 = 1$ case, the operators form the tilt multiplet are

$$
\begin{pmatrix} \mathbb{F} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} \mathbb{O}^a & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} \mathbb{\Lambda}_4 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.15}
$$

The tlit multiplet is

$$
\begin{pmatrix} 0 & 0 \\ 0 & \reflectbox{$\mathbb{F}$} \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & \reflectbox{$\mathbb{O}$}_a \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & \reflectbox{$\mathbb{\Lambda}$}^1 \end{pmatrix}, \tag{3.16}
$$

and the displacement multiplet is

$$
\mathbb{O} = \begin{pmatrix} \mathbb{O}^4 & 0 \\ 0 & \reflectbox{$\mathbb{O}$}_1 \end{pmatrix}, \qquad \mathbb{A}_a = \begin{pmatrix} \mathbb{\Lambda}_a & 0 \\ 0 & \epsilon_{ab}\reflectbox{$\mathbb{\Lambda}$}^b \end{pmatrix}, \qquad \mathbb{D} = \begin{pmatrix} \mathbb{D} & 0 \\ 0 & \reflectbox{$\mathbb{D}$} \end{pmatrix}. \tag{3.17}
$$

We do not introduce different notation for the matrices in (3.15) and (3.16) and at times below refer to the entire larger matrices with the same letter as the operator inside. It should be clear from the context, which of those is meant.

The action of the preserved generators on the different operators are presented in appendix D. Explicit expressions for these operators in terms of the ABJM fields are presented in appendix E.2.

In addition to the multiplets inherited from the 1/2 BPS displacement multiplet, we have the permutation multiplet constructed in appendix E.1

$$
\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Sigma^a = \begin{pmatrix} 0 & \bar{G}^a - G^a \\ G^a - \bar{G}^a & 0 \end{pmatrix}, \qquad o\,. \tag{3.18}
$$

For the expression for o, see (E.10). We can also think of them more abstractly as a short representation of the 1/3 BPS algebra with a primary with labels $[0]^0_0$ in the $\mathbf{A_1\bar{A}_1}$ multiplet the notations of [41]. Unlike the fields in the other multiplets, this operator is real.

We can of course form composites of these operators, which include the combinations like $\mathbb{F}\mathbb{\Lambda}$ as well as off-diagonal entries arising from σ times another operator. As usual, if two operators do not share supercharges, the composite would not be protected. An example of that is σ and $\mathbb{O}$.

We can also endow the operators with Chan-Paton factors

$$\reflectbox{$\mathbb{F}$}_1{}^1 \simeq \begin{pmatrix} \reflectbox{$\mathbb{F}$} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \reflectbox{$\mathbb{F}$}_1{}^2 \simeq \begin{pmatrix} 0 & \reflectbox{$\mathbb{F}$} \\ 0 & 0 \end{pmatrix}, \qquad \reflectbox{$\mathbb{F}$}_2{}^1 \simeq \begin{pmatrix} 0 & 0 \\ \reflectbox{$\mathbb{F}$} & 0 \end{pmatrix}, \qquad \reflectbox{$\mathbb{F}$}_2{}^2 \simeq \begin{pmatrix} 0 & 0 \\ 0 & \reflectbox{$\mathbb{F}$} \end{pmatrix}. \tag{3.19}$$

In particular operators like $\reflectbox{$\mathbb{F}$}_1{}^1 = \sigma \reflectbox{$\mathbb{F}$}_2{}^2 \sigma$ are inserted into the $\mathcal{L}_1^+$ line and enable our construction in the next subsection.

3.5 1/3 BPS multiplets of the 1/2 BPS Wilson line

The 1/2 BPS line W_1^+ has a displacement multiplet, as presented in section 3.1. It combines the 1/3 BPS displacement multiplet with $\mathbb{O}^4$, $\mathbb{\Lambda}_a$ and $\mathbb{D}$ as well as the 1/3 BPS tilt with $\mathbb{F}$, $\mathbb{O}^a$ and $\mathbb{\Lambda}$ (3.8).

As presented in section 2.2, The operators σ^+ and σ^- (or σ and $\bar{\sigma}$) (2.5) are also natural insertions in W_1^+. They are in the 1/3 BPS multiplet (3.18) and as σ changes the superconnection from $\mathcal{L}_1^+$ to $\mathcal{L}_4^-$, we can then insert operators naturally living on W_4^-. In particular this applies to the tlit operators, (3.10) as

$$\sigma \reflectbox{$\mathbb{F}$}, \qquad \sigma \reflectbox{$\mathbb{O}$}_a, \qquad \sigma \reflectbox{$\mathbb{\Lambda}$}. \tag{3.20}$$

We can adjoin to all these operators $\bar{\sigma}$ from the right, so they become insertions into the W_1^+ loop without a change in connection.

Some care is required in analysing σ and composites like $\sigma \reflectbox{$\mathbb{F}$}$ or $\sigma \reflectbox{$\mathbb{F}$} \bar{\sigma}$. Recall that a special feature of the ABJM Wilson lines is that the preserved supercharges do not annihilate the connections in appendix A but give total derivatives (E.2). When acting on the entire line, these total derivative integrate to zero, hence the loops are BPS. This is rather similar to the insertion of BPS protected operator into the BPS cusp in $\mathcal{N} = 4$ SYM in 4d [42].

When acting on the line with insertions, we find extra boundary terms. As discussed in appendix E.4, the action of a supercharge on an odd supermatrix insertion in W_1^- is covariantised to

$$\tilde{Q}_+^{1a} \bullet = Q_+^{1a} \bullet - \{\bar{G}^a, \bullet\}, \qquad \tilde{Q}_-^{a4} \bullet = Q_-^{a4} \bullet - \{G^a, \bullet\}, \tag{3.21}$$

with $\bar{G}^a$ and G^a in (E.3). In the case of W_4^- the roles of $\bar{G}^a$ and G^a are reversed (E.2).

In evaluating the variation of σ, the direct action by Q_α^{IJ} is trivial and we only have the covariant part, with that from W_1^+ on the left and from W_4^- on the right. This is the source of the terms in the expression for $\mathbb{\Sigma}$ (3.18).

An $\reflectbox{$\mathbb{F}$}$ insertion into W_4^- is annihilated by three supercharges of which two are shared by W_1^+. Yet, when inserting it as $\sigma \reflectbox{$\mathbb{F}$} \bar{\sigma}$ into W_1^+, there are different total derivative terms. In fact, no supercharges annihilate it and only the combination of $Q_+^{1a} + Q_-^{a4}$ acting on it gives $\epsilon^{ab} \sigma \reflectbox{$\mathbb{O}$}_b \bar{\sigma}$.

This construction seems to introduces several new marginal operators into the 1/2 BPS loop: $\sigma \reflectbox{$\mathbb{O}$}_a \bar{\sigma}$, $o \bar{\sigma}$ and $(\mathcal{D}_x \sigma) \bar{\sigma}$ (E.9). Unlike $\mathbb{O}^2$, $\mathbb{O}^3$ and $\mathbb{O}^4$, these operators do not arise from broken global symmetries, so it is not guaranteed that they are indeed exactly marginal. We leave this question for further study.

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Of particular note is the operator $(\mathcal{D}_x\sigma)\bar{\sigma}$, the descendant of σ (E.9), which is an infinitesimal deformation in the direction of the ABJM version of the loops described in section 6.3.2 of [13]. The loops described there are classically conformal, but conformality is not guaranteed by the preserved supercharges. The question raised in the last paragraph is another avatar of the question of whether these loops are truly conformal.

For explicit expressions in terms of the ABJM fields, see appendix E.2.

4 Two point functions

For the operators arising from broken symmetries, as in 3.4, their normalisations are fixed by the normalisation of the conserved currents. We study here the relations between the normalisations of the different operators and their relation to the bremsstrahlung functions of these loops.

4.1 Ward identities

From conformal symmetry we know that the correlators of the operators in the displacement multiplet take the form

$$\langle\!\langle \mathbb{D}(0)\bar{\mathbb{D}}(x)\rangle\!\rangle = \frac{C_{\mathbb{D}}}{x^4}\,, \tag{4.1}$$

$$\langle\!\langle \mathbb{A}_a(0)\bar{\mathbb{A}}^b(x)\rangle\!\rangle = \frac{C_{\mathbb{A}_a}\delta_a^b}{x^3}\,, \tag{4.2}$$

$$\langle\!\langle \mathbb{O}(0)\bar{\mathbb{O}}(x)\rangle\!\rangle = \frac{C_{\mathbb{O}}}{x^2}\,. \tag{4.3}$$

The notation $\langle\!\langle \bullet \cdots \bullet \rangle\!\rangle$ represents the expectation value of the $\bullet$ insertions into the line normalized by the expectation value of the line without insertions.

The coefficients $C_{\mathbb{D}}$ and $C_{\mathbb{O}}$ are fixed from the definition of the operators and the normalisation of the broken currents in (3.14). They are also related to the bremsstrahlung functions of the line operators, as in (B.7), (B.8). The relations between them can be found from the ward identity for supersymmetry, following [29, 30].

Starting with the vanishing correlator $\langle\!\langle \mathbb{A}_2(0)\,\bar{\mathbb{D}}(x)\rangle\!\rangle = 0$ and acting with the preserved supercharge Q_+^{12}, using (3.2), we find

$$-\,2\langle\!\langle \mathbb{D}(0)\,\bar{\mathbb{D}}(x)\rangle\!\rangle = \langle\!\langle \mathbb{A}_2(0)\,\mathcal{D}_x\bar{\mathbb{A}}^2(x)\rangle\!\rangle\,, \tag{4.4}$$

where $\mathcal{D}_x$ is an appropriate covariant derivative along the Wilson line. This gives $2C_{\mathbb{D}} = 3C_{\mathbb{A}_a}$.

Likewise starting with $\langle\!\langle \mathbb{O}(0)\,\bar{\mathbb{A}}^3(x)\rangle\!\rangle = 0$ and acting with the preserved supercharge Q_+^{12} as in (3.3), we find

$$-\,\langle\!\langle \mathbb{A}_3(0)\,\bar{\mathbb{A}}^3(x)\rangle\!\rangle = 2\langle\!\langle \mathbb{O}\,\mathcal{D}_x\bar{\mathbb{O}}(x)\rangle\!\rangle\,, \tag{4.5}$$

or $C_{\mathbb{A}_a} = 4C_{\mathbb{O}}$. Combining the two, we find $C_{\mathbb{D}} = 6C_{\mathbb{O}}$.

These expressions were already derived in [23] from a superspace representation of the displacement multiplet in the case of the 1/2 BPS loop and they are not modified in the 1/3 BPS case.

For the operators in the tilt and tlit multiplets

$$\langle\!\langle \mathbb{\Lambda}(0)\bar{\mathbb{\Lambda}}(x)\rangle\!\rangle = \frac{C_{\mathbb{\Lambda}}}{x^3}\,, \qquad \langle\!\langle \reflectbox{$\mathbb{\Lambda}$}(0)\,\bar{\reflectbox{$\mathbb{\Lambda}$}}(x)\rangle\!\rangle = \frac{C_{\reflectbox{$\mathbb{\Lambda}$}}}{x^3}\,, \tag{4.6}$$

$$\langle\!\langle \mathbb{O}^a(0)\bar{\mathbb{O}}_b(x)\rangle\!\rangle = \frac{C_{\mathbb{O}^a}\delta^a_b}{x^2}\,, \qquad \langle\!\langle \reflectbox{$\mathbb{O}$}_a(0)\,\bar{\reflectbox{$\mathbb{O}$}}^b(x)\rangle\!\rangle = \frac{C_{\reflectbox{$\mathbb{O}$}_a}\delta^b_a}{x^3}\,, \tag{4.7}$$

$$\langle\!\langle \mathbb{F}(0)\bar{\mathbb{F}}(x)\rangle\!\rangle = \frac{C_{\mathbb{F}}}{x}\,. \qquad \langle\!\langle \reflectbox{$\mathbb{F}$}(0)\,\bar{\reflectbox{$\mathbb{F}$}}(x)\rangle\!\rangle = \frac{C_{\reflectbox{$\mathbb{F}$}}}{x^3}\,. \tag{4.8}$$

To find relations among those, we start with the vanishing correlator $\langle\!\langle \mathbb{O}^3(0)\,\bar{\mathbb{\Lambda}}(x)\rangle\!\rangle$ and act with the preserved supercharge Q_+^{12}, which yields

$$-\langle\!\langle \mathbb{\Lambda}(0)\,\bar{\mathbb{\Lambda}}(x)\rangle\!\rangle = 2\langle\!\langle \mathbb{O}^3(0)\,\mathcal{D}_x\bar{\mathbb{O}}_3(x)\rangle\!\rangle \quad\Rightarrow\quad C_{\mathbb{\Lambda}} = 4C_{\mathbb{O}^a}\,. \tag{4.9}$$

Then taking $\langle\!\langle \mathbb{F}(0)\,\bar{\mathbb{O}}_2(x)\rangle\!\rangle$ and acting with the preserved supercharge Q_+^{12}, we find

$$-\langle\!\langle \mathbb{O}(0)\,\bar{\mathbb{O}}(x)\rangle\!\rangle = 2\langle\!\langle \mathbb{F}^2(0)\,\mathcal{D}_x\bar{\mathbb{F}}_2(x)\rangle\!\rangle\,. \tag{4.10}$$

This gives $C_{\mathbb{O}^a} = 2C_{\mathbb{F}}$ and finally $C_{\mathbb{\Lambda}} = 8C_{\mathbb{F}}$.

The expressions for the tlit multiplet are identical, but $C_{\reflectbox{$\mathbb{O}^{a} does not have to be equal to $C_{\mathbb{O}^a}$. Likewise, for the 1/2 BPS loop we know that $C_{\mathbb{\Lambda}} = C_{\mathbb{A}_a}$ and $C_{\mathbb{O}^a} = C_{\mathbb{O}}$, but this does not necessarily hold for 1/3 BPS operators, as we discuss in the next section.

4.2 Relations accross multiplets

We can go further and relate the different multiplets to each-other, using the explicit representation of $W_{1/3}$ (2.2) and its expression in terms of 1/2 BPS loops. We consider the case of n_1 copies of $\mathcal{L}_1^+$ and n_4 copies of $\mathcal{L}_4^-$, but for simplicity write them as 2×2 matrices.

Using the representation in (3.15), the two point function of the operators from the tilt multiplets can be related to those of the 1/2 BPS loop W_1^+ as

$$C^{1/3}_{\mathbb{O}^a}\delta^a_b = x^2\left\langle\!\!\left\langle \begin{pmatrix} \mathbb{O}^a(0) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\mathbb{O}}_b(x) & 0 \\ 0 & 0 \end{pmatrix} \right\rangle\!\!\right\rangle = \frac{n_1\delta^a_b}{n_1+n_4} C^{1/2}_{\mathbb{O}}\,. \tag{4.11}$$

This is a simple consequence of having n_1 insertions. Likewise for $\reflectbox{$\mathbb{O}$}_a$ and $\mathbb{O}$, we have

$$C^{1/3}_{\reflectbox{$\mathbb{O}$}_a} = \frac{n_4}{n_1+n_4} C^{1/2}_{\mathbb{O}}\,, \qquad C^{1/3}_{\mathbb{O}} = C^{1/2}_{\mathbb{O}}\,. \tag{4.12}$$

In particular

$$C^{1/3}_{\mathbb{O}^a} + C^{1/3}_{\reflectbox{$\mathbb{O}$}_a} = C^{1/3}_{\mathbb{O}}\,. \tag{4.13}$$

Though we derived this from the expressions in (2.2), we expect this relation to hold for any 1/3 BPS loop.

The story is very different when considering the operators inserted into the 1/2 BPS Wilson loop with the aid of σ. In that case $C_{\mathbb{O}^a} = C_{\mathbb{O}^4} = C^{1/2}_{\mathbb{O}}$, as these are simply the usual tilt operators of the 1/2 BPS line. If we look at $\langle\!\langle \sigma\reflectbox{$\mathbb{O}$}_a\bar{\sigma}(0)\,\sigma\bar{\reflectbox{$\mathbb{O}$}}^b\bar{\sigma}(x)\rangle\!\rangle$, assume $C_\sigma = 1$ to cancel the middle $\sigma(0)\bar{\sigma}(x)$ and move the other σ's far away, then it is natural to expect that this too is $C^{1/2}_{\mathbb{O}}$. This indicates that in this case all three normalisation constants are equal to $C^{1/2}_{\mathbb{O}}$, though this deserves more careful study.

4.3 Bremsstrahlung functions of $W_{1/3}$

As reviewed in appendix B, the normalisation constants are related to the bremsstrahlung functions arising from nearly straight cusps.

We can characterise cusps of the 1/3 BPS loop (2.2) by an angle ϕ and an R symmetry SU(4) matrix U. When this matrix is close to the identity we can write it in terms of the broken symmetry generators as

$$U = I + i\theta {J_1}^4 + i\theta'_a {J_1}^a + i\theta''^b {J_b}^4 \,. \tag{4.14}$$

Then the cusp anomalous dimension takes the form (where we omit the indices from the θ)

$$\Gamma(\phi, U) \simeq B_\theta^{1/3}\theta^2 + \theta'^2 B_{\theta'}^{1/3} + \theta''^2 B_{\theta''}^{1/3} - \phi^2 B_\phi^{1/3} \,. \tag{4.15}$$

$W_{1/3}$ has therefore four brensstrahlung functions and the usual relations (B.7) and (B.8) give

$$B_\phi^{1/3} = \frac{1}{24} C_{\mathbb{D}}^{1/3} \,, \qquad B_\theta^{1/3} = \frac{1}{4} C_{\mathbb{O}}^{1/3} \,, \qquad B_{\theta'}^{1/3} = \frac{1}{4} C_{\mathbb{O}^a}^{1/3} \,, \qquad B_{\theta''}^{1/3} = \frac{1}{4} C_{\mathbb{O}_a}^{1/3} \,. \tag{4.16}$$

The relation after (4.5) and (4.13) then lead to the equalities

$$B_\phi^{1/3} = B_\theta^{1/3} = B_{\theta'}^{1/3} + B_{\theta''}^{1/3} \,. \tag{4.17}$$

This allows us to write (4.15) in terms of only two independent functions

$$\Gamma(\phi, U) \simeq (\theta^2 + \theta'^2 - \phi^2) B_{\theta'}^{1/3} + (\theta^2 + \theta''^2 - \phi^2) B_{\theta''}^{1/3} \,. \tag{4.18}$$

Furthermore, we can rely on the relation to the 1/2 BPS loop (4.12) to write this in terms of the 1/2 BPS bremsstrahlung function $B_\phi^{1/2}$ and n_1, n_4 as

$$\Gamma(\phi, U) \simeq \left(\theta^2 - \phi^2 + \frac{n_1}{n_1 + n_4}\theta'^2 + \frac{n_4}{n_1 + n_4}\theta''^2\right) B_\phi^{1/2} \,. \tag{4.19}$$

This relation can be seen as a generalisation of that found for the 1/6 BPS bosonic loop, where $2B_\theta^{\text{bos}} = B_\phi^{\text{bos}}$ [39, 43]. To see the relation, take $\theta = \theta'' = 0$ and $n_1 = n_4$ in (4.19) and then identify θ' with the angle in the 1/6 BPS cusp.

5 Defect conformal manifolds

We identified multiple marginal operators living on the 1/3 BPS line as well as possible new marginal operators on the 1/2 BPS line. Such operators allow to deform the defect along a defect conformal manifold, the space of all connected conformal defects. For complex marginal operators Φ_i and $\bar\Phi_{\bar\imath}$ we define the coordinates ζ^i and $\bar\zeta^{\bar\imath}$ and express the deformed line as

$$W_{\zeta,\bar\zeta}[\bullet \cdots \bullet] = W\left[\bullet \cdots \bullet \, \exp \int dx \left(\zeta^i \Phi_i(x) + \bar\zeta^{\bar\imath}\bar\Phi_{\bar\imath}(x)\right)\right] . \tag{5.1}$$

This space of dCFTs is endowed with the Zamolodchikov metric

$$g_{i\bar\jmath}(\zeta, \bar\zeta) = \frac{\langle W_{\zeta,\bar\zeta}[\Phi_i(0)\bar\Phi_{\bar\jmath}(\infty)]\rangle}{\langle W_{\zeta,\bar\zeta}\rangle} \,, \qquad \bar\Phi_{\bar\jmath}(\infty) = \lim_{x\to\infty} x^2 \bar\Phi_{\bar\jmath}(x) \,. \tag{5.2}$$

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Clearly at $\zeta = \bar{\zeta} = 0$ the metric is given by expressions like $C_{\mathbb{O}^a}$, $C_{\mathbb{O}_a}$ and $C_{\mathbb{O}}$ (4.3), (4.7). According to [44, 45] the curvature is as in Riemann normal coordinates; the second derivative of the metric with respect to the coordinates, leading to the integrated 4-point function

$$\begin{aligned} R_{i\bar{j}k\bar{l}} &= \int_{-\infty}^{+\infty} dx_1\, dx_2 \Big(\langle\!\langle \bar{\Phi}_{\bar{j}}(x_1)\Phi_k(x_2)\Phi_i(0)\bar{\Phi}_{\bar{l}}(\infty)\rangle\!\rangle_c - \langle\!\langle \Phi_i(x_1)\Phi_k(x_2)\bar{\Phi}_{\bar{j}}(0)\bar{\Phi}_{\bar{l}}(\infty)\rangle\!\rangle_c \Big) \\ &= -\mathrm{RV} \int_{-\infty}^{+\infty} d\eta \log|\eta| \Big(\langle\!\langle \Phi_i(1)\bar{\Phi}_{\bar{j}}(\eta)\Phi_k(\infty)\bar{\Phi}_{\bar{l}}(0)\rangle\!\rangle_c + \langle\!\langle \Phi_i(0)\bar{\Phi}_{\bar{j}}(1-\eta)\Phi_k(\infty)\bar{\Phi}_{\bar{l}}(1)\rangle\!\rangle_c \Big)\,, \end{aligned} \tag{5.3}$$

and likewise for other components of the curvature. The subscript c indicates the connected component, RV is a regularisation prescription detailed in [45] and the order of the operators is not as indicated in the second line, but should be by increasing argument which depends on the value of x_1 and x_2 or η. Explicitly

$$\langle\!\langle \bar{\Phi}_{\bar{j}}(x_1)\Phi_k(x_2)\Phi_i(0)\bar{\Phi}_{\bar{l}}(\infty)\rangle\!\rangle_c = \begin{cases} \langle\!\langle \bar{\Phi}_{\bar{j}}(x_1)\Phi_k(x_2)\Phi_i(0)\bar{\Phi}_{\bar{l}}(\infty)\rangle\!\rangle_c & \text{for } x_1 < x_2 < 0\,, \\ \langle\!\langle \bar{\Phi}_{\bar{j}}(x_1)\Phi_i(0)\Phi_k(x_2)\bar{\Phi}_{\bar{l}}(\infty)\rangle\!\rangle_c & \text{for } x_1 < 0 < x_2\,, \\ \langle\!\langle \Phi_i(0)\bar{\Phi}_{\bar{j}}(x_1)\Phi_k(x_2)\bar{\Phi}_{\bar{l}}(\infty)\rangle\!\rangle_c & \text{for } 0 < x_1 < x_2\,, \\ \text{other three similar cases} & \text{for } x_2 < x_1. \end{cases} \tag{5.4}$$

Equation (5.3) was further simplified in [31], by using crossing symmetry. See the expressions there.

5.1 The case of $W_{1/3}$

Of all the marginal operators, the simplest ones are those that arise from global symmetry breaking. In the case of 1/3 BPS line operators, they break the global symmetry group $\mathrm{OSp}(6|4)$ to $\mathrm{SU}(1,1|2) \times \mathrm{U}(1) \times \mathrm{U}(1)$. The $\mathrm{SU}(4)$ R-symmetry group is broken to $\mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)$. This indicates that the space of allowed 1/3 BPS loops is (at least) the quotient

$$\mathcal{M} = \mathrm{SU}(4)/S(\mathrm{U}(2) \times \mathrm{U}(1) \times \mathrm{U}(1))\,. \tag{5.5}$$

This is a 10 dimensional manifold (or 5 complex-dimensional).

Symmetry breaking gives rise to the tilt, tlit and displacement multiplets and they contain five complex operators of dimension one, $\mathbb{O}^a$, $\mathbb{O}_a$ and $\mathbb{O}$. To conform with the notation in (5.1), we label the marginal operators collectively as

$$\Phi_i \simeq \{\mathbb{O}^2, \mathbb{O}^3, \mathbb{O}, \mathbb{O}_2, \mathbb{O}_3\}\,, \qquad \bar{\Phi}_{\bar{\imath}} \simeq \{\bar{\mathbb{O}}_2, \bar{\mathbb{O}}_3, \bar{\mathbb{O}}, \bar{\mathbb{O}}^2, \bar{\mathbb{O}}^3\}\,, \qquad i, \bar{\imath} = 1, \cdots, 5\,. \tag{5.6}$$

For finite ζ^1, ζ^2, the $\mathcal{L}_1^+$ entries in the line (2.2) are rotated into another one with $\mathcal{L}_{1'}^+$. Finite ζ^4 and ζ^5 change the $\mathcal{L}_4^+$ block.

The nonvanishing components of the metric are (4.11), (4.12)

$$g_{i\bar{j}} = \begin{cases} C^{1/3}_{\mathbb{O}^a}\delta_{i\bar{j}} = \frac{n_1}{n_1+n_4} C^{1/2}_{\mathbb{O}}\delta_{i\bar{j}}\,, & i, \bar{j} = 1, 2\,. \\ C^{1/3}_{\mathbb{O}} = C^{1/2}_{\mathbb{O}}\,, & i, \bar{j} = 3\,, \\ C^{1/3}_{\mathbb{O}_a}\delta_{i\bar{j}} = \frac{n_4}{n_1+n_4} C^{1/2}_{\mathbb{O}}\delta_{i\bar{j}}\,, & i, \bar{j} = 4, 5\,. \end{cases} \tag{5.7}$$

To calculate the curvature we use (5.3), where we insert the operators (3.15), (3.16) and (3.17) into the superconnection. For example, for $i = k = 1$ and $\bar{\imath} = \bar{l} = 1$

$$
\begin{aligned}
R_{1\bar{1}1\bar{1}} = \int_{-\infty}^{+\infty} dx_1\, dx_2 \Bigg[&\left\langle\!\!\left\langle \begin{pmatrix} \bar{\mathbb{O}}_2(x_1) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbb{O}^2(x_2) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbb{O}^2(0) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\mathbb{O}}_2(\infty) & 0 \\ 0 & 0 \end{pmatrix} \right\rangle\!\!\right\rangle_c \\
&- \left\langle\!\!\left\langle \begin{pmatrix} \mathbb{O}^2(x_1) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbb{O}^2(x_2) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\mathbb{O}}_2(0) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\mathbb{O}}_2(\infty) & 0 \\ 0 & 0 \end{pmatrix} \right\rangle\!\!\right\rangle_c \Bigg] .
\end{aligned}
\tag{5.8}
$$

We write here 2×2 matrices, but they should be larger, as appropriate. This expression involves only the insertions of the tilt operators of W_1^+ into W_1^+, so is the same as in [31], except for the normalisation, which is $n_1/(n_1 + n_4)$, because there are no insertions into the W_4^- block. The 4-point function in the case of the 1/2 BPS Wilson loop was calculated in [23] and the integral was evaluated in [31] with the final expression (accounting for the normalisation) being

$$
R_{1\bar{1}1\bar{1}} = 2g_{1\bar{1}} = 2C_{\mathbb{O}^a}^{1/3} = \frac{2n_1}{n_1 + n_4} C_{\mathbb{O}}^{1/2} . \tag{5.9}
$$

In appendix F we calculate the Riemann tensor of the quotient (5.5) in a matching coordinate system and write down all its nonzero components. Indeed $R_{1\bar{1}1\bar{1}} = 2g_{1\bar{1}}$, as in the CFT calculation above. In the same way we can match all the components of the curvature for $c = a + b$ (F.11) except for terms mixing $1, 2$ and $4, 5$ indices, such as $R_{1\bar{4}4\bar{1}}$ and $R_{1\bar{2}4\bar{5}}$.

In those cases, plugging the expressions from (3.15) and (3.16) into (5.3) would give something like (5.8), but with two non-zero entries at the top left and two on the bottom right, which seems to vanish.

To fix that, we need another ingredient ignored so far.[3] The expression for the tilt and tlit in (3.15), (3.16) are the terms arising from symmetry breaking, as in (3.14). If symmetries are not broken, then there should be a conserved current along the line. In the case of the preserved supercharges these are the total derivatives in (E.2), where $\bar{G}^a$ and G^a can be considered as supercurrents along the line. For the R-symmetry charges

$$
[J_1{}^a, W_{1/3}] = \int dx\, W_{1/3} \left[\begin{pmatrix} \mathbb{O}^a & \partial_x \Gamma_1{}^a \\ \partial_x \Gamma_1{}^a & \partial_x \Gamma_1{}^a \end{pmatrix} (x) \right] , \tag{5.10}
$$

where $\Gamma_1{}^a$ are SU(4) generators and serve as 1d conserved currents (they should be written as $\Gamma^x{}_1{}^a$, but we omit the repetitive superscript). Their derivative vanishes, since this symmetry is preserved for those three entries, but these expressions are important to reproduce the missing components of the curvature.

Looking at the first term in the first line of (5.3) in the case of $R_{1\bar{4}4\bar{1}}$, we get the integrand

$$
\left\langle\!\!\left\langle \begin{pmatrix} \partial_x \Gamma_2{}^4 & \partial_x \Gamma_2{}^4 \\ \partial_x \Gamma_2{}^4 & \mathbb{O}_2 \end{pmatrix} (x_1) \begin{pmatrix} \partial_x \Gamma_4{}^2 & \partial_x \Gamma_4{}^2 \\ \partial_x \Gamma_4{}^2 & \bar{\mathbb{O}}^2 \end{pmatrix} (x_2) \begin{pmatrix} \mathbb{O}^2 & \partial_x \Gamma_1{}^2 \\ \partial_x \Gamma_1{}^2 & \partial_x \Gamma_1{}^2 \end{pmatrix} (0) \begin{pmatrix} \bar{\mathbb{O}}_2 & \partial_x \Gamma_2{}^1 \\ \partial_x \Gamma_2{}^1 & \partial_x \Gamma_2{}^1 \end{pmatrix} (\infty) \right\rangle\!\!\right\rangle_c . \tag{5.11}
$$

[3] We thank V. Schomerus for clarifying this point to us.

The derivatives $\partial_x\Gamma$ vanish at the points 0 and ∞, so we keep there only $\mathbb{O}^2$ and $\bar{\mathbb{O}}_2$. We then ignore $\mathbb{O}_2$ and $\bar{\mathbb{O}}^2$ from the first two terms, since they give disconnected contributions. Integration over the remaining $\partial_x\Gamma$, for the ordering $0 < x_1 < x_2$ gives

$$\begin{aligned}
&-\left\langle\!\!\left\langle \begin{pmatrix}\mathbb{O}^2 & 0\\ 0 & 0\end{pmatrix}(0)\begin{pmatrix}{\Gamma_2}^4 & {\Gamma_2}^4\\ {\Gamma_2}^4 & 0\end{pmatrix}(0)\begin{pmatrix}{\Gamma_4}^2 & {\Gamma_4}^2\\ {\Gamma_4}^2 & 0\end{pmatrix}(\infty)\begin{pmatrix}\bar{\mathbb{O}}_2 & 0\\ 0 & 0\end{pmatrix}(\infty)\right\rangle\!\!\right\rangle_c \\
&= -\left\langle\!\!\left\langle \begin{pmatrix}\mathbb{O}^2{\Gamma_2}^4 & \mathbb{O}^2{\Gamma_2}^4\\ 0 & 0\end{pmatrix}(0)\begin{pmatrix}{\Gamma_4}^2\bar{\mathbb{O}}_2 & 0\\ {\Gamma_4}^2\bar{\mathbb{O}}_2 & 0\end{pmatrix}(\infty)\right\rangle\!\!\right\rangle_c \sim -\left\langle\!\!\left\langle \begin{pmatrix}\mathbb{O}^4(0) & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\bar{\mathbb{O}}_4(\infty) & 0\\ 0 & 0\end{pmatrix}\right\rangle\!\!\right\rangle .
\end{aligned} \tag{5.12}$$

It is natural to consider only the off-diagonal terms as contributing to the connected part of the correlator, as the other terms would arise also in the case of the 1/2 BPS loop in [31].

For a general $W_{1/3} = n_1 W_1^+ + n_4 W_4^-$, we would get contributions from $n_1 n_4$ off-diagonal entries, giving the answer

$$R_{1\bar{4}4\bar{1}} = -n_4 C^{1/3}_{\mathbb{O}^a} = -n_1 C^{1/3}_{\mathbb{O}_a} = -\frac{n_1 n_4}{n_1 + n_4} C^{1/2}_{\mathbb{O}}\,, \tag{5.13}$$

in agreement with (F.11). One would expect another contribution from the rotations of $\mathbb{O}_2$ and $\bar{\mathbb{O}}^2$, but one can see that there is no such term in (5.3). In that expression, symmetry was used to reduce four terms to two, so we could recover the other contribution and divide them both by 2. In any case, they are identical, so the expression in (5.13) is correct.

Another case is $R_{1\bar{5}4\bar{2}}$, where the same calculation yields

$$-\left\langle\!\!\left\langle \begin{pmatrix}\mathbb{O}^2{\Gamma_2}^4 & \mathbb{O}^2{\Gamma_2}^4\\ 0 & 0\end{pmatrix}(0)\begin{pmatrix}{\Gamma_4}^3\bar{\mathbb{O}}_3 & 0\\ {\Gamma_4}^3\bar{\mathbb{O}}_3 & 0\end{pmatrix}(\infty)\right\rangle\!\!\right\rangle_c . \tag{5.14}$$

with the same result as in (5.13), in agreement with (F.11). Terms like $R_{1\bar{5}5\bar{1}}$ vanish in (F.11) and this is true also from the field theory side, since ${\Gamma_4}^3$ does not act on $\bar{\mathbb{O}}_2$. It is easy to verify then that such arguments exactly reproduces all terms in (F.11).

The results presented above are for the marginal operators arising from broken global symmetries. Those are guaranteed to be marginal. We mentioned above possible other marginal operators, like insertions of $\sigma\mathbb{O}_a\bar{\sigma}$ in W_1^+ or o and $(\mathcal{D}_x\sigma)$ from the σ multiplet in $W_{1/3}$ or for $n_1 > 1$ an insertion of $\mathbb{O}_a$ into only one of the $\mathcal{L}_1^+$ blocks. We postpone the question of whether they are exactly marginal as well as the resulting conformal manifolds to future work.

6 Discussion

We found an explicit realisation of a 1/3 BPS Wilson line operator in ABJM theory in terms of a large superconnection, combining two 1/2 BPS Wilson lines, and discussed general properties of 1/3 BPS line operators. Many of these results are valid for any 1/3 BPS loop, including the vortex loop of [16]. We have not attempted to verify them by detailed microscopic calculations in that setting, as the explicit forms of defect operators on the vortex loop may be subtle, given that there is a singularity along the line.

The entire discussion was for the straight line operator, but it carries over to the case of the circle. The preserved and broken symmetries are related by conjugation and we do

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not think that there are any subtleties in our calculation due to the difference between compact and non-compact loops. Of course, when we consider only one σ insertion in W_1^+, we should remember the $\bar{\sigma}$ at infinity, when mapping to the circle.

The circular Wilson loop has a finite expectation value that can be calculated using localization [46–49]. Given that $W_{1/3} = n_1 W_1^+ + n_4^- W_4^-$, the expression for the 1/3 BPS loop is exactly the same as the 1/2 BPS one.

The operators σ, $\bar{\sigma}$ are a side product of our construction and should be studied more fully on their own right. They are presented in section 2.1, section 2.2 and appendix E.1. There is also the τ operator presented under (2.4) and T (studied already in [32]) and the relations among them should be examined more closely. For example, whether anything changes if we replace σ by $\sigma\tau$. With those operators under control, one could then try to study operators like $\sigma \mathbb{O}_a \bar{\sigma}$ and whether they are truly marginal.

Our analysis of the relation between the normalisation constants in sections 4.1 and 4.2 is modeled closely after [29]. There this was done for the 1/6 BPS bosonic Wilson loop, preserving the superalgebra $\mathfrak{su}(1,1|1)$ and in addition a bosonic $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. The supermultiplets are much shorter, one has the complex displacement $\mathbb{D}$ and a superpartner $\mathbb{\Lambda}$ (and their descendents). There are also four complex twist operators in the $(\mathbf{2}, \bar{\mathbf{2}})$ representation and each has a superpartner.

In that case, the symmetry guaranties that the normalisation factors of all of the displacements $C_{\mathbb{O}}$ are equal, and a similar result to section 4.2 shows that they are half what they would be if they were in the same multiplet as $\mathbb{D}$ (as below (4.5)), so $C_{\mathbb{D}} = 12 C_{\mathbb{O}}$ and the two bremsstrahlung functions are related by this factor of 2.

The defect conformal manifold constructed in section 5 is a generalisation of that in [31]. It is higher dimensional and not a symmetric space. Technically we also had to take care of the seemingly vanishing mixed curvature terms, which required the inclusion of the conserved R-symmetry currents on the line.

For the bosonic loops and their four tilts, the conformal manifold is four complex dimensional, and should be $SU(4)/S(U(2) \times U(2)) = \mathbb{Gr}_2(\mathbb{C}^4)$, the Grassmannian for complex 2-planes in $\mathbb{C}^4$. Since the preserved symmetry includes the $S(U(2) \times U(1) \times U(1))$ of the 1/3 BPS loops studied here, our conformal manifold is a $\mathbb{CP}^1$ bundle over this Grassmannian. Shrinking the fibers would give the base, in the same way we can reduce our conformal manifold in section 5.1 to that of the 1/2 BPS loop, $\mathbb{CP}^3$, by simply taking $n_1 \to 0$ or $n_4 \to 0$.

In the case of our 5 complex dimensional conformal manifold the size of the $\mathbb{CP}^1$, which is fixed by $C_{\mathbb{O}}$ is related to the other two length scales via $C_{\mathbb{O}} = C_{\mathbb{O}^a} + C_{\mathbb{O}_a}$ (4.13), so it cannot be shrunk, without also shrinking the base.

Interestingly, this shrinking can be realised with the aid of the 1/6 BPS fermionic loops [6, 9]. They all preserve the same $\mathfrak{su}(1,1|1)$ superalgebra of the bosonic loop, but the bosonic symmetry is only $SU(2) \times U(1) \times U(1)$, enhancing to $SU(3) \times U(1)$ at the 1/2 BPS points and $S(U(2) \times U(2))$ at the bosonic point.

The general 1/6 BPS loop still has one complex displacement and a superpartner. There should then be five complex tilts, as in the case of the 1/3 BPS loop. The two doublets form multiplets $\{\mathbb{O}^a, \mathbb{\Lambda}^a\}$, $\{\mathbb{O}^a, \mathbb{\Lambda}^a\}$ and the singlet is now in a different multiplet $\{\mathbb{F}, \mathbb{O}\}$.

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This last multiplet is not in the spectrum of the bosonic loop and this tilt generates motion along the $\mathbb{CP}^1$, so we expect its normalisation $C_{\mathbb{O}}$, which starts as $C_{\mathbb{D}}/6$ at the 1/2 BPS point, to vanish as we approach the bosonic loop. Presumably there are still relations like $C_{\mathbb{O}^a} + C_{\mathbb{O}_a} = C_{\mathbb{D}}/6$, as in the case of the 1/3 BPS loop studied here.

The fact that the singlet tilt is in the same multiplet with $\mathbb{F}$ is consistent with the breaking the 1/2 BPS multiplet (3.2), but cannot arise from the 1/3 BPS loop, where there are a pair $\mathbb{F}$ and $\reflectbox{$\mathbb{F}$}$ in different multiplets without the singlet tilt. This is another indication that there is no 1/3 BPS loop in the same muduli space of 1/6 BPS loops based on 2×2 superconnections unrelated to 1/2 BPS loops.

Another family of 1/6 BPS loops that have previously not been studied are based on superconnections $\mathcal{L}_1^+$ and $\mathcal{L}_2^+$, where the latter, unlike $\mathcal{L}_4^-$, is the direct SU(4) rotation of $\mathcal{L}_1^+$. Unlike the 1/3 BPS loops, one can continuously rotate W_1^+ into W_2^+ while preserving 4 supercharges. These have the same bosonic symmetries as the generic 1/6 BPS loops, so are a simpler setting to study this system and one can redo our calculation in section 5, again relying on the 4-point functions that were calculated for the 1/2 BPS loop.

A better understanding of the space of line operators in the field theory could help in identifying the holographic duals, which is still an open question [3, 9, 50, 51]. To this we now add the puzzle of the holographic dual of the 1/3 BPS Wilson line. This is unlikely to be the 1/3 BPS solution of [16], but rather a superposition of two strings at different points on $\mathbb{CP}^3$.

Other natural questions are the values of the normalisation constants (bremsstrahlung functions) for arbitrary 1/6 BPS loops, which are not bosonic. Likewise, one could push the analysis here to theories with $\mathcal{N} = 4$ supersymmetry [52, 53], and their equally rich spectrum of line operators [6, 7, 12–14, 17, 18, 54–56]. We hope to address some of these questions, and many more that arise from this work, in the near future.

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A Some 1/6 and 1/2 BPS Wilson lines

We present here the BPS Wilson loops that are used in our analysis. All of them are straight lines along the x_3 axis denoted as x. The first is the bosonic Wilson loop [3–5] which is the ABJM analogue of the Gaiotto-Yin loop in $\mathcal{N} = 2$ theories [57]. It is 1/6 BPS, preserving the four supercharges Q_+^{12}, Q_-^{34}, S_+^{12}, S_-^{34}. It is given as

$$W_{\rm bos} = {\rm Tr}\,\mathcal{P}\exp\left(\int i\mathcal{A}_{\rm bos}\,dx\right), \tag{A.1}$$

where in the case of a loop in the first gauge group

$$\mathcal{A}_{\rm bos} = A_x^{(1)} - \frac{2\pi i}{k} M^I{}_J C_I \bar{C}^J\,, \qquad M = {\rm diag}(-1,-1,1,1)\,, \tag{A.2}$$

and similarly for the second group.

There is a large moduli space of Wilson loops preserving these supercharges [9, 12]. One first needs to elevate the bosonic Wilson loop to couple to both gauge groups as

$$W_{\rm bos} = {\rm Tr}\,\mathcal{P}\exp\left(\int i\mathcal{L}_{\rm bos}\,dx\right), \qquad \mathcal{L}_{\rm bos} = \begin{pmatrix} \mathcal{A}^{(1)}_{\rm bos} & 0 \\ 0 & \mathcal{A}^{(2)}_{\rm bos} \end{pmatrix}. \tag{A.3}$$

Then we can deform it as (w_I and $\bar{w}^I$ are not necessarily complex conjugate)

$$\mathcal{L} = \mathcal{L}_{\rm bos} - i(Q_+^{12} + Q_-^{34})\mathcal{G} + 2\mathcal{G}^2\,, \qquad \mathcal{G} = \begin{pmatrix} 0 & \bar{w}^I C_I \\ w_I \bar{C}^I & 0 \end{pmatrix}. \tag{A.4}$$

The explicit action of the supercharges on the fields is given in appendix A.1. The resulting loop is 1/6 BPS for arbitrary constant w_1, w_2, $\bar{w}^1$, $\bar{w}^2$ and the other vanishing or vice versa. Modding out by a $\mathbb{C}^*$ action discussed in section 2, the moduli space is two copies of the conifold [9, 11].

In the case with $w_3 = w_4 = \bar{w}^3 = \bar{w}^4 = 0$ we write those loops as in (A.3) with

$$\mathcal{L} = \begin{pmatrix} A_x^{(1)} + \alpha\bar{\alpha} M^I{}_J C_I \bar{C}^J & -i\bar{w}^1\bar{\psi}_+^2 + i\bar{w}^2\bar{\psi}_+^1 \\ i w_1^+\psi_2^+ - i w_2\psi_1^+ & A_x^{(2)} + \alpha\bar{\alpha} M^I{}_J C_I \bar{C}^J \end{pmatrix}, \qquad M^I{}_J = (M_{\rm bos})^I{}_J + \frac{2}{\alpha\bar{\alpha}}\bar{w}^I w_J\,, \tag{A.5}$$

with $\alpha\bar{\alpha} = -2\pi i/k$. In the other case we have

$$\mathcal{L} = \begin{pmatrix} A_x^{(1)} + \frac{\alpha\bar{\alpha}}{2} k M^I{}_J C_I \bar{C}^J & -i\bar{w}^3\bar{\psi}_-^4 + i\bar{w}^4\bar{\psi}_-^3 \\ -i w_3\psi_4^- + i w_4^-\psi_3^- & A_x^{(2)} + \frac{\alpha\bar{\alpha}}{2} M^I{}_J C_I \bar{C}^J \end{pmatrix}, \qquad M^I{}_J = (M_{\rm bos})^I{}_J + \frac{2}{\alpha\bar{\alpha}}\bar{w}^I w_J\,. \tag{A.6}$$

Within this space, the loops with $w_1\bar{w}^1 + w_2\bar{w}^2 = \alpha\bar{\alpha}$ are 1/2 BPS as are those with $w_3\bar{w}^3 + w_4\bar{w}^4 = -\alpha\bar{\alpha}$. The particular cases that are used in the body of the paper are:

$\boldsymbol{W_1^+}$: Taking $w_2 = \alpha$ and $\bar{w}^2 = \bar{\alpha}$ satisfying $\bar{\alpha}\alpha = -2\pi i/k$, and all others vanishing we get a loop with SU(3) symmetry among indices 2, 3, 4

$$\mathcal{L}_1^+ = \begin{pmatrix} A_x^{(1)} + \alpha\bar{\alpha}(M_1)^I{}_J C_I \bar{C}^J & i\bar{\alpha}\bar{\psi}_+^1 \\ -i\alpha\psi_1^+ & A_x^{(2)} + \alpha\bar{\alpha}(M_1)^I{}_J \bar{C}^J C_I \end{pmatrix}, \qquad M_1 = {\rm diag}(-1,1,1,1)\,. \tag{A.7}$$

Explicitly, W_1^+ preserves Q_+^{12}, Q_+^{13}, Q_+^{14}, Q_-^{34}, Q_-^{24}, Q_-^{23} and the corresponding superconformal generators.

W_4^-: The loop with SU(3) symmetry among indices 1, 2, 3 has

$$\mathcal{L}_4^- = \begin{pmatrix} A_x^{(1)} + \alpha\bar{\alpha}(M_4)^I{}_J C_I \bar{C}^J & i\bar{\alpha}\bar{\psi}_-^4 \\ -i\alpha\psi_4^- & A_x^{(2)} + \alpha\bar{\alpha}(M_4)^I{}_J \bar{C}^J C_I \end{pmatrix}, \qquad M_4 = \mathrm{diag}(-1,-1,-1,1)\,. \tag{A.8}$$

W_4^- preserves Q_+^{12}, Q_+^{13}, Q_+^{23}, Q_-^{34}, Q_-^{24}, Q_-^{14} and the corresponding S's. It shares 8 supercharges with W_1^+.

A.1 Supersymmetry transformations of the fields

Based on the conventions in [13] with some factors of 2 to be compatible with the algebra in appendix C and with [23], the variations of the fields are

$$\begin{aligned}
Q_\alpha^{IJ} C_K &= \delta_K^I \bar{\psi}_\alpha^J - \delta_K^J \bar{\psi}_\alpha^I\,, \\
Q_\alpha^{IJ} \bar{C}^K &= -\epsilon^{IJKL}\epsilon_{\alpha\beta}\psi_L^\beta\,, \\
Q_\alpha^{IJ}\psi_K^\beta &= -2\delta_K^I \left(i(\gamma^\mu)_\alpha{}^\beta D_\mu \bar{C}^J + 2\alpha\bar{\alpha}\delta_\alpha^\beta \bar{C}^{[J} C_L \bar{C}^{L]}\right) \\
&\quad + 2\delta_K^J \left(i(\gamma^\mu)_\alpha{}^\beta D_\mu \bar{C}^I + 2\alpha\bar{\alpha}\delta_\alpha^\beta \bar{C}^{[I} C_L \bar{C}^{L]}\right) - 8\alpha\bar{\alpha}\delta_\alpha^\beta \bar{C}^{[I} C_K \bar{C}^{J]}\,, \\
Q_\alpha^{IJ}\bar{\psi}_\beta^K &= 2\epsilon^{IJKL}\left(i\epsilon_{\alpha\gamma}(\gamma^\mu)_\beta{}^\gamma D_\mu C_L + 2\alpha\bar{\alpha}\epsilon_{\alpha\beta} C_{[L}\bar{C}^M C_{M]}\right) \\
&\quad + 4\alpha\bar{\alpha}\epsilon^{IJLM}\epsilon_{\alpha\beta} C_{[L}\bar{C}^K C_{M]}\,, \\
Q_\alpha^{IJ} A_\mu^{(1)} &= -\alpha\bar{\alpha}\epsilon^{IJKL}\epsilon_{\alpha\gamma}(\gamma_\mu)_\beta{}^\gamma C_K \psi_L^\beta - 2\alpha\bar{\alpha}(\gamma^\mu)_\alpha{}^\beta \bar{\psi}_\beta^{[I}\bar{C}^{J]}\,, \\
Q_\alpha^{IJ} A_\mu^{(2)} &= \alpha\bar{\alpha}\epsilon^{IJKL}\epsilon_{\alpha\gamma}(\gamma_\mu)_\beta{}^\gamma \psi_K^\beta C_L + 2\alpha\bar{\alpha}(\gamma_\mu)_\alpha{}^\beta \bar{C}^{[I}\bar{\psi}_\beta^{J]}\,.
\end{aligned} \tag{A.9}$$

The anti-symmetrisation symbol is normalised with a factor of 1/2, the gamma matrices given by the Pauli matrices with $(\gamma^3)_+{}^+ = 1$ and $\epsilon^{+-} = \epsilon_{-+} = 1$.

B Cusps, bremsstrahlungs and displacements

In this appendix we review the necessary background on cusped Wilson loops, the small angle limit giving the bremsstrahlung functions and their relation to displacement and tilt operators.

B.1 Cusped Wilson loops

A cusped Wilson loop is comprised of two semi-infinite rays meeting at an angle ϕ such that $\phi = 0$ is a straight line. We can parametrise the curve as

$$x^\mu(x) = \begin{cases} (0,0,x)\,, & x < 0\,, \\ (0, x\sin\phi, x\cos\phi)\,, & x > 0\,. \end{cases} \tag{B.1}$$

Generically such loops suffer from logarithmic divergences [34, 35], which means that the singular point obtains an anomalous dimension $\Gamma(\phi)$. For small angles this should be an even function, so to lowest order

$$\Gamma(\phi) = -B_\phi \phi^2 + \mathcal{O}(\phi^4)\,, \tag{B.2}$$

and B^ϕ is known as the bremsstrahlung function.

For loops coupling to scalar fields or fermions, we can also change those at the same point. With the structure of the 1/2 BPS loops, we can use the expressions in (A.5) and take

$$\Big(w_1(x), w_2(x), \bar{w}^1(x), \bar{w}^2(x)\Big) = \begin{cases} \alpha(0,-1,0,-1)\,, & x<0\,, \\ \bar{\alpha}(\sin\frac{\theta}{2}, -\cos\frac{\theta}{2}, \sin\frac{\theta}{2}, -\cos\frac{\theta}{2})\,, & x>0\,. \end{cases} \tag{B.3}$$

This would also lead to an anomalous dimension

$$\Gamma(\theta) = B_\theta \theta^2 + \mathcal{O}(\theta^4)\,. \tag{B.4}$$

More generally we have a function of both ϕ and θ.

One may wonder why the case of a straight line with a nonzero θ there is an anomaly, given that all the loops in (A.5) share four supercharges. The reason is that the supersymmetry variation of the superconnection $\mathcal{L}$ does not vanish, but is a total derivative cf. (E.2). For different θ, these total derivatives are different, so leave a boundary term at the location the change occures. In the special case of $\theta = \phi$ the two rays also share supercharges and the boundary terms cancel, so the combined system is BPS [26]. In this case there should not be any anomaly and from the small angle expansions (B.2) and (B.4) we conclude that $B_\phi^{1/2} = B_\theta^{1/2}$.

For the bosonic loops (A.1), (A.2) the ϕ cusp is as above and for the θ cusp we can take $M^I{}_J$ along the second ray to be

$$M_{\text{bos}}^{\theta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -\cos\theta & -\sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{B.5}$$

In this case, $\phi = \theta$ is not a BPS configuration so the relation between B_ϕ^{bos} and B_θ^{bos} remained unclear until it was proven [27–29] that

$$B_\phi^{\text{bos}} = 2B_\theta^{\text{bos}}\,. \tag{B.6}$$

We are left with two independent functions B_ϕ^{bos} and $B_\phi^{1/2}$. The first is expressed as the derivative of the n-wound Wilson loop, which can be evaluated using localisation [58, 59]. The second can be related to the so-called latitude Wilson loop [39, 43, 60, 61]. They are also related via the framing anomaly factor that arises in calculating Wilson loops in Chern-Simons theory [60, 62].

B.2 Displacement and twist

The bremsstrahlung function of $\mathcal{N} = 4$ SYM in 4d was defined in [24], where it was also related to the exact expectation value of the circular Wilson loop [46, 63, 64] via the exact expectation value of other BPS Wilson loops [65–67]. The bremsstrahlung function is related to the two point functions of the displacement operator, and with enough supersymmetry, also of its superpartner, the tilt [30].

In the context of ABJM theory, the relation between the bremsstrahlung function and the two point functions of displacement operators was presented in [39]. We do not repeat the derivation here, but the result is that the normalisation $C_{\mathbb{D}}$ in (4.1) is related to B^ϕ as[4]

$$C_{\mathbb{D}} = 24 B_\phi \,. \tag{B.7}$$

Such expression are valid for any BPS conformal loop, so the 1/2 BPS one, the bosonic loop and the 1/3 BPS loop as well.

A similar argument relates the two point function of the tilt $\mathbb{O}$ to B^θ. Specifically [23],

$$C_{\mathbb{O}} = 4 B_\theta \,. \tag{B.8}$$

In the case of the 1/2 BPS loop, this is consistent with $B^\phi_{1/2} = B^\theta_{1/2}$ and $C_{\mathbb{D}} = 6 C_{\mathbb{O}}$.

C Algebras and subalgebras

We present here the superconformal algebra of ABJM and the subalgebras preserving various Wilson loops. We follow closely the notations in [39] so do not impose reality conditions. In [39] some factors of i were introduced in describing the $\mathfrak{su}(1,1|3)$ algebra. We refrain from doing that to avoid confusion and also do not introduce separate notations for the subalgebras associated to W_1^+ and w_4^- or spell out their commutation relations, as they are all directly inherited from the original algebra. We could have imposed the reality condition on $\mathfrak{osp}(6|4)$ that would be appropriate for a theory in $\mathbb{R}^{2,1}$, but the benefit of that extra works seems marginal compared with consistency with [23, 39].

C.1 Full $\mathfrak{osp}(6|4)$ algebra

The symmetry algebra of ABJM theory in flat space includes the conformal group $\mathfrak{so}(4,1)$, the R-symmetry $\mathfrak{su}(4)$ and the supersymmetry generators forming the $\mathfrak{osp}(6|4)$ superalgebra. The first is comprised of the Lorentz generators $M_{\mu\nu}$, translations P_μ, special conformal transformations K_μ and dilation D. The R-symmetry generators can be written as $J_I{}^J$ and we write them in a redundant notation allowing $I, J = 1, \dots, 4$, with the constraint $J_I{}^I = 0$. The supercharges are Q_α^{IJ} and S_α^{IJ} and satisfy the reality constraint $Q_\alpha^{IJ} = \frac{1}{2}\epsilon^{IJKL}\bar{Q}_{KL\alpha}$ and $S_\alpha^{IJ} = \frac{1}{2}\epsilon^{IJKL}\bar{S}_{KL\alpha}/2$.

The nonzero commutators in the conformal algebra are [68, 69]

$$\begin{aligned}
&[P_\mu, K_\nu] = 2\delta_{\mu\nu} D + 2M_{\mu\nu} \,, && [D, P_\mu] = P_\mu \,, && [D, K_\mu] = -K_\mu \,, \\
&[M_{\mu\nu}, M_{\rho\sigma}] = \delta_{\mu[\sigma} M_{\rho]\nu} - \delta_{\nu[\sigma} M_{\rho]\mu} \,, && [P_\mu, M_{\nu\rho}] = \delta_{\mu[\nu} P_{\rho]} \,, && [K_\mu, M_{\nu\rho}] = \delta_{\mu[\nu} K_{\rho]} \,.
\end{aligned} \tag{C.1}$$

For the R-symmetry generators

$$[J_I{}^J, J_K{}^L] = \delta_I^L J_K{}^J - \delta_K^J J_I{}^L \,. \tag{C.2}$$

The anticommutators of the fermionic generators are

$$\begin{aligned}
&\{Q_\alpha^{IJ}, Q^{KL\beta}\} = 2\epsilon^{IJKL} (\gamma^\mu)_\alpha{}^\beta P_\mu \,, \qquad \{S_\alpha^{IJ}, S^{KL\beta}\} = 2\epsilon^{IJKL} (\gamma^\mu)_\alpha{}^\beta K_\mu \,, \\
&\{Q_\alpha^{IJ}, S^{KL\beta}\} = \epsilon^{IJKL} (\gamma^{\mu\nu})_\alpha{}^\beta M_{\mu\nu} + 2\delta_\alpha^\beta \left(\epsilon^{IJKL} D - \epsilon^{NJKL} J_N{}^I - \epsilon^{INKL} J_N{}^J\right) \,,
\end{aligned} \tag{C.3}$$

[4] This is for the complex $\mathbb{D} = \mathbb{D}_1 + i\mathbb{D}_2$, so double the usual expression in [24].

Finally the mixed commutators are

$$\begin{aligned}
[D, Q_\alpha^{IJ}] &= \frac{1}{2} Q_\alpha^{IJ} , & [D, S_\alpha^{IJ}] &= -\frac{1}{2} S_\alpha^{IJ} , \\
[M_{\mu\nu}, Q_\alpha^{IJ}] &= -\frac{1}{2} (\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta^{IJ} , & [M_{\mu\nu}, S_\alpha^{IJ}] &= -\frac{1}{2} (\gamma_{\mu\nu})_\alpha{}^\beta S_\beta^{IJ} , \\
[K_\mu, Q_\alpha^{IJ}] &= (\gamma_\mu)_\alpha{}^\beta S_\beta^{IJ} , & [P_\mu, S_\alpha^{IJ}] &= (\gamma_\mu)_\alpha{}^\beta Q_\beta^{IJ} , \\
[J_I{}^J, Q_\alpha^{KL}] &= \delta_I^K Q_\alpha^{JL} + \delta_I^L Q_\alpha^{KJ} - \frac{1}{2} \delta_I^J Q_\alpha^{KL} , & [J_I{}^J, S_\alpha^{KL}] &= \delta_I^K S_\alpha^{JL} + \delta_I^L S_\alpha^{KJ} - \frac{1}{2} \delta_I^J S_\alpha^{KL} .
\end{aligned} \tag{C.4}$$

C.2 1/2 BPS $\mathfrak{su}(1,1|3)$ subalgebras

For the 1/2 BPS loop W_1^+, the preserved supercharges are Q_+^{12}, Q_+^{13}, Q_+^{14}, Q_-^{23}, Q_-^{24}, Q_-^{34}, and likewise S_+^{12}, etc. Choosing $\gamma_3 = \sigma_3$ and $(\sigma_3)_+{}^+ = 1$, their anticommutators give the bosonic generators

$$P_3 , \quad K_3 , \quad M_{12} + 2D , \quad J_i{}^j - \frac{1}{3} \delta_i^j J_k{}^k , \quad i, j, k \in \{2, 3, 4\} . \tag{C.5}$$

Since $[P_3, K_3] = 2D$, we get separately this generator and M_{12}. The full algebra can be easily read off from the commutators in appendix C.1 and can also be found in [39].

Inside $\mathfrak{osp}(6|4)$ there is an extra $\mathfrak{u}(1)$ symmetry $J_1{}^1$ (or being pedantic about the tracelessness condition $J_1{}^1 - J_i{}^i/3$) that commutes with this $\mathfrak{su}(1,1|3)$. This generator acts nontrivially on the off-diagonal entries in $\mathcal{L}_1^+$, the fermionic fields $\bar{\psi}_+^1$ and ψ_1^+. It's action on $\mathcal{L}_1^+$ is the commutator with the supermatrix $T = \mathrm{diag}(I, -I)$ (2.3) (studied recently in [32]). M_{12} has a similar action on the fermions, so the combination $M_{12} + J_1{}^1/2$ acts trivially on the superconnection. Still each generator is a symmetry of the Wilson loop W_1^+, since their action either vanishes, or can be expressed as a total derivative of τ, which integrates to zero [2, 13].

For the second 1/2 BPS loop, W_4^-, the preserved supercharges are Q_+^{12}, Q_+^{13}, Q_+^{23}, Q_-^{14}, Q_-^{24}, Q_-^{34}, and likewise S_+^{12}, etc. Their algebra closes onto the bosonic generators

$$P_3 , \quad K_3 , \quad D , \quad M_{12} , \quad J_{\hat\imath}{}^{\hat\jmath} - \frac{1}{3} \delta_{\hat\imath}^{\hat\jmath} J_{\hat k}{}^{\hat k} , \quad \hat\imath, \hat\jmath, \hat k \in \{1, 2, 3\} . \tag{C.6}$$

And again, $M_{12} - J_4{}^4/2$ generates an extra central $\mathfrak{u}(1)$ symmetry.

C.3 1/3 BPS $\mathfrak{su}(1,1|2)$ subalgebra

The supercharges preserved by both W_1^+ and W_4^- are Q_+^{12}, Q_+^{13}, Q_-^{24}, Q_-^{34}, and likewise S_+^{12}, etc. Their algebra close onto $\mathfrak{su}(1,1|2)$ and in particular the bosonic generators

$$P_3 , \quad K_3 , \quad D , \quad M_{12} , \quad J_2{}^3 , \quad J_3{}^2 , \quad J_2{}^2 - J_3{}^3 . \tag{C.7}$$

Though not generated separately by the supercharges, the intersection of the two algebras (C.5) and (C.6) includes also $M_{1/2} + J_1{}^1/2$ and $J_1{}^1 - J_4{}^4$.

C.4 Broken and unbroken generators

The table 1 lists all generators in $\mathfrak{osp}(6|4)$ and whether they are broken by W_1^+, W_4^- and/or $W_{1/3}$.

Generator	W_1^+	W_4^-	$W_{1/3}$
$P_3,\, K_3,\, D$	✓	✓	✓
$P_1,\, P_2,\, K_1,\, K_2$	✗	✗	✗
M_{12}	✓	✓	✓
$M_{13},\, M_{23}$	✗	✗	✗
${J_2}^3,\, {J_3}^2,\, {J_2}^2 - {J_3}^3$	✓	✓	✓
${J_1}^1 - \frac{1}{2}{J_2}^2 - \frac{1}{2}{J_3}^3,\, {J_4}^4 - \frac{1}{2}{J_2}^2 - \frac{1}{2}{J_3}^3$	✓	✓	✓
${J_1}^2,\, {J_1}^3,\, {J_2}^1,\, {J_3}^1$	✓	✗	✗
${J_4}^2,\, {J_4}^3,\, {J_2}^4,\, {J_3}^4$	✗	✓	✗
${J_1}^4,\, {J_4}^1$	✗	✗	✗
$Q_+^{12},\, Q_+^{13},\, Q_-^{24},\, Q_-^{34}$	✓	✓	✓
$Q_+^{14},\, Q_-^{23}$	✓	✗	✗
$Q_-^{14},\, Q_+^{23}$	✗	✓	✗
$Q_-^{12},\, Q_-^{13},\, Q_+^{24},\, Q_+^{34}$	✗	✗	✗
$S_+^{12},\, S_+^{13},\, S_-^{24},\, S_-^{34}$	✓	✓	✓
$S_+^{14},\, S_-^{23}$	✓	✗	✗
$S_-^{14},\, S_+^{23}$	✗	✓	✗
$S_-^{12},\, S_-^{13},\, S_+^{24},\, S_+^{34}$	✗	✗	✗

Table 1. Broken and unbroken generators of $\mathfrak{osp}(6|4)$ by W_1^+, W_4^- and $W_{1/3}$.

D Multiplet structure

We list here the explicit action of the preserved supercharges on the tilt and displacement multiplets

D.1 The tilt multiplets

with $a = 2, 3$ and $\epsilon^{23} = -\epsilon_{23} = 1$

$$\begin{aligned} \{Q_+^{1a}, \mathbb{F}\} &= \mathbb{O}^a\,, \qquad [Q_+^{1a}, \mathbb{O}^b] = \epsilon^{ab}\mathbb{\Lambda}\,, \\ &\qquad [Q_-^{a4}, \mathbb{O}^b] = 2i\epsilon^{ab}\mathcal{D}_x\mathbb{F}\,, \quad \{Q_-^{a4}, \mathbb{\Lambda}\} = -2i\mathcal{D}_x\mathbb{O}^a\,, \end{aligned} \tag{D.1}$$

$$\begin{aligned} \{Q_-^{a4}, \bar{\mathbb{F}}\} &= \epsilon^{ab}\bar{\mathbb{O}}_b\,, \qquad [Q_-^{a4}, \bar{\mathbb{O}}_b] = -\delta_b^a\bar{\mathbb{\Lambda}}\,, \\ &\qquad [Q_+^{1a}, \bar{\mathbb{O}}_b] = 2i\delta_b^a\mathcal{D}_x\bar{\mathbb{F}}\,, \quad \{Q_+^{1a}, \bar{\mathbb{\Lambda}}\} = 2i\epsilon^{ab}\mathcal{D}_x\bar{\mathbb{O}}_b\,, \end{aligned} \tag{D.2}$$

D.2 The tlit multiplets

$$\begin{aligned} \{Q_+^{1a}, \reflectbox{$\mathbb{F}$}\} &= \epsilon^{ab}\reflectbox{$\mathbb{O}$}_b\,, \qquad [Q_+^{1a}, \reflectbox{$\mathbb{O}$}_b] = -\delta_b^a\reflectbox{$\mathbb{\Lambda}$}\,, \\ &\qquad [Q_-^{a4}, \reflectbox{$\mathbb{O}$}_b] = -2i\delta_b^a\mathcal{D}_x\reflectbox{$\mathbb{F}$}\,, \quad \{Q_-^{a4}, \reflectbox{$\mathbb{\Lambda}$}\} = -2i\epsilon^{ab}\mathcal{D}_x\reflectbox{$\mathbb{O}$}_b\,, \end{aligned} \tag{D.3}$$

$$\begin{aligned} \{Q_-^{a4}, \bar{\reflectbox{$\mathbb{F}$}}\} &= \bar{\reflectbox{$\mathbb{O}$}}^a\,, \qquad [Q_-^{a4}, \bar{\reflectbox{$\mathbb{O}$}}^b] = -\epsilon^{ab}\bar{\reflectbox{$\mathbb{\Lambda}$}}\,, \\ &\qquad [Q_+^{1a}, \bar{\reflectbox{$\mathbb{O}$}}^b] = -2i\epsilon^{ab}\mathcal{D}_x\bar{\reflectbox{$\mathbb{F}$}}\,, \quad \{Q_+^{1a}, \bar{\reflectbox{$\mathbb{\Lambda}$}}\} = -2i\mathcal{D}_x\bar{\reflectbox{$\mathbb{O}$}}^a\,, \end{aligned} \tag{D.4}$$

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D.3 Displacement multiplets

$$\begin{aligned}
&[Q_+^{1a},\mathbb{O}] = -\epsilon^{ab}\mathbb{A}_b\,, \qquad \{Q_+^{1a},\mathbb{A}_b\} = -2\delta_b^a\mathbb{D}\,,\\
&\qquad\qquad\qquad\qquad\qquad \{Q_-^{a4},\mathbb{A}_b\} = 2i\delta_b^a\mathcal{D}_x\mathbb{O}\,, \quad [Q_-^{a4},\mathbb{D}] = -i\epsilon^{ab}\mathcal{D}_x\mathbb{A}_b\,,
\end{aligned} \tag{D.5}$$

$$\begin{aligned}
&[Q_-^{a4},\bar{\mathbb{O}}] = \bar{\mathbb{A}}^a\,, \qquad \{Q_-^{a4},\bar{\mathbb{A}}^b\} = -2\epsilon^{ab}\bar{\mathbb{D}}\,,\\
&\qquad\qquad\qquad\qquad \{Q_+^{1a},\bar{\mathbb{A}}^b\} = -2i\epsilon^{ab}\mathcal{D}_x\bar{\mathbb{O}}\,, \quad [Q_+^{1a},\bar{\mathbb{D}}] = -i\mathcal{D}_x\bar{\mathbb{A}}^a\,,
\end{aligned} \tag{D.6}$$

E Explicit expressions in terms of ABJM fields

E.1 The σ multiplet

The permutation operator σ in the 1/3 BPS loop is given in (2.4). Let us start with a more general GL(2) matrix g (or more precisely $g \in I_{2\times 2} \otimes \mathrm{GL}(2)_{\mathbb{C}}$) acting by conjugation as

$$\begin{pmatrix} \mathcal{L}_1^+ & 0 \\ 0 & \mathcal{L}_4^- \end{pmatrix} \to g \begin{pmatrix} \mathcal{L}_1^+ & 0 \\ 0 & \mathcal{L}_4^- \end{pmatrix} g^{-1} \tag{E.1}$$

To examine its variation under supersymmetry, we recall that [2]

$$\begin{aligned}
&[Q_+^{1a}, i\mathcal{L}_1^+] = \mathcal{D}_x^{\mathcal{L}_1^+}\bar{G}^a\,, \qquad && [Q_+^{1a}, i\mathcal{L}_4^-] = \mathcal{D}_x^{\mathcal{L}_4^-}G^a\,,\\
&[Q_-^{a4}, i\mathcal{L}_1^+] = \mathcal{D}_x^{\mathcal{L}_1^+}G^a\,, && [Q_-^{a4}, i\mathcal{L}_4^-] = \mathcal{D}_x^{\mathcal{L}_4^-}\bar{G}^a\,,
\end{aligned} \tag{E.2}$$

where

$$G^a = \begin{pmatrix} 0 & 2i\bar{\alpha}\epsilon^{ab}C_b \\ 0 & 0 \end{pmatrix}, \qquad \bar{G}^a = \begin{pmatrix} 0 & 0 \\ -2i\alpha\bar{C}^a & 0 \end{pmatrix}. \tag{E.3}$$

Integrating the total derivaties, we find the boundary terms

$$Q_+^{1a} W[g] = W\left[\begin{pmatrix} \bar{G}^a & 0 \\ 0 & G^a \end{pmatrix} g - g \begin{pmatrix} \bar{G}^a & 0 \\ 0 & G^a \end{pmatrix}\right], \tag{E.4}$$

as this is a local action, we can identify the action of the preserved supercharges on g as

$$[Q_+^{1a}, g] = \left[\begin{pmatrix} \bar{G}^a & 0 \\ 0 & G^a \end{pmatrix}, g\right], \qquad [Q_-^{a4}, g] \;=\left[\begin{pmatrix} G^a & 0 \\ 0 & \bar{G}^a \end{pmatrix}, g\right]. \tag{E.5}$$

A nicer action arises from the sum and difference of the supercharges

$$\begin{aligned}
&[Q_+^{1a} + Q_-^{a4}, g] = \left[\begin{pmatrix} \bar{G}^a + G^a & 0 \\ 0 & \bar{G}^a + G^a \end{pmatrix}, g\right] = 0\\
&[Q_+^{1a} - Q_-^{a4}, g] = \left[\begin{pmatrix} \bar{G}^a - G^a & 0 \\ 0 & -(\bar{G}^a - G^a) \end{pmatrix}, g\right] = (\bar{G}^a - G^a) \otimes [\tau, g].
\end{aligned} \tag{E.6}$$

In the last expression we view g as a 2×2 matrix and use the tensor symbol explicitly. We also use

$$\tau = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{E.7}$$

Clearly for a diagonal g all variations cancel, so we can focus on off-diagonal g, which are linear combinations of σ and $\tau\sigma$. We find then the descendents of σ and $\tau\sigma$ as

$$\begin{aligned} \mathbb{\Sigma}^a &= \frac{1}{2}[Q_+^{1a} - Q_-^{a4}, \sigma] = (\bar{G}^a - G^a) \otimes \tau\sigma \,, \\ \rotatebox[origin=c]{180}{$\mathbb{\Sigma}$}^a &= \frac{1}{2}[Q_+^{1a} - Q_-^{a4}, \tau\sigma] = (\bar{G}^a - G^a) \otimes \sigma \,. \end{aligned} \tag{E.8}$$

Looking at the second variation, first acting with the sum, then with the proper covariant derivative (E.32) we find

$$\begin{aligned} \frac{1}{2}\{Q_+^{1a} + Q_-^{a4}, \mathbb{\Sigma}^b\} &= \epsilon^{ab} \begin{pmatrix} -2\bar{\alpha}\alpha C_c \bar{C}^c & i\bar{\alpha}(\bar{\psi}_-^4 - \bar{\psi}_+^1) \\ -i\alpha(\psi_4^+ - \psi_1^+) & -2\bar{\alpha}\alpha\bar{C}^c C_c \end{pmatrix} \otimes \tau\sigma \\ &= \epsilon^{ab} \begin{pmatrix} 0 & \mathcal{L}_4^- - \mathcal{L}_1^+ \\ \mathcal{L}_1^+ - \mathcal{L}_4^- & 0 \end{pmatrix} = \epsilon^{ab} \mathcal{D}_x \sigma \,. \end{aligned} \tag{E.9}$$

Here $C_c\bar{C}^c = C_2\bar{C}^2 + C_3\bar{C}^3$ and the result is the covariant derivative of σ, in agreement with the algebra (C.3). We find a similar result for $\rotatebox[origin=c]{180}{$\mathbb{\Sigma}^{a}.

Acting with the other combinations of supercharges we find the descendant

$$\begin{aligned} \epsilon^{ab} o = \frac{1}{2}\{Q_+^{1a} - Q_-^{a4}, \mathbb{\Sigma}^b\} &= \epsilon^{ab} \begin{pmatrix} 2\bar{\alpha}\alpha C_c\bar{C}^c & 0 \\ 0 & -2\bar{\alpha}\alpha\bar{C}^c C_c \end{pmatrix} \otimes \sigma \\ &\quad - \epsilon^{ab} \begin{pmatrix} 0 & i\bar{\alpha}(\bar{\psi}_+^1 + \bar{\psi}_-^4) \\ i\alpha(\bar{\psi}_+^1 + \bar{\psi}_-^4) & 0 \end{pmatrix} \otimes \tau\sigma \,. \end{aligned} \tag{E.10}$$

The expressions become a bit easier when starting with $\sigma^\pm = (\sigma \pm \tau\sigma)/2$

$$\begin{aligned} o^+ &= \frac{1}{8}\{Q_+^{12} - Q_-^{24}, \mathbb{\Sigma}^3 + \rotatebox[origin=c]{180}{$\mathbb{\Sigma}$}^3\} = \begin{pmatrix} 2\bar{\alpha}\alpha C_c\bar{C}^c & -i\bar{\alpha}(\bar{\psi}_+^1 + \bar{\psi}_-^4) \\ -i\alpha(\bar{\psi}_+^1 + \bar{\psi}_-^4) & -2\bar{\alpha}\alpha\bar{C}^c C_c \end{pmatrix} \otimes \sigma^+ \,, \\ o^- &= \frac{1}{8}\{Q_+^{12} - Q_-^{24}, \mathbb{\Sigma}^3 - \rotatebox[origin=c]{180}{$\mathbb{\Sigma}$}^3\} = \begin{pmatrix} 2\bar{\alpha}\alpha C_c\bar{C}^c & i\bar{\alpha}(\bar{\psi}_+^1 + \bar{\psi}_-^4) \\ i\alpha(\bar{\psi}_+^1 + \bar{\psi}_-^4) & -2\bar{\alpha}\alpha\bar{C}^c C_c \end{pmatrix} \otimes \sigma^- \,. \end{aligned} \tag{E.11}$$

E.2 The tilt multiplets

We can act by the broken generators ${J_1}^a$, ${J_a}^1$, ${J_a}^4$ and ${J_4}^a$ on $\mathcal{L}_1^+$ and $\mathcal{L}_4^-$ to find the tilt operators

$$\begin{aligned} \mathbb{O}^a &= \begin{pmatrix} -2\bar{\alpha}\alpha C_1\bar{C}^a & i\bar{\alpha}\bar{\psi}_+^a \\ 0 & -2\bar{\alpha}\alpha\bar{C}^a C_1 \end{pmatrix}, & \bar{\mathbb{O}}_a &= \begin{pmatrix} 2\bar{\alpha}\alpha C_a\bar{C}^1 & 0 \\ i\alpha\psi_a^+ & 2\bar{\alpha}\alpha\bar{C}^1 C_a \end{pmatrix}, \\ \bar{\mathbb{O}}^a &= \begin{pmatrix} 2\bar{\alpha}\alpha C_4\bar{C}^a & i\bar{\alpha}\bar{\psi}_-^a \\ 0 & 2\bar{\alpha}\alpha\bar{C}^a C_4 \end{pmatrix}, & \mathbb{O}_a &= \begin{pmatrix} -2\bar{\alpha}\alpha C_a\bar{C}^4 & 0 \\ i\alpha\psi_a^- & -2\bar{\alpha}\alpha\bar{C}^4 C_a \end{pmatrix}. \end{aligned} \tag{E.12}$$

By matching the fermionic parts of (see appendix D)

$$\begin{aligned} \{\tilde{Q}_+^{1a}, \mathbb{F}\} &= \mathbb{O}^a \,, & \{\tilde{Q}_-^{a4}, \bar{\mathbb{F}}\} &= \epsilon^{ab}\bar{\mathbb{O}}_b \,, \\ \{\tilde{Q}_+^{1a}, \rotatebox[origin=c]{180}{$\mathbb{F}$}\} &= \epsilon^{ab}\mathbb{O}_b \,, & \{\tilde{Q}_-^{a4}, \bar{\rotatebox[origin=c]{180}{$\mathbb{F}$}}\} &= \bar{\mathbb{O}}^a \,. \end{aligned} \tag{E.13}$$

we get

$$\begin{aligned} \mathbb{F} &= \begin{pmatrix} 0 & i\bar{\alpha}C_1 \\ 0 & 0 \end{pmatrix}, & \bar{\mathbb{F}} &= \begin{pmatrix} 0 & 0 \\ i\alpha\bar{C}^1 & 0 \end{pmatrix}, \\ \bar{\reflectbox{$\mathbb{F}$}} &= \begin{pmatrix} 0 & -i\bar{\alpha}C_4 \\ 0 & 0 \end{pmatrix}, & \reflectbox{$\mathbb{F}$} &= \begin{pmatrix} 0 & 0 \\ -i\alpha\bar{C}^4 & 0 \end{pmatrix}. \end{aligned} \tag{E.14}$$

To make these expressions work, one needs to use the form of the variation on odd supermatrices in (E.32).

The covariant supercharges acting on Grassmann odd matrices like $\mathbb{F}$ and $\bar{\mathbb{F}}$ inserted into W_1^+ as (see the discussion in appendix E.4)

$$\tilde{Q}_+^{1a}\bullet = Q_+^{1a}\bullet - \{\bar{G}^a, \bullet\}, \qquad \tilde{Q}_-^{a4}\bullet = Q_-^{a4}\bullet - \{G^a, \bullet\}, \tag{E.15}$$

with $\bar{G}^a$ and G^a in (E.3). For the tlit operators inserted into W_4^- we need to use the corresponding covariantisation with the roles of $\bar{G}^a$ and G^a reversed.

We can then check that conversely (with a mixed anti-commutator for the even and odd entries in $\mathcal{L}_1^+$)

$$[\tilde{Q}_-^{a4}, \mathbb{O}^b] = 2i\epsilon^{ab}(\partial_x\mathbb{F} + i[\mathcal{L}_1^+, \mathbb{F}\}), \tag{E.16}$$

and likewise should be the case for the other operators, in accordance with (D.1) and (D.2).

We can carry over the tlit operators $\reflectbox{$\mathbb{F}$}$, $\mathbb{O}_a$ and $\mathbb{\Lambda}$ to be insertions in W_1^+. We denote those operators as $\sigma\reflectbox{$\mathbb{F}$}\bar{\sigma}$, etc., but in terms of the field expressions, they have the same form as above. The difference is that when acting on them with a preserved charge, we need to use instead the appropriate covariantisation for W_1^+.

Since $Q_+^{1a} + Q_-^{a4}$ annihilates σ (E.6), these operators have the same covariantisation with $G^a - \bar{G}^a$ inside both W_1^+ and W_4^-, so acting with them on $\sigma\reflectbox{$\mathbb{F}$}\bar{\sigma}$ we find

$$\begin{aligned} [\tilde{Q}_+^{1a} + \tilde{Q}_-^{a4}, \sigma\reflectbox{$\mathbb{F}$}\bar{\sigma}] &= \epsilon^{ab}\sigma\mathbb{O}_b\bar{\sigma}, \\ [\tilde{Q}_+^{1a} + \tilde{Q}_-^{a4}, \sigma\bar{\reflectbox{$\mathbb{F}$}}\bar{\sigma}] &= \sigma\bar{\mathbb{O}}^a\bar{\sigma}, \end{aligned} \tag{E.17}$$

which is the appropriate covariantisation for operators in W_4^-.

Acting with Q_+^{1a} and Q_-^{a4} according to (D.1), (D.2) and the corresponding covariant derivatives (E.29), we get the remainging operators in the tilt multiplets

$$\begin{aligned} \mathbb{\Lambda} &= -2\bar{\alpha}\alpha \begin{pmatrix} \bar{\psi}_+^2\bar{C}^3 - \bar{\psi}_+^3\bar{C}^2 + C_1\psi_4^- & -\frac{1}{\alpha}(D_1 - iD_2)C_4 \\ 2i\alpha(\bar{C}^3C_1\bar{C}^2 - \bar{C}^2C_1\bar{C}^3) & \psi_4^-C_1 - \bar{C}^2\bar{\psi}_+^3 + \bar{C}^3\bar{\psi}_+^2 \end{pmatrix}, \\ \bar{\mathbb{\Lambda}} &= -2\bar{\alpha}\alpha \begin{pmatrix} \bar{\psi}_-^4\bar{C}^1 + C_2\psi_3^+ - C_3\psi_2^+ & 2i\bar{\alpha}(C_3\bar{C}^1C_2 - C_2\bar{C}^1C_3) \\ \frac{1}{\bar{\alpha}}(D_1 + iD_2)\bar{C}^4 & \bar{C}^1\bar{\psi}_-^4 + \psi_3^+C_2 - \psi_2^+C_3 \end{pmatrix}. \end{aligned} \tag{E.18}$$

D_1 and D_2 are covariant derivatives (with the usual connections $A_\mu^{(1)}$ and $A_\mu^{(2)}$ in the transverse $\mu = 1, 2$ directions.

Likewise from $[Q_+^{1a}, \mathbb{O}_b] = -\delta_b^a\mathbb{\Lambda}$ and $[Q_-^{a4}, \bar{\mathbb{O}}^b] = \epsilon^{ab}\bar{\mathbb{\Lambda}}$ we get

$$\begin{aligned} \mathbb{\Lambda} &= -2\bar{\alpha}\alpha \begin{pmatrix} \bar{\psi}_+^1\bar{C}^4 + C_2\psi_3^- - C_3\psi_2^- & 2i\bar{\alpha}(C_2\bar{C}^4C_3 - C_3\bar{C}^4C_2) \\ -\frac{1}{\bar{\alpha}}(D_1 - iD_2)\bar{C}^1 & \bar{C}^4\bar{\psi}_+^1 + \psi_3^-C_2 - \psi_2^-C_3 \end{pmatrix}, \\ \bar{\mathbb{\Lambda}} &= -2\bar{\alpha}\alpha \begin{pmatrix} \bar{\psi}_-^2\bar{C}^3 + C_4\psi_1^+ - \bar{\psi}_-^3\bar{C}^2 & \frac{1}{\alpha}(D_1 + iD_2)C_1 \\ 2i\alpha(\bar{C}^2C_4\bar{C}^3 - \bar{C}^3C_4\bar{C}^2) & \bar{C}^3\bar{\psi}_-^2 + \psi_1^+C_4 - \bar{C}^2\bar{\psi}_-^3 \end{pmatrix}. \end{aligned} \tag{E.19}$$

E.3 The dispalcement multiplet

It is easy to get the tilt operator in the displacement multiplet by replacing a in (E.12) with 4 and 1 we find

$$
\begin{aligned}
&\mathbb{O}^4 = \begin{pmatrix} -2\bar{\alpha}\alpha C_1\bar{C}^4 & i\bar{\alpha}\bar{\psi}_+^4 \\ 0 & -2\bar{\alpha}\alpha\bar{C}^4 C_1 \end{pmatrix}, \qquad \bar{\mathbb{O}}_4 = \begin{pmatrix} 2\bar{\alpha}\alpha C_4\bar{C}^1 & 0 \\ i\alpha\psi_4^+ & 2\bar{\alpha}\alpha\bar{C}^1 C_4 \end{pmatrix}, \\
&\bar{\mathbb{O}}^1 = \begin{pmatrix} 2\bar{\alpha}\alpha C_4\bar{C}^1 & i\bar{\alpha}\bar{\psi}_-^1 \\ 0 & 2\bar{\alpha}\alpha\bar{C}^1 C_4 \end{pmatrix}, \qquad \mathbb{O}_1 = \begin{pmatrix} -2\bar{\alpha}\alpha C_1\bar{C}^4 & 0 \\ i\alpha\psi_1^- & -2\bar{\alpha}\alpha\bar{C}^4 C_1 \end{pmatrix}.
\end{aligned}
\tag{E.20}
$$

$\mathbb{O}$ and $\bar{\mathbb{O}}$ are then expressed in terms of those as in (3.17).

Then we find the explicit expressions for $\mathbb{\Lambda}_a$, $\bar{\mathbb{\Lambda}}^a$, $\mathbb{\Lambda}^a$ and $\bar{\mathbb{\Lambda}}_a$ as

$$
\begin{aligned}
\mathbb{\Lambda}_a &= -2\bar{\alpha}\alpha \begin{pmatrix} C_1\psi_a^- - \epsilon_{ab}(\bar{\psi}_+^b\bar{C}^4 - \bar{\psi}_+^4\bar{C}^b) & -\frac{1}{\alpha}(D_1 - iD_2)C_a \\ -2i\alpha\epsilon_{ab}(\bar{C}^4 C_1\bar{C}^b - \bar{C}^b C_1\bar{C}^4) & \psi_a^- C_1 - \epsilon_{ab}(\bar{C}^4\bar{\psi}_+^b - \bar{C}^b\bar{\psi}_+^4) \end{pmatrix} \\
\bar{\mathbb{\Lambda}}^a &= -2\bar{\alpha}\alpha \begin{pmatrix} \bar{\psi}_-^a\bar{C}^1 - \epsilon^{ab}(C_4\psi_b^+ + C_b\psi_4^+) & 2i\bar{\alpha}\epsilon^{ab}(C_4\bar{C}^1 C_b - C_b\bar{C}^1 C_4) \\ \frac{1}{\bar{\alpha}}(D_1 + iD_2)\bar{C}^a & \bar{C}^1\bar{\psi}_-^a - \epsilon^{ab}(\psi_b^+ C_4 + \psi_4^+ C_b) \end{pmatrix}
\end{aligned}
\tag{E.21}
$$

and

$$
\begin{aligned}
\mathbb{\Lambda}^a &= -2\bar{\alpha}\alpha \begin{pmatrix} \bar{\psi}_+^a\bar{C}^4 - \epsilon^{ab}(C_1\psi_b^- + C_b\psi_1^-) & 2i\bar{\alpha}\epsilon^{ab}(C_b\bar{C}^4 C_1 - C_1\bar{C}^4 C_b) \\ -\frac{1}{\bar{\alpha}}(D_1 - iD_2)\bar{C}^a & \bar{C}^4\bar{\psi}_+^a - \epsilon^{ab}(\psi_b^- C_1 + \psi_1^- C_b) \end{pmatrix} \\
\bar{\mathbb{\Lambda}}_a &= -2\bar{\alpha}\alpha \begin{pmatrix} C_4\psi_a^+ - \epsilon_{ab}(\bar{\psi}_-^b\bar{C}^1 - \bar{\psi}_-^1\bar{C}^b) & \frac{1}{\alpha}(D_1 + iD_2)C_a \\ -2i\alpha\epsilon_{ab}(\bar{C}^b C_4\bar{C}^1 - \bar{C}^1 C_4\bar{C}^b) & \psi_a^+ C_4 - \epsilon_{ab}(\bar{C}^1\bar{\psi}_-^b - \bar{C}^b\bar{\psi}_-^1) \end{pmatrix}
\end{aligned}
\tag{E.22}
$$

The expressions for $\mathbb{D}$ and $\bar{\mathbb{D}}$ can then be found by further action with the supercharges.

E.4 Subtlety in covariant derivatives of supermatrices

For a Grassmann-even matrix $\mathcal{O}$ inserted into a Wilson line

$$
W[\mathcal{O}(0)] = \operatorname{Tr}\mathcal{P}\left[\left(\exp\int_{-\infty}^0 i\mathcal{L}(x)dx\right)\mathcal{O}(0)\left(\exp\int_0^\infty i\mathcal{L}(x)dx\right)\right],
\tag{E.23}
$$

as well as a Grassmann-even symmetry generator δ with $\delta(i\mathcal{L}) = \mathcal{D}_x^{\mathcal{L}}\mathcal{G}$, the variation of the Wilson line is

$$
\begin{aligned}
\delta W[\mathcal{O}(0)] &= W\left[\left(\int_{-\infty}^0 \mathcal{D}_x^{\mathcal{L}}\mathcal{G}(x')\,dx'\,\mathcal{O}(0) + \delta\mathcal{O}(0) + \mathcal{O}(0)\int_0^\infty \mathcal{D}_x^{\mathcal{L}}\mathcal{G}(x')\,dx'\right)\right] \\
&= W\left[\delta\mathcal{O} + \mathcal{G}\mathcal{O} - \mathcal{O}\mathcal{G})(0)\right].
\end{aligned}
\tag{E.24}
$$

So that we can define a covariant symmetry $\tilde{\delta}$ acting by

$$
\tilde{\delta}\mathcal{O} = \delta\mathcal{O} + \mathcal{G}\mathcal{O} - \mathcal{O}\mathcal{G}.
\tag{E.25}
$$

Turning to the case of Grassmann-odd operators, for example, the Grassmann-odd Q and G

$$
G = \begin{pmatrix} 0 & g_{12} \\ g_{21} & 0 \end{pmatrix},
\tag{E.26}
$$

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with even g_{ij}. We can use a unit Grassmannian θ to repackage them into even objects $\delta = \theta Q$ and $\mathcal{G} = \theta G$.

In the case where $\mathcal{O}$ is an even supermatrix (like $\mathbb{O}^a$ and $\mathbb{D}$)

$$\mathcal{O} = \begin{pmatrix} B_{11} & F_{12} \\ F_{21} & B_{22} \end{pmatrix}, \tag{E.27}$$

so the covariant action of Q can be found from (E.25) to be

$$\tilde{Q}\mathcal{O} = \begin{pmatrix} QB_{11} + g_{12}F_{21} + F_{12}g_{21} & QF_{12} + g_{12}B_{22} - B_{11}g_{12} \\ QF_{21} + g_{21}B_{11} - B_{22}g_{21} & QB_{22} + g_{21}F_{12} + F_{21}g_{12} \end{pmatrix}. \tag{E.28}$$

In other words,

$$\tilde{Q}\mathcal{O} = Q\mathcal{O} + \{G, \mathcal{O}_F\} + [G, \mathcal{O}_B], \tag{E.29}$$

where $\mathcal{O}_B$ and $\mathcal{O}_F$ are the bosonic and fermionic parts of $\mathcal{O}$.

The other case is for an odd supermatrix

$$\mathcal{O}' = \begin{pmatrix} F_{11} & B_{12} \\ B_{21} & F_{22} \end{pmatrix}. \tag{E.30}$$

We take an odd ϵ such that $\mathcal{O} = \epsilon\mathcal{O}'$ is an even supermatrix. Then plugging this into (E.25), we get

$$\tilde{Q}\mathcal{O}' = \begin{pmatrix} QF_{11} - (g_{12}B_{21} + B_{12}g_{21}) & QB_{12} - (g_{12}F_{22} - F_{11}g_{12}) \\ QB_{21} - (g_{21}F_{11} - F_{22}g_{21}) & QF_{22} - (g_{21}B_{12} + B_{21}g_{12}). \end{pmatrix} \tag{E.31}$$

In short

$$\tilde{Q}\mathcal{O}' = Q\mathcal{O}' - \{G, \mathcal{O}'_B\} - [G, \mathcal{O}'_F]. \tag{E.32}$$

F The geometry of SU(4)/S(U(2) × U(1) × U(1))

As explained in section 5, the defect conformal manifold is the quotient SU(4)/S(U(2) × U(1) × U(1)) and the integrated 4-point functions of the tilt operators are related to the curvature of this manifold. We follow [70] (see also [71]) to describe this quotient and evaluate the Riemann tensor.

We start by choosing explict generators of SU(4) in terms of the 4×4 matrices, α_{ab} with entry 1 at location ab. The generators of S(U(2) × U(1) × U(1)) are the three diagonal ones and $\sqrt{2}\alpha_{12}$ and $\sqrt{2}\alpha_{21}$. Note that this is not a Hermitian basis, but we normalise them such multiplying by the hermitian conjugate and tracing gives 2. We denote them collectively as h_A with $A = 1, \dots, 5$.

The remaining generators are

$$\begin{aligned} &m_1 = \sqrt{2}\alpha_{13}\,, && m_2 = \sqrt{2}\alpha_{23}\,, && m_3 = \sqrt{2}\alpha_{41}\,, && m_4 = \sqrt{2}\alpha_{42}\,, && m_5 = \sqrt{2}\alpha_{43}\,, \\ &m_{\bar{1}} = \sqrt{2}\alpha_{31}\,, && m_{\bar{2}} = \sqrt{2}\alpha_{32}\,, && m_{\bar{3}} = \sqrt{2}\alpha_{14}\,, && m_{\bar{4}} = \sqrt{2}\alpha_{24}\,, && m_{\bar{5}} = \sqrt{2}\alpha_{34}\,. \end{aligned} \tag{F.1}$$

We denote them collectively as m_i. One can then define structure constants such that

$$[h_A, h_B] = f_{AB}{}^C h_C\,, \qquad [h_A, m_i] = f_{Ai}{}^j m_j\,, \qquad [m_i, m_j] = f_{ij}{}^A h_A + f_{ij}{}^k m_k\,. \tag{F.2}$$

We do not wish to write explicit coordinates on the quo, but in any (local) representation in terms of group elemets g, the Maurer-Cartan form on the quotient can then be decomposed as $g^{-1}dg = \ell^i m_i + \Omega^A h_A$. The metric on the quotient can then be written as

$$ds^2 = g_{ij}\ell^i \ell^j \,, \tag{F.3}$$

and this metric is SU(4) invariant if g_{AB} are constants and satisfy

$$f_{Ai}{}^{k} g_{jk} + f_{Aj}{}^{k} g_{ik} = 0 \,. \tag{F.4}$$

In our case the possible solutions are

$$g_{1\bar{1}} = g_{\bar{1}1} = g_{2\bar{2}} = g_{\bar{2}2} = a \,, \qquad g_{3\bar{3}} = g_{\bar{3}3} = c \,, \qquad g_{4\bar{4}} = g_{\bar{4}4} = g_{5\bar{5}} = g_{\bar{5}5} = b \,. \tag{F.5}$$

In terms of the dCFT data (4.3), (4.7), those are

$$a = C_{\mathbb{O}_a} \,, \qquad b = C_{\mathbb{O}^a} \,, \qquad c = C_{\mathbb{O}} \,. \tag{F.6}$$

The Levi-Civita connection is then given as

$$C_k{}^i{}_j = \frac{1}{2}\left(g^{il} f_{lj}{}^{m} g_{km} + g^{il} f_{lk}{}^{m} g_{jm} + f_{kj}{}^{i} \right) . \tag{F.7}$$

And the Riemann tensor is

$$R^i{}_{jkl} = \left(C_k{}^i{}_m C_l{}^m{}_j - C_l{}^i{}_m C_k{}^m{}_j - C_m{}^i{}_j f_{kl}{}^m - f_{Aj}{}^i f_{kl}{}^A \right) . \tag{F.8}$$

The full explanation of these expressions and their implementation for other quotients can be found in [70].

Lowering the first index and plugging in the metric and structure constants, we find that up to the usual symmetries of the Riemann tensor, the nonzero components of the form $R_{i\bar{j}k\bar{l}}$ are

$$\begin{aligned}
&R_{1\bar{1}1\bar{1}} = R_{2\bar{2}2\bar{2}} = 2a \,, \qquad R_{1\bar{1}2\bar{2}} = R_{1\bar{2}2\bar{1}} = a \,, \\
&R_{1\bar{1}3\bar{3}} = R_{2\bar{2}3\bar{3}} = -\frac{(a+b-c)^2 - 4ab}{4b} \,, \\
&R_{1\bar{1}4\bar{4}} = R_{1\bar{2}4\bar{5}} = R_{2\bar{1}5\bar{4}} = R_{2\bar{2}5\bar{5}} = \frac{(a+b-c)^2 - 4ab}{4c} \,, \\
&R_{1\bar{4}4\bar{1}} = R_{1\bar{5}4\bar{2}} = R_{2\bar{4}5\bar{1}} = R_{2\bar{5}5\bar{2}} = \frac{(a-b+c)(a-b-c)}{4c} \,, \\
&R_{3\bar{3}4\bar{4}} = R_{3\bar{3}5\bar{5}} = -\frac{(a+b-c)^2 - 4ab}{4a} \,, \\
&R_{1\bar{3}3\bar{1}} = R_{2\bar{3}3\bar{2}} = \frac{a+c-b}{2} \,, \qquad R_{3\bar{4}4\bar{3}} = R_{3\bar{5}5\bar{3}} = \frac{b+c-a}{2} \,, \\
&R_{3\bar{3}3\bar{3}} = 2c \,, \qquad R_{4\bar{4}4\bar{4}} = R_{5\bar{5}5\bar{5}} = 2b \,, \qquad R_{4\bar{4}5\bar{5}} = R_{4\bar{5}5\bar{4}} = b \,.
\end{aligned} \tag{F.9}$$

There are also nonvanishing $R_{ij\bar{k}\bar{l}}$ components

$$\begin{aligned}
&R_{1\bar{3}1\bar{3}} = R_{2\bar{3}2\bar{3}} = \frac{(a+b-c)(b+c-a)}{4b} \,, \qquad R_{3\bar{4}3\bar{4}} = R_{3\bar{5}3\bar{5}} = \frac{(a+b-c)(a+c-b)}{4a} \,. \\
&R_{14\bar{1}\bar{4}} = R_{14\bar{2}\bar{5}} = R_{25\bar{1}\bar{4}} = R_{25\bar{2}\bar{5}} = -\frac{a+b-c}{2} \,.
\end{aligned} \tag{F.10}$$

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Such terms are incompatible with a Kähler structure and they all vanish for $\gamma = \alpha + \beta$, which is in fact the case for the 1/3 BPS loop (4.13). With that condition also (F.9) simplifies to

$$\begin{aligned}
&\frac{1}{2}R_{1\bar{1}1\bar{1}} = \frac{1}{2}R_{2\bar{2}2\bar{2}} = R_{1\bar{1}2\bar{2}} = R_{1\bar{2}2\bar{1}} = R_{1\bar{1}3\bar{3}} = R_{2\bar{2}3\bar{3}} = R_{1\bar{3}3\bar{1}} = R_{2\bar{3}3\bar{2}} = a\,, \\
&R_{1\bar{1}4\bar{4}} = R_{1\bar{2}4\bar{5}} = R_{2\bar{1}5\bar{4}} = R_{2\bar{2}5\bar{5}} = R_{1\bar{4}4\bar{1}} = R_{1\bar{5}4\bar{2}} = R_{2\bar{4}5\bar{1}} = R_{2\bar{5}5\bar{2}} = -\frac{ab}{a+b}\,, \\
&R_{3\bar{3}3\bar{3}} = 2(a+b)\,, \\
&R_{3\bar{3}4\bar{4}} = R_{3\bar{3}5\bar{5}} = R_{3\bar{4}4\bar{3}} = R_{3\bar{5}5\bar{3}} = R_{4\bar{4}5\bar{5}} = R_{4\bar{5}5\bar{4}} = \frac{1}{2}R_{4\bar{4}4\bar{4}} = \frac{1}{2}R_{5\bar{5}5\bar{5}} = b\,.
\end{aligned} \tag{F.11}$$

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