

Two-loop hard thermal loops for vector bosons in general models

Andreas Ekstedt

*II. Institute of Theoretical Physics, Universität Hamburg,
22761 Hamburg, Germany*

*Deutsches Elektronen-Synchrotron DESY,
Notkestr. 85, 22607 Hamburg, Germany*

*Department of Physics and Astronomy, Uppsala University,
P.O. Box 256, SE-751 05 Uppsala, Sweden*

E-mail: andreas.ekstedt@desy.de

ABSTRACT: Hard thermal loops describe how soft gauge fields are screened and damped in hot plasmas. As such they are used to calculate transport coefficients, Sphaleron rates, equations of state, and particle production. However, most calculations are done using one-loop hard thermal loop self-energies. And two-loop contributions can be large. To that end this paper provides vector two-loop self-energies for generic models: any scalar, fermion, or vector representation; and all possible renormalizable terms. Several examples are given to showcase the results. Two-loop results for higher-point functions are also given.

KEYWORDS: Finite Temperature or Finite Density, Thermal Field Theory, Effective Field Theories of QCD, Resummation

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Contents

1	Introduction	1
2	The real-time formalism	1
2.1	Hard thermal loops from transport equations	2
2.2	Using kinetic theory beyond leading order	3
2.3	Two-loop hard thermal loops	3
2.4	Procedure for general diagrams	4
3	Generic models	5
3.1	Conventions and structure of the calculation	8
3.2	One-loop hard thermal loops	8
3.3	Two-loop hard thermal loops	9
3.4	Power corrections from one-loop diagrams	10
3.5	Transverse and longitudinal self-energies	11
3.6	Examples	11
4	Higher-point hard thermal loops	12
5	Conclusions	13
A	Derivatives of resummed distributions	14
A.1	Momentum integrals	14
B	Non-abelian gauge theories	14
B.1	Vector current	14
B.2	Ghost diagrams	15
B.3	Total contribution from non-abelian diagrams	15
C	Fermion diagrams	16
C.1	Fermion current	16
C.2	Vector current	16
C.3	Yukawa diagrams	17
D	Scalar Diagrams	17
D.1	Scalar current	17
D.2	Vector current	18
E	Counter-term contributions	19
E.1	Vector loops	20
E.2	Fermion loops	20
E.3	Scalar loops	20
E.4	Power corrections before field redefinitions	20

1 Introduction

Be it phase transitions [1–3]; baryon number violation [4, 5]; photon emission from heavy-ion collisions [6, 7]; or axion production [8–10]; thermal field theory is indispensable all the same. Whilst the picture is quite complicated for generic systems, the physics is considerably simpler if we look at length-scales of the order $L \gg T^{-1}$. For in that case high-energy modes with $E \sim T$ behave as quasiparticles [11, 12]. And much intuition from plasma physics directly carries over. So as charged particles move in, say an electric field, they redistribute themselves to screen the field. Likewise, free charges resist a changing magnetic field in accordance with Lenz’s law. In both cases the field is screened by high-energy modes.

The best way to incorporate this screening depends on the situation. In equilibrium, for example, the scalar potential (A^0) effectively obtains a thermal Debye mass [1]. Since there is no time dependence, it is useful to describe such systems with a three-dimensional field theory [13, 14]. A different effective description, known as hard thermal loops, can be used when fields vary slowly in time.

These hard thermal loops are particularly important when the system is pushed from equilibrium. This is because deviations from equilibrium are driven back by scattering processes; and the characteristic momentum transfer, and thus the cross-section, is set by the screening length. As such hard thermal loops are key for calculating transport coefficients [15–19], particle production [20–22], and colour conductivity [23–25].

Though hard thermal loops are important, little is known about them beyond leading order. Existing studies are limited to quantum electrodynamics at high temperatures [26, 27] and at finite chemical potential [28]. Reason being that direct evaluations are hampered by an increased complexity at two loops. Nevertheless, in this paper we use kinetic theory to simplify the calculations. Our method of choice is rather compact and admits neat expressions for generic model — including non-abelian theories.

The first section of the paper describes the calculation; section 3 provides results for general models; section 4 provides higher-point correlators; and additional details are given in the appendices.

2 The real-time formalism

Throughout this article we use the mostly-plus metric: $P^2 = -(p^0)^2 + \vec{p}^2$, and all four-vectors are denoted by capitalized letters, while spatial vectors are denoted by lowercase ones. To save ink we also use the notation $p^2 \equiv \vec{p}^2$.

Because we are interested in real-time dynamics we have to double the field content [29, 30]: here we follow [6, 31, 32] and use retarded and advanced fields, otherwise known as the r/a basis. In this basis there are three propagators for each field. For a free theory

these are¹

$$\Delta_{B/F}^{rr}(P) = 2\pi\delta(P^2) \left\{ \theta(p^0) N_{B/F}^+(p^0, \vec{p}) + \theta(-p^0) N_{B/F}^-(p^0, -\vec{p}) \right\}, \quad (2.1)$$

$$\Delta^R(P) = \frac{-i}{P^2 - i\eta p^0}, \quad \Delta^A(P) = \frac{-i}{P^2 + i\eta p^0}, \quad N_{B/F}(p^0, \vec{p}) = \frac{1}{2} \pm n_{B/F}(p^0). \quad (2.2)$$

To condense the notation we denote rr propagators by

$$\Delta_X(P) = 2\pi\delta(P^2) \left\{ \theta(p^0) N_X(p^0, \vec{p}) + \theta(-p^0) \bar{N}_X(-p^0, -\vec{p}) \right\}, \quad (2.3)$$

where $X = V, F, S$ depending on the particle. In this case the rr propagator for vectors, fermions, and scalars is

$$D_{\mu\nu}^{rr}(P) = g_{\mu\nu} \Delta_V(P), \quad S_F^{rr} = -\not{P} \Delta_F(P), \quad D_S^{rr}(P) = \Delta_S(P), \quad (2.4)$$

where we have used Feynman gauge.

To handle divergences we use dimensional regularization. This means that our integration measures are

$$\int_P \equiv \left(\frac{\mu^2 e^\gamma}{4\pi} \right)^\epsilon \int \frac{d^D P}{(2\pi)^D}, \quad \int_p \equiv \left(\frac{\mu^2 e^\gamma}{4\pi} \right)^\epsilon \int \frac{d^d p}{(2\pi)^d}, \quad (2.5)$$

where $D = 4 - 2\epsilon$ and $d = 3 - 2\epsilon$.

2.1 Hard thermal loops from transport equations

As of yet, two-loop hard thermal loops are only known for quantum electrodynamics [26–28]. These calculations are quite involved and have so far been done using Feynman diagrams.² To make our calculations tractable, we instead use transport equations. This method is well-known, and is a clean way to derive hard thermal loops at leading order [12, 35–37]. Here we extend the method to the next order. Essentially we use that fields with typical momenta $p \sim T$ can be treated as quasiparticles. For example, we can describe electrons with the Vlasov equation:

$$\dot{N}_F^\pm + \vec{v} \cdot \vec{\nabla} N_F^\pm \pm e \left(\vec{E} + \vec{v} \times \vec{B} \right) \cdot \vec{\nabla}^p N_F^\pm = 0. \quad (2.6)$$

If we now assume that the electrons are driven slightly away from equilibrium by the electric field, we can expand the electron distribution as

$$N_F^\pm = \frac{1}{2} - n_F - \delta n_F^\pm, \quad v \cdot \partial \delta n_F^\pm(\vec{p}, x) = \mp e \vec{v} \cdot \vec{E} \frac{d}{dp} n_F(p), \quad (2.7)$$

$$v^\mu = (1, \vec{v}), \quad \vec{v} \equiv \frac{\vec{p}}{p^0}, \quad n_F(p) = \left(e^{p/T} + 1 \right)^{-1}. \quad (2.8)$$

The photon self-energy then follows from the electron current [12, 35]:

$$\partial_\mu F^{\nu\mu} = e \left\langle \bar{\Psi} \gamma^\nu \Psi \right\rangle \sim e \int_p v^\mu (N_F^+ - N_F^-), \quad (2.9)$$

¹See [32, 33] for a clear diagrammatic representation of Feynman rules in this basis.

²See [34] for results with general external momenta.

where $\langle \dots \rangle$ denotes the average over hard modes with characteristic momenta $p \sim T$. That is

$$\partial_\mu F^{\nu\mu} = -e \int \frac{d^4 p}{(2\pi)^4} \text{Tr} [\not{p} \gamma^\nu] \Delta_F(p) = 2e \int \frac{d^3 p}{(2\pi)^3} v^\nu [N^+(p, x) - N^-(p, x)] \quad (2.10)$$

$$= 4e^2 \int \frac{d^3 p}{(2\pi)^3} \frac{v^\nu \vec{v} \cdot \vec{E}(x)}{v \cdot K} n'_F(p) = -\frac{e^2 T^2}{3} \int \frac{d\Omega_v}{4\pi} \frac{v^\nu \vec{v} \cdot \vec{E}(x)}{v \cdot K} \equiv \Pi^{\nu\mu} A_\mu, \quad (2.11)$$

where $\vec{E} = -\vec{\nabla} A^0 - \dot{\vec{A}}$.

Note that the kinetic approach works because quantum fields with $p \sim eT$ behave classically at high temperatures. In generic situations we have no right to expect classical equations of motion. We should also remember that scattering processes become important at time scales of order $t \sim (e^4 T)^{-1}$ [11, 23–25], and that our results only hold for soft fields: $\dot{\vec{A}} \sim \vec{\nabla} A \sim (eT)A$.

2.2 Using kinetic theory beyond leading order

There are two ways that we can go about applying the kinetic approach at two loops. First, we can include resummed self-energies directly in the transport equations and use this to derive effective particle distributions [37]. While possible, this approach involves evaluating self-energies at finite external momentum. Instead we elect to only use leading-order transport equations — two-loop results are then obtained by calculating corrections to the fermion current $\langle \bar{\Psi} \gamma^\nu \Psi \rangle$. At first glance it seems like we are back to brute-force evaluating diagrams. Be that as it may, working with currents is considerably easier than calculating self energies. And as we shall see, the results for different kinds of particles involve the same compact expressions.

2.3 Two-loop hard thermal loops

Let us demonstrate our approach for quantum electrodynamics. The two-loop contribution to the electron current is shown in figure 1a:

$$\begin{aligned} \langle \bar{\Psi} \gamma^\mu \Psi \rangle_{2\text{-loop}} &= e^2 \int_{PQ} F^\mu \left\{ \Delta_F(P) \Delta_V(Q) \left[\Delta^R(P) \Delta^R(P+Q) + \Delta^A(P) \Delta^A(P+Q) \right] \right. \\ &\quad + \Delta_F(P) \Delta_F(P+Q) \left[\Delta^R(P) \Delta^A(Q) + \Delta^A(P) \Delta^R(Q) \right] \\ &\quad \left. + \Delta_F(P+Q) \Delta_V(Q) \left[\Delta^R(P) \Delta^A(P) \right] \right\}, \end{aligned} \quad (2.12)$$

where $F^\mu = \text{Tr} \not{P} \gamma^\mu \not{P} \gamma^\alpha (\not{P} + \not{Q}) \gamma_\alpha = -4(D-2) [(P+Q)^2 p^\mu - P^2 q^\mu - Q^2 p^\mu]$.

To evaluate the integrals we have to define

$$\delta(p^2) \Delta^{R/A}(P). \quad (2.13)$$

This expression contains terms with two delta functions — these must be regulated. To do so we use the original approach [30]:

$$\pi \delta(P^2) \Delta^{R/A}(P) = \frac{-i}{P^2 \mp i\eta p^0} \frac{\eta}{P^4 + \eta^2} = \pm p^0 \left[\frac{\eta}{P^4 + \eta^2} \right]^2 - i \frac{P^2 \eta}{(P^4 + \eta^2)^2} \quad (2.14)$$

$$= \pm p^0 \left[\pi \delta(P^2) \right]^2 - \frac{i}{2} \frac{\partial}{\partial p_0^2} \left[\pi \delta(P^2) \right]. \quad (2.15)$$

For a given topology all $[\pi\delta(P^2)]^2$ terms cancel, while the remaining pieces can be handled by integration-by-parts.

As an example, consider

$$\int_{PQ} F^\mu \Delta_F(P) \Delta_V(Q) \left[\Delta^R(P) \Delta^R(P+Q) + \Delta^A(P) \Delta^A(P+Q) \right]. \quad (2.16)$$

After using equation (2.14) we find

$$\begin{aligned} & \pi\delta(P^2) \left[\Delta^R(P) \Delta^R(P+Q) + \Delta^A(P) \Delta^A(P+Q) \right] \\ &= p^0 \left[\pi\delta(P^2) \right]^2 \left[\Delta^R(P+Q) - \Delta^A(P+Q) \right] - \frac{i}{2} \frac{\partial}{\partial p_0^2} \left[\pi\delta(P^2) \right] \left[\Delta^R(P+Q) + \Delta^A(P+Q) \right]. \end{aligned} \quad (2.17)$$

The first term vanishes, so we are left with the second term. Now, for the $P^2 q^\mu$ term to contribute, the $\frac{\partial}{\partial p_0^2}$ derivative must hit P^2 . So this term is proportional to

$$\int_{PQ} q^\mu \Delta_F(P) \Delta_V(Q) \left[\frac{1}{(P+Q)^2} \right]. \quad (2.18)$$

Naively we expect a collinear ($\vec{p} \parallel \vec{q}$) divergence from the angular integration, but these cancel once we sum all contributions.

The $p^\mu(P+Q)^2$ factor results in a term proportional to

$$\int_{PQ} \frac{N_V(q)}{qp} \left\{ \left[\partial_0^p N_F - \partial_0^p \bar{N}_F \right] v_p^\mu - \frac{v_p^\mu - n^\mu}{p} (N_F - \bar{N}_F) \right\}, \quad n^\mu = (1, \vec{0}). \quad (2.19)$$

Finally, the $Q^2 p^\mu$ term does not contribute as $\Delta_V(Q)$ sets $Q^2 = 0$.

The remaining terms in $\langle \bar{\Psi} \gamma^\mu \Psi \rangle_{2\text{-loop}}$ are obtained in the same way. After performing the integrals and using the formulas in appendix A, we find

$$\Pi_{\text{NLO}}^{\mu\nu}(K) = -\frac{e^4 T^2}{8\pi^2} \int \frac{d\Omega_v}{4\pi} \left\{ v^\mu v^\nu \left[\frac{(k^0)^2}{(v \cdot K)^2} - \frac{2k^0}{v \cdot K} \right] + [v^\mu n^\nu + n^\mu v^\nu] \frac{k^0}{v \cdot K} - n^\mu n^\nu \right\}, \quad (2.20)$$

which can be compared with the leading-order self-energy

$$\Pi_{\text{LO}}^{\mu\nu}(K) = -\frac{e^2 T^2}{3} \int \frac{d\Omega_v}{4\pi} \left[n^\mu n^\nu + v^\mu v^\nu \frac{k_0}{v \cdot K} \right]. \quad (2.21)$$

This result is in agreement with previous calculations [27, 28]. For completeness we have to add power-corrections. This is done in section 3.4.

2.4 Procedure for general diagrams

Irrespective of the diagram or particle, the only terms that contribute are of the form

$$\int_{PQ} (ap^\mu + bq^\mu) (P+Q)^2 \Delta_X(P) \Delta_Y(Q) \left[\Delta^R(P) \Delta^R(P+Q) + \Delta^A(P) \Delta^A(P+Q) \right]. \quad (2.22)$$

The piece going with p^μ give terms proportional to

$$\int_{PQ} \frac{N_Y(q)}{qp} \left\{ \left[\partial_0^p N_X(p) - \partial_0^p \bar{N}_X(p) \right] v_p^\mu - \frac{v_p^\mu - n^\mu}{p} (N_X(p) - \bar{N}_X(p)) \right\}, \quad (2.23)$$

and the term multiplying q^μ give terms of the form

$$\int_{PQ} v_q^\mu \left(N_Y(q) - \bar{N}_Y(q) \right) \frac{1}{p^2} \left\{ \partial_0^p \left[N_X(p) + \bar{N}_X(p) \right] - \frac{1}{p} \left(N_X(p) + \bar{N}_X(p) \right) \right\}. \quad (2.24)$$

In our example we only had the first type, but the second type of terms appears in non-abelian theories. Physically the first Lorentz structure corresponds to deviations from the ballistic approximation:

$$v_p^\mu = \frac{p^\mu}{p^0} \rightarrow v_p^\mu - \frac{m^2}{2p^2} \left[v_p^\mu - n^\mu \right] + \dots, \quad (2.25)$$

where $m^2 \sim \int p^{-1} n_{B/F}(p)$ represents hard charges obtaining a thermal mass.

The second structure, on the other hand, represents a renormalization of the hard distributions themselves. Connected with this the momentum integral in equation (2.24) contain divergences.³

We also note that the calculation is simpler in Feynman gauge. In particular, scalar and vector currents contain terms of the form

$$\langle A_\mu^a R^i R^j \rangle, \quad \langle A_\mu^a A^{b,\nu} A_\nu^c \rangle, \quad (2.26)$$

which at two loops give the diagrams shown in figures 2a and 2b. However, in Feynman gauge these diagrams vanish. In addition, the ghost-current shown in figure 1d does not contribute at two loops in Feynman gauge.

3 Generic models

We denote scalar particles by i, j, k, \dots ; vector particles by a, b, c, \dots ; and fermions by I, J, K, \dots . To parametrize a general model we use the Lagrangian [38–41]

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} R_i (-\delta_{ij} \partial_\mu \partial^\mu + \mu_{ij}) R_j - \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu,b} \delta_{ab} - \frac{1}{2\xi_a} (\partial_\mu A^{a,\mu})^2 \\ & - \partial^\mu \bar{\eta}^a \partial_\mu \eta^a + i \psi^{\dagger,I} \bar{\sigma}^\mu \partial_\mu \psi_I - \frac{1}{2} (M^{IJ} \psi_I \psi_J + \text{h.c.}) + \mathcal{L}_{\text{int}} \\ \mathcal{L}_{\text{int}} = & -\frac{1}{4!} \lambda^{ijklm} R_i R_j R_k R_m - \frac{1}{2} (Y^{iIJ} R_i \psi_I \psi_J + \text{h.c.}) \\ & + g_J^{a,I} A_\mu^a \psi^{\dagger,J} \bar{\sigma}^\mu \psi_I - g_{jk}^a A_\mu^a R_j \partial^\mu R_k - \frac{1}{2} g_{jn}^a g_{kn}^b A_\mu^a A^{\mu,b} R_j R_k - g^{abc} A^{\mu,a} A^{\nu,b} \partial_\mu A_\nu^c \\ & - \frac{1}{4} g^{abe} g^{cde} A^{\mu a} A^{\nu b} A_\mu^c A_\nu^d + g^{abc} A_\mu^a \eta^b \partial^\mu \bar{\eta}^c. \end{aligned} \quad (3.1)$$

In this notation R_i are scalar fields in a real basis; A_μ^a are vector bosons; η^a are ghosts; and ψ_I are Weyl fermions [42]. The sigma matrices are defined as

$$\sigma^\mu = (\mathbb{1}, \sigma^i), \quad \bar{\sigma}^\mu = (-\mathbb{1}, \sigma^i), \quad (3.2)$$

³These cancel once counter-term insertions are added. Note that there is a subtlety with treating this divergence as discussed in appendix B.

and satisfy

$$\{\sigma_\mu, \bar{\sigma}_\nu\} = -2g_{\mu\nu}, \quad g_{\mu\nu} = \text{diag}(-1, \vec{1}). \quad (3.3)$$

The couplings are normalized such that for the Standard-model we have

$$\begin{aligned} \delta_{ab} \text{Tr} [g_V^a g_V^b] &= -24g_s^2 - 6g_w^2, & \delta_{ab} \text{Tr} [g_S^a g_S^b] &= -3g_w^2 - g_Y^2, \\ \delta_{ab} \text{Tr} [g_F^a g_F^b] &= N_F \left(16g_s^2 + 6g_w^2 + \frac{10}{3}g_Y^2 \right). \end{aligned}$$

For a generic model these coupling tensors can be calculated by hand, but they are also straightforward to find from GroupMath [43].

To calculate hard thermal loops we use resummed distributions. For a general model these are [12, 44]

$$N_V^\pm \rightarrow N_V^{ab,\pm} = \delta^{ab} \left[\frac{1}{2} + n_B(p^0) \right] + \delta N_V^{ab,\pm}(p^0, \vec{p}), \quad (3.4)$$

$$N_S^\pm \rightarrow N_S^{ij,\pm} = \delta^{ij} \left[\frac{1}{2} + n_B(p^0) \right] + \delta N_S^{ij,\pm}(p^0, \vec{p}), \quad (3.5)$$

$$N_F^\pm \rightarrow N_{F,J}^{I,\pm} = \delta_J^I \left[\frac{1}{2} - n_F(p^0) \right] - \delta N_{F,J}^{I,\pm}(p^0, \vec{p}), \quad (3.6)$$

where

$$n_B(p) = \left(e^{p/T} - 1 \right)^{-1}, \quad n_F(p) = \left(e^{p/T} + 1 \right)^{-1}. \quad (3.7)$$

We can condense the notation further:

$$\delta N_V^{ab} \equiv -ig^{abc} \delta N_V^c, \quad \delta N_S^{ij} \equiv -ig_{ij}^c \delta N_S^c, \quad \delta N_{F,J}^I \equiv g_J^{cI} \delta N_F^c, \quad (3.8)$$

where the distributions satisfy⁴

$$v \cdot \partial \delta N_X^{\pm,a} = \mp \vec{v} \cdot \vec{E}^a n'_X(p), \quad \vec{E}^a = -\dot{\vec{A}}^a - \vec{\nabla} A^{0,a}. \quad (3.9)$$

⁴We are for the moment omitting higher-point functions. These are calculated in section 4.

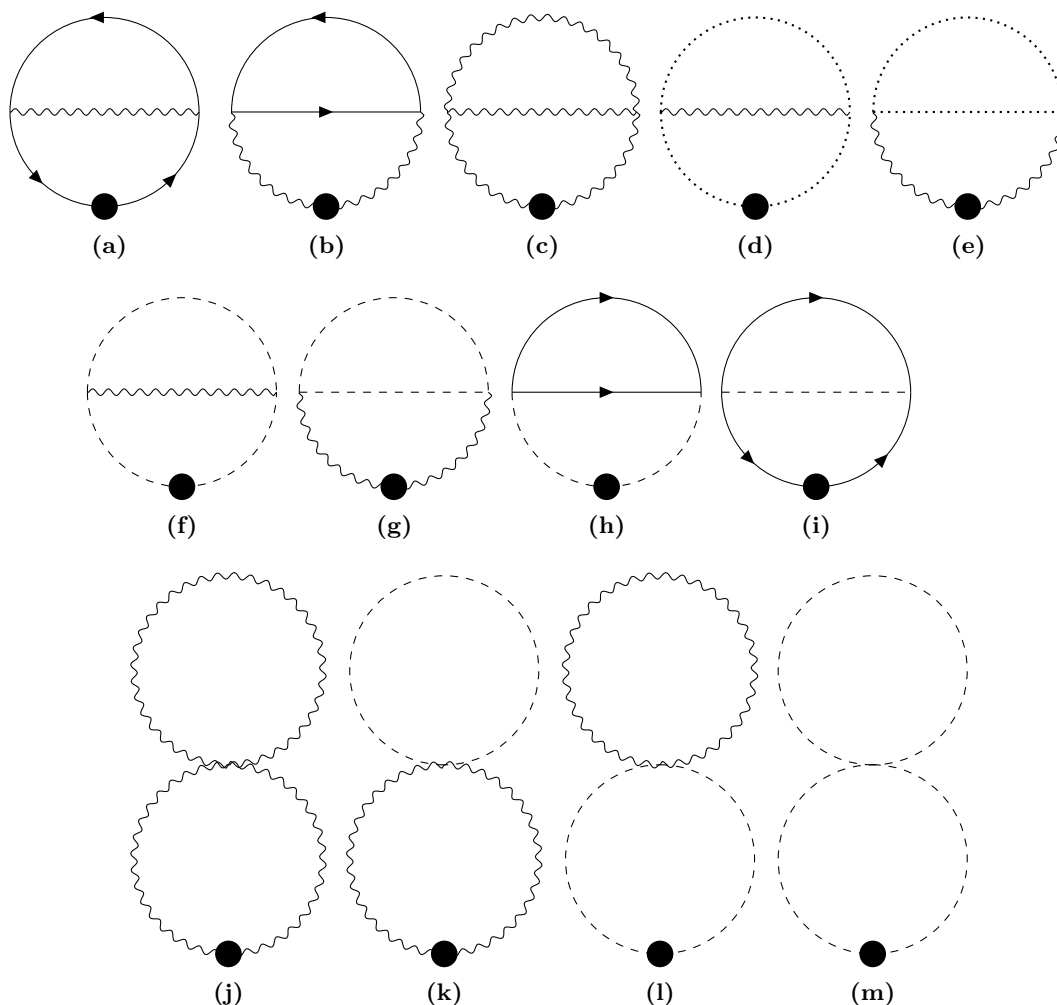


Figure 1. Figures (a) and (i) represent corrections to the fermion current; figures (b), (c), (e), (k), (j), and (e) represent corrections to the vector current; figure (d) represents corrections to the ghost current; figures (f), (l), (m), and (h) represent corrections to the scalar current.

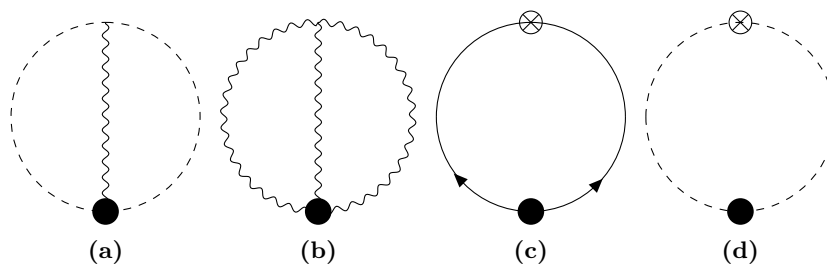


Figure 2. Additional diagrams that contribute to the vector self-energy at next-to-leading order. Diagrams (c) and (d) correspond to mass insertions, and diagrams (a) and (b) vanish in Feynman gauge.

3.1 Conventions and structure of the calculation

All correlators that contribute at next-to-leading order are shown in figures 1 and 2, and the details are given in appendices B, C, D, and E. We use

$$\Pi_1^{\mu\nu} = \int \frac{d\Omega_v}{4\pi} \left(n^\mu n^\nu + k^0 \frac{v^\mu v^\nu}{v \cdot K} \right), \quad n^\mu = (1, \vec{0}) \quad (3.10)$$

$$\Pi_2^{\mu\nu} = \int \frac{d\Omega_v}{4\pi} \left\{ v^\mu v^\nu \left[\frac{(k^0)^2}{(v \cdot K)^2} - \frac{2k^0}{v \cdot K} \right] + [v^\mu n^\nu + n^\mu v^\nu] \frac{k^0}{v \cdot K} - n^\mu n^\nu \right\}, \quad (3.11)$$

to signify the two Lorentz structures that appear. Note that these satisfy $K_\mu \Pi^{\mu\nu} = 0$, so the self-energy is automatically transverse.

To derive the self-energy we need various currents:

$$\partial_\mu F^{\nu\mu,a} = j_F^{a,\nu} + j_S^{a,\nu} + j_g^{a,\nu} + j_V^{a,\nu}. \quad (3.12)$$

The fermion current is given by

$$j_F^{a,\nu} = g_I^{a,J} \langle \psi^\dagger, I \bar{\sigma}^\nu \psi_J \rangle. \quad (3.13)$$

The scalar current is

$$j_S^{a,\nu} = \frac{1}{2!} g_{ij}^a \langle \partial^\nu R_i R_j - R_i \partial^\nu R_j \rangle. \quad (3.14)$$

The ghost current is

$$j_g^{a,\nu} = g^{abc} \langle \eta^b \partial^\mu \bar{\eta}^c \rangle. \quad (3.15)$$

Finally, the vector current is

$$j_V^{a,\nu} = -g^{abc} \langle \partial_\mu A^{\nu,b} A^{\mu,c} + A^{\nu,b} \partial \cdot A^c + A_\mu^b \partial^\nu A^{\mu,c} - A^b \cdot \partial A^{\nu,c} \rangle. \quad (3.16)$$

3.2 One-loop hard thermal loops

As mentioned, one-loop results are well known [12, 35, 45, 46]. With our notation the results are

$$\Pi_{\text{LO}}^{\mu\nu} = \Pi_V^{\mu\nu} + \Pi_F^{\mu\nu} + \Pi_S^{\mu\nu}, \quad (3.17)$$

where

$$\Pi_V^{\mu\nu} = -(D-2) \text{Tr} \left[g_V^a g_V^b \right] \int_p n'_B(p) \Pi_1^{\mu\nu} = \frac{T^2}{3} \text{Tr} \left[g_V^a g_V^b \right] \Pi_1^{\mu\nu} + \mathcal{O}(\epsilon), \quad (3.18)$$

$$\Pi_F^{\mu\nu} = 2 \text{Tr} \left[g_F^a g_F^b \right] \int_p n'_F(p) \Pi_1^{\mu\nu} = -\frac{T^2}{6} \text{Tr} \left[g_F^a g_F^b \right] \Pi_1^{\mu\nu} + \mathcal{O}(\epsilon), \quad (3.19)$$

$$\Pi_S^{\mu\nu} = -\text{Tr} \left[g_S^a g_S^b \right] \int_p n'_B(p) \Pi_1^{\mu\nu} = \frac{T^2}{6} \text{Tr} \left[g_S^a g_S^b \right] \Pi_1^{\mu\nu} + \mathcal{O}(\epsilon). \quad (3.20)$$

3.3 Two-loop hard thermal loops

At two loops various diagrams introduce factors of $D=4-2\epsilon$; below we have only kept the $\mathcal{O}(\epsilon^0)$ contribution, but the full results are given in the appendices. We separate the result as

$$\Pi_{\text{NLO}}^{\mu\nu,ab} = - \left[\Pi_{\text{V}}^{\mu\nu,ab} + \Pi_{\text{SV}}^{\mu\nu,ab} + \Pi_{\text{FV}}^{\mu\nu,ab} + \Pi_{\text{SF}}^{\mu\nu,ab} \right], \quad (3.21)$$

signifying pure vector, scalar-vector, fermion-vector, and scalar-fermion-vector type interactions respectively. All repeated indices are summed.

Let us start with the pure-vector contribution:

$$\Pi_{\text{V}}^{\mu\nu,ab} = T^2 \frac{11 \log\left(\frac{\mu e^\gamma}{4\pi T}\right) + 6k^0 L[k] - \frac{1}{2}}{36\pi^2} g_V^{adc} g_V^{cef} g_V^{dfn} g_V^{enb} \Pi_1^{\mu\nu} - \frac{T^2}{12\pi^2} g_V^{adc} g_V^{cef} g_V^{dfn} g_V^{enb} \Pi_2^{\mu\nu}. \quad (3.22)$$

The scalar-vector contribution is

$$\begin{aligned} \Pi_{\text{SV}}^{\mu\nu,ab} &= \frac{T^2}{192\pi^2} \text{Tr} \left[g_S^a g_S^b \right]_{jl} \lambda^{jlmn} \Pi_2^{\mu\nu} + \frac{1}{8\pi^2} \text{Tr} \left[g_S^a g_S^b \right]_{ij} \mu^{ij} \Pi_2^{\mu\nu} \\ &\quad - T^2 \frac{\log\left(\frac{\mu e^\gamma}{4\pi T}\right) + 1}{288\pi^2} \text{Tr} \left[g_S^a g_S^c \right] \text{Tr} \left[g_S^c g_S^b \right] \Pi_1^{\mu\nu} - \frac{T^2}{32\pi^2} \text{Tr} \left[g_S^a g_S^b g_S^c g_S^c \right] \Pi_2^{\mu\nu} \\ &\quad - T^2 \frac{\log\left(\frac{\mu e^\gamma}{4\pi T}\right) + k^0 L[K]}{24\pi^2} g_V^{aec} g_V^{bdc} \text{Tr} \left[g_S^d g_S^e \right] \Pi_1^{\mu\nu} + \frac{T^2}{48\pi^2} g_V^{aec} g_V^{bdc} \text{Tr} \left[g_S^d g_S^e \right] \Pi_2^{\mu\nu} \\ &\quad + T^2 \frac{\log\left(\frac{\mu e^\gamma}{4\pi T}\right) - 3}{72\pi^2} \left\{ \text{Tr} \left[g_S^a g_S^c \right] \text{Tr} \left[g_V^c g_V^b \right] + \text{Tr} \left[g_V^a g_V^c \right] \text{Tr} \left[g_S^c g_S^b \right] \right\} \Pi_1^{\mu\nu}. \end{aligned} \quad (3.23)$$

The fermion-vector contribution is

$$\begin{aligned} \Pi_{\text{FV}}^{\mu\nu,ab} &= T^2 \frac{\log\left(\frac{\mu e^\gamma}{4\pi T}\right) + k^0 L[K]}{24\pi^2} g_V^{ace} g_V^{bcd} \text{Tr} \left[g_F^d g_F^e \right] \Pi_1^{\mu\nu} \\ &\quad - T^2 \frac{\log\left(\frac{\mu e^\gamma}{4\pi T}\right) - \frac{1}{2} + \log(4)}{72\pi^2} \text{Tr} g_F^a g_F^c \text{Tr} g_F^c g_F^b \Pi_1^{\mu\nu} \\ &\quad - T^2 \frac{\log\left(\frac{\mu e^\gamma}{4\pi T}\right) + \frac{3}{2} - 8 \log(2)}{288\pi^2} \left\{ \text{Tr} \left[g_F^a g_F^c \right] \text{Tr} \left[g_V^c g_V^b \right] + \text{Tr} \left[g_V^a g_V^c \right] \text{Tr} \left[g_F^c g_F^b \right] \right\} \Pi_1^{\mu\nu} \\ &\quad + \frac{T^2}{16\pi^2} \text{Tr} g_F^c g_F^c g_F^a g_F^b \Pi_2^{\mu\nu} - \frac{T^2}{48\pi^2} g_V^{ace} g_V^{bcd} \text{Tr} \left[g_F^d g_F^e \right] \Pi_2^{\mu\nu} + \frac{1}{8\pi^2} \text{Tr} \left[g_F^a M_F M_F^\dagger g_F^b \right] \Pi_2^{\mu\nu}. \end{aligned} \quad (3.24)$$

And finally, the mixed fermion-scalar contribution is

$$\begin{aligned} \Pi_{\text{SF}}^{\mu\nu,ab} &= T^2 \frac{5 \log\left(\frac{\mu e^\gamma}{4\pi T}\right) - 1 + 8 \log(2)}{576\pi^2} \left\{ \text{Tr} \left[g_S^a g_S^c \right] \text{Tr} \left[g_F^c g_F^b \right] + \text{Tr} \left[g_F^a g_F^c \right] \text{Tr} \left[g_S^c g_S^b \right] \right\} \Pi_1^{\mu\nu} \\ &\quad + \frac{T^2}{32\pi^2} \left[g_F^a g_F^b \right]_J^I (Y Y^c)_I^J \Pi_2^{\mu\nu} + \frac{T^2}{192\pi^2} \left[g_S^a g_S^b \right]_{ij} (Y Y^c + Y^c Y)^{ij} \Pi_2^{\mu\nu}. \end{aligned} \quad (3.25)$$

In the traces over generators the contractions are made with the conventions

$$\text{Tr} \left[g_V^a g_V^b \right] = g^{acd} g^{bdc}, \quad \text{Tr} \left[g_S^a g_S^b \right] = g_{ij}^a g_{ji}^b, \quad \text{Tr} \left[g_F^a g_F^b \right] = g_J^{aI} g_I^{bJ}. \quad (3.26)$$

Note that our two-loop results in equation (3.22) ensures that

$$\Pi^{\mu\nu} = \Pi_{\text{LO}}^{\mu\nu} + \Pi_{\text{NLO}}^{\mu\nu}, \quad (3.27)$$

is renormalization-scale invariant. As such one should choose $\mu \sim T$ to ensure that no large logarithms are present.

3.4 Power corrections from one-loop diagrams

Power corrections modify the kinetic terms and are, for example, responsible for anomalous dimensions. We forgo using transport equations since the diagrams are straightforward to evaluate [27, 28, 47].

We use a convention where the Debye mass is given by

$$(m_D^2)^{ab} = - \lim_{k^0 \rightarrow 0} \Pi_{\text{NLO}}^{\mu\nu, ab}, \quad (3.28)$$

with $\Pi_{\text{NLO}}^{\mu\nu, ab}$ defined by equation (3.21). This means that we have rescaled our vector fields to make the A^0 kinetic term canonical when $k^0 = 0$. To wit, we have moved all renormalization-scale dependence (and some finite pieces) away from the power corrections.⁵ The original results — before field-redefinitions — are given in appendix E.4.

That said, the scalar loop gives

$$\Pi_S^{\mu\nu, ab}(K) = -\text{Tr} \left[g_S^a g_S^b \right] \int_P (2P+K)^\mu (2P+K)^\nu \Delta_S(P) \Delta^R(P+K). \quad (3.29)$$

We are only interested in the sub-leading correction scaling as $K^2 \sim k^2 \sim (gT)^2$. After expanding the integral, and adding counter-terms, we find

$$g_{\mu\nu} \Pi_S^{\mu\nu, ab}(K) = \text{Tr} \left[g_S^a g_S^b \right] \frac{K^2}{16\pi^2} \left\{ k^0 L(K) - \frac{3}{3} \right\}, \quad (3.30)$$

$$\Pi_S^{00, ab}(K) = -\text{Tr} \left[g_S^a g_S^b \right] \frac{k^2}{16\pi^2} \left\{ \frac{1}{3} \frac{(k^0)^2}{k^2} (k^0 L(K) - 1) \right\} \quad (3.31)$$

The fermion loop gives

$$\Pi_F^{\mu\nu, ab}(K) = \text{Tr} \left[g_F^a g_F^b \right] \int_P F^{\mu\nu} \Delta_F(P) \Delta^R(P+K), \quad (3.32)$$

$$F^{\mu\nu} = 2 \left[-g^{\mu\nu} (K \cdot P + P^2) + 2p^\mu p^\nu + k^\mu p^\nu + k^\nu p^\mu \right]. \quad (3.33)$$

After expanding the integral, and adding counter-terms, we find

$$g_{\mu\nu} \Pi_F^{\mu\nu, ab}(K) = \text{Tr} \left[g_F^a g_F^b \right] \frac{K^2}{16\pi^2} \left\{ 4k^0 L(K) + \frac{4}{3} \right\}, \quad (3.34)$$

$$\Pi_F^{00, ab}(K) = -\text{Tr} \left[g_F^a g_F^b \right] \frac{k^2}{16\pi^2} \left\{ \frac{2}{3} k^0 \left(3 - \frac{(k^0)^2}{k^2} \right) L(K) + \frac{2}{3} \frac{(k^0)^2}{k^2} \right\}.$$

⁵Our convention makes the Lorentz structure easy at two loops, in addition, the result is manifestly renormalization-scale invariant.

For non-abelian diagrams we group ghosts and vectors together. After adding counter-terms we find

$$g_{\mu\nu}\Pi_V^{\mu\nu,ab}(K) = \text{Tr} \left[g_V^a g_V^b \right] \frac{K^2}{16\pi^2} \left\{ 10k^0 L(K) + \frac{4}{3} \right\}, \quad (3.35)$$

$$\Pi_V^{00,ab}(K) = -\text{Tr} \left[g_V^a g_V^b \right] \frac{k^2}{16\pi^2} \left\{ \frac{2}{3}k^0 \left(6 - \frac{(k^0)^2}{k^2} \right) L(K) + \frac{2(k^0)^2}{3k^2} \right\}.$$

3.5 Transverse and longitudinal self-energies

It is useful to write the vector self-energy in terms of transverse and longitudinal components [48]:

$$\Pi^{\mu\nu} = \Pi_T P_T^{\mu\nu} + \Pi_L P_L^{\mu\nu}, \quad P_T^{ij} = \delta^{ij} - \frac{p^i p^j}{p^2}, \quad P_L^{\mu\nu} = g^{\mu\nu} - \frac{K^\mu K^\nu}{K^2} - P_T^{\mu\nu}. \quad (3.36)$$

We then find

$$\Pi_T = \frac{1}{d-1} \left[g_{\mu\nu} \Pi^{\mu\nu} + \frac{K^2}{k^2} \Pi^{00} \right], \quad \Pi_L = -\frac{K^2}{k^2} \Pi^{00}. \quad (3.37)$$

Since our results are built from the Lorentz structures $\Pi_1^{\mu\nu}$ and $\Pi_2^{\mu\nu}$ defined in equation (3.10), we only need to find the traces of these. To wit

$$\Pi_1^{00} = 1 - k^0 L[K], \quad g_{\mu\nu} \Pi_1^{\mu\nu} = -1, \quad (3.38)$$

$$\Pi_2^{00} = -1 - \frac{(k^0)^2}{K^2}, \quad g_{\mu\nu} \Pi_2^{\mu\nu} = 1 + 2k^0 L[K], \quad (3.39)$$

$$L[K] \equiv \frac{1}{2k} \log \frac{k^0 + k + i\eta}{k^0 - k + \eta}, \quad \eta = 0^+, \quad (3.40)$$

where we have used known results for the angular integrals [28, 49].

3.6 Examples

Consider now the gluon self-energy with N_q fundamental quarks:⁶

$$\Pi_{\text{NLO}}^{\mu\nu} = \frac{g_s^4 (N_q + 6) T^2 \left[(4N_q - 66) \log \frac{\mu e^\gamma}{4\pi T} - 2N_q + 8N_q \log(2) + 3 - 32k^0 L[K] \right]}{288\pi^2} \Pi_1^{\mu\nu} \quad (3.41)$$

$$- \frac{g_s^4 (N_q - 18) T^2}{48\pi^2} \Pi_2^{\mu\nu}. \quad (3.42)$$

Next, the Standard-Model. The gluon self-energy is

$$\Pi_{\text{NLO}}^{\mu\nu} = - \frac{g_s^4 T^2 \left[14 \log \frac{\mu e^\gamma}{4\pi T} + 3 - 16 \log(2) + 12k^0 L[K] \right]}{8\pi^2} \Pi_1^{\mu\nu} \quad (3.43)$$

$$- \frac{g_s^2 T^2 \left[-48g_s^2 + 27g_w^2 + 11g_Y^2 + 12y_t^2 \right]}{192\pi^2} \Pi_2^{\mu\nu}. \quad (3.44)$$

⁶This result agrees with a recent independent calculation [50].

Here g_s is the strong coupling constant, g_w the weak one, g_Y is the hypercharge coupling, and y_t is the top-Yukawa coupling.

Finally, take an SO(10) gauge theory with N_F fermions in the spinor (16) representation, and a $45 \oplus 16$ Higgs. The gauge self-energy is

$$\Pi_{\text{NLO}}^{\mu\nu} = \frac{g_x^4(N_F+14)T^2 \left[(2N_F-41) \log \frac{\mu e^\gamma}{4\pi T} + N_F(\log(16)-1) + 5 - 24k^0 L[K] \right]}{36\pi^2} \Pi_1^{\mu\nu} \quad (3.45)$$

$$- \frac{g_x^4(71N_F-1415)T^2}{192\pi^2} \Pi_2^{\mu\nu}. \quad (3.46)$$

4 Higher-point hard thermal loops

So far we have focused on the self-energy, but higher-point correlators can be extracted from the results in section 3.3. In particular, it is well-known that at one loop all higher-point functions can be derived by using [12, 35, 44]

$$\left[v \cdot D, \delta N_X^\pm \right]^a = \mp \vec{v} \cdot \vec{E}^a n'_X(p), \quad (4.1)$$

where the covariant derivative is $[D_\mu N]^a = \partial_\mu N^a + g^{abc} A_\mu^b N^c$. We can then expand the currents as

$$j_\mu^a = \Pi_{\mu\nu}^{ab} A^{\nu,b} + \frac{1}{2} \Gamma_{\mu\nu\rho}^{abc} A^{\nu,b} A^{\rho,c} + \dots \quad (4.2)$$

To find these higher-point functions we can use the results in section 3.1 together with the replacements:

$$C^{ab} \Pi_1^{\mu\nu} \rightarrow C^{ab} \int \frac{d\Omega_v}{4\pi} \left[\frac{v^\mu \vec{v} \cdot \vec{E}}{v \cdot D} \right]^b, \quad (4.3)$$

$$D^{ab} \Pi_2^{\mu\nu} \rightarrow D^{ab} \int \frac{d\Omega_v}{4\pi} \left\{ v^\mu \left(-\frac{D_0}{(v \cdot D)^2} - \frac{1}{v \cdot D} \right) \vec{v} \cdot \vec{E} - \frac{v^\mu - n^\mu}{v \cdot D} \vec{v} \cdot \vec{E} \right\}^b, \quad (4.4)$$

where now $E^{a,i} = \partial^i A^{0,a} - \partial^0 A^{i,a} + g^{abc} A^{i,b} A^{0,c}$.

Consider the first Lorentz-structure, which coincides with the one-loop one. The corresponding three-point vertex is well-known [35, 45, 51]:

$$C^{ab} \Pi_1^{\mu\nu} \rightarrow -i C^{ae} g^{bc} \Gamma_1^{\mu\nu\rho}(P, Q, R), \quad \Gamma_1^{\mu\nu\rho}(P, Q, R) = \int \frac{d\Omega_v}{4\pi} \frac{v^\mu v^\nu v^\rho}{v \cdot P} \left[\frac{q^0}{v \cdot Q} - \frac{r^0}{v \cdot R} \right]. \quad (4.5)$$

Likewise, it is possible to find the three-point vertex corresponding to $\Pi_2^{\mu\nu}$ by expanding the covariant derivatives. Yet it is easier to exploit that this new Lorentz structure arises because the ballistic approximation ceases to hold:

$$v_p^\mu \rightarrow v_p^\mu - \frac{m^2}{2p^2} (v_p^\mu - n^\mu) + \dots \quad (4.6)$$

As such we can use equation (4.5) — together with the correction above⁷ — and collect all terms proportional to m^2 :

$$D^{ab}\Pi_2^{\mu\nu} \rightarrow -iD^{ae}g^{ebc}\Gamma_2^{\mu\nu\rho}(P,Q,R), \quad (4.7)$$

$$\Gamma_2^{\mu\nu\rho}(P,Q,R) = \int \frac{d\Omega_v}{4\pi} \frac{-2v^\mu v^\nu v^\rho + (n^\mu v^\nu v^\rho + \text{perm})}{v \cdot P} \left[\frac{q^0}{v \cdot Q} - \frac{r^0}{v \cdot R} \right] \quad (4.8)$$

$$+ \int \frac{d\Omega_v}{4\pi} \frac{v^\mu v^\nu v^\rho}{v \cdot P} \left[\frac{p^0}{v \cdot P} \left(\frac{q^0}{v \cdot Q} - \frac{r^0}{v \cdot R} \right) + \left(\frac{(q^0)^2}{(v \cdot Q)^2} - \frac{(r^0)^2}{(v \cdot R)^2} \right) \right]. \quad (4.9)$$

Note that the Ward identity is automatically satisfied since

$$P_\mu \Gamma_1^{\mu\nu\rho}(P,Q,R) = \Pi_1^{\nu\rho}(Q) - \Pi_1^{\nu\rho}(R), \quad (4.10)$$

$$P_\mu \Gamma_2^{\mu\nu\rho}(P,Q,R) = \Pi_2^{\nu\rho}(Q) - \Pi_2^{\nu\rho}(R). \quad (4.11)$$

The same procedure can be applied to four-point interactions, which at one-loop are given in [45, 51].

5 Conclusions

In this paper we have provided hard thermal loops, for vector boson self-energies, at two-loops for any renormalizable model. This was made possible by using transport equations to simplify the calculations — thus extending known one-loop methods [12, 35–37]. In particular, this approach provides compact expression for each particle type; the result is independent of the matching scale; and known results for Debye masses are reproduced in the appropriate limit. We also demonstrated how higher-point functions can be extracted from the results.

The results of this paper can be used to study particle production in the early universe; transport coefficients; and wall speeds in first-order phase transitions [52]. The effect of including two-loop contributions is likely significant for the strong interaction, since the coupling constant is rather large $N\alpha_S \sim 0.3$ when $T \sim 100$ GeV.

The next step is to provide two-loop hard thermal loops for fermion propagators. Performing these calculations for quarks, by using Feynman diagrams, is likely arduous beyond leading order. However, we expect that similar methods as used in this paper will prove useful in this endeavour.

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⁷We have to remember that the original term depends on $\int dp p^2 n'(E)$, which when $E \approx p + \frac{m^2}{2p}$ becomes $\int dp \left[p^2 n'(p) - \frac{m^2}{2} n'(p) \right]$.

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A Derivatives of resummed distributions

The distributions satisfy

$$\delta N_X^\pm = \mp \frac{ep_\alpha F^{\alpha\beta}}{p \cdot \partial} \partial_p^\beta n_X(p^0). \quad (\text{A.1})$$

So taking the derivative $\frac{\partial}{\partial p^0}$ and going to momentum space we find

$$\int p dp \frac{\partial}{\partial p^0} \delta N_X^\pm \rightarrow \mp \int dp \left[\frac{k^0}{(v \cdot K)^2} - \frac{1}{v \cdot K} \right] \vec{v} \cdot \vec{E}(K) n'_X(p). \quad (\text{A.2})$$

A.1 Momentum integrals

We use dimensional regularization where $d = 3 - 2\epsilon$. When evaluating the self-energy we encounter the integrals

$$T^{2\epsilon} \int dpp^{d-1} n'_B(p) = -\frac{1}{3} \pi^2 T^2 + \frac{1}{3} \pi^2 T^2 (-24 \log(A) + 3 + \log(4) + 2 \log(\pi)) \epsilon, \quad (\text{A.3})$$

$$T^{2\epsilon} \int dpp^{d-1} n'_F(p) = -\frac{1}{6} \pi^2 T^2 + \frac{1}{6} \pi^2 T^2 (-24 \log(A) + 3 + \log(16) + 2 \log(\pi)) \epsilon, \quad (\text{A.4})$$

$$T^{2\epsilon} \int dpp^{d-3} n_B(p) = -\frac{T}{2\epsilon} + \mathcal{O}(\epsilon), \quad T^{2\epsilon} \int dpp^{d-3} n_F(p) = T \log(2) + \mathcal{O}(\epsilon), \quad (\text{A.5})$$

$$T^{2\epsilon} \int dpp^{d-3} n'_B(p) = \frac{1}{2} + \mathcal{O}(\epsilon), \quad T^{2\epsilon} \int dpp^{d-3} n'_F(p) = -\frac{1}{2} + \mathcal{O}(\epsilon) \quad (\text{A.6})$$

Here $A \approx 1.28243$ is the Glaisher constant.

B Non-abelian gauge theories

Note that all (collinear) divergences resulting from angular integrations cancel. We will however obtain divergences — real ones — from radial integrations: $\int_p \frac{n_B(p)}{p^2} \sim -\frac{T}{2\epsilon}$. The ϵ poles from these terms cancel once zero-temperature counterterms are used.

Throughout this and the following sections we keep factors of $D = 4 - 2\epsilon$ explicit. There are four contributions. Corrections to the vector current are shown in figures 1c, 1e, and 1j; sunset corrections to the ghost current are shown in figure 1d. We also note that diagram 2b vanishes.

B.1 Vector current

We start with the vector current. The sunset diagram gives

$$\begin{aligned} \Pi_{1c}^\mu = & -\frac{1}{4} g^{abc} g^{hgn} g^{def} \int_{PQ} F^\mu \left\{ \delta^{en} \delta^{cd} \Delta_V^{bh}(P) \Delta_V^{gf}(Q) \left[\Delta^R(P) \Delta^R(P+Q) + \Delta^A(P) \Delta^A(P+Q) \right] \right. \\ & \delta^{cd} \delta^{gf} \Delta_V^{bh}(P) \Delta_V^{en}(P+Q) \left[\Delta^R(P) \Delta^A(Q) + \Delta^A(P) \Delta^R(Q) \right] \\ & \left. \delta^{cd} \delta^{bh} \Delta_V^{gf}(Q) \Delta_V^{en}(P+Q) \Delta^A(P) \Delta^R(P) \right\} \quad (\text{B.1}) \end{aligned}$$

where

$$F^\mu(P, Q) = (P+Q)^2 [(5-4D)p^\mu + 2(2D-3)q^\mu] + P^2 [(5-6D)p^\mu] + Q^2 [(11-8D)p^\mu + 2(3-2D)q^\mu]. \quad (\text{B.2})$$

We can rewrite the terms so that they all multiply $\Lambda^{ab} = g_V^{anc} g_V^{cef} g_V^{nfn} g_V^{enb}$. Explicitly,

$$\begin{aligned} \Pi_{1c}^\mu &= \Lambda^{ab} \frac{(4D-5)}{4} \int_{PQ} n_B(q) \frac{1}{pq} \left\{ \partial_0^p [N(p)_V - \bar{N}_V(p)] v_p^\mu - \frac{v_p^\mu - n^\mu}{p} (N_V(p) - \bar{N}_V(p)) \right\}^b \\ &\quad + \Lambda^{ab} \frac{(2D-3)}{4} \int_{PQ} v_p^\mu \frac{1}{q^2} (n_B(q) - qn'_B(q)) (N_V(p) - \bar{N}_V(p))^b \end{aligned} \quad (\text{B.3})$$

We now turn to the bubble diagram. We find

$$\Pi_{1j}^\mu = \frac{1}{4} g_V^{abl} g_V^{ecg} g_V^{fdg} \delta^{dl} \int_{PQ} F^\mu \Delta_V^{bc}(P) (\Delta^R(P) + \Delta^A(p)) \Delta_V^{fg}(Q), \quad F^\mu = -4(D-1)^2 p^\mu.$$

The result is

$$\Pi_{1j}^\mu = -\frac{(D-1)^2}{2} \Lambda^{ab} \int_{PQ} n_B(q) \frac{1}{pq} \left\{ \partial_0^p [N_V(p) - \bar{N}_V(p)] v_p^\mu - \frac{v_p^\mu - n^\mu}{p} (N_V(p) - \bar{N}_V(p)) \right\}^b.$$

B.2 Ghost diagrams

We now consider diagrams with internal ghosts. There are two contributions: the ghost-current with an internal vector and the vector current with a ghost loop — shown in figures 1d and 1e respectively. The latter diagram vanish, so we only need the former one:

$$\begin{aligned} \Pi_{1e}^\mu &= \frac{1}{2} g_V^{abc} g_V^{hgn} g_V^{def} \int_{PQ} F^\mu \left\{ \delta^{en} \delta^{cd} \Delta_V^{bh}(P) \Delta_V^{gf}(Q) [\Delta^R(P) \Delta^R(P+Q) + \Delta^A(P) \Delta^A(P+Q)] \right. \\ &\quad \delta^{cd} \delta^{gf} \Delta_V^{bh}(P) \Delta_V^{en}(P+Q) [\Delta^R(P) \Delta^A(Q) + \Delta^A(P) \Delta^R(Q)] \\ &\quad \left. \delta^{cd} \delta^{bh} \Delta_V^{gf}(Q) \Delta_V^{en}(P+Q) \Delta^A(P) \Delta^R(P) \right\}, \end{aligned} \quad (\text{B.4})$$

$$F^\mu = (P+Q)^2 (q^\mu - p^\mu / 2) + P^2 p^\mu / 2 - Q^2 \left(q^\mu + \frac{3}{2} p^\mu \right).$$

We find

$$\begin{aligned} \Pi_{1e}^\mu &= -\Lambda^{ab} \frac{1}{4} \int_{PQ} \frac{n_B(q)}{pq} \left\{ \partial_0^p [N_V(p) - \bar{N}_V(p)] v_p^\mu - \frac{v_p^\mu - n^\mu}{p} (N_V(p) - \bar{N}_V(p)) \right\}^b \\ &\quad - \frac{1}{4} \Lambda^{ab} \int_{PQ} v_p^\mu \frac{n_B(q) - qn'_B(q)}{q^2} (N_V(p) - \bar{N}_V(p))^b. \end{aligned} \quad (\text{B.5})$$

B.3 Total contribution from non-abelian diagrams

We find

$$\Pi_{1c}^{\mu\nu} + \Pi_{1j}^{\mu\nu} + \Pi_{1e}^{\mu\nu} = T^2 \frac{(D-2)^2}{48\pi^2} \Lambda^{ab} \Pi_2^{\mu\nu} - 2(D-2) \Lambda^{ab} I_{VV} \Pi_1^{\mu\nu}, \quad (\text{B.6})$$

where

$$I_{VV} = \int_{PQ} n'_B(q) \frac{1}{p^2} \left(n'_B(p) - \frac{n_B(p)}{p} \right) = T^2 \left\{ \frac{1}{48\pi^2 \epsilon} + \frac{(24 \log(A) + 4 \log \frac{\mu}{4\pi T} + 2\gamma - 1)}{48\pi^2} \right\},$$

and $\Lambda^{ab} = g_V^{anc} g_V^{cef} g_V^{nfn} g_V^{enb}$.

Besides the above terms, the term going with I_{VV} produces an additional term. In particular, the terms proportional to I_{VV} involves momenta flowing through the diagram before it is absorbed by the resummed distribution. So formally we have to use retarded and advanced propagators that depend on the background field. However, in practice it is much easier to just shift $\Delta^{R/A}(P) \rightarrow \Delta^{R/A}(P+K)$. Essentially the only change is that $\log \frac{\mu}{4\pi T} \rightarrow \log \frac{\mu}{4\pi T} + k^0 L[K]$ for this contribution. The same replacement holds for all other divergent terms of this type.

C Fermion diagrams

C.1 Fermion current

We now turn diagrams with fermions. We will omit collinear divergences as they cancel once we sum fermion and vector currents. The fermion current gives

$$\begin{aligned} \Pi_{1a}^\mu = & g_I^{aN} g_K^{cJ} g_M^{dL} \int_{PQ} F^\mu(P, Q) \left\{ \delta_N^M \delta_L^K \Delta_{F,J}^I(P) \Delta_V^{cd}(Q) \left[\Delta^R(P) \Delta^R(P+Q) + \Delta^A(P) \Delta^A(P+Q) \right] \right. \\ & \delta^{cd} \delta_N^M \Delta_{F,J}^I(P) \Delta_{F,L}^K(P+Q) \left[\Delta^R(P) \Delta^A(Q) + \Delta^A(P) \Delta^R(Q) \right] \\ & \left. \delta_N^M \delta_J^I \Delta_{F,L}^K(P+Q) \Delta_V^{cd}(Q) \left[\Delta^R(P) \Delta^A(P) \right] \right\}, \\ F^\mu = & 2(D-2) \left[(P+Q)^2 p^\mu - P^2 q^\mu - Q^2 p^\mu \right] \end{aligned} \quad (C.1)$$

We find

$$\begin{aligned} \Pi_{1a}^\mu = & -\frac{D-2}{2} \text{Tr} g_F^c g_F^c g_F^a g_F^b \\ & \int_{PQ} \frac{N_B(q) - N_F(q)}{pq} \left\{ \partial_0^p \left[N_F^+(p) - N_F^-(p) \right] v_p^\mu - \frac{v_p^\mu - n^\mu}{p} (N_F^+(p) - N_F^-(p)) \right\}^b. \end{aligned}$$

Here we should use the leading-order relation: $N_B(q) - N_F(q) = n_B(q) + n_F(q)$. After inserting the resummed propagators and performing the integrals we find

$$\Pi_{1a}^{\mu\nu} = -\frac{(D-2)T^2}{32\pi^2} \text{Tr} g_F^c g_F^c g_F^a g_F^b \Pi_2^{\mu\nu}. \quad (C.2)$$

C.2 Vector current

Consider now fermion corrections to the vector current:

$$\begin{aligned} \Pi_{1b}^\mu = & -\frac{1}{2} g_V^{ace} g_J^{d,I} g_L^{f,K} \int_{PQ} F^\mu \delta_K^J \delta^{ef} \Delta_V^{cd}(P) \Delta_{F,I}^L(Q) \left[\Delta^R(P) \Delta^R(P+Q) + \Delta^A(P) \Delta^A(P+Q) \right] \\ & + \delta_L^I \delta^{ef} \Delta_V^{cd}(P) \Delta_{F,K}^J(P+Q) \left[\Delta^R(P) \Delta^A(Q) + \Delta^A(P) \Delta^R(Q) \right] \\ & + \delta^{ef} \delta^{cd} \Delta_{F,K}^J(P+Q) \Delta_{F,I}^L(Q) \left[\Delta^R(P) \Delta^A(P) \right], \end{aligned} \quad (C.3)$$

where

$$F^\mu = -2i \left\{ (P+Q)^2 [(D-2)p^\mu + 2q^\mu] - (D-2)P^2 p^\mu + Q^2 [(D-4)p^\mu - 2q^\mu] \right\}. \quad (\text{C.4})$$

We are left with

$$\begin{aligned} \Pi_{1b}^\mu &= -\frac{(D-2)}{2} g_V^{ace} g_V^{bcd} \text{Tr} \left[g_F^d g_F^e \right] \int_{PQ} \frac{n_F(q)}{qp} \\ &\quad \left\{ \partial_0^p \left[N_V(p) - \bar{N}_V(p) \right] v_p^\mu - \frac{(v_p^\mu - n^\mu)}{p} \left(N_V(p) - \bar{N}_V(p) \right) \right\}^b \\ &\quad + i g_V^{ace} \text{Tr} \left[(g_F^c g_F^e - g_F^e g_F^c) g_F^b \right] \int_{PQ} (N_F(q) - \bar{N}_F(q))^b \frac{v_q^\mu}{p^2} \left\{ n'_B(p) - \frac{n_B(p)}{p} \right\} \end{aligned} \quad (\text{C.5})$$

This result can be further simplified because $-i g_V^{ace} \text{Tr} \left[(g_F^c g_F^b - g_F^b g_F^c) g_F^e \right] = g_V^{ace} g_V^{cbd} \text{Tr} \left[g_F^d g_F^e \right]$, so the entire diagram is proportional to the structure $g_V^{ace} g_V^{bcd} \text{Tr} \left[g_F^d g_F^e \right]$. In any case, after inserting the resummed propagators and performing the integrals we find

$$\Pi_{1b}^{\mu\nu} = (D-2) \frac{T^2}{96\pi^2} g_V^{ace} g_V^{bcd} \text{Tr} \left[g_F^d g_F^e \right] \Pi_2^{\mu\nu} + T^2 g_V^{ace} g_V^{bcd} \text{Tr} \left[g_F^d g_F^e \right] I_{\text{FV}} \Pi_1^{\mu\nu}, \quad (\text{C.6})$$

where

$$I_{\text{FV}} = \int_{PQ} n'_F(q) \frac{1}{p^2} \left(n'_B(p) - \frac{n_B(p)}{p} \right) = T^2 \left\{ \frac{1}{96\pi^2 \epsilon} - \frac{-24 \log A - 4 \log \frac{\mu}{4\pi T} - 2\gamma + 1 + \log(4)}{96\pi^2} \right\}, \quad (\text{C.7})$$

contains divergences that cancel against counter-term insertions.

C.3 Yukawa diagrams

There are two diagrams with Yukawa couplings, one from the fermion current and one from the scalar current. The sum of the two gives

$$\Pi_{1h}^{\mu\nu} + \Pi_{1i}^{\mu\nu} = -\frac{T^2}{32\pi^2} \left[g_F^a g_F^b \right]_J^I (Y Y^c)_I^J \Pi_2^{\mu\nu} - \frac{T^2}{192\pi^2} \left[g_S^a g_S^b \right]_{ij} (Y Y^c + Y^c Y)^{ij} \Pi_2^{\mu\nu} \quad (\text{C.8})$$

D Scalar Diagrams

D.1 Scalar current

The vector sunset gives

$$\begin{aligned} \Pi_{1f}^\mu &= \frac{1}{2} g_{ni}^a g_{jk}^c g_{lm}^d \int_{PQ} F^\mu \left\{ \delta^{kl} \delta^{mn} \Delta_S^{ij}(P) \Delta_V^{cd}(Q) \left[\Delta^R(P) \Delta^R(P+Q) + \Delta^A(P) \Delta^A(P+Q) \right] \right. \\ &\quad + \delta^{mn} \delta^{cd} \Delta_S^{ij}(P) \Delta_S^{kl}(P+Q) \left[\Delta^R(P) \Delta^A(Q) + \Delta^A(P) \Delta^R(Q) \right] \\ &\quad \left. + \delta^{ij} \delta^{nm} \Delta_V^{cd}(Q) \Delta_S^{kl}(P+Q) \left[\Delta^R(P) \Delta^A(P) \right] \right\}, \end{aligned} \quad (\text{D.1})$$

$$F^\mu = 4ip^\mu \left\{ (P+Q)^2 + P^2 - \frac{1}{2} Q^2 \right\}.$$

Since the scalar-vector bubble give the same combination of couplings we can group the diagrams together. We find

$$\begin{aligned} \Pi_{1f}^\mu + \Pi_{1l}^\mu &= -\frac{D+2}{8} \text{Tr} \left[g_S^a g_S^b g_S^c g_S^c \right] \\ &\int_{PQ} \frac{n_B(q)}{qp} \left\{ \partial_0^p \left[N_S(p) - \bar{N}_S(p) \right] v_p^\mu - \frac{(v_p^\mu - n^\mu)}{p} \left(N_S(p) - \bar{N}_S(p) \right) \right\}^b. \end{aligned}$$

After performing the integrals we obtain

$$\Pi_{1f}^{\mu\nu} + \Pi_{1l}^{\mu\nu} = \frac{T^2(D+2)}{192\pi^2} \text{Tr} \left[g_S^a g_S^b g_S^c g_S^c \right] \Pi_2^{\mu\nu}. \quad (\text{D.2})$$

The scalar-bubble gives

$$\begin{aligned} \Pi_{1m}^\mu &= \frac{1}{4} g_{li}^a \lambda^{jlmn} \int_{PQ} F^\mu \delta^{kl} \Delta_S^{ij}(P) \Delta_S^{mn}(Q) \left[\Delta^R(P) + \Delta^A(P) \right] \\ F^\mu &= 2p^\mu, \end{aligned} \quad (\text{D.3})$$

which simplify to

$$\Pi_{1m}^\mu = \frac{1}{8} \left[g_S^a g_S^b \right]_{jl} \lambda^{jlmn} \int_{PQ} \frac{n_B(q)}{qp} \left\{ \partial_0^p \left[N_S(p) - \bar{N}_S(p) \right] v_p^\mu - \frac{(v_p^\mu - n^\mu)}{p} \left(N_S(p) - \bar{N}_S(p) \right) \right\}^b.$$

After performing the integrals we find

$$\Pi_{1m}^{\mu\nu} = -\frac{T^2}{192\pi^2} \text{Tr} \left[g_S^a g_S^b \right]_{jl} \lambda^{jlmn} \Pi_2^{\mu\nu}. \quad (\text{D.4})$$

We can also have scalar-mass insertions from one loop diagrams:

$$\Pi_{2d}^\mu = i g_{ki}^a \mu^{jk} \int_P \Delta_S^{ij}(P) \left(\Delta^R(P) + \Delta^A(P) \right), \quad (\text{D.5})$$

which gives

$$\Pi_{2d}^\mu = \left[g_S^a g_S^b \right]_{ij} \mu^{jj} \int_P \frac{1}{p} \left\{ \partial_0^p \left[N_S(p) - \bar{N}_S(p) \right] v_p^\mu - \frac{(v_p^\mu - n^\mu)}{p} \left(N_S(p) - \bar{N}_S(p) \right) \right\}^b.$$

After performing the integral we find

$$\Pi_{2d}^{\mu\nu} = -\frac{1}{8\pi^2} \text{Tr} \left[g_S^a g_S^b \right]_{ij} \mu^{jj} \Pi_2^{\mu\nu}. \quad (\text{D.6})$$

D.2 Vector current

The scalar sunset gives

$$\begin{aligned} \Pi_{1g}^\mu &= \frac{1}{2} g_V^{ace} g_{jn}^d g_{mi}^f \int_{PQ} F^\mu \left\{ \delta^{nm} \delta^{ef} \Delta_V^{cd}(P) \Delta_S^{ij}(Q) \left[\Delta^R(P) \Delta^R(P+Q) + \Delta^A(P) \Delta^A(P+Q) \right] \right. \\ &\quad + \delta^{ef} \delta^{ij} \Delta_V^{cd}(P) \Delta_S^{nm}(P+Q) \left[\Delta^R(P) \Delta^A(Q) + \Delta^A(P) \Delta^R(Q) \right] \\ &\quad \left. + \delta^{ef} \delta^{cd} \Delta_S^{ij}(Q) \Delta_S^{nm}(P+Q) \left[\Delta^R(P) \Delta^A(P) \right] \right\}, \end{aligned} \quad (\text{D.7})$$

$$F^\mu = i \left[(4q^\mu - 2p^\mu)(P+Q)^2 + 2P^2 p^\mu - Q^2(6p^\mu + 2q^\mu) \right],$$

or after simplifying

$$\begin{aligned} \Pi_{1g}^{\mu\nu} = & -g_V^{aec} g_V^{bdc} \text{Tr} \left[g_S^d g_S^f \right] \int_{PQ} \frac{n_B(q)}{qp} \left\{ \partial_0^p \left[N_S(p) - \bar{N}_S(p) \right] v_p^\mu - \frac{(v_p^\mu - n^\mu)}{p} \left(N_S(p) - \bar{N}_S(p) \right) \right\}^b \\ & + \frac{1}{4} g_V^{aec} g_V^{bdc} \text{Tr} \left[g_S^d g_S^f \right] \int_{PQ} (N_V(q) - \bar{N}_V(q))^b \frac{v_q^\mu}{p^2} \left\{ d_0(N_S + \bar{N}_S) - \frac{1}{p} (N_S + \bar{N}_S) \right\} \end{aligned} \quad (\text{D.8})$$

After performing the integrals we find

$$\Pi_{1g}^{\mu\nu} = \frac{T^2}{24\pi^2} g_V^{aec} g_V^{bdc} \text{Tr} \left[g_S^d g_S^e \right] \Pi_2^{\mu\nu} - T^2 I_{SV} g_V^{aec} g_V^{bdc} \text{Tr} \left[g_S^d g_S^e \right] \Pi_1^{\mu\nu}, \quad (\text{D.9})$$

where

$$I_{SV} = \int_{PQ} n'_B(q) \frac{1}{p^2} \left(n'_B(p) - \frac{n_B(p)}{p} \right) = \left\{ \frac{1}{48\pi^2 \epsilon} + \frac{T^2 (24 \log(A) + 4 \log \frac{\mu}{4\pi T} + 2\gamma - 1)}{48\pi^2} \right\}, \quad (\text{D.10})$$

Finally, the scalar bubble gives

$$\Pi_{1k}^\mu = \frac{1}{2} g_V^{ace} H_{V,ij}^{df} \int_{PQ} F^\mu \delta^{ef} \Delta_V^{cd}(P) \Delta_S^{ij}(Q) \left[\Delta^R(P) + \Delta^A(P) \right], \quad F^\mu = -2(D-1)p^\mu,$$

which after simplifying gives

$$\begin{aligned} \Pi_{1k}^\mu = & \frac{(D-1)}{2} g_V^{aec} g_V^{bdc} \text{Tr} \left[g_S^d g_S^e \right] \\ & \int_{PQ} \frac{n_B(q)}{qp} \left\{ \partial_0^p \left[N_V(p) - \bar{N}_V(p) \right] v_p^\mu - \frac{(v_p^\mu - n^\mu)}{p} \left(N_V(p) - \bar{N}_V(p) \right) \right\}^b. \end{aligned}$$

After doing the integrals we find

$$\Pi_{1g}^{\mu\nu} + \Pi_{1k}^{\mu\nu} = -T^2 \frac{(D-3)}{48\pi^2} g_V^{aec} g_V^{bdc} \text{Tr} \left[g_S^d g_S^e \right] \Pi_2^{\mu\nu} - T^2 I_{SV} g_V^{aec} g_V^{bdc} \text{Tr} \left[g_S^d g_S^e \right] \Pi_1^{\mu\nu}. \quad (\text{D.11})$$

E Counter-term contributions

To renormalize we need wave-function and coupling counter-terms. These are all well known [38–40]. The anomalous dimensions are⁸

$$\gamma_J^I = \frac{1}{16\pi^2} \left\{ -[g_F^c g_F^c]_J^I \right\}, \quad (\text{E.1})$$

$$\gamma_V^{ab} = \frac{1}{16\pi^2} \left\{ -\frac{5}{3} \text{Tr} [g_V^a g_V^b] - \frac{2}{3} \text{Tr} [g_F^a g_F^b] + \frac{1}{6} \text{Tr} [g_S^a g_S^b] \right\}, \quad (\text{E.2})$$

$$\gamma_g^{ab} = -\frac{1}{16\pi^2} \left\{ \frac{1}{2} \text{Tr} [g_V^a g_V^b] \right\}, \quad \gamma_S^{ij} = \frac{1}{16\pi^2} \left\{ -2 [g^c g^c]^{ij} \right\}. \quad (\text{E.3})$$

⁸We here omit all Yukawa couplings since their counter-term contributions cancel.

Next the vector and fermion-vector trilinear couplings:

$$\delta g_J^{a,I} = \frac{1}{32\pi^2\epsilon} \left\{ -2 [g_F^c g_F^a g_F^c]_J^I + 6i g_V^{abc} [g_F^b g_F^c]_J^I - \gamma_V^{ab} g_{F,J}^{b,I} - g_{F,K}^{a,I} \gamma_J^K - g_{F,J}^{a,K} \gamma_K^{*,I} \right\}. \quad (\text{E.4})$$

$$\delta g^{abc} = \frac{1}{32\pi^2\epsilon} \left\{ -2 \text{Tr} [g_V^a g_V^b g_V^c] - g_V^{abe} \gamma_V^{ec} - \gamma_g^{ae} g_V^{ebc} - \gamma_g^{be} g_V^{aec} \right\}. \quad (\text{E.5})$$

For the scalar-coupling we only need the combination $H_{ij}^{ab} = g_{ik}^a g_{kj}^b + g_{ik}^b g_{kj}^a$. The counter-term is

$$\delta H_{ij}^{ab} = -\frac{1}{16\pi^2\epsilon} \left\{ \frac{8}{3} g_V^{ace} g_V^{bef} H_{ij}^{cf} + 2 [g_S^c (g_S^a g_S^b + g_S^b g_S^a) g_S^c]_{ij} \right\} \quad (\text{E.6})$$

$$-\frac{1}{2} \left[\gamma_S^{\text{in}} H_{nj}^{ab} + \gamma_S^{jn} H_{\text{in}}^{ab} + \gamma_V^{ac} H_{ij}^{cb} + \gamma_V^{bc} H_{ij}^{ac} \right] \quad (\text{E.7})$$

E.1 Vector loops

Using the counter-terms from section E we find

$$\Pi_{\text{CT,V}}^{\mu\nu} = -\frac{(D-2)}{4\pi^2\epsilon} g^{acd} g^{def} g^{cfn} g^{bnx} \int_P n'_B(p) \Pi_1^{\mu\nu}, \quad (\text{E.8})$$

where

$$\int_P n'_B(p) = -\frac{T^2}{6} - \frac{T^2}{6} \left(24 \log(A) + 2 \log \frac{\mu}{4\pi T} - 1 \right) \epsilon + \mathcal{O}(\epsilon^2). \quad (\text{E.9})$$

E.2 Fermion loops

Using the counter-terms from section E we find

$$\Pi_{\text{CT,F}}^{\mu\nu} = -\frac{1}{4\pi^2\epsilon} g_V^{ace} g_V^{bcd} \text{Tr} [g_F^d g_F^e] \int_P n'_F(p) \Pi_1^{\mu\nu}, \quad (\text{E.10})$$

where

$$\int_P n'_F(p) = -\frac{T^2}{12} - \frac{T^2}{12} \left(24 \log(A) + 2 \log \frac{\mu}{4\pi T} - 1 - \log 4 \right) \epsilon + \mathcal{O}(\epsilon^2). \quad (\text{E.11})$$

E.3 Scalar loops

Using the counter-terms from appendix E we find

$$\Pi_{\text{CT,S}}^{\mu\nu} = \frac{1}{8\pi^2\epsilon} g_V^{aec} g_V^{bdc} \text{Tr} [g_S^d g_S^e] \int_P n'_B(p) \Pi_1^{\mu\nu}. \quad (\text{E.12})$$

E.4 Power corrections before field redefinitions

The scalar loop gives

$$g_{\mu\nu} \Pi_S^{\mu\nu,ab}(K) = \text{Tr} [g_S^a g_S^b] \frac{K^2}{16\pi^2} \left\{ \frac{1}{2\epsilon} + \log \frac{\mu e^\gamma}{4\pi T} + k^0 L(K) \right\}, \quad (\text{E.13})$$

$$\Pi_S^{00,ab}(K) = -\text{Tr} [g_S^a g_S^b] \frac{1}{3} \frac{k^2}{16\pi^2} \left\{ \frac{1}{2\epsilon} + \log \frac{\mu e^\gamma}{4\pi T} + 1 + \frac{(k^0)^2}{k^2} (k^0 L(K) - 1) \right\} \quad (\text{E.14})$$

The fermion loop gives

$$g_{\mu\nu}\Pi_F^{\mu\nu,ab}(K) = \text{Tr} \left[g_F^a g_F^b \right] \frac{K^2}{16\pi^2} \left\{ \frac{2}{\epsilon} + 4 \left(\log \frac{\mu e^\gamma}{4\pi T} + \log 4 \right) - 2 + 4k^0 L(K) \right\}, \quad (\text{E.15})$$

$$\begin{aligned} \Pi_F^{00,ab}(K) = -\text{Tr} \left[g_F^a g_F^b \right] \frac{1}{3} \frac{k^2}{16\pi^2} & \left\{ \frac{2}{\epsilon} + 4 \left(\log \frac{\mu e^\gamma}{4\pi T} + \log 4 \right) - 2 \right. \\ & \left. + 2k^0 \left(3 - \frac{(k^0)^2}{k^2} \right) L(K) + 2 \frac{(k^0)^2}{k^2} \right\}. \end{aligned}$$

For non-abelian diagrams we group ghosts and vectors together, the result is

$$g_{\mu\nu}\Pi_V^{\mu\nu,ab}(K) = \text{Tr} \left[g_V^a g_V^b \right] \frac{K^2}{16\pi^2} \left\{ \frac{5}{\epsilon} + 10 \log \frac{\mu e^\gamma}{4\pi T} - 3 + 10k^0 L(K) \right\}, \quad (\text{E.16})$$

$$\Pi_V^{00,ab}(K) = -\text{Tr} \left[g_V^a g_V^b \right] \frac{1}{3} \frac{k^2}{16\pi^2} \left\{ \frac{5}{\epsilon} + 10 \log \frac{\mu e^\gamma}{4\pi T} - 1 + 2k^0 \left(6 - \frac{(k^0)^2}{k^2} \right) L(K) + 2 \frac{(k^0)^2}{k^2} \right\}.$$

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