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# Color-factor symmetry of the amplitudes of Yang-Mills and biadjoint scalar theory using perturbiner methods

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Abstract: Color-factor symmetry is a property of tree-level gauge-theory amplitudes containing at least one gluon. BCJ relations among color-ordered amplitudes follow directly from this symmetry. Color-factor symmetry is also a feature of biadjoint scalar theory amplitudes as well as of their equations of motion. In this paper, we present a new proof of color-factor symmetry using a recursive method derived from the perturbiner expansion of the classical equations of motion.

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## Contents



# 1 Introduction

The discovery of *color-kinematic duality* in the amplitudes of Yang-Mills theory, and subsequently in the amplitudes of a much broader class of field theories (see ref. [1] for a review), has unleashed a tool of great power, particularly in the calculation of gravitational amplitudes through the double-copy procedure [2–4]. In 2008, Bern, Carrasco, and Johansson (BCJ) showed that the assumption of color-kinematic duality in tree-level amplitudes of Yang-Mills theory implies a set of linear relations among the color-ordered amplitudes. The subsequent proof of these BCJ relations using string-theory techniques [5, 6] and BCFW on-shell recursion [7, 8] provided evidence for the conjecture of tree-level color-kinematic duality. Bern et al. also conjectured that color-kinematic duality applies to integrands of loop-level amplitudes [2, 3]; while not proven, this conjecture has been tested for amplitudes of various multiplicities and loop levels in supersymmetric Yang-Mills theories, which have been used to construct supergravity amplitudes [1].

In 2016, R. W. Brown and the current author observed that tree-level gauge-theory amplitudes possess a *color-factor symmetry*, which acts as a momentum-dependent shift on the color factors of an amplitude, leaving the full amplitude invariant [9–11]. This symmetry was proved for both Yang-Mills theory and for gauge theories with massive particles of various spins using the radiation vertex expansion [12]. The BCJ relations follow as an immediate consequence of color-factor symmetry [9, 10].

Color-kinematic duality and color-factor symmetry are closely related features of gauge theories: the former implies the latter (as proved using the cubic vertex expansion), but the latter implies a less stringent (but gauge-invariant) constraint than the former on the kinematic numerators of a tree-level amplitude [9]. Similarly color-kinematic duality implies color-factor symmetry for loop-level amplitudes, but no independent proof of the latter has yet been developed.

Color-factor symmetry is also a property of tree-level amplitudes of the biadjoint scalar (BAS) theory [13], whose fields transform in the adjoint representation of $\mathrm{U}(N) \times \mathrm{U}(\tilde{N})$, as was proved using the cubic vertex expansion [9]. Cheung and Mangan [14] observed that the classical equations of motion of the BAS theory also possess color-factor symmetry, and that this implies the invariance of the tree-level amplitudes. They demonstrated a relation between the $\mathrm{U}(N)$ color-factor symmetry of the equations of motion and the conservation of current associated with the dual $\mathrm{U}(\tilde{N})$ symmetry. In ref. [15], these results were generalized to curved symmetric spacetime.

In this paper, we offer a new proof of color-factor symmetry based on a recursive approach. In 1987, Berends and Giele [16] introduced a method for computing tree-level QCD amplitudes using a set of partially off-shell amplitudes (subsequently known as Berends-Giele currents), which were then computed recursively. Rosly and Selivanov [17–19] later showed that a perturbative solution of the classical equations of motion (dubbed the *perturbiner expansion*) acts as a generating function for Berends-Giele currents. Mafra, Schlotterer, et al. [20–23] also used classical equations of motion to generate Berends-Giele currents in various theories. Mizera and Skrzypek [24] introduced the *color-dressed perturbiner expansion*, which, as we will see in this paper, is well adapted for the demonstration of color-factor symmetry of tree-level amplitudes.

Further developments in this subject include the work of Lopez Arcos, Quintero Vélez, et al., who related the $L_\infty$-algebra that appears in Batalin-Vilkovisky quantization [25] to the perturbiner expansion for biadjoint scalar and Yang-Mills theories [26] as well as in gauge theories with matter [27]. Berends-Giele currents in BCJ gauge were constructed using Bern-Kosower rules [28], with this work extended to gravity using the double-copy procedure [29]. Gomez and Jusinskas have applied perturbiner methods to gravity coupled to matter [30], and in ref. [31], perturbiner methods were used to compute tree-level boundary correlators in anti-de Sitter space. The perturbiner approach has also been found effective for computing one-loop integrands [32]. The connection between tree-level Berends-Giele recursion relations and the $L_\infty$-algebra uncovered in ref. [25] was extended to loop-level recursion relations and the quantum homotopy algebra $A_\infty$ in ref. [33]. For connections between the homotopy algebra and the double copy, see refs. [34–36].

We present this alternative proof of color-factor symmetry because the recursive methods employed may be more familiar to modern readers than the radiation vertex expansion used in ref. [9] to prove color-factor symmetry. Moreover, this recursive approach may be easier to generalize to the exploration of color-factor symmetry in other theories.

The outline of this paper is as follows. In section 2, we recall how color-factor symmetry acts on amplitudes, and how the BCJ relations follow as a consequence. In section 3, we show how the color-dressed perturbiner expansion is used to compute tree-level amplitudes, first in the biadjoint scalar theory, and then in Yang-Mills theory. In section 4, we then use the color-dressed perturbiner expansion to prove color-factor symmetry for tree-level amplitudes of the biadjoint scalar theory and of Yang-Mills theory. Section 5 contains our conclusions.



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## 2 Color-factor symmetry and BCJ relations

Tree-level scattering amplitudes of a gauge theory are given by a sum of Feynman diagrams, and can be expressed as [37]

$$\mathcal{A}_n = \sum_i a_i c_i \tag{2.1}$$

where $c_i$ are *color factors*, consisting of the contraction of various color tensors $f^{\mathsf{abc}}$ and $(T^{\mathsf{a}})^{\mathsf{i}}{}_{\mathsf{j}}$ appearing in the Feynman diagrams, and $a_i$ depends on kinematic and spin factors. Each color factor can itself be represented as a Feynman diagram [38, 39], one that contains only trivalent vertices. (If the full Feynman diagram contains only trivalent vertices, then it contributes to the color factor with the same Feynman diagram. If the full Feynman diagram also contains quartic vertices, then its contribution is parcelled out among different color factors by expressing the quartic vertex as products of trivalent vertices.) Note that, due to the group theory identities

$$0 = f^{\mathsf{bae}} f^{\mathsf{ecd}} + f^{\mathsf{cae}} f^{\mathsf{edb}} + f^{\mathsf{dae}} f^{\mathsf{ebc}}\,, \tag{2.2}$$

$$0 = (T^{\mathsf{a}})^{\mathsf{i}}{}_{\mathsf{k}} (T^{\mathsf{c}})^{\mathsf{k}}{}_{\mathsf{j}} - (T^{\mathsf{c}})^{\mathsf{i}}{}_{\mathsf{k}} (T^{\mathsf{a}})^{\mathsf{k}}{}_{\mathsf{j}} - f^{\mathsf{ace}} (T^{\mathsf{e}})^{\mathsf{i}}{}_{\mathsf{j}} \tag{2.3}$$

there exist (Jacobi) relations among the various color factors $c_i$. Since the $c_i$ are not independent, there is some choice about how the coefficients $a_i$ are defined.

### 2.1 Color-factor symmetry

There exists a color-factor symmetry associated with each external gluon $a$ contributing to the amplitude [9, 10]. This symmetry acts on each color factor $c_i$ appearing in eq. (2.1) by a momentum-dependent shift $\delta_a c_i$. For each color factor $c_i$, the gluon leg $a$ divides the associated tree-level diagram in two at its point of attachment. Let $S_{a,i}$ denote the subset of the remaining legs on one side of this point; it does not matter which side we choose. The shift of the color factor $c_i$ associated with gluon $a$ then satisfies[1]

$$\delta_a c_i \;\propto \sum_{d \in S_{a,i}} k_a \cdot k_d \tag{2.4}$$

where $k_a^\mu$ is the outgoing momentum of gluon $a$ (satisfying $k_a^2 = 0$), and $k_d^\mu$ are the outgoing momenta of the legs belonging to $S_{a,i}$. The color-factor shift also respects the group theory identities, as we will see below.

We may regard the color-factor symmetry as acting directly on the color tensors appearing in $c_i$. If gluon $a$ (with color $\mathsf{a}$) is attached to a gluon line, so that the color factor contains $f^{\mathsf{bac}}$, then the color-factor symmetry acts as [14]

$$\delta_a f^{\mathsf{bac}} = \alpha_a \delta^{\mathsf{bc}} (k_c^2 - k_b^2) \tag{2.5}$$

[1]Choosing to sum over the complement of $S_{a,i}$ gives the same result (up to sign) due to momentum conservation.

where $k_b^\mu$ and $k_c^\mu$ are the momenta flowing out of the vertex associated with $f^{\mathsf{bac}}$ and $\alpha_a$ is a constant parameter. If gluon $a$ is attached to a line corresponding to a particle in some other representation, so that the color factor contains $(T^{\mathsf{a}})^{\mathsf{i}}{}_{\mathsf{j}}$, then the color-factor symmetry acts as

$$\delta_a\,(T^{\mathsf{a}})^{\mathsf{i}}{}_{\mathsf{j}} = \alpha_a \delta^{\mathsf{i}}{}_{\mathsf{j}}(k_j^2 - k_i^2) \tag{2.6}$$

where $k_i^\mu$ and $k_j^\mu$ are the momenta flowing out of the vertex associated with $(T^{\mathsf{a}})^{\mathsf{i}}{}_{\mathsf{j}}$. Using momentum conservation at each vertex, we may express these shifts as

$$\delta_a f^{\mathsf{bac}} = \alpha_a \delta^{\mathsf{bc}}(2k_a \cdot k_b)\,, \qquad\qquad \delta_a\,(T^{\mathsf{a}})^{\mathsf{i}}{}_{\mathsf{j}} = \alpha_a \delta^{\mathsf{i}}{}_{\mathsf{j}}(2k_a \cdot k_i)\,. \tag{2.7}$$

The relations (2.7) guarantee that the color-factor shifts satisfy eq. (2.4). We must also check that the color-factor shifts leave the group theory identities invariant. Using eq. (2.7) in eqs. (2.2) and (2.3), we find

$$\begin{aligned} \delta_a\left[f^{\mathsf{bae}}f^{\mathsf{ecd}} + f^{\mathsf{cae}}f^{\mathsf{edb}} + f^{\mathsf{dae}}f^{\mathsf{ebc}}\right] &= 2\alpha_a k_a \cdot (k_b + k_c + k_d) f^{\mathsf{bcd}} \\ &= -2\alpha_a k_a^2 f^{\mathsf{bcd}} = 0\,, \\ \delta_a\left[(T^{\mathsf{a}})^{\mathsf{i}}{}_{\mathsf{k}}\,(T^{\mathsf{c}})^{\mathsf{k}}{}_{\mathsf{j}} - (T^{\mathsf{c}})^{\mathsf{i}}{}_{\mathsf{k}}\,(T^{\mathsf{a}})^{\mathsf{k}}{}_{\mathsf{j}} - f^{\mathsf{ace}}\,(T^{\mathsf{e}})^{\mathsf{i}}{}_{\mathsf{j}}\right] &= 2\alpha_a k_a \cdot (k_i + k_j + k_c)\,(T^{\mathsf{c}})^{\mathsf{i}}{}_{\mathsf{j}} \\ &= -2\alpha_a k_a^2\,(T^{\mathsf{c}})^{\mathsf{i}}{}_{\mathsf{j}} = 0 \end{aligned} \tag{2.8}$$

using momentum conservation and the masslessness of the gluon.

In ref. [9], the $n$-point amplitude eq. (2.1) was proved to be invariant under the color-factor shift associated with any of the external gluons it contains

$$\delta_a \mathcal{A}_n = 0 \tag{2.9}$$

by rewriting the amplitude using the radiation vertex expansion. In section 4 we give an alternative proof of this fact using the recursive perturbiner approach.

### 2.2 BCJ relations

In the remainder of this section, we recall the demonstration [9] that color-factor symmetry of the amplitude implies the fundamental BCJ relation [5, 7, 40] among the color-ordered amplitudes. As mentioned above, the color factors $c_i$ are not independent due to group theory identities (2.2) and (2.3). It is useful to identity an independent basis of color factors, whose coefficients will be unambiguously specified [39]. For tree-level $n$-gluon amplitudes, such a basis consists of half-ladder color factors

$$\mathbf{c}_{1\gamma n} \equiv \sum_{\mathsf{b}_1,\ldots,\mathsf{b}_{n-3}} f^{\mathsf{a}_1\mathsf{a}_{\gamma(2)}\mathsf{b}_1} f^{\mathsf{b}_1\mathsf{a}_{\gamma(3)}\mathsf{b}_2}\cdots f^{\mathsf{b}_{n-3}\mathsf{a}_{\gamma(n-1)}\mathsf{a}_n} \tag{2.10}$$

in terms of which the amplitude may be written as [41, 42]

$$\mathcal{A}_n = \sum_{\gamma\in S_{n-2}} \mathbf{c}_{1\gamma n} A(1,\gamma(2),\cdots,\gamma(n-1),n) \tag{2.11}$$

where $\gamma$ runs over all permutations of $\{2, \cdots, n-1\}$, and $A(1, \gamma(2), \cdots, \gamma(n-1), n)$ are color-ordered amplitudes. Singling out one of the external gluons ($a = 2$) and letting $\sigma$ denote an arbitrary permutation of $\{3, \cdots, n-1\}$, we may reexpress eq. (2.11) as

$$\mathcal{A}_n = \sum_{\sigma \in S_{n-3}} \left[ \sum_{e=3}^{n} \mathbf{c}_{1\sigma(3)\cdots\sigma(e-1)2\sigma(e)\cdots\sigma(n-1)n} A(1, \sigma(3), \cdots, \sigma(e-1), 2, \sigma(e), \cdots, \sigma(n-1), n) \right] \tag{2.12}$$

where

$$\begin{aligned} &\mathbf{c}_{1\sigma(3)\cdots\sigma(e-1)2\sigma(e)\cdots\sigma(n-1)n} \\ &\quad = \sum_{\mathsf{b}_1, \ldots, \mathsf{b}_{n-3}} f^{\mathsf{a}_1 \mathsf{a}_{\sigma(3)} \mathsf{b}_1} \cdots f^{\mathsf{b}_{e-4} \mathsf{a}_{\sigma(e-1)} \mathsf{b}_{e-3}} f^{\mathsf{b}_{e-3} \mathsf{a}_2 \mathsf{b}_{e-2}} f^{\mathsf{b}_{e-2} \mathsf{a}_{\sigma(e)} \mathsf{b}_{e-1}} \cdots f^{\mathsf{b}_{n-3} \mathsf{a}_{\sigma(n-1)} \mathsf{a}_n} . \end{aligned} \tag{2.13}$$

The color-factor symmetry associated with gluon $a = 2$ acts on eq. (2.13) as

$$\delta_2\, \mathbf{c}_{1\sigma(3)\cdots\sigma(e-1)2\sigma(e)\cdots\sigma(n-1)n} = 2\alpha_2 k_2 \cdot \left( k_1 + \sum_{d=3}^{e-1} k_{\sigma(d)} \right) \mathbf{c}_{1\sigma(3)\cdots\sigma(e-1)\sigma(e)\cdots\sigma(n-1)n} \tag{2.14}$$

and therefore

$$\begin{aligned} \delta_2\, \mathcal{A}_n = 2\alpha_2 \sum_{\sigma \in S_{n-3}} \mathbf{c}_{1\sigma n} \sum_{e=3}^{n} k_2 \cdot \left( k_1 + \sum_{d=3}^{e-1} k_{\sigma(d)} \right) \\ \times A(1, \sigma(3), \cdots, \sigma(e-1), 2, \sigma(e), \cdots, \sigma(n-1), n) . \end{aligned} \tag{2.15}$$

Since $\delta_2 \mathcal{A}_n = 0$ by color-factor symmetry, and since the half-ladder color factors $\mathbf{c}_{1\sigma n}$ are independent, this establishes that

$$\sum_{e=3}^{n} \left( k_2 \cdot k_1 + \sum_{d=3}^{e-1} k_2 \cdot k_{\sigma(d)} \right) A(1, \sigma(3), \cdots, \sigma(e-1), 2, \sigma(e), \cdots, \sigma(n-1), n) = 0 \tag{2.16}$$

which is the fundamental BCJ relation, from which the rest of the BCJ relations may be derived [5, 7, 40]. This argument may be generalized to the amplitudes of the BAS theory in curved symmetric spacetime [15].

For tree-level amplitudes containing fields in other representations (e.g. quarks) in addition to gluons, an independent basis of color factors is given by the Melia basis [43–45]. The independent amplitudes corresponding to this basis also satisfy BCJ relations that follow from the assumption of color-kinematic duality [46, 47]. Color-factor symmetry can also be used to derive BCJ relations for these amplitudes [10].

## 3 Color-dressed perturbiner expansion

In this section, we review the color-dressed perturbiner expansion [24] of the solutions to the classical equations of motion for the biadjoint scalar theory and Yang-Mills theory, and how its coefficients (Berends-Giele currents) are used to obtain tree-level $n$-point amplitudes in those theories.

### 3.1 Biadjoint scalar theory

The biadjoint scalar theory is a theory of a massless scalar field $\phi^{\mathsf{aa'}}$ transforming in the adjoint representation of $\mathrm{U}(N) \times \mathrm{U}(\tilde{N})$, with Lagrangian [13]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^{\mathsf{aa'}})(\partial^\mu \phi^{\mathsf{aa'}}) \; - \; \frac{1}{6}\lambda f^{\mathsf{abc}} \tilde{f}^{\mathsf{a'b'c'}} \phi^{\mathsf{aa'}} \phi^{\mathsf{bb'}} \phi^{\mathsf{cc'}} \tag{3.1}$$

where $f^{\mathsf{abc}}$ and $\tilde{f}^{\mathsf{a'b'c'}}$ are the structure constants[2] of $\mathrm{U}(N)$ and $\mathrm{U}(\tilde{N})$ respectively. This Lagrangian yields the equation of motion

$$\partial^2 \phi^{\mathsf{aa'}} \; = \; -\frac{1}{2}\lambda f^{\mathsf{abc}} \tilde{f}^{\mathsf{a'b'c'}} \phi^{\mathsf{bb'}} \phi^{\mathsf{cc'}} \,. \tag{3.2}$$

Rosly and Selivanov [17–19] introduced the *perturbiner ansatz*, which is a solution to the nonlinear classical equation of motion obtained by first solving the free equation of motion $\partial^2 \phi^{\mathsf{aa'}} = 0$ with an arbitrary linear combination of plane waves[3]

$$\phi^{\mathsf{aa'}}(x) = \sum_{i=1}^{M} \phi_i^{\mathsf{aa'}} e^{ik_i \cdot x} + \mathcal{O}(\lambda)\,, \qquad \phi_i^{\mathsf{aa'}} = \varepsilon_i \delta^{\mathsf{aa}_i} \delta^{\mathsf{a'a}'_i}\,, \qquad k_i^2 = 0 \tag{3.3}$$

and then using this as a seed in eq. (3.2) to generate corrections higher order in $\lambda$. The coefficients of this expansion are used to compute tree-level amplitudes of the theory.

For the purpose of proving the color-factor symmetry of tree-level amplitudes in section 4, we find it convenient to use a version of the perturbiner ansatz developed by Mizera and Skrzypek [24], called the *color-dressed perturbiner expansion* (in distinction from the color-stripped perturbiner expansion). For the BAS theory, the ansatz can be written[4]

$$\phi^{\mathsf{aa'}}(x) = \sum_i \phi_i^{\mathsf{aa'}} \, e^{ik_i \cdot x} + \sum_{i<j} \phi_{ij}^{\mathsf{aa'}} \, e^{ik_{ij} \cdot x} + \sum_{i<j<k} \phi_{ijk}^{\mathsf{aa'}} \, e^{ik_{ijk} \cdot x} + \cdots \tag{3.4}$$

where $k_{ij}^\mu = k_i^\mu + k_j^\mu$, etc. Equation (3.4) is expressed compactly as

$$\phi^{\mathsf{aa'}}(x) = \sum_P \phi_P^{\mathsf{aa'}} e^{ik_P \cdot x} \tag{3.5}$$

summing over all non-empty ordered words $P = p_1 p_2 \cdots p_m$ with $1 \le p_1 < p_2 < \cdots < p_m \le M$, where $k_P = \sum_{j=1}^m k_{p_j}$. By inserting eq. (3.5) into eq. (3.2) one obtains [24]

$$\phi_P^{\mathsf{aa'}} = \frac{\lambda}{2k_P^2} f^{\mathsf{abc}} \tilde{f}^{\mathsf{a'b'c'}} \sum_{P=Q\cup R} \phi_Q^{\mathsf{bb'}} \phi_R^{\mathsf{cc'}} \tag{3.6}$$

where $P = Q \cup R$ denotes all possible divisions of $P$ into two non-empty ordered words $Q$ and $R$. The coefficients $\phi_P^{\mathsf{aa'}}$ are Berends-Giele currents[5] of the BAS theory, computed recursively using eq. (3.6). One sees that $\phi_P^{\mathsf{aa'}}$ has a pole at $k_P^2 = 0$.

[2] Normalized by $f^{\mathsf{abc}} = \mathrm{Tr}([T^{\mathsf{a}}, T^{\mathsf{b}}]T^{\mathsf{c}})$ with $\mathrm{Tr}(T^{\mathsf{a}} T^{\mathsf{b}}) = \delta^{\mathsf{ab}}$, so that $[T^{\mathsf{a}}, T^{\mathsf{b}}] = f^{\mathsf{abc}} T^{\mathsf{c}}$. We use $\eta_{00} = 1$.

[3] The number $M$ of plane waves is arbitrary, but the light-like momenta $k_i^\mu$ will eventually be taken as the momenta of external states in an $n$-point amplitude, so $M$ should at least equal the multiplicity.

[4] In the higher order terms, one suppresses terms in which some of the indices $i$, $j$, $k$ coincide. This may be achieved formally [17–19] by setting $\varepsilon_i^2 = 0$.

[5] Berends and Giele [16] originally defined these currents for Yang-Mills theory, which Mafra [22] adapted to the BAS theory.



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To obtain the tree-level $n$-point amplitude $\mathcal{A}_n$, one first computes $\phi_P^{\mathsf{aa}'}$ for $P = 12\cdots(n-1)$. Since momentum conservation for the $n$-point amplitude implies $k_P = -k_n$, and an on-shell amplitude has $k_n^2 = 0$, one extracts the residue of the $k_P^2$ pole of $\phi_P^{\mathsf{aa}'}$ and contracts with $\phi_n^{\mathsf{aa}'}$ to get [24]

$$\mathcal{A}_n = \lim_{k_P^2 \to 0} \phi_n^{\mathsf{aa}'} k_P^2 \phi_P^{\mathsf{aa}'} \,. \tag{3.7}$$

To illustrate this procedure for the four-point amplitude we first use eqs. (3.3) and (3.6) to compute the rank-2 perturbiner coefficient

$$\phi_{ij}^{\mathsf{aa}'} = \frac{\lambda}{2k_{ij}^2} f^{\mathsf{abc}} \tilde{f}^{\mathsf{a}'\mathsf{b}'\mathsf{c}'} \left( \phi_i^{\mathsf{bb}'} \phi_j^{\mathsf{cc}'} + \phi_j^{\mathsf{bb}'} \phi_i^{\mathsf{cc}'} \right) = \frac{\lambda \varepsilon_i \varepsilon_j}{k_{ij}^2} f^{\mathsf{a}\mathsf{a}_i\mathsf{a}_j} \tilde{f}^{\mathsf{a}'\mathsf{a}'_i\mathsf{a}'_j} \tag{3.8}$$

and from this the rank-3 coefficient

$$\begin{aligned} \phi_{123}^{\mathsf{aa}'} &= \frac{\lambda}{2k_{123}^2} f^{\mathsf{abc}} \tilde{f}^{\mathsf{a}'\mathsf{b}'\mathsf{c}'} \left( \phi_{12}^{\mathsf{bb}'} \phi_3^{\mathsf{cc}'} + \phi_3^{\mathsf{bb}'} \phi_{12}^{\mathsf{cc}'} + \phi_{13}^{\mathsf{bb}'} \phi_2^{\mathsf{cc}'} + \phi_2^{\mathsf{bb}'} \phi_{13}^{\mathsf{cc}'} + \phi_{23}^{\mathsf{bb}'} \phi_1^{\mathsf{cc}'} + \phi_1^{\mathsf{bb}'} \phi_{23}^{\mathsf{cc}'} \right) \\ &= \frac{\lambda^2 \varepsilon_1 \varepsilon_2 \varepsilon_3}{k_{123}^2} \left[ \frac{c_{123}^{\mathsf{a}} \tilde{c}_{123}^{\mathsf{a}'}}{k_{12}^2} + (\text{cyclic permutations of } 123) \right] \end{aligned} \tag{3.9}$$

where $c_{123}^{\mathsf{a}} = f^{\mathsf{a}_1\mathsf{a}_2\mathsf{c}} f^{\mathsf{c}\mathsf{a}_3\mathsf{a}}$ and $\tilde{c}_{123}^{\mathsf{a}'} = \tilde{f}^{\mathsf{a}'_1\mathsf{a}'_2\mathsf{c}'} \tilde{f}^{\mathsf{c}'\mathsf{a}'_3\mathsf{a}'}$. Then eq. (3.7) is used to obtain the four-point amplitude (setting $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = 1$)

$$\mathcal{A}_4 = \lambda^2 \left[ \frac{c_{1234} \tilde{c}_{1234}}{k_{12}^2} + (\text{cyclic permutations of } 123) \right] \tag{3.10}$$

where $c_{1234} = f^{\mathsf{a}_1\mathsf{a}_2\mathsf{c}} f^{\mathsf{c}\mathsf{a}_3\mathsf{a}_4}$ and $\tilde{c}_{1234} = \tilde{f}^{\mathsf{a}'_1\mathsf{a}'_2\mathsf{c}'} \tilde{f}^{\mathsf{c}'\mathsf{a}'_3\mathsf{a}'_4}$. This result agrees with four-point amplitude found in ref. [13].

### 3.2 Yang-Mills theory

We now describe the color-dressed perturbiner expansion for Yang-Mills theory [24]. The Yang-Mills Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^{\mathsf{a}} F^{\mu\nu\,\mathsf{a}} \tag{3.11}$$

implies the equation of motion[6]

$$\partial_\nu F^{\nu\mu\,\mathsf{a}} = i g f^{\mathsf{abc}} A_\nu^{\mathsf{b}} F^{\nu\mu\,\mathsf{c}} \tag{3.12}$$

where the Yang-Mills field strength is given by

$$F_{\mu\nu}^{\mathsf{a}} = \partial_\mu A_\nu^{\mathsf{a}} - \partial_\nu A_\mu^{\mathsf{a}} - i g f^{\mathsf{abc}} A_\mu^{\mathsf{b}} A_\nu^{\mathsf{c}} \,. \tag{3.13}$$

Choosing Lorenz gauge

$$\partial_\nu A^{\nu\,\mathsf{a}} = 0 \tag{3.14}$$

[6] See footnote 2.

we can write eq. (3.12) as

$$\partial^2 A^{\mu\,\mathsf{a}} = igf^{\mathsf{abc}} A_\nu^{\mathsf{b}} \left(\partial^\nu A^{\mu\,\mathsf{c}} + F^{\nu\mu\,\mathsf{c}}\right) . \tag{3.15}$$

For convenience we define $G^{\nu\mu\mathsf{a}} \equiv -i\left(\partial^\nu A^{\mu\,\mathsf{a}} + F^{\nu\mu\,\mathsf{a}}\right)$, which becomes

$$G^{\nu\mu\mathsf{a}} = -i\left(2\partial^\nu A^{\mu\,\mathsf{a}} - \partial^\mu A^{\nu\,\mathsf{a}}\right) - gf^{\mathsf{abc}} A^{\nu\,\mathsf{b}} A^{\mu\,\mathsf{c}} \tag{3.16}$$

so that eq. (3.15) is expressed as

$$\partial^2 A^{\mu\,\mathsf{a}} = -gf^{\mathsf{abc}} A_\nu^{\mathsf{b}} G^{\nu\mu\,\mathsf{c}} . \tag{3.17}$$

The advantage of using two fields, $A^{\mu\mathsf{a}}$ and $G^{\nu\mu\mathsf{a}}$, rather than just $A^{\mu\mathsf{a}}$ is that eqs. (3.16) and (3.17) contain only quadratic (not cubic) terms, simplifying the recursion relations derived below [20–24].

We now solve these equations with the perturbiner ansatz. As in the previous subsection, we begin by solving the free equation

$$\partial^2 A^{\mu\,\mathsf{a}} = 0, \qquad\qquad \partial_\nu A^{\nu\,\mathsf{a}} = 0 \tag{3.18}$$

with an arbitrary linear combination of plane waves

$$A^{\mu\,\mathsf{a}}(x) = \sum_{i=1}^{M} A_i^{\mu\,\mathsf{a}} e^{ik_i\cdot x} + \mathcal{O}(g) \qquad \text{with} \qquad k_i^2 = 0 \tag{3.19}$$

where

$$A_i^{\mu\,\mathsf{a}} = \varepsilon_i^\mu \delta^{\mathsf{a}\mathsf{a}_i} \qquad \text{with} \qquad \varepsilon_i \cdot k_i = 0 . \tag{3.20}$$

Also to this order we have

$$G^{\nu\mu\,\mathsf{a}}(x) = \sum_{i=1}^{M} G_i^{\nu\mu\,\mathsf{a}} e^{ik_i\cdot x} + \mathcal{O}(g) \tag{3.21}$$

where

$$G_i^{\nu\mu\,\mathsf{a}} = g_i^{\nu\mu} \delta^{\mathsf{a}\mathsf{a}_i} \qquad \text{with} \qquad g_i^{\nu\mu} = 2k_i^\nu \varepsilon_i^\mu - k_i^\mu \varepsilon_i^\nu . \tag{3.22}$$

As before, the lowest-order solution is the first term of the color-dressed perturbiner expansion[7]

$$A^{\mu\,\mathsf{a}}(x) = \sum_P A_P^{\mu\,\mathsf{a}} e^{ik_P\cdot x}, \qquad\qquad G^{\nu\mu\,\mathsf{a}}(x) = \sum_P G_P^{\nu\mu\,\mathsf{a}} e^{ik_P\cdot x} . \tag{3.23}$$

The coefficients $A_P^{\mu\,\mathsf{a}}$ are the (color-dressed) Berends-Giele currents of the Yang-Mills theory [16]. To obtain the tree-level $n$-gluon amplitude, one first computes $A_P^{\mu\mathsf{a}}$ for

[7]In refs. [20–24] the plane wave factor is written $e^{k_P\cdot x}$, with the momentum taken imaginary, in order to avoid a proliferation of factors of $i$. With the conventions of this paper, it is simpler to write $e^{ik_P\cdot x}$ and use real momenta.

$P = 12 \cdots (n-1)$, then extracts the residue of the $k_P^2$ pole, and finally contracts[8] with the Berends-Giele current $A_n^{\mu\,\mathsf{a}}$ of the last gluon [16]

$$\mathcal{A}_n = \lim_{k_P^2 \to 0} A_n^{\mu\,\mathsf{a}} k_P^2 A_P^{\mu\,\mathsf{a}} . \tag{3.24}$$

The recursion relations for the Berends-Giele currents $A_P^{\mu\mathsf{a}}$ are obtained by plugging eq. (3.23) into eq. (3.17) to obtain

$$k_P^2 A_P^{\mu\,\mathsf{a}} = g f^{\mathsf{abc}} \sum_{P=Q\cup R} A_Q^{\nu\,\mathsf{b}} G_R^{\nu\mu\,\mathsf{c}} . \tag{3.25}$$

Similarly, plugging eq. (3.23) into eq. (3.16) we find

$$G_P^{\nu\mu\,\mathsf{a}} = 2k_P^{\nu} A_P^{\mu\,\mathsf{a}} - k_P^{\mu} A_P^{\nu\,\mathsf{a}} - H_P^{\nu\mu\,\mathsf{a}} \tag{3.26}$$

where

$$H_P^{\nu\mu\,\mathsf{a}} \equiv g f^{\mathsf{abc}} \sum_{P=Q\cup R} A_Q^{\nu\,\mathsf{b}} A_R^{\mu\,\mathsf{c}} . \tag{3.27}$$

We combine the three previous equations to obtain

$$k_P^2 G_P^{\nu\mu\,\mathsf{a}} = g f^{\mathsf{abc}} \sum_{P=Q\cup R} \Big[ 2k_P^{\nu} A_Q^{\lambda\,\mathsf{b}} G_R^{\lambda\mu\,\mathsf{c}} - k_P^{\mu} A_Q^{\lambda\,\mathsf{b}} G_R^{\lambda\nu\,\mathsf{c}} - k_P^2 A_Q^{\nu\,\mathsf{b}} A_R^{\mu\,\mathsf{c}} \Big] . \tag{3.28}$$

Eqs. (3.25) and (3.28) play a key role in the proof of color-factor symmetry in the next section. In the remainder of this section, we illustrate how they are used recursively to compute the four-gluon amplitude. We first use eqs. (3.25) and (3.28) together with eqs. (3.20) and (3.22) to obtain the rank-2 coefficients

$$A_{ij}^{\mu\,\mathsf{a}} = \frac{g}{k_{ij}^2} f^{\mathsf{abc}} \Big[ A_i^{\nu\,\mathsf{b}} G_j^{\nu\mu\,\mathsf{c}} + (i \leftrightarrow j) \Big] = \frac{g}{k_{ij}^2} f^{\mathsf{a}\mathsf{a}_i\mathsf{a}_j} \Big[ \varepsilon_i^{\nu} g_j^{\nu\mu} - (i \leftrightarrow j) \Big] , \tag{3.29}$$

$$G_{ij}^{\nu\mu\,\mathsf{a}} = \frac{g}{k_{ij}^2} f^{\mathsf{a}\mathsf{a}_i\mathsf{a}_j} \Big[ 2k_{ij}^{\nu} \varepsilon_i^{\lambda} g_j^{\lambda\mu} - k_{ij}^{\mu} \varepsilon_i^{\lambda} g_j^{\lambda\nu} - k_{ij}^2 \varepsilon_i^{\nu} \varepsilon_j^{\nu} - (i \leftrightarrow j) \Big] . \tag{3.30}$$

These are then used in eq. (3.25) to determine the rank-3 coefficient

$$\begin{aligned} A_{123}^{\mu\,\mathsf{a}} &= \frac{g}{k_{123}^2} f^{\mathsf{abc}} \Big[ A_{12}^{\nu\,\mathsf{b}} G_3^{\nu\mu\,\mathsf{c}} + A_3^{\nu\,\mathsf{b}} G_{12}^{\nu\mu\,\mathsf{c}} + A_{13}^{\nu\,\mathsf{b}} G_2^{\nu\mu\,\mathsf{c}} + A_2^{\nu\,\mathsf{b}} G_{13}^{\nu\mu\,\mathsf{c}} + A_{23}^{\nu\,\mathsf{b}} G_1^{\nu\mu\,\mathsf{c}} + A_1^{\nu\,\mathsf{b}} G_{23}^{\nu\mu\,\mathsf{c}} \Big] \\ &= \frac{g^2}{k_{123}^2} \left[ \frac{c_{123}^{\mathsf{a}} n_{123}^{\mu}}{k_{12}^2} + \text{cyc}(1,2,3) \right] \end{aligned} \tag{3.31}$$

where

$$c_{123}^{\mathsf{a}} = f^{\mathsf{a}_1\mathsf{a}_2\mathsf{c}} f^{\mathsf{c}\mathsf{a}_3\mathsf{a}} , \tag{3.32}$$

$$n_{123}^{\mu} = \Big[ \varepsilon_1^{\lambda} g_2^{\lambda\nu} g_3^{\nu\mu} - \varepsilon_3^{\nu} \Big( 2k_{12}^{\nu} \varepsilon_1^{\lambda} g_2^{\lambda\mu} - k_{12}^{\mu} \varepsilon_1^{\lambda} g_2^{\lambda\nu} - k_{12}^2 \varepsilon_1^{\nu} \varepsilon_2^{\mu} \Big) \Big] - (1 \leftrightarrow 2) . \tag{3.33}$$

[8] For notational clarity, we resort here and below to the regrettable practice of writing all Lorentz indices upstairs. Repeated indices are of course contracted with the Minkowski metric.

We may obtain a more explicit form for $n^\mu_{123}$ by using eq. (3.22) in eq. (3.33) and simplifying

$$\begin{aligned}
n^\mu_{123} = k_1^\mu &[2\varepsilon_2\cdot\varepsilon_3\; k_2\cdot\varepsilon_1 - 2\varepsilon_1\cdot\varepsilon_3\; k_1\cdot\varepsilon_2 - 3\varepsilon_1\cdot\varepsilon_2\; k_2\cdot\varepsilon_3 - \varepsilon_1\cdot\varepsilon_2\; k_1\cdot\varepsilon_3] \\
&+ k_2^\mu[2\varepsilon_2\cdot\varepsilon_3\; k_2\cdot\varepsilon_1 - 2\varepsilon_1\cdot\varepsilon_3\; k_1\cdot\varepsilon_2 + \varepsilon_1\cdot\varepsilon_2\; k_2\cdot\varepsilon_3 + 3\varepsilon_1\cdot\varepsilon_2\; k_1\cdot\varepsilon_3] \\
&+ k_3^\mu[-2\varepsilon_2\cdot\varepsilon_3\; k_2\cdot\varepsilon_1 + 2\varepsilon_1\cdot\varepsilon_3\; k_1\cdot\varepsilon_2 + \varepsilon_1\cdot\varepsilon_2\; k_2\cdot\varepsilon_3 - \varepsilon_1\cdot\varepsilon_2\; k_1\cdot\varepsilon_3] \\
&+ \varepsilon_1^\mu[4k_1\cdot\varepsilon_2\; k_1\cdot\varepsilon_3 + 4k_1\cdot\varepsilon_2\; k_2\cdot\varepsilon_3 - 2\varepsilon_2\cdot\varepsilon_3\; k_1\cdot k_2] \\
&+ \varepsilon_2^\mu[-4k_2\cdot\varepsilon_1\; k_1\cdot\varepsilon_3 - 4k_2\cdot\varepsilon_1\; k_2\cdot\varepsilon_3 + 2\varepsilon_1\cdot\varepsilon_3\; k_1\cdot k_2] \\
&+ \varepsilon_3^\mu[4k_2\cdot\varepsilon_1\; k_3\cdot\varepsilon_2 - 4k_3\cdot\varepsilon_1\; k_1\cdot\varepsilon_2 - 2\varepsilon_1\cdot\varepsilon_2\; k_2\cdot k_3 + 2\varepsilon_1\cdot\varepsilon_2\; k_1\cdot k_3]\,.
\end{aligned} \tag{3.34}$$

We observe that color factors appearing in the Berends-Giele current (3.31) satisfy the Jacobi identity $c^{\mathsf{a}}_{123} + c^{\mathsf{a}}_{231} + c^{\mathsf{a}}_{312} = 0$, but the kinematic numerators do not

$$n^\mu_{123} + n^\mu_{231} + n^\mu_{312} = (k_1^\mu + k_2^\mu + k_3^\mu)\Big[\varepsilon_1\cdot\varepsilon_2(k_1 - k_2)\cdot\varepsilon_3 + \mathrm{cyc(1,2,3)}\Big]\,. \tag{3.35}$$

Finally, eq. (3.24) yields the well known four-gluon amplitude

$$\mathcal{A}_4 = g^2\left[\frac{c_{1234}n_{1234}}{k_{12}^2} + \mathrm{cyc(1,2,3)}\right], \qquad c_{1234} = f^{\mathsf{a}_1\mathsf{a}_2\mathsf{c}}f^{\mathsf{c}\mathsf{a}_3\mathsf{a}_4}, \qquad n_{1234} = n^\mu_{123}\varepsilon^\mu_4\,. \tag{3.36}$$

Both color factors and kinematic numerators in the four-gluon amplitude satisfy the Jacobi identity

$$c_{1234} + c_{2314} + c_{3124} = 0\,, \tag{3.37}$$

$$n_{1234} + n_{2314} + n_{3124} = \varepsilon_4\cdot(k_1^\mu + k_2^\mu + k_3^\mu)\Big[\varepsilon_1\cdot\varepsilon_2(k_1 - k_2)\cdot\varepsilon_3 + \mathrm{cyc(1,2,3)}\Big] = 0 \tag{3.38}$$

because $k_1 + k_2 + k_3 = -k_4$ and $\varepsilon_4\cdot k_4 = 0$.

## 4 Recursive proof of color-factor symmetry

In this section, we present proofs that the tree-level $n$-point amplitudes of the BAS theory and Yang-Mills theory are invariant under the color-factor shifts described in section 2 using the recursion relations derived from the color-dressed perturbiner expansions in section 3.

### 4.1 Biadjoint scalar theory

We begin by combining eq. (3.7) with eqs. (3.3) and (3.6) to obtain the following expression for the tree-level $n$-point amplitude of the BAS theory

$$\mathcal{A}_n = \frac{1}{2}\lambda\varepsilon_n \sum_{P=Q\cup R} f^{\mathsf{a}_n\mathsf{b}\mathsf{c}}\tilde{f}^{\mathsf{a}'_n\mathsf{b}'\mathsf{c}'}\phi_Q^{\mathsf{b}\mathsf{b}'}\phi_R^{\mathsf{c}\mathsf{c}'}\,, \qquad P = 12\cdots(n-1)\,. \tag{4.1}$$

We now determine how this amplitude transforms under the color-factor symmetry associated with scalar $n$. The color-factor symmetry acts only on the U($N$) structure constants $f^{\mathsf{abc}}$, with the $\mathrm{U}(\tilde{N})$ structure constants $\tilde{f}^{\mathsf{a'b'c'}}$ behaving as spectators.[9] From eq. (2.5) we have

$$\begin{aligned}\delta_n\, \mathcal{A}_n &= \frac{1}{2}\lambda\varepsilon_n \tilde{f}^{\mathsf{a}_n'\mathsf{b'c'}} \sum_{P=B\cup C} (\delta_n f^{\mathsf{a}_n\mathsf{bc}})\phi_B^{\mathsf{bb'}}\phi_C^{\mathsf{cc'}} \\ &= \frac{1}{2}\lambda\alpha_n\varepsilon_n \tilde{f}^{\mathsf{a}_n'\mathsf{b'c'}} \sum_{P=B\cup C} \delta^{\mathsf{bc}}(k_B^2 - k_C^2)\phi_B^{\mathsf{bb'}}\phi_C^{\mathsf{cc'}}\,. \end{aligned} \tag{4.2}$$

Since the sum over divisions of $P$ into words $B$ and $C$ is symmetric under $B \leftrightarrow C$, we may relabel $B, \mathsf{b}, \mathsf{b'} \leftrightarrow C, \mathsf{c}, \mathsf{c'}$ in the first term

$$\tilde{f}^{\mathsf{a}_n'\mathsf{b'c'}} \sum_{P=B\cup C} \delta^{\mathsf{bc}} k_B^2 \phi_B^{\mathsf{bb'}}\phi_C^{\mathsf{cc'}} = \tilde{f}^{\mathsf{a}_n'\mathsf{c'b'}} \sum_{P=B\cup C} \delta^{\mathsf{cb}} k_C^2 \phi_C^{\mathsf{cc'}}\phi_B^{\mathsf{bb'}} \tag{4.3}$$

so that using $\tilde{f}^{\mathsf{a}_n'\mathsf{c'b'}} = -\tilde{f}^{\mathsf{a}_n'\mathsf{b'c'}}$ we have

$$\delta_n\, \mathcal{A}_n = -\lambda\alpha_n\varepsilon_n \tilde{f}^{\mathsf{a}_n'\mathsf{b'c'}} \sum_{P=B\cup C} \delta^{\mathsf{bc}}\phi_B^{\mathsf{bb'}} k_C^2 \phi_C^{\mathsf{cc'}}\,. \tag{4.4}$$

Now we again use eq. (3.6) to obtain

$$\delta_n\, \mathcal{A}_n = -\frac{1}{2}\lambda^2\alpha_n\varepsilon_n \tilde{f}^{\mathsf{a}_n'\mathsf{b'c'}}\tilde{f}^{\mathsf{c'd'e'}} \left( f^{\mathsf{bde}} \sum_{P=B\cup D\cup E} \phi_B^{\mathsf{bb'}}\phi_D^{\mathsf{dd'}}\phi_E^{\mathsf{ee'}} \right). \tag{4.5}$$

Using the invariance of the sum over $P = B \cup D \cup E$ under any permutation of the words $B$, $D$, and $E$, we observe that the term in parentheses is invariant under cyclic permutations

$$B, \mathsf{b}, \mathsf{b'} \to D, \mathsf{d}, \mathsf{d'} \to E, \mathsf{e}, \mathsf{e'} \to B, \mathsf{b}, \mathsf{b'}$$

so that we may cyclically symmetrize $\tilde{f}^{\mathsf{a}_n'\mathsf{b'c'}}\tilde{f}^{\mathsf{c'd'e'}}$ to obtain

$$\begin{aligned}\delta_n\, \mathcal{A}_n = &-\frac{1}{6}\lambda^2\alpha_n\varepsilon_n \left( \tilde{f}^{\mathsf{a}_n'\mathsf{b'c'}}\tilde{f}^{\mathsf{c'd'e'}} + \tilde{f}^{\mathsf{a}_n'\mathsf{d'c'}}\tilde{f}^{\mathsf{c'e'b'}} + \tilde{f}^{\mathsf{a}_n'\mathsf{e'c'}}\tilde{f}^{\mathsf{c'b'd'}} \right) \\ &\times \left( f^{\mathsf{bde}} \sum_{P=B\cup D\cup E} \phi_B^{\mathsf{bb'}}\phi_D^{\mathsf{dd'}}\phi_E^{\mathsf{ee'}} \right). \end{aligned} \tag{4.6}$$

Since the term in the left parenthesis vanishes by the Jacobi identity, the $n$-point amplitude is invariant under the color-factor symmetry associated with scalar $n$. Since the amplitude is Bose symmetric, it is invariant under the color-factor symmetry associated with any of the external fields

$$\delta_a\, \mathcal{A}_n = 0 \tag{4.7}$$

as was previously established using the cubic vertex expansion [9].

[9] Naturally, one could alternatively define color-factor shifts that act on $\tilde{f}^{\mathsf{a'b'c'}}$.

### 4.2 Yang-Mills theory

To prove that the tree-level $n$-gluon amplitude is invariant under color-factor shifts, we begin by combining eq. (3.24) with eqs. (3.20) and (3.25) to obtain

$$\mathcal{A}_n = g\varepsilon_n^\mu \sum_{P=Q\cup R} f^{\mathsf{a}_n\mathsf{bc}} A_Q^{\nu\,\mathsf{b}} G_R^{\nu\mu\,\mathsf{c}}\,, \qquad\qquad P = 12\cdots(n-1)\,. \tag{4.8}$$

The color-factor symmetry associated with gluon $n$ acts only on the explicit factor $f^{\mathsf{a}_n\mathsf{bc}}$ in the equation above, giving

$$\begin{aligned}\delta_n\,\mathcal{A}_n &= g\varepsilon_n^\mu \sum_{P=Q\cup R} (\delta_n f^{\mathsf{a}_n\mathsf{bc}}) A_Q^{\nu\,\mathsf{b}} G_R^{\nu\mu\,\mathsf{c}} \\ &= g\alpha_n\varepsilon_n^\mu \sum_{P=Q\cup R} \delta^{\mathsf{bc}}(k_Q^2 - k_R^2) A_Q^{\nu\,\mathsf{b}} G_R^{\nu\mu\,\mathsf{c}} \\ &= g\alpha_n\varepsilon_n^\mu \sum_{P=Q\cup R} \left[(k_Q^2 A_Q^{\lambda\,\mathsf{c}}) G_R^{\lambda\mu\,\mathsf{c}} - A_Q^{\nu\,\mathsf{a}}(k_R^2 G_R^{\nu\mu\,\mathsf{a}})\right]\,.\end{aligned} \tag{4.9}$$

Our goal is to show that the right hand side of this equation vanishes, so we must first compute

$$S_P^\mu \equiv \sum_{P=Q\cup R} \left[(k_Q^2 A_Q^{\lambda\,\mathsf{c}}) G_R^{\lambda\mu\,\mathsf{c}} - A_Q^{\nu\,\mathsf{a}}(k_R^2 G_R^{\nu\mu\,\mathsf{a}})\right] \tag{4.10}$$

which unfortunately is a bit more complicated than the biadjoint scalar case. First we use eqs. (3.25) and (3.28) to find

$$\begin{aligned}S_P^\mu &= g f^{\mathsf{abc}} \sum_{P=A\cup B\cup C} \left[\left(A_A^{\nu\,\mathsf{a}} G_B^{\nu\lambda\,\mathsf{b}}\right) G_C^{\lambda\mu\,\mathsf{c}} - A_A^{\nu\,\mathsf{a}}\left(2k_{BC}^\nu A_B^{\lambda\,\mathsf{b}} G_C^{\lambda\mu\,\mathsf{c}} - k_{BC}^\mu A_B^{\lambda\,\mathsf{b}} G_C^{\lambda\nu\,\mathsf{c}} - k_{BC}^2 A_B^{\nu\,\mathsf{b}} A_C^{\mu\,\mathsf{c}}\right)\right] \\ &= g f^{\mathsf{abc}} \sum_{P=A\cup B\cup C} A_A^{\nu\,\mathsf{a}} \left[\left(G_B^{\nu\lambda\,\mathsf{b}} - 2k_B^\nu A_B^{\lambda\,\mathsf{b}} - 2k_C^\nu A_B^{\lambda\,\mathsf{b}}\right) G_C^{\lambda\mu\,\mathsf{c}} + k_{BC}^\mu A_B^{\lambda\,\mathsf{b}} G_C^{\lambda\nu\,\mathsf{c}} + k_{BC}^2 A_B^{\nu\,\mathsf{b}} A_C^{\mu\,\mathsf{c}}\right]\end{aligned} \tag{4.11}$$

where $k_{BC}^\mu = k_B^\mu + k_C^\mu$. We use eq. (3.26) to reexpress this as

$$\begin{aligned}S_P^\mu = g f^{\mathsf{abc}} \sum_{P=A\cup B\cup C} A_A^{\nu\,\mathsf{a}} \Big[ &\left(-k_B^\lambda A_B^{\nu\,\mathsf{b}} - 2k_C^\nu A_B^{\lambda\,\mathsf{b}} - H_B^{\nu\lambda\,\mathsf{b}}\right)(2k_C^\lambda A_C^{\mu\,\mathsf{c}} - k_C^\mu A_C^{\lambda\,\mathsf{c}} - H_C^{\lambda\mu\,\mathsf{c}}) \\ &+ (k_C^\mu + k_B^\mu) A_B^{\lambda\,\mathsf{b}}\left(2k_C^\lambda A_C^{\nu\,\mathsf{c}} - k_C^\nu A_C^{\lambda\,\mathsf{c}} - H_C^{\lambda\nu\,\mathsf{c}}\right) + \left(k_C^2 + 2k_B\cdot k_C + k_B^2\right) A_B^{\nu\,\mathsf{b}} A_C^{\mu\,\mathsf{c}}\Big]\,.\end{aligned} \tag{4.12}$$

Equation (4.12) can be split into two contributions

$$S_P^\mu = S_{P,1}^\mu + S_{P,2}^\mu\,, \tag{4.13}$$

$$\begin{aligned}S_{P,1}^\mu = g f^{\mathsf{abc}} \sum_{P=A\cup B\cup C} \Big[ &\left(k_B\cdot A_C^{\mathsf{c}}\ A_A^{\mathsf{a}}\cdot A_B^{\mathsf{b}} + 2k_C\cdot A_A^{\mathsf{a}}\ A_B^{\mathsf{b}}\cdot A_C^{\mathsf{c}} + A_A^{\nu\,\mathsf{a}} H_B^{\nu\lambda\,\mathsf{b}} A_C^{\lambda\,\mathsf{c}}\right) k_C^\mu \\ &+ \left(2k_C\cdot A_B^{\mathsf{b}}\ A_A^{\mathsf{a}}\cdot A_C^{\mathsf{c}} - k_C\cdot A_A^{\mathsf{a}}\ A_B^{\mathsf{b}}\cdot A_C^{\mathsf{c}} - A_A^{\nu\,\mathsf{a}} A_B^{\lambda\,\mathsf{b}} H_C^{\lambda\nu\,\mathsf{c}}\right) k_C^\mu \\ &+ \left(2k_C\cdot A_B^{\mathsf{b}}\ A_A^{\mathsf{a}}\cdot A_C^{\mathsf{c}} - k_C\cdot A_A^{\mathsf{a}}\ A_B^{\mathsf{b}}\cdot A_C^{\mathsf{c}} - A_A^{\nu\,\mathsf{a}} A_B^{\lambda\,\mathsf{b}} H_C^{\lambda\nu\,\mathsf{c}}\right) k_B^\mu\Big]\,,\end{aligned} \tag{4.14}$$

$$
\begin{aligned}
S^{\mu}_{P,2} = g f^{\mathsf{abc}} &\sum_{P=A\cup B\cup C} \left[-4k_C\cdot A^{\mathsf{a}}_A\; k_C\cdot A^{\mathsf{b}}_B + k_C^2\; A^{\mathsf{a}}_A\cdot A^{\mathsf{b}}_B\right] A^{\mu\,\mathsf{c}}_C \\
&+ g f^{\mathsf{abc}} \sum_{P=A\cup B\cup C} \left[-2A^{\nu\,\mathsf{a}}_A H^{\nu\lambda\,\mathsf{b}}_B k^{\lambda}_C + A^{\nu\,\mathsf{a}}_A (k^2_B A^{\nu\,\mathsf{b}}_B)\right] A^{\mu\,\mathsf{c}}_C \\
&+ g f^{\mathsf{abc}} \sum_{P=A\cup B\cup C} A^{\nu\,\mathsf{a}}_A \left(k^{\lambda}_B A^{\nu\,\mathsf{b}}_B + 2k^{\nu}_C A^{\lambda\,\mathsf{b}}_B + H^{\nu\lambda\,\mathsf{b}}_B\right) H^{\lambda\mu\,\mathsf{c}}_C
\end{aligned}
\tag{4.15}
$$

where $S^{\mu}_{P,1}$ contains the terms in which the free index $\mu$ labels a momentum $k$ and $S^{\mu}_{P,2}$ contains those in which it labels a field $A$ or $H$.

First we examine $S^{\mu}_{P,1}$. Relabelling $B,\mathsf{b} \leftrightarrow C,\mathsf{c}$ in the last line (the $k^{\mu}_B$ term) of eq. (4.14) and using $f^{\mathsf{acb}} = -f^{\mathsf{abc}}$, we obtain two terms

$$
S^{\mu}_{P,1} = S^{\mu}_{P,1a} + S^{\mu}_{P,1b}\,,
\tag{4.16}
$$

$$
\begin{aligned}
S^{\mu}_{P,1a} = g f^{\mathsf{abc}} &\sum_{P=A\cup B\cup C} [-k_B\cdot A^{\mathsf{c}}_C\; A^{\mathsf{a}}_A\cdot A^{\mathsf{b}}_B + k_C\cdot A^{\mathsf{a}}_A\; A^{\mathsf{b}}_B\cdot A^{\mathsf{c}}_C \\
&+ 2k_C\cdot A^{\mathsf{b}}_B\; A^{\mathsf{a}}_A\cdot A^{\mathsf{c}}_C + k_B\cdot A^{\mathsf{a}}_A\; A^{\mathsf{b}}_B\cdot A^{\mathsf{c}}_C] k^{\mu}_C\,,
\end{aligned}
\tag{4.17}
$$

$$
S^{\mu}_{P,1b} = g f^{\mathsf{abc}} \sum_{P=A\cup B\cup C} A^{\nu\,\mathsf{a}}_A \left[H^{\nu\lambda\,\mathsf{b}}_B A^{\lambda\,\mathsf{c}}_C + A^{\lambda\,\mathsf{c}}_C H^{\lambda\nu\,\mathsf{b}}_B - A^{\lambda\,\mathsf{b}}_B H^{\lambda\nu\,\mathsf{c}}_C\right] k^{\mu}_C\,.
\tag{4.18}
$$

For $S^{\mu}_{P,1a}$, we relabel $A,\mathsf{a} \leftrightarrow B,\mathsf{b}$ in the first two terms of eq. (4.17) and use $f^{\mathsf{bac}} = -f^{\mathsf{abc}}$ to obtain

$$
S^{\mu}_{P,1a} = g f^{\mathsf{abc}} \sum_{P=A\cup B\cup C} [k_A\cdot A^{\mathsf{c}}_C\; A^{\mathsf{a}}_A\cdot A^{\mathsf{b}}_B + k_C\cdot A^{\mathsf{b}}_B\; A^{\mathsf{a}}_A\cdot A^{\mathsf{c}}_C + k_B\cdot A^{\mathsf{a}}_A\; A^{\mathsf{b}}_B\cdot A^{\mathsf{c}}_C] k^{\mu}_C\,.
\tag{4.19}
$$

Then we cyclically relabel the last two terms of eq. (4.19) and use $f^{\mathsf{bca}} = f^{\mathsf{cab}} = f^{\mathsf{abc}}$ to obtain

$$
\begin{aligned}
S^{\mu}_{P,1a} &= g f^{\mathsf{abc}} \sum_{P=A\cup B\cup C} k_A\cdot A^{\mathsf{c}}_C\; A^{\mathsf{a}}_A\cdot A^{\mathsf{b}}_B\; (k^{\mu}_C + k^{\mu}_A + k^{\mu}_B) \\
&= g f^{\mathsf{abc}} k^{\mu}_P \sum_{P=A\cup B\cup C} k_A\cdot A^{\mathsf{c}}_C\; A^{\mathsf{a}}_A\cdot A^{\mathsf{b}}_B
\end{aligned}
\tag{4.20}
$$

where we have used $k^{\mu}_A + k^{\mu}_B + k^{\mu}_C = k^{\mu}_P$.

Next, we turn to $S^{\mu}_{P,1b}$, observing that the first two terms in eq. (4.18) cancel (since $H^{\nu\lambda\,\mathsf{b}}_B = -H^{\lambda\nu\,\mathsf{b}}_B$), leaving

$$
S^{\mu}_{P,1b} = -g f^{\mathsf{abe}} \sum_{P=A\cup B\cup E} A^{\nu\,\mathsf{a}}_A A^{\lambda\,\mathsf{b}}_B H^{\lambda\nu\,\mathsf{e}}_E k^{\mu}_E\,.
\tag{4.21}
$$

Using eq. (3.27) with $E = D\cup C$ we have

$$
S^{\mu}_{P,1b} = g^2 f^{\mathsf{abe}} f^{\mathsf{ecd}} \sum_{P=A\cup B\cup C\cup D} A^{\mathsf{a}}_A\cdot A^{\mathsf{c}}_C\; A^{\mathsf{b}}_B\cdot A^{\mathsf{d}}_D\; (k^{\mu}_C + k^{\mu}_D)\,.
\tag{4.22}
$$

Since $f^{\mathsf{abe}} f^{\mathsf{ecd}} A_A^{\mathsf{a}} \cdot A_C^{\mathsf{c}}\ A_B^{\mathsf{b}} \cdot A_D^{\mathsf{d}}$ is invariant under $(A, \mathsf{a} \leftrightarrow C, \mathsf{c}; B, \mathsf{b} \leftrightarrow D, \mathsf{d})$, we replace this with

$$\begin{aligned} S^\mu_{P,1b} &= \frac{1}{2} g^2 f^{\mathsf{abe}} f^{\mathsf{ecd}} \sum_{P=A\cup B\cup C\cup D} A_A^{\mathsf{a}} \cdot A_C^{\mathsf{c}}\ A_B^{\mathsf{b}} \cdot A_D^{\mathsf{d}}\ (k_A^\mu + k_B^\mu + k_C^\mu + k_D^\mu) \\ &= \frac{1}{2} g^2 f^{\mathsf{abe}} f^{\mathsf{ecd}} k_P^\mu \sum_{P=A\cup B\cup C\cup D} A_A^{\mathsf{a}} \cdot A_C^{\mathsf{c}}\ A_B^{\mathsf{b}} \cdot A_D^{\mathsf{d}} \end{aligned} \tag{4.23}$$

using $k_P^\mu = k_A^\mu + k_B^\mu + k_C^\mu + k_D^\mu$. Combining eqs. (4.20) and (4.23), we have

$$S^\mu_{P,1} = k_P^\mu \left[ g f^{\mathsf{abc}} \sum_{P=A\cup B\cup C} k_A \cdot A_C^{\mathsf{c}}\ A_A^{\mathsf{a}} \cdot A_B^{\mathsf{b}} + \frac{1}{2} g^2 f^{\mathsf{abe}} f^{\mathsf{ecd}} \sum_{P=A\cup B\cup C\cup D} A_A^{\mathsf{a}} \cdot A_C^{\mathsf{c}}\ A_B^{\mathsf{b}} \cdot A_D^{\mathsf{d}} \right]. \tag{4.24}$$

Now we turn to $S^\mu_{P,2}$. The two terms on the first line of eq. (4.15) vanish using $f^{\mathsf{abc}} = -f^{\mathsf{bac}}$, leaving

$$S^\mu_{P,2} = S^\mu_{P,2a} + S^\mu_{P,2b}\,, \tag{4.25}$$

$$S^\mu_{P,2a} = g f^{\mathsf{aic}} \sum_{P=A\cup I\cup C} \left[ -2 A_A^{\nu\,\mathsf{a}} H_I^{\nu\lambda\,\mathsf{i}} k_C^\lambda + A_A^{\nu\,\mathsf{a}} (k_I^2 A_I^{\nu\,\mathsf{i}}) \right] A_C^{\mu\,\mathsf{c}}\,, \tag{4.26}$$

$$S^\mu_{P,2b} = g f^{\mathsf{abj}} \sum_{P=A\cup B\cup J} A_A^{\nu\,\mathsf{a}} \left( k_B^\lambda A_B^{\nu\,\mathsf{b}} + 2 k_J^\nu A_B^{\lambda\,\mathsf{b}} + H_B^{\nu\lambda\,\mathsf{b}} \right) H_J^{\lambda\mu\,\mathsf{j}}\,. \tag{4.27}$$

For $S^\mu_{P,2a}$, we use eqs. (3.25) and (3.27) with $I = B\cup D$ in eq. (4.26) to obtain

$$\begin{aligned} S^\mu_{P,2a} &= g^2 f^{\mathsf{aic}} f^{\mathsf{ibd}} \sum_{P=A\cup B\cup C\cup D} \left[ -2 k_C \cdot A_D^{\mathsf{d}}\ A_A^{\mathsf{a}} \cdot A_B^{\mathsf{b}} + A_A^{\nu\,\mathsf{a}} A_B^{\lambda\,\mathsf{b}} G_D^{\lambda\nu\,\mathsf{d}} \right] A_C^{\mu\,\mathsf{c}} \\ &= g^2 f^{\mathsf{cai}} f^{\mathsf{ibd}} \sum_{P=A\cup B\cup C\cup D} \Big[ -2 k_C \cdot A_D^{\mathsf{d}}\ A_A^{\mathsf{a}} \cdot A_B^{\mathsf{b}} + 2 k_D \cdot A_B^{\mathsf{b}}\ A_A^{\mathsf{a}} \cdot A_D^{\mathsf{d}} \\ &\qquad - k_D \cdot A_A^{\mathsf{a}}\ A_B^{\mathsf{b}} \cdot A_D^{\mathsf{d}} - A_A^{\nu\,\mathsf{a}} A_B^{\lambda\,\mathsf{b}} H_D^{\lambda\nu\,\mathsf{d}} \Big] A_C^{\mu\,\mathsf{c}}\,. \end{aligned} \tag{4.28}$$

For $S^\mu_{P,2b}$, we use eq. (3.27) with $J = D\cup C$ in eq. (4.27) to obtain

$$\begin{aligned} S^\mu_{P,2b} &= g^2 f^{\mathsf{abj}} f^{\mathsf{jdc}} \sum_{P=A\cup B\cup C\cup D} \Big[ k_B \cdot A_D^{\mathsf{d}}\ A_A^{\mathsf{a}} \cdot A_B^{\mathsf{b}} + 2 k_D \cdot A_A^{\mathsf{a}}\ A_B^{\mathsf{b}} \cdot A_D^{\mathsf{d}} \\ &\qquad + 2 k_C \cdot A_A^{\mathsf{a}}\ A_B^{\mathsf{b}} \cdot A_D^{\mathsf{d}} + A_A^{\nu\,\mathsf{a}} H_B^{\nu\lambda\,\mathsf{b}} A_D^{\lambda\,\mathsf{d}} \Big] A_C^{\mu\,\mathsf{c}}\,. \end{aligned} \tag{4.29}$$

Letting $A, \mathsf{a} \leftrightarrow D, \mathsf{d}$ in eq. (4.29) and using $f^{\mathsf{dbj}} f^{\mathsf{jac}} = f^{\mathsf{cai}} f^{\mathsf{ibd}}$, we obtain

$$\begin{aligned} S^\mu_{P,2b} &= g^2 f^{\mathsf{cai}} f^{\mathsf{ibd}} \sum_{P=A\cup B\cup C\cup D} \Big[ k_B \cdot A_A^{\mathsf{a}}\ A_D^{\mathsf{d}} \cdot A_B^{\mathsf{b}} + 2 k_A \cdot A_D^{\mathsf{d}}\ A_B^{\mathsf{b}} \cdot A_A^{\mathsf{a}} \\ &\qquad + 2 k_C \cdot A_D^{\mathsf{d}}\ A_B^{\mathsf{b}} \cdot A_A^{\mathsf{a}} + A_D^{\nu\,\mathsf{d}} H_B^{\nu\lambda\,\mathsf{b}} A_A^{\lambda\,\mathsf{a}} \Big] A_C^{\mu\,\mathsf{c}}\,. \end{aligned} \tag{4.30}$$



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Recombining eqs. (4.28) and (4.30) and symmetrizing on $B, \mathsf{b} \leftrightarrow D, \mathsf{d}$ we find

$$S^{\mu}_{P,2} = S^{\mu}_{P,2a} + S^{\mu}_{P,2b} = S^{\mu}_{P,2c} + S^{\mu}_{P,2d}\,, \tag{4.31}$$

$$\begin{aligned} S^{\mu}_{P,2c} = g^2 f^{\mathsf{cai}} f^{\mathsf{ibd}} \sum_{P=A\cup B\cup C\cup D} \Big[\, & k_A \cdot A^{\mathsf{d}}_D\ A^{\mathsf{b}}_B \cdot A^{\mathsf{a}}_A + k_B \cdot A^{\mathsf{a}}_A\ A^{\mathsf{d}}_D \cdot A^{\mathsf{b}}_B + k_D \cdot A^{\mathsf{b}}_B\ A^{\mathsf{a}}_A \cdot A^{\mathsf{d}}_D \\ & - k_A \cdot A^{\mathsf{b}}_B\ A^{\mathsf{d}}_D \cdot A^{\mathsf{a}}_A - k_B \cdot A^{\mathsf{d}}_D\ A^{\mathsf{a}}_A \cdot A^{\mathsf{b}}_B - k_D \cdot A^{\mathsf{a}}_A\ A^{\mathsf{b}}_B \cdot A^{\mathsf{d}}_D \Big] A^{\mu\,\mathsf{c}}_C\,, \end{aligned} \tag{4.32}$$

$$S^{\mu}_{P,2d} = -2g^2 f^{\mathsf{cai}} f^{\mathsf{ibj}} \sum_{P=A\cup B\cup C\cup J} A^{\nu\,\mathsf{a}}_A A^{\lambda\,\mathsf{b}}_B A^{\mu\,\mathsf{c}}_C H^{\lambda\nu\,\mathsf{j}}_J\,. \tag{4.33}$$

For $S^{\mu}_{P,2c}$, we cyclically relabel $ABD$ in four of the six terms in eq. (4.32) to obtain

$$\begin{aligned} S^{\mu}_{P,2c} &= g^2 \left( f^{\mathsf{cai}} f^{\mathsf{ibd}} + f^{\mathsf{cdi}} f^{\mathsf{iab}} + f^{\mathsf{cbi}} f^{\mathsf{ida}} \right) \\ &\times \sum_{P=A\cup B\cup C\cup D} [k_A \cdot A^{\mathsf{d}}_D\ A^{\mathsf{b}}_B \cdot A^{\mathsf{a}}_A - k_A \cdot A^{\mathsf{b}}_B\ A^{\mathsf{d}}_D \cdot A^{\mathsf{a}}_A] A^{\mu\,\mathsf{c}}_C \end{aligned} \tag{4.34}$$

which vanishes by the Jacobi identity.

For $S^{\mu}_{P,2d}$, we use eq. (3.27) with $J = D \cup E$ in eq. (4.33) to obtain

$$S^{\mu}_{P,2d} = -2g^3 f^{\mathsf{cai}} f^{\mathsf{ibj}} f^{\mathsf{jde}} \sum_{P=A\cup B\cup C\cup D\cup E} A^{\mathsf{a}}_A \cdot A^{\mathsf{e}}_E\ A^{\mathsf{b}}_B \cdot A^{\mathsf{d}}_D\ A^{\mu\,\mathsf{c}}_C\,. \tag{4.35}$$

Using the symmetries of $A^{\mathsf{a}}_A \cdot A^{\mathsf{e}}_E\ A^{\mathsf{b}}_B \cdot A^{\mathsf{d}}_D$, we may replace

$$f^{\mathsf{cai}} f^{\mathsf{ibj}} f^{\mathsf{jde}} \to \frac{1}{8} \left\{ \left[ \left( f^{\mathsf{cai}} f^{\mathsf{ibj}} f^{\mathsf{jde}} + f^{\mathsf{cei}} f^{\mathsf{idj}} f^{\mathsf{jba}} \right) + (\mathsf{a} \leftrightarrow \mathsf{e}) \right] + (\mathsf{b} \leftrightarrow \mathsf{d}) \right\} \tag{4.36}$$

which vanishes identically.[10] Hence we also have that $S^{\mu}_{P,2d} = 0$.

In sum, we have shown that $S^{\mu}_{P,2} = 0$, leaving $S^{\mu}_P = S^{\mu}_{P,1}$ as given in eq. (4.24)

$$S^{\mu}_P = k^{\mu}_P \left[ g f^{\mathsf{abc}} \sum_{P=A\cup B\cup C} k_A \cdot A^{\mathsf{c}}_C\ A^{\mathsf{a}}_A \cdot A^{\mathsf{b}}_B + \frac{1}{2} g^2 f^{\mathsf{abe}} f^{\mathsf{ecd}} \sum_{P=A\cup B\cup C\cup D} A^{\mathsf{a}}_A \cdot A^{\mathsf{c}}_C\ A^{\mathsf{b}}_B \cdot A^{\mathsf{d}}_D \right]. \tag{4.37}$$

Note that for $P = 123$ the first term is given by $g f^{\mathsf{abc}} (n^{\mu}_{123} + n^{\mu}_{231} + n^{\mu}_{312})$ as written in eq. (3.35), and the second term is absent.

Combining eq. (4.9) with eq. (4.37) we therefore have that the change in the amplitude under the color-factor shift associated with gluon $n$ is

$$\delta_n \mathcal{A}_n = g^2 \alpha_n \varepsilon^{\mu}_n k^{\mu}_P \left[ f^{\mathsf{abc}} \sum_{P=A\cup B\cup C} k_A \cdot A^{\mathsf{c}}_C\ A^{\mathsf{a}}_A \cdot A^{\mathsf{b}}_B + \frac{1}{2} g f^{\mathsf{abe}} f^{\mathsf{ecd}} \sum_{P=A\cup B\cup C\cup D} A^{\mathsf{a}}_A \cdot A^{\mathsf{c}}_C\ A^{\mathsf{b}}_B \cdot A^{\mathsf{d}}_D \right] \tag{4.38}$$

where $P = 12\cdots(n-1)$. Momentum conservation $\sum_{i=1}^{n} p^{\mu}_i = 0$ implies $k^{\mu}_P = -k^{\mu}_n$. Since $\varepsilon_n \cdot k_n = 0$, we have established that the $n$-gluon amplitude is invariant under the color-factor shift associated with gluon $n$. Since the $n$-gluon amplitude is Bose symmetric, it is therefore invariant under a color-factor shift associated with any of the external gluons

$$\delta_a \mathcal{A}_n = 0 \tag{4.39}$$

which is what we set out to prove.

[10] This may most directly be seen by expressing $f^{\mathsf{cai}} f^{\mathsf{ibj}} f^{\mathsf{jde}} = \mathrm{Tr}\left( T^{\mathsf{c}} [T^{\mathsf{a}}, [T^{\mathsf{b}}, [T^{\mathsf{d}}, T^{\mathsf{e}}]]] \right)$ and expanding.



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## 5 Conclusions

We began by reviewing the color-factor symmetry of tree-level amplitudes of the BAS and Yang-Mills theories. This symmetry acts as a momentum-dependent shift on the color factors, leaving the amplitude invariant. The BCJ relations follow as a direct consequence of this symmetry.

Tree-level amplitudes can be obtained from Berends-Giele currents, which are computed recursively. The recursions relation for the currents can be derived from the classical equations of motion of the theory using the color-dressed perturbiner formalism. We used these recursion relations, together with a variety of group theory relations, to prove the invariance of tree-level amplitudes under a color-factor shift. This proof is a (somewhat) easier alternative to the proof of color-factor symmetry using the radiation vertex expansion given in ref. [9], and is amenable to generalization to other theories.

Cheung and Mangan [14] have shown that the color-factor symmetry of the BAS theory, with scalars transforming in the adjoint of $\mathrm{U}(N) \times \mathrm{U}(\tilde{N})$, also applies to the equations of motion of the theory, and that the color-factor invariance of the equations of motion associated with $\mathrm{U}(N)$ is related to the conservation of current of the global symmetry of the Lagrangian under the dual group $\mathrm{U}(\tilde{N})$. It would be interesting to find a similar relation for Yang-Mills and other theories possessing color-kinematic duality.

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