Page curve from non-Markovianity

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ABSTRACT: In this paper, we use the exactly solvable Sachdev-Ye-Kitaev model to address the issue of entropy dynamics when an interacting quantum system is coupled to a non-Markovian environment. We find that at the initial stage, the entropy always increases linearly matching the Markovian result. When the system thermalizes with the environment at a sufficiently long time, if the environment temperature is low and the coupling between system and environment is weak, then the total thermal entropy is low and the entanglement between system and environment is also weak, which yields a small system entropy in the long-time steady state. This manifestation of non-Markovian effects of the environment forces the entropy to decrease in the later stage, which yields the Page curve for the entropy dynamics. We argue that this physical scenario revealed by the exact solution of the Sachdev-Ye-Kitaev model is universally applicable for general chaotic quantum many-body systems and can be verified experimentally in near future.

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1 Introduction

Studying open quantum many-body systems is of fundamental importance for understanding quantum matters and for future applications of quantum technology because all systems are inevitably in contact with environments, and decoherence due to coupling with environments is a major obstacle for future applications of quantum devices [1]. So far, most studies of open quantum systems are limited to either situations in which the systems are small or weakly correlated, or situations that the environment is treated by the Born-Markovian approximation [2, 3]. Little effort has been made on strongly correlated quantum many-body systems coupled to a non-Markovian environment. This is simply because both strong correlation and non-Markovianity are difficult to handle theoretically.

Open systems are also of interest to gravity studies, and the best-known problem is the black hole information paradox [4]. The central issue of the black hole information paradox is whether the black hole evaporation can be considered as undergoing unitary dynamics. If so, the entropy should first increase and then decrease as the black hole evaporates. Such an entropy curve as shown in figure 1(a) is known as the Page curve [5]. Here the entanglement entropy is the entropy of the reduced density matrix of the radiation part $\mathcal{A}$ after tracing out the remaining black hole part $\mathcal{B}$. As shown in figure 1(a), this entanglement entropy reaches the maximum when half of the black hole is evaporated, giving rise to the Page curve. Reproducing the Page curve from gravity theory is a challenging part of the black hole information problem, and progresses have been made recently using the semi-classical gravity calculations [6–10].

Figure 1. (a) Illustration of the Page curve: the entanglement entropy between two sub-systems with length xL and $(1-x)L$. The total length is L and $0 \leq x \leq 1$. (b) Illustration of the setup: an SYK₄ system with N Majorana fermions serves as the system and an SYK₂ system with M Majorana fermions serves as the environment. Here $M \gg N$.

In this paper, we explore the Sachdev-Ye-Kitaev model [11, 12] with random four-Majorana fermions interactions (SYK₄), and this SYK₄ model is coupled to a system with random quadratic Majorana fermions couplings (SYK₂). The SYK₂ part contains a lot more degrees-of-freedom compared with the SYK₄ part such that the SYK₂ part can be viewed as the environment. The motivations for considering such a model are two folds. First, the SYK₄ model is exactly solvable in the large- N limit and its solution gives rise to a strongly correlated non-Fermi liquid state [11]. Recently, techniques related to SYK₄ model has been widely used to construct exact solutions to address open issues of strongly correlated quantum matters [13–31]. Here the situation we explored is also exactly solvable and we can use the solution to understand how a non-Markovian environment affects strongly correlated phases [23–25]. Secondly, the SYK₄ model is holographically dual to the Jackiw-Teitelboim gravity theory in the AdS₂ geometry with a black hole [32–37]. Thus, the entropy dynamics of the SYK₄ system coupled to an environment [22, 26, 31] resembles the black hole evaporation process [6–10] and it will be interesting to study when a Page-like curve can emerge after turning on the coupling between the system and the environment. Remarkably, the main findings of this work bring these two aspects together, that is, we show that the Page curve emerges because of the non-Markovian effect of the environment.

2 Model

The system under consideration is illustrated in figure 1(b) and the total Hamiltonian is given by

$$\begin{aligned} \hat{H} &= \hat{H}_S + \hat{H}_E + \hat{H}_{SE} \\ &= \sum_{i < j < k < l} J^S_{ijkl} \psi_i \psi_j \psi_k \psi_l + i \sum_{a < b} J^E_{ab} \chi_a \chi_b + i \sum_{i,a} V_{ia} \psi_i \chi_a. \end{aligned} \quad (2.1)$$

Here ψ_i ($i = 1, \dots, N$) denotes N Majorana fermions in the system and χ_a ($a = 1, \dots, M$) denotes M Majorana fermions in the environment, with $\{\psi_i, \psi_j\} = \delta_{ij}$ and $\{\chi_a, \chi_b\} = \delta_{ab}$.

Throughout the manuscript, we will use the subscript “S” and “E” to denote the system part and the environment part respectively. $\hat{H}_S$ and $\hat{H}_E$ are then SYK₄ and SYK₂ Hamiltonians. J_{ijkl}^S , J_{ab}^E and V_{ia} are independent random Gaussian variables with variances given by

$$\overline{(J_{ijkl}^S)^2} = \frac{3!J_S^2}{N^3}, \quad \overline{(J_{ab}^E)^2} = \frac{2!J_E^2}{M}, \quad \overline{(V_{ia})^2} = \frac{V^2}{M}. \quad (2.2)$$

Throughout the paper, we take $J_S = 1$ as the energy unit. We focus on the limit $M \gg N \gg 1$ in which the Schwinger-Dyson equation for χ contains no contribution from ψ , justifying that the χ part can be viewed as the environment. Therefore, the Green’s function $G_\chi(\tau) = \langle T_\tau \chi_i(\tau) \chi_i(0) \rangle$ of the environment takes the standard form of the SYK₂ model as [11]

$$G_\chi(\omega) = -\frac{2}{i\omega + i\text{sgn}(\omega)\sqrt{4J_E^2 + \omega^2}}. \quad (2.3)$$

The fact that this Green’s function has frequency dependence means that the environment is treated beyond the Markovian approximation.

We consider the situation in which the system and the environment are initially decoupled, and both are in thermal equilibrium with inverse temperatures β_S and β_E respectively. The initial density matrix is therefore given by $\rho(0) = \frac{1}{Z_S Z_E} e^{-\beta_S \hat{H}_S} \otimes e^{-\beta_E \hat{H}_E}$ with Z_S and Z_E being the corresponding partition functions. Evolving the system with the Hamiltonian eq. (2.1) and tracing out the environment, one obtains the reduced density matrix of the system $\rho_S(t)$ at time t as

$$\rho_S(t) = \text{tr}_E \left[e^{-i\hat{H}t} \rho(0) e^{i\hat{H}t} \right]. \quad (2.4)$$

The corresponding second Rényi entropy $S^{(2)}$ of the system is then given by

$$e^{-S_S^{(2)}(t)} = \text{tr}_S \left[\rho_S(t)^2 \right]. \quad (2.5)$$

Under the disorder replica diagonal assumption, $S_S^{(2)}(t)$ can be expressed in terms of path-integral over bilocal fields. In the large- N limit, the integral is dominated by the saddle point solution, and the entropy can be obtained by evaluating the on-shell action. We leave the details of the derivation into the appendix.

3 Recovering the Markovian results

Below we will first discuss situations where the entropy dynamics of our model can recover the Markovian results. Here, by Markovian results we mean dynamics obtained by solving the following Lindblad master equation [2, 3]

$$\partial_t \hat{\rho} = -i[\hat{H}_S, \hat{\rho}] + \sum_i \kappa_i \left(\hat{L}_i \hat{\rho} \hat{L}_i^\dagger - \frac{1}{2} \{ \hat{L}_i^\dagger \hat{L}_i, \hat{\rho} \} \right). \quad (3.1)$$

By treating the environment with the Markovian approximation, we only need to consider the Hamiltonian of the system and the dissipation operators, without having to explicitly include the environment. To make comparison with our model, we take $\kappa_i = \kappa$ and

$\hat{L}_i = \psi_i$. Similar to previous procedures, we consider the initial thermal density matrix $\rho_S(0) = \frac{1}{Z_S} e^{-\beta_S \hat{H}_S}$, and we then evolve the system with eq. (3.1). The second Rényi entropy can also be represented as a path-integral where the saddle points approximation is applicable. In the Markovian case, the Rényi entropy first grows linearly in time and then saturates to its maximum value $(N/2) \log 2$, forbidding the possibility of any page-like behaviors [38]. For reasons that will become clear below, we consider the large J_E limit and fix V^2/J_E as κ in our model. Our discussions below will then identify the following conditions as sufficient for our model to recover the Markovian results.

- (i) Infinite Environment Temperature. The SYK₂ environment becomes a Markovian one when $\beta_E = 0$. This is because only the two-point function enters the effective action for entropy dynamics, and in the large J_E limit, the Fourier transformation of the real-time Green's function of the environment $G_E^>(t, \beta) = \langle \chi_a(t) \chi_a(0) \rangle_\beta / Z_B$ gives

$$G_E^>(\omega, \beta_E) = \frac{1}{J_E} \frac{1}{1 + e^{-\beta_E \omega}}. \quad (3.2)$$

When $\beta_E = 0$, the second term vanishes and the Green's function of the environment becomes frequency independent, which is equivalent to the Markovian approximation. Under this situation, the standard derivation of the master equation eq. (3.1) through a second-order perturbation theory yields the dissipation strength $\kappa = V^2/J_E$. As one can see in figure 2(a), when β_E is set to zero, the entropy curve becomes independent of κ . This universal curve also coincides with the results from the Markovian approximation.

- (ii) Short Time. With the Markovian approximation, it can be shown by the perturbation theory that the entropy grows linearly at the initial stage, and when $\kappa t \ll 1$ [38, 39], the growth rate derived analytically in the appendix gives

$$\frac{dS_S^{(2)}(t)}{dt} = \kappa N \left(1 - 2G_S^W(0, 2\beta_S) \right). \quad (3.3)$$

Here we have defined the Wightman Green's function for the single SYK₄ model as

$$G_S^W(t, \beta) = \frac{1}{Z_S} \left\langle \psi_i \left(t - i\frac{\beta}{2} \right) \psi_i(0) \right\rangle_\beta. \quad (3.4)$$

For our model, similar perturbative calculation [31] for short-time yields

$$\begin{aligned} \frac{dS_S^{(2)}}{dt} &= 2V^2 N \int_{-t}^t dt' \left[\left(G_S^>(t', 2\beta_S) - G_S^W(t', 2\beta_S) \right) \right. \\ &\quad \left. \times G_E^>(t', \beta_B) \right]. \end{aligned} \quad (3.5)$$

By approximating $t' = 0$ in the integrand, eq. (3.5) becomes

$$\frac{dS_S^{(2)}}{dt} = \frac{V^2}{J_E} N \left(1 - 2G_S^W(0, 2\beta_S) \right). \quad (3.6)$$

Figure 2. (a) Entropy curves for different κ and β_E . Here $\kappa = 0.1$ for solid lines and $\kappa = 0.2$ for dashed lines. Three different $\beta_E = (0, 3, 6)$ are plotted. The red dashed straight line represents the same initial slope for all curves. Two curves with $\beta_E = 0$ in (a) coincide with the Markovian results. (b) Entropy curves with different κ and a fixed $\beta_E = 6$. The curve with large κ in (b) coincides with $\beta_E = 0$ curves in (a) and the Markovian results.

By equalling $V^2/J_E = \kappa$, eq. (3.6) is the same as eq. (3.3). This can also be seen in figure 2(a) that the initial growth is linear in κt and the slope is a constant for varying β_E with fixed β_S . In other words, although the Green's function of the environment eq. (3.2) contains the frequency dependent part, it is not important for initial time and the short-time behavior is always dominated by the frequency independent part.

- (iii) Large System-Environment Coupling. The entropy curve of our model also matches the Markovian results when κ is sufficiently large compared to J_S . Since the short time limit is always Markovian as discussed in ii), here we focus on the long-time limit. Physically, when the coupling between system and environment is strong enough compared with the internal energy scales of the system, all Majorana fermions in the system tend to be maximally entangled with the environment, because the environment contains more degrees of freedom. Consequently, the entropy is expected to saturate to the maximum value $(N/2) \log 2$ in the long-time limit, which is the same as the Markovian case. This can also be shown more rigorously using the path-integral formalism by relating the Rényi entropy to the inner product of Kourkoulou-Maldacena pure states [27, 29]. In figure 2(b), one can see that the entropy curve gradually approaches the Markovian result as κ increases.

Figure 3. The entropy curve for varying β_S with fixed $\kappa = 0.1$ and $\beta_E = 3$. Three different $\beta_S = (2, 3, 8)$ are plotted. Page-like behaviours is guaranteed when $\beta_S = 2$ because the initial entropy is higher than the saturated entropy.

4 The Page curve

Above we have shown that, under these three situations, our model recovers the Markovian results, and the Markovian results do not display the Page curve for entropy dynamics. Hence, to reveal effects beyond the Markovian approximation, the following three conditions should be satisfied simultaneously, which are: i) the bath temperature should not be too high; ii) the evolution time should not be too short; iii) the coupling between system and environment should not be too large. Under these conditions, we find that Page curve for entropy dynamics is often observed, as was shown in figure 3. Thus, this attributes the emergence of the Page curve to the *beyond Markovian* effect. Since we have discussed that the entropy always increases at the initial time, it is essential to understand the decreasing behavior of the entropy curve at long time to understand the Page curve. Below we offer two physical understandings.

The first understanding again relies on perturbation theory. When κ is small, the entropy dynamics can be obtained by doing perturbation in κ , which yields the same perturbative results as eq. (3.5). Here, since we focus on the long-time behavior, we can simply replace t by infinity and the range of integration in eq. (3.5) is set to be $(-\infty, \infty)$. By expressing the Green's functions $G^>$ and G^W in terms of the spectral function ρ [11]

$$G^>(\omega, \beta) = \rho(\omega) \frac{1}{1 + e^{-\beta\omega}}, \quad (4.1)$$

$$G^W(\omega, \beta) = \rho(\omega) \frac{1}{2 \cosh(\beta\omega/2)}, \quad (4.2)$$

and making use of the fact that $\rho(\omega)$ is even in ω , we obtain the following expression:

$$\frac{dS_S^{(2)}}{dt} = 2V^2N \int_0^\infty \left[\frac{d\omega}{2\pi} \frac{\rho_S(\omega, 2\beta_S)\rho_E(\omega, \beta_E)}{2 \cosh \beta_S\omega} \times \frac{(e^{\beta_S\omega} - 1)(1 - e^{(\beta_E - \beta_S)\omega})}{1 + e^{\beta_E\omega}} \right]. \quad (4.3)$$

Note that in eq. (4.3), all terms are positive definite except for the $1 - e^{(\beta_E - \beta_S)\omega}$ term. When the temperature of the environment is lower than the temperature of the system,

$\beta_E > \beta_S$ and $1 - e^{(\beta_E - \beta_S)\omega} < 0$. Therefore, $dS_S^{(2)}/dt < 0$ and the entropy decreases at long time, which yields the Page curve. This gives a sufficient condition for the emergence of the Page curve, that is, κ being small and the environment temperature lower than system temperature, which also agrees with the three aforementioned conditions.

The second understanding replies on inspecting how the system entropy saturates at sufficiently long times. It is reasonable to assume that the system eventually reaches thermal equilibrium with the environment, and since the environment contains much more degrees-of-freedom, the saturation entropy is determined by κ and β_E and is independent of β_S . The saturation entropy is smaller when the environment temperature is lower, which corresponds to a decrease in thermal entropy. When the coupling κ is smaller the saturation entropy is also smaller because this lowers the entanglement entropy. On the other hand, the initial entropy of the system is solely determined by parameters J_S and β_S of the system itself. When this saturation entropy is smaller than the initial entropy, the entropy has to decrease at a later stage, which also leads to a sufficient condition for the emergence of the Page curve.

5 CFT bath and holography

Now we discuss possible gravity interpretation of our results. It is known that the low-energy sector of the SYK₄ model is described by a Schwarzian action [11], which is equivalent to the AdS₂ Jackiw-Teitelboim gravity [32]. Consequently, for $\beta_S J_S \gg 1$, the initial thermal ensemble of the SYK system can be viewed as a Rindler patch of the AdS₂ space-time with gravity dynamics lying on the boundary. On the gravity side, previous works [6–9] usually introduce the bath as a (1+1)-D boundary conformal field theory (bCFT), which is glued to the AdS₂ space-time with a transparent boundary condition.

In our model (2.1), we have chosen an SYK₂ bath for simplicity, which is 0 + 1-D. However, it is known that the SYK₂ model shares a similar behavior to a higher-dimensional Fermi liquid [11]. In particular, we can introduce an alternative bath as in [26]:

$$\hat{H}' = \sum_{i < j < k < l} J_{ijkl}^S \psi_i \psi_j \psi_k \psi_l + \sum_i \left(\Lambda \sum_{x \geq 0} i \chi_{i,x} \chi_{i,x+1} + iV \psi_i \chi_{i,0} \right). \quad (5.1)$$

Now the bath $\chi_{i,x}$ becomes N copies of half-infinite Majorana chains, with $x = 0, 1, 2 \dots$ labeling different sites with lattice constant a . Although (5.1) and (2.1) describe very different microscopic systems, after integrating out the bath χ , we find that the effective action of the system takes the same form (see the first and the third line of (A.8)), and only depends on the bath two-point function. Taking the limit of $\Lambda/J \rightarrow \infty$ with fixed $\Lambda a = 2\pi$, we find that in the new model we have $\rho_E(\omega, \beta_E) \sim 1/\Lambda$, which is also a constant. Consequently, the Majorana chain bath model (5.1) is equivalent to the SYK₂ bath model with $\Lambda = J_E$ and we can identify $\kappa = V^2/\Lambda$.

The holographic interpretation requires that the system can be described purely by the reparametrization modes in the low-energy manifold of the SYK₄ model. This is valid in the regime $\kappa \sqrt{\beta_S/J_S} \ll \alpha_S/\beta_S J_S$ or $\kappa \beta_S^{3/2} J_S^{1/2} \ll 1$ [22], α_S being the coefficient of the

Figure 4. A sketch for (a). the chain model (5.1), and (b) its gravity dual. Here the orange color labels the system, which is an SYK₄ model or a AdS₂ region with gravity dynamics. The blue color labels the bath, described by a Majorana chain or CFT on a flat space-time.

Schwarzian action in the SYK model [11]. Since we expect the system to ultimately thermalize, $\beta_S J_S$ will eventually become small enough to spoil the holographic interpretation if the bath is initially prepared at a high temperature. Nevertheless, this can be avoided by either considering a low-temperature bath or focusing on the short time physics. Under these conditions, (5.1) then admits a direct gravity interpretation. The bulk system contains two regions (see figure 4), one with dynamical gravity, and one with flat space-time. This is similar to the setup in previous works [6–9]. The main difference is that our model contains no CFT degrees of freedom in the gravity region: using the holographic dictionary, the Majorana fermion with scaling dimension $\Delta = 1/4$ is mapped to a two-component Majorana fermion satisfying the Dirac equation with mass $m = 1/2 - \Delta = 1/4$. Fortunately, previous studies find this difference does not change the qualitative behavior of the entropy dynamics [26]. Our work reveals the effect of gluing AdS₂ to the flat space-time beyond Markovian approximation. We find only in the short time limit, the effect of the flat space-time region can be approximated by the Lindblad master equation. During this short period, the entanglement builds up due to free excitations passing through the boundary between the AdS₂ and the flat space-time, long before the thermalization takes place. The Page curve only appears at later time when the Markovian approximation breaks down and the temperature of the bath is low enough. This is consistent with previous studies for CFT matters in bulk [7, 8]. However, in these calculations, the entropy curve for Von Neumann entropy contains a transition for the RT surfaces, which is smoothed in our case. Whether having transitions or crossovers in the entropy dynamics in fact depends on the details of the system-bath coupling, as was noted in [26].

6 Summary

To summarize, we address the issue of when a Page curve can emerge in entropy dynamics of a system coupled to the environment. Although we use SYK model as an exactly solvable model to study this problem, the lesson we learn from our model reveals a general physical picture that should be applicable to generic chaotic quantum many-body systems. This physical picture contains two ingredients. First, at the initial stage, the entropy dynamics is always dominated by the Markovian process which leads to a linear increase of entropy in time. Secondly, a chaotic system thermalizes with the environment in the

long-time limit. After thermalization, a low environment temperature and a weak system-environment coupling respectively suppress the thermal and the entanglement contributions to the system entropy, which ensures a lower system entropy at long time and forces the entropy to decrease at the later stage. The long-time decreasing behavior is essential for the emergence of the Page curve. This long-time behavior is distinct from the Markovian case where the system is often heated to infinite temperature and the long-time steady state is described by a density matrix given by the identity matrix. Therefore, the Page curve is a consequence of the non-Markovian environment. Since the entanglement entropy can now be measured experimentally [40–42] and the coupling to the environment can be also highly controllable, for instance, in ultracold atomic systems, the physical picture revealed in this work can be experimentally verified in near future.

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Note added. When finishing the manuscript, we become aware of a work by Chen in which the Rényi entropy dynamics has been studied in a similar model by the perturbation theory [43].

A Path-integral representation of the Rényi entropy

The method we use to calculate the entropy curve is the path-integral formalism [22, 26–31]. Here we briefly illustrate how this method can be tailored to solve our models.

A.1 Non-Markovian bath

We first write the initial thermal ensemble of the decoupled ψ/χ system with the inverse temperature β_S/β_B as path-integrals. The initial density matrix is

$$\rho(t=0) = \frac{1}{Z_B Z_S} e^{-\beta_S H_S} \otimes e^{-\beta_B H_B} = \frac{1}{Z_B Z_S} \text{ (diagram) } . \quad (\text{A.1})$$

Here the solid arc represents the path integral on the imaginary time contour and the end points represent the bra or the ket. We then evolve the system ρ under the total Hamiltonian eq. (2.1) in the main text, and obtain the reduced density matrix of ψ by tracing out the bath χ . This procedure is represented by the contour below:

$$\rho_S(t) = \frac{1}{Z_B Z_S} \text{tr}_B \left[e^{-iHt} \left(e^{-\beta_S H_S} \otimes e^{-\beta_B H_B} \right) e^{iHt} \right] = \frac{1}{Z_B Z_S} \text{ (diagram) } . \quad (\text{A.2})$$

$f(s)$, $g(l)$ and $p(s)$ are introduced to distinguish the forward or the backward real-time or imaginary-time evolutions. We have $f(s) = 1$, $g(l) = 1$ and $p(s) = 0$ if s or l lies on the imaginary-time evolution and $f(s) = \pm i$, $g(l) = \pm i$ and $p(s) = \pm i$ if s or l lies on the forward/backward real-time evolution. Explicitly, for s we have

$$\begin{aligned} \text{imaginary} \quad & s \in [t, t + \beta_S] \cup [3t + \beta_S, 3t + 2\beta_S], \\ \text{forward} \quad & s \in [t + \beta_S, 2t + \beta_S] \cup [3t + 2\beta_S, 4t + 2\beta_S], \\ \text{backward} \quad & s \in [0, t] \cup [2t + \beta_S, 3t + \beta_S]. \end{aligned} \quad (\text{A.10})$$

For l , we just need to replace β_S by β_B . In the large- N limit, we could take the saddle-point equation. For system ψ , we have

$$\begin{aligned} G_\psi(s, s') &= (\partial_s - \Sigma_\psi)_S^{-1}(s, s'), \\ \Sigma_\psi(s, s') &= J_S^2 G_\psi^3(s, s') f(s) f(s') + V^2 G_\chi(l(s), l(s')) p(s) p(s'), \end{aligned} \quad (\text{A.11})$$

and for the bath χ , we have

$$G_\chi(l, l') = (\partial_l - \Sigma_\chi)_B^{-1}(l, l'), \quad \Sigma_\chi(l, l') = J_B^2 G_\chi(l, l') g(l) g(l'). \quad (\text{A.12})$$

Note that the Schwinger-Dyson equation for the bath χ does not contain the Green's function of ψ , further indicating that the bath is unaffected by the system. Consequently, the Green's function of χ is diagonal in the two replica space and takes the form of the Green's functions on the Keldysh contour.

After solving (A.11), we can compute the on-shell action with eq. (A.8) and obtain the second Rényi entropy

$$\overline{S_S^{(2)}(t)} = \frac{N}{2} \log \left[\det_S G_\psi \right] + \frac{N}{2} \int ds ds' f(s) f(s') \frac{3J_S^2}{4} G_\psi^4 + 2 \log Z_S. \quad (\text{A.13})$$

Here we have used the fact that the contribution from χ cancels $2 \log Z_B$.

A.2 Markovian bath

Similarly, we can consider the evolution described by the master equation eq. (3.1) in the main text. Now the contour can be drawn as

$$\text{tr}_S [\rho_S(t)^2] = \frac{1}{Z_S^2} \left(\text{Diagram} \right), \quad (\text{A.14})$$

where the dotted lines now represent the insertion of the Lindblad operators. Again, we can relate the purity to a path-integral and introduce bi-local fields. We then arrive at

$$Z_S^2 \text{tr}_S [\overline{\rho_S(t)^2}] = \int \mathcal{D}G_\psi \mathcal{D}\Sigma_\psi \exp(-S_{\text{eff}}^{(2)}), \quad (\text{A.15})$$

with the action $S_{\text{eff}}^{(2)}$

$$S_{\text{eff}}^{(2)} = -\frac{N}{2} \log \left[\det_S (\partial_s - \Sigma_\psi) \right] + \frac{N}{2} \int ds ds' \left(\Sigma_\psi G_\psi - f(s) f(s') \frac{J_S^2}{4} G_\psi^4 \right) - \frac{N\kappa}{2} \int ds ds' G_\psi(s, s') g(s, s') + N\kappa t. \quad (\text{A.16})$$

Here $g(s, s') = -g(s', s)$ is an anti-symmetric function. For $s > s'$, it is non-zero when

$$\begin{aligned} g(s, s') &= \delta(\beta + 2t - s - s') & \text{if } s' \in [0, t], \\ g(s, s') &= \delta(3\beta + 6t - s - s') & \text{if } s' \in [\beta_S + 2t, \beta_S + 3t]. \end{aligned} \quad (\text{A.17})$$

The ψ system again satisfies the boundary condition $\psi(0^+) = -\psi(2\beta_S + 4t^-)$ when we compute the entropy. The saddle-point equation now reads

$$\begin{aligned} G_\psi(s, s') &= (\partial_s - \Sigma_\psi)_S^{-1}(s, s'), \\ \Sigma_\psi(s, s') &= J_S^2 G_\psi^3(s, s') f(s) f(s') + \kappa g(s, s'). \end{aligned} \quad (\text{A.18})$$

The second Rényi entropy $\overline{S_S^{(2)}(t)}$ can again be calculated similar to eq. (A.13) as:

$$\overline{S_S^{(2)}(t)} = \frac{N}{2} \log \left[\det_S G_\psi \right] + \frac{N}{2} \int ds ds' f(s) f(s') \frac{3J_S^2}{4} G_\psi^4 + N\kappa t + 2 \log Z_S. \quad (\text{A.19})$$

B Perturbation results

In this section, we elaborate on details of the perturbation theory introduced in the main text. The parameter that controls the perturbation is κt , so our results apply both for small κ and short t . Below we illustrate these two aspects respectively.

B.1 The short-time perturbation

We first consider the case where t is small and illustrate our perturbation scheme. This result is crucial for understanding the universal initial growth in the main text. At $t = 0$, the system is in a thermal ensemble, and $\overline{S_S^{(2)}(0)}$ is given by the thermal entropy with the inverse temperature β_S . When the real-time evolution begins, the system entangles with the bath, and the entropy grows. In this early-time regime, we could treat the coupling to bath perturbatively.

We firstly consider the Markovian case where the early-time result can be computed analytically. To the leading order of κ , we need to evaluate the second line of eq. (A.16) directly using the Green's function at $\kappa = 0$. We then have:

$$\overline{S_S^{(2)}(t)} \approx \overline{S_S^{(2)}(0)} + N\kappa t \left(1 - 2G_\psi^W(0, 2\beta_S) \right). \quad (\text{B.1})$$

This formula works for $\kappa t \lesssim 1$. Here we have defined the Wightman Green's function

$$G_\psi^W(t, \beta) = \frac{1}{Z(\beta)} \left\langle \psi_i \left(t - i\frac{\beta}{2} \right) \psi_i(0) \right\rangle_\beta$$

for a single SYK₄ model. In the low-energy limit $\beta_S J_S \gg 1$, we have $G_\psi^W(0, \beta) = \left(\frac{\pi}{4J_S^2\beta^2}\right)^{1/4}$ and

$$\overline{S_S^{(2)}(t)} \approx \overline{S_S^{(2)}(0)} + N\kappa t \left[1 - \left(\frac{\pi}{J_S^2\beta^2}\right)^{1/4} \right]. \quad (\text{B.2})$$

We now consider the SYK₂ bath. We similarly perform a perturbative calculation in terms of V^2 . We then need to compute

$$\overline{S_S^{(2)}(t)} \approx \overline{S_S^{(2)}(0)} - \frac{N}{2} \int ds ds' V^2 p(s) p(s') G_\psi(s, s') G_\chi(l(s), l(s')), \quad (\text{B.3})$$

using the Green's function with $V = 0$. Instead of using the parameter s , we could now directly focus on a single replica and write the integral in terms of the real-time t :

$$2V^2 N \int_0^t \int_0^t dt_1 dt_2 \left[G_\psi^T(t_{12}, 2\beta_S) G_\chi^T(t_{12}, \beta_B) - G_\psi^W(t_{12}, 2\beta_S) G_\chi^>(t_{12}, \beta_B) \right]. \quad (\text{B.4})$$

Here G^T is the real-time time-ordered Green's function and $G_\chi^>(t, \beta_B) = \langle \chi_i(t) \chi_i(0) \rangle_{\beta_B}$. On the other hand, the perturbation of the partition function should be strictly zero. Then, it yields an equality as

$$2V^2 N \int_0^t \int_0^t dt_1 dt_2 \left[G_\psi^T(t_{12}, \beta_S) G_\chi^T(t_{12}, \beta_B) - G_\psi^>(t_{12}, \beta_S) G_\chi^>(t_{12}, \beta_B) \right] = 0, \quad (\text{B.5})$$

Substituting this equality into eq. (B.4) we obtain

$$2V^2 N \int_0^t \int_0^t dt_1 dt_2 \left[G_\psi^>(t_{12}, 2\beta_S) G_\chi^>(t_{12}, \beta_B) - G_\psi^W(t_{12}, 2\beta_S) G_\chi^>(t_{12}, \beta_B) \right], \quad (\text{B.6})$$

We now focus on the initial time behavior of this perturbation term. Specifically, we take its derivative with respect to t , which gives

$$\frac{d\Delta S_S^{(2)}(t)}{dt} = 2V^2 N \int_{-t}^t dt_{12} \left(G_\psi^>(t_{12}, 2\beta_S) - G_\psi^W(t_{12}, 2\beta_S) \right) G_\chi^>(t_{12}, \beta_B). \quad (\text{B.7})$$

B.2 Long time asymptotic behavior with small κ

The small κ perturbation theory is utilized to illustrate the emergence of the page curves. Revisiting (B.7) we note that there are two limits where we can obtain analytical results. The first limit is when $t \rightarrow 0$, which gives the initial entropy growth. The second limit is when $t \rightarrow \infty$, revealing the asymptotic entropy growth rate. We find that the sign of the asymptotic growth rate only depends on whether the bath temperature is higher or lower than the system temperature. To find the asymptotic growth rate we simply take t to infinity in eq. (B.7). We now show that this expression can be cast into a form in which the temperature dependence is manifest.

Writing the Green's functions in terms of the spectral function with the help of the fluctuation-dissipation theorem, we obtain the following expression

$$\begin{aligned}
 R &= 2V^2 N \int_{-\infty}^{\infty} dt_{12} \left(G_{\psi}^>(t_{12}, 2\beta_S) - G_{\psi}^W(t_{12}, 2\beta_S) \right) G_{\chi}^>(t_{12}, \beta_B) \\
 &= 2V^2 N \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G_{\chi}^>(\omega, \beta_B) \left(G_{\psi}^>(-\omega, 2\beta_S) - G_{\psi}^W(-\omega, 2\beta_S) \right) \\
 &= 2V^2 N \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\rho_{\psi}(\omega, 2\beta_S) \rho_{\chi}(\omega, \beta_B)}{2 \cosh \beta_S \omega} \left(\frac{e^{-\beta_S \omega} - 1}{1 + e^{-\beta_B \omega}} + \frac{e^{\beta_S \omega} - 1}{1 + e^{\beta_B \omega}} \right) \times \frac{1}{2} \\
 &= 2V^2 N \int_0^{\infty} \frac{d\omega}{2\pi} \frac{\rho_{\psi}(\omega, 2\beta_S) \rho_{\chi}(\omega, \beta_B)}{2 \cosh \beta_S \omega} \frac{(e^{\beta_S \omega} - 1)(1 - e^{(\beta_B - \beta_S)\omega})}{1 + e^{\beta_B \omega}}.
 \end{aligned} \tag{B.8}$$

From this expression we can easily see that the sign of the long time derivative is solely determined by $\beta_B - \beta_S$. When the bath temperature is lower than that of the system, the perturbation term decreases linearly in the long time limit. This result is general in the sense that it only requires the coupling between the system and the bath to be linear. The bath can in principle be any SYK-like bath and this would still apply. We leave more complex coupling terms, such as those involving higher order terms, for future discussions.

C Numerical discretization

Equations (A.11) and (A.18) solely determine the Rényi entropy of our system in the large- N limit at any given t . It is however difficult to study them analytically since we do not have time-translational symmetry. Instead, we resort to a numerical study, the details of which will be described below [26–30]. We illustrate it with eq. (A.18) as an example.

We will use an iterative approach to solve eq. (A.18). The contour variable $s \in [0, 2\beta + 4t)$ can be discretized with interval β/L , then both G_{ψ} and Σ_{ψ} become $(2 + 4t')L \times (2 + 4t')L$ matrices, where $t' = t/\beta$. We typically take $L = 80 \sim 160$. After the discretization, the Schwinger-Dyson equation is written as:

$$\begin{aligned}
 (G_{\psi})_{ij} &= ((G_{\psi}^0)^{-1} - \Sigma_{\psi})_{ij}^{-1}, \\
 (\Sigma_{\psi})_{ij} &= J_S^2 (G_{\psi})_{ij}^3 f_i f_j + V^2 (G_{\chi})_{ij} p_i p_j.
 \end{aligned} \tag{C.1}$$

To obtain $(G_{\psi})_{ij}^{-1}$ we need to find the matrix of $(G_{\psi})_{ij}$. Taking into account the boundary condition for the bare Green's function, we have:

$$(G_{\psi}^0)_{ij} = \frac{1}{2} \text{sgn}(i - j), \quad \text{for } i, j \in \{1, 2, \dots, (2\beta + 4t')L\}. \tag{C.2}$$

After the iteration converges, we can plug the solution matrix back into the action and then obtain the entropy value. To further solidify our results we have performed an extrapolation for different L .

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