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Octagon with finite bridge: free fermions and determinant identities

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Abstract: We continue the study of the octagon form factor which helps to evaluate a class of four-point correlation functions in $\mathcal{N} = 4$ SYM theory. The octagon is characterised, besides the kinematical parameters, by a "bridge" of ℓ propagators connecting two non-adjacent operators. In this paper we construct an operator representation of the octagon with finite bridge as an expectation value in the Fock space of free complex fermions. The bridge ℓ appears as the level of filling of the Dirac sea. We obtain determinant identities relating octagons with different bridges, which we derive from the expression of the octagon in terms of discrete fermionic oscillators. The derivation is based on the existence of a previously conjectured similarity transformation, which we find here explicitly.

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Contents

1 Introduction

A new non-perturbative approach for the computation of the correlation functions of single-trace operators in the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory has been developed in the past five years [1–6]. The method, mostly referred to as hexagonalisation, is based on the world-sheet integrability of $\mathcal{N} = 4$ SYM [7]. The hexagonalisation prescribes to decompose the correlation function into elementary blocks called hexagon form factors, or shortly hexagons, which are almost uniquely determined by the huge symmetry of the theory. Being formulated in terms if infinite-volume form factors, the prescription involves divergencies and, in spite of some important progress [8], it still awaits an appropriate regularisation procedure.

Remarkably, a class of four-point functions of half-BPS operators with large R-charges and specially tuned polarisations, discovered in [9, 10], are free of divergencies and can be evaluated exactly for any value of the 't Hooft coupling. In these correlation functions the hexagons couple only pairwise. The composite form factors representing two paired hexagons, named *octagons*, completely factorise. The factorisation was shown to take place in all orders of the $1/N_c$ expansion [11]. If there are ℓ propagators sandwiched between the two hexagons, one speaks of octagon with *bridge* ℓ.

The two constituent hexagons are bound by exchanging virtual particles in the mirror channel. In [12, 13], the octagon was represented as a Fredholm pfaffian and was also given a more tractable representation as the pfaffian of a discrete kernel $\mathbf{K}$ representing a complex semi-infinite anti-symmetric matrix $\mathbf{K}_{m,n}$ with $m, n \geq 0$. It was also conjectured that the octagon kernel can be rotated to a simplified kernel $\mathring{\mathbf{K}}$ which is a real half-sparse matrix and as such can be split into two equivalent diagonal blocks. Based on this conjecture, the pfaffian was expressed as the determinant of one of the blocks. The simplified kernel $\mathring{\mathbf{K}}$ was defined in [13] by its perturbative series and then non-perturbatively in [14–16]. This second representation of the octagon allowed the authors of [14–16] to reformulate the latter as Fredholm determinant of a generalised Bessel kernel, for which powerful mathematical methods have been developed previously. However the existence of a similarity transformation turning $\mathbf{K}$ into $\mathring{\mathbf{K}}$ has not been established. One of the goals of this paper is to construct explicitly such a transformation.[1]

The octagon is the simplest of a family of computable observables in $\mathcal{N} = 4$ SYM, such as the cusp anomalous dimension [17] and the MHV six-gluon amplitude in the collinear [18] and close-to-the origin [19] limits. As emphasised in [19], these objects exhibit similar mathematical structures involving semi-infinite matrices.

In this paper we propose an operator description for the octagon based on a pair of complex fermionic fields, $\psi(x)$ and $\psi^*(x)$, with the holomorphic variable x being the Zhukovsky parametrisation of the rapidities of the virtual particles. Similar descriptions exist for all observables mentioned above. Below we present, for reader's convenience, a short summary of our main results.

The operator formalism proposed here is a Fock space realisation of the description with real fermions presented in [13]. The Fock space for the complex fermions is a direct sum of sectors characterised by the U(1) charge of the vacuum or, in other words, by the level of filling of the Dirac sea. The octagon with bridge ℓ is constructed as the expectation value of a product of exponential operators in the sector of charge ℓ,

$$\mathbb{O}_\ell = \langle\, \ell |\exp\left[\frac{1}{2}\psi\mathbf{K}\psi\right]\ \exp\left[-\frac{1}{2}\psi^*\mathbf{C}\psi^*\right] |\ell\,\rangle, \tag{1.1}$$

[1]While we were working on this manuscript, we learned that Andrey Belitsky and Gregory Korchemsky found another solution for the similarity transformation, to be published as appendix to v2 of [16]. We comment on their solution in our appendix B. The two solutions are related by a transformation which leaves the kernel $\mathbf{K}$ invariant.

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$\mathbf{K}$ is the octagon kernel and $\mathbf{C}$ is a standard quasi-diagonal symplectic matrix. The right exponential imposes non-trivial correlation of the modes of ψ and resembles the operators of boundary states in CFT, hence the notation

$$|\,\ell\,\rangle\!\rangle \stackrel{\text{def}}{=} \exp\left[-\frac{1}{2}\psi^*\mathbf{C}\psi^*\right]|\ell\,\rangle. \tag{1.2}$$

As the two exponents contain either two creation operators, or two annihilation operators, the U(1) charge is not preserved and the expectation value is given by a Fredholm pfaffian [13].

The Fock-space realisation (1.1) gives a nice interpretation of the bridge as an operator composed of the ℓ lowest fermion oscillator modes. Based on this we show that the octagon with non-zero bridge ℓ is obtained by multiplying the octagon with $\ell = 0$ by a pfaffian of a $2\ell \times 2\ell$ matrix of fermionic correlators.

We give an explicit solution for the similarity transformation mentioned above and explore its consequences for the fermionic oscillator model. For any $\ell \geq 0$, the similarity transformation acts only on the oscillators ψ_n, ψ_n^* above the Fermi level, $n \geq \ell$, by a semi-infinite matrix $\mathbf{U}_\ell$,

$$\tilde{\psi}_j = \sum_{k\geq\ell}[\mathbf{U}_\ell]_{jk}\;\psi_k, \quad \tilde{\psi}^*_j = \sum_{k\geq\ell}[\mathbf{U}^{-1}]_{kj}\psi^*_k\,, \quad j \geq \ell\,. \tag{1.3}$$

The canonical transformation (1.3) preserves the matrix $\mathbf{C}$ and transforms the octagon kernel $\mathbf{K}$ into the simplified kernel $\mathring{\mathbf{K}}$. We will give its explicit formula for any ℓ, but what is important is the very fact of its existence. The operator expression for the octagon then takes the form

$$\mathbb{O}_\ell = \langle\,\ell|\exp\left[\frac{1}{2}\psi\mathring{\mathbf{K}}\psi\right]\exp\left[-\frac{1}{2}\psi^*\mathbf{C}\psi^*\right]|\ell\,\rangle. \tag{1.4}$$

Here we replaced $\{\tilde{\psi}, \tilde{\psi}^*\}$ by $\{\psi, \psi^*\}$, as the existence of a transformation (1.3) for any ℓ then guarantees that the vacuum states have the same form for the original and the transformed fermions.

Both the simplified kernel $\mathring{\mathbf{K}}$ and the matrix $\mathbf{C}$ relate only modes of different parity. Thanks to this property, half of the modes in (1.4) can be eliminated and the resulting operator expression is exponential of a fermion bilinear which, unlike the exponential operators in (1.4), preserves the U(1) charge. This leads to the Fredholm determinant formula for the octagon and to finite determinant relations between octagons with different bridges.

For even/odd bridge we expressed the octagon as an expectation value in the Fock space built on the odd/even oscillators, $\psi^{\rm e}_j = \psi_{2j}, \psi^{*\rm e}_j = \psi^*_{2j}$ and $\psi^{\rm o}_j = \psi_{2j+1}, \psi^{*\rm o} = \psi^*_{2j+1}$,

$$\mathbb{O}_\ell = \begin{cases} \langle\,m,\mathrm{o}|\; e^{-\psi^{\rm o}\mathbb{K}^{\rm oo}\psi^{*\rm o}}|m,\mathrm{o}\,\rangle = \det[(1-\mathbb{K}^{\rm oo})_{\geq m}] & \text{if } \ell = 2m, \\ \langle\,m,\mathrm{e}|e^{-\psi^{\rm e}\,\mathbb{K}^{\rm ee}\psi^{*\rm e}}|m,\mathrm{e}\,\rangle = \det[(1-\mathbb{K}^{\rm ee})_{\geq m}] & \text{if } \ell = 2m-1. \end{cases} \tag{1.5}$$

The vacuum states $|m,\mathrm{o}\,\rangle$ and $|m,\mathrm{e}\;>$ in (1.5) are the standard vacuum vectors of charge m respectively for the ensembles of odd and the even oscillator modes. By $\mathbb{K}^{\rm ee}$ and $\mathbb{K}^{\rm oo}$ we denoted respectively the even-even and the odd-odd blocks of the block diagonal product

$\mathring{\mathbf{K}}\mathbf{C} = \mathbb{K}^{\rm ee} \oplus \mathbb{K}^{\rm oo}$ and 1 stays for the identity matrix. Finally, for any semi-infinite matrix $\mathbf{A} = \{\mathbf{A}_{i,j}\}_{i,j\geq 0}$, the symbol $(\mathbf{A})_{\geq m}$ denotes the semi-infinite matrix obtained by deleting the first m rowes and columns, $(\mathbf{A})_{\geq m} = \{\mathbf{A}_{i,j}\}_{i,j\geq m}$. The determinants in (1.5) are equivalent to those formulated in [13, 14], only the matrix elements are indexed differently.

The operator representations in the form (1.5) give rise to $m\times m$ determinant identities, presented in section 4, which relate the octagons with finite bridge $\ell = 2m-1$ or $\ell = 2m$ to the octagon with zero bridge. Hence the ratio $\mathbb{O}_{2m}$ and $\mathbb{O}_0$ as an $m\times m$ determinant,

$$\frac{\mathbb{O}_{2m}}{\mathbb{O}_0} = \det\left[(1+\mathbb{R}^{\rm oo})_{<m}\right], \qquad \frac{\mathbb{O}_{2m-1}}{\mathbb{O}_0} = \det\left[(1+\mathbb{R}^{\rm ee})_{<m}\right], \tag{1.6}$$

where $\mathbb{R}^{\alpha\alpha}$ are the even ($\alpha = \rm e$) and odd ($\alpha = \rm o$) resolvent matrices

$$\mathbb{R}^{\rm ee} = \frac{\mathbb{K}^{\rm ee}}{1-\mathbb{K}^{\rm ee}}, \qquad \mathbb{R}^{\rm oo} = \frac{\mathbb{K}^{\rm oo}}{1-\mathbb{K}^{\rm oo}}, \tag{1.7}$$

and the symbol $(\mathbf{A})_{<m}$ denotes the $m\times m$ diagonal block $\{\mathbf{A}_{i,j}\}_{0\leq i,j\leq m-1}$ of the semi-infinite matrix $\mathbf{A}$. For example,

$$\begin{aligned} &\frac{\mathbb{O}_2}{\mathbb{O}_0} = 1 + \mathbb{R}^{\rm oo}_{0,0}, \qquad \frac{\mathbb{O}_1}{\mathbb{O}_0} = 1 + \mathbb{R}^{\rm ee}_{0,0}, \\ &\frac{\mathbb{O}_3}{\mathbb{O}_0} = (1+\mathbb{R}^{\rm ee}_{0,0})(1+\mathbb{R}^{\rm ee}_{1,1}) - \mathbb{R}^{\rm ee}_{0,1}\mathbb{R}^{\rm ee}_{1,0}, \\ &\frac{\mathbb{O}_4}{\mathbb{O}_0} = (1+\mathbb{R}^{\rm oo}_{0,0})(1+\mathbb{R}^{\rm oo}_{1,1}) - \mathbb{R}^{\rm oo}_{0,1}\mathbb{R}^{\rm oo}_{1,0}. \end{aligned} \tag{1.8}$$

The organisation of the paper is as follows. In section 2 we derive, starting from the expression of the octagon as a sum over virtual particles, the operator representation in terms of fermion oscillators. From the fermionic representation we re-derive the expression for the octagon as semi-infinite pfaffian found in [13] as well as new finite pfaffian formulas relating octagons with different bridges. In section 3 we give an explicit expression for the similarity transformation $\mathbf{U}_\ell$ relating the original and the improved octagon kernels for any ℓ. The details of the proof are relegated to appendices A and C. For $\ell = 0$, we give an alternative exponential expression for the similarity transformation, the derivation of which is presented in appendix D. In section 4 we derive the operator representations (1.5) and the finite determinant formulas that follow from them. Section 5 contains some comments on the results.

2 The octagon from free fermions

2.1 The sum over virtual particles as a Coulomb gas

The role of this subsection is to remind the notations and make the presentation self-consistent. The octagon $\mathbb{O}_\ell = \mathbb{O}_\ell(z,\bar{z},\alpha,\bar{\alpha})$ is characterised by four points $x_1,\ldots,x_4$ in the Euclidean space and four polarisations $y_1,\ldots,y_4$, as well as by the length ℓ of the bridge separating the two hexagons which should be crossed by the virtual particles. The bridge summarises the effect of a stack of ℓ tree-level propagators connecting the operators $\mathcal{O}_1$ and

$\mathcal{O}_4$. The octagon is also a function of the 't Hooft coupling g. The trivial dependence of the (large) R-charges of the four half-BPS operators is factored out. Thanks to the conformal symmetry, the dependence on x_i, y_i is only through the cross ratios in the coordinate and in the flavour spaces

$$\begin{aligned} z\bar{z} &= \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \\ \alpha\bar{\alpha} &= \frac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2}, \quad (1-\alpha)(1-\bar{\alpha}) = \frac{y_{14}^2 y_{23}^2}{y_{13}^2 y_{24}^2}. \end{aligned} \tag{2.1}$$

where $x_{ij}^2 = (x_i - x_j)^2, y_{ij}^2 = (y_i - y_j)^2$ and $y_i^2 = 0$. For the cross ratios in the Euclidean space we adopt the exponential parametrisation

$$z = e^{-\xi + i\phi}, \quad \bar{z} = e^{-\xi - i\phi}, \quad \alpha = e^{\varphi - \xi + i\theta}, \quad \bar{\alpha} = e^{\varphi - \xi - i\theta}. \tag{2.2}$$

The parameters ϕ and ξ, respectively φ and θ, characterise the rotation aligning the two hexagons in the Euclidean, respectively flavour, space. We consider Euclidean metric where the angle ϕ is real. In Minkowski space the angle ϕ should be taken complex, $\phi = \pi + iy$ with y real.

The octagon represents two hexagons glued together by inserting a complete set of virtual states in the Hilbert space associated with the common mirror edge. An n-particle virtual state is characterised by the rapidities u_i and the bound-state numbers a_i of its particles. The contribution of such virtual state factorises into one-particle factors $W_{a_j}(u_j)$ and two-particle interactions $W_{a_j,a_k}(u_j,u_k)$ accounting for the hexagon weights. The octagon thus is expanded as a series of multiple integrals with integrand given by a product of local and bi-local weights [9]

$$\mathbb{O}_\ell = \frac{1}{2}\sum_{\pm}\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{a_1,\dots,a_n\geq 1}\int\prod_{j=1}^{n}\frac{du_j}{2\pi i}\, W^{\pm}_{a_j}(u_j)\, \prod_{j<k}^{n} W_{a_j,a_k}(u_j,u_k). \tag{2.3}$$

Bi-local weights. The bi-local weights are defined in terms of a single function

$$W(u,v) = \frac{x(u) - x(v)}{x(u)x(v) - 1} \tag{2.4}$$

where the function $x(u)$ is defined by the Zhukovsky map

$$u/g = x + 1/x \tag{2.5}$$

transforming the physical sheet in the rapidity plane into the exterior of the unit circle. Namely

$$W_{a,b}(u,v) = W\left(u+\frac{i}{2}a, v+\frac{i}{2}b\right) W\left(u+\frac{i}{2}a, v-\frac{i}{2}b\right) W\left(u-\frac{i}{2}a, v+\frac{i}{2}b\right) W\left(u-\frac{i}{2}a, v-\frac{i}{2}b\right). \tag{2.6}$$

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Local weights. The one-particle factors are

$$W_a^{\pm}(u,\xi) = \frac{1}{g}(-1)^a \chi_a^{\pm}\,\Omega_\ell\left(u+\frac{i}{2}a,\xi\right)\Omega_\ell\left(u-\frac{i}{2}a,\xi\right)\times W\left(u+\frac{i}{2}a,u-\frac{i}{2}a\right). \tag{2.7}$$

where

$$\Omega_\ell(u,\xi) = \frac{1}{x^\ell}\frac{e^{ig\xi\,[x-1/x]}}{x-1/x} = g\,\frac{e^{ig\xi\,[x-1/x]}}{x^\ell}\,\frac{d\log x}{du}\,, \tag{2.8}$$

$\chi_a^{\pm}$ is essentially the character of the a-th antisymmetric representation of $\mathfrak{psu}(2|2)$

$$\chi_a^{\pm}(\phi,\varphi,\theta) = (-1)^a\,\frac{\sin(a\phi)}{\sin\phi}\,\left[2\cos\phi - 2\cosh(\varphi\pm i\theta)\right]. \tag{2.9}$$

For simplicity we will assume that $\theta=0$. The function $\Omega_\ell(u,\xi)$ reflects the form of the momentum and the energy of the mirror magnons as functions of the rapidity u,

$$\begin{aligned}\tilde p_a(u) &= \frac{1}{2}g\left(x-\frac{1}{x}\right)_{u+ia/2} + \frac{1}{2}g\left(x-\frac{1}{x}\right)_{u-ia/2}\,,\\ \tilde E_a(u) &= \log x|_{u+ia/2} + \log x|_{u-ia/2}.\end{aligned} \tag{2.10}$$

The $\mathfrak{psu}(2|2)$ characters are determined by the generating function

$$\mathcal{W}(t) = 1+\sum_{a=1}^{\infty}(-1)^a\chi_a\,e^{-at} = 1-\frac{\cosh\varphi-\cos\phi}{\cosh t-\cos\phi}. \tag{2.11}$$

2.2 Free complex fermions

The fermionic representation we give here was sketched in [20]. Let us first give our conventions, mostly following the conventions of [21], with $\psi_{\rm here} = \psi^*_{\rm there}$, $\psi^*_{\rm here} = \psi_{\rm there}$. The pair of fermionic fields is defined as

$$\begin{aligned}\psi(x) &= \sum_{n\in\mathbb{Z}}\psi_n x^{-n}, \qquad & \psi^*(x) &= \sum_{n\in\mathbb{Z}}\psi^*_n\,x^n,\\ [\psi_m,\psi^*_n]_+ &= \delta_{m,n}, & m,n&\in\mathbb{Z}.\end{aligned} \tag{2.12}$$

The operators ψ_n,ψ^*_n act in the standard fermionic Fock space $\mathcal{H}$, which splits as a sum of Fock spaces with given U(1) charge ℓ,

$$\mathcal{H} = \underset{\ell\in\mathbb{Z}}{\oplus}\mathcal{H}_\ell. \tag{2.13}$$

The Fock space $\mathcal{H}_\ell$ is built on the highest-weight state $|\ell\,\rangle$ and its dual $\langle\ell|$, constructed for $\ell\geq 0$ as

$$\langle\ell| = \langle 0|\prod_{n=0}^{\ell-1}\psi_n, \quad |\ell\rangle = \prod_{n=0}^{\ell-1}\psi^*_n|0\rangle. \tag{2.14}$$

The two vacua satisfy

$$\begin{aligned}\psi^*_n|\ell\rangle &= 0, \qquad \langle\ell|\psi_n = 0 \qquad (n<\ell),\\ \langle\ell|\psi^*_n &= 0, \qquad \psi_n|\ell\rangle = 0 \qquad (n\geq\ell).\end{aligned} \tag{2.15}$$

The non-vanishing correlators are

$$\langle \ell | \psi_m \psi_n^* | \ell \rangle = \begin{cases} \delta_{m,n} & \text{if } m \geq \ell, \\ 0 & \text{if } m < \ell \end{cases} \tag{2.16}$$

and the two-point function is

$$G(x,y) \equiv \langle \ell | \psi(x)\ \psi^*(y) | \ell \rangle_{|y|<|x|} = \frac{(y/x)^\ell}{1 - y/x}. \tag{2.17}$$

The correlation function of a product of fermions is given by the determinant of the two-point correlators.

In [13], the bi-local weights in the expansion (2.3) of section 1 were expressed in terms of the two-point function of the field $\psi(x)$ whose form was postulated. On the present interpretation the two-point function of the field ψ results from replacing the right vacuum by a coherent state[2]

$$|\, \ell \,\rangle\!\rangle \stackrel{\text{def}}{=} e^{-\frac{1}{2}\psi^* \mathbf{C} \psi^*} |\ell\,\rangle, \quad \psi^* \mathbf{C} \psi^* = \sum_{m,n \geq 0} \psi_m^* \mathbf{C}_{mn} \psi_n^* \tag{2.18}$$

where $\mathbf{C}$ is the skew-symmetric matrix with elements

$$\mathbf{C}_{m,n} = \delta_{m+1,n} - \delta_{m,n+1} \qquad (m, n \geq 0). \tag{2.19}$$

For the action of the fermionic oscillators ψ_n on the coherent state one obtains

$$(\psi_m + [\mathbf{C}\psi^*]_m)\, |\, \ell \,\rangle\!\rangle = 0\,, \qquad m \geq \ell\,. \tag{2.20}$$

With the ket vacuum replaced by the coherent state, the ψ-oscillators have a non-vanishing correlation

$$\langle\, \ell | \psi_m \psi_n |\, \ell \,\rangle\!\rangle = \mathbf{C}_{m,n} \tag{2.21}$$

and their two-point function takes the desired form in the x-representation[3]

$$\langle \ell |\ \psi(x)\psi(y) |\, \ell \,\rangle\!\rangle = \sum_{m,n \geq \ell} \mathbf{C}_{mn}\ x^{-m} y^{-n} = (xy)^{-\ell} \frac{x - y}{xy - 1}. \tag{2.22}$$

As in any ensemble of fermions, the $2n$-point correlator is the pfaffian of the matrix of the two-point correlators:

$$\langle\, \ell \,|\psi(x_1) \ldots \psi(x_{2n}) |\, \ell \,\rangle\!\rangle = \operatorname{Pf}\left(\left[\frac{1}{(x_j x_k)^\ell} \frac{x_j - x_k}{x_j x_k - 1} \right]_{i,j=1}^{2n} \right) = \prod_{i=1}^{2n} \frac{1}{x_i^\ell} \prod_{j<k}^{2n} \frac{x_j - x_k}{x_j x_k - 1}. \tag{2.23}$$

[2] I.K. is obliged to Y. Matsuo for a discussion on this way to introduce fermions.

[3] eq. (2.18) gives a fermionic operator realisation of the twisted vertex operators introduced in [13],

$$\langle \ell |\ \psi(x)\psi(y) |\, \ell \,\rangle\!\rangle = \frac{1}{(xy)^\ell} \frac{x - y}{xy - 1} = \langle 0 | : e^{\phi(x)} :: e^{\phi(y)} : e^{-\frac{\ell}{\sqrt{2}}\hat{q}} | 0 \rangle.$$

The r.h.s. represents an expectation on the bosonic vacuum, with $\phi(x) = \frac{1}{\sqrt{2}}(\varphi(x) - \varphi(1/x))$, where $\varphi(x)$ being the standard bosonic oscillator with mode expansion $\varphi(x) = \hat{q} + \hat{p} \log x - \sum_{n \neq 0} \frac{J_n}{n} x^{-n}$ with $[J_n, J_m] = n\delta_{n+m,0}$, $[\hat{p}, \hat{q}] = 1$, and the action of the bosonic oscillators on the bosonic vacua is $\langle\, 0 | J_{n<0} = \langle\, 0 | \hat{q} = J_{>0} | 0 \,\rangle = \hat{p} | 0 \,\rangle = 0$.

Applying (2.23), we can sum up the expansion (2.3). For that we take the fermion in the rapidity plane by replacing $x \to x(u)$ in the expansion (2.12). A virtual particle of type a is represented by the fermion pair $\psi[x(u+ia/2)]\psi[x(u-ia/2)]$. Its expectation value yields the last factor in local weights (2.7). All the bi-local weights are nicely reproduced by the correlation functions of these fermion pairs and the expansion takes the form

$$\begin{aligned}\mathbb{O}_\ell = \sum_{n=0}^{\infty} \frac{g^{-n}}{n!} \sum_{a_1,\ldots,a_n \geq 1} \int \prod_{j=1}^{n} \frac{du_j}{2\pi i} \, (-1)^{a_j} \chi_{a_j} \; \Omega_0\left(u_j + \frac{1}{2} i a_j, \xi\right) \; \Omega_0\left(u_j - \frac{1}{2} i a_j, \xi\right) \\ \times \langle\, \ell \,|\prod_{j=1}^{n} \psi\left[x\left(u_j + \frac{1}{2} i a_j\right)\right] \; \psi\left[x\left(u_j - \frac{1}{2} i a_j\right)\right] |\, \ell \,\rangle\!\rangle.\end{aligned} \tag{2.24}$$

The series (2.24) sums up into an exponential,

$$\begin{aligned}\mathbb{O}_\ell &= \langle\, \ell \,| e^{\frac{1}{2}\psi K \psi} |\, \ell \,\rangle\!\rangle, \\ \psi K \psi &= \frac{2}{g} \sum_{a\geq 1} (-1)^a \chi_a(\phi, \varphi, \theta) \int_{\mathbb{R}} \frac{du}{2\pi i} [\Omega_0 \psi]_{u+ia/2} [\Omega_0 \psi]_{u-ia/2}.\end{aligned} \tag{2.25}$$

By Fourier transformation the summation in a is separated and gives the generating function (2.11) as a function of the Fourier variable t. The Fourier transforms of the two factors in (2.25) are given by integrals over real variables u and v running below and above the real axis respectively. They are transformed into contour integrals in Zhukovsky variables $x(u)$ and $y(v)$ which can be deformed to integrals on the unit circle imposing a bound from below on the t-integration

$$\begin{aligned}\mathbb{O}_\ell &= \langle\, \ell \,| \exp\left(\frac{1}{2} \frac{1}{(2\pi i)^2} \oint \frac{dx}{x} \oint \frac{dy}{y} \; \psi(x) K(x,y) \psi(y)\right) |\ell \,\rangle\!\rangle, \\ K(x,y) &= 2\; e^{ig\xi\,(x - \frac{1}{x} + y - \frac{1}{y})} g \int_{|\xi|}^{\infty} dt \sin\left[gt\left(x + \frac{1}{x} - y - \frac{1}{y}\right)\right] \mathrm{X}(t)\,, \\ X(t) &= \frac{\cos\phi - \cosh\xi}{\cos\phi - \cosh t}\,.\end{aligned} \tag{2.26}$$

In terms of the fermionic oscillators the quadratic form is represented by the semi-infinite matrix $\mathbf{K} = \{\mathbf{K}_{m,n}\}_{m,n\geq 0}$. Using the integration formula

$$\frac{1}{2\pi i} \oint \frac{dx}{x} \; x^{-n} \; e^{ig\xi(x-1/x) \pm igt(x+1/x)} = \left(i \sqrt{\frac{t+\xi}{t-\xi}}\right)^{\pm n} J_n\left(2g\sqrt{t^2 - \xi^2}\right) \; \theta(t \pm \xi). \tag{2.27}$$

where the contour integration goes along the unit circle, the discrete kernel $\mathbf{K}$ can be expressed in terms of Bessel finctions,

$$\mathbb{O}_\ell = \langle\, \ell \,| \exp\left(\frac{1}{2} \sum_{m,n\geq 0} \psi_m \mathbf{K}_{m,n} \psi_n\right) |\ell \,\rangle\!\rangle, \tag{2.28}$$

$$\begin{aligned}\mathbf{K}_{m,n} = \frac{g}{i} \int_{|\xi|}^{\infty} dt \; X(t) \left[\left(i\sqrt{\frac{t+\xi}{t-\xi}}\right)^{m-n} - \left(i\sqrt{\frac{t+\xi}{t-\xi}}\right)^{n-m}\right] \\ \times J_m\left(2g\sqrt{t^2 - \xi^2}\right) J_n\left(2g\sqrt{t^2 - \xi^2}\right)\,,\end{aligned} \tag{2.29}$$

For vacuum states of charge ℓ, the sum in the exponential in (2.28) is effectively restricted to $m, n \geq \ell$.

2.3 Pfaffian formula for the octagon

The computation of the expectation value (2.28) is straightworward and reproduces the pfaffian formula of [13],

$$\mathbb{O}_\ell = \mathrm{Pf}\begin{pmatrix} \mathbf{C}_{\geq\ell} & 1 \\ -1 & -\mathbf{K}_{\geq\ell} \end{pmatrix} = \exp\left(\frac{1}{2}\mathrm{Tr}\log\left[1 - \mathbf{C}_{\geq\ell}\mathbf{K}_{\geq\ell}\right]\right). \tag{2.30}$$

In this expression the semi-infinite matrices $\mathbf{C}_{\geq\ell}$ and $\mathbf{K}_{\geq\ell}$ are obtained from $\mathbf{C}$ and $\mathbf{K}$ by deleting the first ℓ rows and columns.[4] For example, $\mathbf{K}_{\geq\ell} = \{\mathbf{K}_{m,n}\}_{m,n\geq\ell}$. The r.h.s. of (2.30) is defined rigorously by first truncating the semi-infinite matrices $\mathbf{C}_{\geq\ell}$ and , $\mathbf{K}_{\geq\ell}$ to $N \times N$ matrices[5] and then taking the limit $N \to \infty$. The limit is convergent for any finite g because $\mathbf{K}_{m,n}$ decay exponentially when $m, n \to \infty$. A more direct derivation of the pfaffian is based on the formulation of the expectation value as an integral over the Grassmann variables [22],

$$\begin{aligned} \mathbb{O}_\ell &= \int \prod_{m\geq\ell} d\zeta_m d\zeta_m^* \; e^{\mathcal{S}(\zeta,\zeta^*)}, \\ \mathcal{S}(\zeta,\zeta^*) &= -\frac{1}{2}\sum_{m,n\geq\ell} \zeta_m \mathbf{C}_{m,n}\zeta_n + \sum_{n\geq\ell}\zeta_n^*\zeta_n + \frac{1}{2}\sum_{m,n\geq\ell}\zeta_m^*\mathbf{K}_{m,n}\zeta_n^*. \end{aligned} \tag{2.31}$$

2.4 Finite pfaffian relations

Take the operator representation of the octagon with bridge ℓ, eq. (2.28) and consider the right and left vacua as the result of the action of the ℓ lowest fermion oscillators as in eq. (2.14),

$$\mathbb{O}_\ell = \langle\, 0|\psi_1 \dots \psi_{\ell-1}\; e^{\frac{1}{2}\psi\mathbf{K}\psi}\; e^{-\frac{1}{2}\psi^*\mathbf{C}\psi^*}\; \psi^*_{\ell-1}\dots\psi^*_0|0\,\rangle. \tag{2.32}$$

Hence one can obtain $\mathbb{O}_\ell$ by inserting in the expectation value for $\mathbb{O}_0$ an operator creating ℓ pairs of fermions,

$$\mathbb{O}_\ell = \langle\, 0|e^{\frac{1}{2}\psi\mathbf{K}\psi}\; B_\ell\; |\,0\,\rangle\!\rangle, \quad B_\ell = \psi_1\dots\psi_{\ell-1}\psi^*_{\ell-1}\dots\psi^*_0. \tag{2.33}$$

This can be used to derive an expression for the octagon with bridge ℓ in terms of the expectation value of the operator B_ℓ,

$$\frac{\mathbb{O}_\ell}{\mathbb{O}_0} = \langle\, B_\ell\,\rangle, \tag{2.34}$$

where the expectation value of an operator $\mathcal{O}$ is defined as

$$\langle\,\mathcal{O}\,\rangle \overset{\text{def}}{=} \frac{\langle\, 0|e^{\frac{1}{2}\psi\mathbf{K}\psi}\;\mathcal{O}\;|0\,\rangle\!\rangle}{\langle\, 0|e^{\frac{1}{2}\psi\mathbf{K}\psi}\;|0\,\rangle\!\rangle}. \tag{2.35}$$

[4] Of course $\mathbf{C}_{\geq\ell}$ and $\mathbf{C}$ are identical as matrices, but considered as functions of two discrete variables they are related by a shift by ℓ in both arguments.

[5] If the semi-infinite matrix is truncated to a $N \times N$-dimensional matrix, there will be an extra sign factor $(-1)^{N(N-1)/2}$ multiplying the pfaffian.

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As any expectation value of free fermions, $\langle B_\ell \rangle$ is equal to the pfaffian of the two-point correlation functions of the fermions involved. A direct calculation gives, for $j,k = 0,1,2,\ldots$,

$$\begin{aligned} \langle\, \psi_j^* \psi_k^* \,\rangle &= -[\mathbf{K}(1+\mathbf{R})]_{j,k}, \\ \langle\, \psi_j \psi_k^* \,\rangle &= [1+\mathbf{R}]_{j,k}, \\ \langle\, \psi_j^* \psi_k \,\rangle &= -[1+\mathbf{R}]_{k,j}, \\ \langle\, \psi_j \psi_k \,\rangle &= [(1+\mathbf{R})\mathbf{C}]_{j,k} \end{aligned} \tag{2.36}$$

where

$$\mathbf{R} = \frac{\mathbf{CK}}{1-\mathbf{CK}}. \tag{2.37}$$

The matrix of all correlators is the inverse of the quadratic form in the representation as integral over grassman variables, as it should,

$$\begin{pmatrix} (1+\mathbf{R})\mathbf{C} & 1+\mathbf{R} \\ -(1+\mathbf{R})^{\mathrm{T}} & -\mathbf{K}(1+\mathbf{R}) \end{pmatrix} = \begin{pmatrix} -\mathbf{K} & -\mathbf{I} \\ \mathbf{I} & \mathbf{C} \end{pmatrix}^{-1}. \tag{2.38}$$

Now we can express the ratio $\mathbb{O}_\ell/\mathbb{O}_0$ as an $2\ell \times 2\ell$ pfaffian

$$\frac{\mathbb{O}_\ell}{\mathbb{O}_0} = (-1)^{\frac{\ell(\ell-1)}{2}} \operatorname{Pf}\left[\begin{pmatrix} (1+\mathbf{R})\mathbf{C} & 1+\mathbf{R} \\ -(1+\mathbf{R})^{\mathrm{T}} & -\mathbf{K}(1+\mathbf{R}) \end{pmatrix}_{<\ell}\right]. \tag{2.39}$$

Here we introduced the symbol $\mathbf{X}_{<\ell}$ which represents the truncation of the semi-infinite matrix $\mathbf{X}$ to an $\ell \times \ell$ matrix $\{\mathbf{X}_{m,n}\}_{0\le m,n<\ell}$. The truncation is applied to all four blocks of the matrix.

We have checked the finite pfaffian relation (2.39) for $\ell \le 4$ up to g^{16}. As another-consistency check let us consider the limit $\ell \to \infty$ where $\mathbb{O}_\ell \to 1$. Then after taking into account (2.38), the identity (2.39) reproduces the original pfaffian formula (2.30) for $\ell = 0$.

An obvious generalisation of (2.39) relates two octagons with bridges $\ell < \ell_1$,

$$\frac{\mathbb{O}_{\ell_1}}{\mathbb{O}_\ell} = (-1)^{\frac{(\ell_1-\ell)(\ell_1-\ell-1)}{2}} \operatorname{Pf}\left[\begin{pmatrix} (1+\mathbf{R}_{\ge\ell})\mathbf{C}_{\ge\ell} & 1+\mathbf{R}_{\ge\ell} \\ -1-\mathbf{R}_{\ge\ell}^{\mathrm{T}} & -\mathbf{K}_{\ge\ell}(1+\mathbf{R}_{\ge\ell}) \end{pmatrix}_{<\ell_1}\right]. \tag{2.40}$$

In particular, the octagons with bridges ℓ and $\ell+1$ are related as

$$\frac{\mathcal{O}_{\ell+1}}{\mathcal{O}_\ell} = \left[\frac{1}{1-\mathbf{C}_{\ge\ell}\mathbf{K}_{\ge\ell}}\right]_{\ell,\ell} \tag{2.41}$$

which provides a factorised form for the relation (2.39).

3 The similarity transformation

3.1 The original and the simplified octagon kernels

In this section we give explicit expression for the similarity transformation relating the original and the simplified octagon kernels, which corresponds to the canonical transformation (1.3) of the fermion oscillators. The operator representation based on the new set

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of oscillators has the advantage that it preserves the U(1) charge and therefore leads to a determinant instead of a pfaffian.

In this subsection we remind the definition of the two kernels. It is convenient to change the variables in (2.29) as

$$\xi \equiv \frac{\sigma}{g}, \quad t \equiv \frac{1}{g}\sqrt{\tau^2+\sigma^2}. \tag{3.1}$$

so that the integration now spreads on the whole positive real axis and the dependence on the 't Hooft coupling is carried only by the weight function X. In the new variables, the weight function takes the form

$$\chi(\tau,\sigma) \equiv X\left(\frac{\sqrt{\tau^2+\sigma^2}}{g}\right) = \frac{\cos\phi - \cosh\varphi}{\cos\phi - \cosh\frac{\sqrt{\tau^2+\sigma^2}}{g}} \tag{3.2}$$

and the integral formula for the matrix elements (2.29) becomes

$$\mathbf{K}_{m,n} \;= 2\int_0^\infty d\tau\;\chi(\tau,\sigma)\;\mathcal{K}_{m,n}(\tau;\sigma) \qquad (m,n\geq 0), \tag{3.3}$$

with

$$\mathcal{K}_{m,n}(\sigma,\tau) = \Pi_{m-n}(\sigma/\tau)\;J_m(2\tau)\;J_n(2\tau), \tag{3.4}$$

$$\Pi_n(z) \;\stackrel{\text{def}}{=}\; \frac{i^n\left(\sqrt{z^2+1}+z\right)^n - i^{-n}\left(\sqrt{z^2+1}-z\right)^n}{2i\sqrt{z^2+1}} = -\Pi_{-n}(z). \tag{3.5}$$

Importantly, Π_n is a polynomial,

$$\Pi_0(z)=0,\;\;\Pi_1(z)=1, \Pi_2(z)=2iz,\;\;\Pi_3(z)=-1-4z^2,\;\;\text{etc.} \tag{3.6}$$

It equals the $(n-1)$-th Chebyshev polynomial of second kind with imaginary argument. We give the explicit expression for the coefficients of this polynomial, which will be needed in the following,

$$\Pi_n(z) = U_{n-1}(iz) = \sum_{p=0}^{n-1}\sin\pi\tfrac{n-p}{2}\;A_n^{(p)}\;\frac{(-iz)^p}{p!} \qquad\qquad (n\geq 1) \tag{3.7}$$

with

$$A_n^{(p)} = (-2)^p\frac{\Gamma\left[\frac{1}{2}(n+1+p)\right]}{\Gamma\left[\frac{1}{2}(n+1-p)\right]}. \tag{3.8}$$

Summarising, the integrand in (3.3) is given by a sum of products of Bessel functions,

$$\mathcal{K}_{m,n}(\tau;\sigma) = \sum_{j=0}^{|m-n|-1}\frac{1-(-1)^{m-n-j}}{2}\;A_{m-n}^{(j)}\;\frac{(i\sigma)^j}{j!}\;\frac{i^{m-n-1+j}\,J_m(2\tau)\;J_n(2\tau)}{\tau^j}\,. \tag{3.9}$$

The octagon kernel (3.3) depends on the cross ratios of the spacetime coordinates (the parameters σ and ϕ) through the weight function χ and also through the polynomials

$\Pi_{m-n}(\sigma/\tau)$. It was noticed [13] that the second dependence is redundant in the sense that only the constant terms $\Pi_{m-n}(0) = \sin(\frac{m-n}{2}\pi)$ of these polynomials contribute. Based on this observation, it was conjectured that in the pfaffian formula (2.30), the kernel $\mathbf{K}$ can be replaced with a simplified kernel whose matrix elements $\mathring{\mathbf{K}}_{m,n}$ are real and vanish if m and n have the same parity. The last property implies that the pfaffian (2.30) can be written as a determinant. The simplified kernel was found as a perturbative series in [13] and in integral form in [15],

$$\begin{aligned}\mathring{\mathbf{K}}_{m,n} &= 2\int_0^\infty d\tau\ \chi(\tau,\sigma)\ \mathring{\mathcal{K}}_{m,n}(\tau) \qquad (m,n\geq 0)\\ \mathring{\mathcal{K}}_{m,n}(\tau) &= \sin\left(\frac{m-n}{2}\pi\right)\ J_m(2\tau)\ J_n(2\tau),\end{aligned} \tag{3.10}$$

where $\chi(\tau,\sigma)$ is the weight function defined in (3.2).

The conjecture of [13] states, with the interpretation of the bridge we adopted here, that for any $\ell \geq 0$, the matrices $\mathbf{C}_{\geq\ell}\mathbf{K}_{\geq\ell}$ and $\mathbf{C}_{\geq\ell}\mathring{\mathbf{K}}_{\geq\ell}$ are related by a similarity transformation. (We remind that $\mathbf{X}_{\geq\ell}$ denotes the matrix $\mathbf{X}$ with its first ℓ rows and columns deleted.) This is equivalent to claiming that there exists a symplectic transformation preserving $\mathbf{C}$ and relating $\mathbf{K}_{\geq\ell}$ and $\mathring{\mathbf{K}}_{\geq\ell}$,

$$\mathbf{C}_{\geq\ell}\mathbf{K}_{\geq\ell} = \mathbf{U}_\ell^{-1}\mathbf{C}_{\geq\ell}\mathring{\mathbf{K}}_{\geq\ell}\mathbf{U}_\ell \quad\Leftrightarrow\quad \begin{cases}\mathbf{K}_{\geq\ell} = \mathbf{U}_\ell^{\mathrm{T}}\mathring{\mathbf{K}}_{\geq\ell}\mathbf{U}_\ell,\\ \mathbf{C}_{\geq\ell} = \mathbf{U}_\ell\mathbf{C}_{\geq\ell}\mathbf{U}_\ell^{\mathrm{T}}.\end{cases} \tag{3.11}$$

In terms of the ensemble of fermions, the above statements mean, first, that the operators in the expectation values (1.1) and (1.4) are related by the canonical transformation (1.3), and second, that the canonical transformation in question leaves the bra and ket vacua of charge ℓ invariant.

3.2 Explicit solution for the similarity transformation as a power series

The solution for the matrix $\mathbf{U}_\ell$ in (3.11) is not unique. We found a particular solution of the first equation (3.11) in the form of a power series in σ,

$$\mathbf{U}_\ell = \sum_{p=0}^\infty \frac{(-i\sigma)^p}{p!}\ \left(\mathbf{C}_{\geq\ell}\mathbf{M}_{\geq\ell}\right)^p \mathbf{Q}_\ell^{(p)} \tag{3.12}$$

where the diagonal matrices $\mathbf{M}_{\geq\ell}$ and $\mathbf{Q}_\ell^{(p)}$ are defined as[6]

$$\mathbf{M}_{\geq\ell} = \mathrm{diag}\left\{\frac{1}{n}\right\}_{n\geq\ell} = \mathrm{diag}\left\{\frac{1}{\ell},\frac{1}{\ell+1},\frac{1}{\ell+2},\dots\right\} \tag{3.13}$$

$$\mathbf{Q}_\ell^{(p)} = \mathrm{diag}\left\{\theta_{n+1-p-\ell}\ \left(\alpha_{n-\ell-p}\ A_{n-\ell}^{(p)} + \alpha_{n-1-\ell-p}(-1)^p\ B_{n-\ell}^{(p)}\right)\right\}_{n\geq\ell}. \tag{3.14}$$

Here

$$\alpha_k \equiv \frac{1}{2}(1-(-1)^k), \qquad \theta_k = \begin{cases}0 & \text{if } k\leq 0,\\ 1 & \text{if } k>0\end{cases} \tag{3.15}$$

[6]The lowest matrix element $[\mathbf{M}_{\geq\ell}]_{\ell,\ell}$ is singular for $\ell=0$, but it does not appear neither in (3.12) nor in the matrix relations further on.

the coefficients $A_n^{(p)}$ are defined above in (3.8), and the coefficients $B_m^{(p)}$ are given by

$$B_m^{(p)} = m(-1)^{p-1} A_m^{(p-1)} = \frac{2^{p-1} m \Gamma\left(\frac{1}{2}(m+p)\right)}{\Gamma\left(\frac{1}{2}(m+2-p)\right)} . \tag{3.16}$$

These coefficients appear in the Taylor expansions

$$\frac{\left(\sqrt{z^2+1}-z\right)^m}{\sqrt{z^2+1}} = \sum_{p\geq 0} A_m^{(p)} \frac{z^p}{p!}, \qquad \left(\sqrt{z^2+1}+z\right)^m = \sum_{p\geq 0} B_m^{(p)} \frac{z^p}{p!} . \tag{3.17}$$

For fixed $m, n \geq \ell$, the matrix element $(\mathbf{U}_\ell)_{m,n}$ is a polynomial in σ of degree $n-\ell$ for $m-\ell$ even, or of degree $n-1-\ell$ for $m-\ell$ odd. The coefficients of this polynomial depend explicitly on the bridge length ℓ. The lowest matrix elements $\ell \leq m, n \leq \ell+3$ are

$$\mathbf{U}_\ell = \begin{pmatrix} 1 & \frac{i\sigma}{\ell+1} & -\frac{2\sigma^2}{(\ell+1)(\ell+2)} & -\frac{4i\sigma^3}{(\ell+1)(\ell+2)(\ell+3)} & * \\ 0 & 1 & \frac{2i\sigma}{\ell+2} & -\frac{4\sigma^2}{(\ell+2)(\ell+3)} & * \\ 0 & -\frac{i\sigma}{\ell+1} & 1+\frac{4\sigma^2}{(\ell+1)(\ell+3)} & \frac{3i\sigma(\ell+1)(\ell+4)+12i\sigma^2}{(\ell+1)(\ell+3)(\ell+4)} & * \\ 0 & 0 & -\frac{2i\sigma}{\ell+2} & 1+\frac{8\sigma^2}{(\ell+2)(\ell+4)} & * \\ 0 & 0 & -\frac{2\sigma^2}{(\ell+2)(\ell+3)} & -\frac{3i\sigma\left(4\sigma^2+\ell(\ell+7)+10\right)}{(\ell+2)(\ell+3)(\ell+5)} & * \\ 0 & 0 & 0 & * & * \end{pmatrix} . \tag{3.18}$$

We give the idea of the derivation of the symplectic transformation in appendix C. The proof is based on a linear relation between $\mathbf{K}_{\geq\ell}$ and $\mathring{\mathbf{K}}_{\geq\ell}$,

$$\begin{aligned} \mathbf{K}_{m,n} &= \sum_{k=0}^{m-n-1} \frac{(i\sigma)^k}{k!} \alpha_{m-n-k} \; A_{m-n}^{(k)} [(\mathbf{MC})^k \mathbf{K}^\circ]_{m,n} && (m>n) \\ \mathbf{K}_{m,n} &= -\sum_{k=0}^{n-m-1} \frac{(-i\sigma)^k}{k!} \alpha_{m-n-k} \; A_{n-m}^{(k)} [(\mathbf{K}^\circ (\mathbf{CM})^k]_{m,n} && (m<n), \end{aligned} \tag{3.19}$$

which follows from the expansion (3.9) and the recurrence relation for the Bessel functions

$$J_{m+1}(2\tau) + J_{m-1}(2\tau) = m \, \frac{J_m(2\tau)}{\tau}, \tag{3.20}$$

see appendix A. Concerning the second relation in (3.11), we checked that it is satisfied by the series (3.12) for the first several orders in σ, but we do not know how to prove it analytically in general. In the next subsection we give another form of the solution (3.12) for $\ell = 0$, for which this property comes out naturally.

As we mentioned before, the similarity transformation is not unique, and another solution was independently obtained by Belitsky and Korchemsky [16]. In appendix B we re-derive their result as a solution of an ordinary differential equation describing the operator flow connecting $\mathbf{K}$ and $\mathring{\mathbf{K}}$.

3.3 Exponential form of the solution for $\ell = 0$

When $\ell = 0$, the solution (3.12) for the similarity transformation can be written in a quasi exponential form,

$$\mathbf{U}_{\ell=0} = \sum_{j=0}^{\infty} \Big[\Big(\mathbf{P}_{\rm e}\; e^{-\frac{1}{2}\sigma^2 \mathbf{CMS}} + \mathbf{P}_{\rm o}\; e^{-\frac{1}{2}\sigma^2 \mathbf{SCM}} \Big)\, e^{i\sigma \mathbf{C}} \Big]_j \mathbf{P}^{(j)}. \tag{3.21}$$

Here $[\ldots]_j$ denotes the coefficient of the power σ^j in the expansion of the expression in the brackets, $\mathbf{M} \equiv \mathbf{M}_{\ell=0}$ is given by (3.13), the matrix $\mathbf{S}$ is defined as

$$\mathbf{S}_{m,n} = \delta_{m+1,n} + \delta_{m,n+1} \qquad (m, n \geq 0)\,, \tag{3.22}$$

$\mathbf{P}^{(j)}$ is as in (3.14) the projector to the matrices with the first j columns vanishing,

$$\mathbf{P}^{(j)} = \mathrm{diag}\{\theta_{n+1-j}\}_{n\geq 0}\,, \tag{3.23}$$

and $\mathbf{P}_{\rm e}$ and $\mathbf{P}_{\rm o}$ are the projectors respectively to the even and odd subsets,

$$\mathbf{P}_{\rm e} = \mathrm{diag}\{\alpha_{m+1}\}_{m\geq 0}\,, \quad \mathbf{P}_{\rm o} = \mathrm{diag}\{\alpha_m\}_{m\geq 0} \tag{3.24}$$

with α_k given by (3.15).

To get some intuition on the origin of the two exponential factors in (3.21), let us write the simplified kernel for $\ell = 0$ in x-representation,

$$\mathring{K}(x, y) = \sum_{m,n\in\mathbb{Z}} x^m y^n\, \mathring{\mathbf{K}}_{m,n} = 2 \int_0^{\infty} d\tau\; \chi(\tau, \sigma) \sin\Big[\Big(x + \frac{1}{x} - y - \frac{1}{y} \Big) \tau \Big]\,, \tag{3.25}$$

and compare it with the original kernel (2.26), written in terms of the variables (3.1),

$$K(x,y) = 2\; e^{i\sigma\,(x-\frac{1}{x}+y-\frac{1}{y})} \int_0^{\infty} d\tau\; \chi(\tau,\sigma)\; \frac{\tau}{\sqrt{\tau^2+\sigma^2}} \sin\Big[\Big(x+\frac{1}{x}-y-\frac{1}{y}\Big)\sqrt{\tau^2+\sigma^2}\Big]. \tag{3.26}$$

The first expression is obtained from the second by setting $\sigma = 0$ everywhere but in the factor $\chi(\tau, \sigma)$. In (3.21), the right exponential factor $e^{i\sigma\mathbf{C}}$ accounts for the factor $e^{i\sigma\,(x-1/x+y-1/y)}$ in (3.26). Indeed, in x-representation, the operator $\mathbf{C}$ acts as a multiplication by $x - 1/x$. The second factor in (3.26) originates from the σ^2-dependence of the integrand of (3.26). The latter expands as a series in σ^2, with the constant term given by the integrand of (3.25).

4 Determinant identities

Since in the simplified kernel (3.10) the matrix elements with the same parity vanish, the $2\ell \times 2\ell$ pfaffians in the finite pfaffian formulas obtained in section 2.4 can now be written as $\ell \times \ell$ determinants. It turns out that these determinants can be simplified further and written as determinants of approximately twice less size. More precisely, for $\ell = 2m - 1$ and $\ell = 2m$, the ratio $\mathbb{O}_\ell/\mathbb{O}_0$ is an $m \times m$ determinant.

To obtain the reduced determinant identities, we first notice that in the Fock space representation (1.4) the exponents are bilinear forms of the even and odd modes,

$$\mathbb{O}_\ell = \langle\, \ell \,|\exp\left(\sum_{j,k\geq 0} \psi_{2j+1}\mathring{\mathbf{K}}_{2j+1,2k}\,\psi_{2k}\right)\ \exp\left(-\sum_{j,k\geq 0}\psi^*_{2j}\mathbf{C}_{2j,2k+1}\,\psi^*_{2k+1}\right)|\ell\,\rangle. \tag{4.1}$$

We will show that, depending on the parity of ℓ, one can eliminate either the even or the odd modes from the expectation value taking into account the identification (2.20).

4.1 Even bridge

Let us assume that the length of the bridge is even, $\ell = 2m$. In the operator expression (4.1), we can commute all the even modes of ψ^* to the left and all the even modes of ψ to the right until they both are annihilated by the corresponding vacua. As a result we obtain an operator expression only in terms of the odd modes,

$$\begin{aligned}\mathbb{O}_{\ell=2m} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\sum_{\substack{j_1,\dots,j_n\geq m\\ k_1,\dots,n_n\geq m}}\langle\,\ell\,|\prod_{a=1}^{n}\psi_{2j_a+1}\prod_{a,b=1}^{n}\mathbb{K}^{\text{oo}}_{j_a,k_b}\prod_{b=1}^{n}\psi^*_{2k_b+1}|\ell\,\rangle\\ &= \langle\,\ell|\,\mathring{\mathring{}}\,\exp\left(-\sum_{k,j\geq 0}\psi_{2j+1}\,\mathbb{K}^{\text{oo}}_{j,k}\,\psi^*_{2k+1}\right)\mathring{\mathring{}}\,|\ell\,\rangle.\end{aligned} \tag{4.2}$$

In the last line $\mathring{\mathring{}}\ \mathring{\mathring{}}$ denotes the anti-normal ordering where all ψ^* are on the right of all ψ. By $\mathbb{K}^{\text{oo}}$ we denoted odd-odd diagonal block of the matrix $\mathring{\mathbf{K}}\mathbf{C}$,

$$\mathbb{K}^{\text{oo}}_{j,k} \stackrel{\text{def}}{=} [\mathring{\mathbf{K}}\mathbf{C}]_{2j+1,2k+1}\,, \tag{4.3}$$

whose matrix elements are given explicitly by

$$\mathbb{K}^{\text{oo}}_{i,j} = 2(2j+1)\,(-1)^{i-j}\int_0^\infty d\tau\ \chi(\tau,\sigma)\ \frac{J_{2i+1}(2\tau)\ J_{2j+1}(2\tau)}{\tau}\qquad (i,j\geq 0). \tag{4.4}$$

To obtain the r.h.s. of (4.2) we used the identity

$$\left(\mathring{\mathbf{K}}_{\geq\ell}\mathbf{C}_{\geq\ell}\right)_{2j+1,2k+1} = \left([\mathring{\mathbf{K}}\mathbf{C}]_{\geq\ell}\right)_{2j+1,2k+1}\qquad (\ell = 2m) \tag{4.5}$$

which follows from the fact that the matrix $\mathbf{C}$ is quasi-diagonal. Evaluating the expectation value with the correlators (2.16), we obtain the determinant formula for the octagon

$$\mathbb{O}_{\ell=2m} = \det\left[(1-\mathbb{K}^{\text{oo}})_{\geq m}\right]. \tag{4.6}$$

Now we would like to evaluate the ratio of the octagons with $\ell = 2m$ and $\ell = 0$ as an expectation value, as in section 2.4. The identity (4.5) also guarantees that the even fermion modes in the vacuum states can be removed without altering the result,

$$\begin{aligned}|\ell\,\rangle &\to \psi^*_{2m-1}\psi^*_{2m-3}\cdots\psi^*_1|0\,\rangle \equiv |m,\text{o}\,\rangle,\\ \langle\,\ell| &\to \langle\,0|\psi_1\psi_3\cdots\psi_{2m-1} \equiv \langle\,m,\text{o}|\end{aligned}\qquad (\ell = 2m,\ m\geq 1). \tag{4.7}$$

We express the octagon with bridge $\ell = 2m$ as the result of the insertion of m pairs of odd fermionic modes, and divide by the octagon with bridge zero,

$$\begin{aligned}\frac{\mathbb{O}_{2m}}{\mathbb{O}_0} &= \frac{\langle 0|\psi_1\psi_3\cdots\psi_{2m-1} \,\mathring{\circ}\, e^{\psi\mathbb{K}^{\text{oo}}\psi^*} \mathring{\circ}\, \psi^*_{2m-1}\psi^*_{2m-3}\cdots\psi^*_1|0\rangle}{\langle 0|\mathring{\circ}\, e^{\psi\mathbb{K}^{\text{oo}}\psi^*}\mathring{\circ}|0\rangle} \\ &\equiv \langle \prod_{j=0}^{m-1}\psi_{2j+1}\prod_{j=0}^{m-1}\psi^*_{2j+1}\rangle.\end{aligned} \tag{4.8}$$

The expectation value is equal to the determinant of the two-point correlators

$$\langle \psi_{2j+1}\psi^*_{2k+1}\rangle = \delta_{j,k} + \mathbb{R}^{\text{oo}}_{j,k}, \tag{4.9}$$

where the semi-infinite matrix $\mathbb{R}^{\text{oo}}$ is related to $\mathbb{K}^{\text{oo}}$ by

$$(1+\mathbb{R}^{\text{oo}})(1-\mathbb{K}^{\text{oo}}) = 1. \tag{4.10}$$

Hence the ratio $\mathbb{O}_{2m}$ and $\mathbb{O}_0$ is an $m\times m$ determinant,

$$\frac{\mathbb{O}_{2m}}{\mathbb{O}_0} = \det\left[(1+\mathbb{R}^{\text{oo}})_{<m}\right]. \tag{4.11}$$

Since $\mathbb{O}_\ell \to 1$ when $\ell\to\infty$, eq. (4.11) reproduces in the large ℓ limit the determinant formula for the octagon with zero bridge,

$$\frac{1}{\mathbb{O}_0} = \det\left[1+\mathbb{R}^{\text{oo}}\right] = \frac{1}{\det[1-\mathbb{K}^{\text{oo}}]}. \tag{4.12}$$

Since we never used the specific form of $\mathbb{K}^{\text{oo}}$, the identity (4.11) is in fact an identity in the linear algebra.[7] Namely, for any non-singular matrix A,

$$\frac{\det\left[A_{\geq k}\right]}{\det A} = \det\left[\{(A^{-1})_{i,j}\}_{i,j=0,\dots,k-1}\right], \qquad A = \{A_{i,j}\}_{i,j=0,\dots,k-1}. \tag{4.13}$$

Indeed, the octagon with bridge $2m$ is given by the determinant of the kernel $\mathbb{K}^{\text{oo}}$ with the first m rows and columns deleted,

$$\mathbb{O}_{\ell=2m} = \langle m,\text{o}|\mathring{\circ}\, e^{-\psi\mathbb{K}^{\text{oo}}\psi^*}\mathring{\circ}|m,\text{o}\rangle = \det\left[(1-\mathbb{K}^{\text{oo}})_{\geq m}\right], \tag{4.14}$$

and (1.6) follows from the general identity (4.13).

4.2 Odd bridge

In a similar way, in the case $\ell = 2m-1$ one can eliminate all odd modes in (4.1). As a result we obtain an operator expression only in terms of the even modes,

$$\begin{aligned}\mathbb{O}_{\ell=2m-1} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\sum_{\substack{j_1,\dots,j_n\geq m\\ k_1,\dots,n_n\geq m}}\langle \ell\,|\prod_{a=1}^{n}\psi_{2j_a}\prod_{a,b=1}^{n}\mathbb{K}^{\text{ee}}_{j_a,k_b}\prod_{b=1}^{n}\psi^*_{2k_b}|\ell\rangle \\ &= \langle \ell|\,\mathring{\circ}\exp\left(-\sum_{k,j\geq 0}\psi_{2j}\,\mathbb{K}^{\text{ee}}_{j,k}\,\psi^*_{2k}\right)\mathring{\circ}\,|\ell\rangle.\end{aligned} \tag{4.15}$$

[7]We thank G. Korchemsky for making this point.

The matrix elements of

$$\mathbb{K}^{\rm ee}_{i,j} \equiv [\mathring{\mathbf{K}}\mathbf{C}]_{2j,2k} \tag{4.16}$$

are given explicitly by $(j,k \geq 0)$

$$\mathbb{K}^{\rm ee}_{i,j} = 2\int_0^\infty d\tau\ \chi(\tau,\sigma)\ \left[(1-\delta_{j,0})(-1)^{i-j}2j\frac{J_{2i}(2\tau)J_{2j}(2\tau)}{\tau} + \delta_{j,0}(-1)^j J_{2i}J_1\right]. \tag{4.17}$$

Thanks to the identity $[\mathring{\mathbf{K}}_{\geq\ell}\mathbf{C}_{\geq\ell}]_{2j,2k} = [[\mathring{\mathbf{K}}\mathbf{C}]_{\geq\ell}]_{2j,2k}$ for ℓ odd, we can eliminate the odd modes also from the left and the right vacuum and formulate the octagon as an expectation value in the ensemble of the even oscillators,

$$\mathbb{O}_{\ell=2m-1} = \langle\, m,{\rm e}|\circ\!\!\!\!\circ\, e^{-\psi^{\rm e}\,\mathbb{K}^{\rm ee}\psi^{*\rm e}} \circ\!\!\!\!\circ |m,{\rm e}\,\rangle = \det\left[(1-\mathbb{K}^{\rm ee})_{\geq m}\right] \qquad (m\geq 1). \tag{4.18}$$

The bra and ket vacuum states here are the m-charged vacua for the even modes,

$$|m,{\rm e}\,\rangle = \psi^*_{2m-2}\psi^*_{2m-4}\cdots\psi^*_0|0\ \ \rangle, \quad \langle\, m,{\rm e}| = \langle\, 0|\psi_0\psi_2\cdots\psi_{2m-2}. \tag{4.19}$$

Again, the ratio $\mathbb{O}_{2m-1}/\mathbb{O}_0$ can be computed as an expectation value of m fermion pairs which gives an $m\times m$ determinant

$$\frac{\mathbb{O}_{2m-1}}{\mathbb{O}_0} = \langle\prod_{j=0}^{m-1}\psi^{\rm e}_{2j}\psi^{*\rm e}_{2j}\,\rangle = \det\left[(1+\mathbb{R}^{\rm ee})_{<m}\right], \tag{4.20}$$

where the two-point correlator $\langle\,\psi^{\rm e}_j\psi^{*\rm e}_k\,\rangle = \delta_{j,k} + \mathbb{R}^{\rm ee}_{j,k}$ is related to the even-even kernel (4.17) by[8]

$$(1+\mathbb{R}^{\rm ee})(1-\mathbb{K}^{\rm ee}) = 1. \tag{4.21}$$

Taking the large ℓ limit of (4.20), we reproduce the determinant formula for the octagon with zero bridge in terms of the even kernel. Thus the octagon with zero bridge can be expressed as a determinant in either of the sectors

$$\mathbb{O}_0 = \det\left(1-\mathbb{K}^{\rm oo}\right) = \det\left(1-\mathbb{K}^{\rm ee}\right). \tag{4.22}$$

We have checked that the finite determinant identities obtained in this section are fulfilled within the weak coupling expansion of the octagon. Let us stress that although the expressions (4.6) and (4.18) look differently, they are both identical to the determinant of the odd block which was the starting point for the studies in [14–16]. The difference is in the interpretation: in [13–16] the bridge appeared as a parameter while here it is the effect of truncating the semi-infinite matrix.

4.3 The octagon as a Fredholm determinant of a holomorphic kernel

We will show that the operator representation with the simplified kernel (1.4) can be expressed as the expectation value of an exponential operator which commutes with the U(1) charge. For that we will represent the ordered exponentials in the expectation values (4.2)

[8]The relation with the full resolvent is $\mathbb{R}^{\rm e,e}_{j,k} = [\mathbf{C}\mathring{\mathbf{K}}/(1-\mathbf{C}\mathring{\mathbf{K}}]_{2j,2k}$ and $\mathbb{R}^{\rm oo}_{j,k} = [\mathbf{C}\mathring{\mathbf{K}}/(1-\mathbf{C}\mathring{\mathbf{K}}]_{2j+1,2k+1}$.

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and (4.15) as ordinary exponentials. This can be done at the expense of extending the sum in the exponents to all possible modes, positive and negative, after having extended the semi-infinite matrix $\mathbb{K}^{\alpha\alpha}$ to a doubly infinite matrix.

Let us choose an even bridge $\ell = 2m$ so that $\alpha = \mathrm{o}$. The expectation value (4.2) can be expressed as that of an ordinary exponential as

$$\begin{aligned} \mathbb{O}_{\ell=2m} &= \langle\, \ell|\exp\left(-\sum_{j,k\geq 0} \mathbb{K}^{\alpha\alpha}_{j,k}\ \left(\psi_{2j+1}+\psi_{-2j-1}\right)\left(\psi^*_{2k+1}-\psi^*_{-2k-1}\right)\right)|\ell\,\rangle \\ &= \langle\, \ell|\exp\left(-\sum_{j,k\in\mathbb{Z}} \mathbb{K}^{\alpha\alpha}_{j,k}\ \psi_{2j+1}\ \psi^*_{2k+1}\right)|\ell\,\rangle, \end{aligned} \tag{4.23}$$

where in the second line the octagon kernel is extended to negative values of the indices by the symmetries $\mathbb{K}^{\mathrm{oo}}_{j,k} = \mathbb{K}^{\mathrm{oo}}_{-j-1,k} = -\mathbb{K}^{\mathrm{oo}}_{j,-k-1}$. To illustrate why the negative modes are necessary, compare the quadratic terms in the expansion of (4.23) with that of (4.2),

$$\begin{aligned} (4.23) \Rightarrow &= \sum_{j,k\geq 0}\sum_{i,r\geq 0}\big(\mathbb{K}^{\mathrm{oo}}_{i,j}\mathbb{K}^{\mathrm{oo}}_{k,r}\,\langle\, 0|\psi_{2i+1}\psi^*_{2j+1}\psi_{2k+1}\psi^*_{2r+1}|0\,\rangle \\ &\quad + \mathbb{K}^{\mathrm{oo}}_{i,-j-1}\mathbb{K}^{\mathrm{oo}}_{-k-1,r}\,\langle\, 0|\psi_{2i+1}\psi^*_{-2j-1}\psi_{-2k-1}\psi^*_{2r+1}|0\,\rangle\big) \\ &= \sum_{i,j\geq 0}\mathbb{K}^{\mathrm{oo}}_{i,i}\mathbb{K}^{\mathrm{oo}}_{j,j} + \sum_{i,j\geq 0}\mathbb{K}^{\mathrm{oo}}_{i,-j-1}\mathbb{K}^{\mathrm{oo}}_{-j-1,i} \\ &= \sum_{i,j\geq 0}\mathbb{K}^{\mathrm{oo}}_{i,i}\mathbb{K}^{\mathrm{oo}}_{j,j} - \sum_{i,j\geq 0}\mathbb{K}^{\mathrm{oo}}_{i,j}\mathbb{K}^{\mathrm{oo}}_{j,i} \ \Leftarrow\ (4.2). \end{aligned} \tag{4.24}$$

By rewriting the exponent in (4.23) as a double contour integral, we get

$$\mathbb{O}_{\ell=2m} = \langle\, \ell|\exp\left(-\frac{1}{4\pi^2}\oint_{\mathcal{C}}\frac{dx}{x}\oint_{\mathcal{C}^*}\frac{dy}{y}\psi(x)\mathbb{K}^{\mathrm{oo}}(x,y)\psi^*(y)\right)|\ell\,\rangle, \tag{4.25}$$

where the contour $\mathcal{C}^*$ contains the origin and is contained in the contour $\mathcal{C}$, and the holomorphic kernel is given by

$$\begin{aligned} \mathbb{K}^{\mathrm{oo}}(x,y) &= \sum_{i,j\in\mathbb{Z}}\mathbb{K}^{\mathrm{oo}}_{i,j}\ x^{2i+1}y^{-2j-1} \\ &= -2\left(y-\frac{1}{y}\right)\int_0^\infty d\tau\chi(\tau,\sigma)\sin\left[\tau\left(x+\frac{1}{x}\right)\right]\ \cos\left[\tau\left(y+\frac{1}{y}\right)\right] \\ &= -2\int_0^\infty\frac{d\tau}{\tau}\chi(\tau,\sigma)\ \sin\left[\tau\left(x+\frac{1}{x}\right)\right]\ y\partial_y\sin\left[\tau\left(y+\frac{1}{y}\right)\right]. \end{aligned} \tag{4.26}$$

In a similar way, for ℓ odd we can write

$$\mathbb{O}_{\ell=2m-1} = \langle\, \ell|\exp\left(-\oint_{\mathcal{C}}\frac{dx}{2\pi i x}\oint_{\mathcal{C}^*}\frac{dy}{2\pi i y}\psi(x)\mathbb{K}^{\mathrm{ee}}(x,y)\psi^*(y)\right)|\ell\,\rangle, \tag{4.27}$$

$$\begin{aligned} \mathbb{K}^{\mathrm{ee}}(x,y) &= \sum_{i,j\in\mathbb{Z}}\mathbb{K}^{\mathrm{ee}}_{i,j}\ x^{2i}y^{-2j} \\ &= 2\left(y-\frac{1}{y}\right)\int_0^\infty d\tau\chi(\tau,\sigma)\cos\left[\tau\left(x+\frac{1}{x}\right)\right]\ \sin\left[\tau\left(y+\frac{1}{y}\right)\right] \\ &= 2\int_0^\infty\frac{d\tau}{\tau}\chi(\tau,\sigma)\cos\left[\tau\left(x+\frac{1}{x}\right)\right]y\partial_y\cos\left[\tau\left(y+\frac{1}{y}\right)\right]. \end{aligned} \tag{4.28}$$

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The sum of two kernels, (4.26) and (4.28), gives the x-representation of the full operator $\mathbb{K} \equiv \mathring{\mathbf{K}}\mathbf{C} = \mathbb{K}^{\rm ee} \otimes \mathbb{K}^{\rm oo}$. The factor $y - 1/y$ results from the action of the operator $\mathbf{C}$ which diagonalises in the x-space, and the rest reproduces the r.h.s. of (3.25).

The operator representaton of the octagon in x-space, eqs. (4.25)–(4.26), was derived for even bridge $\ell = 2m$. It is possible to extend each of the representations, (4.25)–(4.26) and (4.27)–(4.28), to any value of the bridge length, odd and even.[9] Thus we have, for any ℓ, two operator representations of the octagon,

$$\mathbb{O}_\ell = \langle\,\ell|\exp\left(-\frac{1}{4\pi^2}\oint_{\mathcal{C}}\frac{dx}{x}\oint_{\mathcal{C}^*}\frac{dy}{y}\psi(x)\mathbb{K}^{\alpha\alpha}(x,y)\psi^*(y)\right)|\ell\,\rangle \qquad (\alpha = \rm o, e) \tag{4.29}$$

with $\mathbb{K}^{\alpha\alpha}$ given by (4.26) for $\alpha = \rm o$ and by (4.28) for $\alpha = \rm e$. The expectation values (4.25) and (4.27) are evaluated by the series

$$\mathbb{O}_\ell = \sum_{N=0}^{\infty}\frac{(-1)^N}{N!}\frac{1}{(2\pi)^{2N}}\prod_{k=1}^{N}\oint_{\mathcal{C}}\frac{dx_k}{x_k}\oint_{\mathcal{C}^*}\frac{dy_k}{y_k}\,\left(\frac{y_j}{x_j}\right)^{\ell}\mathbb{K}^{\alpha\alpha}(x_j,y_j)\ \det_{jk}\frac{x_j}{x_j-y_k} \tag{4.30}$$

The series (4.30) is the expansion of the Fredholm determinant of the operator $\mathbb{K}^{\rm oo}$ acting in the space of the functions, odd for $\alpha = \rm o$ and even for $\alpha = \rm e$, which are analytic inside the unit circle,

$$\mathbb{O}_{2m} = \mathrm{Det}\left(1 - \mathbb{K}^{\alpha\alpha}\right), \qquad [\mathbb{K}^{\alpha\alpha}f](x) = \oint\frac{dy}{2\pi i y}\mathbb{K}^{\alpha\alpha}(x,y)f(y). \tag{4.31}$$

The x-representation (4.29) simplifies a lot in the strong coupling limit, where the leading order of the strong coupling expansion can be evaluated using the clustering method [23], but it is not known how to obtain the subleading orders. In this respect the τ-representation is more efficient because it allowed the authers of [15, 16] to obtain the whole strong coupling expansion. On the other hand, the x-representation allows one to find the strong coupling limit in the case of more general local weights.

5 Conclusion

In this paper we completed the study of the octagon form factor started in [12, 13] in the following two aspects. First, we gave a precise Fock space description of the fermionic representation outlined there, in which the length of the bridge determines the level of the Dirac sea. Second, we found explicitly the similarity transformation conjectured there, which leads, by simplifying the octagon kernel, to the determinant formula for the octagon. Such similarity transformation is not unique and, as we already mentioned, another solution has been found independently by Belitsky and Korchemsky [16]. The interpretation we found for the bridge length allowed us to express the ratio of two octagons with different bridges as a determinant of finite size involving the resolvent of the octagon kernel.

The free fermions proposed here as a device to handle the diagonal symmetric part of the weights of the virtual particles might be useful for studying other observables which

[9] The derivation of (4.25) for odd bridge and of (4.27) for even bridge is slightly more complicated because the discrete octagon kernel should be modified by a term similar to the last term in (4.17).

can take the form (4.29), that is, a vacuum expectation value of an element of GL(∞). Such objects can be deformed by an infinite set of commuting flows associated with the modes of the fermion current, following the recipe of [21], turning them into τ-functions of the Toda lattice hierarchy. Half of these flows can be associated with the conserved charges in the spin chain description of $\mathcal{N}=4$ SYM.

In the case of the octagon, the kinematical parameters ξ and ϕ can be associated, at least in the light-like limit, with the "times" coupled to the modes $J_{\pm 1}$ of the fermion current. Indeed, it was shown in [16] that certain scaling in the light-like limit the octagon satisfies the radial 2D Toda lattice equation. Slightly generalising, we can consider the scaling limit

$$\phi = \pi + i\frac{s}{g}, \quad \xi = \frac{\sigma}{g}, \quad g \to 0 \qquad (s > \sigma) \tag{5.1}$$

where their solution takes the form

$$\mathbb{O}_\ell = e^{-\hat{s}^2} \ \det \left[I_{j-k}(2\hat{s}) \right]_{j,k=1,\dots,\ell}, \tag{5.2}$$

where I_n are modified Bessel functions and $\hat{s} = \sqrt{s^2 - \sigma^2}$. The octagon in this limit satisfies the full 2D Toda lattice equation with time variable s and space variable σ,

$$\frac{1}{4}\left(\partial_s^2 - \partial_\sigma^2\right)\Phi_\ell + e^{\Phi_{\ell+1}-\Phi_\ell} - e^{\Phi_\ell - \Phi_{\ell-1}} = 0\,, \quad \Phi_\ell = \log\frac{\mathbb{O}_{\ell+1}}{\mathbb{O}_\ell}\,, \quad \ell \geq 1\,. \tag{5.3}$$

A clue about the general validity of (5.3) would be a direct proof using the fermion representation, which is still to be found.

Finally, let us mention two intriguing recent observations which suggest to look for a unified formalism working both for the BMN and the GKP vacua. First, in the null square limit, the authors of [14] noticed that the anomalous dimension characterising the light-like octagon has an alternative representation similar to the cusp anomalous dimension. By still unclear reasons, the light-like limit of the octagon can also be obtained by choosing as a weight function $\chi(\tau) = 2/(e^{\tau/g} - 1)$. Second, it was shown in [19] that the six-gluon amplitude in certain kinematical limit can be expressed in terms of the so called tilted BES kernel which becomes the BES kernel or the (light-like) octagon kernel for particular values of the tilting angle. The tilted kernel by angle α is obtained by replacing in (2.29) $i \to i\,e^{-i\alpha}$ which leads to (3.10) with $\sin\left[(m-n)\pi/2\right]$ replaced by $\sin\left[(m-n)(\pi/2-\alpha)\right]$. The fermionic representation and the subsequent analysis (except for section 4 which is relevant only to the case $\alpha = 0$) can be generalised to a generic angle α. In particular, we checked that the flow equation obtained in appendix B holds for the tilted kernel as well.

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A Proof of the linear relation (3.19) between the original and the simplified kernels

We will show that the linear relation (3.9) holding at the level of the integrands (3.3), (3.10), leads to the relation (3.19) for the integral kernels. Consider the bilinear of Bessel functions

$$\mathbf{J}_{m,n}(\tau) = i^{m-n-1} J_m(2\tau) J_n(2\tau)\,. \tag{A.1}$$

With this normalization the functional relation (3.20) for the Bessel function is rewritten with the help of the matrix $\mathbf{C}$ in (2.19) and $\mathbf{M}_{m,n} = \frac{1}{m}\delta_{m,n}$ as

$$i\frac{\mathbf{J}_{m,n}(\tau)}{\tau} = (\mathbf{MC})_{m,m'}\mathbf{J}_{m',n}(\tau) = \mathbf{J}_{m,n'}(\tau)(\mathbf{CM})_{n',n}\,, \quad m,n \neq 0\,. \tag{A.2}$$

Repeated for an arbitrary power of τ this gives, in matrix notations,

$$\left(\frac{i}{\tau}\right)^j \mathbf{J}(\tau) = (\mathbf{MC})^j \mathbf{J}(\tau)\,, \tag{A.3}$$

where the explicit expressions for the matrix powers of $\mathbf{MC}$ are

$$\begin{aligned} \left[(\mathbf{MC})^{2s}\right]_{m,m'} &= \sum_{r=-s}^{s} \binom{2s}{s-|r|} \frac{\Gamma(m-s+r)(m+2r)}{\Gamma(m+s+r+1)} (-1)^{r-s} \delta_{m+2r,m'}\,, \\ \left[(\mathbf{MC})^{2s+1}\right]_{m,m'} &= \sum_{r=-s}^{s} \binom{2s}{s-|r|} \frac{\Gamma(m-s+r)}{\Gamma(m+s+r+1)} (-1)^{r-s} (\delta_{m+2r+1,m'} - \delta_{m+2r-1,m'})\,. \end{aligned} \tag{A.4}$$

The expression (A.4) is defined for positive integer m with the power j of the matrix $(\mathbf{MC})$ restricted to

$$j \leq m\,. \tag{A.5}$$

The second m' index in (A.4) runs between $m-j$ and $m+j$ (mod 2). For the even power $j = 2s$, the formula (A.4) has sense for $j = 0$, reproducing the identity $[(\mathbf{MC})^0]_{m,m'} = \delta_{m,m'}$. Furthermore in this case the formula extends for $m = 0$, taking into account (A.5), i.e., $[(\mathbf{MC})^{2s}]_{0,m'} = \delta_{s,0}\delta_{0,m'}$. The powers of $\mathbf{CM}$ are obtained by transposition,

$$[(\mathbf{CM})^j]_{n',n} = (-1)^j [(\mathbf{MC})^j]_{n,n'}\,. \tag{A.6}$$

Next we observe that for odd values $j+m-n$, eq. (A.3) turns into

$$\left(\frac{i}{\tau}\right)^j \mathbf{J}_{m,n}(\tau) = [(\mathbf{MC})^j]_{m,m'}\, \mathring{\mathcal{K}}(\tau)]_{m',n} \quad (j+m = n-1 \ (\mathrm{mod}\ 2))\,, \tag{A.7}$$

where $\mathring{\mathcal{K}}(\tau)_{m,n} = \frac{1-(-1)^{m-n}}{2}\mathbf{J}_{m,n}$ (3.10). Inserting (A.7) in (3.9) we obtain after integration (3.19)

$$\begin{aligned} \mathbf{K}_{m,n} &= \sum_{\substack{j=0,\ldots,m-n-1 \\ j+m+n=\mathrm{odd}}} \frac{(i\sigma)^j}{j!} A^{(j)}_{m-n} \left[(\mathbf{MC})^j \mathring{\mathbf{K}}\right]_{m,n} \qquad (m>n) \\ &= \sum_{\substack{j=0,\ldots,n-m-1 \\ j+m+n=\mathrm{odd}}} \frac{(-i\sigma)^j}{j!} A^{(j)}_{n-m} \left[\mathring{\mathbf{K}}(\mathbf{CM})^j\right]_{m,n} \qquad (n>m) \end{aligned} \tag{A.8}$$

The inequality (A.5) is fulfilled in (A.8). In our problem the indices of the Bessel functions J_m take values $m \geq \ell$. This implies that the power j of the matrices $(\mathbf{M}_{\geq \ell}\mathbf{C}_{\geq \ell})$ (cf. (3.13)) is restricted to

$$j \leq m - \ell\,, \tag{A.9}$$

$[(\mathbf{M}_{\geq \ell}\mathbf{C}_{\geq \ell})^{2s}]_{\ell,m'} = \delta_{s,0}\delta_{\ell,m'}$. The relation (A.8) then holds with $\mathbf{M}, \mathbf{C}$ replaced by $\mathbf{M}_{\geq \ell}, \mathbf{C}_{\geq \ell}$. The restriction of the indices of the kernel $\mathbf{K}_{m,n}$ given by (A.8) to $m, n \geq \ell$ projects, taking into account the upper bound (A.9), the second index of $[(\mathbf{MC})^j]_{m,m'}$ in the r.h.s. to $m' \geq \ell$.

B Flow equation

The matrix elements $\mathbf{K}_{m,n}$ satisfy the following differential equation and its conjugate,

$$m\partial_\sigma \mathbf{K}_{m,n} - i\sigma\partial_\sigma \left(\mathbf{K}_{m+1,n} + \mathbf{K}_{m-1,n}\right) + i(m-n)\left(\mathbf{K}_{m+1,n} - \mathbf{K}_{m-1,n}\right) = 0\,, \tag{B.1}$$

for any $m, n \geq 0$. Here the weight function χ is treated as a functional parameter and the derivative in σ does not act on it. The equations follow straightforwardly from (3.19) using the relations for the coefficients

$$A^{j+1}_{m-n} = -(m-n\mp j)A^{j}_{m\pm 1-n}, \quad A^{m-n}_{m+1-n} = 0. \tag{B.2}$$

Introducing the diagonal matrix $\mathbf{N}_{m,n} = n\,\delta_{m,n}\,, m, n \geq 0$ the equation (B.1) and its conjugate can be cast in a matrix form

$$\begin{aligned} (\mathbf{N} - i\sigma\mathbf{S})\partial_\sigma \mathbf{K} + i[\mathbf{N}, \mathbf{CK}] &= 0 \\ \partial_\sigma \mathbf{K}\;(\mathbf{N} - i\sigma\mathbf{S}) + i[\mathbf{N}, \mathbf{KC}] &= 0. \end{aligned} \tag{B.3}$$

The flow equation (B.3) determines the evolution of the full kernel $\mathbf{K}$ which characterises the octagon with zero bridge. The equation for non-zero bridge is obtained by replacing the kernel and the matrices involved with $\mathbf{K}_{\geq \ell}\,, \mathbf{N}_{\geq \ell}\,, \mathbf{S}_{\geq \ell}\,, \mathbf{C}_{\geq \ell}\,.$

Remark. After substituting $\mathbf{K}_{\geq \ell} \to \mathbf{U}_\ell^{\mathrm{T}} \mathring{\mathbf{K}}_{\geq \ell} \mathbf{U}_\ell$, the equation (B.1) turns into an equation for the similarity operator $\mathbf{U}_\ell$. In general, it is not straightforward to integrate it. Since there is a continuum of solutions, one can impose additional conditions on the solution. Belitsky and Korchemsky imposed [16] the condition that the semi-infinite similarity matrix $\mathbf{\Omega} = \{\mathbf{\Omega}_{i,k}\}_{k,j\geq 0}$ acts trivially on the first two columns,

$$\mathbf{\Omega}_{k,0} = \delta_{k,0}\,, \mathbf{\Omega}_{k,1} = \delta_{k,1}\,. \tag{B.4}$$

In our conventions their matrix $\mathbf{\Omega}$ corresponds to a matrix $\tilde{\mathbf{U}}_\ell$ with elements

$$[\tilde{\mathbf{U}}_\ell]_{k+\ell,j+\ell} = [\mathbf{\Omega}]_{k,j}. \tag{B.5}$$

Under these conditions $[\tilde{\mathbf{U}}_\ell]_{k+\ell,\ell} = \delta_{k,0}$ and $[\tilde{\mathbf{U}}_\ell]_{k+\ell,1+\ell} = \delta_{k,1}$ one obtains

$$\mathbf{K}_{m+\ell,\ell} = [\tilde{\mathbf{U}}_\ell^{\mathrm{T}}\,\mathring{\mathbf{K}}]_{m+\ell,\ell}\,, \quad \mathbf{K}_{m+\ell,1+\ell} \;= [\tilde{\mathbf{U}}_\ell^{\mathrm{T}}\,\mathring{\mathbf{K}}]_{m+\ell,1+\ell} \qquad (m \geq 0). \tag{B.6}$$

In this way the differential equation (B.1) reduces to an equation for $\tilde{\mathbf{U}}_\ell$. Accordingly the first/second relation (B.6) determines the matrix elements of $\mathbf{\Omega}^{\mathrm{T}}$ with odd/even first index directly from (3.19)

$$\begin{aligned}
\mathbf{\Omega}_{2k+1,n} &= \sum_{\substack{j=0,\ldots,n-1 \\ j-n=\text{ odd}}} \frac{(-i\sigma)^j}{j!}[(\mathbf{CM})^j]_{2k+1+\ell,n+\ell} A_n^{(j)} = [\mathbf{U}_\ell]_{2k+1+\ell,n+\ell} \\
\mathbf{\Omega}_{2k,n} &= \delta_{n,0}\delta_{k,0} + \sum_{\substack{j=0,\ldots,n-2 \\ j-n=\text{ even}}} \frac{(-i\sigma)^j}{j!}[(\mathbf{CM})^j]_{2k+\ell,n+\ell} A_{|n-1|}^{(j)} \\
&= \delta_{n,0}\delta_{k,0} + [\mathbf{U}_{\ell+1}]_{2k-1+(\ell+1),n-1+(\ell+1)} \,.
\end{aligned} \tag{B.7}$$

The last equality relates the matrix elements of $\mathbf{\Omega}$ to a different projection of the solution $[\mathbf{U}_\ell]_{n',n}$ (3.12) to odd $n'-\ell$.

C Proof of the similarity transformation (3.11)–(3.12)

In this appendix we will omit the index ℓ in $\mathbf{U}_\ell$ and $\mathbf{M}_\ell$ in order to avoid ugly formulas. We will show that the linear transformation (3.19) can be written as adjoint action matrix relation (3.11)

$$\mathbf{K}_{m+\ell,n+\ell} = \sum_{\substack{m',n'\geq 0 \\ n'-m'=\text{odd}}} \mathbf{U}^{\mathrm{T}}_{m+\ell,m'+\ell} \mathring{\mathbf{K}}_{m'+\ell,n'+\ell} \mathbf{U}_{n'+\ell,n+\ell} \,. \tag{C.1}$$

The matrix elements of $\mathbf{U} = \mathbf{U}_\ell$ in (3.12) read more explicitly

$$\begin{aligned}
\mathbf{U}_{2k+1+\ell,n+\ell} &= \sum_{\substack{j=0,\ldots,n-1 \\ j-n=\text{odd}}} \frac{(-i\sigma)^j}{j!} \; A_n^{(j)} \; [(\mathbf{CM})^j]_{2k+1+\ell,n+\ell} \\
\mathbf{U}_{2k+\ell,n+\ell} &= \sum_{\substack{j=0,\ldots,n \\ j-n=\text{even}}} \frac{(i\sigma)^j}{j!} \; B_n^{(j)} [(\mathbf{CM})^j]_{2k+\ell,n+\ell} \,.
\end{aligned} \tag{C.2}$$

They satisfy the relations

$$\mathbf{U}_{k+\ell,\ell} = \delta_{k,0} \,, \quad \mathbf{U}_{2k+1+\ell,1+\ell} = \delta_{k,0} \,, \tag{C.3}$$

which differ from (B.4) since $\mathbf{U}_{2k+\ell,1+\ell} = i\sigma(\delta_{k,0} - \delta_{k,1})$. Note that the zeros of the coefficients $A_n^{(j)}$ and $B_n^{(j)}$ in (3.16) compensate the poles in the expressions (A.4) for the matrix powers $(\mathbf{CM})^j_{n'+\ell,n+\ell}$ and one can write regularised closed expressions for the operators (C.2). For example[10]

$$\begin{aligned}
\mathbf{U}_{2m+\ell,2n+\ell} &= \delta_{m,0} \; \sigma^{2n} (2^{2n-1}(-1)^n \, (1-\delta_{n,0}) + \delta_{n,0}) \frac{\Gamma(1+\ell)}{\Gamma(2n+\ell+1)} \\
&\quad + \theta_m \sum_{s=0}^{n} \frac{(2\sigma)^{2s}(-1)^{m+n}(2m+\ell)\, n\, \Gamma(n+s) \prod_{k=1}^{m+\ell-1}(n-s+k)}{\Gamma(s-m+n+1)\Gamma(s+m-n+1)\Gamma(n+m+\ell+s+1)} .
\end{aligned} \tag{C.4}$$

[10]This expression does not change if the upper bound of the summation is extended to $s \leq n+m+\ell-1$.

We want to prove (C.1) with the operators given in (C.2). The key ingredient of the proof is the intertwining relation

$$[(\mathbf{MC})^j\mathring{\mathbf{K}}]_{m+\ell,n+\ell} = [(\mathbf{MC})^{j-p}\mathring{\mathbf{K}}(\mathbf{CM})^p]_{m+\ell,n+\ell} \qquad (j \le m\,,\ p \le n). \tag{C.5}$$

The latter allows to redistribute the matrix powers in (A.8) on both sides of $\mathring{\mathbf{K}}$. E.g.,

$$\begin{aligned} -A^{(1)}_{m-n}[\mathbf{MC}\,\mathring{\mathbf{K}}]_{m,n} &= (m-n)[\mathbf{MC}\,\mathring{\mathbf{K}}]_{m,n} = [m\,\mathbf{MC}\;\mathring{\mathbf{K}}) - n\,\mathring{\mathbf{K}}\,\mathbf{CM}]_{m,n} \\ A^{(2)}_{m-n}[(\mathbf{MC})^2\,\mathring{\mathbf{K}}]_{m,n} &= [m^2(\mathbf{MC})^2\,\mathring{\mathbf{K}} - 2m(\mathbf{MC})\mathring{\mathbf{K}}(\mathbf{CM})n + (n^2-1)\mathring{\mathbf{K}}(\mathbf{CM})^2]_{m,n}\,. \end{aligned} \tag{C.6}$$

In what follows we assume, for the sake of simplicity of the presentation, that $\ell = 0$. To illustrate the general procedure, consider the matrix element $\mathbf{K}_{1,n}$. We can split the coefficient in (3.19) as

$$A^{(j)}_{n-1} = \sum_{p=0}^{j} \binom{j}{p} B^{(p)}_1 A^{(j-p)}_n = j\,B^{(1)}_1 A^{(j-1)}_n + (-1)^j B^{(j)}_n A^{(0)}_1 \tag{C.7}$$

taking into account that $B^{(2s+1)}_1 = 0$ for $s \ge 1$, while the contributions of $B^{(2s)}_1$, $s \ge 0$ sum to the second term in the r.h.s. Next we distribute accordingly the matrix powers in agreement with the inequality (A.5)

$$\begin{aligned} \mathbf{K}_{1,n} &= \mathring{\mathbf{K}}_{1,n'} \sum_{\substack{j=0,\dots,n-2\\ j-n=\text{even}}} \frac{(-i\sigma)^j}{j!}[(\mathbf{CM})^j]_{n',n} A^{(j)}_{|n-1|} \\ &= \sum_{\substack{j=0,\dots,n-2\\ j-n=\text{even}}} \frac{(-i\sigma)^j}{j!}\left(j\,B^{(1)}_1[(\mathbf{MC})\,\mathring{\mathbf{K}}(\mathbf{CM})^{(j-1)}]_{1,n}\,A^{(j-1)}_n + (-1)^j[\mathring{\mathbf{K}}(\mathbf{CM})^{(j)}]_{1,n}B^{(j)}_n\right). \end{aligned} \tag{C.8}$$

This is almost (C.1) in this particular case, with odd intermediate summation index n' in the first term in the r.h.s. of (C.8) and n'- even in the second. The two terms reproduce the corresponding expressions for the operators in (C.2) up to the upper bounds. In fact the upper bound in (C.8) extends from $n-2$ to n. In the first line this is so due the vanishing of $A^{(n)}_{n-1} = 0$. Equivalently for $j = n$ the two terms in (C.7) and their contributions to (C.8) compensate each other due to the relation (3.16). Hence, taking into account (C.3) which implies $\mathbf{U}^{\mathrm{T}}_{1,2k+1} = \delta_{k,0}$, we reproduce (C.1) for this particular example

$$\mathbf{K}_{1,n} = [\mathbf{U}^{\mathrm{T}}\mathring{\mathbf{K}}\mathbf{U}]_{1,n} = \sum_{n'-\ \text{odd}} \mathbf{U}^{\mathrm{T}}_{1,m'}\mathring{\mathbf{K}}_{m',n'}\mathbf{U}_{n',n} + \sum_{n'-\ \text{even}} \mathbf{U}^{\mathrm{T}}_{1,m'}\mathring{\mathbf{K}}_{m',n'}\mathbf{U}_{n',n}\,. \tag{C.9}$$

The generalization of (C.7) for $m < n$ reads

$$A^{(j)}_{n-m} = \sum_{\substack{s=0,\dots,m\\ s-m=\text{even}}} \binom{j}{s} B^{(s)}_m A^{(j-s)}_n + (-1)^j \sum_{\substack{s=0,\dots,m-1\\ s-m=\text{odd}}} \binom{j}{s} A^{(s)}_m B^{(j-s)}_n\,. \tag{C.10}$$

For general $\mathbf{K}_{m,n}$ one proceeds as in the example $\mathbf{K}_{1,n}$ considered above, distributing accordingly the matrix powers to the left and right of $\mathring{\mathbf{K}}$. The last step is to ensure the upper bounds as in (C.2). E.g., for odd m the upper bound in the initial expression (A.8) can be lifted to n adding $m+1$ terms without violating (A.5) exploiting the vanishing of the coefficients $A^{(j)}_{n-m}$ for $j \geq n-m+1+2k\,, k \in \mathbb{Z}_+$. Moreover, the upper bound can be extended further to $n+m-1$ moving to the left the additional matrix powers of $(\mathbf{CM})$, so that to comply with the inequalities (A.5). In the process, some zeros may appear also in each of the two terms in (C.10). Altogether this ensures the correct upper bounds of the operators $\mathbf{U}$ in the transformed expresssion obtained using (C.10).

D Proof of the exponential representation for $\ell = 0$

Here we show that the matrix $\mathbf{U}$ defined in (3.12) is given by the series (3.21) for $\ell = 0$. The matrix $\mathbf{U}$ and its transposed $\mathbf{U}^{\mathrm{T}}$ factorise as

$$\mathbf{U} = \hat{\mathbf{U}}\mathbf{P}, \qquad \hat{\mathbf{U}} = \left(\mathbf{P}_{\mathrm{e}}\, e^{-\frac{1}{2}\sigma^2\mathbf{CMS}} + \mathbf{P}_{\mathrm{o}}\, e^{-\frac{1}{2}\sigma^2\mathbf{SCM}}\right) e^{i\sigma\mathbf{C}} = \sum_{j\geq 0}[\hat{\mathbf{U}}]_j\sigma^j\,; \tag{D.1}$$

$$\mathbf{U}^{\mathrm{T}} = \mathbf{P}\,\hat{\mathbf{U}}^{\mathrm{T}}, \qquad \hat{\mathbf{U}}^{\mathrm{T}} = e^{-i\sigma\mathbf{C}}\left(e^{\frac{1}{2}\sigma^2\mathbf{SMC}}\mathbf{P}_{\mathrm{e}} + e^{\frac{1}{2}\sigma^2\mathbf{MCS}}\,\mathbf{P}^{\mathrm{o}}\right) = \sum_{j\geq 0}[\hat{\mathbf{U}}^{\mathrm{T}}]_j\sigma^j\,, \tag{D.2}$$

where $\mathbf{P}$ is the projector restricting the power j of σ of the matrix element $[\hat{\mathbf{U}}]_{n',n}$ to n or $n-1$ as in (3.21).

We will give the idea of the proof for the transposed matrix $\hat{\mathbf{U}}^{\mathrm{T}}$, restricting ourselves to the piece acting in the even sector. The coefficients $\mathbf{X}^{(j)} = [\hat{\mathbf{U}}^{\mathrm{T}}]_j\mathbf{P}_{\mathrm{e}}$ in the expansion of the first term in (D.2) read

$$j!\mathbf{X}^{(j)} = j!\sum_{k=0}^{[\frac{j}{2}]}\beta^j_k\mathbf{C}^{j-2k}(\mathbf{SMC})^k\,, \quad \beta^j_k = \frac{(-1)^k}{2^k(j-2k)!k!}\,. \tag{D.3}$$

On the other hand, the piece of the transposed matrix (3.12) restricted to the even sector, $\mathbf{U}^{\mathrm{T}}_{\mathrm{e}} = \mathbf{U}^{\mathrm{T}}\mathbf{P}_{\mathrm{e}}$, is expanded as

$$[\mathbf{U}^{\mathrm{T}}_{\mathrm{e}}]_{m,m'} = \sum_{\substack{j=0,\ldots,m\\ j-m=\mathrm{even}}}\frac{(-i\sigma)^j}{j!}B^{(j)}_m[(\mathbf{MC})^j]_{m,m'}\,, \qquad m' - \ \text{even}. \tag{D.4}$$

We have to show that

$$B^{(j)}_m[(\mathbf{MC})^j]_{m,m'} = j!\mathbf{X}^{(j)}_{m,m'}\,. \tag{D.5}$$

Let us see how this works with the lowest coefficients $j = 1, 2$. By the expression (3.17) for the coefficients $B^{(j)}_m$, for $j = 1, 2$ we have

$$\begin{aligned} B^{(1)}_m(\mathbf{MC})_{m,m'} &= m(\mathbf{MC})_{m,m'} = \mathbf{C}_{m,m'}\,, \\ B^{(2)}_m[(\mathbf{MC})^2]_{m,m'} &= m^2[(\mathbf{MC})^2]_{m,m'} = m\sum_{\pm}\pm[\mathbf{MC}]_{m\pm1,m'} \\ &= \sum_{\pm}\pm\left((m\pm1)\mp1\right)[\mathbf{MC}]_{m\pm1,m'} = [\mathbf{C}^2 - \mathbf{SMC}]_{m,m'}\,. \end{aligned} \tag{D.6}$$

We see that the computation for $j = 2$ in the last line of (D.6) is reduced to that for $j = 1$. The general proof of (D.5) can be done by induction. For that we will use the following recursive formula for $B_m^{(j)}$

$$
\begin{aligned}
B_m^{(j+1)} = (-1)^j m A_{m\pm1\mp1}^{(j)} &= m\left(B_{m\pm1}^{(j)} \pm \sum_{\substack{p=1,\ldots,j\\p=\text{odd}}} \binom{j}{p} B_{m\pm1}^{(j-p)} A_1^{(p)}\right) \\
&= m\left(B_{m\pm1}^{(j)} \mp \sum_{\substack{p=1,\ldots,j\\p=\text{odd}}} \binom{j}{p} \prod_{s=0}^{\frac{p-1}{2}} (1-(2s)^2)\, B_{m\pm1}^{(j-p)}\right)
\end{aligned} \tag{D.7}
$$

derived from the expansion

$$
A_{m-n}^{(j)} = (-1)^j \sum_{p=0}^{j} \binom{j}{p} B_m^{(p)} A_n^{(j-p)} \qquad (m > n) \tag{D.8}
$$

and we have taken into account that $A_{-1}^{(p)} = -A_1^{(p)}$ and $A_1^{(2p)} = \delta_{p,0}$. From (D.7) we obtain a recursive formula for the corresponding matrices,

$$
\begin{aligned}
B_m^{(j+1)}((\mathbf{MC})^{j+1})_{m,n} = &\sum_{\pm} \pm (B_{m\pm1}^{(j)}[(\mathbf{MC})^j]_{m\pm1,n} \\
&- \sum_{\substack{p=1,\ldots,j\\p=\text{odd}}} \binom{j}{p} \prod_{s=0}^{\frac{p-1}{2}} (1-(2s)^2) \sum_{\pm} (B_{m\pm1}^{(j-p)}[(\mathbf{MC})^{j-p}(\mathbf{MC})^p]_{m\pm1,n}
\end{aligned} \tag{D.9}
$$

If we assume the relation (D.5), then (D.9) implies that the coefficients $\mathbf{X}^{(j)}$ satisfy the recurrence relation (D.9) which takes the form

$$
(j+1)!\; \mathbf{X}^{(j+1)} = j!\mathbf{C}\mathbf{X}^{(j)} - j(j-1)!\; \mathbf{X}^{(j-1)}(\mathbf{SMC})^1 + \mathcal{A}, \tag{D.10}
$$

where we wrote explicitly only the first two terms. If we can prove independently that (D.10) is satisfied, this would imply (D.5). To do that, let us first notice that the l.h.s. of (D.10) equals the sum of the first two terms in the r.h.s. This follows from the explicit form of $\mathbf{X}^{(j)}$, eq. (D.3). Therefore to prove (D.5) it is sufficient to show that $\mathcal{A} = 0$. One can check, after tedious algebra, that this is indeed the case. Let us only write down a basic commutator used:

$$
\begin{aligned}
[\mathbf{S}, (\mathbf{SMC})^k] &= -\sum_{r=0}^{k-1} a_r^k (\mathbf{SMC})^{k-r} (\mathbf{MC})^{2r+1}\,, \\
a_r^k &= \prod_{s=1}^{r} (2s-1) \binom{k}{r+1}\,,\;\; a_0^k = k.
\end{aligned} \tag{D.11}
$$

One of the nice features of the exponential form (D.1) is that it renders the symplectic property $\mathbf{C} = \mathbf{UCU}^{\mathrm{T}}$ almost obvious,

$$
\begin{aligned}
\mathbf{UCU}^{\mathrm{T}} &= \left(\mathbf{P}_{\mathrm{e}}\; e^{-\frac{1}{2}\sigma^2\mathbf{CMS}} + \mathbf{P}_{\mathrm{o}}\; e^{-\frac{1}{2}\sigma^2\mathbf{CSM}}\right) \mathbf{PCP} \left(e^{\frac{1}{2}\sigma^2\mathbf{SMC}}\mathbf{P}_{\mathrm{e}} + e^{\frac{1}{2}\sigma^2\mathbf{MCS}}\; \mathbf{P}_{\mathrm{o}}\right) \\
&= \mathbf{P}_{\mathrm{e}}\; e^{-\frac{1}{2}\sigma^2\mathbf{CMS}}\; \mathbf{P}\; e^{\frac{1}{2}\sigma^2\mathbf{CMS}}\mathbf{C} + \mathbf{P}_{\mathrm{o}}\; e^{-\frac{1}{2}\sigma^2\mathbf{CSM}}\; \mathbf{P}\; e^{\frac{1}{2}\sigma^2\mathbf{CSM}}\mathbf{C} \\
&= (\mathbf{P}_{\mathrm{e}} + \mathbf{P}_{\mathrm{o}})\mathbf{PC} = \mathbf{C}.
\end{aligned} \tag{D.12}
$$

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