Unified non-metric $(1, 0)$ tensor-Einstein supergravity theories and $(4, 0)$ supergravity in six dimensions

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ABSTRACT: The ultrashort unitary $(4, 0)$ supermultiplet of $6d$ superconformal algebra $\mathrm{OSp}(8^*|8)$ reduces to the CPT-self conjugate supermultiplet of $4d$ superconformal algebra $\mathrm{SU}(2, 2|8)$ that represents the fields of maximal $N = 8$ supergravity. The graviton in the $(4, 0)$ multiplet is described by a mixed tensor gauge field which can not be identified with the standard metric in $6d$. Furthermore the $(4, 0)$ supermultiplet can be obtained as a double copy of $(2, 0)$ conformal supermultiplet whose interacting theories are non-Lagrangian. It had been suggested that an interacting non-metric $(4, 0)$ supergravity theory might describe the strongly coupled phase of $5d$ maximal supergravity. In this paper we study the implications of the existence of an interacting non-metric $(4, 0)$ supergravity in $6d$. The $(4, 0)$ theory can be truncated to non-metric $(1, 0)$ supergravity coupled to 5,8 and 14 self-dual tensor multiplets that reduce to three of the unified magical supergravity theories in $d = 5$. This implies that the three infinite families of unified $N = 2$, $5d$ Maxwell-Einstein supergravity theories (MESGTs) plus two sporadic ones must have uplifts to unified non-metric $(1, 0)$ tensor Einstein supergravity theories (TESGT) in $d = 6$. These theories have non-compact global symmetry groups under which all the self-dual tensor fields including the gravitensor transform irreducibly. Four of these theories are uplifts of the magical supergravity theories whose scalar manifolds are symmetric spaces. The scalar manifolds of the other unified theories are not homogeneous spaces. We also discuss the exceptional field theoretic formulations of non-metric unified $(1, 0)$ tensor-Einstein supergravity theories and conclude with speculations concerning the existence of higher dimensional non-metric supergravity theories that reduce to the $(4, 0)$ theory in $d = 6$.

KEYWORDS: Conformal Field Models in String Theory, Field Theories in Higher Dimensions, Supergravity Models

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Contents

1 Introduction

Conformal supergravity theories with local Lagrangians based on the conformal superalgebras $SU(2,2|N)$ have long been known to exist for $N \leq 4$. It was generally believed that one could not go beyond $N = 4$ without having higher spins (> 2). In [1] it was shown that the fields of maximal $N = 8$ supergravity of Cremmer and Julia [2] can be fitted into an ultra short CPT-self-conjugate unitary supermultiplet of $N = 8$ superconformal algebra $SU(2,2|8)$ referred to as the doubleton supermultiplet. The corresponding ultra short supermultiplet of $SU(2,2|4)$ is the Yang-Mills supermultiplet in $d = 4$ [3]. The $N = 4$ Yang-Mills theory of doubleton supermultiplets of $SU(2,2|4)$ is conformally invariant both classically and quantum mechanically. This led the authors of [1] to pose the question whether a conformal supergravity theory based on the doubleton supermultiplet of $SU(2,2|8)$ exists which is closely related to the maximal $N = 8$ supergravity theory of Cremmer, Julia and Scherk. Since the latter theory is not conformally invariant any superconformal theory based on the doubleton supermultiplet of $SU(2,2|8)$ must be unconventional or exotic.

The superalgebra $SU(2,2|8)$ was used to classify the counterterms in maximal supergravity in [4]. Furthermore, it is known that amplitudes of maximal supergravity are $SU(8)$ covariant even though the Lagrangian does not have $SU(8)$ symmetry. This and above mentioned results provided part of the motivation for the work of Chiodaroli, Roiban and the current author [5] who studied the connection between maximal supergravity and superconformal symmetry in all dimensions that admit simple superconformal algebras as classified by Nahm [6]. They showed that the six dimensional counterpart of the doubleton supermultiplet of $SU(2,2|8)$ is the $(4,0)$ supermultiplet of the superconformal algebra $OSp(8^*|8)$ with the even subalgebra $SO^*(8) \oplus USp(8)$, where $USp(8)$ is the R-symmetry group, which reduces to the CPT-self-conjugate doubleton supermultiplet of $SU(2,2|8)$ under dimensional reduction. They also showed that the $(4,0)$ theory can be obtained as a double copy of the $(2,0)$ theory based on the CPT-self-conjugate doubleton supermultiplet of $OSp(8^*|4)$.[1] The $(2,0)$ supermultiplet first appeared in the work of [8] who constructed the entire Kaluza-Klein spectrum of 11-dimensional supergravity over $AdS_7 \times S^4$ by simple tensoring of the $(2,0)$ doubleton supermultiplet. In the mid 1990s interacting $(2,0)$ supersymmetric theories in $6d$ were investigated within the framework of M/Superstring theory [9–12]. In particular, Seiberg pointed out the existence of four infinite series of new quantum theories with super-Poincare symmetry in six dimensions, which are not local quantum field theories [12]. Later an interacting $(2,0)$ superconformal theory was proposed by Maldacena as being dual to M-theory on $AdS_7 \times S^4$ [13].

The $(4,0)$ supermultiplet was studied earlier by Hull using the formalism of double gravitons whose equivalence to the $(4,0)$ supermultiplet obtained using the twistorial oscillators was shown in [5]. Hull argued that an interacting $(4,0)$ theory in $d = 6$ might arise as the effective theory of the strongly coupled phase of five dimensional maximal supergravity when one of the dimensions decompactifies [14–16]. On the other hand the interacting $(2,0)$

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[1]Double copy construction of the $(4,0)$ theory in terms of the $(2,0)$ theory was later reformulated by showing that it can be obtained as the square of $(2,0)$ Abelian theory by using an involutive field-theoretic product at the linearized level [7].

theory in six dimensions is believed to describe the strong coupling limit of $5d$ maximal super Yang-Mills theory. Since the maximal supergravity can be obtained as double copy of maximal super Yang-Mills theory in $5d$ these two proposals are consistent with the result that $(4,0)$ theory can also be obtained as double copy of $(2,0)$ theory in $6d$ [5, 7]. More recently, the action for the free $(4,0)$ theory was written down by Henneaux, Lekeu and Leonard using the formalism of prepotentials in [17] based on their earlier work on $(2,2)$ mixed chiral tensor describing the graviton [18]. The most unorthodox property of the $(4,0)$ doubleton supermultiplet of $OSp(8^*|8)$ is the fact that the field strength of the graviton does not arise from a metric and hence the corresponding theory in $6d$ is sometimes referred to as non-metric , exotic or generalized supergravity. However under dimensional reduction it reduces to the standard maximal supergravity in five and four dimensions.

Independently of the work on maximal supergravity, five dimensional $N = 2$ supergravity theories coupled to vector multiplets (MESGT) were constructed in [19–21] and their gaugings were studied in [22–27]. Among these MESGTs four are very special in the sense that they are unified theories with symmetric scalar manifolds G/H such that G is a symmetry of the Lagrangian. They were called magical supergravity theories since their symmetry groups in five , four and three dimensions coincide with the symmetry groups of the famous Magic Square of Freudenthal, Rosenfeld and Tits [19]. Later it was shown that there exist three infinite families of unified MESGTs and two isolated ones [28]. Three of the magical supergravities belong to the three infinite families. The scalar manifolds of unified MESGTs outside the magical ones are not homogeneous. One infinite family of unified MESGTs can be gauged to obtain an infinite family of unified Yang-Mills Einstein supergravity theories in $d = 5$ with the gauge group $SU(N,1)$ [28].

In this paper we study some of the implications of the existence of an interacting $6d$, non-metric $(4,0)$ supergravity theory. We show that the $(4,0)$ supergravity can be truncated consistently to non-metric $(1,0)$ supergravity coupled to 14, 8 and 5 self-dual tensor multiplets such that the resulting non-metric tensor-Einstein supergravity theories are unified theories in the sense that all the tensor fields including the gravitensor transform irreducibly under a simple global symmetry group. This in turn implies that all the three infinite families of unified $5d$ MESGTs as well as the two sporadic ones must also admit uplifts to non-metric unified $(1,0)$ tensor-Einstein supergravity theories in $d = 6$. We conclude with some speculations about the possible extensions of the non-metric supergravity theories to higher dimensions with exotic spacetime signatures and the role of generalized superconformal algebras of these spacetimes.

The plan of the paper is as follows. In section 2 we review the $5d$, $N = 2$ Maxwell-Einstein supergravity theories and their gaugings. Section 3 reviews the truncations of $5d$, $N = 8$ supergravity to three of the magical supergravity and the symmetries of octonionic magical supergravity which can not be obtained from maximal supergravity. Section 4 reviews the uplifts of magical supergravity theories to six dimensions as Poincare supergravities. In subsection 5.1 we review the on-shell superfield formulation of $(4,0)$ supermultiplet and the gauge potentials in the "first order formalism" following [5] and give the gauge potential of the graviton field strength in the "second order formalism". In subsection 5.2 we review the exceptional field theoretic formulation of linearized $(4,0)$

supergravity following the recent work of [29]. Subsection 5.3 is devoted to the question whether interacting conformal supergravity theories with $SU(2, 2|8)$ symmetry in $d = 4$ and $OSp(8^*|8)$ symmetry in $d = 6$ exist. In section 6 we give the truncations of $(4, 0)$ supergravity to non-metric $(3, 0)$ supergravity, to non-metric $(2, 0)$ supergravity coupled to $(2, 0)$ tensor multiplets and to non-metric $(1, 0)$ supergravity coupled to $(1, 0)$ tensor multiplets. In section 7 we discuss the metric and non-metric $(1, 0)$ magical supergravity theories. Section 8 is devoted general unified non-metric $(1, 0)$ tensor-Einstein supergravity theories in six dimensions and their exceptional field theoretic formulation. In section 9 we speculate about the possible extensions of non-metric supergravity theories to higher dimensions with non-standard space-time signatures. Appendix A reproduces the CPT-self-conjugate doubleton supermultiplet of $SU(2, 2|8)$ [1].

2 Review of $5D$, $\mathcal{N} = 2$ Maxwell-Einstein supergravity theories and their gaugings

$\mathcal{N} = 2$ MESGTs in five dimensions describes the coupling of $\mathcal{N} = 2$ supergravity to an arbitrary number, n, of vector multiplets. The supergravity multiplet consists of the fünfbein e_μ^m, two gravitini Ψ_μ^i ($i = 1, 2$) and one vector field A_μ (the "bare graviphoton"). On the other hand a $\mathcal{N} = 2$ vector multiplet consists of a vector field A_μ, two symplectic Majorana spinor fields λ^i and one real scalar field φ. The fermions in these theories transform as doublets under the R-symmetry group $USp(2)_R \cong SU(2)_R$ while all the bosonic fields are $SU(2)_R$ singlets.

Hence the fields of an $\mathcal{N} = 2$ MESGT can be labelled as

$$\{e_\mu^m, \Psi_\mu^i, A_\mu^I, \lambda^{ia}, \varphi^x\} \tag{2.1}$$

with

$$I = 0, 1, \ldots, n$$
$$a = 1, \ldots, n$$
$$x = 1, \ldots, n.$$

where we labelled the bare graviphoton as A_μ^0. The indices $a, b, \ldots$ and $x, y, \ldots$ correspond to the flat and curved indices on the scalar manifold, $\mathcal{M}$, respectively.

The bosonic part of the Lagrangian is given by [20]

$$e^{-1}\mathcal{L}_{\text{bosonic}} = -\frac{1}{2}R - \frac{1}{4}\overset{\circ}{a}_{IJ}F_{\mu\nu}^I F^{J\mu\nu} - \frac{1}{2}g_{xy}(\partial_\mu \varphi^x)(\partial^\mu \varphi^y) +$$
$$+ \frac{e^{-1}}{6\sqrt{6}}C_{IJK}\varepsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}^I F_{\rho\sigma}^J A_\lambda^K, \tag{2.2}$$

where e is the determinant of the fünfbein , R is the scalar curvature and $F_{\mu\nu}^I$ are the field strengths of Abelian vector fields A_μ^I.

The completely symmetric tensor C_{IJK}, with lower indices is constant and determines the corresponding $\mathcal{N} = 2$ MESGT uniquely [20]. The global symmetries of the Lagrangian

are the same as symmetries of C_{IJK}. The n dimensional scalar manifold can be identified with a hypersurface in an $(n + 1)$ dimensional ambient space with the metric

$$a_{IJ}(h) := -\frac{1}{3} \frac{\partial}{\partial h^I} \frac{\partial}{\partial h^J} \ln \mathcal{V}(h) \,. \tag{2.3}$$

where

$$\mathcal{V}(h) := C_{IJK} h^I h^J h^K \,. \tag{2.4}$$

with real variables h^I $(I = 0, 1, \ldots, n)$ representing the coordinates of the ambient space.

The scalar manifold $\mathcal{M}$ is simply the hypersurface $\mathcal{V}(h) = 1$ and the metric , $\overset{\circ}{a}_{IJ}(\varphi)$, of the kinetic energy term of vector fields is simply the restriction a_{IJ} to $\mathcal{M}$:

$$\overset{\circ}{a}_{IJ}(\varphi) = a_{IJ}|_{\mathcal{V}=1} \,.$$

The physical requirements of unitarity and positivity of the MESGT restrict the possible C-tensors. The most general C_{IJK} that satisfy these constraints can be brought to the form

$$C_{000} = 1, \quad C_{0ij} = -\frac{1}{2} \delta_{ij}, \quad C_{00i} = 0, \tag{2.5}$$

where C_{ijk} $(i, j, k = 1, 2, \ldots, n)$ are completely arbitrary. This is referred to as the canonical basis. Arbitrariness of C_{ijk} implies that for a given number n of vector multiplets, there exist, in general, MESGTs with different scalar manifolds and different global symmetries.

2.1 Unified Maxwell-Einstein supergravity theories

Unified Maxwell-Einstein supergravity theories in $d = 5$ are those theories with a simple global symmetry group under which *all* the vector fields A_μ^I, including the graviphoton, form a single irreducible representation. With a combination of supersymmetry and global noncompact symmetry group any field can be transformed into any other field within this class of theories.

Among MESGTs whose scalar manifolds are homogeneous spaces only four are unified theories. They are defined by the four simple Euclidean Jordan algebras $J_3^{\mathbb{A}}$ of degree three defined by 3×3 Hermitian matrices over the four division algebras $\mathbb{A}$, namely the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$ and octonions $\mathbb{O}$. The cubic norm defined by the C-tensor in these theories is identified with the cubic norm of the underlying Jordan algebra. They are referred to as magical supergravity theories because of the deep connection between their geometries and the geometries associated with the "magic square" of Freudenthal, Rosenfeld and Tits [30–32].

In $N = 2$ MESGTs defined by Euclidean Jordan algebras , J , of degree three the scalar manifold is a symmetric space of the form

$$\mathcal{M}(J) = Str_0(J)/Aut(J) \tag{2.6}$$

where $Str_0(J)$ and $Aut(J)$ are the reduced structure and automorphism group of J, respectively.[2] Below we list the corresponding scalar manifolds:

$$\mathcal{M}(J_3^{\mathbb{R}}) = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3) \qquad (n=5)$$
$$\mathcal{M}(J_3^{\mathbb{C}}) = \mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3) \qquad (n=8)$$
$$\mathcal{M}(J_3^{\mathbb{H}}) = \mathrm{SU}^*(6)/\mathrm{USp}(6) \qquad (n=14)$$
$$\mathcal{M}(J_3^{\mathbb{O}}) = E_{6(-26)}/F_4 \qquad (n=26)$$

We should note that for MESGTs defined by Euclidean Jordan algebras of degree three such as the magical theories the C-tensor is an invariant tensor of the isometry group $Str_0(J)$ of the scalar manifold and we have

$$C_{IJK} = C^{IJK} \tag{2.7}$$

where the indices $I, J, K, \ldots$ are raised by the inverse $\overset{\circ}{a}^{IJ}(\varphi)$ of the metric of kinetic energy term of vector fields.

In addition to four unified MESGTs defined by four simple Euclidean Jordan algebras of degree three there exist three infinite families of unified theories whose scalar manifolds are not homogeneous as was shown in [28]. These three infinite families are defined by Lorentzian Jordan algebras of arbitrary degree.

Now $(n \times n)$ Hermitian matrices over various division algebras form Euclidean Jordan algebras with the symmetric Jordan product defined as 1/2 the anticommutator. Their automorphism groups are compact groups. Non-compact analogs of these algebras, denoted as $J_{(q,n-q)}^{\mathbb{A}}$, are generated by matrices over various division algebras , $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ for $n \geq 3$ and over $\mathbb{O}$ for $n \leq 3$,[3] that are Hermitian with respect to a non-Euclidean "metric" η with signature $(q, n-q)$:

$$(\eta X)^{\dagger} = \eta X \qquad \forall\ X \in J_{(q,n-q)}^{\mathbb{A}}. \tag{2.8}$$

It was shown in [28] that the structure constants (d-symbols) of traceless elements T_I of noncompact Jordan algebras $J_{(1,N)}^{\mathbb{A}}$ with Lorentzian metric η of signature $(1, N)$ defined as

$$d_{IJK} \equiv \frac{1}{2}\mathrm{tr}(T_I\{T_J, T_K\}) = \mathrm{tr}(T_I \circ (T_J \circ T_K)) \tag{2.9}$$

satisfy the unitarity and positivity requirements and can be identified with the C-tensor of a MESGT:

$$C_{IJK} = d_{IJK} \tag{2.10}$$

The resulting MESGTs are all unified (for $N \geq 2$) since all the vector fields including the graviphoton transform in a single irrep of the simple automorphism groups of the underlying Jordan algebras $\mathrm{Aut}(J_{(1,N)}^{\mathbb{A}})$ which are also the symmetry groups of their Lagrangians.

In table 2.1 we list all the simple Lorentzian Jordan algebras of type $J_{(1,N)}^{\mathbb{A}}$, their automorphism groups and the number of vector and scalar fields in the MESGTs defined by them.

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[2]Reduced structure group is the invariance group of the norm form of underlying Jordan algebra.

[3]The Hermitian $(n \times n)$ matrices over the octonions do not form Jordan algebras for $n > 3$.

J	D	$\mathrm{Aut}(J)$	No. of vector fields	No. of scalars
$J^{\mathbb{R}}_{(1,N)}$	$\frac{1}{2}(N+1)(N+2)$	$\mathrm{SO}(N,1)$	$\frac{1}{2}N(N+3)$	$\frac{1}{2}N(N+3)-1$
$J^{\mathbb{C}}_{(1,N)}$	$(N+1)^2$	$\mathrm{SU}(N,1)$	$N(N+2)$	$N(N+2)-1$
$J^{\mathbb{H}}_{(1,N)}$	$(N+1)(2N+1)$	$\mathrm{USp}(2N,2)$	$N(2N+3)$	$N(2N+3)-1$
$J^{\mathbb{O}}_{(1,2)}$	27	$F_{4(-20)}$	26	25

Table 1. Simple Lorentzian Jordan algebras $J^{\mathbb{A}}_{(1,N)}$. Second and third columns list their dimensions D and automorphism groups $\mathrm{Aut}(J^{\mathbb{A}}_{(1,N)})$. Third and fourth columns list the number of vector fields $(D-1)$ and the number of scalars $(D-2)$ in the $N=2$ MESGTs defined by them.

Remarkably the structure constants of the Lorentzian Jordan algebras of degree four $J^{\mathbb{R}}_{(1,3)}$, $J^{\mathbb{C}}_{(1,3)}$ and $J^{\mathbb{H}}_{(1,3)}$ coincide with the C-tensors of the magical MESGTs defined by the Euclidean Jordan algebras $J^{\mathbb{C}}_3$, $J^{\mathbb{H}}_3$ and $J^{\mathbb{O}}_3$, respectively [28]. Hence the magical MESGTs based on Euclidean Jordan algebras $J^{\mathbb{C}}_3$, $J^{\mathbb{H}}_3$ and $J^{\mathbb{O}}_3$ [19] are *equivalent* to the MESGTs defined by the Minkowskian algebras $J^{\mathbb{R}}_{(1,3)}$, $J^{\mathbb{C}}_{(1,3)}$ and $J^{\mathbb{H}}_{(1,3)}$, respectively. As a consequence the global symmetries of these theories get extended to the reduced structure groups $\mathrm{SL}(3,\mathbb{C})$, $\mathrm{SU}^*(6)$ and $E_{6(-26)}$ of the Euclidean Jordan algebras $J^{\mathbb{C}}_3$, $J^{\mathbb{H}}_3$ and $J^{\mathbb{O}}_3$, respectively. They have the automorphism groups $\mathrm{SO}(3,1)$, $\mathrm{SU}(3,1)$ and $\mathrm{USp}(6,2)$ of $J^{\mathbb{R}}_{(1,3)}$, $J^{\mathbb{C}}_{(1,3)}$ and $J^{\mathbb{H}}_{(1,3)}$ as subgroups, respectively. Furthermore their scalar manifolds are symmetric spaces as reviewed above, while the scalar manifolds of all the other MESGTs defined by Lorentzian Jordan algebras are not even homogeneous. The smallest magical MESGT defined by the Euclidean Jordan algebra $J^{\mathbb{R}}_3$, does not belong to the three infinite families. In addition the octonionic Lorentzian Jordan algebra $J^{\mathbb{O}}_{(1,2)}$ defines a unified MESGT which does not belong to an infinite family. It has the global symmetry group $F_{4(-20)}$ with the maximal compact subgroup $\mathrm{SO}(9)$.

2.2 Unified $\mathcal{N}=2$ Yang-Mills-Einstein supergravity theories in five dimensions

A unified $N=2$ Yang-Mills Einstein supergravity (YMESGT) theory is defined as a theory in which all the vector fields including the graviphoton transform in the adjoint representation of a simple non-Abelian subgroup of the global symmetry group that is gauged. Turning off the gauge coupling constant yields a unified MESGT under whose global symmetry group all the vectors transform irreducibly.

In [28] the *complete* list of unified $N = 2$ YMESGTs in $d = 5$ was given. They are obtained by gauging the global $\mathrm{SU}(N,1)$ symmetry groups of unified MESGTs defined by complex Lorentzian Jordan algebras $J^{\mathbb{C}}_{(1,N)}$ under which all the vector fields transform in the adjoint representation of $\mathrm{SU}(N,1)$. As stated above the MESGT defined by $J^{\mathbb{C}}_{(1,3)}$ is equivalent to the MESGT defined by the Euclidean Jordan algebra $J^{\mathbb{H}}_3$ whose global symmetry is $\mathrm{SU}^*(6)$. Gauging the $\mathrm{SU}(3,1) = \mathrm{SO}^*(6)$ subgroup of $\mathrm{SU}^*(6)$ leads to the unique unified $5d$ YMESGT whose scalar manifold is a symmetric space [23]. Again in [23] it was shown that the dimensionless ratio $\frac{g^3}{\kappa}$ involving the non-Abelian gauge coupling constant g and the gravitational constant κ must be quantized in the quantum theory by invariance under large gauge transformations. The same argument extends to all unified YMESGTs since

$$\Pi_5(\mathrm{SU}(N,1)) = \Pi_5(\mathrm{U}(N)) = \Pi_5(\mathrm{SU}(N)) = \mathbb{Z},$$

where Π_5 stands for the fifth homotopy group.

Pure YMESGTs in $d = 5$ without tensor or hypermultiplets do not have a scalar potential. By expanding around the base point

$$T_0 = \begin{pmatrix} a & 0 \\ 0 & -\frac{a}{N}\mathbf{1}_{(N)} \end{pmatrix}, \tag{2.11}$$

where a is some real number fixed by the condition $d_{000} = 1$, one can show that the non-compact gauge fields transforming in $N \oplus \bar{N}$ of $\mathrm{U}(N)$ become massive by eating scalar fields and around this ground state $\mathrm{U}(1) \times \mathrm{SU}(N)$ remains unbroken with the $\mathrm{U}(1)$ gauge field corresponding to the graviphoton. Spin $1/2$ fields transforming in the symplectic $N \oplus \bar{N}$ also become massive and together with massive gauge fields form short BPS multiplets, with the central charge generated by the $\mathrm{U}(1)$ factor.

2.3 $\mathcal{N} = 2$ Yang-Mills-Einstein supergravity theories coupled to tensor fields

Unified YMESGTs are obtained by gauging the $\mathrm{SU}(N,1)$ global symmetry groups of unified MESGTs defined by the Jordan algebras $J^{\mathbb{C}}_{(1,N)}$. Since the Jordan algebras $J^{\mathbb{C}}_{(1,N)}$ are subalgebras of the quaternionic Jordan algebras $J^{\mathbb{H}}_{(1,N)}$ one can also gauge the $\mathrm{SU}(N,1)$ subgroups of the global symmetry groups $\mathrm{USp}(2N,2)$ of the MESGTs defined by $J^{\mathbb{H}}_{(1,N)}$. Under the automorphism group $\mathrm{USp}(2N,2)$ of $J^{\mathbb{H}}_{(1,N)}$, the vector fields transform in the $(2N^2 + 3N)$ dimensional anti-symmetric symplectic traceless tensor representation. They decompose as

$$[(N+1)^2 - 1] \oplus \frac{N(N+1)}{2} \oplus \overline{\frac{N(N+1)}{2}}$$

under the $\mathrm{SU}(N,1)$ subgroup of $\mathrm{USp}(2N,2)$ for $N \geq 2$. Therefore in gauging the $\mathrm{SU}(N,1)$ subgroup the $N(N+1)$ non-adjoint vector fields must be dualized to massive tensor fields satisfying odd dimensional self-duality conditions [25].

As for the family of unified MESGTs defined by the real Jordan algebras $J^{\mathbb{R}}_{(1,N)}$, the vector fields transform in the symmetric tensor representation of $\mathrm{SO}(N,1)$. For even $N =$

$2n$ with $N > 3$ one can gauge the U(n) subgroup of SO($2n, 1$) by dualizing the non-adjoint vector fields transforming in the reducible symplectic representation

$$\frac{n(n+1)}{2} \oplus \overline{\frac{n(n+1)}{2}}$$

of U(n) to tensor fields. For odd $N = 2n + 1$ ($N > 3$) in gauging the U(n) subgroup of SO($2n + 1, 1$) the remaining vector fields in the reducible representation

$$(n \oplus \bar{n}) \oplus \left(\frac{n(n+1)}{2} \oplus \overline{\frac{n(n+1)}{2}} \right) \oplus (1 \oplus \bar{1})$$

of U(n) must be dualized to tensor fields.

In the MESGT defined by the octonionic Jordan algebra $J_{(2,1)}^{\mathbb{O}}$ with the global symmetry group $F_{4(-20)}$ one can gauge the SU($2, 1$) subgroup with the remaining vector fields transforming in the reducible representation

$$(3 \oplus \bar{3}) \oplus (3 \oplus \bar{3}) \oplus (3 \oplus \bar{3}) \,.$$

of SU($2, 1$) dualized to tensor fields.

3 Magical supergravity theories and maximal supergravity

The magical Maxwell-Einstein supergravity theories defined by the real, complex and quaternionic Jordan algebras $J_3^{\mathbb{A}}$ ($\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$) can all be obtained by a consistent truncation of the maximal supergravity in $d = 5, 4$ and 3 dimensions [19]. The same is true for their 6 dimensional uplifts as Poincare supergravities [33]. The exceptional supergravity defined by the exceptional Jordan algebra $J_3^{\mathbb{A}}$ on the other hand can not be obtained by a truncation of maximal supergravity. In five dimensions the U-duality group of maximnal supergravity is $E_{6(6)}$ and that of exceptional supergravity is $E_{6(-26)}$. They can both be truncated to the $N = 2$ MESGT defined by the quaternionic Jordan algebra with the U-duality group SU*(6). Maximal supergravity can be gauged in $d = 5$ with the gauge group SU($3, 1$) and 12 tensor fields which admits an $N = 2$ supersymmetric vacuum with vanishing cosmological constant [34]. Similarly the exceptional supergravity theory can be gauged with the gauge group SU($3, 1$) and 12 tensor fields. The common sector of these two gauged supergravity theories is the unique unified $N = 2$ YMESGT with the gauge group SU($3, 1$) and whose scalar manifold is SU*(6)/USp(6).

Both real forms $E_{6(6)}$ and $E_{6(-26)}$ have SU*(6) $\times$ SU(2) as subgroups under which they decompose as

$$27 = (15, 1) \oplus (\bar{6}, 2) \tag{3.1}$$

$$78 = (35, 1) \oplus (1, 3) \oplus (20, 2) \tag{3.2}$$

Under the maximal compact subgroup USp(6) the above representations of SU*(6) decompose as

$$6 = 6 \tag{3.3}$$
$$15 = 14 \oplus 1 \tag{3.4}$$
$$20 = 6 \oplus 14' \tag{3.5}$$
$$35 = 21 \oplus 14 \tag{3.6}$$

Under the maximal compact subgroup USp(8) of $E_{6(6)}$ we have the decompositions

$$27 = 27 \tag{3.7}$$
$$78 = 36 \oplus 42 \tag{3.8}$$

which further decompose under the USp(6) × USp(2) as

$$27 = (15, 1) \oplus (6, 2) \tag{3.9}$$
$$36 = (21, 1) \oplus (1, 3) \oplus (6, 2) \tag{3.10}$$
$$42 = (14, 1) \oplus (14', 2) \tag{3.11}$$

On the other hand maximal compact subgroup of $E_{6(-26)}$ is F_4 under which we have the decompositions

$$27 = 26 \oplus 1 \tag{3.12}$$
$$78 = 52 \oplus 26 \tag{3.13}$$

Under the USp(6) × USp(2) subgroup the above representations of F_4 decompose as

$$26 = (14, 1) \oplus (6, 2) \tag{3.14}$$
$$52 = (21, 1) \oplus (1, 3) \oplus (14', 2) \tag{3.15}$$

The above decompositions show that restricting to the USp(2) invariant subsector the spectra coincide with that of quaternionic magical theory defined by $J_3^{\mathbb{H}}$. The global symmetry group SU*(6) of the quaternionic magical theory has the subgroup $SL(3, \mathbb{C}) \times SO(2)$ where $SL(3, \mathbb{C})$ is the global symmetry group of the complex magical MESGT

$$SU^*(6) \supset SL(3, \mathbb{C}) \times U(1)_C \tag{3.16}$$

The $U(1)_C$ invariant sector of the quaternionic theory corresponds to the consistent truncation to the complex magical theory. Similarly the global symmetry group of the complex magical theory decomposes as

$$SL(3, \mathbb{C}) \supset SL(3, \mathbb{R}) \times \mathbb{Z}_2 \tag{3.17}$$

and $\mathbb{Z}_2$ invariant subsector describes the consistent truncation to the real magical supergravity defined by $J_3^{\mathbb{R}}$.

G_T	$\mathcal{R}_{\rm v}$	A_μ^A	Γ^I_{AB}	$\mathcal{R}_{\rm ten}$
$\mathrm{SO}(9,1)$	$\mathbf{16}_c$	MW	Γ^I_{AB}	$\mathbf{10}$
$\mathrm{SO}(5,1) \times \mathrm{USp}(2)$	$(\mathbf{4}_c, \mathbf{2})$	SMW, $A = (\alpha r)$	$\Gamma^I_{\alpha r, \beta s} = \Gamma^I_{\alpha\beta} \epsilon_{rs}$	$(\mathbf{6}, \mathbf{1})$
$\mathrm{SO}(3,1) \times \mathrm{U}(1)$	$(\mathbf{2}, \mathbf{1})_+ + (\mathbf{1}, \mathbf{2})_-$	W, $A = \{\alpha, \dot\beta\}$	$\begin{pmatrix} 0 & \Gamma^I_{\alpha\dot\beta} \\ \bar\Gamma^I_{\dot\alpha\beta} & 0 \end{pmatrix}$	$(\mathbf{2}, \mathbf{2})_{\mathbf{0}}$
$\mathrm{SO}(2,1)$	$\mathbf{2}$	M	Γ^I_{AB}	$\mathbf{3}$

Table 2. Above we reproduce the field content and symmetries of magical supergravity theories in six dimensions. The first column lists their global symmetry groups G_T and the second column lists the representations $\mathcal{R}_{\rm v}$ of the vector fields A_μ^A under G_T. The reality properties of these representations are given in the third column: Majorana (M), Weyl (W), Majorana-Weyl (MW), symplectic Majorana-Weyl (SMW). The last column lists the representations $\mathcal{R}_{\rm ten}$ of the tensor fields under G_T. Γ^I are the gamma matrices in the respective dimensions.

4 Magical Poincare supergravity theories in six dimensions

Six dimensional magical supergravity theories coupled to hypermultiplets and their gaugings were studied in [33]. Magical supergravities in six dimensions describe the coupling of $(1,0)$ Poincare supergravity to $n_T = 2, 3, 5, 9$ tensor fields and vector fields in a definite spinor representation of $\mathrm{SO}(n_T, 1)$. The coupling between vector fields and tensors involve $\mathrm{SO}(n_T, 1)$ invariant tensors Γ^I_{AB} that are the Dirac Γ-matrices for $n_T = 2, 3$, and the Van der Waerden symbols for $n_T = 5, 9$. They satisfy the identities

$$\Gamma_{I(AB} \Gamma^I_{C)D} = 0 \,. \tag{4.1}$$

which are simply the Fierz identities for the existence supersymmetric Yang-Mills theories in 3,4,6 and 10 dimensions. These identities follow from the adjoint identity satisfied by the elements of simple Euclidean Jordan algebras of degree three [35]. We reproduce their field contents in table 2.

Since the vector fields transform in a spinor representation which belong to a unique orbit of the isometry group of the scalar manifold one finds that six dimensional magical supergravity theories admit a unique gauge group which is a centrally extended Abelian nilpotent group. For the octonionic magical theory the unique gauge group is the maximal centrally extended Abelian subgroup of $F_{4(-20)}$ which is the automorphism group of the Lorentzian octonionic Jordan algebra $J^{\mathbb{O}}_{(2,1)}$. For the quaternionic (complex) magical theory the unique gauge group is the maximal centrally extended Abelian subgroup of $\mathrm{USp}(4,2)$ ($\mathrm{SU}(2,1)$) which is the automorphism group of the Lorentzian quaternionic (complex) Jordan algebra $J^{\mathbb{H}}_{(2,1)}$ $(J^{\mathbb{C}}_{(2,1)})$.

They satisfy the inclusions

$$F_{4(-20)} \supset \mathrm{USp}(4,2) \times \mathrm{USp}(2) \supset \mathrm{SU}(2,1) \times \mathrm{U}(1) \tag{4.2}$$

These results show that semisimple gaugings of the 5d magical supergravity theories do not admit uplifts to six dimensions as standard Lagrangian Poincare supergravities. On the other hand it is known that the 5d , $N = 4$ super Yang-Mills theory can be obtained from $(1, 1)$ Poincare supersymmetric Yang-Mills theory in six dimensions by dimensional reduction and it is generally believed that it can also be obtained from an interacting $(2,0)$ superconformal field theory. Similarly the $N = 2$ super Yang-Mills theory in five dimensions can be obtained from $(1, 0)$ supersymmetric Yang-Mills theory or from a $(1, 0)$ superconformal theory of self-dual tensor multiplets in $d = 6$. The standard Yang-Mills theories in $d = 6$ involving vector fields are not conformally invariant. The $(2, 0)$ conformal supermultiplet decomposes as a $(1, 0)$ tensor multiplet plus a conformal hypermultiplet in $d = 6$

$$(2, 0) = (1, 0) \text{ tensor multiplet} \oplus (1, 0) \text{ hypermultiplet} \tag{4.3}$$

Therefore an interacting $(2, 0)$ theories can be viewed as a special family of interacting $(1, 0)$ tensor multiplets coupled to hypermultiplets. Similarly the $N = 4$ super Yang-Mills theories that descend from the interacting $(2, 0)$ theories in $d = 6$ can be viewed as a special class of $N = 2$ super Yang-Mills theories coupled to $N = 2$ hypermultiplets in the adjoint representation of the gauge group.

5 Superconformal symmetry and non-metric (4,0) supergravity in six dimensions

5.1 On-shell superfields of 6d , $(4, 0)$ supermultiplet of OSp$(8^*|8)$ in twistorial formulation and first versus second order formalism

The physical degrees of freedom corresponding to the fields of maximal $N = 8$ supergravity in $d = 4$ were shown to belong to the CPT-self-conjugate unitary representation (doubleton) of the conformal superalgebra SU$(2, 2|8)$ in [1]. Formulation of this unitary supermultiplet in terms of constrained on-shell superfields was given in [5] which we review in the appendix. Even though the physical degrees of freedom form a unitary supermultiplet of the conformal superalgebra SU$(2, 2|8)$ interactions in maximal supergravity break the conformal symmetry down to Poincare subgroup. Whether a conformal supergravity based on this supermultiplet exists, as contemplated in [1], is still an open problem as discussed below.

In six dimensions the unique superconformal algebra with 64 supersymmetry generators is OSp$(8^*|8)$ with the maximal even subalgebra SO$^*(8) \oplus$USp(8). Explicit construction of the CPT self-conjugate doubleton supermultiplet of OSp$(8^*|8)$ in terms of twistorial oscillators was given in [5] and shown to reduce to the doubleton supermultiplet of SU$(2, 2|8)$ under dimensional reduction. In table 3 we reproduce the doubleton supermultiplet of OSp$(8^*|8)$ with R-symmetry group USp(8) given in [5].[4] This multiplet is referred to as the $(4, 0)$ conformal supermultiplet in $d = 6$ and was studied earlier by Hull using the

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[4]We should note that in the manifestly unitary construction using twistorial oscillators the field strengths corresponding to physical degrees of freedom and not the corresponding gauge fields form the unitary supermultiplets. With that caveat we will use the terms fields and field strengths interchangeably.

Field Strengths	$\mathrm{SU}^*(4)_D$	$\mathrm{USp}(8)$
$\phi^{[ABCD]\|}(x)$	$(0,0,0)$	42
$\lambda_{\hat{\alpha}}^{[ABC]\|}(x)$	$(1,0,0)$	48
$h_{(\hat{\alpha}\hat{\beta})}^{[AB]\|}(x)$	$(2,0,0)$	27
$\psi_{(\hat{\alpha}\hat{\beta}\hat{\gamma})}^{A}(x)$	$(3,0,0)$	8
$R_{(\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta})}(x)$	$(4,0,0)$	1

Table 3. $(4,0)$ doubleton supermultiplet of $\mathrm{OSp}(8^*|8)$. First column lists the field strengths corresponding to actual physical degrees of freedom. Second column gives their Dynkin labels under the Lorentz group $\mathrm{SU}^*(4)$. Third column gives the dimensions of their $\mathrm{USp}(8)$ representations. The field strengths are labelled by chiral $\mathrm{SU}^*(4)$ spinor indices $\hat{\alpha}, \hat{\beta}, \dots$ and $\mathrm{USp}(8)$ indices $A, B, \dots$.

formalism of double gravitons who argued that an interacting theory based on this supermultiplet may describe a strongly coupled phase of $5d$ maximal supergravity when one of the dimensions decompactifies [14–16].

The fields belonging to the $(4,0)$ supermultiplet can be fitted into an on-shell superfield satisfying an algebraic and a differential constraint [5]. For this it turns out to be very convenient to represent the coordinates of the six-dimensional extended superspace as anti-symmetric tensors in spinorial indices [36, 37]

$$\left(x^{\hat{\alpha}\hat{\beta}} = -x^{\hat{\beta}\hat{\alpha}}, \theta_A^{\hat{\alpha}} \right) \qquad \hat{\alpha}, \hat{\beta} = 1, \dots, 4; \quad A = 1, \dots, 8; \tag{5.1}$$

where the spinorial indices of the Lorentz group $\mathrm{SU}^*(4)$ in $d = 6$ are labelled by hatted Greek indices $\hat{\alpha}, \hat{\beta} \dots$ and the $\mathrm{USp}(8)$ indices by $A, B, C \dots$. Defining the superspace covariant derivative

$$D_{\hat{\alpha}}^A = \partial_{\hat{\alpha}}^A + i\Omega^{AB}\theta_B^{\hat{\beta}}\partial_{\hat{\alpha}\hat{\beta}}, \tag{5.2}$$

where $\partial_{\hat{\alpha}}^A \theta_B^{\hat{\beta}} = \delta_B^A \delta_{\hat{\alpha}}^{\hat{\beta}}$ and $\Omega_{AB} = -\Omega_{BA}$ one finds

$$\{D_{\hat{\alpha}}^A, D_{\hat{\beta}}^B\} = 2i\Omega^{AB}\partial_{\hat{\alpha}\hat{\beta}}, \qquad \{D_{\hat{\alpha}A}, D_{\hat{\beta}}^B\} = 2i\delta_A^B\partial_{\hat{\alpha}\hat{\beta}}. \tag{5.3}$$

The symplectic metric satisfies

$$\Omega_{AB}\Omega^{BC} = \delta_A^C. \tag{5.4}$$

and is used to raise or lower indices,

$$\theta^{A\hat{\alpha}} = \Omega^{AB}\theta_B^{\hat{\alpha}}, \qquad D_{A\hat{\alpha}} = \Omega_{AB}D_{\hat{\alpha}}^B. \tag{5.5}$$

The scalar superfield of the $(4,0)$ supermultiplet

$$\Phi^{ABCD}\left(x^{\hat{\alpha}\hat{\beta}}, \theta_A^{\hat{\alpha}} \right) \tag{5.6}$$

is completely anti-symmetric in its indices and is symplectic traceless i.e.

$$\Phi^{ABCD}\Omega_{CD} = 0, \tag{5.7}$$

and satisfies the differential constraint

$$D^A_{\hat{\alpha}} \Phi^{BCDE} + \frac{1}{21} D_{\hat{\alpha}F} \left\{ \Omega^{A[B} \Phi^{CDE]F} + \frac{3}{4} \Omega^{[BC} \Phi^{DE]AF} \right\} = 0 \,. \tag{5.8}$$

The mapping between twistorial formulation of $(4,0)$ supermultiplet and formulation in terms of vectorial indices $M, N, \ldots = 0, 1, \ldots, 5$ of $SO(5,1)$ as was done by Hull was given in [5]. The field strength $R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$ of the non-metric graviton corresponds to the $(3,3)$ tensor $R_{[MNO][PQR]} = R_{[PQR][MNO]}$ which satisfies self-duality conditions in the first as well as the last 3 indices

$$* R = R* = R \,, \tag{5.9}$$

where $*$ operation is performed with the Levi-Civita tensor in six dimensional Minkowskian spacetime

$$(*R)_{[MNO][PQR]} = \frac{1}{6} \epsilon_{MNOSTU} R^{STU}{}_{PQR} \,.$$

In [5] the gauge potential for the non-metric graviton field strength $R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$ transforming in the $(4,0,0)_D$ representation of $SU^*(4)$ was chosen as a tensor field $C^{\hat{\alpha}}_{(\hat{\beta}\hat{\gamma}\hat{\delta})}$ transforming in the $(3,0,1)_D$ representation such that the field strength involves a single derivative

$$R_{(\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta})} = \partial_{\hat{\lambda}(\hat{\alpha}} C^{\hat{\lambda}}_{\hat{\beta}\hat{\gamma}\hat{\delta})} \tag{5.10}$$

where

$$C^{\hat{\lambda}}_{(\hat{\beta}\hat{\gamma}\hat{\delta})} = C^{\hat{\lambda}}_{\hat{\beta}\hat{\gamma}\hat{\delta}} \,. \tag{5.11}$$

It is invariant under the gauge transformations

$$C^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}\hat{\delta}} \to C^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}\hat{\delta}} + \partial_{\hat{\omega}(\hat{\beta}} \chi^{\hat{\omega}\hat{\alpha}}_{\hat{\gamma}\hat{\delta})} \tag{5.12}$$

where the gauge parameters $\chi^{\hat{\alpha}\hat{\beta}}_{\hat{\gamma}\hat{\delta}}$ satisfy

$$\chi^{\hat{\alpha}\hat{\beta}}_{\hat{\gamma}\hat{\delta}} = -\chi^{\hat{\beta}\hat{\alpha}}_{\hat{\gamma}\hat{\delta}} = \chi^{\hat{\alpha}\hat{\beta}}_{\hat{\delta}\hat{\gamma}} \,. \tag{5.13}$$

However, since the standard Riemann tensor involves two derivatives of the metric one can also choose a gauge potential such that the non-metric graviton field strength involves two derivatives of that gauge potential as was done in [15, 17]. In terms of spinorial indices such a gauge potential must transform as a mixed tensor $C^{\hat{\alpha}\hat{\beta}}_{\hat{\gamma}\hat{\delta}}$ satisfying the conditions

$$C^{\hat{\alpha}\hat{\beta}}_{\hat{\gamma}\hat{\delta}} = C^{\hat{\beta}\hat{\alpha}}_{\hat{\gamma}\hat{\delta}} = C^{\hat{\alpha}\hat{\beta}}_{\hat{\delta}\hat{\gamma}} \tag{5.14}$$

$$C^{\hat{\alpha}\hat{\beta}}_{\hat{\gamma}\hat{\beta}} = 0 \tag{5.15}$$

such that the non-metric graviton field strength is given by

$$R_{(\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta})} = \partial_{\hat{\lambda}(\hat{\alpha}} \partial_{\hat{\epsilon}\hat{\beta}} C^{\hat{\lambda}\hat{\epsilon}}_{\hat{\gamma}\hat{\delta})} \tag{5.16}$$

which is invariant under the gauge transformations

$$C^{\hat{\alpha}\hat{\beta}}_{\hat{\gamma}\hat{\delta}} \longrightarrow C^{\hat{\alpha}\hat{\beta}}_{\hat{\gamma}\hat{\delta}} + \partial_{\hat{\lambda}(\hat{\gamma}} \chi^{\hat{\alpha}\hat{\beta}\hat{\lambda}}_{\hat{\delta})} \tag{5.17}$$

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where the gauge parameters satisfy

$$\chi_{\hat{\delta}}^{\hat{\alpha}\hat{\beta}\hat{\lambda}} = -\chi_{\hat{\delta}}^{\hat{\alpha}\hat{\lambda}\hat{\beta}} \tag{5.18}$$

$$\chi_{\hat{\gamma}}^{\hat{\alpha}\hat{\beta}\hat{\gamma}} = 0 \tag{5.19}$$

which imply that they transform in the 64 dimensional representation of $SU^*(4)$ with Dynkin label $(1,1,1)_D$. The formulation given in [5] and the formulation in terms of a gauge potential of the form 5.16 given in [15, 17] are the analogs of first and second order formalisms in ordinary supergravity [38].[5] The underlying unitary $(4,0)$ supermultiplet that describes the physical degrees of freedom is the same for both formulations just as is the case for first and second order formalism of Poincaré supergravity.

The gauge potential of the non-metric gravitino field strength is a traceless tensor $\psi_{(\hat{\beta}\hat{\gamma})}^{\hat{\alpha}}$. Under a gauge transformation it transforms as

$$\psi_{\hat{\beta}\hat{\gamma}}^{\hat{\alpha}} \to \psi_{\hat{\beta}\hat{\gamma}}^{\hat{\alpha}} + \partial_{\hat{\omega}(\hat{\beta}}\chi_{\hat{\gamma})}^{\hat{\omega}\hat{\alpha}} \,; \tag{5.20}$$

with the gauge parameter $\chi_{\hat{\gamma}}^{[\hat{\alpha}\hat{\beta}]}$ such that $\chi_{\hat{\beta}}^{\hat{\alpha}\hat{\beta}} = 0$. We should note that in contrast to standart local supersymmetry gauge parameter which involves a single spinor index, the gauge parameter $\chi_{\hat{\gamma}}^{\hat{\alpha}\hat{\beta}}$ of this local gauge symmetry of the non-metric gravitino field transforms as a spinor vector under $SU^*(4)$.

In terms of vectorial indices the non-metric gravitino field can be written as $\psi_{[MN]\hat{\alpha}}$ such that its field strength $\psi_{[MNO]\hat{\alpha}}$ satisfies the self-duality condition

$$* \psi = \psi \tag{5.21}$$

Similarly the tensor field with field strength $h_{(\hat{\alpha}\hat{\beta})}$ is $b_{\hat{\beta}}^{\hat{\alpha}}$ so that

$$h_{(\hat{\alpha}\hat{\beta})} = \partial_{\hat{\gamma}(\hat{\alpha}}b_{\hat{\beta})}^{\hat{\gamma}}, \quad b_{\hat{\alpha}}^{\hat{\alpha}} = 0 \tag{5.22}$$

It undergoes gauge transformations with parameters $\chi^{\hat{\alpha}\hat{\beta}} = -\chi^{\hat{\beta}\hat{\alpha}}$. In terms of vectorial indices it is described by an anti-symmetric tensor field $b_{[MN]}$ whose field strength $h_{[MNP]}$ satisfies the selfduality condition

$$*h = h$$

5.2 A review of exceptional field theoretic formulation of the linearized $(4,0)$ supergravity

The fact that the ultrashort $(4,0)$ supermultiplet of $N = 8$, $d = 6$ superconformal algebra $OSp(8|8^*)$ reduces to the ultrashort doubleton supermultiplet of $N = 8$ $4d$ superconformal algebra $SU(2,2|8)$ that describes the fields of $4d$ maximal supergravity raises several interesting questions. First is the question whether there exist interacting supergravity theory or theories based on this supermultiplet. In four dimensions interactions in maximal

Poincare supergravity break the superconformal symmetry $SU(2,2|8)$ down to its Poincare subalgebra. On the other hand $N = 4$ super Yang-Mills theory based on the corresponding ultrashort unitary supermultiplet of $PSU(2,2|4)$ is conformally invariant even at the quantum level. Remarkably the amplitudes of maximal $N = 8$ Poincare supergravity can be obtained by double copy methods of Bern, Carrasco and Johansson (BCJ) [39, 40] from those of $N = 4$ super Yang-Mills theory that is conformally invariant [41].[6]

Existence of an interacting $4d$ Poincare supergravity based on the conformal ultrashort CPT self-conjugate supermultiplet of $SU(2,2|8)$ suggests that an interacting Poincare supergravity in $6d$ based on the conformal $(4,0)$ supermultiplet of $OSp(8^*|8)$ also exists in which the interactions break the superconformal symmetry $OSp(8^*|8)$ down to its Poincare subgroup. On the other hand the existence of conformal supergravity based on the $(4,0)$ supermultiplet in six dimensions is an open question just like the existence of an interacting $N = 8$ conformal supergravity in $d = 4$ as will be discussed below.

In six dimensions the $(2,0)$ supermultiplet of $OSp(8^*|4)$ is the conformal analog of $4d$ $N = 4$ Yang-Mills supermultiplet [8]. It is generally believed that the interacting theories of $(2,0)$ supermultiplets in six dimensions are not conventional field theories and may only exist as quantum theories [43]. Nonetheless they reduce to conventional field theories in lower dimensions. In [5] it was pointed out that the $(4,0)$ supermultiplet can be obtained by tensoring the $(2,0)$ supermultiplets. This raises the possibility that the "amplitudes" or correlation functions of an interacting superconformal $(2,0)$ theory could yield the amplitudes of an interacting non-metric Poincare supergravity based on the $(4,0)$ supermultiplet via some generalization of double copy methods of BCJ.

Even though the supermultiplet of fields of $(4,0)$ supergravity and their free equations were known for a long time the action for linearized non-metric $(4,0)$ Poincare supergravity in six dimensions was first written down rather recently in [17]. The authors of [17] use the formalism of prepotentials adapted to the self-duality properties of the fields of the $(4,0)$ supermultiplet. They show that the resulting action is invariant under $(4,0)$ Poincaré supersymmetry in $d = 6$ but not manifestly. The reason for loss of manifest Poincaré covariance is due to the fact that to write down the action they split the $6d$ spacetime coordinates as $5+1$ with the singlet coordinate being timelike. In their Lagrangian formulations of the bosonic self-dual tensors which they refer to as chiral two-forms [44] as well as bosonic chiral $(2,2)$-tensor corresponding to the gauge potential of non-metric graviton [18] involve only spatial tensors, and their temporal components are pure gauge.[7] They also present a similar formulation for the non-metric gravitensorino ("chiral spinorial two-form"). The resulting action of free $(4,0)$ supergravity in terms of prepotentials is fourth order in spatial derivatives.[8]

$(4,0)$ supergravity as well as the $(3,1)$ supergravity were also studied within the so-called exceptional field theory (ExFT) formalism by the authors of [29] recently. Before giving the exceptional field theoretic formulation of these theories they first present novel

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[6]For a review and references on the subject see [42].

[7]With the caveat that equations of motion do not yield the self-duality conditions directly but an equivalent differential form.

[8]The authors of [17] also present the action for the free $(3,1)$ supergravity. Unlike the $(4,0)$ supermultiplet $(3,1)$ Poincare supermultiplet does not extend to a conformal supermultiplet in six dimensions.

actions for the bosonic sectors of linearized $(4,0)$ and $(3,1)$ using the $5+1$ split of six dimensional spacetime coordinates such that the singlet coordinate y is space-like. These actions are two-derivative actions that reduce to the bosonic sector of linearized maximal supergravity in five dimensions.

The exceptional field theory formalism is a particular extension of the double field theory formalism and is an outgrowth of the attempts to make the hidden U-duality groups of lower dimensional supergravity theories manifest in higher dimensions from which they can be obtained by toroidal compactification.[9] To achieve this one intoduces an auxiliary "internal" spacetime with coordinates Y^M motivated by the U-duality group with which to extend the standard external d-dimensional spacetime with coordinates x^μ and imposes a section constraint such that the resulting theory describes the higher dimensional supergravity theory. For 5d maximal supergravity one introduces a 27 dimensional auxiliary internal space-time extending the 5 dimensional external spacetime and imposes a section constraint of the form

$$C^{IJK}\partial_J \otimes \partial_K = 0 \tag{5.23}$$

where C^{IJK} is the $E_{6(6)}$ invariant symmetric tensor. The above equation is to be interpreted as differentials acting on functions $f(x,Y), g(x,Y)$ of extended coordinates x^μ, Y^I such that

$$C^{IJK}\partial_J \partial_K f(x,Y) = 0\,, \qquad C^{IJK}\partial_J f(x,Y)\partial_K g(x,Y) = 0 \tag{5.24}$$

To recover 11-d supergravity corresponding to the $5+6$ split of the coordinates one decomposes the 27 internal coordinates Y^I with respect to $\mathrm{SL}(6,\mathbb{R}) \times \mathrm{GL}(1)$ subgroup of $E_{6(6)}$

$$27 = 6^{+1} + 15^0 + 6^{-1} \Leftrightarrow Y^I = (y^m, y^{mn} = -y^{nm}, \bar{y}^m) \tag{5.25}$$

where $m, n, \ldots = 1, \ldots, 6$. By restricting the dependence on Y^I only to the 6 coordinates y^m one obtains a solution to section constraints that leads to the 11 dimensional supergravity [46].[10]

To construct the linearized 6d $(4,0)$, $(3,1)$ and standard $(2,2)$ supergravity theories as ExFTs in a unified manner describing the maximal $N=8$ supergravity in $5d$ the authors of [29] extend the 27 dimensional internal space-time with an extra singlet coordinate $Y^\bullet$ and impose the more general section constraint

$$C^{IJK}\partial_J \otimes \partial_K - \frac{1}{\sqrt{10}}\Delta^{IJ}(\partial_J \otimes \partial_\bullet + \partial_\bullet \otimes \partial_J) = 0 \tag{5.26}$$

where Δ^{IJ} is a constant tensor describing the background spacetime. Setting $\partial_\bullet = 0$ one has the standart section constraint of the formulation of the maximal $N=8$ supergravity as an ExFT whose solutions include the 11d sugra, type IIB supergravity and maximal $(2,2)$ Poincare supergravity theory in $6d$. On the other hand setting $\partial_I = 0$ the generalized section constraint 5.26 is satisfied trivially and by identifying the extra coordinate $Y^\bullet$ with

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[9]For a review and references on the subject see [45].

[10]Type IIB supergravity corresponds to the decomposition of 27 of $E_{6(6)}$ with respect to the $\mathrm{SL}(5,\mathbb{R}) \times$ $\mathrm{SL}(2,\mathbb{R} \times \mathrm{GL}(1)$ subgroup given by $27 = (5,1)^{-4} + (5',2)^{-1} + (10,1)^{-2} + (1,2)^{-5}$ and restricting dependence on internal coordinates to the coordinates y^a in $(5,1)$.

the space-like coordinate in the $(5+1)$ split of $6d$ coordinates x^μ they show that one obtains the bosonic sector of linearized $(4,0)$ supergravity. In addition to $(2,2)$ and linearized $(4,0)$ supergravity in $6d$ the generalized section constraint admits a solution corresponding to $(3,1)$ supergravity at the linearized level as well. They leave to future work the supersymmetric extension of the bosonic sector of $(4,0)$ supergravity. Furthermore the fact that a unified framework exists for ExFT formulations of $6d$ $(2,2)$, $(4,0)$ and $(3,1)$ supergravity theories and the $(2,2)$ theory can be extended to the full nonlinear theory is interpreted by the authors of [29] as evidence that the same may be true for the $(4,0)$ and $(3,1)$ theories. If the interacting non-metric $(4,0)$ supergravity exists as a Lagrangian theory it will admit a formulation as an ExFT describing the uplift of maximal Poincare supergravity to six dimensions. However if the interacting $(4,0)$ theory is non-Lagrangian we shall assume that an appropriate generalization of the ExFT formalism exists with which to uplift the $5d$ maximal supergravity to $6d$ as a non-metric $(4,0)$ Poincare supersymmetric theory.

5.3 On the question of existence of interacting $N = 8$ conformal supergravity theory in $d = 4$ and $(4,0)$ conformal supergravity in $d = 6$

The existence of an interacting conformal supergravity based on the $(4,0)$ supermultiplet in $d = 6$ would suggest the existence of an interacting conformal supergravity in $d = 4$ based on the CPT self-conjugate doubleton supermultiplet of $SU(2,2|8)$ whose existence was posed as an open problem in [3]. To this date no such supergravity theory has been constructed. Conformal supergravity theories were first studied in the pioneering papers of [47, 48].[11] Recently they have been studied as massless limits of Einstein-Weyl supergravity theories.[12] The standard conformal supergravity theories in $d = 4$ based on the conformal superalgebras $SU(2,2|N)$ exist only for $N \leq 4$. All N =4 conformal supergravities in $d = 4$ have recently been constructed in [51, 52]. In addition to a massless graviton they contain a massive spin two ghost field and hence are not unitary. The constraint $N \leq 4$ arises from the fact that for $N \geq 4$ the conformal supermultiplets containing the massive spin two field must necessarily contain fields of spin greater that two. Furthermore the vector fields associated with the gauge fields of $SU(n)$ and $U(1)$ subgroups inside $U(n) \subset SU(2,2|n)$ have kinetic energy terms that are of opposite sign and hence are not all positive definite. Hence if an interacting $N = 8$ superconformal theory exists that is unitary its formulation must go beyond the standard local gauging of the underlying conformal superalgebras. It may exist purely at the quantum level without a Lagrangian formulation or its Lagrangian formulation may be non-local. In fact there are non-local formulations of conformal gravity that are both unitary and ultraviolet finite.[13] Their Lagrangians typically involve infinite number of terms that are bilinear in scalar curvature R, Ricci tensor $R_{\mu\nu}$ and Riemann tensor $R_{\mu\nu\rho\lambda}$ with powers of the D'Alembertian sandwiched between them. These results are consistent with the findings of [55] who studied the four

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[11]For an older review see [49].

[12]For a review see [50] and the references therein.

[13]For reviews we refer to [53, 54] and references therein.

derivative action of Stelle [56] of the form

$$S = M_{Pl}^2 \int d^4x \sqrt{-g} \left[-2\Delta + R + \frac{1}{\mu^2} \left(\frac{1}{3} R^2 - R^{\mu\nu} R_{\mu\nu} \right) \right] \tag{5.27}$$

where M_{Pl} is the Planck mass, Δ is the cosmological constant and μ indicates the mass scale. This theory has a massive spin two ghost of mass μ whose kinetic energy has opposite sign to that of the massless mode. They show that there exists a ghost free completion of this theory which requires an infinite series of higher derivative terms. The resulting theory is classically equivalent to ghost-free bimetric theory of two symmetric tensor fields studied in [57]. Whether these ultraviolet finite unitary nonlocal (higher derivative) theories admit supersymmetric extensions that go beyond the $N = 4$ bound and result in a unitary theory whose massless physical spectrum coincides with the CPT-self-conjugate doubleton supermultiplet of $SU(2,2|8)$ is an open problem.

The question about the existence of interacting conformal supergravity theory with $OSp(8^*|8)$ symmetry in $d = 6$ is more subtle. Such a theory would necessarily involve a non-metric "graviton" and no interacting non-metric gravity theories have been constructed to date. The metric conformal supergravity theories in $d = 6$ with superconformal symmetry $OSp(8^*|2N)$ exist only for $N \leq 2$). Whether there exist non-local (higher derivative) and non-metric conformal supergravity theories with $OSp(8^*|8)$ symmetry and are unitary is an open problem.

6   Truncations of $(4, 0)$ supergravity

Assuming that there exists an interacting $(4, 0)$ supergravity that reduces to the maximal $N = 8$ supergravity one can consider its truncations to interacting theories with lower number of supersymmetries.

6.1 $(2, 0)$ supersymmetric truncations

The decomposition of the $(4, 0)$ supermultiplet of $OSp(8^*|8)$ into supermultiplets of $N = 4$ superconformal algebra $OSp(8^*|4)$ in $d = 6$ and some of its implications were studied in [5]. One finds that $(4, 0)$ multiplet decomposes into one $(2, 0)$ non-metric graviton supermultiplet [5]

$$R_{(\hat\alpha\hat\beta\hat\gamma\hat\delta)}(x), \quad \psi^a_{(\hat\alpha\hat\beta\hat\gamma)}(x), \quad h^{[ab]|}_{(\hat\alpha\hat\beta)}(x) \oplus h_{(\hat\alpha\hat\beta)}(x), \quad \lambda^{[abc]|}_{\hat\alpha}(x), \quad \phi(x) \tag{6.1}$$

four $(2, 0)$ non-metric gravitino supermultiplets

$$\psi_{(\hat\alpha\hat\beta\hat\gamma)}(x), \quad h^a_{(\hat\alpha\hat\beta)}(x), \quad \lambda^{[ab]|}_{\hat\alpha}(x) \oplus \lambda_{\hat\alpha}(x), \quad \phi^{[abc]|}(x) \tag{6.2}$$

and five $(2, 0)$ self-dual tensor multiplets (doubletons)

$$h_{(\hat\alpha\hat\beta)}(x), \quad \lambda^a_{\hat\alpha}(x), \quad \phi^{[ab]|}(x) \tag{6.3}$$

where $a, b, c = 1, 2, 3, 4$ are the $USp(4)$ indices. Therefore the interacting $(4, 0)$ theory must admit a consistent truncation that describes the coupling of the non-metric $(2, 0)$ graviton supermultiplet to five $(2, 0)$ doubleton supermultiplets.

The $(2,0)$ supersymmetric truncation of the $(4,0)$ conformal supergravity reduces to $N = 4$ supergravity coupled to 5 vector multiplets in $d = 5$ with the global symmetry group $\mathrm{SO}(5,5) \times \mathrm{SO}(1,1)$ which is also the global symmetry group of the 6 dimensional theory with the moduli space

$$\mathrm{SO}(5,5) \times \mathrm{SO}(1,1)/\mathrm{SO}(5) \times \mathrm{SO}(5)$$

The tensor fields of the six dimensional theory form the $(10+1)$ representation of $\mathrm{SO}(5,5) \times \mathrm{SO}(1,1)$. The resulting supergravity can be gauged in $d = 5$ to obtain Yang-Mills Einstein supergravity theories with various possible gauge groups, in particular $\mathrm{SU}(2) \times \mathrm{U}(1)$. Since it is generally believed that strongly coupled phase of $5d$ maximal super Yang-Mills is described by an interacting $(2,0)$ theory in $d = 6$ this raises the question whether there exist deformations of the non-linear $(4,0)$ theory that correspond to various gaugings of maximal supergravity in $d = 5$ or in $d = 4$. In general not all gaugings of maximal supergravity are expected to have uplifts to higher dimensions since gaugings in general introduce potentials with anti-de Sitter as well as de Sitter vacua.

6.2 $(1,0)$ supersymmetric truncations of $(4,0)$ conformal supergravity

The $(4,0)$ supermultiplet of $\mathrm{OSp}(8^*|8)$ can be decomposed into $(1,0)$ supermultiplets. Splitting the $\mathrm{USp}(8)$ indices $A, B, \ldots$ as $A = (a,i), B = (b,j), \ldots$ where $i, j, \ldots = 1, 2$ and $a, b, \ldots = 3, 4, \ldots, 8$ we find

- non-metric graviton supermultiplet

$$R_{(\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta})}(x), \quad \psi^i_{(\hat{\alpha}\hat{\beta}\hat{\gamma})}(x), \quad h^{[ij]}_{(\hat{\alpha}\hat{\beta})}(x)$$

- 6 non-metric gravitino (gravitensorino) supermultiplets

$$\psi^a_{(\hat{\alpha}\hat{\beta}\hat{\gamma})}(x), \quad h^{ai}_{(\hat{\alpha}\hat{\beta})}(x), \quad \lambda^{a[ij]}_{\hat{\alpha}}(x)$$

- 14 (1,0) self-dual tensor multiplets

$$h^{[ab]|}_{(\hat{\alpha}\hat{\beta})}(x), \quad \lambda^{[ab]|i}_{\hat{\alpha}}(x), \quad \phi^{[ab]|ij}(x)$$

transforming in the antisymmetric symplectic traceless representation of $\mathrm{USp}(6)$.

- 14 (1,0) hypermultiplets

$$\lambda^{[abc]|}_{\hat{\alpha}}(x), \quad \phi^{[abc]|i}(x)$$

transforming as symplectic traceless anti-symmetric tensor of rank three which we will denote as $14'$.

In parallel to full $5d$, $N = 8$ supergravity we expect the interacting $(4,0)$ superconformal theory to admit two maximal $(1,0)$ supersymmetric truncations. By discarding the 6 generalized gravitino multiplets and 14 hyper multiplets one obtains an interacting theory describing the coupling of non-metric $(1,0)$ graviton supermultiplet to 14 tensor multiplets with the global symmetry group $\mathrm{SU}^*(6)$ under which 15 tensor fields, including the

gravitensor transform irreducibly. 14 scalar fields belong to the coset $\mathrm{SU}^*(6)/\mathrm{USp}(6)$ in the quaternionic magical theory in $d = 5$. On the other hand by discarding the 6 non-metric gravitino multiplets together with the 14 tensor multiplets one obtains a consistent truncation to a $6d$ theory describing the coupling of non-metric $(1,0)$ graviton supermultiplet to 14 hypermultiplets which reduces in five dimensions to $N = 2$ supergravity coupled to 14 hypermultiplets with the scalar manifold $F_{4(4)}/\mathrm{USp}(6) \times \mathrm{SU}(2)$ [19].

The $(1,0)$ truncation of $(4,0)$ theory that reduces to the quaternionic magical theory in 5d can be further truncated such that the resulting theory describes a unified theory in $d = 5$. First, by restricting to the U(1) invariant sector in the decomposition of $\mathrm{SU}^*(6)$ with respect to its subgroup $\mathrm{SL}(3,\mathbb{C}) \times \mathrm{U}(1)$, which requires discarding 6 tensor multiplets, one obtains a theory describing the coupling of non-metric $(1,0)$ graviton supermultiplet to eight tensor multiplets that reduces to the complex magical supergravity theory in $d = 5$ whose moduli space is $\mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3)$. Second, by a further restriction to $\mathbb{Z}_2$ invariant sector under the decomposition $\mathrm{SL}(3,\mathbb{C}) \supset \mathrm{SL}(3,\mathbb{R}) \times \mathbb{Z}_2$ one obtains a 6d theory describing the coupling to 5 tensor multiplets that reduces to the real magical supergravity in $5d$ with the moduli space $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$.

In all the above truncations of the interacting $(4,0)$ theory to a $(1,0)$ supersymmetric theory describing the coupling of generalized graviton multiplet to tensor multiplets all the tensor fields, including the gravitensor, transform in an irrep of the global symmetry group, which are $\mathrm{SU}^*(6), \mathrm{SL}(3,\mathbb{C})$ and $\mathrm{SL}(3,\mathbb{R})$, respectively. The quaternionic theory with global $\mathrm{SU}^*(6)$ symmetry can be extended to the octonionic theory with the global symmetry group $E_{6(-26)}$ symmetry by coupling additional 12 tensor multiplets and reduces to the octonionic magical supergravity theory with 27 vector fields in $d = 5$.

6.3 $(3,0)$ supersymmetric truncation of $(4,0)$ supergravity

Using the labelling of indices as in the previous subsection one can show that by discarding two $N = 6$ gravitensorino multiplets consisting of the fields

$$\psi^i_{(\hat\alpha\hat\beta\hat\gamma)}(x), \quad h^{ia}_{(\hat\alpha\hat\beta)}(x), \quad \lambda^{i[ab]|}_{\hat\alpha}(x), \quad \phi^{i[abc]|}(x)$$

one obtains the non-metric $(3,0)$ graviton supermultiplet consisting of the fields

$$R_{(\hat\alpha\hat\beta\hat\gamma\hat\delta)}(x), \quad \psi^a_{(\hat\alpha\hat\beta\hat\gamma)}(x), \quad h^{[ab]|}_{(\hat\alpha\hat\beta)}(x) + h^{[ij]}_{(\hat\alpha\hat\beta)}(x), \quad \lambda^{[abc]|}_{\hat\alpha}(x) + \lambda^{[ij]a}_{\hat\alpha}(x), \quad \phi^{[ij][ab]|}(x)$$

The truncation to the non-metric $(3,0)$ supergravity theory has the same bosonic field content as the truncation to the maximal $(1,0)$ tensor Einstein supergravity with 14 self-dual tensor multiplets and the 14 scalars. The corresponding result for the $5d$ supergravity, namely that $N = 6$ supergravity has the same bosonic content as the quaternionic magical theory with the symmetric target space $\mathrm{SU}^*(6)/\mathrm{USp}(6)$ as was shown in [19].

7 ExFT formulation of metric and non-metric $(1,0)$ magical supergravity theories in six dimensions

The unified ExFT formulation of the bosonic sector of linearized $(4,0)$ and $(2,2)$ Poincare supergravities as formulated in [29] can be readily extended to a unified construction of

$(1, 0)$ metric and non-metric supergravity theories in six dimensions that descend to the four magical supergravity theories in five dimensions. For the octonionic magical supergravity which can not be obtained from maximal supergravity by truncation one imposes the section constraint

$$C^{IJK}\partial_J \otimes \partial_K - \frac{1}{\sqrt{10}}\Delta^{IJ}(\partial_J \otimes \partial_\bullet + \partial_\bullet \otimes \partial_J) = 0 \qquad (7.1)$$

where the C-tensor C^{IJK} is the one given by the cubic norm of the real exceptional Jordan algebra $J_3^{\mathbb{O}}$ which is Euclidean. It an invariant tensor of $E_{6(-26)}$. To obtain the ExFT formulation of the metric $(1, 0)$ supergravity in $d = 6$ one first decomposes the indices of 27 dimensional representation of $E_{6(-26)}$ with respect ot its SO(9, 1) subgroup

$$C^{0ab} = \frac{1}{\sqrt{10}}\eta^{ab}, \qquad C^{a\alpha\beta} = \frac{1}{2\sqrt{5}}(\Gamma^a)^{\alpha\beta} \qquad (7.2)$$

where η^{ab} is the SO(9, 1) invariant metric and $(\Gamma^a)^{\alpha\beta}$ are the gamma matrices of SO(9, 1) with $(a, b, \ldots = 0, 1, \ldots 9)$ and $(\alpha, \beta, \ldots = 1, 2, \ldots, 16)$. Imposing the conditions

$$\partial_\alpha = \partial_a = \partial_\bullet = 0 \qquad (7.3)$$

solves the section constraint and the resulting ExFT describes the metric octonionic magical supergravity in $d = 6$ with 9 tensor multiplets and 16 vector multiplets and scalar manifold SO(9, 1)/SO(9) [33].

For constructing the non-metric $(1, 0)$ supergravity theory one imposes the condition $\partial_I = 0$ and identifies the coordinate $Y^\bullet$ with the spatial singlet component y in the $(5 + 1)$ split of external $6d$ spacetime coordinates x^μ as was done for the maximal supergravity in [29].

To obtain the unified ExFT formulations of metric and non-metric $(1, 0)$ supergravity theories that reduce to the linearized quaternionic, complex and magical supergravity theories in $d = 5$ one needs to simply substitute the C-tensors of these theories in the section constraint 5.26 and decompose the indices with respect to subgroups of their $5d$ U-duality groups listed in the first column of table 2 and the gamma matrices by those listed in column 4 of that table. For the quaternionic magical theory the tensor C^{IJK} becomes the invariant tensor of the internal Lorentz (reduced structure) group $SU^*(6)$ of $J_3^{\mathbb{H}}$. The resulting metric $(1, 0)$ supergravity describes the coupling of 6 tensor multiplets and eight vector multiplets to metric $(1, 0)$ supergravity. The corresponding non-metric $(1, 0)$ supergravity theory describes the coupling of 14 tensor multiplets to non-metric $(1, 0)$ supergravity. In contrast to the metric theory the non-metric supergravity theory describes a unified theory since the 15 tensor multiplets transform irreducibly under the global symmetry group $SU^*(6)$. Since the quaternionic magical supergravity can be embedded both in maximal supergravity and octonionic magical supergravity in five dimensions the corresponding ExFts describing the metric and non-metric $(1, 0)$ supergravity theories in $6d$ can also be obtained by truncation of the unified formulation of ExFTs describing $(4, 0)$ and $(2, 2)$ theories in $6d$.

The ExFT defined by $J_3^{\mathbb{H}}$ can be consistently truncated to a $(1, 0)$ unified non-metric tensor-Einstein supergravity described by the Euclidean Jordan algebra $J_3^{\mathbb{C}}$ describing the

coupling of 8 tensor multiplet to non-metric $(1,0)$ supergravity. It is an invariant tensor of the Lorentz (reduced structure) group $\mathrm{SL}(3,\mathbb{C})$ of $J_3^{\mathbb{C}}$. The latter theory can be further truncated to an ExFT corresponding to the real magical supergravity in $d=5$ defined by $J_3^{\mathbb{R}}$. Its C-tensor is invariant under $\mathrm{SL}(3,\mathbb{R})$ and describes the coupling of 5 self-dual tensors to non-metric $(1,0)$ supergravity in six dimensions.

8 Unified non-metric tensor-Einstein supergravity theories in $d=6$

As we reviewed in section 2 there exist three infinite families of unified $N=2$ MESGTs in $d=5$. Their C-tensors are given by the structure constants d_{IJK} of the traceless elements of Lorentzian Jordan algebras $J_{(1,N)}^{\mathbb{A}}$ of degree three over the associative division algebras $\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H}$ which are invariant under their automorphism groups of $J_{(1,N)}^{\mathbb{A}}$. Since the structure constants d_{IJK} of three Lorentzian Jordan algebras of degree four, namely $J_{(1,3)}^{\mathbb{R}}$, $J_{(1,3)}^{\mathbb{C}}$ and $J_{(1,3)}^{\mathbb{H}}$, coincide with the C-tensors defined by the norms of the Euclidean Jordan algebras of degree three, $J_3^{\mathbb{C}}$, $J_3^{\mathbb{H}}$ and $J_3^{\mathbb{O}}$ respectively [28], they have accidental enlarged hidden symmetries:

$$Aut[J_{(1,3)}^{\mathbb{R}}] = \mathrm{SO}(3,1) \qquad \Longrightarrow \qquad Str_0(J_3^{\mathbb{C}}) = \mathrm{SL}(3,\mathbb{C}) \qquad (8.1)$$

$$Aut[J_{(1,3)}^{\mathbb{C}}] = \mathrm{SU}(3,1) \qquad \Longrightarrow \qquad Str_0(J_3^{\mathbb{H}}) = \mathrm{SU}^*(6) \qquad (8.2)$$

$$Aut[J_{(1,3)}^{\mathbb{H}}] = \mathrm{USp}(6,2) \qquad \Longrightarrow \qquad Str_0(J_3^{\mathbb{O}}) = E_{6(-26)} \qquad (8.3)$$

The enlarged symmetry groups $\mathrm{SL}(3,\mathbb{C}), \mathrm{SU}^*(6)$ and $E_{6(-26)}$ are the global symmetry groups of the $5d$ MESGTs defined by the corresponding Euclidean Jordan algebras. As we discussed above these three $5d$ unified MESGTs theories can be obtained from unified tensor-Einstein supergravity theories in $6d$. For the non-metic tensor-Einstein supergravity theories to be unified theories all the tensor fields including the gravitensor must transform irreducibly under the global symmetry group. Remarkably the corresponding irreducible representations of $\mathrm{SL}(3,\mathbb{C}), \mathrm{SU}^*(6)$ and $E_{6(-26)}$ remain irreducible under the restriction to the manifest symmetry subgroups $\mathrm{SO}(3,1), \mathrm{SU}(3,1)$ and $\mathrm{USp}(6,2)$, respectively.

For the other members of the three infinite families of unified MESGTs there is no symmetry enhancement beyond the automorphism groups of the underlying Lorentzian Jordan algebras. Nonetheless we expect them to descend from unified tensor-Einstein supergravity theories in $6d$ in a similar fashion. In addition to the 3 infinite families there exist a unified MESGT in $d=5$ defined by the Lorentzian octonionic Jordan algebra of degree three $J_{(1,2)}^{\mathbb{O}}$. This isolated theory is also expected to descend from a unified tensor-Einstein supergravity in $d=6$. In tables 4 and 5 we give the list of unified tensor-Einstein supergravity theories in $d=6$, their field content and global symmetry groups, under which all the tensor fields including the gravitensor form an irrep.

As is evident from the tables 4 and 5 in the decomposition of the irreducible representation of the global symmetry group with respect to its maximal compact subgroup there is a unique singlet which is to be identified with the "bare gravitensor". Furthermore all the bosonic fields in tensor Einstein supergravity theories are singlets of the R-symmetry group $\mathrm{USp}(2)$.

J	$\mathrm{Aut}(J) \supset K$	Decompositon of self-dual tensor fields under K	Number of scalars
$J^{\mathbb{R}}_{(1,N)}$	$\mathrm{SO}(N,1) \supset \mathrm{SO}(N)$	$\frac{1}{2}N(N+3) = [\frac{1}{2}N(N+1)-1] \oplus N \oplus 1$	$\frac{1}{2}N(N+3)-1$
$J^{\mathbb{C}}_{(1,N)}$	$\mathrm{SU}(N,1) \supset \mathrm{SU}(N) \times \mathrm{U}(1)$	$N(N+2) = [N^2-1]^0 \oplus N^{+1} \oplus N^{-1} \oplus 1^0$	$N(N+2)-1$
$J^{\mathbb{H}}_{(1,N)}$	$\mathrm{USp}(2N,2) \supset \mathrm{USp}(2N) \times \mathrm{USp}(2)$	$N(2N+3) = [(N(2N-1)-1,1)] \oplus (2N,2) + (1,1)$	$N(2N+3)-1$
$J^{\mathbb{O}}_{(1,2)}$	$F_{4(-20)} \supset \mathrm{SO}(9)$	$26 = 9 \oplus 16 \oplus 1$	25

Table 4. Unified tensor-Einstein supergravity theories in $d=6$ that reduce to unified MESGTs in $d=5$ defined by simple Lorentzian Jordan algebras $J^{\mathbb{A}}_{(1,N)}$ whose scalar manifolds are not homogeneous spaces for $N \neq 3$. Second column lists their global symmetry groups $\mathrm{Aut}(J^{\mathbb{A}}_{(1,N)})$ and their maximal compact subgroups K. Third column lists dimension of the irrep of tensor fields under $\mathrm{Aut}(J^{\mathbb{A}}_{(1,N)})$ and their decomposition with respect to the maximal compact subgroup K. The last column lists the number of scalars. The theories with $N=3$ have hidden larger symmetries and correspond to three of the magical supergravity theories.

J	Global symmetry group	Number of self-dual tensor fields	Scalar Manifold
$J^{\mathbb{R}}_3$	$\mathrm{SL}(3,\mathbb{R}) \supset \mathrm{SO}(3)$	$6 = 5 \oplus 1$	$\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$
$J^{\mathbb{C}}_3$	$\mathrm{SL}(3,\mathbb{C}) \supset \mathrm{SU}(3)$	$9 = 8 \oplus 1$	$\mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3)$
$J^{\mathbb{H}}_3$	$\mathrm{SU}^*(6) \supset \mathrm{USp}(6)$	$15 = 14 \oplus 1$	$\mathrm{SU}^*(6)/\mathrm{USp}(6)$
$J^{\mathbb{O}}_{(3)}$	$E_{6(-26)} \supset F_4$	$27 = 26 \oplus 1$	$E_{6(-26)}/F_4$

Table 5. Unified tensor-Einstein supergravity theories in $d=6$ that reduce to magical supergravity theories in $d=5$ defined by simple Euclidean Jordan algebras $J^{\mathbb{A}}_3$ of degree three where $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. Second column lists their global symmetry groups and their maximal compact subgroups. Third column lists the number of tensor fields and their decomposition with respect to the maximal compact subgroup. The last column lists their scalar manifolds which are symmetric spaces.

One can gauge certain subgroup of the global symmetry groups of the unified MESGTs in $d = 5$. This naturally leads to the question whether the corresponding six dimensional tensor-Einstein supergravity theories admit interactions among tensor fields that reduce to the non-Abelian gauge interactions in five dimensions. It is generally believed that the interacting superconformal $(2, 0)$ theories do not admit a Lagrangian formulation since they decribe multiple M-5 branes that are strongly coupled with no free parameter for formulating a perturbative Lagrangian theory. Nonetheless a novel method of introducing such non-Abelian couplings in certain $(1, 0)$ superconformal field theories in $d = 6$ was developed by Samtleben and collaborators [58–60]. It is an open problem whether some of these theories can be coupled to non-metric $(1, 0)$ supergravity in $6d$ that upon dimensional reduction reduce to $5d$, $N = 2$ Yang-Mills-Einstein supergravity theories.

8.1 Dimensional reduction of non-metric $(4, 0)$ and $(1, 0)$ tensor Einstein supergravity theories

Dimensional reduction of $(4, 0)$ supergravity multiplet using vectorial indices was performed by Hull who showed that the resulting field content coincides with that of maximal Poincare supergravity in five dimensions. Dimensional reduction of the $(4, 0)$ unitary supermultiplet using the twistorial spinor indices was given in [5]. Twistorial oscillator method yields manifestly unitary supermultiplets which involve only the physical degrees of freedom. Hence the resulting supermultiplets involve only the field strengths and not the gauge fields. For the doubleton supermultiplet of $OSp(8^*|8)$ of the $(4, 0)$ non-metric supergravity in $d = 6$ the field strengths correspond to symmetric tensors in the spinor indices $\hat{\alpha}, \hat{\beta}, \ldots$ of the Lorentz group $SU^*(4)$. Since the spinor representation $S_{\hat{\alpha}}$ of 6 dimensional Lorentz group $SU^*(4)$ and symmetric tensor representations $S_{\hat{\alpha}\hat{\beta}\hat{\delta},\ldots}$ remain irreducible under the restriction to the five dimensional Lorentz group $USp(2, 2)$, the dimensional reduction to $d = 5$ in the twistorial formulation as given in [5] is much simpler than in the formulation involving vectorial indices. There is a one-to-one correspondence between the field strengths belonging to the doubleton supermutiplet of $OSp(8^*|8)$ and the field strengths of the fields of maximal supergravity in $d = 5$. In particular non-metric graviton in $6d$ with the field strength $R_{(\hat{\alpha}\hat{\beta}\hat{\delta}\hat{\gamma})}$ reduces to the five dimensional graviton field strength without an extra vector or scalar field [5, 15]. This is to be contrasted with the $6d$ metric graviton which reduces to a graviton plus a vector and a scalar. Similarly the gravitensor field strength $\psi^A_{(\hat{\alpha}\hat{\beta}\hat{\gamma})}$ remains irreducible under restriction to $USp(2, 2)$ subgroup of $SU^*(4)$ and becomes the field strength of a gravitino in $5d$:

$$\psi^A_{(\alpha\beta\gamma)} = \partial_{[\mu}\psi^A_{\nu](\alpha} * \gamma^{\mu\nu}_{\beta\gamma)}$$

where $*\gamma^{\mu\nu} = \frac{1}{3!}\epsilon^{\mu\nu\lambda\epsilon\delta}\gamma_{\lambda\epsilon\delta}$. Self-dual tensor fields reduce to vector fields in $d = 5$ and the symplectic Majorana Weyl spinors go over to symplectic Majorana spinors in $d = 5$.

Therefore non-metric $(1, 0)$ supersymmetric tensor Einstein supergravity theory in $d = 6$ will reduce to a five dimensional $N = 2$ supergravity with the same number of $5d$ vector fields as self-dual tensors in $6d$. The bare gravitensor in $6d$ will reduce to the bare graviphoton in $5d$. What distinguishes unified tensor-Einstein supergravity theories from

others is the fact that gravity sector can not be decoupled from the tensorial matter sector without breaking their global symmetry groups since the gravitensor together with the other tensor fields transform irreducibly under them. Ungauged tensor Einstein theories reduce to Maxwell-Einstein supergravity theories in $d = 5$.

Unified MESGT theories, in particular the magical supergravity theories , admit gaugings with simple gauge groups with or without tensor fields in five dimensions. It was shown in [33] that Poincare uplifts of magical supergravity theories to six dimensions do not admit gaugings with simple groups. Furthermore Poincare uplifts of magical supergravity theories in 6d are no longer unified since some of the vector fields uplift to selfdual tensors while the others uplift to vector fields in 6d. In addition one finds that magical Poincare supergravity theories in 6d admit a unique gauge group which is a nilpotent Abelian group with (n_T-1) translation generators, where n_T is the number of tensor multiplets coupled to $(1, 0)$ metric supergravity. These $(n_t - 1)$ generators can not lie within the isometry group $\mathrm{SO}(n_T, 1)$ of the scalar manifold due to appearance of central extensions of the gauge algebra. The gauge algebra with the central extension can be embedded within the isometry group of the corresponding five dimensional magical supergravity. On the other hand in the uplift of the magical supergravity theories to six dimensions as non-metric $(1, 0)$ tensor Einstein supergravity the isometry group of the scalar manifold of the five dimensional theory becomes a global symmetry group in 6d.

8.2 On the general ExFT formulation of unified non-metric tensor-Einstein supergravity theories in six dimensions

The 27 dimensional internal space-time that is intoduced in ExFT formulation of the 5d maximal Poincare supergravity can be identified with the generalized spacetime coordinatized by the split exceptional Jordan algebra $J_3^{\mathbb{O}_s}$ of 3×3 Hermitian matrices over the split octonions O_s. This generalized space-time was first intoduced in the early days of spacetime supersymmetry before any supergravity theories was written down [61]. Its automorphism, reduced structure and linear fractional groups were identified with the rotation, Lorentz and conformal groups of this space-time which are $F_{4(4)}, E_{6(6)}$ and $E_{7(7)}$ respectively. The generalized space-times defined by Jordan algebras were later studied further in [62–65]. For the maximal supergravity in $d = 5$ the Lorentz group $E_{6(6)}$ is the invariance of the C-tensor as well as of the Lagrangian.

For the 5d octonionic magical supergravity theory the internal space-time is the generalized spacetime defined by the real exceptional Jordan algebra $J_3^{\mathbb{O}}$ of 3×3 Hermitian matrices over the division algebra of octonions whose rotation, Lorentz and conformal groups are $F_4, E_{6(-26)}$ and $E_{7(-25)}$, respectively. The invariance groups of the C-tensors of magical supergravity theories are given by the Lorentz groups of underlying Euclidean Jordan algebras of degree three. As was discussed above the complex, quaternionic and octonionic magical supergravity theories can equivalently be described by the Lorentzian Jordan algebras of degree four over the reals $\mathbb{R}$, complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$. The C-tensor C^{IJK} given by the cubic norm of $J_3^{\mathbb{C}}, J_3^{\mathbb{H}}$ and $J_3^{\mathbb{O}}$ coincides with the structure constants of the traceless elements of the Lorentzian Jordan algebra degree four $J_{(1,3)}^{\mathbb{R}}$, $J_{(1,3)}^{\mathbb{C}}$ and $J_{(1,3)}^{\mathbb{H}}$, respectively. As such they belong to three infinite families of unified

MESGTs in $5d$ defined by Lorentzian Jordan algebras $J_{(1,n)}^{\mathbb{A}}$ of arbitrary degree over the reals $\mathbb{R}$, complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$.

The unified ExFT formulation of linearized $6d$, $(4,0),(3,1)$ and standard maximal $(2,2)$ supergravity all reduce to the linearized maximal supergravity in $d = 5$ [29]. At the linearized level only the maximal compact symmetry group $\mathrm{USp}(8)$ of $E_{6(6)}$ is manifest in five dimensions. The maximal compact symmetries groups of $(4,0),(3,1)$ and $(2,2)$ theories in $d = 6$ are $\mathrm{USp}(8), \mathrm{USp}(6) \times \mathrm{USp}(2)$ and $\mathrm{USp}(4) \times \mathrm{USp}(4)$, respectively. The fact that they all lead to the same linearized theory in five dimensions and the standard maximal supergravity theory has a non-linear extension in both five and six dimensions suggests that the generalized ExFT formulation of the maximal $(2,2)$ supergravity theory in $d = 6$ could be extended to a nonlinear interacting $(4,0)$ and $(3,1)$ non-metric supergravity theories in six dimensions as was argued in [29].

The maximal $(2,2)$ supergravity theory in six dimensions can be truncated to $(1,0)$ supersymmetric Poincare supergravity that reduces to the magical supergravity defined by the Euclidean Jordan algebra $J_3^{\mathbb{H}}$ in five dimensions. Its global symmetry group in six dimensions $\mathrm{SU}^*(4) \times \mathrm{USp}(2)$ which is a subgroup of the$5d$ global symmetry group $\mathrm{SU}^*(6)$.

$$\mathrm{SO}(5,1) \times \mathrm{USp}(2) \subset \mathrm{SO}(5,1) \times \mathrm{SO}(4) = \mathrm{SU}^*(4) \times \mathrm{USp}(2) \times \mathrm{USp}(2) \subset \mathrm{SO}(5,5) \quad (8.4)$$

and describes the coupling of 5 self-dual and 8 vector multiplets to $(1,0)$ metric supergravity. The global symmetry group of this theory gets enlarged to $\mathrm{SU}^*(6)$ in five dimensions with the scalars parametrizing the symmetric space $\mathrm{SU}^*(6)/\mathrm{USp}(6)$. This theory can be further truncated to the complex and real magical supergravity theories. The dimensional reduction of unified ExFT formulation of linearized metric and non-metric magical supergravity theories to five dimensions the resulting theories are invariant only under the maximal compact subgroups of their global symmetry groups which are $F_4, \mathrm{USp}(6), \mathrm{SU}(3)$ and $\mathrm{SO}(3)$.

As summarized in section 2.1 $5d$ MESGTs described by the Euclidean Jordan algebras $J_3^{\mathbb{H}}$ and $J_3^{\mathbb{C}}$ can be equivalently formulated using the Lorentzian Jordan algebras $J_{(1,3)}^{\mathbb{C}}$ and $J_{(1,3)}^{\mathbb{R}}$, respectively. Since the section constraint for a unified ExFT formulation of these theories depends only on the C-tensor it extends to the formulation in terms of Lorentzian Jordan algebras. Interestingly the extra singlet coordinate $Y^\bullet$ in the ExFT formalism can now be identified with the identity element of the corresponding Lorentzian Jordan algebra. Therefore for the three infinite families as well as the sporadic unified non-metric $(1,0)$ TESGTs in $d = 6$ one can give an ExFT formulation of their linearized bosonic sectors using the section constraint

$$C^{IJK} \partial_J \otimes \partial_K - \frac{1}{\sqrt{10}} \Delta^{IJ}(\partial_J \otimes \partial_\bullet + \partial_\bullet \otimes \partial_J) = 0$$

where C^{IJK} are now the structure constants of the underlying Lorentzian Jordan algebras which are invariant tensors of their global symmetry groups given by their automorphism groups. In all cases the extra singlet coordinate $Y^\bullet$ can be identified with the identity element of the underlying Lorentzian Jordan algebra. Imposing the condition $\partial_I = 0$ solves

the section constraint trivially and leads to the bosonic sector of non-metric $(1,0)$ TESGTs in $d = 6$.

Apart from the magical supergravity theories the higher dimensional origins of the three infinite families of $5d$ unified MESGTs as Poincare supergravity theories is not known. Whether the unified section constraint admits solutions that lead to three infinite families of $(1,0)$ supersymmetric metric Poincare supergravities in six dimensions will be left for future investigations.

9 Conformal path to higher dimensions and non-metric supergravity theories

If an interacting non-metric $(4,0)$ supergravity exists in $d = 6$ that reduces to the maximal supergravity in lower dimensions it raises the question as to whether $d = 6$ is the maximal dimension for the existence of non-metric supergravity theories. Now six is the maximal dimension for the existence of conformal superalgebras that extend the conformal algebra of $\mathrm{SO}(d, 2)$ to a simple conformal superalgebra [6]. Here the isomorphism of $\mathrm{SO}(6,2)$ to $\mathrm{SO}^*(8)$ plays a key role for satisfying spin and statistics constraints. The extensions of the Lie algebra of $\mathrm{SO}(d, 2)$ to simple Lie superalgebras for $d > 6$ do not satisfy the correct spin and statistics connection. Therefore it is believed that conformal metric supergravity theories based on simple superconformal algebras exist only in $d \leq 6$.

On the other hand Poincare superalgebras, which are not simple, exist in any dimension. However maximal dimension for Poincare supergravity is $d = 11$ [6]. The Poincare superalgebra in $d = 11$ with 32 supercharges can be embedded in the simple Lie superalgebra $\mathrm{OSp}(1|32, \mathbb{R})$ with the even subalgebra $\mathrm{Sp}(32, \mathbb{R})$ [66–69]. Its contraction to 11 dimensional Poincare superalgebra involves tensorial central charges

$$\{Q_A, Q_B\} = (C\Gamma^M)_{AB}P_M + \frac{1}{2}(C\Gamma^{MN})_{AB}Z_{MN} + \frac{1}{5!}(C\Gamma^{M_1\dots M_5})_{AB}Y_{M_1\dots M_5} \qquad (9.1)$$

where $\Gamma^M (M, N = 0, 1, \dots, 10)$ are the Dirac gamma matrices and C is the charge conjugation matrix. The embedding is such that the fundamental representation of $\mathrm{Sp}(32, \mathbb{R})$ is identified with the Majorana spinor of the Lorentz group $\mathrm{SO}(10, 1)$. The Z_{MN} and $Y_{M_1,\dots,M_5}$ are the tensorial central charges with P_M representing translations.

Werner Nahm's classification of spacetime superalgebras assumed that the spacetime has Minkowskian signature and gravity is described by a spacetime metric or a corresponding spin connection. In the $(4,0)$ supermultiplet of $\mathrm{OSp}(8^*|8)$ the gauge field of the graviton field strength is not a spacetime metric but rather a mixed tensor both in the first order as well as the second order formalism as reviewed above. As was shown by Hull the gauge symmetries of the mixed tensor reduce to diffeomorphisms in five dimensions and the interacting theory should yield the standard maximal supergravity in five dimensions. This raises the question whether there could be higher dimensional non-metric supergravity theories which reduce to the interacting $(4,0)$ theory in $d = 6$ or standard maximal supergravity in five and lower dimensions. At the level of Lie superalgebras the answer appears to be yes. Namely there exist superalgebras of the form $\mathrm{OSp}(2n^*|2m)$ which extend the

Lie superalgebra $OSp(8^*|8)$ to higher dimensions. For 64 supercharges the possibilities are $OSp(16^*|4)$ and $OSp(32^*|2)$.

In this context we should point out that M-theory was studied in space-times with exotic signatures by Hull [70]. Later M-theory and superstring theory in exotic signature space-times were studied within the framework of negative branes in [71]. Typically the Lorentz groups of these exotic spacetimes are of the form $SO(p, 11-p)$ or $SO(p-10-p)$ for formulations of M-theory and IIB superstring theory, respectively and the corresponding gravitational theories are of metric type. To our knowledge formulation of M/superstring theory on exotic spacetimes with conformal groups of type $SO^*(2n)$ that admit interpretation as non-metric gravity theories has not yet been investigated.

One natural framework for going beyond standard Minkowskian spacetimes with metric gravity is that of generalized spacetimes coordinatized by Euclidean Jordan algebras [61, 63, 64, 72]. The conformal groups of space-times defined by Euclidean Jordan algebras all admit positive energy unitary representations [63] and were shown to be causal space-times in [65]. The standart critical Minkowskian space-times that admit supersymmetric Yang-Mills theories can be coordinatized by Euclidean Jordan algebras $J_2^{\mathbb{A}}$ of degree two generated by Hermitian 2×2 matrices over the four division algebras $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. Their conformal groups are $Sp(4, \mathbb{R}), SU(2, 2), SO^*(8)$ and $SO(10, 2)$, respectively. Except for the octonionic case they all admit extensions to simple Lie superalgebras, namely $OSp(n|4, \mathbb{R}), SU(2, 2|n)$ and $OSp(8^*|2n)$ which all satisfy the usual spin and statistics connection.[14] The natural extension of these space-times is to consider those defined by Euclidean Jordan algebras $J_3^{\mathbb{A}}$ of degree 3 over the four division algebras $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ which were studied in [64]. These spacetimes correspond to extensions of the Minkowskian space-times by twistorial coordinates given in the second column of table 2 and an extra singlet coordinate. The C-tensors given by the norm forms of Jordan algebras of degree three satisfy the so-called adjoint identity

$$C^{IJK}C_{J(MN}C_{PQ)K} = \delta^I_{(M}C_{NPQ)}$$

(9.2)

It is a remarkable but little known fact that for simple Jordan algebras of degree three the adjoint identity implies the Fierz identities for the existence of supersymmetric Yang-Mills theories in the critical dimensions [35]. The conformal groups of $J_3^{\mathbb{A}}$ are $Sp(6, \mathbb{R}), SU(3, 3), SO^*(12)$ and $E_{7(-25)}$, respectively. Again except for the octonionic case they admit extensions to simple superalgebras $OSp(n|6, \mathbb{R}), SU(3, 3|n)$ and $OSp(12^*|2n)$, respectively. The conformal superalgebras in critical dimensions 3, 4 and 6 are subalgebras of these superalgebras.

We should note that $SO^*(4n)$ is the conformal group of a spacetime coordinatized by the Euclidean Jordan algebra $J_n^{\mathbb{H}}$ of $n \times n$ Hermitian matrices over the division algebra of quaternions [63]. Its Lorentz and rotation groups are $SU^*(2n)$ and $USp(2n)$, respectively. The quaternionic Jordan algebra $J_2^{\mathbb{H}}$ of degree two describes the standard $6d$ Minkowskian space-time with the Lorentz group $SU^*(4)$ and conformal group $SO^*(8)$, which are isomorphic to $Spin(5, 1)$ and $SO(6, 2)$, respectively. The conformal superalgebras $OSp(8^*|2n)$

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[14]The octonionic conformal algebra $SO(10, 2)$ has an extension to the simple Lie superalgebra $OSp(1|32, \mathbb{R})$ which requires its extensions to the Lie algebra $Sp(32, \mathbb{R})$.

come in three versions due to triality. The usual spin and statistics connection requires the version for which the supersymmetry generators transform in the spinor representation of $SO(6,2)$ which decomposes as $(4 + \bar{4})$ with respect to the Lorentz subgroup $SU^*(4)$. The six dimensional Minkowski coordinates are anti-symmetric tensors $x^{\alpha\beta} = -x^{\beta\alpha}$ in spinorial indices α, β as discussed earlier. For $n > 2$ the supersymmetry generators of $OSp(4n^*|2m)$ decompose as $(2n) + \overline{(2n)}$ with respect to the "Lorentz" group $SU^*(2n)$. The 2n-dimensional representation of $SU^*(2n)$ is the analog of spinor representation of $SU^*(4)$ and may be interpreted as generalized twistorial coordinates.[15]

If indeed there exist non-metric supergravity theories that reduce to standard metric supergravity theories one would have to extend the analysis of Werner Nahm with regard to classification of physically relevant spacetime superalgebras. One possible way to look for such exotic supergravity theories is to study solutions of ExFT formulation of maximal supergravity in 5,4 and 3 dimensions that admit interpretation as non-metric supergravity theories in higher dimensions. Investigations of these questions and their implications for M/superstring theory will be left to future investigations.

Note added to the revised version. The results of the first version of this paper were obtained before the work of [29] and [74] on $(4,0)$ supergravity in $6d$ appeared in the arXiv. The authors of [74] study the anomalies in $(4,0)$ supergravity and its truncations and argue that only their dimensional reduction to three dimensions make contact with standard metric supergravity theories. In [29] the free $(4,0)$ theory is studied from the point of view of exceptional field theory. In this revised version we review briefly the work of [29] on ExFT formulation of linearized $(4,0)$ theory (subsection 5.2) and extend it to ExFT formulation of linearized unified $(1,0)$ tensor Einstein supergravity theories in $6d$ (section 7).

Acknowledgments

A CPT-self-conjugate supermultiplet of $SU(2,2|8)$

In [1] it was shown that the fields of maximal $N = 8$ supergravity can be fitted into the CPT-self-conjugate doubleton supermultiplet of the conformal superalgebra $SU(2,2|8)$ with 64 supersymmetry generators. We reproduce this supermultiplet in table 6.

In the full nonlinear $\mathcal{N} = 8$ supergravity superconformal symmetry $SU(2,2|8)$ is broken down to $\mathcal{N} = 8$ Poincaré supersymmetry. Whether an interacting superconformal theory based on the supermultiplet of table 6 exists was posed as an open problem in [1]. To date

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[15]The corresponding generalized twistorial coordinates for the space-times defined by complex Jordan algebras $J_n^{\mathbb{C}}$ appeared in the work of [73] without using the language of Jordan algebras as generalizations of the twistor theory of Penrose and were called hyperspinors.

$SU(2)_{j_1} \times SU(2)_{j_2}$	E_0	$SU(8)$	$U(1)_R$	Fields
$(0,0)$	1	$\mathbf{70}$	0	$\phi^{[ABCD]}$
$(\frac{1}{2},0)$	$\frac{3}{2}$	$\mathbf{56}$	$1/2$	$\lambda_+^{[ABC]} \equiv \lambda_\alpha^{[ABC]}$
$(0,\frac{1}{2})$	$\frac{3}{2}$	$\overline{\mathbf{56}}$	$-1/2$	$\lambda_{-[ABC]} \equiv \lambda_{\dot\alpha[ABC]}$
$(1,0)$	2	$\mathbf{28}$	1	$h_{\mu\nu}^{+[AB]} \equiv h_{(\alpha\beta)}^{[AB]}$
$(0,1)$	2	$\overline{\mathbf{28}}$	-1	$h_{\mu\nu[AB]}^{-} \equiv h_{(\dot\alpha\dot\beta)[AB]}$
$(\frac{3}{2},0)$	$\frac{5}{2}$	$\mathbf{8}$	$3/2$	$\partial_{[\mu}\psi_{\nu]}^{+A} \equiv \psi_{(\alpha\beta\gamma)}^{A}$
$(0,\frac{3}{2})$	$\frac{5}{2}$	$\bar{\mathbf{8}}$	$-3/2$	$\partial_{[\mu}\psi_{\nu]A}^{-} \equiv \psi_{(\dot\alpha\dot\beta\dot\gamma)A}$
$(2,0)$	3	1	2	$R_{(\alpha\beta\gamma\delta)}$
$(0,2)$	3	1	-2	$R_{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}$

Table 6. The CPT-self-conjugate doubleton supermultiplet of the superconformal algebra $SU(2,2|8)$ The first two columns list the $SU(2)_{j_1} \times SU(2)_{j_2}$ and $U(1)_E$ labels of the positive energy unitary representation of $SU(2,2)$ with respect to its maximal compact subgroup. Third column lists their $SU(8)_R$ transformations and fourth column lists the $U(1)_R$ labels corresponding to their helicities. The last column lists the Lorentz covariant fields as well as their $SU(8)$ labelling. The indices $A, B, C, \ldots = 1, 2, \ldots, 8$ are the $SU(8)$ R-symmetry indices. The indices $\alpha, \beta, \ldots$ and $\dot\alpha, \dot\beta, \ldots$ denote the left-handed and right-handed spinorial indices of $SL(2,\mathbb{C})$. Round (square) brackets denote complete symmetrization (antisymmetrization) of indices.

no conformal supergravity with the same field content as maximal $N = 8$ supergravity in four dimensions has been constructed. It is clear that such a supergravity would have to have some unusual properties such as non-locality and higher derivaties. On the other hand the superalgebra $SU(2,2|8)$ and the above supermultiplet play a key role in the construction and classification of potential higher-loop counterterms in maximal supergravity [4].

The physical fields of the CPT-self-conjugate doubleton supermultiplet of $SU(2,2|8)$ can be represented as a scalar superfield W_{abcd} [5, 75, 76] in the superspace with coordinates,

$$\left(x^{\alpha\dot\beta}, \theta^{\alpha a}, \bar\theta_b^{\dot\beta}\right) \tag{A.1}$$

where $\alpha, \dot\beta = 1, 2$ and $a = 1, 2, \ldots 8$. The covariant derivatives in this superspace satisfy

$$\{D_{\alpha a}, \bar{D}_{\dot\beta}^b\} = 2i\delta_a^b \partial_{\alpha\dot\beta}. \tag{A.2}$$

The superfield W_{abcd} is completely anti-symmetric and obeys the self-duality condition

$$\bar{W}^{abcd} = \frac{1}{4!}\epsilon^{abcdefgh}W_{efgh} \tag{A.3}$$

as well as the differential constraint

$$\bar{D}_{\dot\alpha}^a W_{bcde} + \frac{4}{5}\delta_{[b}^a \bar{D}_{\dot\alpha}^f W_{cde]f} = 0. \tag{A.4}$$

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