Published for SISSA by Springer

Received: May 7, 2021
Accepted: May 19, 2021
Published: June 9, 2021

Colored HOMFLY-PT for hybrid weaving knot $\hat{W}_3(m, n)$

Vivek Kumar Singh,[a] **Rama Mishra**[a] **and P. Ramadevi**[b]

[a] *Department of Mathematics, Indian Institute of Science Education and Research, Homi Bhabha Rd, Pashan, Pune 411008, India*

[b] *Department of Physics, Indian Institute of Technology Bombay, Mumbai 400076, India*

E-mail: vivek.singh@fuw.edu.pl, r.mishra@iiserpune.ac.in, ramadevi@phy.iitb.ac.in

Abstract: Weaving knots $W(p, n)$ of type (p, n) denote an infinite family of hyperbolic knots which have not been addressed by the knot theorists as yet. Unlike the well known (p, n) torus knots, we do not have a closed-form expression for HOMFLY-PT and the colored HOMFLY-PT for $W(p, n)$. In this paper, we confine to a hybrid generalization of $W(3, n)$ which we denote as $\hat{W}_3(m, n)$ and obtain closed form expression for HOMFLY-PT using the Reshitikhin and Turaev method involving $\mathcal{R}$-matrices. Further, we also compute $[r]$-colored HOMFLY-PT for $W(3, n)$. Surprisingly, we observe that trace of the product of two dimensional $\hat{\mathcal{R}}$-matrices can be written in terms of infinite family of Laurent polynomials $\mathcal{V}_{n,t}[q]$ whose absolute coefficients has interesting relation to the Fibonacci numbers $\mathcal{F}_n$. We also computed reformulated invariants and the BPS integers in the context of topological strings. From our analysis, we propose that certain refined BPS integers for weaving knot $W(3, n)$ can be explicitly derived from the coefficients of Chebyshev polynomials of first kind.

Keywords: Quantum Groups, Topological Strings, Wilson, 't Hooft and Polyakov loops, Chern-Simons Theories

ArXiv ePrint: 2103.10228

https://doi.org/10.1007/JHEP06(2021)063

Contents

1 Introduction **1**

2 Knot invariants from quantum groups **2**
- 2.1 $\hat{\mathcal{R}}$-matrices with block structure 4
  - 2.1.1 $[r]$-colored HOMFLY-PT polynomial for closure of two strand braids 5
- 2.2 $\hat{\mathcal{R}}$- matrices with block structure for three strand braids 7

3 Hybrid weaving knot $\hat{W}_3(m,n)$ **8**
- 3.1 HOMFLY-PT for hybrid weaving knot $\hat{W}_3(m,n)$ 10
- 3.2 Examples 12

4 Colored HOMFLY-PT for weaving knot type $W(3,n)$ **13**
- 4.1 Representation $[r]=[2]$ 13
- 4.2 Representation $[3]$ 15

5 Integrality structures in topological strings **16**

6 Conclusion and discussion **19**

A Unitary matrices **20**

B Colored HOMFLY-PT polynomials **22**

C Refined BPS integers **26**

1 Introduction

Distinguishing knots and links up to ambient isotopy is the central problem in knot theory. The main technique that a knot theorist uses is to compute some knot invariants and see if one of them can be of help. Over the last 35 years tremendous progress has been made in the development of several new knot invariants, starting with the Jones polynomial and the HOMFLY-PT polynomial [1–3]. Recently even more sophisticated invariants such as Heegard-Floer homology groups [4] and Khovanov homology groups [5] have been added to the toolkit. In the 1980s William Thurston's seminal result [6, corollary 2.5] that most knot complements have the structure of a hyperbolic manifold, combined with Mostow's rigidity theorem [6, theorem 3.1] giving uniqueness of such structures, establishes a strong connection between hyperbolic geometry and knot theory, since knots are determined by their complements. Indeed, any geometric invariant of a knot complement, such as the hyperbolic volume, becomes a topological invariant of the knot. Thus, investigating if data derived from the new knot invariants is related to natural differential geometric invariants becomes another natural problem. In this direction 'volume conjecture' is one of the most challenging open problem. This conjecture has been tested for torus knots but for hyperbolic knots it has been verified only for a handful of knots. Weaving knots $W(p,n)$ of type (p,n) for a pair of co-prime integers p and n are doubly infinite family of alternating, hyperbolic knots and share the same projection with torus knots. They can be thought of a prototype of hyperbolic knots. Thus an extensive study of this family of knots will provide an insight to 'volume conjecture.' One of us in an earlier work [7] have attempted recursive method of relating the HOMFLY-PT of $W(3,n)$. In a parallel paper [8], the closed form of HOMFLY-PT for $W(3,n)$ with explicit proof is provided. In this paper, we study the *hybrid family of weaving knots* denoted by $(\hat{W}_3(m,n))$. Here, we use the approach of Reshitikhin and Turaev to evaluate the colored polynomials for knots and obtained the closed form expression for HOMFLY-PT polynomial for hybrid weaving knots. Further, we have computed the $[r]$-colored HOMFLY-PT polynomial for $W(3,n)$ which agrees when $[r]=[1]$ with the results in [8]. Further we study the reformulated invariants in the context of topological string dualities and validate Oogur-Vafa conjecture [9–11]. Interestingly, we show that certain BPS integers of weaving knot $W(3,n)$ can be written in the Chebyshev coefficients of first kind.

The paper is organized as follows.

In section 2, we will review Reshitikhin and Turaev (RT) method of constructing knot and link invariants which involves $\mathcal{R}$-matrices. This is followed by the subsection 2.2 where we present the $\hat{\mathcal{R}}$-matrices in a block structure form for a three strands braid. In section 3, we used the properties of quantum $\hat{\mathcal{R}}_{2\times 2}$ matrices, we succeeded in writing a closed form expression of HOMFLY-PT polynomial for $(\hat{W}_3(m,n))$. As a consequence, we showed the relation to the infinite set of Laurent polynomials called $\mathcal{V}_{n,t}[q]$ whose absolute coefficients are related to Fibonacci numbers. Section 4 deals with $[r]$-colored HOMFLY-PT for weaving knots $W(3,n)$. Particularly, we could express the trace of product of 2 dimensional matrices as a Laurent polynomial. We explicitly calculate colored polynomials for weave knot up to representation $[r]=[3]$. In section 5, we verify that the reformulated invariants from these weave knot invariants indeed respect Ooguri-Vafa conjecture. The concluding section 6 contains summary and related challenging open problems. There are two appendices B and C with explicit data on colored HOMFLY-PT and reformulated invariants for $W(3,n)$.

2 Knot invariants from quantum groups

Recall Alexander theorem which states that any knot or link can be viewed as closure of m-strand braid. Hence the knot invariants can be constructed from the braid group $\mathcal{B}_m$ representations. The representations of the generators σ_i's of $\mathcal{B}_m$:

$$\mathcal{R}_1, \mathcal{R}_2, \ldots \mathcal{R}_i, \ldots \mathcal{R}_{m-1}$$

are derivable from the well-known universal $\check{\mathcal{R}}$-matrix of $U_q(\mathfrak{sl}_N)$ defined as

$$\check{\mathcal{R}} = q^{\sum_{i,j} C^{-1}_{ij} H_i \otimes H_j} \prod_{\text{positive root } \alpha} \exp_q[(1-q^{-1}) E_\alpha \otimes F_\alpha]\,, \tag{2.1}$$

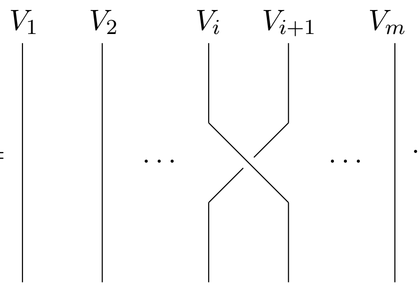
where q is complex number, (C_{ij}) is the Cartan matrix and $\{H_i, E_i, F_i\}$ are generators of $U_q(sl_N)$. Braid group generators $\mathcal{R}_i$'s, depicted in (2.4), in terms of (2.1) is

$$\mathcal{R}_i = 1_{V_1} \otimes 1_{V_2} \otimes \ldots \otimes P\check{\mathcal{R}}_{i,i+1} \otimes \ldots \otimes 1_{V_m} \;\in \mathrm{End}(V_1 \otimes \ldots, \otimes V_m)\,, \tag{2.2}$$

where P denotes the permutation operation: $P(x \otimes y) = y \otimes x$. Notice that the subscript $i, i{+}1$ on the universal quantum $\check{\mathcal{R}}$ in the above equation implies $\check{\mathcal{R}}$ acts only on the modules V_i and V_{i+1} of the $U_q(sl_N)$. The quantum $\mathcal{R}_i$ matrices discussed in [12–15] provides a braid group $\mathcal{B}_m$ representation. That is.,

$$\begin{aligned} \pi : \mathcal{B}_m &\rightarrow \mathrm{End}(V_1 \otimes \ldots, \otimes V_m)\,, \\ \pi(\sigma_i) &= \mathcal{R}_i. \end{aligned} \tag{2.3}$$

Graphically the braid group generator $\mathcal{R}_i$ as follows:

$\mathcal{R}_i \;=$  . (2.4)

Algebriacally these generators in terms of (2.1). These operators $\mathcal{R}_i$ obeys the following relations:

$$\mathcal{R}_i \mathcal{R}_j = \mathcal{R}_j \mathcal{R}_i \qquad \text{for } |i-j| > 1\,, \tag{2.5}$$

$$\mathcal{R}_i \mathcal{R}_{i+1} \mathcal{R}_i = \mathcal{R}_{i+1} \mathcal{R}_i \mathcal{R}_{i+1}, \quad \text{for } \; i = 1, \ldots, m-2. \tag{2.6}$$

Graphically, the equation (b) is equivalent to the third Reidemeister move. According to Reshetikhin-Turaev approach [16, 17] the quantum group invariant, known as $[r]$-colored HOMFLY polynomial of the knot $\mathcal{K}$ denoted by $H_{[r]}^{\mathcal{K}}$ is defined as follows:

$$H_{[r]}^{\mathcal{K}} = {}_q\mathrm{tr}_{V_1 \otimes \cdots \otimes V_m} \left(\pi(\alpha_{\mathcal{K}})\right), \tag{2.7}$$

where ${}_q\mathrm{tr}$ is the quantum trace ([18]) defined as follows:

$${}_q\mathrm{tr}_V(z) = \mathrm{tr}_V(z K_{2\rho}) \quad \forall z \in \mathrm{End}(V), \tag{2.8}$$

where $\vec{\rho}$ is the Weyl vector that can expressed in terms of simple roots $\vec{\alpha}_i$ is $2\vec{\rho} = \sum_i a_i \vec{\alpha}_i$ and the $K_{2\rho}$ is defined as

$$K_{2\rho} = K_1^{a_1}\, K_2^{a_2} \ldots K_{N-1}^{a_{N-1}}$$

where $K_p = q^{\vec{\alpha}_p \cdot \mathbf{H}}$ having Cartan generators $H_1, H_2, \ldots H_{N-1}$.

Note that the universal $\check{R}$ matrix is not diagonal and makes the computations of knot invariants very cumbersome. There is a modified RT-approach [19–21] where the braiding generators can be written in a block structure form. This methodology gives a better control and simplify the computation of knot invariants. We will present the details of this modified RT method in the following section.

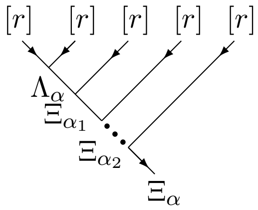
2.1 $\hat{\mathcal{R}}$-matrices with block structure

The modified RT approach fixes the block structure form for $\hat{\mathcal{R}}_i$'s from the study of the irreducible representation in the tensor product of symmetric representations $\underbrace{[r]\otimes[r]\otimes\ldots\otimes[r]}_{m}$:

$$[r]^{\bigotimes m} = \bigoplus_{\alpha,\ \Xi_\alpha \vdash m|r|} (\dim \mathcal{M}^{1,2\ldots m}_{\Xi_\alpha})\ \Xi_\alpha\,, \tag{2.9}$$

where Ξ_α denote the irreducible representations labeled by index α. The repetition in the irreducible representation called multiplicity (an irreducible representation occurs more than once) denoted by $\mathcal{M}^{1,2,\ldots m}_{\Xi_\alpha}$ that keep track of the subspace of the highest weight vectors[1] sharing same highest weights corresponding to Young diagram $\Xi_\alpha \vdash m|r|$,[2] which we indicate as $\Xi_{\alpha,\mu}$ with the index μ, takes values $1, 2, \ldots \dim\mathcal{M}^{1,2,\ldots m}_{\Xi_\alpha}$, keep track of the different highest weight vectors sharing the same highest weight $\vec{\omega}_{\Xi_\alpha}$.

To evaluate quantum trace (2.8), we need to write the states in weight space incorporating the multiplicity as well. There are several paths leading to the state corresponding to the irreducible representations Ξ_α. Pictorially depicted one such state in the weight space (see in (2.10))



(2.10)

and algebraically it can written as

$$|\Big(\ldots(([r]\otimes[r])_{\Lambda_\alpha}\otimes[r])_{\Xi_{\alpha_1}}\ldots[r]\Big)_{\Xi_\alpha}\rangle^{(\mu)} \equiv |\Xi_\alpha;\Xi_{\alpha,\mu},\Lambda_\alpha\rangle \cong\ |\Xi_{\alpha,\mu},\Lambda_\alpha\rangle\otimes|\Xi_\alpha\rangle\,, \tag{2.11}$$

where $[r]\otimes[r] = \oplus^r_{\alpha=0}\Lambda_\alpha \equiv [2r-\alpha,\alpha]$. For clarity, in this paper, we denote the $\hat{\mathcal{R}}_i$'s-matrices corresponding to Ξ_α as $\hat{\mathcal{R}}^{\Xi_\alpha}_i$. Incidentally, the choice of state (2.11) is an eigenstate of quantum $\hat{\mathcal{R}}^{\Xi_\alpha}_1$ matrix:

$$\hat{\mathcal{R}}^{\Xi_\alpha}_1|\Xi_\alpha;\Xi_{\alpha,\mu},\Lambda_\alpha\rangle = \lambda_{\Lambda_{\alpha,\mu}}([r],[r])|\Xi_\alpha;\Xi_{\alpha,\mu},\Lambda_\alpha\rangle\rangle.$$

Hence we will denote the $\hat{\mathcal{R}}^{\Xi_\alpha}_1$ matrix which is diagonal in the above basis and the elements denoted by $\lambda_{\Lambda_{\alpha,\mu}}([r],[r])$. These elements are the braiding eigenvalues whose explicit form is [18, 22]

$$\lambda_{\Lambda_{\alpha,\mu}}([r],[r]) = \epsilon_{\Lambda_{\alpha,\mu}} q^{\varkappa(\Lambda_\alpha)-4\varkappa([r])-rN}\,, \tag{2.12}$$

where $\varkappa(\Lambda_\alpha) = \frac{1}{2}\sum_j \alpha_j(\alpha_j+1-2j)$,[3] is cut-and-join-operator eigenvalue of Young tableaux representation Λ_α that does not depend on the braid representation of the knot $\mathcal{K}$ [23, 24]

[1]Note that the Young diagram Ξ_α represented as $[\xi^\alpha_1,\xi^\alpha_1,\ldots,\xi^\alpha_l]$ partitioned by $\{\xi^\alpha_1 \geq \xi^\alpha_2 \geq \ldots, \xi^\alpha_{l-1} \geq \xi^\alpha_l \geq 0\}$, then the highest weights $\vec{\omega}_{\Xi_\alpha}$ of the corresponding representation are $\omega^\alpha_i = \xi^\alpha_i - \xi^\alpha_{i+1}\ \forall\, i = 1,\ldots,l$, and vice versa $\xi^\alpha_i = \sum^l_{k=i}\omega^\alpha_k$.

[2]$\Xi_\alpha \vdash m|r|$ means a sum over all Young diagrams Ξ_α of the size equal to $m|r|$. Here,$|r|$ is total number of boxes in the Young diagram $[r] = \underbrace{\square\square\ldots\square\square}_{r}$.

[3]The representation Λ_α whose Young diagram is denoted by $\alpha_1 \geq \alpha_2 \ldots, \geq \alpha_{N-1}$.

JHEP06(2021)063

and $\epsilon_{\Lambda_\alpha,\mu}$ will be ± 1.[4] From the eq. (2.11), (2.7), incorporating all the facts of $\hat{\mathcal{R}}$-matrix and the decomposition of states (2.11),[5] the unreduced [r]-colored HOMFLY-PT will become

$$H^{\star\mathcal{K}}_{[r]}\left(q,\, A=q^N\right) = \mathrm{tr}_{V_1\otimes\cdots\otimes V_m}\left(\pi(\alpha_\mathcal{K})\, K_{2\rho}\right) \;=\; \sum_\alpha \mathrm{tr}_{\mathcal{M}^{1,2\ldots m}_{\Xi_\alpha}}\left(\pi(\alpha_\mathcal{K})\right)\cdot \mathrm{tr}_{\Xi_\alpha}\left(K_{2\rho}\right)$$
$$= \sum_\alpha \mathrm{tr}_{\mathcal{M}^{1,2\ldots m}_{\Xi_\alpha}}\left(\pi(\alpha_\mathcal{K})\right)\cdot S^*_{\Xi_\alpha} \;=\; \sum_{\alpha,\Xi_\alpha\vdash m|r|} S^*_{\Xi_\alpha} C^{\mathcal{K}}_{\Xi_\alpha}, \tag{2.13}$$

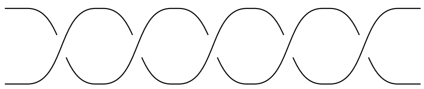
where Ξ_α represent the irreducible representations in the product $[r]^{\otimes m}$, m stands for number of braid strands, $[r]$ denotes the representation on each strand, C_{Ξ_α} having the trace of product of all $\hat{\mathcal{R}}$-matrices, and $S^*_{\Xi_\alpha}$ is the quantum dimension of the representation Ξ_α whose explicit form is given in terms of Schur polynomials [25, 26]. Note that the notation $H^{\star\mathcal{K}}_{[r]}$ denote unreduced HOMFLY-PT of knot $\mathcal{K}$. The reduced [r]- colored HOMFLY-PT ($H^{\mathcal{K}}_{[r]}$) is obtained by dividing the $[r]$-colored unknot invariant ($H^{\text{unknot}}_{[r]}$) i.e.

$$H^{\mathcal{K}}_{[r]} = \frac{H^{\star\mathcal{K}}_{[r]}}{H^{\text{unknot}}_{[r]}} = \frac{H^{\star\mathcal{K}}_{[r]}}{S^*_{[r]}}. \tag{2.14}$$

For clarity, we present the invariants of knots obtained from the simplest two-strand braids using this method in the following subsection.

2.1.1 $[r]$-colored HOMFLY-PT polynomial for closure of two strand braids

We will illustrate the $[r]$-colored HOMFLY-PT for knot $\mathcal{K}$ carrying symmetric representation $[r]$ obtained from the braid word σ_1^n (2.15) where n is odd integer. These knots known as torus knot $T_{(2,n)}$ and we have drawn as a example in figure 1. The irreducible representation Λ_α in the tensor product of $[r]\bigotimes[r] = \bigoplus_{\beta=0}^{r}\Lambda_\alpha = \bigoplus_{\beta=0}^{r}[2r-\alpha,\alpha]$ has no multiplicity. Note that each irreducible representation occurs only once. So the $\hat{\mathcal{R}}$ are only eigenvalues and not matrices.

 $\ldots n\,.$ (2.15)

Hence, using eqs. (2.13), (2.12), we can obtain colored HOMFLY-PT $H^{\mathcal{K}}_{[r]}(q, A = q^N)$ which involves the single diagonal $\hat{\mathcal{R}}$-matrix i.e. $\hat{\mathcal{R}}_1^n$ whose explicit entries depicted from eq. (2.12).

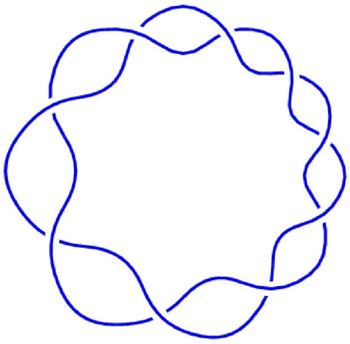
$$\lambda_{\Lambda_\alpha}([r],[r]) = (-1)^\alpha q^{(2r^2-r(2\alpha+1)+\alpha(\alpha-1)-rN)}. \tag{2.16}$$

The HOMFLY-PT for torus knot $T_{(2,n)}$

$$H^{\mathcal{K}}_{[r]}(q,A) = \frac{\sum_\alpha \mathrm{tr}_{\Lambda_\alpha}\hat{\mathcal{R}}^n S^*_{\Lambda_\alpha}}{S^*_{[r]}} = \frac{1}{S^*_{[r]}}\sum_\alpha \lambda_{\Lambda_\alpha}([r],[r])^n\; S^*_{\Lambda_\alpha},$$
$$= \frac{A^{(-rn)}}{S^*_{[r]}}\sum_{\alpha=0}^{r}(-1)^{n\alpha}q^{(2r^2-r(2\alpha+1)+\alpha(\alpha-1))n}\; S^*_{\Lambda_\alpha}. \tag{2.17}$$

[4]The multiplicity subspace state $\Xi_{\alpha,\mu}$ is connected by Λ_α and zero otherwise.

[5]In facts, the action of $\hat{\mathcal{R}}$-matrix acts an identity operator on $|\Xi_\alpha\rangle$ and non-trivially on the subspace $\mathcal{M}^{1,2,\ldots m}_{\Xi_\alpha}$ and similarly on other way, the element $K_{2\rho}$ acts diagonally on $|\Xi_\alpha\rangle$ but as identity operator on subspace $\mathcal{M}^{1,2\ldots m}_{\Xi_\alpha}$ as this space represent all possible highest weight vectors $\Xi_{\alpha,\mu}$ with the same weight $\vec{\omega}_{\Xi_\alpha}$.



$\mathbf{9_1}$ Knot

Figure 1. Torus knot $T_{(2,9)} = \mathbf{9_1}$ knot.

The explicit form of quantum dimension $S^*_{\Lambda_\alpha}$

$$S^*_{\Lambda_\alpha} = \frac{[N+\alpha-2]_q!\,[N+2r-\alpha-1]_q!\,[2r-2\alpha+1]_q}{[\alpha]_q!\,[2r-\alpha+1]_q!\,[N-1]_q!\,[N-2]_q!},$$

where the factorial is defined as $[n]_q! = \prod_{i=1}^{n}[i]_q$ with $[0]_q! = 1$ and the q-numbers for our computation will be given as,

$$[n]_q = \frac{q^n - q^{-n}}{q^1 - q^{-1}}. \tag{2.18}$$

The explicit polynomial form for colors $[r] = [1]$ and $[r] = [2]$ for this knot $T(2,9) = \mathbf{9_1}$ are

$$H^{\mathbf{9_1}}_{[1]}(q,A) = A^8 + \frac{A^8}{q^8} - \frac{A^{10}}{q^6} + \frac{A^8}{q^4} - \frac{A^{10}}{q^2} - A^{10}q^2 + A^8q^4 - A^{10}q^6 + A^8q^8,$$

$$\begin{aligned} H^{\mathbf{9_1}}_{[2]}(q,A) = \Bigg(& A^{16} - A^{18} + A^{20} + \frac{A^{16}}{q^{16}} - \frac{A^{18}}{q^{12}} + \frac{A^{16}}{q^{10}} - \frac{A^{18}}{q^{10}} + \frac{A^{16}}{q^8} - \frac{A^{18}}{q^6} + \frac{A^{20}}{q^6} + \frac{A^{16}}{q^4} \\ & - \frac{2A^{18}}{q^4} + \frac{A^{16}}{q^2} - \frac{A^{18}}{q^2} + A^{16}q^2 - 2A^{18}q^2 + A^{20}q^2 + A^{16}q^4 - 2A^{18}q^4 + A^{16}q^6 \\ & - 2A^{18}q^6 + A^{20}q^6 + 2A^{16}q^8 - 2A^{18}q^8 + A^{20}q^8 + A^{16}q^{10} - 2A^{18}q^{10} + A^{20}q^{10} \\ & + A^{16}q^{12} - 3A^{18}q^{12} + A^{20}q^{12} + 2A^{16}q^{14} - 3A^{18}q^{14} + A^{20}q^{14} + 2A^{16}q^{16} \\ & - 2A^{18}q^{16} + A^{20}q^{16} + A^{16}q^{18} - 3A^{18}q^{18} + 2A^{20}q^{18} + 2A^{16}q^{20} - 3A^{18}q^{20} + \\ & A^{20}q^{20} + A^{16}q^{22} - 2A^{18}q^{22} + A^{20}q^{22} + A^{16}q^{24} - 2A^{18}q^{24} + A^{20}q^{24} + A^{16}q^{26} \\ & - 2A^{18}q^{26} + A^{20}q^{26} + A^{16}q^{28} - A^{18}q^{28} - A^{18}q^{30} + A^{20}q^{30} + A^{16}q^{32} - A^{18}q^{32}\Bigg). \end{aligned}$$

If we go beyond two-strand braids, we need to deal with quantum $\hat{\mathcal{R}}_i's$ which could be matrices depending on the multiplicity sub-spaces. As our focus is on weaving knots $W(3,n)$ and their hybrid generalization, we will elaborate the steps of the modified RT method for three strands braid in the following section. Notice that, the braiding property eq. (2.6) means that both $\hat{\mathcal{R}}_1$ and $\hat{\mathcal{R}}_2$ cannot be simultaneously diagonal but related by a unitary matrix which can be identified with the $U_q(sl_N)$ Racah matrices.

2.2 $\hat{\mathcal{R}}$- matrices with block structure for three strand braids

For three strand braids and each strands carrying the symmetric representation, the tensor product of representations $\underbrace{[r] \otimes [r] \otimes \ldots \otimes [r]}_{m}$ into the direct sum of irreducible representations (Ξ_α) is shown:

$$
\begin{aligned}
\bigotimes^3 [1] &= [3,0,0] \bigoplus [1,1,1] \bigoplus 2[2,1,0], \\
\bigotimes^3 [2] &= [6,0,0] \bigoplus [3,3,0] \bigoplus [4,1,1] \bigoplus 2[5,1,0] \bigoplus 2[3,2,1] \bigoplus 3[4,2,0], \\
\bigotimes^3 [3] &= [9,0,0] \bigoplus [7,1,1] \bigoplus [5,2,2] \bigoplus [4,4,1] \bigoplus [3,3,3] \bigoplus 3[8,1,0] \bigoplus \\
& \quad 2[4,3,2] \bigoplus 2[6,2,1] \bigoplus 2[5,4,0] \bigoplus 3[7,2,0] \bigoplus 2[5,3,1] \bigoplus 4[6,3,0], \\
& \ldots \\
\bigotimes^3 [r] &= \sum_\alpha (\dim \mathcal{M}^{1,2,3}_{\Xi_\alpha}) \Xi_\alpha,
\end{aligned}
$$

where $\Xi_\alpha \equiv [\xi_1{}^\alpha, \xi_2{}^\alpha, \xi_3{}^\alpha]$ is such that $\xi_1{}^\alpha + \xi_2{}^\alpha + \xi_3{}^\alpha = 3r$ and $\xi_1{}^\alpha \geq \xi_2{}^\alpha \geq \xi_3{}^\alpha \geq 0$. Let us discuss the path and block structure of $\hat{\mathcal{R}}$-matrix for irreducible representation $[4,2,0]$. Note that the multiplicity of the representation $[4,2,0]$ is equal to three which means there are three possible paths:

$$
\begin{aligned}
&(i) \quad [2] \to [2] \to [4] \quad \to [4,2,0]\,, \\
&(ii) \quad [2] \to [2] \to [3,1] \to [4,2,0]\,, \\
&(iii) \; [2] \to [2] \to [2,2] \to [4,2,0].
\end{aligned}
\tag{2.19}
$$

Let us choose $\hat{\mathcal{R}}_1^{[4,2,0]}$ to be diagonal whose entries defined by (2.12)

$$
\lambda_{[2,2]}([2],[2]) = A^{-2},\ \lambda_{[3,1]}([2],[2]) = -A^{-2}q^2,\ \lambda_{[4]}([2],[2]) = A^{-2}q^6. \tag{2.20}
$$

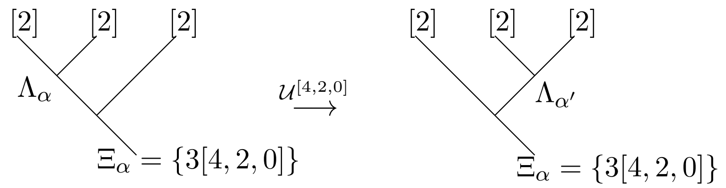
The explicit form of $\hat{\mathcal{R}}_1^{[4,2,0]}$

$$
\hat{\mathcal{R}}_1^{[4,2,0]} = A^{-2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -q^2 & 0 \\ 0 & 0 & q^6 \end{pmatrix}. \tag{2.21}
$$

$\hat{\mathcal{R}}_2$ is defined as

$$
\hat{\mathcal{R}}_2^{\Xi_\alpha} = \mathcal{U}^{\Xi_\alpha} \hat{\mathcal{R}}_1^{\Xi_\alpha} (\mathcal{U}^{\Xi_\alpha})^\dagger.
$$

Note that $\mathcal{U}^{\Xi_\alpha \dagger}$ denotes the conjugate-transpose of $\mathcal{U}^{\Xi_\alpha}$. This unitary matrix relate two equivalent basis states for irreducible representation $[4,2,0]$ as shown below in:



(2.22)

JHEP06(2021)063

where, Λ_α & $\Lambda'_\alpha \in \{[4],[3,1],[2,2]\}$ and algebraically the transformation state for Ξ_α are:

$$|\Big(([r]\otimes[r])_{\Lambda_\alpha}\otimes[r]\Big)_{\Xi_\alpha}\rangle \xrightarrow{\mathcal{U}^{\Xi_\alpha}} |\Big([r]\otimes([r]\otimes[r])_{\Lambda_{\alpha'}}\Big)_{\Xi_\alpha}\rangle\,,$$

where the elements of the transformation matrix $\mathcal{U}^{\Xi_\alpha}$ related to quantum Racah coefficients discuss in details [26–28]. For completeness, Racah matrix involving $\Xi_\alpha \equiv [{\xi_1}^\alpha, {\xi_2}^\alpha, {\xi_3}^\alpha]$ (whose Young diagram has three rows) can be identified as $U_q(sl_2)$ Racah matrix:

$$\mathcal{U}^{\Xi_\alpha\equiv[{\xi_1}^\alpha,{\xi_2}^\alpha,{\xi_3}^\alpha]} = U_{U_q(sl_2)}\begin{bmatrix}(r-{\xi_3}^\alpha)/2 & (r-{\xi_3}^\alpha)/2\\ (r-{\xi_3}^\alpha)/2 & ({\xi_1}^\alpha-{\xi_2}^\alpha)/2\end{bmatrix}. \tag{2.23}$$

The closed form expression of $U_q(sl_2)$ Racah coefficients [12]:

$$U^{U_q(sl_2)}_{j,l}\begin{bmatrix}j_1 & j_2\\ (j_3 & j_4\end{bmatrix} = \sqrt{[2j+1]_q[2l+1]_q}(-1)^{j_1+j_2+j_3+j_4+j+l+1}\Delta(j_1,j_2,j)\Delta(j_3,\ j_4,j)$$
$$\Delta(\ j_4,\ j_1,l)\Delta(\ j_2,\ j_3,l)F[j_1,j_2,j_3,j_4]\,,$$

where,

$$F[j_1,j_2,j_3,j_4] = \sum_{m\geq 0}(-1)^m[m+1]_q!\{[m-(j+\ j_1+\ j_2)]_q![m-(j+\ j_3+\ j_4)]_q![(m-(\ j_1+\ j_4+l))]_q!$$
$$[m-(\ j_2+\ j_3+l)]_q![(j+\ j_1+\ j_3+l)-m]_q!\ [(j+\ j_2+\ j_4+l)-m]_q!$$
$$[(\ j_1+\ j_2+\ j_3+\ j_4)-m]_q!\}^{-1}$$
$$\Delta(a,b,c) = \sqrt{\frac{[a-b+c]_q![b-a+c]_q![a+b-c]_q!}{[a+b+c+1]_q!}}\,.$$

Hence, from eq. (2.23) the explicit form of unitary matrix $\mathcal{U}^{[4,2,0]}$ defined as

$$\mathcal{U}^{[4,2,0]} = \begin{pmatrix} -\frac{1}{1+\frac{1}{q^2}+q^2} & -\frac{q}{\sqrt{1+q^2+q^4}} & -\frac{\sqrt{1+q^2+q^4+q^6+q^8}}{1+q^2+q^4}\\ -\frac{q}{\sqrt{1+q^2+q^4}} & -1+\frac{q^2}{1+q^4} & \frac{q^4\sqrt{\left(1+\frac{1}{q^2}+q^2\right)\left(1+\frac{1}{q^4}+\frac{1}{q^2}+q^2+q^4\right)}}{(1+q^4)(1+q^2+q^4)}\\ -\frac{\sqrt{1+q^2+q^4+q^6+q^8}}{1+q^2+q^4} & \frac{q^4\sqrt{\left(1+\frac{1}{q^2}+q^2\right)\left(1+\frac{1}{q^4}+\frac{1}{q^2}+q^2+q^4\right)}}{(1+q^4)(1+q^2+q^4)} & -\frac{q^4}{1+q^2+2q^4+q^6+q^8}\end{pmatrix}. \tag{2.24}$$

Hence, the $\hat{\mathcal{R}}_2$ matrix for $[4,2,0]$ is

$$\hat{\mathcal{R}}_2^{[4,2,0]} = \mathcal{U}^{[4,2,0]}\hat{\mathcal{R}}_1^{[4,2,0]}\mathcal{U}^{[4,2,0]\dagger}.$$

Explicit form of quantum $\hat{\mathcal{R}}_i$'s and $\mathcal{U}$ can be similarly worked out for other irreducible representations to compute $[r]$-colored HOMFLY-PT for hybrid weaving knots.

3 Hybrid weaving knot $\hat{W}_3(m,n)$

In this section, we discuss the hybrid weaving knot obtained from closure of three-strand braid whose braid word is

$$(\sigma_1^m\sigma_2^{-m})^n$$

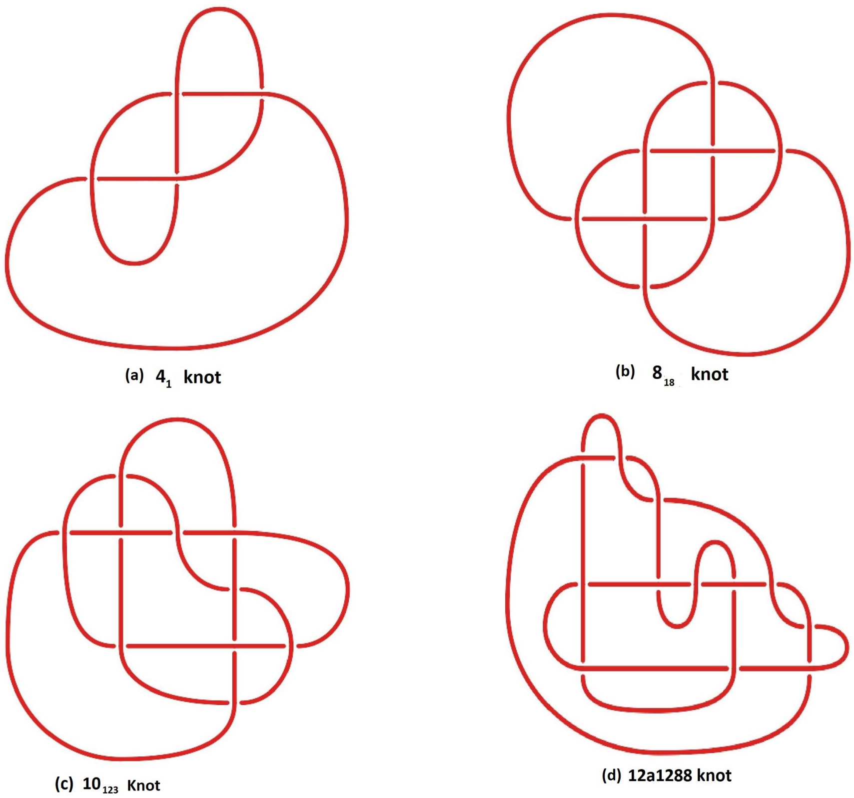
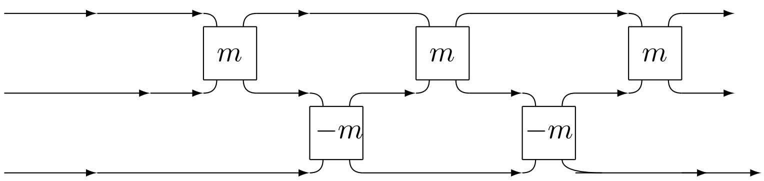


Figure 2. Snappy diagram representation for hybrid knots [29, 30]: (a) $\hat{W}_3(2,1) = \mathbf{4_1}$, (b) $\hat{W}_3(4,1) = \mathbf{8_{18}}$ knot, (c) $\hat{W}_3(5,1) = \mathbf{10_{123}}$ knot, and (d) $\hat{W}_3(3,2) = \mathbf{12a1288}$ knot.

which is pictorially seen in (3.1). Note that the subscript 3 in $\hat{W}_3(m,n)$ indicates three-strand braid.



$\ldots n\,.$ (3.1)

The classification of knots belongs to the hybrid weaving knot $\hat{W}_3(m,n)$ are tabulated below for some values of m and n: where m is odd and $m \neq n > 1$. When $m = 1$, $\hat{W}_3(1,n)$ reduces to the weaving knot $W(3,n)$ discussed in [7, 8]. Well known examples of weaving knots (see in figure 2) are

$$W(3,2) = \mathbf{4_1}, \quad W(3,4) = \mathbf{8_{18}} \quad \text{and} \quad W(3,5) = \mathbf{10_{123}}.$$

For $m > 3$ and $n \geq 2$, the crossing number exceeds 20 whose data are not available in the knot theory literature to validate. Now we will elaborate the modified RT method for hybrid weaving knots and achieve a closed form expression for their HOMFLY-PT polynomial.

JHEP06(2021)063

3.1 HOMFLY-PT for hybrid weaving knot $\hat{W}_3(m,n)$

In this case, tensor product of fundamental representation of three strand braid:

$$[1]^{\bigotimes 3} = [3] \bigoplus [1,1,1] \bigoplus 2[2,1,0]$$

shows that representation $[2,1,0]$ has multiplicity two. Incorporating 2×2 matrix form $\hat{\mathcal{R}}_1$ and $\hat{\mathcal{R}}_2$ for representation $[2,1,0]$ in eq. (2.13), the HOMFLY-PT for $\hat{W}_3(m,n)$ is

$$\begin{aligned}\mathcal{H}_{[1]}^{\hat{W}_3(m,n)} &= \frac{1}{S^*_{[1]}} \sum_{\Xi_\alpha=\{[3],[1,1,1],[2,1,0]\}} S^*_{\Xi_\alpha} \mathrm{Tr}_{\Xi_\alpha} (\hat{\mathcal{R}}_1^{\Xi_\alpha})^m (\hat{\mathcal{R}}_2^{\Xi_\alpha})^{-m} \ldots (\hat{\mathcal{R}}_1^{\Xi_\alpha})^m (\hat{\mathcal{R}}_2^{\Xi_\alpha})^{-m} \\ &= \frac{1}{S^*_{[1]}} (S^*_{[3]} + S^*_{[1,1,1]} + S^*_{[2,1]} \mathrm{Tr}_{[2,1,0]} (\hat{\mathcal{R}}_1^{[2,1,0]})^m (\hat{\mathcal{R}}_2^{[2,1,0]})^{-m})^n \,. \end{aligned} \tag{3.2}$$

Here $S^*_{[1]} = [N]_q$, $S^*_{[3]} = \frac{[N]_q[N+1]_q[N+2]_q}{[2]_q[3]_q}$, $S^*_{[111]} = \frac{[N]_q[N-1]_q[N-2]_q}{[2]_q[3]_q}$, and $S^*_{[21]} = \frac{[N]_q[N+1]_q[N-1]_q}{[3]_q}$.

In order to apply the formula (3.2) to evaluate the HOMFLYPT polynomial for $\hat{W}_3(m,n)$ we need to compute the trace of the matrix $\Psi^{[2,1,0]}[m,n] = ((\hat{\mathcal{R}}_1^{[2,1,0]})^m (\hat{\mathcal{R}}_2^{[2,1,0]})^{-m}))^n$. Using eq. (2.16) and eq. (2.23), we have,

$$\hat{\mathcal{R}}_1 = A^{-1} \begin{pmatrix} q & 0 \\ 0 & -\frac{1}{q} \end{pmatrix} \text{ and } \hat{\mathcal{R}}_2 = A^{-1} \begin{pmatrix} \frac{q^2-[3]_q}{q{[2]_q}^2} & -\frac{\sqrt{[3]_q}}{[2]_q} \\ -\frac{\sqrt{[3]_q}}{[2]_q} & \frac{1-q^2[3]_q}{q{[2]_q}^2} \end{pmatrix}.$$

Thus

$$\hat{\mathcal{R}}_1^m \hat{\mathcal{R}}_2^{-m} = \begin{pmatrix} \frac{1-q^{2m}[3]_q}{([2]_q)^2} & -\frac{(1+q^{2m})\sqrt{[3]_q}}{([2]_q)^2} \\ \frac{(1+q^{2m})\sqrt{[3]_q}}{q^{2m}([2]_q)^2} & \frac{1-q^{-2m}[3]_q}{([2]_q)^2} \end{pmatrix} = \begin{pmatrix} x_1 & -x_2 \\ \frac{x_2}{q^{2m}} & x_3 \end{pmatrix},$$

where $x_1 = \frac{1-q^{2m}[3]_q}{([2]_q)^2}$, $x_2 = \frac{(1+q^{2m})\sqrt{[3]_q}}{([2]_q)^2}$, and $x_3 = \frac{1-q^{-2m}[3]_q}{([2]_q)^2}$. Interestingly, we have succeed in the writing of diagonal entries of the nth power of the above matrix $((\hat{\mathcal{R}}_1^{[2,1,0]})^m (\hat{\mathcal{R}}_2^{[2,1,0]})^{-m})^n$ in a compact form $\Psi_1^{[2,1,0]}[m,n]$ and $\Psi_2^{[2,1,0]}[m,n]$ i.e.

$$\begin{aligned}\Psi_1^{[2,1,0]}[m,n] &= x_1^n + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=1}^{n-i} (-1)^i \binom{k+i-2}{i-1} \binom{n-(k+i-1)}{i} x_1^{n-(2i+k-1)} x_3^{k-1} \left(\frac{x_2}{q^m}\right)^{2i}, \\ \Psi_2^{[2,1,0]}[m,n] &= x_3^n + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=1}^{n-i} (-1)^i \binom{k+i-2}{i-1} \binom{n-(k+i-1)}{i} x_3^{n-(2i+k-1)} x_1^{k-1} \left(\frac{x_2}{q^m}\right)^{2i}. \end{aligned} \tag{3.3}$$

Hence the trace of the matrix $\Psi^{[2,1,0]}[m,n]$

$$\Psi^{[2,1,0]}[m,n] = \Psi_1^{[2,1,0]}[m,n] + \Psi_2^{[2,1,0]}[m,n]. \tag{3.4}$$

JHEP06(2021)063

Using these binomial series for the trace, the closed form expression for HOMFLY-PT for hybrid weaving knot turns out to be

$$\mathcal{H}_{[1]}^{\hat{W}_3(m,n)} = \frac{1}{S^*_{[1]}}\left(S^*_{[3]} + S^*_{[111]} + S^*_{[21]} \cdot \left(x_1^n + x_3^n + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor}\sum_{k=1}^{n-i}(-1)^i \binom{k+i-2}{i-1}\binom{n-(k+i-1)}{i}\right.\right.$$
$$\left.\left.(x_1^{(n+1-2i-k)}x_3^{(k-1)} + x_1^{(k-1)}x_3^{(n-2i-k+1)})\right)\left(\frac{x_2}{q^m}\right)^{2i}\right). \tag{3.5}$$

The closed form expression is an important result providing a useful starting point to investigate $[r]$-colored HOMFLY-PT, knot-quiver correspondence for hybrid weaving knots which we will pursue in future. Incidentally for $m = 1$, $\Psi^{[2,1,0]}[m,n]$ in eq. (3.4) is a Laurent polynomial [8] giving closed form HOMFLY-PT for weaving knots $W(3,n)$. We propose such a Laurent polynomial structure will be seen for all the multiplicity two irreducible representation $\Xi_\alpha \in [r]^{\otimes 3}$ for symmetric colors $[r] > 1$ as well.

Proposition 1. *Given a representation* $\Xi_\alpha \equiv [{\xi_1}^\alpha, {\xi_2}^\alpha, {\xi_3}^\alpha]$ *having multiplicity 2 with* $\hat{\mathcal{R}}_1^{\Xi_\alpha} = \pm q^{m_1} A^{-r}\begin{pmatrix} q^t & 0 \\ 0 & -\frac{1}{q^t}\end{pmatrix}$, *and* $\mathcal{U}^{\Xi_\alpha} = \begin{pmatrix} \frac{1}{[2]_{q^t}} & \frac{\sqrt{[3]_{q^t}}}{[2]_{q^t}} \\ \frac{\sqrt{[3]_{q^t}}}{[2]_{q^t}} & -\frac{1}{[2]_{q^t}}\end{pmatrix}$, *the Laurent polynomial* $\mathcal{V}_{n,t}[q]$ *is defined as*

$$\mathcal{V}_{n,t}[q] = \mathrm{Tr}(\hat{\mathcal{R}}_1^{\Xi_\alpha}\mathcal{U}^{\Xi_\alpha}(\hat{\mathcal{R}}_1^{\Xi_\alpha})^{-1}(\mathcal{U}^{\Xi_\alpha})^\dagger)^n = \sum_{g=-n}^{n}(-1)^g \mathcal{S}_{n,n-|g|}q^{2gt}. \tag{3.6}$$

Here t and m_1 are also an integer dependent on Ξ_α and the coefficients $\mathcal{S}_{n,j}$ are:

$$\mathcal{S}_{n,j} = \sum_{i=0}^{\lfloor \frac{j}{2} \rfloor}\frac{n}{n-i}\binom{n-i}{n-j+i}\binom{j-i-1}{i},$$

where the parameters n &j are positive integers and $|x|$ denote the absolute value of x and $\lfloor x \rfloor$ indicate the greatest integer x. For $m = 1$ and fundamental representation $[r] = [1]$, the trace in eq. (3.4) is

$$\Psi^{[2,1,0]}[1,n] = \mathcal{V}_{n,1}[q],$$

exactly matching with the parallel work [8]. Further, we conjecture the sum of the absolute coefficient $\mathcal{S}_{n,n-|\delta|}$ given by $\mathcal{O}_n$, satisfy the beautiful relation.

Conjecture 1.

$$\mathcal{O}_n = \sum_{\delta=-n}^{n}\mathcal{S}_{n,n-\delta} = 5\mathcal{F}_n^2 + 2(-1)^n\,, \tag{3.7}$$

where $\mathcal{F}_n$ *denotes Fibonacci numbers. The explicit form of* $\mathcal{F}_n$ *is given by [31]*

$$\mathcal{F}_n = \frac{1}{\sqrt{5}}(\phi^n - \cos{(n\pi)}(\phi)^{-n})\,.$$

n	1	2	3	4	5	6	7	8
$\mathcal{F}_n$	1	1	2	3	5	8	13	21
$\mathcal{O}_n$	3	7	18	47	123	322	843	2207

Table 1. $\mathcal{O}_n$ and $\mathcal{F}_n$ for $n \leq 8$.

Notation	Knot
$\hat{W}_3(1,n)$	weaving knot of type $W(3,n)$
$\hat{W}_3(m,1)$	$T_{(2,m)}\#T^*_{(2,m)}$
$\hat{W}_3(3,2)$	$12a1288$ Knot

Table 2. The classification of hybrid weaving knot $\hat{W}_3(m,n)$.

Here $\phi \approx 1.618$ is the golden ratio. We have checked this conjecture for large values of n. For values of $n \leq 8$, we have presented the values of $\mathcal{O}_n, \mathcal{F}_n$ in table 1. For completeness, we will briefly discuss the Fibonacci numbers and its properties. The Fibonacci ($\mathcal{F}_n$) numbers are sequences satisfying the Fibonacci recursion relation

$$\mathcal{F}_{n+1} = \mathcal{F}_n + \mathcal{F}_{n-1},$$

with following initial conditions: $\mathcal{F}_0 = 0, \mathcal{F}_1 = 1$. Here n is integer and it satisfy yhe following relation

$$\mathcal{F}_{-n} = (-1)^{n+1}\mathcal{F}_n.$$

3.2 Examples

For the hybrid weaving knots in table 2, HOMFLY-PT are obtained using our closed form expression for $\hat{W}_3(m,n)$.

- (a) For $m = 1$, the HOMFLY-PT polynomial is for weaving knots $W(3,n)$:

$$\mathcal{H}^{W(3,n)}_{[1]}(A,q) = \frac{1}{S^*_{[1]}}(S^*_{[3]} + S^*_{[111]} + S^*_{[21]}\mathcal{V}_{n,1}[q]). \tag{3.8}$$

  Substituting $A = q^2$, we get the Jones polynomial:

$$\mathcal{J}^{W(3,n)}(q) = q^{-2} + q^2 + \mathcal{V}_{n,1}[q]\,.$$

  These results agree with the results in the parallel paper on weaving knots [8].

- (b) composite knot $T_{(2,m)}\#T^*_{(2,m)}$.
  For odd $m \geq 2$ and $n = 1$, the knot belongs to composite knot of type $T_{(2,m)}\#T^*_{(2,m)}$.[6]
  Hence, the HOMFLY-PT will be

$$\begin{aligned} H^{\hat{W}_3(m,1)}_{[1]} &= H^{T_{(2,m)}}_{[1]}(q,A)H^{T^*_{(2,m)}}_{[1]}(q,A) \\ &= \frac{q^{2-2m}(-1+A^2q^2-A^2q^{2m}+q^{2+2m})(-A^2+q^2-q^{2m}+A^2q^{2+2m})}{A^2(-1+q)^2(1+q)^2(1+q^2)^2}. \end{aligned}$$

  and examples for some values of m shown in table 3.

[6] $T^*_{(2,m)}$ is the mirror of torus knot $T_{(2,m)}$.

m	KNOT	$H_{[1]}^{\hat{W}_3(m,1)}(q,A)$
3	$3_1\#3_1^*$	$(A^{-2}q^{-4})(1-A^2q^2+q^4)(A^2-q^2+A^2q^4)$
9	$9_1\#9_1^*$	$A^{-2}q^{-16}(1-A^2q^2+q^4-A^2q^6+q^8-A^2q^{10}+q^{12}-A^2q^{14}+q^{16})(A^2-q^2+A^2q^4-q^6+A^2q^8-q^{10}+A^2q^{12}-q^{14}+A^2q^{16})$

Table 3. HOMFLY-PT for $\hat{W}_3(m,1)=T_{(2,m)}\#T^*_{(2,m)}$.

$\Xi_\alpha \in [2]^3$	Matrix size	# of matrices
$[6,0,0]$, $[4,1,1]$, $[3,3]$	1	3
$[5,1,0]$,$[3,2,1]$	2	2
$[4,2,0]$	1	3

Table 4. The multiplicity table for $\Xi_\alpha \in [2]^3$.

- (c) The $m=3$ and $n=2$ refers to a 12 crossing knot "12a1288" in the Rolfsen table whose HOMFLY-PT polynomial is

$$H_{[1]}^{W_3(3,2)} = 11+\frac{7}{A^2}+7A^2-\frac{1}{q^{10}}+\frac{1}{q^8}+\frac{1}{A^2q^8}+\frac{A^2}{q^8}-\frac{5}{q^6}-\frac{1}{A^2q^6}-\frac{A^2}{q^6}+\frac{6}{q^4}$$
$$+\frac{4}{A^2q^4}+\frac{4A^2}{q^4}-\frac{11}{q^2}-\frac{5}{A^2q^2}-\frac{5A^2}{q^2}-11q^2-\frac{5q^2}{A^2}-5A^2q^2+6q^4+\frac{4q^4}{A^2}$$
$$+4A^2q^4-5q^6-\frac{q^6}{A^2}-A^2q^6+q^8+\frac{q^8}{A^2}+A^2q^8-q^{10}.$$

In the following section, we will present $[r]$-colored HOMFLYPT for $W(3,n)$ for $[r]=2,3$ and verify our proposition 1.

4 Colored HOMFLY-PT for weaving knot type $W(3,n)$

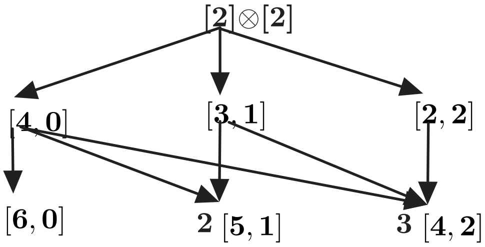
We will use the data on U^{Ξ_α} matrices in section 2.2 for three-strand braid where $\Xi_\alpha \in [2]^3$ and $\Xi_\alpha \in [3]^3$ to compute colored HOMFLY-PT for the weaving knots.

4.1 Representation $[r]=[2]$

In this case, $\bigotimes^3[2]=[6,0,0]\bigoplus[3,3,0]\bigoplus[4,1,1]\bigoplus 2[5,1,0]\bigoplus 2[3,2,1]\bigoplus 3[4,2,0]$.

From the multiplicity, we can see that there one 3×3 matrix, two 2×2 matrices, three 1×1 matrices as shown in the table 4. Also the path and the block structure of $\Xi\in[2]^{\otimes3}$

is shown (4.1)



. (4.1)

The eigenvalues and U^{Ξ_α} matrices in this case are

$$\hat{\mathcal{R}}_1^{[5,1,0]} = A^{-2}q^4 \begin{pmatrix} q^2 & 0 \\ 0 & -q^{-2} \end{pmatrix}, \quad \hat{\mathcal{R}}_1^{[3,2,1]} = A^{-2}q \begin{pmatrix} q^{-1} & 0 \\ 0 & -q \end{pmatrix}, \quad \hat{\mathcal{R}}_1^{[4,2,0]} = A^{-2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -q^2 & 0 \\ 0 & 0 & q^6 \end{pmatrix} \tag{4.2}$$

$$\mathcal{U}^{[5,1,0]} = \begin{pmatrix} \frac{1}{[2]_{q^2}} & \frac{\sqrt{[3]_{q^2}}}{[2]_{q^2}} \\ \frac{\sqrt{[3]_{q^2}}}{[2]_{q^2}} & -\frac{1}{[2]_{q^2}} \end{pmatrix}, \quad \mathcal{U}^{[3,2,1]} = \begin{pmatrix} \frac{1}{[2]_q} & \frac{\sqrt{[3]_q}}{[2]_q} \\ \frac{\sqrt{[3]_q}}{[2]_q} & -\frac{1}{[2]_q} \end{pmatrix}, \tag{4.3}$$

$$\mathcal{U}^{[4,2,0]} = \begin{pmatrix} -\frac{1}{1+\frac{1}{q^2}+q^2} & -\frac{q}{\sqrt{1+q^2+q^4}} & -\frac{\sqrt{1+q^2+q^4+q^6+q^8}}{1+q^2+q^4} \\ -\frac{q}{\sqrt{1+q^2+q^4}} & -1+\frac{q^2}{1+q^4} & \frac{q^4\sqrt{\left(1+\frac{1}{q^2}+q^2\right)\left(1+\frac{1}{q^4}+\frac{1}{q^2}+q^2+q^4\right)}}{(1+q^4)(1+q^2+q^4)} \\ -\frac{\sqrt{1+q^2+q^4+q^6+q^8}}{1+q^2+q^4} & \frac{q^4\sqrt{\left(1+\frac{1}{q^2}+q^2\right)\left(1+\frac{1}{q^4}+\frac{1}{q^2}+q^2+q^4\right)}}{(1+q^4)(1+q^2+q^4)} & -\frac{q^4}{1+q^2+2q^4+q^6+q^8} \end{pmatrix}. \tag{4.4}$$

From eq. (2.13), [2]-HOMFLY-PT for $W(3,n)$:

$$\begin{aligned} \mathcal{H}_{[2]}^{W(3,n)} &= \frac{1}{S^*_{[2]}} \sum_\alpha \mathrm{Tr}_{\Xi_\alpha}(\hat{\mathcal{R}}_1^{\Xi_\alpha}(\hat{\mathcal{R}}_2^{\Xi_\alpha})^{-1} \dots \hat{\mathcal{R}}_1^{\Xi_\alpha}(\hat{\mathcal{R}}_2^{\Xi_\alpha})^{-1}) \\ &= \frac{1}{S^*_{[2]}} \sum_\alpha \mathrm{Tr}_{\Xi_\alpha}(\hat{\mathcal{R}}_1^{\Xi_\alpha}(\hat{\mathcal{R}}_2^{\Xi_\alpha})^{-1})^n \\ &= \frac{1}{S^*_{[2]}} \Big(S^*_{[6]} + S^*_{[3,3]} + S^*_{[4,1,1]} + S^*_{[5,1,0]} \mathrm{Tr}_{[5,1,0]}(\hat{\mathcal{R}}^{[5,1,0]}(\hat{\mathcal{R}}_2^{[5,1,0]})^{-1})^n + S^*_{[3,2,1]} \\ &\qquad \mathrm{Tr}_{[3,2,1]}(\hat{\mathcal{R}}^{[3,2,1]}(\hat{\mathcal{R}}_2^{[3,2,1]})^{-1})^n + S^*_{[4,2,0]} \mathrm{Tr}_{[4,2,0]}(\hat{\mathcal{R}}^{[4,2,0]}(\hat{\mathcal{R}}_2^{[4,2,0]})^{-1})^n \Big). \end{aligned} \tag{4.5}$$

Using eqs. (4.2) to (4.4), and (3.6), we can rewrite the equation (4.5) into neat formula

$$\mathcal{H}_{[2]}^{W(3,n)} = \frac{1}{S^*_{[2]}} \left(S^*_{[6]} + S^*_{[3,3]} + S^*_{[4,1,1]} + S^*_{[5,1,0]} \mathcal{V}_{n,2} + S^*_{[3,2,1]} \mathcal{V}_{n,1} + S^*_{[4,2,0]} \mathrm{Tr}(X^{[4,2,0]})^n \right), \tag{4.6}$$

where,

$$X^{[4,2,0]} = \begin{pmatrix} \frac{1}{q^6+q^8+q^{10}} & -\frac{1}{q^5\sqrt{1+q^2+q^4}} & \frac{\sqrt{1+q^2+q^4+q^6+q^8}}{q^2+q^4+q^6} \\ \frac{1}{q^3\sqrt{1+q^2+q^4}} & \frac{-1+q^2-q^4}{q^2+q^6} & -\frac{q^3\sqrt{\frac{1+q^2+q^4+q^6+q^8}{1+q^2+q^4}}}{1+q^4} \\ \frac{q^4\sqrt{1+q^2+q^4+q^6+q^8}}{1+q^2+q^4} & \frac{q^7\sqrt{\frac{1+q^2+q^4+q^6+q^8}{1+q^2+q^4}}}{1+q^4} & \frac{q^{14}}{1+q^2+2q^4+q^6+q^8} \end{pmatrix}. \tag{4.7}$$

$\Xi_\alpha \in [3]^3$	Matrix size	# of matrices
$[9,0,0]$, $[7,1,1]$, $[5,2,2]$,$[4,4,1]$,$[3,3,3]$	1	5
$[4,3,2]$,$[6,2,1]$, $[5,4,0]$,$[5,3,1]$	2	4
$[8,1,0]$, $[7,2,0]$	3	2
$[6,3,0]$	4	1

Table 5. The multiplicity table for $\Xi_\alpha \in [3]^3$.

Using eq. (3.6), the [2]-colored reduced HOMFLY-PT polynomials for $W(3,n)$. We would like to emphasize that the polynomial form of this algebraic expression for arbitrary n is easily computable. We have listed [2] colored HOMFLY-PT in appendix B for some weaving knots.

4.2 Representation [3]

In this case, $\bigotimes^3[3] = [9,0,0] \bigoplus [7,1,1] \bigoplus [5,2,2] \bigoplus [4,4,1] \bigoplus [3,3,3] \bigoplus 3[8,1,0] \bigoplus 2[4,3,2] \bigoplus 2[6,2,1] \bigoplus 2[5,4,0] \bigoplus 3[7,2,0] \bigoplus 2[5,3,1] \bigoplus 4[6,3,0]$.

Thus, there are two 3×3 matrices, four 2×2 matrices, five 1×1 matrices and one 4×4 matrix tabulated in table 5. The braiding and U^{Ξ_α} matrices in this case are

$$\hat{\mathcal{R}}^{[6,3,0]}=A^{-3}\begin{pmatrix} -q^3 & 0 & 0 & 0 \\ 0 & q^5 & 0 & 0 \\ 0 & 0 & -q^9 & 0 \\ 0 & 0 & 0 & q^{15} \end{pmatrix}, \quad \hat{\mathcal{R}}^{[5,3,1]}=A^{-3}\begin{pmatrix} -q^3 & 0 & 0 \\ 0 & q^5 & 0 \\ 0 & 0 & -q^9 \end{pmatrix}, \quad \hat{\mathcal{R}}^{[7,2,0]}=A^{-3}\begin{pmatrix} q^5 & 0 & 0 \\ 0 & -q^9 & 0 \\ 0 & 0 & q^{15} \end{pmatrix}, \tag{4.8}$$

$$\hat{\mathcal{R}}^{[5,4,0]}=\hat{\mathcal{R}}^{[6,2,1]}=A^{-3}q^7\begin{pmatrix} q^{-2} & 0 \\ 0 & -q^2 \end{pmatrix}, \quad \hat{\mathcal{R}}^{[4,3,2]}=A^{-3}q^4\begin{pmatrix} -q^{-1} & 0 \\ 0 & q \end{pmatrix}, \quad \hat{\mathcal{R}}^{[8,1,0]}=A^{-3}q^{12}\begin{pmatrix} -q^{-3} & 0 \\ 0 & q^3 \end{pmatrix}, \tag{4.9}$$

$$\mathcal{U}^{[5,3,1]}=\begin{pmatrix} -\frac{1}{1+\frac{1}{q^2}+q^2} & -\frac{q}{\sqrt{1+q^2+q^4}} & -\frac{\sqrt{1+q^2+q^4+q^6+q^8}}{1+q^2+q^4} \\ -\frac{q}{\sqrt{1+q^2+q^4}} & -1+\frac{q^2}{1+q^4} & \frac{q^4\sqrt{\left(1+\frac{1}{q^2}+q^2\right)\left(1+\frac{1}{q^4}+\frac{1}{q^2}+q^2+q^4\right)}}{\left(1+q^4\right)\left(1+q^2+q^4\right)} \\ -\frac{\sqrt{1+q^2+q^4+q^6+q^8}}{1+q^2+q^4} & \frac{q^4\sqrt{\left(1+\frac{1}{q^2}+q^2\right)\left(1+\frac{1}{q^4}+\frac{1}{q^2}+q^2+q^4\right)}}{\left(1+q^4\right)\left(1+q^2+q^4\right)} & -\frac{q^4}{1+q^2+2q^4+q^6+q^8} \end{pmatrix}, \tag{4.10}$$

$$\mathcal{U}^{[5,4,0]}=\mathcal{U}^{[3,6,2,1]}=\begin{pmatrix} \frac{1}{[2]_{q^2}} & \frac{\sqrt{[3]_{q^2}}}{[2]_{q^2}} \\ \frac{\sqrt{[3]_{q^2}}}{[2]_{q^2}} & -\frac{1}{[2]_{q^2}} \end{pmatrix}, \tag{4.11}$$

$$\mathcal{U}^{[8,1,0]}=\begin{pmatrix} \frac{1}{[2]_{q^3}} & \frac{\sqrt{[3]_{q^3}}}{[2]_{q^3}} \\ \frac{\sqrt{[3]_{q^3}}}{[2]_{q^3}} & -\frac{1}{[2]_{q^3}} \end{pmatrix}, \qquad \mathcal{U}^{[4,3,2]}=\begin{pmatrix} \frac{1}{[2]_{q}} & \frac{\sqrt{[3]_{q}}}{[2]_{q}} \\ \frac{\sqrt{[3]_{q}}}{[2]_{q}} & -\frac{1}{[2]_{q}} \end{pmatrix}. \tag{4.12}$$

JHEP06(2021)063

We have placed the other 3×3 and also 4×4 matrices in appendix A. From eq. (2.13), [3]-colored HOMFLY-PT for $W(3,n)$:

$$\begin{aligned}
\mathcal{H}_{[3]}^{W(3,n)} &= \frac{1}{S^*_{[3]}} \sum_\alpha \mathrm{Tr}_{\Xi_\alpha}(\hat{\mathcal{R}}_1^{\Xi_\alpha}(\hat{\mathcal{R}}_2^{\Xi_\alpha})^{-1} \dots \hat{\mathcal{R}}_1^{\Xi_\alpha}(\hat{\mathcal{R}}_2^{\Xi_\alpha})^{-1}), \\
&= \frac{1}{S^*_{[3]}} \sum_\alpha \mathrm{Tr}_{\Xi_\alpha}(\hat{\mathcal{R}}_1^{\Xi_\alpha}(\hat{\mathcal{R}}_2^{\Xi_\alpha})^{-1})^n, \\
&= \frac{1}{S^*_{[3]}} \Big(S^*_{[9]} + S^*_{[7,1,1]} + S^*_{[5,2,2]} + S^*_{[4,4,1]} + S^*_{[3,3,3]} + S^*_{[4,3,2]} \mathrm{Tr}_{[4,3,2]}(\hat{\mathcal{R}}^{[4,3,2]}(\hat{\mathcal{R}}_2^{[4,3,2]})^{-1})^n \\
&\quad + S^*_{[6,2,1]} \mathrm{Tr}_{[6,2,1]}(\hat{\mathcal{R}}^{[6,2,1]}(\hat{\mathcal{R}}_2^{[6,2,1]})^{-1})^n + S^*_{[5,4,0]} \mathrm{Tr}_{[5,4,0]}(\hat{\mathcal{R}}^{[5,4,0]}(\hat{\mathcal{R}}_2^{[5,4,0]})^{-1})^n + \\
&\quad S^*_{[8,1,0]} \mathrm{Tr}_{[8,1,0]}(\hat{\mathcal{R}}^{[8,1,0]}(\hat{\mathcal{R}}_2^{[8,1,0]})^{-1})^n + S^*_{[7,2,0]} \mathrm{Tr}_{[7,2,0]}(\hat{\mathcal{R}}^{[7,2,0]}(\hat{\mathcal{R}}_2^{[7,2,0]})^{-1})^n + \\
&\quad S^*_{[6,3,0]} \mathrm{Tr}_{[6,3,0]}(\hat{\mathcal{R}}^{[6,3,0]}(\hat{\mathcal{R}}_2^{[6,3,0]})^{-1})^n \Big).
\end{aligned} \tag{4.13}$$

Using eqs. (4.8) to (4.12), eq. (3.6), and appendix A, we can rewrite the equation (4.13) into neat formula

$$\begin{aligned}
\mathcal{H}_{[3]}^{W(3,n)} = \frac{1}{S^*_{[3]}} \Big(& S^*_{[9]} + S^*_{[7,1,1]} + S^*_{[5,2,2]} + S^*_{[4,4,1]} + S^*_{[3,3,3]} + (S^*_{[6,2,1]} + S^*_{[5,4,0]}) \mathcal{V}_{n,2} \\
& + S^*_{[4,3,2]} \mathcal{V}_{n,1} + S^*_{[8,1,0]} \mathcal{V}_{n,3} + S^*_{[7,2,0]} \mathrm{Tr}(X_1^{[7,2,0]})^n + S^*_{[5,3,1]} \mathrm{Tr}(X_2^{[5,3,1]})^n \\
& + S^*_{[6,3,0]} \mathrm{Tr}(X_3^{[6,3,0]})^n \Big).
\end{aligned} \tag{4.14}$$

where the explicit form of $X_3^{[6,3,0]}$, $X_1^{[7,2,0]}$ and $X_2^{[5,3,1]}$ are given in appendix A and the colored HOMFLY-PT polynomials for $W(3,n)$ for color [3] are presented in appendix B. Even though we have explicitly computed $[r]$-colored HOMFLY-PT upto $[r] = 3$, the method is straightforward. However, it will be interesting if we can write a closed form expression for arbitrary color $[r]$. This is essential to work on volume conjecture for these hyperbolic knots which we plan to pursue in future. As a piece of evidence that our $[r]$-colored HOMFLY-PT for weaving knots are correct, we work out reformulated invariants and BPS integers in the context of topological string duality in the following section.

5 Integrality structures in topological strings

Motivated by the AdS-CFT correspondence, Gopakumar-Vafa conjectured that the SU(N) Chern-Simons theory on S^3 is dual to closed A-model topological string theory on a resolved conifold $\mathcal{O}(-1) + \mathcal{O}\ (-1)$ over $\mathbf{P}^1$. Particularly, the Chern-Simons free energy $\ln Z[S^3]$ was shown to be closed string partition function on the resolved conifold target space:

$$\ln Z[S^3] = -\sum_g \mathcal{F}_g(t) g_s^{2-2g}, \tag{5.1}$$

where $\mathcal{F}_g(t)$ are the genus g topological string amplitude, $g_s = \frac{2\pi}{k+N}$ denotes the string coupling constant and $t = \frac{2\pi i N}{k+N}$ denote the Kähler parameter of $\mathbf{P}^1$. Ooguri-Vafa conjectured that the Wilson loop operators in Chern-Simons theory correspond to the following topological string operator on a deformed conifold T^*S^3:

$$\ln Z(U,V)_{\mathcal{S}^3} = \sum_m \frac{1}{m} Tr_{[1]} U^m Tr_{[1]} V^m, \tag{5.2}$$

where U represent the holonomy of the gauge connection A around the knot $\mathcal{K}$ carrying the fundamental representation ([1]) in the U(N) Chern-Simons theory on S^3, and V is the holonomy of a gauge field $\tilde{A}$ around the same component knot carrying the fundamental representation ([1]) in the U(M) Chern-Simons theory on a Lagrangian sub-manifold $\mathcal{C}$ which intersects S^3 along the knot $\mathcal{K}$. Gopakumar-Vafa duality require integrating the gauge field A on S^3 leading to open topological string amplitude on the resolved conifold background. For unknot, the detailed calculation was performed [9] giving:

$$\langle Z(U,V)\rangle_{S^3} = \exp\left\{\left(i\sum_{m=1}^{\infty} \frac{\exp\{(\frac{mt}{2})\} - \exp\{(\frac{-mt}{2})\}}{2m\sin\left(\frac{mg_s}{2}\right)} \mathrm{Tr} V^{-m}\right)\right\}, \tag{5.3}$$

which was justified using Gopakumar-Vafa duality. Further, Ooguri-Vafa conjectured the generalization of eq. (5.3) for other knots as (also known $LMOV$ integrality conjecture):

$$\begin{aligned}\langle Z(U,V)\rangle_{S^3} &= \sum_{\mathbf{R}} \mathcal{H}^{*\mathcal{K}}_{\mathbf{R}}(q,\mathbf{A}) \mathrm{Tr}_{\mathbf{R}} V \\ &= \exp\left[\sum_{m=1}^{\infty}\left(\sum_R \frac{1}{m} f^{\mathcal{K}}_{\mathbf{R}}(A^m,q^m) \mathrm{Tr}_{\mathbf{R}} V^m\right)\right], \end{aligned}\tag{5.4}$$

where $f^{\mathcal{K}}_{\mathbf{R}}(A,q)$, known as reformulated invariant, obeying the following integrality structure:

$$f^{\mathcal{K}}_{\mathbf{R}}(q,A) = \sum_{i,j} \frac{1}{(q-q^{-1})} \widetilde{\mathbf{N}}^{\mathcal{K}}_{\mathbf{R},i,j} A^i q^j .$$

Here, R denotes the irreducible representation of U(N) and $\widetilde{\mathbf{N}}^{\mathcal{K}}_{\mathbf{R},i,j}$ counts the number of D2-brane intersecting D4-brane (BPS states) where, i and j keeps track of charges and spins respectively [10, 32]. These reformulated invariants can be written in the terms of colored HOMFLY-PT polynomials (5.4). For few lower dimensional representations, the explicit forms are as follows [11, 33, 34]:

$$\begin{aligned} f^{\mathcal{K}}_{[1]}(q,A) &= \mathcal{H}^{*\mathcal{K}}_{[1]}(q,A), \\ f^{\mathcal{K}}_{[2]}(q,A) &= \mathcal{H}^{*\mathcal{K}}_{[2]}(q,A) - \frac{1}{2}\Big(\mathcal{H}^{*\mathcal{K}}_{[1]}(q,A)^2 + \mathcal{H}^{*\mathcal{K}}_{[1]}(q^2,A^2)\Big), \\ f^{\mathcal{K}}_{[1^2]}(q,A) &= \mathcal{H}^{*\mathcal{K}}_{[1^2]}(q,A) - \frac{1}{2}\Big(\mathcal{H}^{*\mathcal{K}}_{[1]}(q,A)^2 - \mathcal{H}^{*\mathcal{K}}_{[1]}(q^2,A^2)\Big), \\ &\ldots \end{aligned}$$

In fact, reformulated invariants obey Ooguri-Vafa conjecture verified for many arborescent knots up to 10 crossings in [35]. Moreover, these reformulated invariant can be equivalently written as [11]:

$$f^{\mathcal{K}}_{\mathbf{R}}(q,A) = \sum_{m,k\geq 0,s} C_{\mathbf{RS}} \hat{\mathbf{N}}^{\mathcal{K}}_{\mathbf{S},m,k} A^m (q-q^{-1})^{2k-1}, \tag{5.5}$$

where $\hat{\mathbf{N}}^{\mathcal{K}}_{\mathbf{S},m,k}$ called refined integers and

$$C_{\mathbf{RS}} = \frac{1}{q-q^{-1}} \sum_{\Delta} \frac{1}{z_\Delta} \psi_{\mathbf{R}}(\Delta) \psi_{\mathbf{S}}(\Delta) \prod_{i=1}^{l(\Delta)} \left(q^{\xi_i} - q^{-\xi_i}\right).$$

Here the sum goes over the Young diagrams Δ with $l(\Delta)$ lines of lengths ξ_i and the number of boxes $|\Delta| = \sum_i^{l(\Delta)} \xi_i$, while $\psi_{\mathbf{R}}(\Delta)$ denote the characters of symmetric groups at $|R| = |\Delta|$ and z_Δ is the standard symmetric factor of the Young diagram [36]. Using our colored HOMFLY-PT form for the weaving knot $W(3,n)$ (listed in appendix B), we computed the reformulated invariants for representations upto length $|\mathbf{R}| = 2$. From our analysis, we propose the following:

Proposition 2. *Refined BPS integer* $\hat{\mathbf{N}}^{W(3,n)}_{[1],\mp 1,k}$ *for weaving knot* $W(3,n)$ *is the coefficient of* z^k *of polynomial* $f^{\mp}_n[z]$ *of degree* $n-1$ *i.e.*

$$f^{\mp}_n[z] = \pm \frac{2(-1)^n T_n\left(\frac{1+z}{2}\right) + 1}{z}\,, \tag{5.6}$$

where $T_n(z)$ represents the nth degree Chebyshev polynomial of the first kind at the point z. Rodrigue's formula to obtain $T_n(z)$ is

$$T_n(z) = \frac{(-2)^n n!}{2n!} \sqrt{1-z^2} \frac{d^n}{dz^n} (1-z^2)^{n-1/2}\,. \tag{5.7}$$

Here we list the polynomial form for some values of n: for completeness,

$$\begin{aligned}
f^-_{11}[z] &= 11 + 22z - 66z^2 - 99z^3 + 77z^4 + 154z^5 + 22z^6 - 66z^7 - 44z^8 - 11z^9 - z^{10} \\
f^-_{10}[z] &= -10 + 15z + 60z^2 - 15z^3 - 98z^4 - 35z^5 + 40z^6 + 35z^7 + 10z^8 + z^9 \\
f^-_5[z] &= 5 + 5z - 5z^2 - 5z^3 - z^4 \\
f^-_4[z] &= -4 + 2z + 4z^2 + z^3\,.
\end{aligned}$$

Unfortunately, we have not managed to write the other integers for fundamental representation $\hat{\mathbf{N}}^{W(3,n)}_{[1],\pm 3,k}$: as a closed form. There are other properties of $\hat{\mathbf{N}}$ which we have checked up to the level $|\mathbf{S}| = 2$ for $W(3,n)$ knot. They are

$$\begin{aligned}
&\sum_m \hat{\mathbf{N}}^{W(3,n)}_{\mathbf{S},m,k} = 0 \\
&\sum_k \hat{\mathbf{N}}^{W(3,n)}_{[\mathbf{1}],\mp 1,k} \;=\; \mp\frac{4}{3} T_{n-1}(-1) \sec\left(\frac{n\pi}{6}\right)^2 \sin\left(\frac{n\pi}{3}\right)^4, \; n \geq 1.
\end{aligned}$$

here $T_{n-1}(z=-1)$ is the Chebyshev polynomial evaluated at $z=-1$. We have tabulated below these refined integers for knot $W(3,4), W(3,5), W(3,10)$, and $W(3,11)$, when $|r=1|$:

$\hat{\mathbf{N}}^{W[3,4]}_{[1]}$:

$k\backslash m =$	-3	-1	1	3
0	1	-4	4	-1
1	-1	2	-2	1
2	-1	4	-4	1
4	0	1	-1	0

, $\hat{\mathbf{N}}^{W[3,5]}_{[1]}$:

$k\backslash m =$	-3	-1	1	3
0	-2	5	-5	2
1	-1	5	-5	$+1$
2	2	-5	5	-2
3	1	-5	5	-1
4	0	-1	1	0

$\hat{\mathbf{N}}^{W[3,10]}_{[1]}$

$k\backslash m =$	-3	-1	1	3
0	3	−10	10	3
1	−6	15	−15	6
2	−18	60	−60	18
3	11	−15	15	−11
4	29	−98	98	−29
5	2	−35	35	−2
6	−14	40	−40	14
7	−7	35	−35	7
8	−1	10	−10	1
9	0	1	−1	0

,

$\hat{\mathbf{N}}^{W[3,11]}_{[1]}$

$k\backslash m =$	-3	-1	1	3
0	−4	11	−11	4
1	−6	22	−22	6
2	24	−66	66	−24
3	25	−99	99	−25
4	−34	77	−77	34
5	−40	154	−154	40
6	6	22	−22	−6
7	20	−66	66	−20
8	8	−44	44	−8
9	1	−11	11	−1
10	0	−1	1	0

.

The table of refined integers for representations whose length $|R| = 2$ are presented in appendix C.

6 Conclusion and discussion

Hybrid weaving knots $\hat{W}_3(m,n)$ obtained from braid word $\left[\sigma_1^m \sigma_2^{-m}\right]^n$ (see figure 3.1) contains weaving knots $W(3,n)$ as subset which are hyperbolic in nature. Finding a closed form expression for $[r]$-colored HOMFLY-PT for such hybrid weaving knots was attempted using the modified Reshtikhin-Turaev approach [16]–[17] method. Using the $\hat{\mathcal{R}}_i$ matrices, we derived the explicit closed form expression of HOMFLY-PT for hybrid weaving knot$\hat{W}_3(m,n)$ (3.5). Motivated by the Laurent polynomial structure studied for HOMFLY-PT of weaving knots [8], we proposed such a structure $\mathcal{V}_{n,t}[q]$ (4.5 and 4.14) for any $[r]$-colored HOMFLY-PT for the weaving knots. Further we showed that the absolute sum of the coefficients in the Laurent polynomial is related to Fibonacci numbers (see conjecture 1 (3.7)). We have computed the colored HOMFLY-PT for $W(3,n)$ upto $[r] = 3$ and presented them in the appendix B. Clearly, writing the polynomial form is computationally simplified by this modified RT method. Using these knot invariants, we computed reformulated invariants and found some of the refined BPS integers can be written in terms of coefficient of Chebyshev polynomials ($T_n(x)$) of first kind for $W(3,n)$ (5.6).

So far, we have managed to write the closed form expression for trace of 2×2 matrices by introducing the $\mathcal{V}_{n,t}[q]$. For higher dimensional matrices, such a Laurent polynomial structure is not obvious. We have seen a concise form for $[r]$-colored HOMFLY-PT for knot $4_1 \equiv W(3,2)$, twist knots and torus knots using q-binomial and q-Pochammer terms. It will be interesting if we can find a similar expression for weaving knots. Such an expression will help us to address volume conjecture, A-polynomials for these weaving knots. We hope to address these problems in future.

Acknowledgments

VKS would like to acknowledge the hospitality of department of mathematics, IISER, Pune (India) where this work was done during his visit as visiting fellow. PR would like to thank SERB (MATRICS) MTR/2019/000956 funding.

A Unitary matrices

$$\mathcal{U}^{[6,3,0]}=\begin{pmatrix} \frac{q^3}{1+q^2+q^4+q^6} & \frac{q^2\sqrt{1+q^2+q^4}}{1+q^2+q^4+q^6} & (u_1)_{13} & (u_1)_{14} \\ \frac{q^2\sqrt{1+q^2+q^4}}{1+q^2+q^4+q^6} & \frac{q+2q^3+2q^5+q^7+2q^9+2q^{11}+q^{13}}{\left(1+q^2+q^4+q^6\right)\left(1+q^2+q^4+q^6+q^8\right)} & (u_1)_{23} & \\ \frac{q\sqrt{1+q^2+q^4+q^6+q^8}}{1+q^2+q^4+q^6} & (u_1)_{32} & -\frac{q+q^3-q^5+q^7+q^9}{1+q^4+q^6+q^{10}} & (u_1)_{34} \\ \frac{\sqrt{1+q^2+q^4+q^6+q^8+q^{10}+q^{12}}}{1+q^2+q^4+q^6} & (u_1)_{42} & (u_1)_{43} & (u_1)_{44} \end{pmatrix} \tag{A.1}$$

$$(u_1)_{14}=\frac{\sqrt{1+q^2+q^4+q^6+q^8+q^{10}+q^{12}}}{1+q^2+q^4+q^6}$$

$$(u_1)_{13}=\frac{q\sqrt{1+q^2+q^4+q^6+q^8}}{1+q^2+q^4+q^6}$$

$$(u_1)_{24}=-\frac{q^5\left(1+q^2+q^4\right)\sqrt{\left(1+\frac{1}{q^2}+q^2\right)\left(1+\frac{1}{q^6}+\frac{1}{q^4}+\frac{1}{q^2}+q^2+q^4+q^6\right)}}{(1+q^2+q^4+q^6)(1+q^2+q^4+q^6+q^8)}$$

$$(u_1)_{23}=\frac{q^3\sqrt{\left(1+\frac{1}{q^2}+q^2\right)\left(1+\frac{1}{q^4}+\frac{1}{q^2}+q^2+q^4\right)}\left(1-q^4+q^8\right)}{(1+q^2)(1+q^4)(1+q^2+q^4+q^6+q^8)}$$

$$(u_1)_{32}=\frac{q^3\sqrt{\left(1+\frac{1}{q^2}+q^2\right)\left(1+\frac{1}{q^4}+\frac{1}{q^2}+q^2+q^4\right)}\left(1-q^4+q^8\right)}{(1+q^2)(1+q^4)(1+q^2+q^4+q^6+q^8)}$$

$$(u_1)_{34}=\frac{q^9\sqrt{\left(1+\frac{1}{q^4}+\frac{1}{q^2}+q^2+q^4\right)\left(1+\frac{1}{q^6}+\frac{1}{q^4}+\frac{1}{q^2}+q^2+q^4+q^6\right)}}{(1+q^4)(1+q^6)(1+q^2+q^4+q^6+q^8)}$$

$$(u_1)_{42}=-\frac{q^5\left(1+q^2+q^4\right)\sqrt{\left(1+\frac{1}{q^2}+q^2\right)\left(1+\frac{1}{q^6}+\frac{1}{q^4}+\frac{1}{q^2}+q^2+q^4+q^6\right)}}{(1+q^2+q^4+q^6)(1+q^2+q^4+q^6+q^8)}$$

$$(u_1)_{44}=-\frac{q^9}{(1+q^4)(1+q^6)(1+q^2+q^4+q^6+q^8)}$$

$$(u_1)_{43}=\frac{q^9\sqrt{\left(1+\frac{1}{q^4}+\frac{1}{q^2}+q^2+q^4\right)\left(1+\frac{1}{q^6}+\frac{1}{q^4}+\frac{1}{q^2}+q^2+q^4+q^6\right)}}{(1+q^4)(1+q^6)(1+q^2+q^4+q^6+q^8)}$$

$$\mathcal{U}^{[7,2,0]}=\begin{pmatrix} -\frac{q^4\left(1+q^2+q^4\right)}{\left(1+q^4\right)\left(1+q^2+q^4+q^6+q^8\right)} & -\frac{q^2\sqrt{\frac{1+q^2+q^4+q^6+q^8+q^{10}+q^{12}}{1+q^4+q^6+q^8+q^{12}}}}{1+q^4} & (u_2)_{13} \\ -\frac{q^2\sqrt{\frac{1+q^2+q^4+q^6+q^8+q^{10}+q^{12}}{1+q^4+q^6+q^8+q^{12}}}}{1+q^4} & -1+\frac{q^4}{\left(1+q^4\right)\left(1-q^2+q^4\right)} & (u_2)_{23} \\ (u_2)_{31} & (u_2)_{32} & -\frac{q^6}{1+q^4+q^6+q^8+q^{12}} \end{pmatrix} \tag{A.2}$$

$$(u_2)_{13}=-\frac{\left(1+q^8\right)\left(1+q^2+q^4+q^6+q^8+q^{10}+q^{12}\right)}{(1+q^2+q^4+q^6+q^8)\sqrt{1+q^4+q^6+2q^8+q^{10}+2q^{12}+q^{14}+2q^{16}+q^{18}+q^{20}+q^{24}}}$$

$$(u_2)_{23}=\frac{q^2\sqrt{\frac{1+q^8}{1+q^2+q^4+q^6+q^8}}}{1-q^2+q^4}$$

$$(u_2)_{32}=\frac{q^2\sqrt{\frac{1+q^8}{1+q^2+q^4+q^6+q^8}}}{1-q^2+q^4}$$

JHEP06(2021)063

$$(u_2)_{31}=-\frac{\left(1+q^8\right)\left(1+q^2+q^4+q^6+q^8+q^{10}+q^{12}\right)}{(1+q^2+q^4+q^6+q^8)\sqrt{1+q^4+q^6+2q^8+q^{10}+2q^{12}+q^{14}+2q^{16}+q^{18}+q^{20}+q^{24}}}$$

$$X_3^{[6,3,0]}=\begin{pmatrix} -\frac{1}{q^{12}\left(1+q^2+q^4+q^6\right)} & \frac{\sqrt{1+q^2+q^4}}{q^{11}\left(1+q^2+q^4+q^6\right)} & (x_3)_{13} & (x_3)_{14} \\ -\frac{\sqrt{1+q^2+q^4}}{q^9\left(1+q^2+q^4+q^6\right)} & \frac{1+2q^2+2q^4+q^6+2q^8+2q^{10}+q^{12}}{q^8\left(1+q^2\right)\left(1+q^4\right)\left(1+q^2+q^4+q^6+q^8\right)} & (x_3)_{23} & (x_3)_{23} \\ -\frac{\sqrt{1+q^2+q^4+q^6+q^8}}{q^2+q^4+q^6+q^8} & (x_3)_{32} & \frac{q^4\left(1+q^2-q^4+q^6+q^8\right)}{1+q^4+q^6+q^{10}} & (x_3)_{34} \\ -\frac{q^9\sqrt{1+q^2+q^4+q^6+q^8+q^{10}+q^{12}}}{1+q^2+q^4+q^6} & (x_3)_{42} & (x_3)_{43} & (x_3)_{44} \end{pmatrix} \tag{A.3}$$

$$(x_3)_{13}=-\frac{\sqrt{1+q^2+q^4+q^6+q^8}}{q^8\,(1+q^2+q^4+q^6)}$$

$$(x_3)_{14}=\frac{\sqrt{1+q^2+q^4+q^6+q^8+q^{10}+q^{12}}}{q^3+q^5+q^7+q^9}$$

$$(x_3)_{23}=\frac{\left(-1+q^4-q^8\right)\sqrt{1+q^2(1+q^2)(2+q^2+2q^4+q^6+q^8)}}{q^5\,(1+q^2)(1+q^4)(1+q^2+q^4+q^6+q^8)}$$

$$(x_3)_{24}=-\frac{q^2\left(1+q^2+q^4\right)\sqrt{1+q^2(1+q^2)(2+q^2+2q^4+q^6+2q^8+q^{10}+q^{12})}}{(1+q^2)(1+q^4)(1+q^2+q^4+q^6+q^8)}$$

$$(x_3)_{32}=\frac{\left(1-q^4+q^8\right)\sqrt{1+q^2(1+q^2)(2+q^2+2q^4+q^6+q^8)}}{q\,(1+q^2)(1+q^4)(1+q^2+q^4+q^6+q^8)}$$

$$(x_3)_{34}=\frac{q^{13}\sqrt{1+q^2(1+q^2)(2+q^2+3q^4+2q^6+3q^8+2q^{10}+2q^{12}+q^{14}+q^{16})}}{(1+q^2)(1+q^4)(1-q^2+q^4)(1+q^2+q^4+q^6+q^8)}$$

$$(x_3)_{42}=-\frac{q^{12}\left(1+q^2+q^4\right)\sqrt{1+q^2(1+q^2)(2+q^2+2q^4+q^6+2q^8+q^{10}+q^{12})}}{(1+q^2)(1+q^4)(1+q^2+q^4+q^6+q^8)}$$

$$(x_3)_{43}=-\frac{q^{19}\sqrt{1+q^2(1+q^2)(2+q^2+3q^4+2q^6+3q^8+2q^{10}+2q^{12}+q^{14}+q^{16})}}{(1+q^2)(1+q^4)(1-q^2+q^4)(1+q^2+q^4+q^6+q^8)}$$

$$(x_3)_{44}=-\frac{q^{30}}{(1+q^2)(1+q^4)(1-q^2+q^4)(1+q^2+q^4+q^6+q^8)}$$

$$X_1^{[5,3,1]}=\begin{pmatrix} \frac{1}{q^6+q^8+q^{10}} & -\frac{1}{q^5\sqrt{1+q^2+q^4}} & \frac{\sqrt{1+q^2+q^4+q^6+q^8}}{q^2+q^4+q^6} \\ \frac{1}{q^3\sqrt{1+q^2+q^4}} & -\frac{1}{q^2}+\frac{1}{1+q^4} & -\frac{q^3\sqrt{\frac{1+q^2+q^4+q^6+q^8}{1+q^2+q^4}}}{1+q^4} \\ \frac{q^4\sqrt{1+q^2+q^4+q^6+q^8}}{1+q^2+q^4} & \frac{q^7\sqrt{\frac{1+q^2+q^4+q^6+q^8}{1+q^2+q^4}}}{1+q^4} & \frac{q^{14}}{1+q^2+2q^4+q^6+q^8} \end{pmatrix}$$

$$X_2^{[7,2,0]}=\begin{pmatrix} \frac{q^{22}}{1+q^4+q^6+q^8+q^{12}} & \frac{q^{12}\sqrt{\frac{1+q^8}{1+q^2+q^4+q^6+q^8}}}{1-q^2+q^4} & (x_2)_{13} \\ -\frac{q^6\sqrt{\frac{1+q^8}{1+q^2+q^4+q^6+q^8}}}{1-q^2+q^4} & \frac{-1+q^2-q^4+q^6-q^8}{q^2\left(1+q^4\right)\left(1-q^2+q^4\right)} & (x_2)_{23} \\ (x_2)_{31} & \frac{\sqrt{\left(1+q^4+q^6+q^8+q^{12}\right)\left(1+q^2+q^4+q^6+q^8+q^{10}+q^{12}\right)}}{q^8\left(1+2q^4+q^6+2q^8+q^{10}+2q^{12}+q^{16}\right)} & \frac{1+q^2+q^4}{q^{10}\left(1+q^4\right)\left(1+q^2+q^4+q^6+q^8\right)} \end{pmatrix} \tag{A.4}$$

JHEP06(2021)063

where

$$
\begin{aligned}
(x_2)_{13} &= -\frac{q^6\sqrt{\frac{1+q^2+q^4+q^6+2q^8+2q^{10}+2q^{12}+q^{14}+q^{16}+q^{18}+q^{20}}{1-q^2+q^4}}}{1+q^2+q^4+q^6+q^8} \\
(x_2)_{23} &= -\frac{\sqrt{(1+q^4+q^6+q^8+q^{12})\,(1+q^2+q^4+q^6+q^8+q^{10}+q^{12})}}{q^4\,(1+2q^4+q^6+2q^8+q^{10}+2q^{12}+q^{16})} \\
(x_2)_{31} &= -\frac{\sqrt{\frac{1+q^2+q^4+q^6+2q^8+2q^{10}+2q^{12}+q^{14}+q^{16}+q^{18}+q^{20}}{1-q^2+q^4}}}{q^4+q^6+q^8+q^{10}+q^{12}}\,.
\end{aligned}
$$

B Colored HOMFLY-PT polynomials

The weaving knot $W(3,n)$ whose $[r]=2,3$ colored HOMFLY-PT worked out in section 4 (see in eq. (4.6) & (4.14)) can be compactly rewritten in the matrix form (q^2, A^2): as example [2]-colored HOMFLY-PT of $W(3,2)$ knot is

$$
\begin{aligned}
H_{[2]}^{W[3,2]} = \frac{1}{A^4q^6}(&-A^2+A^4+q^2-A^2q^2-A^4q^2+A^2q^4-A^6q^4+3A^4q^6-A^2q^8+A^6q^8 \\
&-A^4q^{10}-A^6q^{10}+A^8q^{10}+A^4q^{12}-A^6q^{12}),
\end{aligned}
$$

and it can compactly rewritten in the matrix form (q^2, A^2)

$$
H_{[2]}^{W[3,2]} = A^{-4}q^{-6}\begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}.
$$

Similarly, colored HOMFLY-PT for few other weaving knots listed in the matrix form (q^2, A^2):

$$
H_{[2]}^{W[3,4]} = A^{-4}q^{-18}\begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ 1 & 2 & -3 & 0 & 0 \\ -3 & 3 & 1 & -1 & 0 \\ 0 & -10 & 7 & 2 & 0 \\ 9 & 2 & -14 & 3 & 0 \\ -8 & 19 & 2 & -9 & 1 \\ -7 & -20 & 27 & 3 & -3 \\ 15 & -15 & -26 & 16 & 0 \\ -3 & 31 & -13 & -23 & 8 \\ -9 & -8 & 47 & -8 & -9 \\ 8 & -23 & -13 & 31 & -3 \\ 0 & 16 & -26 & -15 & 15 \\ -3 & 3 & 27 & -20 & -7 \\ 1 & -9 & 2 & 19 & -8 \\ 0 & 3 & -14 & 2 & 9 \\ 0 & 2 & 7 & -10 & 0 \\ 0 & -1 & 1 & 3 & -3 \\ 0 & 0 & -3 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}, \qquad
H_{[2]}^{W[3,5]} = A^{-4}q^{-24}\begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ 1 & 3 & -4 & 0 & 0 \\ -4 & 2 & 3 & -1 & 0 \\ 2 & -15 & 10 & 3 & 0 \\ 12 & 13 & -26 & 2 & 0 \\ -22 & 25 & 11 & -15 & 1 \\ -2 & -58 & 46 & 12 & -4 \\ 45 & 5 & -76 & 24 & 2 \\ -36 & 92 & 1 & -54 & 12 \\ -29 & -87 & 129 & 8 & -21 \\ 66 & -58 & -116 & 85 & -1 \\ -19 & 136 & -65 & -93 & 41 \\ -40 & -40 & 189 & -40 & -40 \\ 41 & -93 & -65 & 136 & -19 \\ -1 & 85 & -116 & -58 & 66 \\ -21 & 8 & 129 & -87 & -29 \\ 12 & -54 & 1 & 92 & -36 \\ 2 & 24 & -76 & 5 & 45 \\ -4 & 12 & 46 & -58 & -2 \\ 1 & -15 & 11 & 25 & -22 \\ 0 & 2 & -26 & 13 & 12 \\ 0 & 3 & 10 & -15 & 2 \\ 0 & -1 & 3 & 2 & -4 \\ 0 & 0 & -4 & 3 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}
$$

$$H_{[2]}^{W[3,10]} = A^{-4} q^{-54} \begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ 1 & 8 & -9 & 0 & 0 \\ -9 & -18 & 28 & -1 & 0 \\ 27 & -25 & -10 & 8 & 0 \\ -3 & 182 & -161 & -18 & 0 \\ -171 & -215 & 410 & -25 & 1 \\ 368 & -452 & -89 & 182 & -9 \\ 60 & 1550 & -1422 & -215 & 27 \\ -1436 & -763 & 2654 & -452 & -3 \\ 2002 & -3631 & 249 & 1550 & -171 \\ 1201 & 6924 & -7730 & -763 & 368 \\ -6748 & 283 & 10046 & -3630 & 60 \\ 5900 & -16340 & 4951 & 6925 & -1436 \\ 7228 & 18695 & -28254 & 274 & 2002 \\ -20384 & 11318 & 24214 & -16348 & 1200 \\ 9951 & -47634 & 25873 & 18733 & -6749 \\ 24582 & 31349 & -73186 & 11346 & 5909 \\ -42876 & 46700 & 36275 & -47744 & 7236 \\ 6553 & -97291 & 79874 & 31285 & -20421 \\ 56419 & 26407 & -138924 & 46951 & 9924 \\ -64446 & 113504 & 23438 & -97179 & 24683 \\ -13101 & -142395 & 171125 & 25948 & -42820 \\ 92427 & -16795 & -195296 & 113324 & 6340 \\ -67075 & 189673 & -35612 & -141595 & 56325 \\ -45008 & -143645 & 269458 & -16752 & -64053 \\ 108753 & -89895 & -196699 & 188829 & -13061 \\ -44308 & 224367 & -128141 & -143909 & 91991 \\ -67241 & -88626 & 313941 & -88626 & -67241 \\ 91991 & -143909 & -128141 & 224367 & -44308 \\ -13061 & 188829 & -196699 & -89895 & 108753 \\ -64053 & -16752 & 269458 & -143645 & -45008 \\ 56325 & -141595 & -35612 & 189673 & -67075 \\ 6340 & 113324 & -195296 & -16795 & 92427 \\ -42820 & 25948 & 171125 & -142395 & -13101 \\ 24683 & -97179 & 23438 & 113504 & -64446 \\ 9924 & 46951 & -138924 & 26407 & 56419 \\ -20421 & 31285 & 79874 & -97291 & 6553 \\ 7236 & -47744 & 36275 & 46700 & -42876 \\ 5909 & 11346 & -73186 & 31349 & 24582 \\ -6749 & 18733 & 25873 & -47634 & 9951 \\ 1200 & -16348 & 24214 & 11318 & -20384 \\ 2002 & 274 & -28254 & 18695 & 7228 \\ -1436 & 6925 & 4951 & -16340 & 5900 \\ 60 & -3630 & 10046 & 283 & -6748 \\ 368 & -763 & -7730 & 6924 & 1201 \\ -171 & 1550 & 249 & -3631 & 2002 \\ -3 & -452 & 2654 & -763 & -1436 \\ 27 & -215 & -1422 & 1550 & 60 \\ -9 & 182 & -89 & -452 & 368 \\ 1 & -25 & 410 & -215 & -171 \\ 0 & -18 & -161 & 182 & -3 \\ 0 & 8 & -10 & -25 & 27 \\ 0 & -1 & 28 & -18 & -9 \\ 0 & 0 & -9 & 8 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix},$$

$$H^{W[3,11]}_{[2]} = A^{-4}q^{-60}\begin{pmatrix}
0 & -1 & 1 & 0 & 0 \\
1 & 9 & -10 & 0 & 0 \\
-10 & -25 & 36 & -1 & 0 \\
35 & -16 & -28 & 9 & 0 \\
-20 & 234 & -189 & -25 & 0 \\
-205 & -397 & 617 & -16 & 1 \\
577 & -434 & -367 & 234 & -10 \\
-158 & 2443 & -1923 & -397 & 35 \\
-2060 & -2239 & 4753 & -434 & -20 \\
3927 & -4832 & -1333 & 2443 & -205 \\
486 & 13585 & -12408 & -2239 & 577 \\
-11853 & -5003 & 21846 & -4832 & -158 \\
14988 & -28037 & 1513 & 13584 & -2060 \\
8635 & 46774 & -54332 & -5004 & 3927 \\
-44032 & 5271 & 66368 & -28027 & 486 \\
35635 & -102619 & 32053 & 46783 & -11852 \\
43315 & 106204 & -169962 & 5224 & 14989 \\
-114157 & 70990 & 137197 & -102655 & 8625 \\
51703 & -260950 & 147528 & 106353 & -44041 \\
130381 & 156476 & -393621 & 71083 & 35681 \\
-216338 & 250236 & 182839 & -261319 & 43350 \\
28407 & -484855 & 414452 & 156292 & -114296 \\
274974 & 114379 & -689801 & 250986 & 51619 \\
-303193 & 555289 & 101777 & -484577 & 130704 \\
-65130 & -665659 & 830585 & 113126 & -216190 \\
427192 & -92394 & -917444 & 554840 & 27806 \\
-305233 & 878875 & -180468 & -663554 & 274736 \\
-205261 & -652190 & 1251930 & -92336 & -302143 \\
494008 & -412593 & -898236 & 876735 & -65072 \\
-203589 & 1021418 & -591159 & -652754 & 426084 \\
-305582 & -409517 & 1435653 & -409517 & -305582 \\
426084 & -652754 & -591159 & 1021418 & -203589 \\
-65072 & 876735 & -898236 & -412593 & 494008 \\
-302143 & -92336 & 1251930 & -652190 & -205261 \\
274736 & -663554 & -180468 & 878875 & -305233 \\
27806 & 554840 & -917444 & -92394 & 427192 \\
-216190 & 113126 & 830585 & -665659 & -65130 \\
130704 & -484577 & 101777 & 555289 & -303193 \\
51619 & 250986 & -689801 & 114379 & 274974 \\
-114296 & 156292 & 414452 & -484855 & 28407 \\
43350 & -261319 & 182839 & 250236 & -216338 \\
35681 & 71083 & -393621 & 156476 & 130381 \\
-44041 & 106353 & 147528 & -260950 & 51703 \\
8625 & -102655 & 137197 & 70990 & -114157 \\
14989 & 5224 & -169962 & 106204 & 43315 \\
-11852 & 46783 & 32053 & -102619 & 35635 \\
486 & -28027 & 66368 & 5271 & -44032 \\
3927 & -5004 & -54332 & 46774 & 8635 \\
-2060 & 13584 & 1513 & -28037 & 14988 \\
-158 & -4832 & 21846 & -5003 & -11853 \\
577 & -2239 & -12408 & 13585 & 486 \\
-205 & 2443 & -1333 & -4832 & 3927 \\
-20 & -434 & 4753 & -2239 & -2060 \\
35 & -397 & -1923 & 2443 & -158 \\
-10 & 234 & -367 & -434 & 577 \\
1 & -16 & 617 & -397 & -205 \\
0 & -25 & -189 & 234 & -20 \\
0 & 9 & -28 & -16 & 35 \\
0 & -1 & 36 & -25 & -10 \\
0 & 0 & -10 & 9 & 1 \\
0 & 0 & 1 & -1 & 0
\end{pmatrix}$$

$$H^{W[3,2]}_{[3]} = A^{-6}q^{-14}\begin{pmatrix}
0 & -1 & 1 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 1 & 0 & 0 \\
0 & 0 & 3 & -3 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 2 & -1 & 0 \\
0 & 0 & 0 & -3 & 3 & 0 & 0 \\
0 & 0 & 1 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & -1 & 0
\end{pmatrix}$$

JHEP06(2021)063

$$H_{[3]}^{W[3,4]} = A^{-6}q^{-38} \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & -2 & -1 & 0 & 0 & 0 \\ -3 & 2 & -2 & 3 & 0 & 0 & 0 \\ 0 & -3 & 4 & -1 & 0 & 0 & 0 \\ 6 & -11 & 8 & -4 & 1 & 0 & 0 \\ 4 & 4 & -6 & 0 & -2 & 0 & 0 \\ -11 & 27 & -24 & 10 & -2 & 0 & 0 \\ -17 & 6 & 8 & 0 & 3 & 0 & 0 \\ 17 & -44 & 48 & -27 & 7 & -1 & 0 \\ 28 & -39 & 11 & 2 & -4 & 2 & 0 \\ -7 & 55 & -82 & 49 & -18 & 2 & 0 \\ -42 & 79 & -59 & 15 & 9 & -2 & 0 \\ -6 & -27 & 98 & -92 & 36 & -9 & 0 \\ 42 & -114 & 126 & -53 & 0 & 3 & 1 \\ 17 & -14 & -73 & 124 & -70 & 19 & -3 \\ -31 & 113 & -190 & 136 & -26 & -2 & 0 \\ -21 & 59 & -10 & -114 & 107 & -36 & 5 \\ 16 & -79 & 200 & -201 & 75 & -14 & 3 \\ 20 & -69 & 71 & 45 & -122 & 64 & -9 \\ -11 & 37 & -151 & 263 & -151 & 37 & -11 \\ -9 & 64 & -122 & 45 & 71 & -69 & 20 \\ 3 & -14 & 75 & -201 & 200 & -79 & 16 \\ 5 & -36 & 107 & -114 & -10 & 59 & -21 \\ 0 & -2 & -26 & 136 & -190 & 113 & -31 \\ -3 & 19 & -70 & 124 & -73 & -14 & 17 \\ 1 & 3 & 0 & -53 & 126 & -114 & 42 \\ 0 & -9 & 36 & -92 & 98 & -27 & -6 \\ 0 & -2 & 9 & 15 & -59 & 79 & -42 \\ 0 & 2 & -18 & 49 & -82 & 55 & -7 \\ 0 & 2 & -4 & 2 & 11 & -39 & 28 \\ 0 & -1 & 7 & -27 & 48 & -44 & 17 \\ 0 & 0 & 3 & 0 & 8 & 6 & -17 \\ 0 & 0 & -2 & 10 & -24 & 27 & -11 \\ 0 & 0 & -2 & 0 & -6 & 4 & 4 \\ 0 & 0 & 1 & -4 & 8 & -11 & 6 \\ 0 & 0 & 0 & -1 & 4 & -3 & 0 \\ 0 & 0 & 0 & 3 & -2 & 2 & -3 \\ 0 & 0 & 0 & -1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix},$$

$$
H_{[3]}^{W[3,5]} = A^{-6} q^{-50} \begin{pmatrix}
0 & 1 & -1 & 0 & 0 & 0 & 0 \\
-1 & -3 & 3 & 1 & 0 & 0 & 0 \\
4 & -1 & 1 & -4 & 0 & 0 & 0 \\
-2 & 6 & -7 & 3 & 0 & 0 & 0 \\
-8 & 9 & -6 & 6 & -1 & 0 & 0 \\
2 & -18 & 17 & -4 & 3 & 0 & 0 \\
21 & -27 & 23 & -18 & 1 & 0 & 0 \\
4 & 40 & -44 & 7 & -7 & 0 & 0 \\
-58 & 73 & -57 & 48 & -7 & 1 & 0 \\
-14 & -49 & 71 & -21 & 16 & -3 & 0 \\
98 & -184 & 146 & -85 & 26 & -1 & 0 \\
72 & 27 & -94 & 25 & -36 & 6 & 0 \\
-145 & 329 & -316 & 179 & -57 & 10 & 0 \\
-167 & 118 & 52 & -35 & 50 & -16 & -1 \\
150 & -481 & 563 & -333 & 126 & -29 & 4 \\
287 & -386 & 157 & -19 & -66 & 29 & -2 \\
-91 & 482 & -793 & 589 & -249 & 64 & -8 \\
-366 & 724 & -571 & 190 & 52 & -30 & 1 \\
-22 & -308 & 872 & -868 & 448 & -142 & 20 \\
386 & -962 & 1111 & -619 & 65 & 27 & 7 \\
116 & -53 & -598 & 1036 & -699 & 248 & -50 \\
-304 & 987 & -1532 & 1169 & -352 & 45 & -13 \\
-180 & 365 & 72 & -863 & 915 & -412 & 79 \\
207 & -763 & 1638 & -1749 & 812 & -197 & 52 \\
174 & -564 & 508 & 334 & -863 & 541 & -130 \\
-114 & 461 & -1314 & 1963 & -1314 & 461 & -114 \\
-130 & 541 & -863 & 334 & 508 & -564 & 174 \\
52 & -197 & 812 & -1749 & 1638 & -763 & 207 \\
79 & -412 & 915 & -863 & 72 & 365 & -180 \\
-13 & 45 & -352 & 1169 & -1532 & 987 & -304 \\
-50 & 248 & -699 & 1036 & -598 & -53 & 116 \\
7 & 27 & 65 & -619 & 1111 & -962 & 386 \\
20 & -142 & 448 & -868 & 872 & -308 & -22 \\
1 & -30 & 52 & 190 & -571 & 724 & -366 \\
-8 & 64 & -249 & 589 & -793 & 482 & -91 \\
-2 & 29 & -66 & -19 & 157 & -386 & 287 \\
4 & -29 & 126 & -333 & 563 & -481 & 150 \\
-1 & -16 & 50 & -35 & 52 & 118 & -167 \\
0 & 10 & -57 & 179 & -316 & 329 & -145 \\
0 & 6 & -36 & 25 & -94 & 27 & 72 \\
0 & -1 & 26 & -85 & 146 & -184 & 98 \\
0 & -3 & 16 & -21 & 71 & -49 & -14 \\
0 & 1 & -7 & 48 & -57 & 73 & -58 \\
0 & 0 & -7 & 7 & -44 & 40 & 4 \\
0 & 0 & 1 & -18 & 23 & -27 & 21 \\
0 & 0 & 3 & -4 & 17 & -18 & 2 \\
0 & 0 & -1 & 6 & -6 & 9 & -8 \\
0 & 0 & 0 & 3 & -7 & 6 & -2 \\
0 & 0 & 0 & -4 & 1 & -1 & 4 \\
0 & 0 & 0 & 1 & 3 & -3 & -1 \\
0 & 0 & 0 & 0 & -1 & 1 & 0
\end{pmatrix} .
$$

C Refined BPS integers

$$
\hat{\mathbf{N}}_{[2]}^{W[3,2]} = \left(\begin{array}{c|ccccccc}
k/m & -6 & -4 & -2 & 0 & 2 & 4 & 6 \\
\hline
0 & 1 & -3 & 4 & -6 & 9 & -7 & 2 \\
1 & 0 & -1 & 2 & -5 & 9 & -6 & 1 \\
2 & 0 & 0 & 0 & -1 & 2 & -1 & 0
\end{array} \right) \tag{C.1}
$$

$$
\hat{\mathbf{N}}_{[2]}^{W[3,5]} = \left(\begin{array}{c|ccccccc}
k/m & -6 & -4 & -2 & 0 & 2 & 4 & 6 \\
\hline
0 & 5 & -15 & 20 & -30 & 45 & -35 & 10 \\
1 & -20 & 85 & -170 & 195 & -135 & 60 & -15 \\
2 & -106 & 441 & -840 & 1125 & -1130 & 676 & -166 \\
3 & -115 & 485 & -865 & 1400 & -1860 & 1245 & -290 \\
4 & 58 & -304 & 893 & -895 & -17 & 319 & -54 \\
5 & 220 & -1112 & 2721 & -3910 & 3407 & -1728 & 402 \\
6 & 190 & -1053 & 2594 & -4360 & 4748 & -2667 & 548 \\
7 & 75 & -504 & 1287 & -2580 & 3268 & -1874 & 328 \\
8 & 14 & -132 & 354 & -901 & 1300 & -737 & 102 \\
9 & 1 & -18 & 51 & -186 & 302 & -166 & 16 \\
10 & 0 & -1 & 3 & -21 & 38 & -20 & 1 \\
11 & 0 & 0 & 0 & -1 & 2 & -1 & 0
\end{array} \right) \tag{C.2}
$$

JHEP06(2021)063

$$\hat{\mathbf{N}}^{W[3,10]}_{[2]} =$$

k/m	-6	-4	-2	0	2	4	6
0	15	−45	60	−90	135	−105	30
1	−60	705	−2310	3285	−2205	630	−45
2	−238	318	30	725	−2340	2223	−718
3	−230	−9535	40480	−59150	35020	−5455	−1130
4	−5276	18038	−10726	−8750	1419	7782	−2487
5	−14765	146944	−442897	622355	−446589	151571	−16619
6	79740	−198204	−1608	289370	−123746	−101606	56054
7	508927	−2324454	4616391	−5906820	5390964	−2958324	673316
8	1028581	−4926087	10581436	−15418579	15626039	−8863136	1971746
9	391793	−1556984	2518630	−6792357	12260745	−8784777	1962950
10	−2012429	10695435	−26645018	34658435	−24266624	9735414	−2165213
11	−3782274	19693607	−47993117	72809354	−69531190	36832395	−8028775
12	−1512571	8341895	−20569240	43380863	−58198567	35897456	−7339836
13	4189988	−21105464	50727994	−60798221	36574589	−13279099	3690213
14	8727473	−46175793	112341954	−166703357	154882064	−80507746	17435405
15	8982577	−49903892	122763600	−201133018	212030754	−116092856	23352835
16	6117458	−36148228	90092284	−161044360	185210485	−103620634	19392995
17	2978166	−19030475	48156926	−94240005	116596085	−65816480	11355783
18	1065351	−7510739	19336164	−41833861	55278128	−31237740	4902697
19	281585	−2245022	5890311	−14299136	20085108	−11298452	1585606
20	54437	−506753	1356948	−3772701	5613530	−3129606	384145
21	7490	−85090	232800	−762716	1198736	−660014	68794
22	695	−10309	28842	−116056	192190	−104205	8843
23	39	−852	2439	−12866	22400	−11932	772
24	1	−43	126	−981	1792	−936	41
25	0	−1	3	−46	88	−45	1
26	0	0	0	−1	2	−1	0

(C.3)

$$\hat{\mathbf{N}}^{W[3,2]}_{[1,1]} =$$

k/m	-6	-4	-2	0	2	4	6
0	−2	7	−9	6	−4	3	−1
1	−1	6	−9	5	−2	1	0
2	0	1	−2	1	0	0	0

(C.4)

$$\hat{\mathbf{N}}^{W[3,5]}_{[1,1]} =$$

k/m	-6	-4	-2	0	2	4	6
0	−10	35	−45	30	−20	15	−5
1	15	−60	135	−195	170	−85	20
2	166	−676	1130	−1125	840	−441	106
3	290	−1245	1860	−1400	865	−485	115
4	54	−319	17	895	−893	304	−58
5	−402	1728	−3407	3910	−2721	1112	−220
6	−548	2667	−4748	4360	−2594	1053	−190
7	−328	1874	−3268	2580	−1287	504	−75
8	−102	737	−1300	901	−354	132	−14
9	−16	166	−302	186	−51	18	−1
10	−1	20	−38	21	−3	1	0
11	0	1	−2	1	0	0	0

(C.5)

JHEP06(2021)063

$\hat{\mathbf{N}}^{W[3,10]}_{[1,1]} =$

k/m	-6	-4	-2	0	2	4	6
0	−30	105	−135	90	−60	45	−15
1	45	−630	2205	−3285	2310	−705	60
2	718	−2223	2340	−725	−30	−318	238
3	1130	5455	−35020	59150	−40480	9535	230
4	2487	−7782	−1419	8750	10726	−18038	5276
5	16619	−151571	446589	−622355	442897	−146944	14765
6	−56054	101606	123746	−289370	1608	198204	−79740
7	−673316	2958324	−5390964	5906820	−4616391	2324454	−508927
8	−1971746	8863136	−15626039	15418579	−10581436	4926087	−1028581
9	−1962950	8784777	−12260745	6792357	−2518630	1556984	−391793
10	2165213	−9735414	24266624	−34658435	26645018	−10695435	2012429
11	8028775	−36832395	69531190	−72809354	47993117	−19693607	3782274
12	7339836	−35897456	58198567	−43380863	20569240	−8341895	1512571
13	−3690213	13279099	−36574589	60798221	−50727994	21105464	−4189988
14	−17435405	80507746	−154882064	166703357	−112341954	46175793	−8727473
15	−23352835	116092856	−212030754	201133018	−122763600	49903892	−8982577
16	−19392995	103620634	−185210485	161044360	−90092284	36148228	−6117458
17	−11355783	65816480	−116596085	94240005	−48156926	19030475	−2978166
18	−4902697	31237740	−55278128	41833861	−19336164	7510739	−1065351
19	−1585606	11298452	−20085108	14299136	−5890311	2245022	−281585
20	−384145	3129606	−5613530	3772701	−1356948	506753	−54437
21	−68794	660014	−1198736	762716	−232800	85090	−7490
22	−8843	104205	−192190	116056	−28842	10309	−695
23	−772	11932	−22400	12866	−2439	852	−39
24	−41	936	−1792	981	−126	43	−1
25	−1	45	−88	46	−3	1	0
26	0	1	−2	1	0	0	0

(C.6)

References

[1] P. Freyd, D. Yetter, J. Hoste, W.R. Lickorish, K. Millett and A. Ocneanu, *A new polynomial invariant of knots and links*, *Bull. Am. Astron. Soc.* **12** (1985) 239 [INSPIRE].

[2] B. Ponsot and J. Teschner, *Clebsch-Gordan and Racah-Wigner coefficients for a continuous series of representations of* $U_q(\mathrm{SL}(2,\mathbb{R}))$, *Commun. Math. Phys.* **224** (2001) 613 [math/0007097] [INSPIRE].

[3] P. de la Harpe, M. Kervaire and C. Weber, *On the Jones polynomial*, *Enseign. Math.* **32** (1986) 271.

[4] C. Manolescu, *An introduction to knot Floer homology*, in *Physics and Mathematics of Link Homology*, *Contemporary Mathematics* **680**, American Mathematical Society, Providence RI U.S.A. (2016), pp. 99–135.

[5] D. Bar-Natan, *On Khovanov's categorification of the Jones polynomial*, *Algebr. Geom. Topol.* **2** (2002) 337 [math.QA/0201043].

[6] W.P. Thurston, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, *Bull. Am. Astron. Soc.* **6** (1982) 357 [INSPIRE].

[7] R. Mishra and R. Staffeldt, *The Jones polynomial and Khovanov homology of weaving knots* $W(3,n)$, arXiv:1704.03982.

[8] R. Mishra, V.K. Singh and R. Staffeldt, *Asymptotics in the invariants of weaving knots* $W(3,n)$, in preparation.

[9] H. Ooguri and C. Vafa, *Knot invariants and topological strings*, *Nucl. Phys. B* **577** (2000) 419 [hep-th/9912123] [INSPIRE].

[10] R. Gopakumar and C. Vafa, *M theory and topological strings. 2*, hep-th/9812127 [INSPIRE].

[11] J.M.F. Labastida, M. Mariño and C. Vafa, *Knots, links and branes at large* N, *JHEP* **11** (2000) 007 [hep-th/0010102] [INSPIRE].

[12] A.N. Kirillov and N.Y. Reshetikhin, *Representations of the algebra* $U_q(sl(2))$*, q-orthogonal polynomials and invariants of links*, in *New Developments in the Theory of Knots*, *Advanced Series in Mathematical Physics*, World Scientific (1990), pp. 202–256.

[13] M. Rosso and V. Jones, *On the invariants of torus knots derived from quantum groups*, *J. Knot Theor. Ramifications* **2** (1993) 97.

[14] X.-S. Lin and H. Zheng, *On the Hecke algebras and the colored HOMFLY polynomial*, *Trans. Am. Math. Soc.* **362** (2010) 1 [math.QA/0601267].

[15] K. Liu and P. Peng, *Proof of the Labastida-Mariño-Ooguri-Vafa conjecture*, *J. Diff. Geom.* **85** (2010) 479 [arXiv:0704.1526] [INSPIRE].

[16] E. Guadagnini, M. Martellini and M. Mintchev, *Chern-Simons Holonomies and the Appearance of Quantum Groups*, *Phys. Lett. B* **235** (1990) 275 [INSPIRE].

[17] N.Y. Reshetikhin and V.G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, *Commun. Math. Phys.* **127** (1990) 1 [INSPIRE].

[18] A. Klimyk and K. Schmüdgen, *Quantum groups and their representations*, in *Texts and Monographs in Physics*, Springer (1997).

[19] A. Mironov, A. Morozov and A. Morozov, *Character expansion for HOMFLY polynomials. II. Fundamental representation. Up to five strands in braid*, *JHEP* **03** (2012) 034 [arXiv:1112.2654] [INSPIRE].

[20] A. Mironov, A. Morozov and A. Morozov, *Character expansion for HOMFLY polynomials. I. Integrability and difference equations*, in *Strings, gauge fields, and the geometry behind: The legacy of Maximilian Kreuzer*, A. Rebhan, L. Katzarkov, J. Knapp, R. Rashkov and E. Scheidegger eds., World Scientific (2011), pp. 101–118 [arXiv:1112.5754] [INSPIRE].

[21] A. Anokhina, A. Mironov, A. Morozov and A. Morozov, *Racah coefficients and extended HOMFLY polynomials for all* 5-, 6- *and* 7-*strand braids*, *Nucl. Phys. B* **868** (2013) 271 [arXiv:1207.0279] [INSPIRE].

[22] M.D. Gould and Y.-Z. Zhang, *Quantum affine Lie algebras, Casimir invariants and diagonalization of Braid generators*, *J. Math. Phys.* **35** (1994) 6757 [hep-th/9311041] [INSPIRE].

[23] A. Mironov, A. Morozov and S. Natanzon, *Complete Set of Cut-and-Join Operators in Hurwitz-Kontsevich Theory*, *Theor. Math. Phys.* **166** (2011) 1 [arXiv:0904.4227] [INSPIRE].

JHEP06(2021)063

[24] A. Mironov, A. Morozov and S. Natanzon, *Algebra of differential operators associated with Young diagrams*, *J. Geom. Phys.* **62** (2012) 148 [arXiv:1012.0433] [INSPIRE].

[25] A. Mironov, A. Morozov and A. Morozov, *Character expansion for HOMFLY polynomials. II. Fundamental representation. Up to five strands in braid*, *JHEP* **03** (2012) 034 [arXiv:1112.2654] [INSPIRE].

[26] S. Dhara et al., *Multi-Colored Links From* 3*-strand Braids Carrying Arbitrary Symmetric Representations*, *Ann. Henri Poincaré* **20** (2019) 4033 [arXiv:1805.03916] [INSPIRE].

[27] H. Itoyama, A. Mironov, A. Morozov and A. Morozov, *Eigenvalue hypothesis for Racah matrices and HOMFLY polynomials for* 3*-strand knots in any symmetric and antisymmetric representations*, *Int. J. Mod. Phys. A* **28** (2013) 1340009 [arXiv:1209.6304] [INSPIRE].

[28] S. Dhara et al., *Eigenvalue hypothesis for multistrand braids*, *Phys. Rev. D* **97** (2018) 126015 [arXiv:1711.10952] [INSPIRE].

[29] D. Bar-Natan, S. Morrison et al., *The knot ATLAS*, (2015) http://katlas.org.

[30] C. Livingston and J.C. Cha, *KnotInfo: Table of knot invariants*, (2020) https://knotinfo.math.indiana.edu/.

[31] L. Peliti, *Fibonacci and Lucas numbers*, (2020).

[32] R. Gopakumar and C. Vafa, *M theory and topological strings. 1*, hep-th/9809187 [INSPIRE].

[33] J.M.F. Labastida and M. Mariño, *Polynomial invariants for torus knots and topological strings*, *Commun. Math. Phys.* **217** (2001) 423 [hep-th/0004196] [INSPIRE].

[34] J.M.F. Labastida and M. Mariño, *A New point of view in the theory of knot and link invariants*, math.QA/0104180 [INSPIRE].

[35] A. Mironov, A. Morozov, A. Morozov, P. Ramadevi, V.K. Singh and A. Sleptsov, *Checks of integrality properties in topological strings*, *JHEP* **08** (2017) 139 [*Addendum JHEP* **01** (2018) 143] [arXiv:1702.06316] [INSPIRE].

[36] W. Fulton, *Young tableaux: with applications to representation theory and geometry*, in *London Mathematical Society Student Texts* **35**, Cambridge University Press, Cambridge U.K. (1996).

JHEP06(2021)063