Fragmentation function of $g \to Q\bar{Q}(^3S_1^{[8]})$ in soft gluon factorization and threshold resummation

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ABSTRACT: We study the fragmentation function of the gluon to color-octet $^3S_1$ heavy quark-antiquark pair using the soft gluon factorization (SGF) approach, which expresses the fragmentation function in a form of perturbative short-distance hard part convoluted with one-dimensional color-octet $^3S_1$ soft gluon distribution (SGD). The short distance hard part is calculated to the next-to-leading order in α_s and all orders in velocity expansion. By deriving and solving the renormalization group equation of the SGD, threshold logarithms are resummed to all orders in perturbation theory. The comparison with gluon fragmentation function calculated in NRQCD factorization approach indicates that the SGF formula resums a series of velocity corrections in NRQCD which are important for phenomenological study.

Contents

1 Introduction

Heavy quarkonium physics has been on the focus of much experimental and theoretical attention since the discovery of the J/ψ in 1974. As the simplest bound state of strong interactions, heavy quarkonium is an ideal system to study both perturbative and non-perturbative aspects of QCD. Our current understanding of the decay and production of quarkonium is mainly based on the non-relativistic QCD (NRQCD) factorization [1], which factorizes physical quantities into summation of perturbatively calculable short-distance coefficients (SDCs) multiplied by nonperturbative long-distance matrix elements (LDMEs).

However, recent studies shown that NRQCD factorization encounters some difficulties in describing inclusive quarkonium production data [2–25]. In ref. [26], it was argued that these difficulties may come from the bad convergence of velocity expansion in NRQCD.

The velocity expansion suffers from large high order relativistic corrections due to ignoring the effects of soft hadrons emitted in the hadronization process. For this reason, the authors proposed a new factorization approach, called soft gluon factorization (SGF), to describe quarkonium production and decay [26]. It was argued in ref. [27] that the SGF is equivalent to the NRQCD factorization, but with a series of important relativistic corrections originated from kinematic effects resummed. As a result, the SGF approach should has a much better convergence in the velocity expansion. Indeed, it was found in ref. [26] that the lowest order result in velocity expansion in NRQCD factorization can deviate from the full SGF result by a factor of 4, but the lowest order velocity expansion in SGF can deviate from its full result by only a small value.

In the SGF approach, the quarkonium production cross section can be expressed in the following factorization formula

$$(2\pi)^3 2P_H^0 \frac{d\sigma_H}{d^3 P_H} = \sum_{n,n'} \int \frac{d^4 P}{(2\pi)^4} d\hat{\sigma}_{[nn']}(P) F_{[nn']\to H}(P, P_H), \tag{1.1}$$

where $d\hat{\sigma}_{[nn']}(P)$ are perturbatively calculable short distance hard parts that produce a $Q\bar{Q}$ pair with quantum numbers $n = {}^{2S+1}L_{J,J_z}^{[c]}$ and $n' = {}^{2S'+1}L'^{[c']}_{J',J'_z}$ in the amplitude and the complex-conjugate of the amplitude, respectively, with $c, c' = 1$ or 8 representing the color-singlet or color-octet state of the $Q\bar{Q}$ pair. In general, n can be different from n'. In the case of producing a polarization-summed quarkonium H, there are constraints $c = c', S = S', J = J', J_z = J'_z$ and $|L - L'| = 0, 2, 4, \cdots$ [26, 28]. In eq. (1.1), P is the total momentum of the intermediate $Q\bar{Q}$ pair, which is different from the momentum of the physical quarkonium P_H. The soft gluon distributions (SGDs), $F_{[nn']\to H}(P, P_H)$, are nonperturbative functions that describe the hadronization of the intermediate $Q\bar{Q}(n, n')$ states to heavy quarkonium H. The factorization formula eq. (1.1) is called 4-dimensional SGF (SGF-4d) in ref. [26], and it can be simplified to the so called SGF-1d and SGF-0d formula by some further expansions. Using these SGF formulas, the authors studied the J/ψ hadroproduction via gluon fragmenting into polarization-summed ${}^3S_1^{[8]}$ intermediate state at tree level, and they found that the SGF-1d formula is a very good approximation of full SGF-4d formula. The SGF approach has also been applied to exclusive quarkonium processes [27, 29], in which the SGDs are reduced to local matrix elements.

In this paper we study quarkonium inclusive production using gluon fragmentation function (FF) $D_{g\to H}(z, \mu)$, where z is the longitudinal momentum fraction of the quarkonium state and μ is the collinear factorization scale. To be concrete, we concentrate on the ${}^3S_1^{[8]}$ state production. In NRQCD framework, the SDC of gluon fragmenting to ${}^3S_1^{[8]}$ has been calculated up to next-to-leading order (NLO) in perturbative expansion [30–32] (but at the lowest order in velocity expansion). It was found that SDC in NRQCD suffers from large (threshold) logarithms of the form

$$\alpha_s^i \left[\frac{\ln^j(1-z)}{1-z}\right]_+, \quad j \le 2i, \tag{1.2}$$

which spoils perturbative expansion in the region $z \to 1$. Similar issues arise when analyzing the J/ψ photoproduction and electroproduction, where the large logarithms were

resummed to all orders in α_s [33–36] by combining the NRQCD with soft collinear effective theory (SCET) [37–40]. Although the same technique can also be used to resum the large logarithms appeared in FF, we choose to apply SGF instead of NRQCD+SCET to deal with this problem. The reason is that, as we will show, the SGF can not only resum the large logarithms but also resum a series of important relativistic corrections, which results in better convergence for relativistic expansion.

The rest of the paper is organized as following. In section 2, we review the SGF formula for quarkonium production, including the definition of SGDs, the perturbative matching procedure and the velocity expansion of short distance hard parts. In section 3, we present the perturbative calculation of the SGDs. In section 4, we discuss the renormalization group equation (RGE) for the $^3S_1^{[8]}$ SGD. In section 5, we study the gluon FF in SGF and calculate the short distance hard part in $^3S_1^{[8]}$ channel up to NLO. We conclude in section 6. In appendix A we provide some details of solving the RGE of SGDs. In appendix B, we provide some integral formulas used in our calculation. Finally, we list the expressions of finite functions that enter the gluon FF in appendix C.

2 Soft gluon factorization

Before studying the gluon fragmentation function, we first briefly review the SGF formula for quarkonium production cross section. We denote M_H and m_Q as the mass of the heavy quarkonium H and the mass of the heavy quark Q, respectively. Lorentz-vector a, denoting as

$$a^\mu = (a^0, a^1, a^2, a^3) = (a^0, \boldsymbol{a}),$$

is sometimes also expressed in light-cone coordinates,

$$a^\mu = (a^+, a^-, a^1, a^2) = (a^+, a^-, a_\perp),$$
$$a^+ = (a^0 + a^3)/\sqrt{2},$$
$$a^- = (a^0 - a^3)/\sqrt{2}.$$

The scalar product of two four-vector a and b then becomes

$$a \cdot b = a^+ b^- + a^- b^+ + a_\perp \cdot b_\perp.$$

We introduce a light-like vector $l^\mu = (0, 1, 0_\perp)$ so that $a \cdot l = a^+$.

Following the discussion in [26], the factorization formula eq. (1.1) for producing a quarkonium H with momentum P_H can be simplified to the following SGF-1d formula

$$(2\pi)^3 2P_H^0 \frac{d\sigma_H}{d^3 P_H} = \sum_{n,n'} \int \frac{dz}{z^2} d\hat{\sigma}_{[nn']}(P_H/z, m_Q, \mu_f) F_{[nn'] \to H}(z, M_H, m_Q, \mu_f), \qquad (2.1)$$

which has similar good convergence in velocity expansion for our purpose, and at the same time is more suitable in practical use. In the above formula, $d\hat{\sigma}_{[nn']}(P_H/z, m_Q, \mu_f)$ are the short-distance hard parts which, roughly speaking, produce a $Q\bar{Q}$ pair with momentum P_H/z and quantum number n in the amplitude and n' in the complex conjugate

of the amplitude, $F_{[nn']\to H}(z, M_H, m_Q, \mu_f)$ are the one-dimensional SGDs which describe the hadronization from the $Q\bar{Q}$ pair to quarkonium H, μ_f is the factorization scale and $z = P_H^+/P^+$ is the longitudinal momentum fraction with P denoting the total momentum of the intermediate $Q\bar{Q}$ pair.

SGDs are defined as

$$F_{[nn']\to H}(z, M_H, m_Q, \mu_f) = P_H^+ \int \frac{db^-}{2\pi} e^{-iP_H^+ b^-/z} \langle 0|[\bar{\Psi}\mathcal{K}_n\Psi]^\dagger(0)[a_H^\dagger a_H][\bar{\Psi}\mathcal{K}_{n'}\Psi](b^-)|0\rangle_{\text{S}},$$
(2.2)

where Ψ stands for Dirac field of heavy quark and the subscript "S" means the field operators in the above definition are the operators obtained in small momentum region. In additional, in this paper we define "S" to select only leading power terms in threshold $(P - P_H)^+ = (1-z)P^+$ expansion, which is sufficient to factorize and then resum leading-power large logarithms as $z \to 1$. In eq. (2.2), the quarkonium state, created by $a_H^\dagger$, has standard relativistic normalization and the projection operators $\mathcal{K}_n$ can be decomposed to a spin operator, a color operator, and a gauge link [26]. For the case $n = {}^3S_{1,\lambda}^{[8]}, {}^3D_{1,\lambda}^{[8]}$ which will be studied in this work, we have

$$\mathcal{K}_{{}^3S_{1,\lambda}^{[8]}}(rb^-) = \frac{\sqrt{M_H}}{M_H + 2m} \frac{M_H + \slashed{P}_H}{2M_H} \mathcal{C}_a^{[8]} \epsilon_\lambda^\mu \gamma_\mu \frac{M_H - \slashed{P}_H}{2M_H},$$
(2.3a)

$$\mathcal{K}_{{}^3D_{1,\lambda}^{[8]}}(rb^-) = \frac{\sqrt{M_H}}{M_H + 2m} \frac{M_H + \slashed{P}_H}{2M_H} \mathcal{C}_a^{[8]} \epsilon_\lambda^\mu \gamma^\nu \left(-\frac{i}{2}\right)^2 \overleftrightarrow{D}^\alpha \overleftrightarrow{D}^\beta \frac{M_H - \slashed{P}_H}{2M_H}$$

$$\times \sqrt{\frac{2(d-1)}{(d-2)(d+1)}} \left(-\mathbb{P}_{\alpha\mu}\mathbb{P}_{\beta\nu} + \frac{1}{d-1}\mathbb{P}_{\alpha\beta}\mathbb{P}_{\mu\nu}\right),$$
(2.3b)

where D_μ is the gauge covariant derivative with $\overline{\Psi}\overleftrightarrow{D}_\mu\Psi = \overline{\Psi}(D_\mu\Psi) - (D_\mu\overline{\Psi})\Psi$ and ϵ_λ are polarization vectors which satisfy the following relations

$$P_H \cdot \epsilon_\lambda = 0, \qquad \epsilon_\lambda \cdot \epsilon_{\lambda'}^* = -\delta_{\lambda\lambda'}, \qquad \sum_\lambda \epsilon_\lambda^\mu \epsilon_\lambda^{*\nu} = \mathbb{P}^{\mu\nu}.$$
(2.4)

The spin projection operator $\mathbb{P}_{\alpha\beta}$ is defined as

$$\mathbb{P}_{\alpha\beta} = -g_{\alpha\beta} + \frac{P_{H\alpha}P_{H\beta}}{M_H^2}.$$
(2.5)

The color operator is defined as

$$\mathcal{C}_a^{[8]} = \sqrt{2}T^{\bar{a}}\Phi_l(rb^-)_{\bar{a}a}.$$
(2.6)

The gauge link $\Phi_l(rb^-)_{\bar{a}a}$ is introduced to enable gauge invariance of SGD, which is defined along the l^μ direction,

$$\Phi_l(rb^-) = \mathcal{P}\exp\left[-ig_s \int_0^\infty \text{d}\xi l \cdot A(rb^- + \xi l)\right],$$
(2.7)

where $\mathcal{P}$ denotes path ordering and $A^\mu(x)$ is the gluon field in the adjoint representation: $[A^\mu(x)]_{ac} = if^{abc}A_b^\mu(x)$.

As the short distance hard parts $d\hat{\sigma}_{nn'}(P_H/z, m_Q, \mu_f)$ do not depend on nonperturbative physics, they can be perturbatively calculated. To this end, we replace the quarkonium H in eq. (2.1) by an on-shell $Q\bar{Q}$ pair with certain quantum number m in the amplitude and m' in the complex-conjugate amplitude, which results in

$$d\bar{\sigma}_{Q\bar{Q}[mm']} = \sum_{n,n'} \int \frac{dz}{z^2} d\hat{\sigma}_{[nn']}(P_H/z, m_Q, \mu_f) F_{[nn'] \to Q\bar{Q}[mm']}(z, M_H, m_Q, \mu_f), \qquad (2.8)$$

where $d\bar{\sigma}_{Q\bar{Q}[mm']} \equiv (2\pi)^3 2P_H^0 d\sigma_{Q\bar{Q}[mm']}/d^3P_H$. We can determine the hard parts by computing both sides of eq. (2.8) in perturbation theory. If the factorization eq. (2.1) holds, the obtained hard parts will be free of infrared divergences. Momenta of the on-shell $Q\bar{Q}$ pair are chosen as

$$p_Q = \frac{1}{2}P_H + q, \qquad p_{\bar{Q}} = \frac{1}{2}P_H - q, \qquad (2.9)$$

where q is half of the relative momentum of the $Q\bar{Q}$ pair. On-shell conditions $p_Q^2 = p_{\bar{Q}}^2 = m_Q^2$ result in

$$P_H \cdot q = 0, \qquad q^2 = m_Q^2 - P_H^2/4. \qquad (2.10)$$

To project the final state $Q\bar{Q}$ pair to quantum number $m(^{2S+1}L_{J,J_z}^{[c]})$, we replace the spinors of $Q\bar{Q}$ by the following projector

$$\Pi[m] = \frac{2}{\sqrt{M_H}(M_H + 2m_Q)}(\not{p}_{\bar{Q}} - m_Q)\frac{M_H - \not{P}_H}{2M_H}\widetilde{\Gamma}_m \frac{M_H + \not{P}_H}{2M_H}(\not{p}_Q + m_Q). \qquad (2.11)$$

For $m = {}^3S_{1,\lambda}^{[8]}$ or ${}^3D_{1,\lambda}^{[8]}$, we have

$$\widetilde{\Gamma}_m = \widetilde{\Gamma}_m^s \widetilde{\mathcal{C}}^{[8],a}. \qquad (2.12)$$

The color operators and spin operator have similar definitions as those in eq. (2.3), which are given by

$$\widetilde{\mathcal{C}}^{[8],a} = \sqrt{\frac{2}{N_c^2 - 1}} T^a, \qquad (2.13a)$$

$$\widetilde{\Gamma}_{{}^3S_{1,\lambda}^{[8]}}^s = \epsilon_\lambda^{*\mu} \gamma_\mu, \qquad (2.13b)$$

$$\widetilde{\Gamma}_{{}^3D_{1,\lambda}^{[8]}}^s = \frac{(d-1)(d+1)}{2} \frac{q^\mu q^\nu}{|\mathbf{q}|^4} \epsilon_\lambda^{*\beta} \gamma^\alpha \sqrt{\frac{d-1}{2(d-2)(d+1)}}$$
$$\times \left(-\mathbb{P}_{\alpha\mu}\mathbb{P}_{\beta\nu} - \mathbb{P}_{\alpha\nu}\mathbb{P}_{\beta\mu} + \frac{2}{d-1}\mathbb{P}_{\alpha\beta}\mathbb{P}_{\mu\nu} \right), \qquad (2.13c)$$

where the factor $\sqrt{N_c^2 - 1}$ is to average over color-octet states.

According to the definition in eq. (2.2), with H replaced by $Q\bar{Q}[mm']$, and the projector in eq. (2.11), it can be found that, at the lowest order in α_s, the free $Q\bar{Q}$ SGDs have the orthogonality relations [26]

$$F_{[nn'] \to Q\bar{Q}[mm']}^{\text{LO}}(z, M_H, m_Q, \mu_f) = \delta_{nm}\delta_{n'm'}\delta(1-z). \qquad (2.14)$$

By inserting perturbative expansions

$$F_{[nn'] \to Q\bar{Q}[mm']} = F^{\text{LO}}_{[nn'] \to Q\bar{Q}[mm']} + F^{\text{NLO}}_{[nn'] \to Q\bar{Q}[mm']} + \cdots,$$

$$d\bar{\sigma}_{Q\bar{Q}[mm']} = d\bar{\sigma}^{\text{LO}}_{Q\bar{Q}[mm']} + d\bar{\sigma}^{\text{NLO}}_{Q\bar{Q}[mm']} + \cdots,$$

$$d\hat{\sigma}_{[nn']} = d\hat{\sigma}^{\text{LO}}_{[nn']} + d\hat{\sigma}^{\text{NLO}}_{[nn']} + \cdots, \tag{2.15}$$

into eq. (2.8) and using the orthogonal relations eq. (2.14), one obtains relations

$$d\hat{\sigma}^{\text{LO}}_{[nn']}(P_H/z, m_Q, \mu_f) = d\bar{\sigma}^{\text{LO}}_{Q\bar{Q}[nn']}(P_H/z, m_Q),$$

$$d\hat{\sigma}^{\text{NLO}}_{[nn']}(P_H/z, m_Q, \mu_f) = d\bar{\sigma}^{\text{NLO}}_{Q\bar{Q}[nn']}(P_H/z, m_Q)$$

$$- \sum_{m,m'} \int \frac{dx}{x^2} d\hat{\sigma}^{\text{LO}}_{[mm']}(P_H/(xz), m_Q, \mu_f) F^{\text{NLO}}_{[mm'] \to Q\bar{Q}[nn']}(x, M_H/z, m_Q, \mu_f), \quad (2.16)$$

and so on, which express short-distance hard parts in terms of perturbative calculated $d\bar{\sigma}_{Q\bar{Q}[nn']}$ and $F_{[nn'] \to Q\bar{Q}[mm']}$.

Finally, we note that perturbative calculation with analytical q dependence are not very easy for complicated processes. There are at least two independent hard scales, m_Q and M_H/z in the hard parts. As suggested in [26], we can further simplify the hard parts by expanding m_Q^2 around $M_H^2/(4z^2)$, i.e.

$$d\hat{\sigma}_{[nn']}(P_H/z, m_Q, \mu_f) = d\hat{\sigma}_{[nn']}(P_H/z, M_H/(2z), \mu_f)$$

$$+ \frac{\partial[d\hat{\sigma}_{[nn']}(P_H/z, m_Q, \mu_f)]}{\partial m_Q^2} \bigg|_{m_Q^2 = M_H^2/(4z^2)} \left(m_Q^2 - \frac{M_H^2}{4z^2} \right) + \cdots$$

$$= \sum_{i=0} d\hat{\sigma}^{(i)}_{[nn']}(P_H/z, \mu_f) \tilde{q}^{2i}, \tag{2.17}$$

where $\tilde{q}^2 = m_Q^2 - M_H^2/(4z^2) \equiv m_Q^2 v^2$ with v being the velocity of the heavy quark in the rest frame of the pair. Then the SGF formula eq. (2.1) can be rewritten as

$$d\bar{\sigma}_H = \sum_{nn'} \int \frac{dz}{z^2} \sum_{i=0} d\hat{\sigma}^{(i)}_{[nn']}(P_H/z, \mu_f) \tilde{q}^{2i} F_{[nn'] \to H}(z, M_H, m_Q, \mu_f), \tag{2.18}$$

which defines a velocity expansion series in SGF. The tree level calculation in ref. [26] shows that the convergence of velocity expansion in SGF is much better than that in NRQCD. We will confirm this conclusion at one-loop level in this paper.

3 Perturbative calculation of soft gluon distributions

3.1 Definition

For the purpose of this paper, we are interested in the intermediate state n or n' equals to state $^3S^{[8]}_{1,\lambda}$ or $^3D^{[8]}_{1,\lambda}$ (denoted as S and D respectively). For convenient, we denote

$$[L\tilde{L}, \lambda] \equiv [^3L^{[8]}_{1,\lambda} {}^3\tilde{L}^{[8]}_{1,\lambda}], \tag{3.1a}$$

$$[L\tilde{L}] \equiv [^3L^{[8]}_1 {}^3\tilde{L}^{[8]}_1]. \tag{3.1b}$$

The notations in the second line denote polarization-summed intermediate states. In general, the SGD $F_{[L\tilde{L},\lambda]\to H}$ is λ dependent even for polarization-summed H. This is different to the polarized NRQCD LDMEs, which can be simplified to the usual unpolarized NRQCD LDMEs due to the rotation invariance [28, 41]. But in the case of SGD, the z-axis direction needs to be specified because the longitudinal momentum of intermediate $Q\bar{Q}$ pair is fixed, thus the rotation invariance is broken. Similar to the polarized NRQCD LDMEs defined in [28], it is convenient to construct the definitions of polarized SGDs as

$$F_{[L\tilde{L},T]\to H}(z, M_H, m_Q, \mu_f) = \frac{1}{d-2}\sum_{|\lambda|=1} F_{[L\tilde{L},\lambda]\to H}(z, M_H, m_Q, \mu_f), \tag{3.2a}$$

$$F_{[L\tilde{L},L]\to H}(z, M_H, m_Q, \mu_f) = \sum_{|\lambda|=0} F_{[L\tilde{L},\lambda]\to H}(z, M_H, m_Q, \mu_f). \tag{3.2b}$$

But for the polarization-summed S-wave quarkonium H, such as J/ψ, the polarized ${}^3S_1^{[8]}$ SGDs can be reduced to following unpolarized SGD:

$$F_{[SS]\to H}(z, M_H, m_Q, \mu_f) = \frac{1}{d-1}\sum_{\lambda} F_{[SS,\lambda]\to H}(z, M_H, m_Q, \mu_f). \tag{3.3}$$

And we have

$$\begin{aligned} F_{[SS,T]\to H}(z, M_H, m_Q, \mu_f) &= F_{[SS,L]\to H}(z, M_H, m_Q, \mu_f) \\ &= F_{[SS]\to H}(z, M_H, m_Q, \mu_f). \end{aligned} \tag{3.4}$$

We will explain this later.

After replacing H with $Q\bar{Q}[L\tilde{L}, \lambda]$, the general form of $F_{[L'\tilde{L}',\lambda']\to Q\bar{Q}[L\tilde{L},\lambda]}$ can be written as

$$\begin{aligned} F_{[L'\tilde{L}',\lambda']\to Q\bar{Q}[L\tilde{L},\lambda]}(z, M_H, m_Q, \mu_f) = T_S\Big[& P_H^+ \int d\Phi\, \mathcal{A}_{L'L}(P_H, P, m_Q, k_i, \lambda', \lambda) \\ & \times \mathcal{A}_{\tilde{L}'\tilde{L}}^*(P_H, P, m_Q, k_i, \lambda', \lambda) \Big], \end{aligned} \tag{3.5}$$

where $d\Phi$ is the final-state phase space, k_i denotes momenta of the final-state gluons or light (anti-)quarks, and $\mathcal{A}_{L'L}(P_H, P, m_Q, k_i, \lambda', \lambda)$ denotes the amplitude of transition from a $Q\bar{Q}$ pair in ${}^3L'^{[8]}_{1,\lambda'}$ state into a $Q\bar{Q}$ pair in ${}^3L^{[8]}_{1,\lambda}$ state which can be written as

$$\mathcal{A}_{L'L}(P_H, P, m_Q, k_i, \lambda', \lambda) = \int \frac{d^{d-2}\Omega}{N_\Omega}\mathrm{Tr}\Big[\mathcal{A}_{L'}^{Q\bar{Q}}\Pi[{}^3L^{[8]}_{1,\lambda}]\Big], \tag{3.6}$$

where $\mathcal{A}_{L'}^{Q\bar{Q}}$ is the amplitude to produce an open $Q\bar{Q}$ pair with spinors of the pair removed, Ω is the solid angle of relative momentum $\boldsymbol{q}$ in the $Q\bar{Q}$ rest frame, and N_Ω is given by

$$N_\Omega = \int d^{d-2}\Omega. \tag{3.7}$$

As pointed out in the last section, in addition to force loop momenta and light-parton momenta k_i in small-momentum region, the operator T_S also selects only the leading power contributions in the threshold expansion.
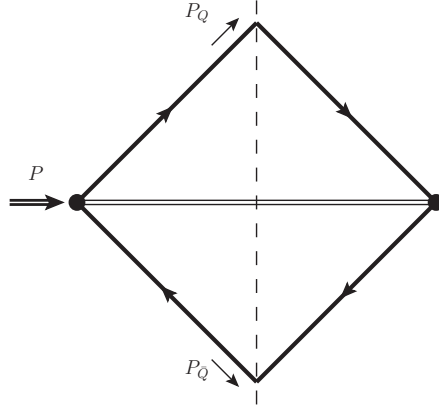
Figure 1. Feynman diagrams for the free $Q\bar{Q}$ SGD at LO. The double solid line represents the gauge link along l direction.

Based on eq. (3.5), we find that, for polarization-summed final state, SGDs can be decomposed as

$$\sum_\lambda F_{[L'\tilde{L}',\lambda']\to Q\bar{Q}[L\tilde{L},\lambda]}(z, M_H, m_Q, \mu_f) = \Big(F_1 \mathbb{P}_{\mu\nu} + F_2 l_\mu l_\nu + F_3(P_{H\mu}l_\nu + P_{H\nu}l_\mu)$$
$$+ F_4 P_{H\mu}P_{H\nu}\Big)\epsilon^\mu_{\lambda'}\epsilon^\nu_{\lambda'}. \qquad (3.8)$$

For general SGDs, F_2 and F_3 are nonzero and the existence of l^μ breaks the rotation invariance. But for the S-wave case, F_2 and F_3 are zero thanks to keeping only LP term in threshold expansion, which results in

$$F_{[SS,T]\to Q\bar{Q}[SS]}(z, M_H, m_Q, \mu_f) = F_{[SS,L]\to Q\bar{Q}[SS]}(z, M_H, m_Q, \mu_f)$$
$$= F_{[SS]\to Q\bar{Q}[SS]}(z, M_H, m_Q, \mu_f). \qquad (3.9)$$

Thus we argue that for the polarization-summed S-wave quarkonium H, we have the relations in eq. (3.4).

3.2 LO calculation

The LO Feynman diagram for $F_{[SS]\to Q\bar{Q}[SS]}$ is shown in figure 1, where the solid circles represent the operator $\overline{\Psi}\mathcal{K}_{{}_{3}S^{[8]}_{1,\lambda}}\Psi$. Based on eqs. (3.5), (2.3) and (2.11) we obtain

$$F^{\rm LO}_{[SS]\to Q\bar{Q}[SS]}(x, M_H, m_Q, \mu_f) = \frac{1}{d-1}\sum_{\lambda',\lambda} F^{\rm LO}_{[SS,\lambda']\to Q\bar{Q}[SS,\lambda]}(x, M_H, m_Q, \mu_f)$$

$$= \frac{P_H^+}{d-1}\int \frac{d^dP}{(2\pi)^d}\delta\left(P^+ - \frac{P_H^+}{x}\right)(2\pi)^d\delta^d(P - P_H)\mathcal{M}_S^{\alpha\beta}\mathcal{M}_S^{*\rho\sigma}\mathbb{P}_{\alpha\rho}\mathbb{P}_{\beta\sigma}, \qquad (3.10)$$

with

$$\mathcal{M}_S^{\alpha\beta} = \int \frac{d^{d-2}\Omega}{N_\Omega}{\rm Tr}[\mathcal{C}_8'^b\Pi_3'^\alpha\widetilde{\mathcal{C}}^{[8],d}\Pi_3^\beta], \qquad (3.11)$$

where

$$\mathcal{C}_8'^b = \sqrt{2}T^b,$$

$$\Pi_3'^\alpha = \frac{\sqrt{M_H}}{M_H + 2m_Q}\frac{M_H + \not{P}_H}{2M_H}\gamma^\alpha\frac{M_H - \not{P}_H}{2M_H},$$

$$\Pi_3^\beta = \frac{2}{\sqrt{M_H}(M_H + 2m_Q)}(\not{p}_{\bar{Q}} - m_Q)\frac{M_H - \not{P}_H}{2M_H}\gamma^\beta\frac{M_H + \not{P}_H}{2M_H}(\not{p}_Q + m_Q). \tag{3.12}$$

Then we obtain

$$F_{[SS]\to Q\bar{Q}[SS]}^{\text{LO}}(x, M_H, m_Q, \mu_f) = \delta(1 - x), \tag{3.13}$$

which is consistent with the result in eq. (2.14). Similarly we can derive

$$F_{[SD,T]\to Q\bar{Q}[SD,T]}^{\text{LO}}(x, M_H, m_Q, \mu_f) = \delta(1 - x),$$

$$F_{[DD,T]\to Q\bar{Q}[DD,T]}^{\text{LO}}(x, M_H, m_Q, \mu_f) = \delta(1 - x). \tag{3.14}$$

To obtain the above results, we have used the relations

$$\sum_{|\lambda|=1}\epsilon_\lambda^\mu\epsilon_\lambda^{*\nu} = -g^{\mu\nu} + \frac{P_H^\mu l^\nu + l^\mu P_H^\nu}{P_H \cdot l} - \frac{M_H^2 l^\mu l^\nu}{(P_H \cdot l)^2}, \tag{3.15a}$$

$$\sum_{|\lambda|=0}\epsilon_\lambda^\mu\epsilon_\lambda^{*\nu} = \frac{P_H^\mu P_H^\nu}{M_H^2} - \frac{P_H^\mu l^\nu + l^\mu P_H^\nu}{P_H \cdot l} + \frac{M_H^2 l^\mu l^\nu}{(P_H \cdot l)^2}. \tag{3.15b}$$

The longitudinally polarized ($\lambda = L$) SGDs can be calculated similarly, but they are irrelevant for our purpose as we will show later, thus we do not consider their contributions in our calculation.

3.3 NLO calculation

At NLO, the corresponding Feynman diagrams are shown in figure 2 and figure 3. Let us first consider virtual corrections in figure 2. Following the calculation in ref. [27], we obtain

$$F_{[SS]\to Q\bar{Q}[SS]}^{\text{NLO}}(x, M_H, m_Q, \mu_f)\Big|_{\text{vir.}}^{a+b} = \frac{\alpha_s}{4\pi}\frac{N_c^2 - 1}{2N_c}\left(\frac{4}{\epsilon_{\text{UV}}} - \frac{4}{\epsilon_{\text{IR}}}\right)\delta(1 - x), \tag{3.16a}$$

$$F_{[SS]\to Q\bar{Q}[SS]}^{\text{NLO}}(x, M_H, m_Q, \mu_f)\Big|_{\text{vir.}}^{c} = \frac{\alpha_s}{4\pi}\frac{1}{2N_c}\left[\left(\frac{4}{\epsilon_{\text{UV}}} - \frac{4}{\epsilon_{\text{IR}}}\right)\frac{1 + \Delta^2}{2\Delta}\ln\left(\frac{1 + \Delta}{1 - \Delta}\right)\right.$$
$$\left. - 2(1 + \Delta^2)\frac{\pi^2}{\Delta}\right]\delta(1 - x), \tag{3.16b}$$

$$F_{[SS]\to Q\bar{Q}[SS]}^{\text{NLO}}(x, M_H, m_Q, \mu_f)\Big|_{\text{vir.}}^{d+e} = \frac{\alpha_s}{4\pi}\frac{N_c}{2}(-1)\left(\frac{4\pi\mu^2 e^{-\gamma_E}}{M_H^2}\right)^\epsilon\left[\frac{1}{\epsilon_{\text{UV}}}\left(\frac{4}{\epsilon_{\text{UV}}} - \frac{4}{\epsilon_{\text{IR}}}\right)\right.$$
$$\left. + 2\left(\frac{1}{2\Delta}\ln\frac{1 + \Delta}{1 - \Delta} - 1\right)\left(\frac{4}{\epsilon_{\text{UV}}} - \frac{4}{\epsilon_{\text{IR}}}\right)\right]$$
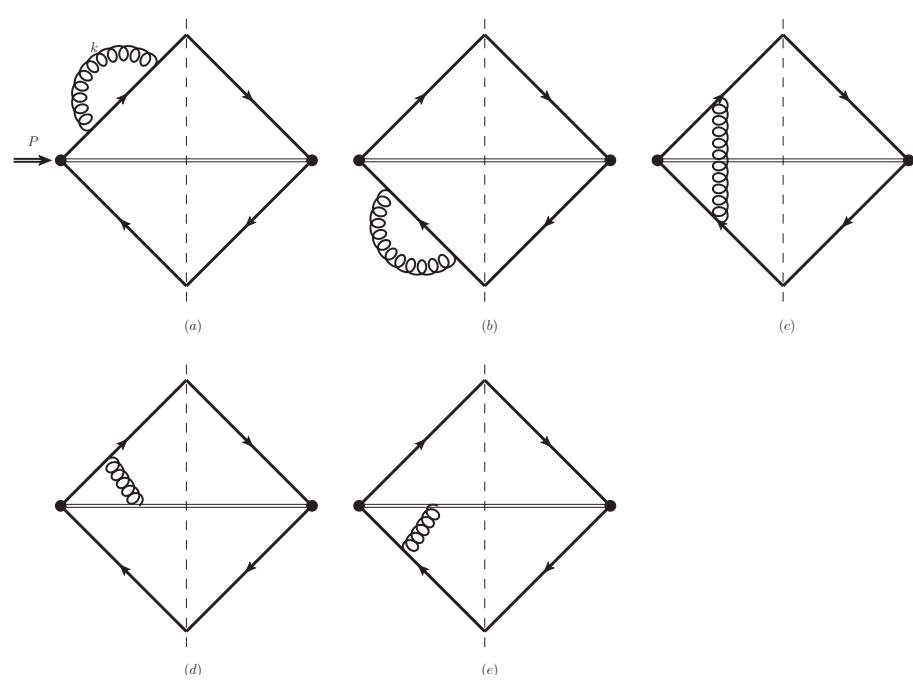$$\times \delta(1 - x). \tag{3.16c}$$

Figure 2. The virtual corrections of SGD at NLO. The complex conjugate diagrams are not shown, but are included in the calculations.
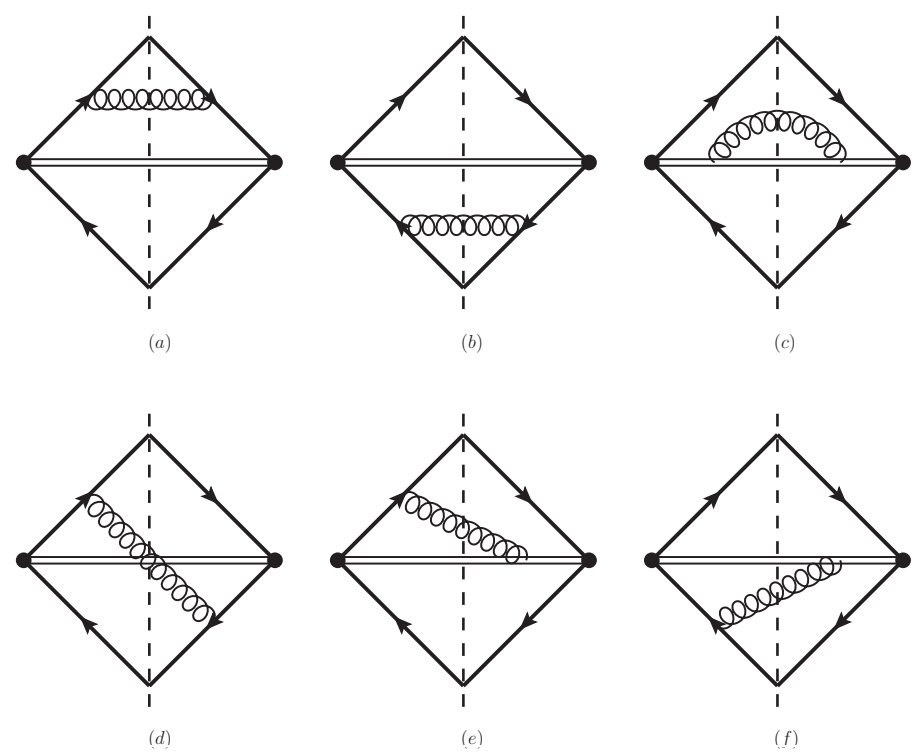

Figure 3. The real corrections of SGD at NLO.

with

$$\Delta = \frac{\sqrt{M_H^2 - 4m_Q^2}}{M_H}. \tag{3.17}$$

For real corrections in figure 3, we have

$$F_{[SS]\to Q\bar{Q}[SS]}^{(\text{NLO})}(z, M_H, m_Q, \mu_f)\Big|_{\text{real}} = T_S\bigg[\int \frac{d^d k}{(2\pi)^d} I'\bigg], \tag{3.18}$$

with

$$I' = \frac{P_H^+}{d-1}\delta(k^+ - w)2\pi\delta(k^2)\theta(k^+)\mathcal{M}_{S,R}^{\alpha\beta\sigma}\mathcal{M}_{S,R}^{*\mu\nu\rho}\mathbb{P}_{\alpha\mu}\mathbb{P}_{\beta\nu}(-g_{\rho\sigma}), \quad w = \frac{1-x}{x}P_H^+, \tag{3.19}$$

where

$$\mathcal{M}_{S,R}^{\alpha\beta\sigma} = \int \frac{d^{d-2}\Omega}{N_\Omega}\Big(\mathcal{M}_{S,R}^{\alpha\beta\sigma,(1)} + \mathcal{M}_{S,R}^{\alpha\beta\sigma,(2)} + \mathcal{M}_{S,R}^{\alpha\beta\sigma,(3)}\Big),$$

$$\mathcal{M}_{S,R}^{\alpha\beta\sigma,(1)} = g_s\mu^\epsilon \frac{\text{Tr}[\gamma^\sigma T^a(\not{p}_Q + \not{k} + m_Q)\mathcal{C}_8'^b\Pi_3'^\alpha\widetilde{\mathcal{C}}^{[8],d}\Pi_3^\beta]}{2p_Q \cdot k + i\varepsilon},$$

$$\mathcal{M}_{S,R}^{\alpha\beta\sigma,(2)} = g_s\mu^\epsilon \frac{\text{Tr}[\mathcal{C}_8'^b\Pi_3'^\alpha(-\not{p}_{\bar{Q}} - \not{k} + m)\gamma^\sigma T^a\widetilde{\mathcal{C}}^{[8],d}\Pi_3^\beta]}{2p_{\bar{Q}} \cdot k + i\varepsilon},$$

$$\mathcal{M}_{S,R}^{\alpha\beta\sigma,(3)} = g_s\mu^\epsilon\text{Tr}[\mathcal{C}_8'^b\Pi_3'^\alpha\widetilde{\mathcal{C}}^{[8],d}\Pi_3^\beta]\bigg(if^{abc}\frac{l^\sigma}{l \cdot k + i\varepsilon}\bigg). \tag{3.20}$$

Here k is the momentum of the emitted real gluon in the final state, which is restricted in the soft domain. Applying T_S to I', we obtain

$$T_S I' = \frac{P_H^+}{d-1}\delta(k^+ - w)2\pi\delta(k^2)\theta(k^+)\mathbb{P}_{\alpha\mu}\mathbb{P}_{\beta\nu}(-g_{\rho\sigma})T_S\Big\{\mathcal{M}_{S,R}^{\alpha\beta\sigma}\mathcal{M}_{S,R}^{*\mu\nu\rho}\Big\}, \tag{3.21}$$

with

$$T_S\mathcal{M}_{S,R}^{\alpha\beta\sigma,(1)} = g_s\mu^\epsilon\text{Tr}[T^a\mathcal{C}_8'^b\Pi_3'^\alpha\widetilde{\mathcal{C}}^{[8],d}\Pi_3^\beta]\sum_j \frac{(p^\sigma + 2q^\sigma)(2\boldsymbol{q}\cdot\boldsymbol{k})^j}{[p_0 k_0 + i\varepsilon]^{1+j}},$$

$$T_S\mathcal{M}_{S,R}^{\alpha\beta\sigma,(2)} = g_s\mu^\epsilon\text{Tr}[\mathcal{C}_8'^b\Pi_3'^\alpha T^a\widetilde{\mathcal{C}}^{[8],d}\Pi_3^\beta]\sum_j \frac{-(p^\sigma - 2q^\sigma)(-2\boldsymbol{q}\cdot\boldsymbol{k})^j}{[p_0 k_0 + i\varepsilon]^{1+j}},$$

$$T_S\mathcal{M}_{S,R}^{\alpha\beta\sigma,(3)} = g_s\mu^\epsilon\text{Tr}[\mathcal{C}_8'^b\Pi_3'^\alpha\widetilde{\mathcal{C}}^{[8],d}\Pi_3^\beta]\bigg(if^{abc}\frac{l^\sigma}{l \cdot k + i\varepsilon}\bigg). \tag{3.22}$$

Performing the Ω and k integration, we obtain the following result

$$F_{[SS]\to Q\bar{Q}[SS]}^{\text{NLO}}(x, M_H, m_Q, \mu_f)\Big|_{\text{real}} = \frac{\alpha_s N_c}{4\pi}\bigg(\frac{4\pi x^2\mu^2 e^{-\gamma_E}}{M_H^2}\bigg)^\epsilon 4x(1-x)^{-1-2\epsilon}$$

$$\times\bigg(\frac{1}{\epsilon_{\text{UV}}} - 1 + \frac{\pi^2}{12}\epsilon\bigg). \tag{3.23}$$

Combing the virtual and real correction contributions shown in eqs. (3.16) and (3.23), we derive the $F_{[SS]\to Q\bar{Q}[SS]}$ before renormalization

$$
\begin{aligned}
F_{[SS]\to Q\bar{Q}[SS]}^{\text{NLO}}(x, M_H, m_Q, \mu_f)\Big|_{\text{bare}} &= \frac{\alpha_s}{4\pi}\left(\frac{4\pi x^2 \mu^2 e^{-\gamma_E}}{M_H^2}\right)^\epsilon \Bigg\{ N_c\Bigg[-\frac{2}{\epsilon_{\text{UV}}^2}\delta(1-x) \\
&+ \frac{2}{\epsilon_{\text{UV}}}\left(\delta(1-x) + \frac{2x}{(1-x)_+}\right) - \left(\frac{4}{\epsilon_{\text{UV}}} - \frac{4}{\epsilon_{\text{IR}}}\right)\left(\frac{1}{2\Delta}\ln\frac{1+\Delta}{1-\Delta} - 1\right)\delta(1-x) \\
&- \frac{4x}{(1-x)_+} - 8x\left(\frac{\ln(1-x)}{1-x}\right)_+ \Bigg] + \frac{1}{N_c}\left(\frac{2}{\epsilon_{\text{UV}}} - \frac{2}{\epsilon_{\text{IR}}}\right)\left(\frac{1+\Delta^2}{2\Delta}\ln\frac{1+\Delta}{1-\Delta} - 1\right) \\
&\times \delta(1-x) - \left(\frac{1}{N_c}(1+\Delta^2)\frac{\pi^2}{\Delta} + N_c\frac{\pi^2}{6}\right)\delta(1-x)\Bigg\}.
\end{aligned}
\tag{3.24}
$$

The ultraviolet divergences in the above result can be removed by using the $\overline{\text{MS}}$ renormalization procedure, and we get the renormalized SGD

$$
\begin{aligned}
F_{[SS]\to Q\bar{Q}[SS]}^{\text{NLO}}(x, M_H, m_Q, \mu_f) &= \frac{\alpha_s}{4\pi}\left(\frac{4\pi x^2 \mu_c^2 e^{-\gamma_E}}{M_H^2}\right)^\epsilon \Bigg\{ N_c\Bigg[\frac{4}{\epsilon_{\text{IR}}}\left(\frac{1}{2\Delta}\ln\frac{1+\Delta}{1-\Delta} - 1\right) \\
&\times \delta(1-x) - \frac{4x}{(1-x)_+} - 8x\left(\frac{\ln(1-x)}{1-x}\right)_+ + 2\left(3 - \frac{1}{\Delta}\ln\frac{1+\Delta}{1-\Delta}\right)\delta(1-x)\ln\left(\frac{\mu_f^2}{M_H^2}\right) \\
&- \delta(1-x)\ln^2\left(\frac{\mu_f^2}{M_H^2}\right) + \frac{4x}{(1-x)_+}\ln\left(\frac{x^2 \mu_f^2}{M_H^2}\right)\Bigg] - \frac{2}{\epsilon_{\text{IR}}}\frac{1}{N_c}\left(\frac{1+\Delta^2}{2\Delta}\ln\frac{1+\Delta}{1-\Delta} - 1\right) \\
&\times \delta(1-x) + \frac{1}{N_c}\left(\frac{1+\Delta^2}{\Delta}\ln\frac{1+\Delta}{1-\Delta} - 2\right)\Delta(1-x)\ln\left(\frac{\mu_f^2}{M_H^2}\right) - \delta(1-x) \\
&\times \left(\frac{1}{N_c}(1+\Delta^2)\frac{\pi^2}{\Delta} + N_c\frac{\pi^2}{6}\right)\Bigg\}.
\end{aligned}
\tag{3.25}
$$

Here we distinguish the dimensional regularization scale μ_c from the factorization scale μ_f.

Similarly we can calculate the other SGDs by projecting the initial and finial $Q\bar{Q}$ pair into corresponding states, and we have

$$
\begin{aligned}
F_{[SD,T]\to Q\bar{Q}[SS]}^{\text{NLO}}(x, M_H, m_Q, \mu_f)\Big|_{\text{vir.}} &= \frac{\alpha_s N_c}{4\pi}\sqrt{\frac{2(d-1)}{(d-2)(d+1)}}\frac{M_H^2}{36}\left(-\frac{1}{\epsilon_{\text{IR}}} + \ln\frac{\mu_f^2}{4\pi\mu_c^2 e^{-\gamma_E}}\right) \\
&\times \left(-3 + 2\Delta^2 + \frac{3(1-\Delta^2)}{2\Delta}\ln\frac{1+\Delta}{1-\Delta}\right)\delta(1-x),
\end{aligned}
\tag{3.26a}
$$

$$
\begin{aligned}
F_{[SD,T]\to Q\bar{Q}[SS]}^{\text{NLO}}(x, M_H, m_Q, \mu_f)\Big|_{\text{real}} &= \frac{\alpha_s N_c}{4\pi}\sqrt{\frac{2(d-1)}{(d-2)(d+1)}}\frac{4m_Q^2(2-3\mathcal{T}) + M_H^2}{18} \\
&\times \left(\frac{4\pi x^2 \mu_c^2 e^{-\gamma_E}}{M_H^2}\right)^\epsilon\Bigg[-\frac{1}{2\epsilon_{\text{IR}}}\delta(1-x) - \frac{4m_Q^2(9\mathcal{T}-8) - M_H^2}{6(4m_Q^2(2-3\mathcal{T}) + M_H^2)}\delta(1-x) \\
&+ \frac{x}{(1-x)_+}\Bigg],
\end{aligned}
\tag{3.26b}
$$

$$
F_{[DD,T]\to Q\bar{Q}[SS]}^{\text{NLO}}(x, M_H, m_Q, \mu_f)\Big|_{\text{vir.}} = 0,
\tag{3.26c}
$$

$$
\begin{aligned}
F^{\mathrm{NLO}}_{[DD,T]\to Q\bar{Q}[SS]}(x,M_H,m_Q,\mu_f)\Big|_{\mathrm{real}} &= \frac{\alpha_s N_c}{4\pi}\frac{2(d-1)}{(d-2)(d+1)}\frac{(4m_Q^2(2-3\mathcal{T})+M_H^2)^2}{108} \\
&\times\left(\frac{4\pi x^2\mu_c^2 e^{-\gamma_E}}{M_H^2}\right)^\epsilon\left[-\frac{1}{2\epsilon_{\mathrm{IR}}}\delta(1-x)+\frac{x}{(1-x)_+}\right. \\
&\left.-\frac{16m_Q^4(189\mathcal{T}^2-204\mathcal{T}+52)-8m_Q^2(87\mathcal{T}-50)M_H^2+37M_H^4}{24(4m_Q^2(2-3\mathcal{T})+M_H^2)^2}\delta(1-x)\right]. \quad (3.26\mathrm{d})
\end{aligned}
$$

Here $\mathcal{T}$ is a hypergeometric function

$$
\mathcal{T} = {}_2F_1\left(\frac{1}{2},1,\frac{3}{2}-\epsilon,\Delta^2\right), \quad (3.27)
$$

which can be expended as

$$
\mathcal{T} = \frac{1}{2\Delta}\left[\ln\frac{1+\Delta}{1-\Delta}+\left(-2\ln\frac{1+\Delta}{1-\Delta}-\mathrm{Li}_2\left(\frac{2\Delta}{\Delta-1}\right)+\mathrm{Li}_2\left(\frac{2\Delta}{\Delta+1}\right)\right)\epsilon\right]+O(\epsilon^2). \quad (3.28)
$$

4 Renormalization group equation and models for SGDs

4.1 Renormalization group equation

Based the factorization formula in eq. (2.1), RGEs for SGDs have the following general form

$$
\begin{aligned}
\frac{d}{d\ln\mu_f}F_{[L'\tilde{L}',\lambda']\to H}(z,M_H,m_Q,\mu_f) &= \sum_{L,\tilde{L},\lambda}\int_z^1\frac{dx}{x}\boldsymbol{K}^{[L\tilde{L},\lambda]}_{[L'\tilde{L}',\lambda']}(\hat{z},M_H/x,m_Q,\mu_f) \\
&\times F_{[L\tilde{L},\lambda]\to H}(x,M_H,m_Q,\mu_f), \quad (4.1)
\end{aligned}
$$

with $\hat{z}\equiv z/x$. The evolution kernel $\boldsymbol{K}$ can be perturbatively calculated by using the matching procedure. At LO in α_s, we have

$$
\boldsymbol{K}^{[L\tilde{L},\lambda],\mathrm{LO}}_{[L'\tilde{L}',\lambda']}(\hat{z},M_H/x,m_Q,\mu_f) = \frac{d}{d\ln\mu_f}F^{\mathrm{NLO}}_{[L'\tilde{L}',\lambda']\to Q\bar{Q}[L\tilde{L},\lambda]}(\hat{z},M_H/x,m_Q,\mu_f). \quad (4.2)
$$

For simplicity, in the following we only consider the evolution equation for $F_{[SS]\to H}$ and ignore contributions from $F_{[SD,T]\to H}$ and $F_{[DD,T]\to H}$ because they are v^4 suppressed.

From eq. (3.25), we have the evolution kernel

$$
\begin{aligned}
\boldsymbol{K}^{[SS],\mathrm{LO}}_{[SS]}(z,M_H,m_Q,\mu_f) &= \frac{\alpha_s}{\pi}\left\{N_c\left[\frac{2z}{(1-z)_+}-\ln\frac{\mu^2 e^{-1}}{M_H^2}\delta(1-z)\right.\right. \\
&\left.\left.-2\delta(1-z)\left(\frac{1}{2\Delta}\ln\frac{1+\Delta}{1-\Delta}-1\right)\right]+\frac{1}{N_c}\left(\frac{1+\Delta^2}{2\Delta}\ln\frac{1+\Delta}{1-\Delta}-1\right)\delta(1-z)\right\}. \quad (4.3)
\end{aligned}
$$

The evolution equation with this kernel is difficult to be solved analytically. However, if we only want to resum large logarithms at leading order in v^2 expansion, we only need to keep the leading power term in the evolution kernel, which reads

$$
\boldsymbol{K}^{[SS],\mathrm{LO}(0)}_{[SS]}(z,M_H,\mu_f) = \frac{\alpha_s}{4\pi}\left[\Gamma_0^F\left(\frac{z}{(1-z)_+}-\ln\frac{\mu_f}{M_H}\delta(1-z)\right)-\gamma_0^F\delta(1-z)\right], \quad (4.4)
$$

where

$$\Gamma_0^F = 8N_c, \quad \gamma_0^F = -4N_c. \tag{4.5}$$

Then the evolution equation for $F_{[SS]\to H}$ becomes

$$\frac{d}{d\ln\mu_f} F_{[SS]\to H}(z, M_H, m_Q, \mu_f) = \int_z^1 \frac{dx}{x} \boldsymbol{K}_{[SS]}^{[SS],\text{LO}(0)}(\hat{z}, M_H/x, \mu_f)$$
$$\times F_{[SS]\to H}(x, P_H, m_Q, \mu_f). \tag{4.6}$$

To solve the above evolution equation, we rewrite it as a function of the variable $\omega = M_H(1/z - 1)$, which results in

$$\frac{d}{d\ln\mu_f} F_{[SS]\to H}(\omega, M_H, m_Q, \mu_f)$$
$$= \int_0^\omega d\omega' \frac{\alpha_s}{4\pi} \left[\Gamma_0^F \left(\left[\frac{1}{\omega - \omega'} \right]_+ - \ln\mu_f \delta(\omega - \omega') \right) - \gamma_0^F \delta(\omega - \omega') \right]$$
$$\times F_{[SS]\to H}(\omega', M_H, m_Q, \mu_f), \tag{4.7}$$

with $\omega' = M_H(1/x - 1)$. To derive the above equation, we have used the rescaling identity for plus functions [42]

$$\kappa \left[\frac{\ln^n(\kappa\omega)}{\kappa\omega} \right]_+ = \sum_{i=0}^n \frac{n!}{(n-i)!i!} \ln^{n-i}(\kappa) \left[\frac{\ln^i(\omega)}{\omega} \right]_+ + \frac{\ln^{n+1}(\kappa)}{n+1} \delta(\omega). \tag{4.8}$$

The plus functions of the dimensionful variable ω are defined as

$$\int_0^\xi d\omega \left[\frac{\ln^n(\omega)}{\omega} \right]_+ f(\omega) = \int_0^\xi d\omega \frac{\ln^n(\omega)}{\omega} \left(f(\omega) - f(0) \right) + \frac{1}{n+1} \ln^{n+1}(\xi) f(0), \tag{4.9}$$

where $f(\omega)$ is a smooth test function. To solve the RGE eq. (4.7), it is convenient to perform a Laplace transformation [43–45], which, together with its inverse, is given by

$$\tilde{f}(s) = \int_0^\infty d\omega e^{-\omega s} f(\omega), \quad \text{and} \quad f(\omega) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds e^{\omega s} \tilde{f}(s), \tag{4.10}$$

where the constant c is chosen to be larger than the real part of the rightmost singularity of $\tilde{f}(s)$. Taking a Laplace transform in eq. (4.7) we have a simple multiplicative RGE

$$\frac{d}{d\ln\mu_f} \tilde{F}_{[SS]\to H}(s, M_H, m_Q, \mu_f) = -\frac{\alpha_s}{4\pi} \left(\Gamma_0^F \ln(\bar{s}\mu_f) + \gamma_0^F \right)$$
$$\times \tilde{F}_{[SS]\to H}(s, M_H, m_Q, \mu_f), \tag{4.11}$$

where $\bar{s} = se^{\gamma_E}$.

Solving the RGE in Laplace space is straight forward, details of which are given in appendix A. We eventually obtain

$$\tilde{F}_{[SS]\to H}(s, M_H, m_Q, M_H) = \exp\left[h_0(\chi^*) \ln(M_H \bar{s}) + h_1(\chi^*) \right] \tilde{F}_{\text{fix}}(M_H) \tilde{F}^{\text{mod}}(s), \tag{4.12}$$

where $\tilde{F}_{\text{fix}}(M_H)$ is given in eq. (A.13) which recovers fixed-order perturbative results at small s region, functions $h_{0,1}$ are given by eq. (A.9), and χ^* is defined as

$$\chi^* = \frac{\beta_0}{2\pi}\alpha_s(M_H)\ln(M_H\bar{s}) + \frac{\beta_0}{2\pi}\alpha_s(M_H)\ln\left(1 + \frac{1}{M_H\bar{s}^*}\right)$$
$$- \frac{\beta_0}{2\pi}\alpha_s(M_H)\ln\left(1 + \frac{\bar{s}}{\bar{s}^*}\right), \tag{4.13}$$

with

$$\beta_0 = \frac{11N_c - 2n_f}{6}, \tag{4.14}$$

where n_f represents the number of light flavors. We have introduced a "frozen" scale $\bar{s}^*$ with

$$\bar{s}^* = \frac{1}{aM_H}\exp\left[\frac{\pi}{\beta_0\alpha_s(M_H)}\right] \tag{4.15}$$

to prevent Landau singularity, where a (with $a \gtrsim 1$) is a parameter and its dependence should be compensated with a change of nonperturbative input. Therefore, we fix it as $a = 1.3$ in the rest of this paper. Finally all nonperturbative information is included in the model $\tilde{F}^{\text{mod}}(s)$. Then SGD can be obtained by transforming eq. (4.12) back to momentum space using

$$F_{[SS]\to H}(x, M_H, m_Q, M_H) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, e^{M_H(1/x-1)s}\tilde{F}_{[SS]\to H}(s, M_H, m_Q, M_H), \tag{4.16}$$

where c is a constant that must be greater than the real part of all singularities of $\tilde{F}_{[SS]\to H}$.

4.2 Sensitivity of nonperturbative model

We will demonstrate that SGD obtained above by solving RGE is only sensitive to two parameters of the nonperturbative model. To this end, we will study various models, with significant different shapes. We choose the model function used in [33–35, 46] as our first class of models,

$$F^{\text{mod}}(\omega') = M_H N_H \frac{b^b}{\Gamma(b)}\frac{\omega'^{b-1}}{\bar{\Lambda}^b}e^{-b\omega'/\bar{\Lambda}}, \quad \omega' = M_H(1/x - 1), \tag{4.17}$$

whose zeroth, first and second moments are $M_H N_H$, $M_H N_H \bar{\Lambda}$ and $M_H N_H \bar{\Lambda}^2(\frac{1}{b} + 1)$, respectively. Here N_H determines the normalization, $\bar{\Lambda}$ characterizes the average radiated momentum, and b is related to the width of the model function. We vary $\bar{\Lambda}$ from $0.5\text{GeV} - 0.7\text{GeV}$,[1] and b from $1 - 3$. Other models are chosen to cover various possible shapes, but with the same normalization ($M_H N_H$) and average radiated momentum

---

[1]There is yet no first-principle way to determine $\bar{\Lambda}$, but we can guess its order of magnitude. For the transition from $^3S_1^{[8]}$ to a 1^{--} quarkonium, where a double color E1 transition should dominate, a good guess is that $\bar{\Lambda}$ is at the order of mass splitting between 2S state and 1S state. Due to $M_{\psi(2S)} - M_{J/\psi} \approx 0.59\text{GeV}$ and $M_{\Upsilon(2S)} - M_{\Upsilon(1S)} \approx 0.56\text{GeV}$, we choose the central value of $\bar{\Lambda}$ at 0.6GeV, although it is adjustable.
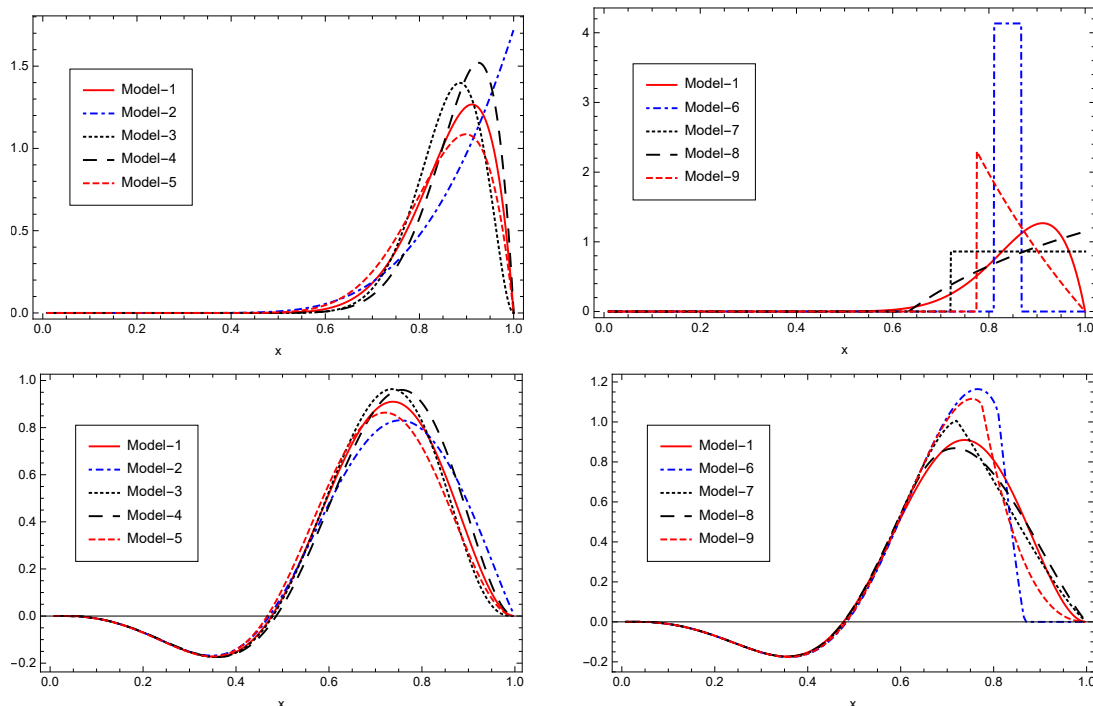
Figure 4. The first row shows the different models, the corresponding SGDs at LO are given in the second row.

(0.6GeV). All models that we will study are listed in the following,

$$
\begin{aligned}
&\text{Model-1:}\quad F^{\text{mod}}(\omega')|_{\bar{\Lambda}=0.6\text{GeV},b=2}, \qquad \text{Model-2:}\quad F^{\text{mod}}(\omega')|_{\bar{\Lambda}=0.6\text{GeV},b=1}, \\
&\text{Model-3:}\quad F^{\text{mod}}(\omega')|_{\bar{\Lambda}=0.6\text{GeV},b=3}, \qquad \text{Model-4:}\quad F^{\text{mod}}(\omega')|_{\bar{\Lambda}=0.5\text{GeV},b=2}, \\
&\text{Model-5:}\quad F^{\text{mod}}(\omega')|_{\bar{\Lambda}=0.7\text{GeV},b=2}, \\
&\text{Model-6:}\quad 4M_H N_H[\theta(w' \geq \tfrac{19}{40}) - \theta(w' > \tfrac{29}{40})], \\
&\text{Model-7:}\quad \tfrac{5}{6}M_H N_H[\theta(w' \geq 0) - \theta(w' > \tfrac{6}{5})], \\
&\text{Model-8:}\quad \begin{cases} M_H N_H(-\tfrac{50}{81}w' + \tfrac{10}{9}), & 0 \leq w' \leq \tfrac{9}{5}, \\ 0, & w' > \tfrac{9}{5}, \end{cases} \\
&\text{Model-9:}\quad \begin{cases} \tfrac{200}{81}M_H N_H w', & 0 \leq w' \leq \tfrac{9}{10}, \\ 0, & w' > \tfrac{9}{10}. \end{cases}
\end{aligned}
\tag{4.18}
$$

These input models are shown in the first row in figure 4, and the corresponding SGDs obtained by solving RGE at LO are given in the second row, where we have taken $M_H = 3.1$GeV, $n_f = 3$, $\Lambda_{QCD}^{(3)} = 0.25$GeV and $N_H = 1/3$. From the plots we find that, comparing with input models, SGDs are much broader and their peaks are shifted to smaller x. More importantly, SGDs at moderate and small x, i.e., $x < 0.65$, are almost independent of input models, as far as the models have the same normalization and average radiated momentum. This is reasonable because small-x region is dominated by perturbative effects.

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Models					
$\langle g_1, g_2 \rangle$	Model-1	Model-2	Model-3	Model-4	Model-5
Model-1	0	0.190	0.107	0.093	0.078
Model-2		0	0.291	0.191	0.210
Model-3			0	0.160	0.122
Model-4				0	0.170
Model-5					0

SGDs					
$\langle g_1, g_2 \rangle$	Model-1	Model-2	Model-3	Model-4	Model-5
Model-1	0	0.060	0.029	0.048	0.045
Model-2		0	0.089	0.061	0.083
Model-3			0	0.062	0.049
Model-4				0	0.093
Model-5					0

Models					
$\langle g_1, g_2 \rangle$	Model-1	Model-6	Model-7	Model-8	Model-9
Model-1	0	0.777	0.204	0.144	0.366
Model-6		0	0.794	0.818	0.633
Model-7			0	0.146	0.371
Model-8				0	0.427
Model-9					0

SGDs					
$\langle g_1, g_2 \rangle$	Model-1	Model-6	Model-7	Model-8	Model-9
Model-1	0	0.129	0.041	0.029	0.084
Model-6		0	0.119	0.148	0.056
Model-7			0	0.036	0.065
Model-8				0	0.098
Model-9					0

Table 1. The differences between the Models and the SGDs.

At larger x region, values of SGDs depend on input models, but the sensitivity after RGE running is much weaker than the original models. To see this more clearly, we introduce a function to describe the difference between two functions $g_1(x)$ and $g_2(x)$

$$\langle g_1, g_2 \rangle = \frac{\int_0^1 |g_1(x) - g_2(x)| dx}{\int_0^1 (|g_1(x)| + |g_2(x)|) dx}. \tag{4.19}$$

Differences between input models and that between corresponding SGDs are shown in table 1. It is clear that differences are significantly decreased after performing resummation

using RGE. Especially for models with the same average radiated momentum, differences have been typically reduced by a factor of $1/5$.

The above phenomenon can be interpreted as following. The overall normalization of a SGD is fully determined by the normalization of the input model. The shape of a SGD is almost fully determined by perturbative calculation with all-order resummation, although its peak can be affected by the average radiated momentum of the input model. In this consideration, in practical use it is sufficient to choose a suitable parameter b in eq. (4.17), and fit the parameters N_H and $\bar{\Lambda}$ by comparing with experiment data.

5 Gluon fragmentation function of quarkonium in SGF

5.1 SGF formula for gluon fragmentation function

The Collins-Soper definition for fragmentation function of a gluon fragmenting into a hadron (quarkonium) H is given by [47]

$$
D_{g \to H}(z, \mu) = \frac{-g_{\mu\nu} z^{d-3}}{P_c^+(N_c^2 - 1)(d-2)} \int \frac{d\xi^-}{2\pi} e^{-iP_c^+ \xi^-} \langle 0| G_c^{+\mu}(0) \Phi_l^\dagger(0)_{cb} \mathcal{P}_{H(P_H)}
$$
$$
\times \Phi_l(\xi^-)_{ba} G_a^{+\nu}(0, \xi^-, 0_\perp)|0\rangle, \tag{5.1}
$$

where $G^{\mu\nu}$ is the gluon field-strength operator, P_H and P_c are the momenta of the hadron and initial virtual gluon, respectively, and z is the "+" momentum fraction defined as $z = P_H^+/P_c^+$. The projection operator $\mathcal{P}_{H(P_H)}$ is given by

$$
\mathcal{P}_{H(P_H)} = \sum_X |H(P_H) + X\rangle\langle H(P_H) + X|, \tag{5.2}
$$

where X sums over all unobserved particles. The gauge link $\Phi_l(\xi^-)$ is the same as the one given in eq. (2.7). The definition of eq. (5.1) is gauge invariant and we use the Feynman gauge in our calculation.

In eq. (5.1), μ is the collinear factorization scale. The dependence of FFs on μ is controlled by the DGLAP evolution equations [48–50],

$$
\frac{d}{d \ln \mu} D_{g \to H}(z, \mu) = \int_z^1 \frac{dx}{x} P_{gg}(z/x, \mu) D_{g \to H}(x, \mu), \tag{5.3}
$$

where we have ignored contributions of quark fragmentation functions because they are usually less important. Evolving FFs from a low scale μ up to a high scale μ_h by solving the DGLAP evolution, one can resum large logarithms of μ^2/μ_h^2 to all orders in perturbation theory. The gluon splitting function P_{gg} has the perturbative expansion

$$
P_{gg}(x, \mu) = \frac{\alpha_s(\mu)}{2\pi} P_{gg}^{\text{LO}}(x) + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2 P_{gg}^{\text{NLO}}(x) + \cdots, \tag{5.4}
$$

where, e.g., $P_{gg}^{\text{LO}}(x)$ is given by

$$
P_{gg}^{\text{LO}}(x) = 2N_c \left[\frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x) + \frac{\beta_0}{2N_c} \delta(1-x) \right]. \tag{5.5}
$$

In the parton (initial virtual gluon) frame, the gluon FF to quarkonium has the form

$$D_{g \to H}(z, M_H, m_Q, \mu) = \int \frac{dP_H^+ d^2 P_{H\perp}}{(2\pi)^3 \, 2P_H^+} \delta \left(z - \frac{P_H^+}{P_c^+} \right) (2\pi)^3 2P_H^0 \frac{d\sigma_H(P_c, P_H, m_Q^2)}{d^3 P_H}. \quad (5.6)$$

Using the SGF-1d formula eq. (2.18) and changing the variables from the parton to the hadron frame, we have

$$\begin{aligned}
D_{g \to H}(z, M_H, m_Q, \mu) &= \int \frac{dP_H^+ d^2 P_{H\perp}}{(2\pi)^3 \, 2P_H^+} \delta \left(z - \frac{P_H^+}{P_c^+} \right) \sum_{n,n'} \int \frac{dx}{x^2} d\hat{\sigma}_{[nn']} (P_H/x, m_Q, \mu_f) \\
&\quad \times F_{[nn'] \to H}(x, M_H, m_Q, \mu_f) \\
&= \sum_{n,n'} \int \frac{dx}{x} \frac{z}{x} P_c^+ \int \frac{dP_c^+ d^2 P_{c\perp}}{(2\pi)^3 \, 2P_c^+} \delta \left(P_c^+ - \frac{P_H^+/x}{z/x} \right) d\hat{\sigma}_{[nn']} (P_H/x, m_Q, \mu_f) \\
&\quad \times F_{[nn'] \to H}(x, M_H, m_Q, \mu_f).
\end{aligned} \quad (5.7)$$

By defining the short distance hard parts $\hat{D}_{[nn']}(\hat{z}, M_H/x, m_Q, \mu, \mu_f)$ as following

$$\hat{D}_{[nn']}(\hat{z}, M_H/x, m_Q, \mu, \mu_f) = \frac{z}{x} P_c^+ \int \frac{dP_c^+ d^2 P_{c\perp}}{(2\pi)^3 \, 2P_c^+} \delta \left(P_c^+ - \frac{P_H^+/x}{z/x} \right) d\hat{\sigma}_{[nn']}(P_H/x, m_Q, \mu_f), \quad (5.8)$$

we obtain

$$D_{g \to H}(z, M_H, m_Q, \mu) = \sum_{n,n'} \int \frac{dx}{x} \hat{D}_{[nn']}(\hat{z}, M_H/x, m_Q, \mu, \mu_f) F_{[nn'] \to H}(x, M_H, m_Q, \mu_f), \quad (5.9)$$

which is the general SGF formula for heavy quarkonium FF. Such a factorization formula holds for any quarkonium state H, and the H dependence only appears in SGDs.

Then for the purpose of this paper, where we are only interested in ${}^3S_{1,\lambda}^{[8]}$ and ${}^3D_{1,\lambda}^{[8]}$ states, we have

$$\begin{aligned}
D_{g \to H}(z, M_H, m_Q, \mu) &= \sum_{\lambda = T,L} \int_z^1 \frac{dx}{x} \\
&\times \Bigg[\hat{D}_{[SS]}(\hat{z}, M_H/x, m_Q, \mu, \mu_f) \times F_{[SS] \to H}(x, M_H, m_Q, \mu_f) \\
&\quad + 2\hat{D}_{[SD,\lambda]}(\hat{z}, M_H/x, m_Q, \mu, \mu_f) \times F_{[SD,\lambda] \to H}(x, M_H, m_Q, \mu_f) \\
&\quad + \hat{D}_{[DD,\lambda]}(\hat{z}, M_H/x, m_Q, \mu, \mu_f) \times F_{[DD,\lambda] \to H}(x, M_H, m_Q, \mu_f) \Bigg], \quad (5.10)
\end{aligned}$$

where we have use the fact that polarized SGDs for S-wave states can be related polarization-summed SGDs, as noted in eq. (3.4).

Similar to eq. (2.8), we replace the quarkonium H in eq. (5.10) by an on-shell $Q\bar{Q}$ pair with specific quantum number and momenta P_H. Then we have following matching relations

$$\hat{D}^{\rm LO}_{[SS]}(\hat{z}, M_H/x, m_Q, \mu, \mu_f) = D^{\rm LO}_{g\to Q\bar{Q}[SS]}(\hat{z}, M_H/x, m_Q, \mu),$$

$$\hat{D}^{\rm LO}_{[SD,\lambda]}(\hat{z}, M_H/x, m_Q, \mu, \mu_f) = D^{\rm LO}_{g\to Q\bar{Q}[SD,\lambda]}(\hat{z}, M_H/x, m_Q, \mu),$$

$$\hat{D}^{\rm LO}_{[DD,\lambda]}(\hat{z}, M_H/x, m_Q, \mu, \mu_f) = D^{\rm LO}_{g\to Q\bar{Q}[DD,\lambda]}(\hat{z}, M_H/x, m_Q, \mu),$$

$$\hat{D}^{\rm NLO}_{[SS]}(\hat{z}, M_H/x, m_Q, \mu, \mu_f) = D^{\rm NLO}_{g\to Q\bar{Q}[SS]}(\hat{z}, M_H/x, m_Q, \mu) - \sum_{\lambda=T,L} \int_0^1 \frac{dy}{y}$$

$$\times \Bigg[\hat{D}^{\rm LO}_{[SS]}(\hat{z}/y, M_H/(xy), m_Q, \mu, \mu_f) \times F^{\rm NLO}_{[SS]\to Q\bar{Q}[SS]}(y, M_H/x, m_Q, \mu_f)$$

$$+ 2\hat{D}^{\rm LO}_{[SD,\lambda]}(\hat{z}/y, M_H/(xy), m_Q, \mu, \mu_f) \times F^{\rm NLO}_{[SD,\lambda]\to Q\bar{Q}[SS]}(y, M_H/x, m_Q, \mu_f)$$

$$+ \hat{D}^{\rm LO}_{[DD,\lambda]}(\hat{z}/y, M_H/(xy), m_Q, \mu, \mu_f) \times F^{\rm NLO}_{[DD,\lambda]\to Q\bar{Q}[SS]}(y, M_H/x, m_Q, \mu_f) \Bigg]. \quad (5.11)$$

These relations enable us to calculate $\hat{D}_{[SS]}$ perturbatively to NLO.

5.2 Perturbative calculation of gluon fragmentation functions

For the fragmentation functions we have

$$D_{g\to Q\bar{Q}[L\tilde{L},\lambda]}(z, M_H, m_Q, \mu) = \frac{-z^{d-3}P_c^+}{(N_c^2-1)(d-2)} \int d\Phi \mathcal{M}_{L\rho}(P_H, P_c, m_Q, k_i, \lambda)$$

$$\times \mathcal{M}^*_{\tilde{L}\sigma}(P_H, P_c, m_Q, k_i, \lambda) \left(g^{\mu\rho} - \frac{P_H^\mu l^\rho}{P_c \cdot l} \right) \left(g^\sigma_\mu - \frac{P_{H\mu} l^\sigma}{P_c \cdot l} \right), \quad (5.12)$$

where $d\Phi$ is the final state phase space, k_i denotes momenta of the final-state gluons, and $\mathcal{M}$ is projected amplitude using projector introduced in eq. (2.11), i.e.

$$\mathcal{M}^\rho_L(P_H, P_c, m_Q, k_i, \lambda) = \int \frac{d^{d-2}\Omega}{N_\Omega} \text{Tr}\left[\mathcal{A}^\rho_{g\to Q\bar{Q}} \Pi[^3L^{[8]}_{1,\lambda}] \right], \quad (5.13)$$

where $\mathcal{A}^\rho_{g\to Q\bar{Q}}$ is the amplitude from a virtual gluon to a $Q\bar{Q}$ pair plus light partons with spinors of $Q\bar{Q}$ removed.

At LO in α_s, the phase space $d\Phi^{\rm LO}$ is given by

$$d\Phi^{\rm LO} = \frac{d^d P_c}{(2\pi)^d} (2\pi)^d \delta^d(P_c - P_H) \delta\left(P_c^+ - \frac{P_H^+}{z} \right)$$

$$= \frac{z}{P_H^+} \delta(1-z). \quad (5.14)$$

Then the calculation of perturbative FFs is straightforward, which gives

$$D^{\rm LO}_{g\to Q\bar{Q}[SS]}(z, M_H, m_Q, \mu) = \bar{D}^{\rm LO}(M_H, m_Q, \mu)\delta(1-z), \quad (5.15a)$$

$$D^{\rm LO}_{g\to Q\bar{Q}[SD,T]}(z, M_H, m_Q, \mu) = D^{\rm LO}_{g\to Q\bar{Q}[DS,T]}(z, M_H, m_Q, \mu)$$

$$= \frac{-2\sqrt{2(d-1)(d+1)(d-2)}}{(M_H + 2m_Q)(M_H(d-2) + 2m_Q)} \bar{D}^{\rm LO}(M_H, m_Q, \mu)\delta(1-z), \quad (5.15b)$$
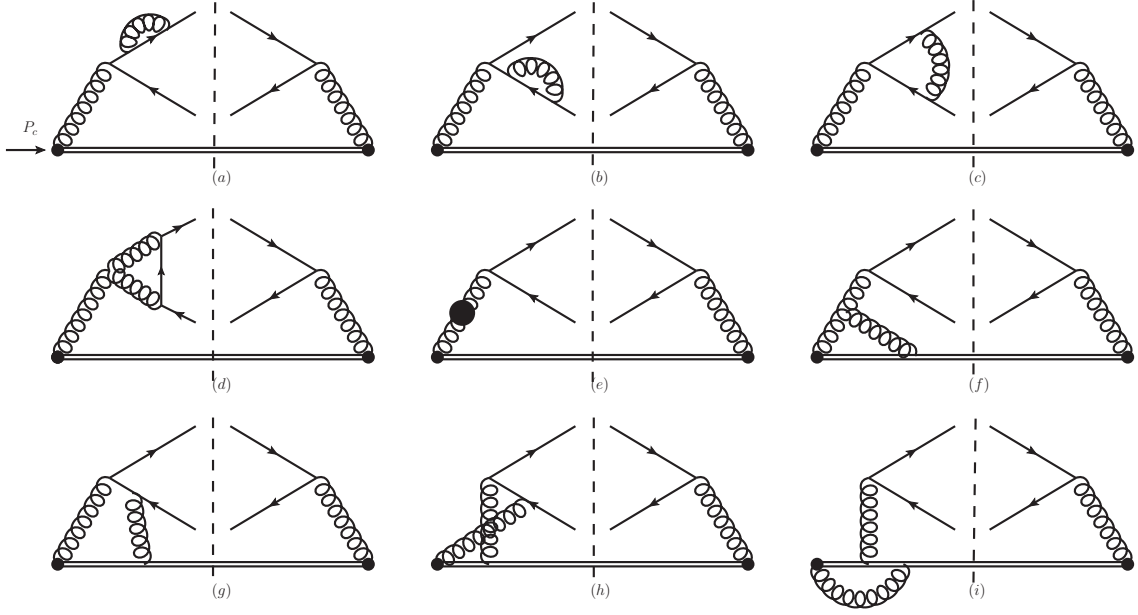
Figure 5. Feynman diagrams for virtual correction of the FF at NLO in Feynman gauge. The double solid line represents the gauge link along l direction. The complex conjugate diagrams are not shown, but are included in the calculations.

$$
\begin{aligned}
D^{\text{LO}}_{g \to Q\bar{Q}[DD,T]}&(z, M_H, m_Q, \mu) \\
&= \frac{8(d-2)(d+1)(d-1)}{(M_H + 2m_Q)^2 (M_H(d-2) + 2m_Q)^2} \bar{D}^{\text{LO}}(M_H, m_Q, \mu)\delta(1-z), \quad (5.15c) \\
D^{\text{LO}}_{g \to Q\bar{Q}[SD,L]}&(z, M_H, m_Q, \mu) = D^{\text{LO}}_{g \to Q\bar{Q}[DD,L]}(z, M_H, m_Q, \mu) \\
&= 0, \quad (5.15d)
\end{aligned}
$$

where

$$
\bar{D}^{\text{LO}}(M_H, m_Q, \mu) \equiv \frac{8\pi\alpha_s \mu^{2\epsilon}}{(N_c^2 - 1)M_H^3} \left(\frac{M_H(d-2) + 2m_Q}{(d-1)M_H} \right)^2. \quad (5.16)
$$

As we have mentioned before, basing on eqs. (5.15) and (5.11), we find the longitudinally polarized ($\lambda = L$) SGDs are irrelevant for calculating $\hat{D}^{\text{NLO}}_{[SS]}$.

At the NLO level, we only consider the FF $D_{g \to Q\bar{Q}[SS]}$, which consists of virtual corrections and real corrections. The diagrams with virtual-gluon corrections are shown in figure 5. To compute the contributions of these diagrams, we rewrite the integral of solid angle as

$$
\int \frac{d^{d-2}\Omega}{N_\Omega} = \frac{2M_H}{|\boldsymbol{q}|^{d-3}} \int \frac{d^d q}{N_\Omega} \delta(q^2 + |\boldsymbol{q}|^2)\delta(P_H \cdot q). \quad (5.17)
$$

We then change all delta functions to propagator denominators [51] by using

$$
\delta(x) = \frac{i}{2\pi} \lim_{\varepsilon \to 0} \left(\frac{1}{x + i\varepsilon} - \frac{1}{x - i\varepsilon} \right). \quad (5.18)
$$

Then the obtained new Feynman integrals can be calculated analytically by using established multi-loop calculation techniques, like the IBP reduction [52–56] and differential equations [57–62], which results in

$$
D_{g \to Q\bar{Q}[SS]}^{\mathrm{NLO}}(z, M_H, m_Q, \mu)\Big|_{\mathrm{vir.}}^{a+b} = \bar{D}^{\mathrm{LO}}(M_H, m_Q, \mu)\delta(1-z)\frac{\alpha_s}{2\pi}\frac{N_c^2-1}{2N_c}\left(\frac{4\pi\mu^2 e^{-\gamma_E}}{m_Q^2}\right)^\epsilon
$$
$$
\times\left[-\frac{3}{\epsilon}-4\right], \tag{5.19a}
$$

$$
D_{g \to Q\bar{Q}[SS]}^{\mathrm{NLO}}(z, M_H, m_Q, \mu)\Big|_{\mathrm{vir.}}^{c} = \bar{D}^{\mathrm{LO}}(M_H, m_Q, \mu)\delta(1-z)\frac{\alpha_s}{2\pi}\left(-\frac{1}{2N_c}\right)\left(\frac{4\pi\mu^2 e^{-\gamma_E}}{m_Q^2}\right)^\epsilon
$$
$$
\times\left[\frac{1}{\epsilon}\left(1+\frac{1+\Delta^2}{\Delta}\ln\frac{1+\Delta}{1-\Delta}\right)+\mathcal{A}(\Delta)\right], \tag{5.19b}
$$

$$
D_{g \to Q\bar{Q}[SS]}^{\mathrm{NLO}}(z, M_H, m_Q, \mu)\Big|_{\mathrm{vir.}}^{d} = \bar{D}^{\mathrm{LO}}(M_H, m_Q, \mu)\delta(1-z)\frac{\alpha_s}{2\pi}\frac{N_c}{2}\left(\frac{4\pi\mu^2 e^{-\gamma_E}}{m_Q^2}\right)^\epsilon
$$
$$
\times\left[\frac{3}{\epsilon}+\mathcal{B}(\Delta)\right], \tag{5.19c}
$$

$$
D_{g \to Q\bar{Q}[SS]}^{\mathrm{NLO}}(z, M_H, m_Q, \mu)\Big|_{\mathrm{vir.}}^{e} = \bar{D}^{\mathrm{LO}}(M_H, m_Q, \mu)\delta(1-z)\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2 e^{-\gamma_E}}{M_H^2}\right)^\epsilon
$$
$$
\times\left[\frac{5N_c-2n_f}{3}\frac{1}{\epsilon}+\frac{31N_c-10n_f}{9}\right], \tag{5.19d}
$$

$$
D_{g \to Q\bar{Q}[SS]}^{\mathrm{NLO}}(z, M_H, m_Q, \mu)\Big|_{\mathrm{vir.}}^{f} = \bar{D}^{\mathrm{LO}}(M_H, m_Q, \mu)\delta(1-z)\frac{\alpha_s}{2\pi}N_c\left(\frac{4\pi\mu^2 e^{-\gamma_E}}{M_H^2}\right)^\epsilon
$$
$$
\times\left[\frac{2}{\epsilon^2}+\frac{1}{\epsilon}+2-\frac{7\pi^2}{6}\right], \tag{5.19e}
$$

$$
D_{g \to Q\bar{Q}[SS]}^{\mathrm{NLO}}(z, M_H, m_Q, \mu)\Big|_{\mathrm{vir.}}^{g+h} = \bar{D}^{\mathrm{LO}}(M_H, m_Q, \mu)\delta(1-z)\frac{\alpha_s}{2\pi}N_c\left(\frac{4\pi\mu^2 e^{-\gamma_E}}{M_H^2}\right)^\epsilon
$$
$$
\times\bigg[-\frac{1}{\epsilon^2}-2\bigg(1-\frac{1}{2\Delta}\ln\frac{1+\Delta}{1-\Delta}+\frac{M_H^2}{12(M_H+2m_Q)(M_H+m_Q)}
$$
$$
\times\bigg(3-2\Delta^2-\frac{3(1-\Delta^2)}{2\Delta}\ln\frac{1+\Delta}{1-\Delta}\bigg)\bigg)\frac{1}{\epsilon}+\mathcal{C}(\Delta)\bigg]. \tag{5.19f}
$$

$$
D_{g \to Q\bar{Q}[SS]}^{\mathrm{NLO}}(z, M_H, m_Q, \mu)\Big|_{\mathrm{vir.}}^{i} = 0. \tag{5.19g}
$$

In the above results we have dropped imaginary parts that are irrelevant for our purpose. The functions $\mathcal{A}(\Delta), \mathcal{B}(\Delta)$ and $\mathcal{C}(\Delta)$ are infrared finite, but $\mathcal{A}(\Delta)$ contains Coulomb divergence.

Feynman diagrams for the real corrections to $D_{g \to Q\bar{Q}[SS]}^{\mathrm{NLO}}$ are shown in figure 6, where only diagrams on the left-hand side of the cut are shown. For real corrections, the final state phase space in eq. (5.12) reads

$$
d\Phi^{\mathrm{NLO}} = \frac{d^d P_c}{(2\pi)^d}\frac{d^d k}{(2\pi)^d}(2\pi)^d\delta^d(P_c-P_H-k)\,\delta\left(P_c^+ - \frac{P_H^+}{z}\right)2\pi\delta(k^2)\theta(k^+), \tag{5.20}
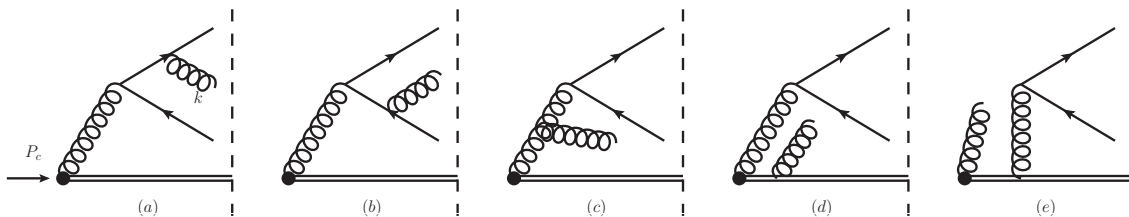$$

Figure 6. Feynman diagrams for real correction of the fragmentation function at NLO.

where k is the momentum of the final-state gluon. In the calculation, we first integrate out the solid angle. To do this, we decompose the amplitude $\mathcal{M}_L^\rho$ into the following form

$$
\mathcal{M}_L^\rho(P_H, P_c, m_Q, k, \lambda) = M_{L0}^\rho \int \frac{d^{d-2}\Omega}{N_\Omega} \frac{1}{(P_H + 2q) \cdot k} + M_{L0}'^\rho \int \frac{d^{d-2}\Omega}{N_\Omega} \frac{1}{(P_H - 2q) \cdot k}
$$
$$
+ M_{L1}^{\rho\mu} \int \frac{d^{d-2}\Omega}{N_\Omega} \frac{q_\mu}{(P_H + 2q) \cdot k} + M_{L1}'^{\rho\mu} \int \frac{d^{d-2}\Omega}{N_\Omega} \frac{q_\mu}{(P_H - 2q) \cdot k}
$$
$$
+ M_{L2}^{\rho\mu\nu} \int \frac{d^{d-2}\Omega}{N_\Omega} \frac{q_\mu q_\nu}{(P_H + 2q) \cdot k} + M_{L2}'^{\rho\mu\nu} \int \frac{d^{d-2}\Omega}{N_\Omega} \frac{q_\mu q_\nu}{(P_H - 2q) \cdot k}
$$
$$
+ M_{L3}^{\rho\mu\nu\tau} \int \frac{d^{d-2}\Omega}{N_\Omega} \frac{q_\mu q_\nu q_\tau}{(P_H + 2q) \cdot k} + M_{L3}'^{\rho\mu\nu\tau} \int \frac{d^{d-2}\Omega}{N_\Omega} \frac{q_\mu q_\nu q_\tau}{(P_H - 2q) \cdot k}, \tag{5.21}
$$

where the functions M_{Li} and M_{Li}' are q independent. Then the integral over Ω can be performed by using equations provided in appendix B. After that, we carry out the integration over the phase space $d\Phi^{\text{NLO}}$, which results in

$$
D_{g \to Q\bar{Q}[SS]}^{\text{NLO}}(z, M_H, m_Q, \mu)\Big|_{\text{real}} = \bar{D}^{\text{LO}}(M_H, m_Q, \mu) \left(\frac{4\pi\mu^2 e^{-\gamma_E}}{M_H^2}\right)^\epsilon \frac{\alpha_s N_c}{\pi}
$$
$$
\times \Bigg[- \frac{\delta(1-z)}{2\epsilon^2} - \frac{1}{2\epsilon} \frac{4\delta(1-z)}{3(M_H + 2m_Q)^2(M_H + m_Q)^2} \Big(m_Q^4(9\mathcal{T}^2 - 9\mathcal{T} - 1)
$$
$$
+ \frac{3}{2} m_Q^3 M_H(3\mathcal{T} - 8) - 10 m_Q^2 M_H^2 - \frac{39}{8} m_Q M_H^3 - \frac{13}{16} M_H^4 \Big)
$$
$$
+ \frac{1}{\epsilon} \frac{1}{(1-z)_+} \frac{(z^2 - z + 1)^2}{z} + \mathcal{R}(z, M_H, \Delta) \Bigg], \tag{5.22}
$$

where $\mathcal{R}(z, M_H, \Delta)$ is infrared finite.

Combining the virtual and real corrections in eq. (5.19) and eq. (5.22), we obtain NLO correction for the FF $D_{g \to Q\bar{Q}[SS]}$, which contains ultraviolet and infrared divergences in the form of single poles in ϵ as shown below

$$
D_{g \to Q\bar{Q}[SS]}^{\text{NLO}}(z, M_H, m_Q, \mu)\Big|_{div.} = \bar{D}^{\text{LO}}(M_H, m_Q, \mu) \frac{\alpha_s}{\pi}(4\pi\mu^2 e^{-\gamma_E})^\epsilon
$$
$$
\times \Bigg\{ \frac{1}{\epsilon} \Bigg[\beta_0 \delta(1-z) + \frac{N_c}{(1-z)_+} \frac{(z^2 - z + 1)^2}{z} \Bigg] - \frac{1}{\epsilon} \delta(1-z) N_c
$$
$$
\times \Bigg[1 - \frac{1}{2\Delta} \ln \frac{1+\Delta}{1-\Delta} + \frac{M_H^2}{12(2m_Q^2 + 3m_Q M_H + M_H^2)} \Big(3 - 2\Delta^2 - \frac{3(1-\Delta^2)}{2\Delta} \ln \frac{1+\Delta}{1-\Delta} \Big) \Bigg]
$$
$$
- \frac{1}{\epsilon} \frac{2N_c \delta(1-z)}{3(M_H + 2m_Q)^2(M_H + m_Q)^2} \Bigg[m_Q^4(9\mathcal{T}^2 - 9\mathcal{T} - 1) + \frac{3}{2} m_Q^3 M_H(3\mathcal{T} - 8) - 10 m_Q^2 M_H^2
$$
$$
- \frac{39}{8} m_Q M_H^3 - \frac{13}{16} M_H^4 \Bigg] - \delta(1-z) \Bigg[\frac{N_c}{2} + \frac{1}{2N_c} \Big(\frac{1+\Delta^2}{2\Delta} \ln \frac{1+\Delta}{1-\Delta} - 1 \Big) \Bigg] \frac{1}{\epsilon} \Bigg\}. \tag{5.23}
$$

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Ultraviolet divergences in the above equation are canceled by the renormalization of the coupling constant α_s and the operator defining the FF. The renormalization of α_s in the $\overline{\text{MS}}$ scheme can be carried out by making the following replacement in eq. (5.15)

$$\alpha_s \to \alpha_s \left(1 - \frac{\alpha_s}{2\pi} (4\pi e^{-\gamma_E})^\epsilon \beta_0 \frac{1}{\epsilon} \right), \tag{5.24}$$

and the operator renormalization in the $\overline{\text{MS}}$ scheme can be carried out by further making the following replacement

$$D^{\text{LO}}_{g \to Q\bar{Q}[SS]}(z, M_H, m_Q, \mu) \to D^{\text{LO}}_{g \to Q\bar{Q}[SS]}(z, M_H, m_Q, \mu) - \frac{\alpha_s}{2\pi}(4\pi e^{-\gamma_E})^\epsilon \frac{1}{\epsilon}$$
$$\times \int_z^1 \frac{dx}{x} P^{\text{LO}}_{gg}(x) D^{\text{LO}}_{g \to Q\bar{Q}[SS]}(z/x, M_H, m_Q, \mu). \tag{5.25}$$

Then the renormalized $D_{g \to Q\bar{Q}[SS]}$ reads

$$D^{\text{NLO}}_{g \to Q\bar{Q}[SS]}(z, M_H, m_Q, \mu) = \bar{D}^{\text{LO}}(M_H, m_Q, \mu) \frac{\alpha_s}{\pi} (4\pi \mu_c^2 e^{-\gamma_E})^\epsilon \left\{ -\frac{1}{\epsilon_{\text{IR}}} N_c \delta(1-z) \right.$$
$$\times \left[1 - \frac{1}{2\Delta} \ln \frac{1+\Delta}{1-\Delta} + \frac{M_H^2}{12(2m_Q^2 + 3m_Q M_H + M_H^2)} \left(3 - 2\Delta^2 - \frac{3(1-\Delta^2)}{2\Delta} \ln \frac{1+\Delta}{1-\Delta} \right) \right]$$
$$- \frac{1}{\epsilon_{\text{IR}}} \frac{2N_c \delta(1-z)}{3(M_H + 2m_Q)^2 (M_H + m_Q)^2} \left[m_Q^4 (9\mathcal{T}^2 - 9\mathcal{T} - 1) + \frac{3}{2} m_Q^3 M_H (3\mathcal{T} - 8) \right.$$
$$\left. - 10m_Q^2 M_H^2 - \frac{39}{8} m_Q M_H^3 - \frac{13}{16} M_H^4 \right] - \frac{1}{\epsilon_{\text{IR}}} \left[\frac{N_c}{2} + \frac{1}{2N_c} \left(\frac{1+\Delta^2}{2\Delta} \ln \frac{1+\Delta}{1-\Delta} - 1 \right) \right]$$
$$\left. \times \delta(1-z) + \mathcal{F}(z, M_H, \Delta, \mu) \right\}. \tag{5.26}$$

Here we have distinguished between the dimensional regularization scale μ_c and the collinear factorization scale μ. All remaining divergences in the above equation are infrared divergences, and they should be absorbed into the definition of SGDs if the SGF is valid at NLO. The infrared finite functions $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{R}, \mathcal{F}$ in above equations are listed in appendix C. They also can be read from the supplementary material.

5.3 Matching the short-distance hard part

Inserting eqs. (5.15), (5.26), (3.13), (3.14), (3.25) and (3.26) into the matching equation eq. (5.11), we obtain the hard part $\hat{D}_{[SS]}$ up to NLO. The result is listed in appendix C and we also provide it in the supplementary material. In the result we have chose $\mu_f = \mu$. We find that all infrared and Coulomb divergences in eq. (5.26) can be correctly subtracted by the SGDs. Thus we conclude that the SGF formula eq. (5.10) holds at least at the one-loop level.

Based on $\hat{D}_{[SS]}$, we can easily obtain the short distance hard part at leading order in the velocity expansion, which reads

$$\hat{D}_{[SS]}^{\text{LO},(0)}(\hat{z}, M_H/x, \mu, \mu_f) = \frac{\pi\alpha_s}{(N_c^2 - 1)}\frac{8x^3}{M_H^3}\delta(1 - \hat{z}), \tag{5.27a}$$

$$\hat{D}_{[SS]}^{\text{NLO},(0)}(\hat{z}, M_H/x, \mu, \mu_f)$$
$$= \frac{4\alpha_s^2 N_c x^3}{(N_c^2 - 1)M_H^3}\left[\frac{1}{2}\delta(1 - \hat{z})\left(2A(\mu, M_H/x) + \frac{2\beta_0}{N_c}\ln\left(\frac{x^2\mu_f^2 e^{-1}}{M_H^2}\right) + \ln^2\left(\frac{x^2\mu_f^2 e^{-1}}{M_H^2}\right)\right.\right.$$
$$\left. + \frac{\pi^2}{6} - 1\right) + \frac{1}{N_c}P_{gg}^{(0)}(\hat{z})\ln\left(\frac{\mu^2}{\mu_f^2}\right) + \left(\frac{2(1 - \hat{z})}{\hat{z}} + \hat{z}(4 + 2\hat{z}^2) + \frac{2\hat{z}^4}{9}(5 + \hat{z})\right)$$
$$\left.\times\left(\ln\left(\frac{x^2\mu_f^2 e^{-1}}{M_H^2}\right) - 2\ln(1 - \hat{z})\right) + \frac{2(1 - \hat{z})}{\hat{z}} - \left(\frac{4\hat{z}^4}{1 - \hat{z}} - \frac{4\hat{z}^4}{9}(5 + \hat{z})\right)\ln\hat{z}\right], \tag{5.27b}$$

where

$$A(\mu, M_H/x) = \frac{\beta_0}{N_c}\left[\ln\left(\frac{x^2\mu^2}{M_H^2}\right) + \frac{13}{3}\right] + \frac{4}{N_c^2} - \frac{\pi^2}{3} + \frac{16}{3}\ln 2. \tag{5.28}$$

The factorization scale μ_f should be chosen at the order of M_H in order to avoid the appearance of large logarithms of M_H/μ_f. Besides, we find the hard part is also free of the threshold logarithms in the $\hat{z} \to 1$ limit with the scale choice $\mu_f = \mu = M_H$. The dependence of μ and μ_f in hard part can be recovered by RGEs obeyed by the gluon FF and the color-octet $^3S_1$ SGD.

We eventually obtain gluon FFs in $^3S_1^{[8]}$ channel

$$D_{g \to H}(z, M_H, m_Q, M_H) = \int_z^1 \frac{dx}{x}\hat{D}_{[SS]}(\hat{z}, M_H/x, m_Q, M_H, M_H)$$
$$\times F_{[SS] \to H}(x, M_H, m_Q, M_H), \tag{5.29a}$$

$$D_{g \to H}^{(0)}(z, M_H, m_Q, M_H) = \int_z^1 \frac{dx}{x}\hat{D}_{[SS]}^{(0)}(\hat{z}, M_H/x, M_H, M_H)$$
$$\times F_{[SS] \to H}(x, M_H, m_Q, M_H). \tag{5.29b}$$

Using the DGLAP equation (5.3) and initial condition eq. (5.29), we can obtain the gluon FFs at any larger scale.

5.4 Numerical results

We now present our numerical results for the gluon FFs given in eq. (5.29). We use the model function eq. (4.17) with overall normalization $N_H = \langle O^H(^3S_1^{[8]})\rangle/3 = 1/3$ and $b = 2$. We choose $m_Q = 1.4\text{GeV}$, $M_H = 3.1\text{GeV}$, $\Lambda_{QCD}^{(4)} = 0.217\text{GeV}$ ($\Lambda_{QCD}^{(4)} = 0.338\text{GeV}$) for LO (NLO), and $n_f = 3$ in β_0 which means contributions from virtual or initial heavy quarks are ignored. In the following, in not specified we refer to fully NLO FF $D_{g \to H}(z, M_H, m_Q, M_H)$ calculated in SGF with $\bar{\Lambda} = 0.6\text{GeV}$. We note that, as NLO evolution kernel for SGD is still not available, all results presented in this paper are obtained by using the LO evolution kernel (4.4).
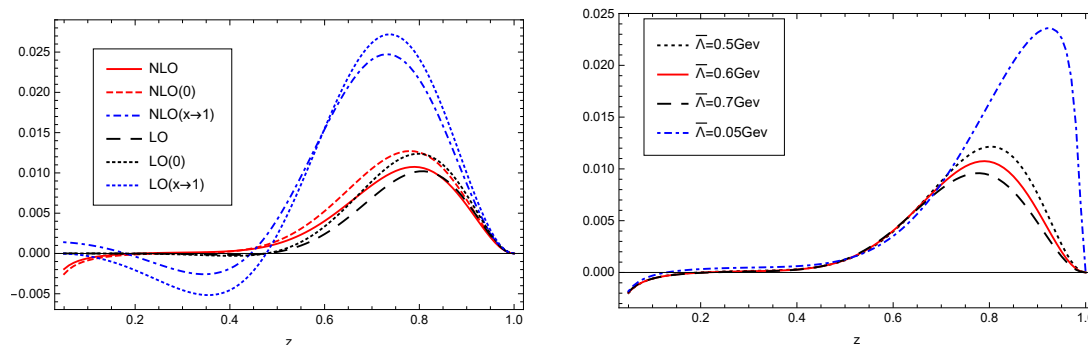
Figure 7. Left figure: comparison of the gluon FF obtained in different approximations. Right figure: $\bar{\Lambda}$ dependence of gluon FF at NLO.

In the left figure of figure 7 we show the gluon FF obtained by different approximations, where LO(0) and NLO(0) represent the lowest order in v^2 expansion $D_{g\to H}^{(0)}$ at the corresponding α_s order. We find that $D_{g\to H}^{(0)}$ is a good approximation to $D_{g\to H}$, with deviation smaller than 20%. This implies that the convergence of velocity expansion in SGF is good and that $D_{g\to H}$ is insensitive to the input of heavy quark mass m_Q. For comparison, we also provide plots of FFs obtained by setting $x = 1$ in hard part (5.27) everywhere except in the delta function $\delta(1 - z/x)$. This is clear a bad approximation because it overshoots original results by at least a factor of 2, which indicates big corrections at high orders in $1 - x$ expansion.[2] This can be easily understood because the hard part are proportional to x^3 and SGD is peaked around $x = 0.75$. We note that, if one uses the NRQCD+SCET method [33–36] to calculate the gluon FF, one should take $x \to 1$ in hard part, which will result in large relativistic corrections as discussed above.

In the right figure of figure 7 we show the $\bar{\Lambda}$ dependence by choosing it to 0.05GeV, 0.5GeV, 0.6GeV and 0.7GeV. We find that the gluon FF at small z is much less affected by the parameter $\bar{\Lambda}$ than those at large z, which indicates that gluon FF at small z is dominated by perturbative effects while that at large z is sensitive to nonperturbative dynamics. This sensitivity provides a possibility to extract $\bar{\Lambda}$ using experimental data.

We note that the plot with $\bar{\Lambda} = 0.05$GeV in figure 7 is to mimic the case with $\bar{\Lambda} \to 0$. Although nonperturbative input models with $\bar{\Lambda} = 0.6$GeV and 0.05GeV have normalized to the same value, they result in significantly different FFs, mainly due to the significantly different $\bar{\Lambda}$. To quantify the difference, we calculate the n-th moment of FFs and define the following ratios,

$$R^X(n) \equiv \frac{\int_0^1 dz\, z^n D_{g\to H}^X(z, M_H, m_Q, \mu)}{\int_0^1 dz\, z^n D_{g\to H}(z, M_H, m_Q, \mu)}, \tag{5.30}$$

with numerical results given in table 2 for $n = 2, 3, 4, 5, 6$. By taking $D_{g\to H}$ as "exact" result, the values of R indicate that the lowest order in velocity expansion in SGF (denoted

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[2]These are also velocity corrections when $1 - x \approx v$ or v^2. Resumming this series of velocity corrections is the main motivation of SGF.

Factor	z^2	z^3	z^4	z^5	z^6
$R^{(0)}$	1.18	1.18	1.17	1.16	1.16
$R^{0.05}$	2.68	2.91	3.16	3.42	3.69
$R^{0.05(\text{pert.})}$	7.81	7.71	7.36	6.70	5.63
R^{NRQCD}	7.54	7.48	7.16	6.49	5.34

Table 2. The ratios of moments of fragmentation functions in different approaches defined in eq. (5.30). $R^{(0)}$ represents FF with lowest order in v^2 expansion, $R^{0.05}$ represents FF with $\bar\Lambda = 0.05\text{GeV}$, $R^{0.05(\text{pert.})}$ represents perturbative expansion of FF with $\bar\Lambda = 0.05\text{GeV}$ and also replacing M_H by $2m_Q$, and R^{NRQCD} represents FF calculated in NRQCD.
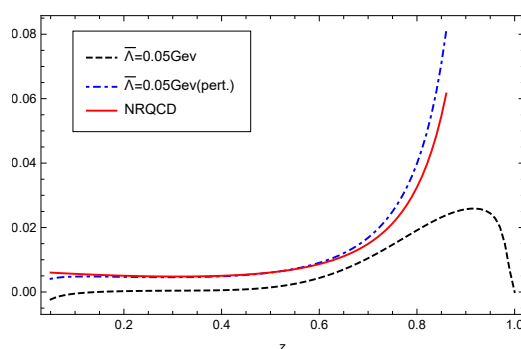

Figure 8. Comparison of the gluon FFs obtained in SGF and NRQCD, where "pert." means expanding $D^{\bar\Lambda=0.05\text{GeV}}_{g\to H}$ to NLO in α_s and replacing M_H by $2m_Q$.

as $R^{(0)}$) is a good approximation, while changing $\bar\Lambda$ from 0.6GeV to 0.05GeV (denoted as $R^{0.05}$) results in large deviation, about a factor of 3.

As shown in figure 8, by perturbatively expanding $D^{\bar\Lambda=0.05\text{GeV}}_{g\to H}$ to NLO in α_s and replacing M_H by $2m_Q$ (denoted as "pert."), we can nicely recover NRQCD result [30–32]. Their deviations at end points of z are due to the nonzero $\bar\Lambda$, but they have almost the same moments as shown in table 2, where R^{NRQCD} means that the numerator of the ratio is the NRQCD FFs. We thus find that the NRQCD results overshoot SGF results with our prefered treatment by a factor about $5-7$, which is consistent with the conclusion in ref. [26]. One of the main reasons for deviation is due to the large logarithm resummation, and the other one is due to the choice of $\bar\Lambda = 0.6\text{GeV}$ instead of zero.

6 Summary

In this paper we studied the FF from the gluon to color-octet $^3S_1$ heavy quark-antiquark pair in SGF approach, which is expressed as the convolution of perturbative short-distance hard part with the one-dimensional color-octet $^3S_1$ SGD in eq. (5.9). We calculated the color-octet $^3S_1$ SGD to NLO in eq. (3.25) and derived a RGE for it in eq. (4.1). By solving the RGE, we resummed the threshold logarithms to all order in perturbation theory. Based

on this, we perturbatively calculated the short distance hard part of the gluon FF up to NLO in α_s and all orders in velocity expansion in eq. (C.6). With a natural scales choice, the hard part is free of threshold logarithms, which has been resummed by the RGE of SGD. This demonstrated the validity of SGF at NLO.

Our numerical results shown that velocity expansion in the SGF has a good convergence, and the lowest order approximation can already capture most physics. By a specific choice of parameters (but not prefered), the SGF can reproduce the NRQCD results. But then we get a big deviation, as large as a factor of 6, from our prefered results. This implies that velocity expansion in NRQCD has a very slow convergence. We also show that if the large threshold logarithms are resumed in NRQCD+SCET approach, there should be still significant velocity corrections as discussed below figure 7.

Our results may provide a new insight to understand the mechanisms of quarkonium production, especially for quarkonium production at high energy colliders and quarkonium production within a jet [63, 64].

Acknowledgments

A Solving RGE of SGD

Here we give the details of solving the RGE in eq. (4.11). Integrating eq. (4.11) from a initial scale μ_0 to μ_f by changing variables to α_s with $d\ln\mu = d\alpha_s/\beta(\alpha_s)$ gives the solution

$$\tilde{F}_{[SS]\to H}(s, M_H, m_Q, \mu_f) = \exp\left[\eta(\mu_f, \mu_0)\ln(\bar{s}\mu_0)\right]e^{V(\mu_f, \mu_0)}$$
$$\times \tilde{F}_{[SS]\to H}(s, M_H, m_Q, \mu_0), \tag{A.1}$$

the evolution functions are given by

$$\eta(\mu_f, \mu_0) = -\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu_f)} \frac{d\alpha}{\beta(\alpha)} \frac{\alpha_s \Gamma_0^F}{4\pi},$$
$$V(\mu_f, \mu_0) = -\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu_f)} \frac{d\alpha}{\beta(\alpha)} \frac{\alpha_s \gamma_0^F}{4\pi} - \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu_f)} \frac{d\alpha}{\beta(\alpha)} \frac{\alpha_s \Gamma_0^F}{4\pi} \int_{\alpha_s(\mu_0)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')}. \tag{A.2}$$

Defining $r = 1 - \alpha_s(\mu_f)/\alpha_s(\mu_0)$ and inserting $\beta(\alpha_s) = -\beta_0\alpha_s^2/\pi + \mathcal{O}(\alpha_s^3)$ into eq. (A.2), we obtain following expression for the evolution functions

$$\eta(\mu_f, \mu_0) = \frac{\Gamma_0^F}{4\beta_0}\ln(1 - r),$$
$$V(\mu_f, \mu_0) = \frac{\Gamma_0^F}{(4\beta_0)^2}\left[-\frac{4\pi}{\alpha_s(\mu)}\left(r + (1 - r)\ln(1 - r)\right)\right] + \frac{\gamma_0^F}{4\beta_0}\ln(1 - r). \tag{A.3}$$

With the help of the relation eq. (A.1), the color-octet $^3S_1$ SGD can be evolved to the scale μ_f from a reference scale μ_0, at which a nonperturbative model for the SGD is provided. And the exponential in eq. (A.1) resums the single and double logarithms of μ_f/μ_0.

To determine the initial scale μ_0, we expand the m_Q^2 in perturbative SGD $F_{[SS]\to Q\bar{Q}[SS]}$ around $M_H^2/4$, we have

$$F_{[SS]\to Q\bar{Q}[SS]}(\omega, M_H, m_Q, \mu_f) = M_H \left\{ \delta(\omega) - \frac{\alpha_s N_c}{4\pi} \left[\left(4\ln^2\mu_f - 4\ln\mu_f + \frac{\pi^2}{6} + \frac{1}{N_c^2}\frac{\pi^2}{\Delta} \right) \right. \right.$$
$$\left. \left. \times \delta(\omega) + \left[\frac{4}{\omega}\right]_+ + 8\left[\frac{\ln(\omega/\mu_f)}{\omega}\right]_+ \right] \right\} + \mathcal{O}(\alpha_s\Delta), \qquad (A.4)$$

which result in

$$\tilde{F}_{[SS]\to Q\bar{Q}[SS]}(s, M_H, m_Q, \mu_0) = M_H \left[1 - \frac{\alpha_s N_c}{4\pi} \left(4\ln^2(\bar{s}\mu_0) - 4\ln(\bar{s}\mu_0) + \frac{5\pi^2}{6} + \frac{1}{N_c^2}\frac{\pi^2}{\Delta} \right) \right]$$
$$+ \mathcal{O}(\alpha_s\Delta). \qquad (A.5)$$

We can find, in the Laplace space, the fixed-order perturbative expansion of $\tilde{F}_{[SS]\to Q\bar{Q}[SS]}$ contains single and double logarithms of $\bar{s}\mu_0$ and these large logarithms can be minimized by the choice of scale

$$\mu_0 = 1/\bar{s}. \qquad (A.6)$$

Making the above characteristic scale choice in eq. (A.1), we can resum those large logarithms of s and obtain the resummed SGD, which reads

$$\tilde{F}_{[SS]\to H}(s, M_H, m_Q, \mu_f = M_H) = \tilde{F}_{[SS]\to H}(s, M_H, m_Q, 1/\bar{s})$$
$$\times \exp\left[h_0(\chi)\ln(M_H\bar{s}) + h_1(\chi) \right], \qquad (A.7)$$

where

$$\chi = \frac{\beta_0}{2\pi}\alpha_s(M_H)\ln(M_H\bar{s}). \qquad (A.8)$$

The functions $h_{0,1}$ are given by

$$h_0(\chi) = \frac{\Gamma_0^F}{8\beta_0\chi}\left(-2\chi - (1-2\chi)\ln(1-2\chi) \right),$$
$$h_1(\chi) = \frac{\gamma_0^F}{4\beta_0}\ln(1-2\chi). \qquad (A.9)$$

Note that the term $\ln(1-2\chi)$ in above expressions gives rise to a cut singularity that start at the branch point

$$\bar{s}_L = \frac{1}{M_H}\exp\left[\frac{\pi}{\beta_0\alpha_s(M_H)} \right] \simeq \frac{1}{\Lambda_{\rm QCD}}. \qquad (A.10)$$

This singularity, as a result of the divergence of the running coupling $\alpha_s(\mu)$ near the Landau pole at $\mu = \Lambda_{\rm QCD}$, signals the onset of nonperturbative phenomena at very large s or,

equivalently, when x is very close to the value $x = 1$. To deal with the Landau singularity, we introduce a cut-off $\bar{s}^*$ above which the functions $h_{0,1}$ remained constant, i.e. "frozen" at $\bar{s}^*$. Additionally, a nonperturbative model was introduced to correct the formalism for nonperturbative effects at very large s. Following [65], we make the replacement

$$\bar{s} \to \bar{s}\frac{1 + 1/(M_H\bar{s}^*)}{1 + \bar{s}/\bar{s}^*}, \qquad \bar{s}^* = \bar{s}_L/a \tag{A.11}$$

in eq. (A.9). Here a is a parameter not smaller than one, but of order one. Such a replacement prevents χ from entering the nonperturbative regime. Then we model the initial SGD $\tilde{F}_{[SS]\to H}(s, M_H, m_Q, 1/\bar{s})$ in following form

$$\tilde{F}_{[SS]\to H}(s, M_H, m_Q, 1/\bar{s}) = \tilde{F}_{\text{fix}}(M_H)\tilde{F}^{\text{mod}}(s), \tag{A.12}$$

with

$$\tilde{F}_{\text{fix}}^{\text{LO}}(M_H) = 1, \tag{A.13a}$$

$$\tilde{F}_{\text{fix}}^{\text{NLO}}(M_H) = -\frac{\alpha_s(M_H)N_c}{4\pi}\frac{5\pi^2}{6}. \tag{A.13b}$$

Here $\tilde{F}_{\text{fix}}(M_H)$ is introduced to recover the perturbative calculated $\tilde{F}_{[SS]\to Q\bar{Q}[SS]}$ with the Coulomb divergences and IR divergences (if they exist) be subtracted, as we have

$$\tilde{F}_{[SS]\to Q\bar{Q}[SS]}(s, M_H, m_Q, 1/\bar{s}) = M_H\left[1 - \frac{\alpha_s N_c}{4\pi}\left(\frac{5\pi^2}{6} + \frac{1}{N_c^2}\frac{\pi^2}{\Delta}\right)\right] + \mathcal{O}(\alpha_s\Delta). \tag{A.14}$$

And the model $\tilde{F}^{\text{mod}}(s)$ is introduced to describe the nonperturbative effects.

B Integration over solid angle

In this appendix, we give the expressions that integrate over the solid angle Ω of q in the $Q\bar{Q}$ rest frame. For the momentum k which satisfies $k^2 = 0$, we have following equations,

$$\int d^{d-2}\Omega\frac{1}{(P_H - 2q)\cdot k} = \int d^{d-2}\Omega\frac{1}{(P_H + 2q)\cdot k} = \frac{\mathcal{T}}{P_H\cdot k}\int d^{d-2}\Omega, \tag{B.1a}$$

$$\int d^{d-2}\Omega\frac{q^\mu}{(P_H - 2q)\cdot k} = \int d^{d-2}\Omega\frac{-q^\mu}{(P_H + 2q)\cdot k} = \mathcal{F}_1\mathbb{P}^{\mu\nu}k_\nu\int d^{d-2}\Omega, \tag{B.1b}$$

$$\int d^{d-2}\Omega\frac{q^\mu q^\rho}{(P_H - 2q)\cdot k} = \int d^{d-2}\Omega\frac{q^\mu q^\rho}{(P_H + 2q)\cdot k}$$
$$= (\mathcal{F}_2\mathbb{P}^{\mu\rho} + \mathcal{G}_2\mathbb{P}^{\mu\nu}\mathbb{P}^{\rho\sigma}k_\nu k_\sigma)\int d^{d-2}\Omega, \tag{B.1c}$$

$$\int d^{d-2}\Omega\frac{q^\mu q^\rho q^\lambda}{(P_H - 2q)\cdot k} = \int d^{d-2}\Omega\frac{-q^\mu q^\rho q^\lambda}{(P_H + 2q)\cdot k}$$
$$= \Big(\mathcal{F}_3(\mathbb{P}^{\mu\rho}\mathbb{P}^{\lambda\sigma}k_\sigma + \mathbb{P}^{\mu\lambda}\mathbb{P}^{\rho\sigma}k_\sigma + \mathbb{P}^{\rho\lambda}\mathbb{P}^{\mu\sigma}k_\sigma)$$
$$+ \mathcal{G}_3\mathbb{P}^{\mu\nu}\mathbb{P}^{\rho\sigma}\mathbb{P}^{\lambda\tau}k_\nu k_\sigma k_\tau\Big)\int d^{d-2}\Omega. \tag{B.1d}$$

Where

$$\mathcal{T} = {}_2F_1\left(\frac{1}{2}, 1, \frac{3}{2} - \epsilon, \Delta^2\right), \tag{B.2a}$$

$$\mathcal{F}_1 = \frac{P_H^2(\mathcal{T} - 1)}{2(P_H \cdot k)^2}, \tag{B.2b}$$

$$\mathcal{F}_2 = -\frac{1}{d-2}\frac{P_H^2(\mathcal{T} - 1) + 4q^2\mathcal{T}}{4P_H \cdot k}, \tag{B.2c}$$

$$\mathcal{G}_2 = \frac{4q^2 P_H^2 \mathcal{T} + (d-1)(\mathcal{T} - 1)P_H^4}{4(d-2)(P_H \cdot k)^3}, \tag{B.2d}$$

$$\mathcal{F}_3 = -\frac{P_H^4}{2(d-2)(P_H \cdot k)^2}\left[\left(\frac{q^2}{P_H^2} + \frac{1}{4}\right)(\mathcal{T} - 1) + \frac{1}{d-1}\frac{q^2}{P_H^2}\right], \tag{B.2e}$$

$$\mathcal{G}_3 = \frac{d+1}{d-2}\frac{P_H^6}{2(P_H \cdot k)^4}\left[\left(\frac{3}{d+1}\frac{q^2}{P_H^2} + \frac{1}{4}\right)(\mathcal{T} - 1) + \frac{1}{d-1}\frac{q^2}{P_H^2}\right]. \tag{B.2f}$$

C Finite results

In this appendix, we provide the expressions of functions $\mathcal{A}(\Delta)$, $\mathcal{B}(\Delta)$, $\mathcal{C}(\Delta)$, $\mathcal{R}(z, M_H, \Delta)$, $\mathcal{F}(z, M_H, \Delta, \mu)$ and the NLO hard part in section 5.2:

$$\begin{aligned}
\mathcal{A}(\Delta) = {}&\frac{1}{4\Delta\left(\sqrt{1-\Delta^2}+1\right)\left(\sqrt{1-\Delta^2}+2\right)^4} \\
&\times \Bigg(\left(\Delta^2 + 1\right)\left(\left(\sqrt{1-\Delta^2}+9\right)\Delta^4 - 2\left(17\sqrt{1-\Delta^2}+37\right)\Delta^2 \right. \\
&+ 81\left(\sqrt{1-\Delta^2}+1\right)\Bigg)\left(8P_1 - 4P_2 + 4\ln\left(4\left(1-\Delta^2\right)\right)\tanh^{-1}(\Delta) + 4\pi^2\right) \\
&- 16\big(\Delta^6\left(-\sqrt{1-\Delta^2} + \sqrt{1-\Delta^2}\ln(2) - 10 + 9\ln(2)\right) \\
&+ \Delta^4\left(41\sqrt{1-\Delta^2} + \left(33\sqrt{1-\Delta^2}+65\right)(-\ln(2)) + 94\right) \\
&+ \Delta^2\left(-108\left(\sqrt{1-\Delta^2}+1\right) + 47\sqrt{1-\Delta^2}\ln(2) + 7\ln(2)\right) \\
&+ 81\left(\sqrt{1-\Delta^2}+1\right)\ln(2)\big)\tanh^{-1}(\Delta)\Bigg).
\end{aligned} \tag{C.1}$$

$$\begin{aligned}
\mathcal{B}(\Delta) = {}&\frac{1}{2\Delta\left(\sqrt{1-\Delta^2}+1\right)\left(\sqrt{1-\Delta^2}+2\right)^3} \\
&\times \Bigg(\left(\Delta^2 - 1\right)\left(-\Delta^2 + 4\sqrt{1-\Delta^2}+5\right)\left(2P_2 - \ln\left(\frac{\Delta+1}{1-\Delta}\right)\ln\left(-\frac{4}{\Delta^2-1}\right)\right) \\
&+ 8\sqrt{1-\Delta^2}\Delta^3\ln\left(\frac{1}{4}\left(1-\Delta^2\right)\right) + 8\Delta\left(\Delta^4 + 36\left(\sqrt{1-\Delta^2}+1\right)\right. \\
&- \Delta^2\left(8\sqrt{1-\Delta^2}+25 + 9\ln(2)\right) + \left(14\sqrt{1-\Delta^2}+13\right)\ln(2)\big) \\
&+ 4\left(9\Delta^2 - 14\sqrt{1-\Delta^2} - 13\right)\Delta\ln\left(1-\Delta^2\right)\Bigg).
\end{aligned} \tag{C.2}$$

$$\mathcal{C}(\Delta)$$

$$= \frac{1}{36\Delta\left(\sqrt{1-\Delta^2}+1\right)\left(\sqrt{1-\Delta^2}+2\right)^4}$$

$$\times \Bigg(-36P_1\left(3\left(\sqrt{1-\Delta^2}+8\right)\Delta^4 - 2\left(41\sqrt{1-\Delta^2}+84\right)\Delta^2 + 175\sqrt{1-\Delta^2}+176\right)$$

$$+ 18P_2\left(\left(\sqrt{1-\Delta^2}+9\right)\Delta^4 - 2\left(18\sqrt{1-\Delta^2}+43\right)\Delta^2 + 107\sqrt{1-\Delta^2}+109\right)$$

$$+ \Delta\Big(33\pi^2\left(\left(\sqrt{1-\Delta^2}+9\right)\Delta^4 - 2\left(17\sqrt{1-\Delta^2}+37\right)\Delta^2 + 81\left(\sqrt{1-\Delta^2}+1\right)\right)$$

$$- 8\left(\left(47\sqrt{1-\Delta^2}+369\right)\Delta^4 - \left(1211\sqrt{1-\Delta^2}+2371\right)\Delta^2 + 2340\sqrt{1-\Delta^2}+2358\right)\Big)$$

$$- 18\tanh^{-1}(\Delta)\Big(-4\left(\left(3\sqrt{1-\Delta^2}+22\right)\Delta^4 - 4\left(16\sqrt{1-\Delta^2}+27\right)\Delta^2\right.$$

$$+ 85\sqrt{1-\Delta^2}+86\big) - \left(5\left(\sqrt{1-\Delta^2}+9\right)\Delta^4 - 2\left(86\sqrt{1-\Delta^2}+191\right)\Delta^2\right.$$

$$+ 431\sqrt{1-\Delta^2}+433\Big)\ln\left(-\frac{4}{\Delta^2-1}\right) + 8\Delta\Big(\left(\sqrt{1-\Delta^2}+9\right)\Delta^4$$

$$- 2\left(17\sqrt{1-\Delta^2}+37\right)\Delta^2 + 81\left(\sqrt{1-\Delta^2}+1\right)\Big)\tanh^{-1}(\Delta)\Big)\Bigg). \tag{C.3}$$

$$\mathcal{R}(z, M_H, \Delta)$$

$$= -\frac{\delta(1-z)}{96\left(M_H+m_Q\right)^3\left(M_H+2m_Q\right)^2}\Big(16m_Q^4\left(3\mathcal{T}(29\mathcal{T}-46)+4\pi^2+60\right)M_H$$

$$+ 4m_Q^3\left(-114\mathcal{T}+25\pi^2+96\right)M_H^2 + 4_Q^2\left(-90\mathcal{T}+19\pi^2+66\right)M_H^3$$

$$+ \left(15+28\pi^2\right)m_Q M_H^4 + \left(15+4\pi^2\right)M_H^5 + 16m_Q^5\left(3\mathcal{T}(5\mathcal{T}-8)+\pi^2+12\right)\Big)$$

$$+ \frac{1}{3\left(M_H+m_Q\right)^2\left(M_H+2m_Q\right)^2}\Big(3m_Q^3 M_H\big(9\mathcal{T}^2(z-3)(z-1)z + z(z(20z-27)+3)$$

$$+ \mathcal{T}(6-3(z-1)z(13z-14))-12\big) + \frac{1}{4}m_Q^2 M_H^2\big(27\mathcal{T}^2(z-1)z(2z-3)$$

$$- 54\mathcal{T}(z-1)(z(5z-2)+2) + z(7z(8z+9)-159)-120\big)$$

$$- \frac{27}{2}m_Q(\mathcal{T}-1)(z-1)\ln(1-z)\left(\frac{M_H}{2}+m_Q\right)\left(2m_Q M_H(\mathcal{T}(-z)+\mathcal{T}+z+1)\right.$$

$$+ M_H^2 + 4m_Q^2(-\mathcal{T}(z-2)+z-1)\big) - \frac{3}{4}m_Q M_H^3\big(3\mathcal{T}(z-1)\left(3z^2+4\right)$$

$$+ z(z(17z-45)+48)+6\big) + \frac{1}{16}(z(13(9-4z)z-141)+24)M_H^4$$

$$+ m_Q^4\big(z(36\mathcal{T}^2((z-3)z+3)-18\mathcal{T}(z(5z-12)+9)+z(50z-99)+57)-12\big)\Big)$$

$$\times \left[\frac{1}{1-z}\right]_+ - \frac{2((z-1)z+1)^2}{z}\left[\frac{\ln(1-z)}{1-z}\right]_+. \tag{C.4}$$

$$\mathcal{F}(z, \Delta, M_H, \mu)$$

$$= \mathcal{R}(z, M_H, \Delta)N_c - \frac{1}{36\Delta N_c\left(M_H+m_Q\right)^2\left(M_H+2m_Q\right)^2}\delta(1-z)$$

$$\times \Big((M_H + m_Q)^2 (M_H + 2m_Q)^2 \Big(\Delta \big(9\mathcal{A}(\Delta) + N_c(20 n_f - N_c(9\mathcal{B}(\Delta) + 18\mathcal{C}(\Delta)$$

$$- 21\pi^2 + 62)) + 18 \ln \left(1 - \Delta^2 \right) + 12 \left((3P_3 + 8)N_c^2 - 2n_f N_c + 3 \right) \ln \left(M_H \right)$$

$$+ 6 N_c \ln \left(\mu^2 \right) (2n_f - 11 N_c) - 36(1 + \ln(2)) \Big)$$

$$- 18 \left(1 + \Delta^2 \right) \tanh^{-1}(\Delta) \left(\ln \left(\frac{1 - \Delta^2}{4} \right) + 2 \ln \left(M_H \right) \right) \Big)$$

$$+ 480 \Delta N_c^2 M_H^2 m_Q^2 \ln \left(M_H \right) + 234 \Delta N_c^2 M_H^3 m_Q \ln \left(M_H \right)$$

$$+ 48 \Delta N_c^2 (1 - 9(\mathcal{T} - 1)\mathcal{T}) m_Q^4 \ln \left(M_H \right) + 72 \Delta N_c^2 (8 - 3\mathcal{T}) M_H m_Q^3 \ln \left(M_H \right)$$

$$+ 39 \Delta N_c^2 M_H^4 \ln \left(M_H \right) \Big) - \frac{(z^2 - z + 1)^2}{z} N_c \ln \left(\frac{M_H^2}{\mu^2} \right) \left[\frac{1}{1 - z} \right]_+ . \tag{C.5}$$

$$\hat{D}_{[SS]}^{\mathrm{NLO}} (z, M_H, m_Q, \mu, \mu_f = \mu)$$

$$= \frac{32 \alpha_s^2 (M_H + m_Q)^2}{9 \left(N_c^2 - 1 \right) M_H^5} \Bigg\{ \mathcal{F}(z, \Delta, M_H, \mu) + \delta(1 - z) \Bigg[\frac{- \ln(4\pi \mu^2 e^{-\gamma_E})}{24 \Delta^2 N_c \left(M_H + m_Q \right)^2 \left(M_H + 2m_Q \right)^2}$$

$$\times \Big(12 \Delta \left(\left(1 + \Delta^2 \right) \tanh^{-1}(\Delta) - \Delta \right) (M_H + m_Q)^2 (M_H + 2m_Q)^2$$

$$- N_c^2 \Big(4\Delta^2 (39 P_3 + 1) M_H^2 m_Q^2 + 6\Delta^2 (12 P_3 + 1) M_H^3 m_Q$$

$$+ 24 \Delta M_H m_Q^3 \left(2\Delta - 3 \tanh^{-1}(\Delta) + 6\Delta P_3 \right) + \Delta^2 (12 P_3 + 1) M_H^4$$

$$+ 16 m_Q^4 \left(9 \left(\Delta - \tanh^{-1}(\Delta) \right) \tanh^{-1}(\Delta) + \Delta^2 (3 P_3 - 2) \right) \Big) \Big)$$

$$+ \frac{1}{288 \Delta^2 \left(\sqrt{1 - \Delta^2} + 1 \right)^2 \left(\sqrt{1 - \Delta^2} + 2 \right)^3 N_c} \Big(24 \Delta \tanh^{-1}(\Delta)$$

$$\times \Big(18 \left(1 - \Delta^2 \right) \left(\left(\sqrt{1 - \Delta^2} + 1 \right) \Delta^2 + \sqrt{1 - \Delta^2} - 1 \right) N_c^2 \ln \left(\frac{\mu^2}{M_H^2} \right)$$

$$+ 6 \ln(4\pi e^{-\gamma_E}) \Big(\left(\Delta^2 + 1 \right) \left(\left(\sqrt{1 - \Delta^2} + 8 \right) \Delta^4 - 27 \left(\sqrt{1 - \Delta^2} + 2 \right) \Delta^2 \right.$$

$$+ 54 \left(\sqrt{1 - \Delta^2} + 1 \right) \Big) - 2 \big(3 \left(\sqrt{1 - \Delta^2} + 4 \right) \Delta^4 - 32 \left(\sqrt{1 - \Delta^2} + 2 \right) \Delta^2$$

$$+ 57 \sqrt{1 - \Delta^2} + 60 \big) N_c^2 \Big) + N_c^2 \left(1 + \Delta^2 \right) \left(2\Delta^2 \left(4 \sqrt{1 - \Delta^2} + 65 \right) \right.$$

$$- 3 \left(23 \sqrt{1 - \Delta^2} + 76 \right) \Big) \Big) + 4\Delta \Big(6 \Delta N_c^2 \ln \left(\frac{\mu^2}{M_H^2} \right) \Big(3 \big(\left(\sqrt{1 - \Delta^2} + 8 \right) \Delta^4$$

$$- 27 \left(\sqrt{1 - \Delta^2} + 2 \right) \Delta^2 + 54 \left(\sqrt{1 - \Delta^2} + 1 \right) \Big) \ln \left(\frac{\mu^2}{M_H^2} \right)$$

$$- 2 \left(\left(\sqrt{1 - \Delta^2} + 26 \right) \Delta^4 - 9 \left(10 \sqrt{1 - \Delta^2} + 19 \right) \Delta^2 + 9 \left(20 \sqrt{1 - \Delta^2} + 19 \right) \right) \Big)$$

$$+ \Delta \Big(12 \ln(4\pi e^{-\gamma_E}) \Big(\Delta^4 \left(2 \left(5 \sqrt{1 - \Delta^2} + 28 \right) N_c^2 - 3 \left(\sqrt{1 - \Delta^2} + 8 \right) \right)$$

$$-3\left(\sqrt{1-\Delta^2}+2\right)\Delta^2\left(58N_c^2-27\right)+9\sqrt{1-\Delta^2}\left(37N_c^2-18\right)\Big)$$

$$+\left(4\left(5\sqrt{1-\Delta^2}+82\right)\Delta^4-12\left(28\sqrt{1-\Delta^2}+89\right)\Delta^2+369\sqrt{1-\Delta^2}\right)N_c^2\Big)$$

$$+18\Delta\left((228\ln(4\pi e^{-\gamma_E})+47)N_c^2-108\ln(4\pi e^{-\gamma_E})\right)+3\pi^2\left(\left(\sqrt{1-\Delta^2}+8\right)\Delta^4\right.$$

$$-27\left(\sqrt{1-\Delta^2}+2\right)\Delta^2+54\left(\sqrt{1-\Delta^2}+1\right)\Big)\left(\Delta\left(6\Delta+N_c^2\right)+6\right)\Big)$$

$$+9\left(\Delta^2-1\right)^2N_c^2\ln^2\left(\frac{1+\Delta}{1-\Delta}\right)\left(5\sqrt{1-\Delta^2}+12\left(\sqrt{1-\Delta^2}+2\right)\ln\left(\frac{4\pi\mu^2 e^{-\gamma_E}}{M_H^2}\right)\right.$$

$$\left.+58\right)\Big)\Big]+\frac{2z^4N_c\left(M_H+zm_Q\right)^2}{\left(M_H+m_Q\right)^2}\left[\frac{\ln(1-z)}{1-z}\right]_++\frac{z^4N_c\left(M_H+zm_Q\right)^2}{12\left(M_H+m_Q\right)^4\left(M_H+2m_Q\right)^2}$$

$$\times\left[\frac{1}{1-z}\right]_+\left[-16m_Q^4\left(3\ln\left(\frac{\mu^2z^2}{M_H^2}\right)+\frac{9}{4\Delta^2}\ln^2\left(\frac{1+\Delta}{1-\Delta}\right)-9\frac{\tanh^{-1}(\Delta)}{\Delta}-1\right)\right.$$

$$-24M_Hm_Q^3\left(6\ln\left(\frac{\mu^2z^2}{M_H^2}\right)+3\frac{\tanh^{-1}(\Delta)}{\Delta}-8\right)-4M_H^2m_Q^2\left(39\ln\left(\frac{\mu^2z^2}{M_H^2}\right)-40\right)$$

$$\left.-(6M_H^3m_Q+M_H^4)\left(12\ln\left(\frac{\mu^2z^2}{M_H^2}\right)-13\right)\right]\Big]\Big\},\tag{C.6}$$

where $\tanh^{-1}(\Delta)=\frac{1}{2}(\ln(1+\Delta)-\ln(1-\Delta))$ and we defined:

$$P_1=\text{Li}_2(-\Delta)-\text{Li}_2(\Delta),\tag{C.7a}$$

$$P_2=\text{Li}_2\left(\frac{1-\Delta}{2}\right)-\text{Li}_2\left(\frac{1+\Delta}{2}\right),\tag{C.7b}$$

$$P_3=\frac{1}{3\Delta^3\left(\Delta^2+3\right)}\left(\left(9\Delta^4+\left(9\sqrt{1-\Delta^2}+6\right)\Delta^2-9\sqrt{1-\Delta^2}+9\right)\tanh^{-1}(\Delta)\right.$$

$$\left.-\Delta\left(8\Delta^4+\left(6\sqrt{1-\Delta^2}+9\right)\Delta^2-9\sqrt{1-\Delta^2}+9\right)\right).\tag{C.7c}$$

In deriving above expressions, we have used the PolyLogTools [66] package.

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