Composite operators in $T\bar{T}$ -deformed free QFTs

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ABSTRACT: We study perturbative renormalization of the composite operators in the $T\bar{T}$ -deformed two-dimensional free field theories. The pattern of renormalization for the stress-energy tensor is different in the massive and massless cases. While in the latter case the canonical stress tensor is not renormalized up to high order in the perturbative expansion, in the massive theory there are induced counterterms at linear order. For a massless theory our results match the general formula derived recently in [1].

KEYWORDS: Conformal Field Theory, Renormalization Regularization and Renormalons, Field Theories in Lower Dimensions, Integrable Field Theories

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1 Introduction

Recently, Smirnov and Zamolodchikov introduced a new class of tractable two-dimensional theories [2]. This class of models is rich. It is defined by the $T\bar{T}$ flow equation of the form

$$\frac{\partial \mathcal{L}(\lambda)}{\partial \lambda} = -4 \left(T_{zz}^\lambda T_{\bar{z}\bar{z}}^\lambda - (T_{z\bar{z}}^\lambda)^2 \right), \quad (1.1)$$

where $\mathcal{L}(\lambda = 0)$ is any Lagrangian and $T_{\mu\nu}^\lambda$ are the components of the energy-momentum tensor of the finite λ theory. The composite operators on both sides of this equation are UV finite. Although this equation has nothing to do with an RG flow equation, it does describe a particular one-parameter family of theories.

Remarkably, the above prescription gives both the Lagrangian [3, 4], as well as the S-matrix [5, 6] and energy spectrum [7, 8] for these models. However, the S-matrices of the $T\bar{T}$ theories do not have a reliable analytic behavior — they grow exponentially at large imaginary momenta. Such a growth is inconsistent with the behavior of a local quantum field theory. However, rather than just discarding these theories, the approach is to interpret them as quantum field theories coupled to gravity [9–13]. Hence, the AdS_3

dual of the $T\bar{T}$ deformed theories is an interesting question studied, for instance, in [14–16], see also [17–21] for recent developments on $T\bar{T}$ and string theory, and [22] for studies of dS. Furthermore, at large N one expects the $T\bar{T}$ deformation to represent a change in boundary conditions [23], whereas the Hagedorn behavior and partition function of the $T\bar{T}$ theories is an interesting question by itself [2, 3, 24–27].

We consider low energy behavior of these theories. In this regime these are quantum field theories which can be studied perturbatively around $\lambda = 0$, see e.g., [28–30]. The renormalization of these theories is highly nontrivial. Indeed, they are non-renormalizable. If these were the standard QFTs, it would mean that one lacks predictive power in the UV. However, integrability gives an infinite number of constraints which uniquely fix all necessary counterterms, see e.g., [29] for perturbative renormalization of the Lagrangian. In this work we focus on renormalization of the composite operators. Our approach is similar in spirit to the renormalization program in a conventional local renormalizable field theory, i.e., we define deformed operators whose correlators are perturbatively finite.

In the first part we demonstrate that, to linear order in the $T\bar{T}$ deformation, renormalization of the composite (non-)primary operator $\mathcal{O}$ in a massless free field theory can be cast in the universal form

$$[\mathcal{O}] = \mathcal{O} - \frac{\lambda}{2\pi\epsilon} \partial^2 \mathcal{O}, \quad (1.2)$$

where $\epsilon = 2-d$ regulates a logarithmic UV divergence, whereas $\mathcal{O}$ and $[\mathcal{O}]$ represent the bare and renormalized operators respectively. The general argument in favour of this universal relation was recently provided in [1]. We explicitly verify it using the examples of scalar and spinning/tensor (non-)primary operators $\mathcal{O}$ in the case of massless Dirac and scalar free field theories deformed by the $T\bar{T}$ operator. In particular, we study renormalization of the composite operators such as ϕ^n , $((\partial\phi)^2)^n$, $(\bar{\psi}\psi)^n$ (for any integer $n \geq 1$), $\partial_\mu\phi\partial_\nu\phi$, $\bar{\psi}\gamma^\mu\psi$, $\bar{\psi}\gamma_\mu\partial_\nu\psi$, and the stress-energy tensor $T_{\mu\nu}$. The renormalization pattern eventually matches (1.2) for all cases.

We argue that the canonical stress-energy tensor for the massless Dirac and scalar free field theories is not renormalized to linear order in the coupling λ . This contrasts significantly with the ϕ^4 theory in $d = 4 - \epsilon$ dimensions, where it is necessary to introduce a non-trivial improvement term $\frac{g}{\epsilon}(\partial_\mu\partial_\nu - \delta_{\mu\nu}\partial^2)\phi^2$ to render the canonical stress-energy tensor finite [31].¹ Since improvement terms can be attributed to the corresponding gravitational counterterms, e.g., $\frac{g}{\epsilon}R\phi^2$ in the case of the ϕ^4 theory [32], they partially elucidate the way in which a quantum field theory couples to a curved background. This is particularly interesting in the context of $T\bar{T}$ theories since little is known about how such theories couple to a curved manifold.

Note that the Ward identity for the correlation functions involving a divergence of the stress-energy tensor (or any other conserved current) implies a non-renormalization of this operator.² However, the potentially singular in the $\epsilon \rightarrow 0$ limit improvement terms, which contribute to the stress-energy tensor, are not in conflict with this statement, because they are separately conserved.

¹Here g is the ϕ^4 coupling constant.

²See [28] for a related discussion in the $T\bar{T}$ context.

In fact, one can list the admissible improvement counterterms. To this end, we recall that such (divergent) terms are local, and can be derived by varying the induced gravitational counterterms with respect to the metric around a flat background. While the coupling of $T\bar{T}$ theories to gravity remains a challenge, there is no obstruction to present an exhaustive list of possible counterterms based on dimensional analysis. For instance, to linear order in the $T\bar{T}$ deformation, we have the following candidates for the massless scalar and Dirac fields

$$I_{(s)}^{\text{g.c.t}} = \frac{\lambda}{\epsilon} \int d^d x \left(b_1^{(s)} R_{\mu\nu} + b_2^{(s)} R g_{\mu\nu} \right) \partial^\mu \phi \partial^\nu \phi, \quad (1.3)$$

$$I_{(f)}^{\text{g.c.t}} = \frac{\lambda}{\epsilon} \int d^d x \left(b_1^{(f)} R_{\mu\nu} + b_2^{(f)} R g_{\mu\nu} \right) \bar{\psi} \gamma^\mu \partial^\nu \psi. \quad (1.4)$$

In what follows we show that the coefficients $b_{1,2}^{(s,f)}$ for the free fields satisfy

$$b_2^{(s,f)} = -\frac{1}{2} b_1^{(s,f)}. \quad (1.5)$$

Hence, the right hand side of (1.3) and (1.4) is proportional to the Einstein tensor, which identically vanishes in two dimensions. Note that the non-renormalization of the stress-energy tensor in the $T\bar{T}$ -deformed free theories is in full agreement with (1.2), because $\partial^2 T_{\mu\nu}$ vanishes in a CFT.³

Finally, in section 4 we study composite operators in the massive free field theories. The renormalization pattern does not match (1.2) in this case. Thus, for instance, to linear order in the deformation there are induced counterterms in the stress-energy tensor which do not obey a simple relation (1.2). This suggests that unlike the case of deformed CFT, presence of a gap in the undeformed theory results in an operator mixing under RG flow for any non-zero λ . It would be interesting to explore its structure in the future.

2 Renormalization of composite operators: scalar field

In this section we consider $T\bar{T}$ deformation of the free massless scalar field. To the linear order in the $T\bar{T}$ coupling λ the action is given by

$$I = \int d^d x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\lambda \mu^\epsilon}{4} \left((\partial\phi)^2 \right)^2 \right), \quad (2.1)$$

where μ is an arbitrary renormalization scale, and the leading order $T\bar{T}$ perturbation is given by

$$T\bar{T}_0 \equiv \det T^{(0)} = -\frac{1}{4} \left((\partial\phi)^2 \right)^2. \quad (2.2)$$

Here label 0 indicates zeroth order in the coupling λ , and we work in dimensional regularization, $d = 2 - \epsilon$. We will perform most of our calculations to the linear order in the $T\bar{T}$ coupling λ , in which case all the counterterms in the action (2.1) vanish, and in

³We are using equations of motion to the zeroth order in the deformation, because (1.2) holds to linear order in the coupling λ .

particular λ is the renormalized dimensionless coupling, and the bare fields coincide with the renormalized fields.

Stress-energy tensor to linear order in λ is given by

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi (1 - \lambda (\partial\phi)^2) - \frac{1}{2} \delta_{\mu\nu} \left((\partial\phi)^2 + 2\lambda T\bar{T} \right). \quad (2.3)$$

To avoid clutter in our notation, here and in what follows we will skip putting a label on the stress-energy tensor to indicate to which order in λ it is written.

Free massless scalar two-point function is given by

$$\langle \phi(x) \phi(0) \rangle = \frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{\frac{d}{2}}} \frac{1}{|x|^{d-2}}. \quad (2.4)$$

In all of our calculations we are looking for logarithmically divergent terms, which (to linear order in the coupling λ) in dimensional regularization correspond to only simple poles in $\epsilon = 2 - d$. Therefore to simplify some of our notations we therefore can and will set $d = 2$ ($\epsilon = 0$) in some of the factors right away.

2.1 $[\phi^n]$

Consider renormalization of the composite operator ϕ^n , $n \geq 2$ in the theory of massless scalar field ϕ to the linear order in the $T\bar{T}$ coupling λ ,

$$\phi^n = [\phi^n] + \Delta\phi^n, \quad \Delta\phi^n = \frac{\lambda\mu^\epsilon}{4} \phi^n \int d^d x \left((\partial\phi)^2 \right)^2, \quad (2.5)$$

where in $\Delta\phi^n$ we implicitly assumed to perform all possible contractions and keep only terms singular in the $d \rightarrow 2$ limit. In this section, just as everywhere else in this paper, we are working in the dimensional regularization, and focus on contributions to $\Delta\phi^n$ which are divergent in the $d \rightarrow 2$ limit. For simplicity and without loss of generality we also consider the operator ϕ^n (and all other composite operators studied in this paper) to sit at the origin $x = 0$.

We begin by writing possible contractions which can contribute to the divergent terms in the $d \rightarrow 2$ limit,⁴

$$\Delta\phi^n = \frac{\lambda\mu^\epsilon}{4} \frac{n(n-1)}{2!} 4\phi^{n-2} \int d^d x \langle \phi(0) \partial^\nu \phi(x) \rangle \quad (2.6)$$

$$\times \left(\langle \phi(0) \partial_\nu \phi(x) \rangle (\partial\phi(x))^2 + 2\partial_\nu \phi(x) \partial_\mu \phi(x) \langle \phi(0) \partial^\mu \phi(x) \rangle \right). \quad (2.7)$$

As mentioned above, to the linear order in the coupling λ we will only encounter at most simple poles in $1/\epsilon$, where $\epsilon = 2 - d$. Therefore we can set $d = 2$ everywhere else, and in particular we substitute here

$$\langle \phi(0) \partial^\nu \phi(x) \rangle = -\frac{1}{2\pi} \frac{x^\nu}{|x|^d}. \quad (2.8)$$

⁴Here the factor of $\frac{n(n-1)}{2!}$ is the combinatorial coefficient due to the choice of two out of n operators $\phi(0)$, and the factor of 4 is due to the possible choices of $\partial^\nu \phi$ in the first line of (2.6).

Since we are interested in the UV divergencies only, we will focus on the structure of the integral (2.7) near the origin $x = 0$. Expanding the integrated operators around $x = 0$, and keeping only the terms logarithmically divergent in $d = 2$ (equivalently, keeping only the terms which exhibit $1/\epsilon$ poles in the $d \rightarrow 2$ limit), we obtain

$$\Delta\phi^n = \frac{\lambda n(n-1)}{8\pi^2} \phi^{n-2} \left(\mathcal{I}(\partial\phi)^2 + 2\mathcal{I}^{\mu\nu} \partial_\mu\phi\partial_\nu\phi \right), \quad (2.9)$$

where $\mathcal{I}$, $\mathcal{I}^{\mu\nu}$ are defined in (A.1). Using (A.3), (A.4) we obtain

$$\Delta\phi^n = \frac{\lambda n(n-1)}{2\pi\epsilon} \phi^{n-2} (\partial\phi)^2. \quad (2.10)$$

Using equations of motion for the field ϕ to the $\mathcal{O}(\lambda^0)$ order, $\partial^2\phi = 0$, we can re-write the order $\mathcal{O}(\lambda)$ expression on the r.h.s. of (2.10) as

$$\Delta\phi^n = \frac{\lambda}{2\pi\epsilon} \partial^2\phi^n + \mathcal{O}(\lambda^2). \quad (2.11)$$

The expression is in agreement with the general form (1.2) stated in Introduction.

2.2 $[\partial_\mu\phi\partial_\nu\phi]$

We next proceed to calculate renormalization of the composite tensor operator $\partial_\mu\phi\partial_\nu\phi$,

$$\partial_\mu\phi\partial_\nu\phi = [\partial_\mu\phi\partial_\nu\phi] + \Delta\partial_\mu\phi\partial_\nu\phi, \quad (2.12)$$

$$\Delta\partial_\mu\phi\partial_\nu\phi = \frac{\lambda\mu^\epsilon}{4} \int d^d x \left(4\partial_\mu\partial_\alpha\langle\phi(0)\phi(x)\rangle\partial_\nu\partial^\alpha\langle\phi(0)\phi(x)\rangle(\partial\phi(x))^2 \right. \quad (2.13)$$

$$\left. + 8\partial_\mu\partial_\alpha\langle\phi(0)\phi(x)\rangle\partial_\nu\partial_\beta\langle\phi(0)\phi(x)\rangle\partial^\alpha\phi(x)\partial^\beta\phi(x) \right), \quad (2.14)$$

where we have kept track of the various degeneracy factors originating from combinatorics. We are ultimately interested in collecting the terms which are singular in the $\epsilon \rightarrow 0$ limit. To this end, using the free correlation function

$$\langle\phi(0)\partial_\mu\partial_\alpha\phi(x)\rangle = \frac{1}{\pi} \frac{1}{|x|^{d+2}} \left(x_\alpha x_\mu - \frac{\delta_{\alpha\mu}}{2} |x|^2 \right) \quad (2.15)$$

and expanding the integrated operators around $x = 0$ we obtain

$$\begin{aligned} \Delta\partial_\mu\phi\partial_\nu\phi = & \frac{\lambda}{2\pi^2} \left[\frac{1}{4} \delta_{\mu\nu} \mathcal{I}_{\alpha\beta} \partial^\alpha\partial^\beta((\partial\phi(0))^2) \right. \\ & \left. + 2\partial^\rho\partial^\sigma(\partial^\alpha\phi\partial^\beta\phi(0)) \left(I_{\rho\sigma\alpha\mu\beta\nu} - \frac{1}{2} \delta_{\beta\nu} \mathcal{I}_{\rho\sigma\alpha\mu} - \frac{1}{2} \delta_{\alpha\mu} \mathcal{I}_{\rho\sigma\beta\nu} + \frac{1}{4} \mathcal{I}_{\rho\sigma} \delta_{\alpha\mu} \delta_{\beta\nu} \right) \right]. \end{aligned} \quad (2.16)$$

This expression can be further simplified using (A.4), (A.5), (A.6). While the calculation is rather tedious, with some insight it can be automatized with the help of *Mathematica*, rendering

$$\begin{aligned} \Delta\partial_\mu\phi\partial_\nu\phi = & \frac{\lambda}{12\pi\epsilon} \left[\left(2\partial^2(\partial\phi)^2 + \partial_\alpha\partial_\beta(\partial^\alpha\phi\partial^\beta\phi) \right) \delta_{\mu\nu} + \partial_\mu\partial_\nu(\partial\phi)^2 + \partial^2(\partial_\mu\phi\partial_\nu\phi) \right. \\ & \left. - \partial_\alpha\partial_\mu(\partial^\alpha\phi\partial_\nu\phi) - \partial_\alpha\partial_\nu(\partial^\alpha\phi\partial_\mu\phi) \right]. \end{aligned} \quad (2.17)$$

As a quick detour, notice that from (2.17) we can immediately deduce renormalization of the composite scalar operator $(\partial\phi)^2$,

$$(\partial\phi)^2 = [(\partial\phi)^2] + \frac{\lambda}{2\pi\epsilon} \partial^2(\partial\phi)^2 + \mathcal{O}(\lambda^2), \quad (2.18)$$

which is in agreement with the general form (1.2) stated in Introduction. Below we will generalize this result to renormalization of $((\partial\phi)^2)^n$ for arbitrary $n \geq 1$.

Returning back to (2.17) we notice that it can be further simplified using tensor identities in $d = 2$. Specifically, using the following variations around the flat metric

$$\begin{aligned} \int d^d x (\partial\phi)^2 g^{\mu\nu} \delta R_{\mu\nu} &= \int d^2 x (\partial\phi)^2 (g_{\mu\nu} \nabla^2 - \nabla_\mu \nabla_\nu) \delta g^{\mu\nu} \\ &= \int d^2 x \delta g^{\mu\nu} (\partial^2 \delta_{\mu\nu} - \partial_\mu \partial_\nu) (\partial\phi)^2, \\ \int d^2 x \partial^\alpha \phi \partial^\beta \phi \delta R_{\alpha\beta} &= \frac{1}{2} \int d^2 x \delta g^{\mu\nu} (\partial_\alpha \partial_\beta (\partial^\alpha \phi \partial^\beta \phi) \delta_{\mu\nu} + \partial^2 (\partial_\mu \phi \partial_\nu \phi) \\ &\quad - \partial_\mu \partial_\alpha (\partial_\nu \phi \partial^\alpha \phi) - \partial_\nu \partial_\alpha (\partial_\mu \phi \partial^\alpha \phi)), \end{aligned}$$

we obtain⁵

$$\begin{aligned} &(-\partial^2(\partial\phi)^2 + \partial_\alpha \partial_\beta (\partial^\alpha \phi \partial^\beta \phi)) \delta_{\mu\nu} + \partial_\mu \partial_\nu (\partial\phi)^2 + \partial^2 (\partial_\mu \phi \partial_\nu \phi) - \partial_\alpha \partial_\mu (\partial^\alpha \phi \partial_\nu \phi) - \partial_\alpha \partial_\nu (\partial^\alpha \phi \partial_\mu \phi) \\ &= \frac{\delta}{\delta g^{\mu\nu}} \int d^2 x (-R g_{\alpha\beta} + 2R_{\alpha\beta}) \partial^\alpha \phi \partial^\beta \phi \equiv 0, \end{aligned} \quad (2.19)$$

where in the last line we took advantage of the fact that the Einstein tensor $R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}$ vanishes identically in two dimensions.

Using (2.19) we can re-write (2.17) as

$$\Delta \partial_\mu \phi \partial_\nu \phi = \frac{\lambda}{4\pi\epsilon} \partial^2 (\partial\phi)^2 \delta_{\mu\nu}. \quad (2.20)$$

Finally, using the $\mathcal{O}(\lambda^0)$ order equations of motion $\partial^2 \phi = 0$, one can show that in $d = 2$ the $\mathcal{O}(\lambda)$ expression (2.20) is equivalent to

$$\Delta \partial_\mu \phi \partial_\nu \phi = \frac{\lambda}{2\pi\epsilon} \partial^2 (\partial_\mu \phi \partial_\nu \phi), \quad (2.21)$$

in agreement with (1.2).

2.3 $[T_{\mu\nu}]$

Above we have accumulated enough results to calculate renormalization of the stress-energy tensor (2.3),

$$T_{\mu\nu} = [T_{\mu\nu}] + \Delta T_{\mu\nu} \quad (2.22)$$

⁵It is possible to verify directly that the l.h.s. of (2.19) vanishes in $2d$, particularly by plugging that expression into *Mathematica*.

to linear order in the $T\bar{T}$ coupling λ . First of all we notice that in massless theory the term $\partial_\mu\phi\partial_\nu\phi(\partial\phi)^2$ does not get renormalized to zeroth order in λ . Then, using (2.20) we obtain (without using equations of motion for the field ϕ)

$$\Delta T_{\mu\nu} = 0 + \mathcal{O}(\lambda^2). \quad (2.23)$$

Notice that while this simple result is in an apparent conflict with the general formula (1.2), the disagreement is superficial. Indeed, as we have argued above, the $\partial_\mu\phi\partial_\nu\phi$ is renormalized according to (1.2) once the equations of motion are used, see discussion leading to (2.21). Since to linear order in λ the renormalization of $\partial_\mu\phi\partial_\nu\phi$ defines renormalization of $T_{\mu\nu}$, we conclude that the $\mathcal{O}(\lambda^0)$ equations of motion imply

$$\partial^2 T_{\mu\nu} = 0, \quad (2.24)$$

which holds in a CFT.

2.4 $[((\partial\phi)^2)^n]$

To conclude our discussion of the $T\bar{T}$ -deformed free massless scalar we generalize our result (2.18) to the case of arbitrary $n \geq 1$

$$\left((\partial\phi)^2\right)^n = \left[\left((\partial\phi)^2\right)^n\right] + \Delta\left((\partial\phi)^2\right)^n. \quad (2.25)$$

Here we have⁶

$$\begin{aligned} \Delta\left((\partial\phi)^2\right)^n &= \frac{\lambda}{4}\left((\partial\phi)^2\right)^n \int d^d x \left((\partial\phi)^2\right)^2 \\ &= n\left((\partial\phi)^2\right)^{n-1} \Delta(\partial\phi)^2 + \lambda n(n-1)\left((\partial\phi)^2\right)^{n-2} \partial_\mu\phi\partial^\alpha\phi \\ &\quad \times \int d^d x \langle\partial^\mu\phi(0)\partial^\nu\phi(x)\rangle \left((\partial\phi(x))^2 \langle\partial_\alpha\phi(0)\partial_\nu\phi(x)\rangle\right. \\ &\quad \left.+ 2\langle\partial_\alpha\phi(0)\partial_\beta\phi(x)\rangle\partial^\beta\phi(x)\partial_\nu\phi(x)\right). \end{aligned} \quad (2.26)$$

Expanding the integrated operator around $x = 0$ and keeping only the logarithmically divergent contributions we obtain

$$\begin{aligned} \Delta\left((\partial\phi)^2\right)^n &= n\left((\partial\phi)^2\right)^{n-1} \Delta(\partial\phi)^2 + \frac{\lambda n(n-1)}{2\pi^2} \left((\partial\phi)^2\right)^{n-2} \left(\frac{1}{2}(\partial\phi)^2 \mathcal{I}^{\alpha\beta} \partial_\alpha\partial_\beta((\partial\phi)^2)\right. \\ &\quad \left.+ 4\partial_\mu\phi\partial^\alpha\phi \left(\mathcal{I}^{\mu\nu\lambda\rho}_{\alpha\beta} - \frac{1}{2}\mathcal{I}^{\mu\nu\lambda\rho}\delta_{\alpha\beta} - \frac{1}{2}\mathcal{I}^{\lambda\rho}_{\alpha\beta}\delta^{\mu\nu} + \frac{1}{4}\mathcal{I}^{\lambda\rho}\delta^{\mu\nu}\delta_{\alpha\beta}\right)\right) \partial_\lambda\partial_\rho(\partial^\beta\phi\partial_\nu\phi). \end{aligned} \quad (2.27)$$

Plugging here (A.4), (A.5), (A.6), and simplifying it in *Mathematica*, we obtain

$$\Delta\left((\partial\phi)^2\right)^n = n\left((\partial\phi)^2\right)^{n-1} \Delta(\partial\phi)^2 + \frac{\lambda n(n-1)}{2\pi\epsilon} \left((\partial\phi)^2\right)^{n-1} \partial^2\left((\partial\phi)^2\right). \quad (2.28)$$

⁶Notice that we contract at most two pairs of fields. It is possible to contract one or two more fields for $n \geq 3$, but such extra terms will not contribute any logarithmic divergencies.

Using (2.18) we can re-write it as

$$\Delta \left((\partial\phi)^2 \right)^n = \frac{\lambda n^2}{2\pi\epsilon} \left((\partial\phi)^2 \right)^{n-1} \partial^2 \left((\partial\phi)^2 \right). \quad (2.29)$$

On the other hand,

$$\partial^2 \left((\partial\phi)^2 \right)^n = n \left((\partial\phi)^2 \right)^{n-2} \left((\partial\phi)^2 \partial^2 (\partial\phi)^2 + (n-1) \partial^\mu (\partial\phi)^2 \partial_\mu (\partial\phi)^2 \right). \quad (2.30)$$

Using equations of motion $\partial^2\phi = 0$ we can demonstrate that in two dimensions $\partial^\mu (\partial\phi)^2 \partial_\mu (\partial\phi)^2 = (\partial\phi)^2 \partial^2 (\partial\phi)^2$. Consequently

$$\partial^2 \left((\partial\phi)^2 \right)^n = n^2 \left((\partial\phi)^2 \right)^{n-1} \partial^2 (\partial\phi)^2. \quad (2.31)$$

Comparing this with (2.29) we conclude

$$\Delta \left((\partial\phi)^2 \right)^n = \frac{\lambda}{2\pi\epsilon} \partial^2 \left((\partial\phi)^2 \right)^n, \quad (2.32)$$

in agreement with (1.2).

3 Renormalization of composite operators: Dirac field

In this section we will recreate analysis performed in section 2 for the theory of free massless Dirac fermion deformed by the $T\bar{T}$ operator. We will be working to the linear order in the $T\bar{T}$ coupling λ . The corresponding Euclidean action to linear order in λ is given by

$$I = \int d^d x \left(\frac{1}{2} (\bar{\psi} \gamma^\mu \partial_\mu \psi - \partial_\mu \bar{\psi} \gamma^\mu \psi) - \frac{\lambda \mu^\epsilon}{2} T_{\mu\nu} T^{\mu\nu} \right). \quad (3.1)$$

Here we have used the fact that in two dimensions for traceless stress-energy tensor one can write down

$$\det T = -\frac{1}{2} T_{\mu\nu} T^{\mu\nu}. \quad (3.2)$$

The canonical stress-energy tensor for the free fermion is given by

$$T_{\mu\nu}^{(c)} = \frac{1}{2} (\bar{\psi} \gamma_\mu \partial_\nu \psi - \partial_\nu \bar{\psi} \gamma_\mu \psi). \quad (3.3)$$

Using the standard Belinfante technique we can symmetrize it, giving

$$T_{\mu\nu} = \frac{1}{4} (\bar{\psi} \gamma_\mu \partial_\nu \psi + \bar{\psi} \gamma_\nu \partial_\mu \psi - \partial_\nu \bar{\psi} \gamma_\mu \psi - \partial_\mu \bar{\psi} \gamma_\nu \psi). \quad (3.4)$$

Free massless fermion two-point function is given by

$$\langle \psi(x) \bar{\psi}(0) \rangle = \frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{\frac{d}{2}}} \frac{x^\mu \gamma_\mu}{|x|^d}. \quad (3.5)$$

3.1 $[(\bar{\psi}\psi)^n]$

In this subsection we consider one-loop renormalization of the composite operator $(\bar{\psi}\psi)^n$ for $n \geq 1$. Let us begin by studying the $n = 1$ case,

$$\bar{\psi}\psi = [\bar{\psi}\psi] + \Delta\bar{\psi}\psi, \quad \Delta\bar{\psi}\psi = \frac{\lambda\mu^\epsilon}{2} \bar{\psi}\psi \int d^d x T_{\lambda\rho} T^{\lambda\rho}, \quad (3.6)$$

where again we imply performing all possible contractions on the r.h.s. of the expression for $\Delta\bar{\psi}\psi$, and keeping only singular terms in the $d \rightarrow 2$ limit. It is convenient to split $\Delta\bar{\psi}\psi$ into two contributions,

$$\Delta\bar{\psi}\psi = \Delta_{(1)}\bar{\psi}\psi + \Delta_{(2)}\bar{\psi}\psi, \quad (3.7)$$

where the first contribution is due to contractions of constituent fermions in $\bar{\psi}\psi$ with any of the two factors $T_{\lambda\rho}$ in the vertex $T_{\lambda\rho} T^{\lambda\rho}$. Taking into account symmetry of stress-energy tensor and real-value-ness of several equal to each other terms, we write down

$$\Delta_{(1)}\bar{\psi}\psi = \frac{\lambda}{4} \int d^d x \text{Tr} \left(\langle \psi(0) \bar{\psi}(x) \rangle \gamma_\lambda \langle \partial_\rho \psi(x) \bar{\psi}(0) \rangle \right) T^{\lambda\rho} \quad (3.8)$$

$$= \frac{\lambda}{16\pi^2} \left(\mathcal{I}^{\mu\alpha} \text{Tr}(\gamma_\mu \gamma_\lambda \gamma_\rho) - 2\mathcal{I}^{\mu\nu\alpha}{}_\rho \text{Tr}(\gamma_\mu \gamma_\lambda \gamma_\nu) \right) \partial_\alpha T^{\lambda\rho}. \quad (3.9)$$

One can verify explicitly that each individual term on the r.h.s. of the last line vanishes.

The second contribution to $\Delta\bar{\psi}\psi$ originates from cross-contraction between constituent fermions of $\bar{\psi}\psi$ and each of the two stress-energy factors in the vertex $T_{\lambda\rho} T^{\lambda\rho}$. Taking into account all possible degeneracy factors, we write down

$$\begin{aligned} \Delta_{(2)}\bar{\psi}\psi = & \frac{\lambda}{16} \left(2 \int d^d x \left(\bar{\psi} \gamma^\lambda \langle \partial^\rho \psi(x) \bar{\psi}(0) \rangle \langle \psi(0) \bar{\psi}(x) \rangle \gamma_\lambda \partial_\rho \psi + \bar{\psi} \gamma^\lambda \langle \partial^\rho \psi(x) \bar{\psi}(0) \rangle \langle \psi(0) \bar{\psi}(x) \rangle \gamma_\rho \partial_\lambda \psi \right) \right. \\ & - \int d^d x \left(\partial^\lambda \bar{\psi} \gamma^\rho \langle \psi(x) \bar{\psi}(0) \rangle \langle \psi(0) \bar{\psi}(x) \rangle \gamma_\lambda \partial_\rho \psi + \partial^\rho \bar{\psi} \gamma^\lambda \langle \psi(x) \bar{\psi}(0) \rangle \langle \psi(0) \bar{\psi}(x) \rangle \gamma_\lambda \partial_\rho \psi \right) \\ & \left. - \int d^d x \left(\bar{\psi} \gamma_\lambda \langle \partial_\rho \psi(x) \bar{\psi}(0) \rangle \langle \psi(0) \partial^\lambda \bar{\psi}(x) \rangle \gamma^\rho \psi + \bar{\psi} \gamma_\rho \langle \partial_\lambda \psi(x) \bar{\psi}(0) \rangle \langle \psi(0) \partial^\lambda \bar{\psi}(x) \rangle \gamma^\rho \psi \right) + \text{c.c.} \right). \end{aligned} \quad (3.10)$$

To explain the calculation we split this into the first, second, and third line contributions (each of them plus their complex conjugate contributions, where we find out that the second and third lines are actually individually real-valued),

$$\Delta_{(2)}\bar{\psi}\psi = \Delta_{(2)}^{(1)}\bar{\psi}\psi + \Delta_{(2)}^{(2)}\bar{\psi}\psi + \Delta_{(2)}^{(3)}\bar{\psi}\psi, \quad (3.11)$$

where

$$\begin{aligned} \Delta_{(2)}^{(1)}\bar{\psi}\psi = & \frac{\lambda}{64\pi^2} \left(-2\partial_\alpha (\bar{\psi} \gamma^\lambda \gamma^\rho \gamma_\nu \gamma_\lambda \partial_\rho \psi) \mathcal{I}^{\alpha\nu} + 4\partial_\alpha (\bar{\psi} \gamma^\lambda \gamma_\mu \gamma_\nu \gamma_\lambda \partial_\rho \psi) \mathcal{I}^{\alpha\mu\nu\rho} \right. \\ & \left. - 2\partial_\alpha (\bar{\psi} \gamma^\lambda \gamma^\rho \gamma_\nu \gamma_\rho \partial_\lambda \psi) \mathcal{I}^{\alpha\nu} + 4\partial_\alpha (\bar{\psi} \gamma^\lambda \gamma_\mu \gamma_\nu \gamma_\rho \partial_\lambda \psi) \mathcal{I}^{\alpha\mu\nu\rho} \right) + \text{c.c.} \end{aligned} \quad (3.12)$$

Using here (A.4), (A.5) we can simplify it to

$$\Delta_{(2)}^{(1)}\bar{\psi}\psi = \frac{\lambda}{8\pi\epsilon} \left(\partial^2(\bar{\psi}\psi) - 2(\partial_\mu \bar{\psi} \gamma^\mu \gamma^\nu \partial_\nu \psi - \partial_\mu \bar{\psi} \partial^\mu \psi) \right). \quad (3.13)$$

Further using $\mathcal{O}(\lambda^0)$ e.o.m in this $\mathcal{O}(\lambda)$ expression we can re-write it as

$$\Delta_{(2)}^{(1)} \bar{\psi} \psi = \frac{\lambda}{4\pi\epsilon} \partial^2 (\bar{\psi} \psi). \quad (3.14)$$

Next, we have

$$\Delta_{(2)}^{(2)} \bar{\psi} \psi = \frac{\lambda}{32\pi^2} \left(\partial^\lambda \bar{\psi} \gamma^\rho \gamma_\lambda \partial_\rho \psi + 2 \partial^\lambda \bar{\psi} \partial_\lambda \psi \right) \mathcal{I} = \frac{\lambda}{8\pi\epsilon} \partial^2 (\bar{\psi} \psi), \quad (3.15)$$

where in the last line we used (A.3) and $\mathcal{O}(\lambda^0)$ equations of motion. Finally,

$$\begin{aligned} \Delta_{(2)}^{(3)} \bar{\psi} \psi = \frac{\lambda}{32\pi^2} & \left(\frac{1}{2} \partial_\alpha \partial_\beta (\bar{\psi} \gamma_\lambda \gamma_\rho \gamma^\lambda \gamma^\rho \psi) \mathcal{I}^{\alpha\beta} - \partial_\alpha \partial_\beta (\bar{\psi} \gamma_\lambda \gamma_\rho \gamma_\nu \gamma^\rho \psi) \mathcal{I}^{\alpha\beta\lambda\nu} \right. \\ & - \partial_\alpha \partial_\beta (\bar{\psi} \gamma_\lambda \gamma_\nu \gamma^\lambda \gamma^\rho \psi) \mathcal{I}^{\alpha\beta\nu\rho} + 2 \partial_\alpha \partial_\beta (\bar{\psi} \gamma_\lambda \gamma_\rho \psi) \mathcal{I}^{\alpha\beta\lambda\rho} \\ & + \frac{1}{2} \partial_\alpha \partial_\beta (\bar{\psi} \gamma_\lambda \gamma_\rho \gamma^\rho \gamma^\lambda \psi) \mathcal{I}^{\alpha\beta} - \partial_\alpha \partial_\beta (\bar{\psi} \gamma_\lambda \gamma_\rho \gamma_\nu \gamma^\lambda \psi) \mathcal{I}^{\alpha\beta\rho\nu} \\ & \left. - \partial_\alpha \partial_\beta (\bar{\psi} \gamma_\lambda \gamma_\nu \gamma_\rho \gamma^\lambda \psi) \mathcal{I}^{\alpha\beta\nu\rho} + 2 \partial_\alpha \partial_\beta (\bar{\psi} \gamma_\lambda \gamma^\lambda \psi) \mathcal{I}^{\alpha\beta\rho}_\rho \right). \end{aligned} \quad (3.16)$$

Using here (A.4), (A.5) only, we can simplify it to

$$\Delta_{(2)}^{(3)} \bar{\psi} \psi = \frac{\lambda}{8\pi\epsilon} \partial^2 (\bar{\psi} \psi). \quad (3.17)$$

Combining everything together we obtain

$$\Delta_{(2)} \bar{\psi} \psi = \frac{\lambda}{2\pi\epsilon} \partial^2 (\bar{\psi} \psi), \quad (3.18)$$

which leads to our final answer

$$\Delta \bar{\psi} \psi = \frac{\lambda}{2\pi\epsilon} \partial^2 (\bar{\psi} \psi), \quad (3.19)$$

which we observe to be once again in perfect agreement with the universal expression (1.2).

Now let us consider $\mathcal{O}(\lambda)$ renormalization of the composite operator $(\bar{\psi} \psi)^n$, generalizing the $n = 1$ result obtained above to arbitrary integer $n \geq 1$,

$$(\bar{\psi} \psi)^n = [(\bar{\psi} \psi)^n] + \Delta(\bar{\psi} \psi)^n, \quad \Delta(\bar{\psi} \psi)^n = \frac{\lambda \mu^\epsilon}{2} (\bar{\psi} \psi)^n \int d^d x T_{\lambda\rho} T^{\lambda\rho}, \quad (3.20)$$

where in the latter expression we imply performing all possible contractions and keeping only the singular terms in the $d \rightarrow 2$ limit. Just as in the analogous calculation of renormalization of the $((\partial\phi)^2)^n$ in subsection 2.4 for the scalar, it turns out that we need to contract at most two pairs of fermions in order to obtain logarithmically divergent contributions.

As in the case of $n = 1$, it is convenient to split terms contributing to $\Delta(\bar{\psi} \psi)^n$ into two groups,

$$\Delta(\bar{\psi} \psi)^n = \Delta_{(1)}(\bar{\psi} \psi)^n + \Delta_{(2)}(\bar{\psi} \psi)^n, \quad (3.21)$$

where $\Delta_{(1)}(\bar{\psi} \psi)^n$ is composed of terms originating from contracting constituent fermions of $(\bar{\psi} \psi)^n$ with one of the two stress-energy tensor factors in the vertex $T_{\lambda\rho} T^{\lambda\rho}$, while

$\Delta_{(2)}(\bar{\psi}\psi)^n$ is composed of terms originating from contracting constituent fermions of $(\bar{\psi}\psi)^n$ with each of the stress-energy tensor factors in that vertex.

First, consider⁷

$$\begin{aligned} \Delta_{(1)}(\bar{\psi}\psi)^n &= n(\bar{\psi}\psi)^{n-1} \Delta_{(1)}(\bar{\psi}\psi) \\ &+ \frac{\lambda}{2} \left(\frac{1}{4}\right)^2 \frac{n(n-1)}{2!} 8(\bar{\psi}\psi)^{n-2} \int d^d x \left(\bar{\psi}(0) \langle \psi(0) \bar{\psi}(x) \rangle \gamma_\lambda \langle \partial_\rho \psi(x) \bar{\psi}(0) \rangle \psi(0) T^{\lambda\rho} + \text{c.c.} \right). \end{aligned} \quad (3.22)$$

As we have obtained above, $\Delta_{(1)}(\bar{\psi}\psi) = 0$. Keeping only singular in the $d \rightarrow 2$ limit terms, we obtain

$$\Delta_{(1)}(\bar{\psi}\psi)^n = \frac{\lambda n(n-1)}{32\pi^2} (\bar{\psi}\psi)^{n-2} \left(\bar{\psi} \gamma_\mu \gamma_\lambda \gamma_\rho \psi \mathcal{I}^{\mu\alpha} - 2\bar{\psi} \gamma_\mu \gamma_\lambda \gamma_\nu \psi \mathcal{I}^{\mu\nu\rho\alpha} \right) \partial_\alpha T^{\lambda\rho} + \text{c.c.} \quad (3.23)$$

One can verify that the first group of terms on the r.h.s. of the last expression is purely imaginary, hence the total is zero.

Next we consider cross-contraction terms

$$\Delta_{(2)}(\bar{\psi}\psi)^n = \Delta_{(2)}^{(1)}(\bar{\psi}\psi)^n + \Delta_{(2)}^{(2)}(\bar{\psi}\psi)^n + \Delta_{(2)}^{(3)}(\bar{\psi}\psi)^n, \quad (3.24)$$

where we denoted

$$\Delta_{(2)}^{(1)}(\bar{\psi}\psi)^n = n(\bar{\psi}\psi)^{n-1} \Delta_{(2)}(\bar{\psi}\psi) = \frac{\lambda n}{2\pi\epsilon} (\bar{\psi}\psi)^{n-1} \partial^2(\bar{\psi}\psi), \quad (3.25)$$

$$\begin{aligned} \Delta_{(2)}^{(2)}(\bar{\psi}\psi)^n &= \frac{\lambda}{2} \left(\frac{1}{4}\right)^2 \frac{n(n-1)}{2!} 4(\bar{\psi}\psi)^{n-2} \int d^d x \\ &\times \left(2 \left(\bar{\psi} \gamma^\lambda \langle \partial^\rho \psi(x) \bar{\psi}(0) \rangle \psi(0) \bar{\psi}(0) \langle \psi(0) \bar{\psi}(x) \rangle \gamma_\lambda \partial_\rho \psi + \bar{\psi} \gamma^\lambda \langle \partial^\rho \psi(x) \bar{\psi}(0) \rangle \psi(0) \bar{\psi}(0) \langle \psi(0) \bar{\psi}(x) \rangle \gamma_\rho \partial_\lambda \psi \right) \right. \\ &- \left(\partial^\lambda \bar{\psi} \gamma^\rho \langle \psi(x) \bar{\psi}(0) \rangle \psi(0) \bar{\psi}(0) \langle \psi(0) \bar{\psi}(x) \rangle \gamma_\lambda \partial_\rho \psi + \partial^\rho \bar{\psi} \gamma^\lambda \langle \psi(x) \bar{\psi}(0) \rangle \psi(0) \bar{\psi}(0) \langle \psi(0) \bar{\psi}(x) \rangle \gamma_\lambda \partial_\rho \psi \right) \\ &- \left(\bar{\psi} \gamma_\lambda \langle \partial_\rho \psi(x) \bar{\psi}(0) \rangle \psi(0) \bar{\psi}(0) \langle \psi(0) \partial^\lambda \bar{\psi}(x) \rangle \gamma^\rho \psi + \bar{\psi} \gamma_\rho \langle \partial_\lambda \psi(x) \bar{\psi}(0) \rangle \psi(0) \bar{\psi}(0) \langle \psi(0) \partial^\lambda \bar{\psi}(x) \rangle \gamma^\rho \psi \right) \\ &\left. + \text{c.c.} \right), \end{aligned} \quad (3.26)$$

$$\begin{aligned} \Delta_{(2)}^{(3)}(\bar{\psi}\psi)^n &= \frac{\lambda}{2} \left(\frac{1}{4}\right)^2 \frac{n(n-1)}{2!} 4(\bar{\psi}\psi)^{n-2} \int d^d x \\ &\times \left(\left(\bar{\psi}(0) \langle \psi(0) \bar{\psi}(x) \rangle \gamma_\lambda \partial_\rho \psi \bar{\psi}(0) \langle \psi(0) \bar{\psi}(x) \rangle \gamma^\lambda \partial^\rho \psi + \bar{\psi}(0) \langle \psi(0) \bar{\psi}(x) \rangle \gamma_\rho \partial_\lambda \psi \bar{\psi}(0) \langle \psi(0) \bar{\psi}(x) \rangle \gamma^\lambda \partial^\rho \psi \right) \right. \\ &- \left(\bar{\psi}(0) \langle \psi(0) \bar{\psi}(x) \rangle \gamma_\lambda \partial_\rho \psi \bar{\psi}(0) \langle \psi(0) \partial^\lambda \bar{\psi}(x) \rangle \gamma^\rho \psi + \bar{\psi}(0) \langle \psi(0) \bar{\psi}(x) \rangle \gamma_\lambda \partial_\rho \psi \bar{\psi}(0) \langle \psi(0) \partial^\rho \bar{\psi}(x) \rangle \gamma^\lambda \psi \right) \\ &+ \left(\bar{\psi} \gamma_\lambda \langle \partial_\rho \psi(x) \bar{\psi}(0) \rangle \psi(0) \bar{\psi} \gamma^\lambda \langle \partial^\rho \psi(x) \bar{\psi}(0) \rangle \psi(0) + \bar{\psi} \gamma_\rho \langle \partial_\lambda \psi(x) \bar{\psi}(0) \rangle \psi(0) \bar{\psi} \gamma^\lambda \langle \partial^\rho \psi(x) \bar{\psi}(0) \rangle \psi(0) \right) \\ &- \left(\bar{\psi} \gamma_\lambda \langle \partial_\rho \psi(x) \bar{\psi}(0) \rangle \psi(0) \partial^\lambda \bar{\psi}(x) \gamma^\rho \langle \psi(x) \bar{\psi}(0) \rangle \psi(0) + \bar{\psi} \gamma_\lambda \langle \partial_\rho \psi(x) \bar{\psi}(0) \rangle \psi(0) \partial^\rho \bar{\psi}(x) \gamma^\lambda \langle \psi(x) \bar{\psi}(0) \rangle \psi(0) \right) \\ &\left. + \text{c.c.} \right) \end{aligned} \quad (3.27)$$

⁷Here $\frac{n(n-1)}{2!}$ is the combinatorial coefficient of picking two operators $\bar{\psi}\psi$ out of n to make contractions, and 8 is the combined factor of symmetry due to contributing factors, such as two possible choices of ψ and $\bar{\psi}$ from $(\bar{\psi}\psi)^2$ to make a contraction, times two possible choices of $T_{\lambda\rho}$ in the vertex $T_{\lambda\rho} T^{\lambda\rho}$ to contract with, times two possible terms in $T_{\lambda\rho}$ to contract with, which result in identical contribution due to the symmetry of $T_{\lambda\rho}$.

Extracting the singular in the $d \rightarrow 2$ limit terms and simplifying these $\mathcal{O}(\lambda)$ expressions using $\mathcal{O}(1)$ equations of motion, we obtain (we relegate details to appendix B)

$$\Delta_{(2)}^{(2)}(\bar{\psi}\psi)^n = \frac{\lambda n(n-1)}{\pi\epsilon} (\bar{\psi}\psi)^{n-2} \partial_\mu \bar{\psi}\psi \bar{\psi} \partial^\mu \psi, \quad (3.28)$$

$$\Delta_{(2)}^{(3)}(\bar{\psi}\psi)^n = \frac{\lambda n(n-1)}{2\pi\epsilon} (\bar{\psi}\psi)^{n-2} \left(\bar{\psi} \partial_\mu \psi \bar{\psi} \partial^\mu \psi + \partial_\mu \bar{\psi}\psi \partial^\mu \bar{\psi}\psi \right). \quad (3.29)$$

Combining (3.25), (3.28), (3.29) and the fact that $\Delta_{(1)}(\bar{\psi}\psi)^n = 0$ we obtain

$$\Delta(\bar{\psi}\psi)^n = \frac{\lambda}{2\pi\epsilon} \partial^2 (\bar{\psi}\psi)^n, \quad (3.30)$$

in agreement with the universal expression (1.2).

3.2 $[\bar{\psi}\gamma_\mu\psi]$

We now briefly discuss renormalization of the conserved U(1) current $\bar{\psi}\gamma_\mu\psi$. As reviewed in Introduction section, Noether currents are expected not to be renormalized due to Ward identity. In case of stress-energy tensor a possible exception to this statement is given by identically conserved improvement counterterms (which we, however, derived to be vanishing for the considered models). In case of a free fermion deformed by $T\bar{T}$ a potential counterpart exception can be written down as

$$\frac{1}{\epsilon} \lambda \left(\partial_\mu \partial_\nu - \delta_{\mu\nu} \partial^2 \right) (\bar{\psi}\gamma^\nu\psi) \sim \frac{1}{\epsilon} \lambda \partial^2 (\bar{\psi}\gamma_\mu\psi). \quad (3.31)$$

Yet we have explicitly verified $\Delta(\bar{\psi}\gamma_\mu\psi) = 0$ at one-loop level.⁸

To avoid burdening the reader with excessive and repetitive details of calculation, we skip detailed explanations, and refrain to pointing out that at every single step the calculation is analogous to calculation of $[\bar{\psi}\psi]$ in subsection 3.1.

We then point out that the statement $\Delta(\bar{\psi}\gamma_\mu\psi) = 0$ still fits the universal expression (1.2). Indeed, in two dimensions one obtains $\lambda \partial^2 (\bar{\psi}\gamma_\mu\psi) = 0$ once the $\mathcal{O}(\lambda^0)$ e.o.m. for ψ is taken into account.

3.3 $[\bar{\psi}\gamma_\mu\partial_\nu\psi]$

Consider renormalization of the composite tensor operator $\bar{\psi}\gamma_\mu\partial_\nu\psi$,

$$\bar{\psi}\gamma_\mu\partial_\nu\psi = [\bar{\psi}\gamma_\mu\partial_\nu\psi] + \Delta\bar{\psi}\gamma_\mu\partial_\nu\psi, \quad \Delta\bar{\psi}\gamma_\mu\partial_\nu\psi = \frac{\lambda\mu^\epsilon}{2} \bar{\psi}\gamma_\mu\partial_\nu\psi \int d^d x T_{\lambda\rho} T^{\lambda\rho}, \quad (3.32)$$

where in $\Delta\bar{\psi}\gamma_\mu\partial_\nu\psi$ we take all possible contractions and retain singular in $d \rightarrow 2$ limit terms only. As before, when calculating renormalization of composite operators, we choose to position the composite operator itself at the origin, $x = 0$, without the loss of generality. To calculate $\Delta\bar{\psi}\gamma_\mu\partial_\nu\psi$ we find it convenient to split possible contributions to it into two groups,

$$\Delta\bar{\psi}\gamma_\mu\partial_\nu\psi = \Delta_{(1)}\bar{\psi}\gamma_\mu\partial_\nu\psi + \Delta_{(2)}\bar{\psi}\gamma_\mu\partial_\nu\psi. \quad (3.33)$$

⁸We established this without the use of equations of motion.

First, we consider singular terms originating from contracting the constituent fermion fields of the composite operator $\bar{\psi}\gamma_\mu\partial_\nu\psi$ with any one of the two factors $T_{\lambda\rho}$ in the interaction vertex $T_{\lambda\rho}T^{\lambda\rho}$. Taking into account symmetric properties of the stress-energy tensor we obtain

$$\Delta_{(1)}\bar{\psi}\gamma_\mu\partial_\nu\psi = -\frac{1}{2}\lambda\mu^\epsilon \int d^d x \text{Tr} \left(\langle \partial_\nu\psi(0)\bar{\psi}(x) \rangle \gamma_\lambda \langle \partial_\rho\psi(x)\bar{\psi}(0) \rangle \gamma_\mu + \text{c.c.} \right) T^{\lambda\rho}(x). \quad (3.34)$$

Using here⁹

$$\langle \partial_\nu\psi(x)\bar{\psi}(0) \rangle = \frac{1}{2\pi} \frac{1}{|x|^d} \left(\frac{2x^\mu x_\nu \gamma_\mu}{|x|^2} - \gamma_\nu \right), \quad (3.35)$$

we notice that on the r.h.s. of (3.34) we have terms of the form

$$\text{Tr}(\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho) = 2(\delta_{\mu\nu}\delta_{\lambda\rho} - \delta_{\rho\nu}\delta_{\mu\lambda} + \delta_{\lambda\nu}\delta_{\mu\rho}), \quad (3.36)$$

which are real-valued, and therefore (3.34) can be re-written as

$$\Delta_{(1)}\bar{\psi}\gamma_\mu\partial_\nu\psi = -\lambda \int d^d x \text{Tr} \left(\langle \partial_\nu\psi(0)\bar{\psi}(x) \rangle \gamma_\lambda \langle \partial_\rho\psi(x)\bar{\psi}(0) \rangle \gamma_\mu \right) T^{\lambda\rho}(x). \quad (3.37)$$

Expanding the integrated $T^{\lambda\rho}(x)$ operator around $x = 0$, and keeping only logarithmically divergent at short distances terms, while taking into account that the stress-energy tensor is symmetric and traceless, we obtain

$$\Delta_{(1)}\bar{\psi}\gamma_\mu\partial_\nu\psi = \frac{\lambda}{2\pi^2} \left(\mathcal{I}^{\alpha\beta\rho}{}_\mu \partial_\alpha\partial_\beta T_{\nu\rho} + \mathcal{I}^{\alpha\beta\rho}{}_\nu \partial_\alpha\partial_\beta T_{\mu\rho} + \delta_{\mu\nu} \mathcal{I}^{\alpha\beta\lambda\rho} \partial_\alpha\partial_\beta T_{\lambda\rho} - 4 \mathcal{I}_{\mu\nu}{}^{\alpha\beta\lambda\rho} \partial_\alpha\partial_\beta T_{\lambda\rho} \right), \quad (3.38)$$

where we have identified the master integrals (A.1). Using (A.5), (A.6) one can demonstrate that

$$\Delta_{(1)}\bar{\psi}\gamma_\mu\partial_\nu\psi = 0. \quad (3.39)$$

The second contribution to $\Delta\bar{\psi}\gamma_\mu\partial_\nu\psi$ originates from all possible cross-contractions between the constituent fermions of the composite operator $\bar{\psi}\gamma_\mu\partial_\nu\psi(0)$ with each of the two factors $T_{\lambda\rho}$ in the interaction vertex $T_{\lambda\rho}T^{\lambda\rho}$,

$$\begin{aligned} \Delta_{(2)}\bar{\psi}\gamma_\mu\partial_\nu\psi &= \frac{\lambda}{2} \left(\frac{1}{4} \right)^2 \cdot 4 \\ &\times \left(\int d^d x \left(\bar{\psi}\gamma^\lambda \langle \partial^\rho\psi(x)\bar{\psi}(0) \rangle \gamma_\mu \langle \partial_\nu\psi(0)\bar{\psi}(x) \rangle \gamma_\lambda \partial_\rho\psi + \bar{\psi}\gamma^\lambda \langle \partial^\rho\psi(x)\bar{\psi}(0) \rangle \gamma_\mu \langle \partial_\nu\psi(0)\bar{\psi}(x) \rangle \gamma_\rho \partial_\lambda\psi \right) \right. \\ &- \int d^d x \left(\bar{\psi}\gamma_\lambda \langle \partial_\rho\psi(x)\bar{\psi}(0) \rangle \gamma_\mu \langle \partial_\nu\psi(0)\bar{\psi}(x) \rangle \gamma^\rho\psi + \partial^\lambda\bar{\psi}\gamma^\rho \langle \psi(x)\bar{\psi}(0) \rangle \gamma_\mu \langle \partial_\nu\psi(0)\bar{\psi}(x) \rangle \gamma_\lambda \partial_\rho\psi \right) \\ &- \int d^d x \left(\bar{\psi}\gamma_\lambda \langle \partial_\rho\psi(x)\bar{\psi}(0) \rangle \gamma_\mu \langle \partial_\nu\psi(0)\bar{\psi}(x) \rangle \gamma^\lambda\psi + \partial^\rho\bar{\psi}\gamma^\lambda \langle \psi(x)\bar{\psi}(0) \rangle \gamma_\mu \langle \partial_\nu\psi(0)\bar{\psi}(x) \rangle \gamma_\lambda \partial_\rho\psi \right) \\ &\left. + \int d^d x \left(\partial^\rho\bar{\psi}\gamma^\lambda \langle \psi(x)\bar{\psi}(0) \rangle \gamma_\mu \langle \partial_\nu\psi(0)\bar{\psi}(x) \rangle \gamma_\lambda\psi + \partial^\lambda\bar{\psi}\gamma^\rho \langle \psi(x)\bar{\psi}(0) \rangle \gamma_\mu \langle \partial_\nu\psi(0)\bar{\psi}(x) \rangle \gamma_\lambda\psi \right) \right). \end{aligned} \quad (3.40)$$

⁹As discussed above, we set $d = 2$ in the factors which are not going to contribute to simple poles, which are expected to occur in the linear order of the perturbation theory.

While this is somewhat laborious, one can show that each of the eight terms contributing to $\Delta_{(2)}\bar{\psi}\gamma_\mu\partial_\nu\psi$ is individually finite in $d \rightarrow 2$ limit. As an example, consider the first term

$$\begin{aligned}\Delta_{(2)}^{(1)}\bar{\psi}\gamma_\mu\partial_\nu\psi &\equiv \frac{\lambda}{8} \int d^d x \bar{\psi}\gamma^\lambda \langle \partial^\rho \psi(x) \bar{\psi}(0) \rangle \gamma_\mu \langle \partial_\nu \psi(0) \bar{\psi}(x) \rangle \gamma_\lambda \partial_\rho \psi \\ &= \frac{\lambda}{64\pi^2} \left(\mathcal{I}^{\alpha\beta} \partial_\alpha \partial_\beta (\bar{\psi}\gamma^\lambda \gamma^\rho \gamma_\mu \gamma_\nu \gamma_\lambda \partial_\rho \psi) - 2\mathcal{I}^{\alpha\beta\rho\sigma} \partial_\alpha \partial_\beta (\bar{\psi}\gamma^\lambda \gamma_\sigma \gamma_\mu \gamma_\nu \gamma_\lambda \partial_\rho \psi) \right. \\ &\quad \left. - 2\mathcal{I}^{\alpha\beta\sigma}{}_\nu \partial_\alpha \partial_\beta (\bar{\psi}\gamma^\lambda \gamma^\rho \gamma_\mu \gamma_\sigma \gamma_\lambda \partial_\rho \psi) + 4\mathcal{I}^{\alpha\beta\rho\sigma\tau}{}_\nu \partial_\alpha \partial_\beta (\bar{\psi}\gamma^\lambda \gamma_\sigma \gamma_\mu \gamma_\tau \gamma_\lambda \partial_\rho \psi) \right).\end{aligned}\tag{3.41}$$

This expression can be quickly simplified in *Mathematica* using (A.4), (A.5), (A.6), rendering $\Delta_{(2)}^{(1)}\bar{\psi}\gamma_\mu\partial_\nu\psi = 0$. Analogous calculation will show that the other contributions to (3.40) vanish as well. Combining this with (3.39) we conclude that

$$\Delta\bar{\psi}\gamma_\mu\partial_\nu\psi = 0.\tag{3.42}$$

The result (3.42) for the linear order correction to the renormalized $[\bar{\psi}\gamma_\mu\partial_\nu\psi]$ actually agrees with our universal expression (1.2). This is due to the fact that once the $\mathcal{O}(\lambda^0)$ equations of motion $\gamma^\mu\partial_\mu\psi = 0$ are imposed, the $\mathcal{O}(\lambda)$ contribution on the r.h.s. of (1.2) can be simplified in two dimensions,

$$\partial^2(\bar{\psi}\gamma_\mu\partial_\nu\psi) = 0.\tag{3.43}$$

3.4 $[T_{\mu\nu}]$

Using expression for the stress-energy tensor (3.4) and eq. (3.42) derived in previous subsection for the renormalization of $[\bar{\psi}\gamma_\mu\partial_\nu\psi]$ we conclude that

$$[T_{\mu\nu}] = T_{\mu\nu} + \mathcal{O}(\lambda^2),\tag{3.44}$$

$$\partial^2 T_{\mu\nu} = 0,\tag{3.45}$$

which agree with the universal relation (1.2). This also agrees with the general argument of [1], which can be applied here to argue that $[T_{\mu\nu}] = T_{\mu\nu}$ to all orders in λ .

4 Massive case

In previous sections, we extensively explored the renormalization of various composite operators in free massless scalar and fermionic theories deformed by the $T\bar{T}$ operator. By carrying our explicit perturbative calculation to linear order in the $T\bar{T}$ coupling, we have demonstrated that all of the considered operators are renormalized by a universal counterterm (1.2). It has been suggested in [1] that such a universal behavior is a general feature in the $T\bar{T}$ -deformed CFTs.

It is therefore crucial to explore the fate of the composite operators in the non-conformal field theories deformed by the $T\bar{T}$ operator. Indeed, dimensional considerations suggest that QFTs with a mass scale could exhibit a richer structure of counterterms compared to (1.2). In this section, we intend to investigate this question by considering $T\bar{T}$ deformation of a free massive scalar theory.

One of the important results which we have obtained for the $T\bar{T}$ -deformed free massless scalar and fermionic theories in sections, 2 3 was that the improvement (‘gravitational counterterm’) contribution, typically appearing in the renormalization of the stress-energy tensor, is trivial in those theories. In fact, we have demonstrated that the stress-energy tensor does not receive any counterterms at the linear order in the $T\bar{T}$ coupling. We expect this conclusion will not persist in the case of $T\bar{T}$ deformed massive theories. In fact, it is well known that even a free massive scalar field requires a cosmological constant counterterm, which contributes the identity operator to the renormalization of the stress-energy tensor.

Recall that due to the Ward identity the possible renormalization of the stress-energy tensor is either a total derivative, i.e., an improvement term (which formally can be seen as arising from the gravitational counterterm), or identity operator, or it is proportional to the stress-energy tensor itself. In this section we will consider the free massive scalar deformed by the $T\bar{T}$ operator, and study it to the linear order in the $T\bar{T}$ coupling λ ,¹⁰

$$I = \int d^d x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \delta_\phi \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \delta_m \phi^2 \right) + \frac{\lambda \mu^\epsilon}{4} \int d^d x \left(m^4 \phi^4 - ((\partial \phi)^2)^2 \right), \quad (4.1)$$

where the $T\bar{T}$ operator at $\mathcal{O}(\lambda^0)$ is given by

$$T\bar{T}_0 = \frac{1}{4} \left(m^4 \phi^4 - ((\partial \phi)^2)^2 \right), \quad (4.2)$$

and $d = 2 - \epsilon$. We argue that the $\mathcal{O}(\lambda)$ renormalization of the stress-energy tensor is given by

$$[T_{\mu\nu}] = \left(1 - \frac{3\lambda m^2}{2\pi\epsilon} \right) \left(T_{\mu\nu} + \frac{m^2}{4\pi\epsilon} \delta_{\mu\nu} \mathbb{I} \right) + \mathcal{O}(\lambda^2). \quad (4.3)$$

Notice that the form (4.3) is consistent with the Ward identity, and it does not include the improvement counterterm contribution. However, the $T\bar{T}$ coupling λ induces a multiplicative renormalization of the stress-energy tensor.

We now proceed to the derivation of (4.3). First of all, the mass and wave-function renormalization counterterm contributions at $\mathcal{O}(\lambda)$ order in the massive action (4.1) are defined by

$$\delta_m = -\frac{3\lambda m^2}{2\pi\epsilon}, \quad \delta_\phi = -\frac{\lambda m^2}{\pi\epsilon}, \quad (4.4)$$

as, e.g., can be readily seen by renormalizing the two-point function

$$\langle \phi(p) \phi(q) \rangle = (2\pi)^d \delta(p+q) \left(\frac{1}{p^2 + m^2} + \frac{-m^2 \delta_m - p^2 \delta_\phi}{(p^2 + m^2)^2} \right) \quad (4.5)$$

$$- \frac{\lambda \mu^\epsilon}{4} \int d^d x \left(m^4 \langle \phi^4(x) \phi(p) \phi(q) \rangle - \langle ((\partial \phi(x))^2)^2 \phi(p) \phi(q) \rangle \right), \quad (4.6)$$

¹⁰This action is written in terms of finite physical mass m and field ϕ , which are related to the bare m_0 , ϕ_0 via the standard equations

$$\phi_0 = \phi \sqrt{1 + \delta_\phi}, \quad m_0^2(1 + \delta_\phi) = m^2(1 + \delta_m),$$

where the counterterms $\delta_{m,\phi}$ are non-trivial in the massive case even at the linear order in coupling λ .

and using

$$\langle \phi^2 \rangle = \frac{(m^2)^{-\frac{\epsilon}{2}}}{2\pi\epsilon}, \quad \langle \partial_\mu \phi \partial_\nu \phi \rangle = -\frac{m^2}{4\pi\epsilon} \delta_{\mu\nu}, \quad \langle (\partial\phi)^2 \rangle = -\frac{m^2}{2\pi\epsilon}. \quad (4.7)$$

The canonical stress-energy tensor to the linear order $\mathcal{O}(\lambda)$, following from the renormalized action (4.1), is given by

$$T_{\mu\nu} = \partial_\mu \phi_0 \partial_\nu \phi_0 (1 - \lambda (\partial\phi_0)^2) - \frac{1}{2} \delta_{\mu\nu} \left((\partial\phi_0)^2 + m_0^2 \phi_0^2 + 2\lambda T\bar{T}_0 \right), \quad (4.8)$$

where we re-absorbed the counterterms into the bare mass m_0 and field ϕ_0 . We will be using the stress-energy tensor (4.8) expressed in terms of bare m_0 , ϕ_0 , but to lighten the notation we will skip putting the zero subscript whenever the resulting expression does not get affected to the linear order in λ .

Using (4.7), (4.8) we immediately notice that to $\mathcal{O}(\lambda^0)$, i.e., in the free theory, due to non-vanishing mass, the renormalized stress-energy tensor is determined by the cosmological constant renormalization,

$$[T_{\mu\nu}] = T_{\mu\nu} + \frac{m^2}{4\pi\epsilon} \delta_{\mu\nu} \mathbb{I} + \mathcal{O}(\lambda), \quad (4.9)$$

in agreement with (4.3). The rest of this section is dedicated to derivation of the $\mathcal{O}(\lambda)$ contribution to (4.3). Specifically, we will extend the derivation in section 2 and find the terms contributing to $[T_{\mu\nu}]$ due to non-vanishing mass m . We will denote the corresponding $\mathcal{O}(\lambda)$ contributions to renormalized stress-energy tensor $[T_{\mu\nu}]$ with $\Delta_{(m)}(T_{\mu\nu})$, that is, $T_{\mu\nu} = [T_{\mu\nu}] + \Delta_{(m)}(T_{\mu\nu}) + \mathcal{O}(\lambda^2) + \dots$, where ellipsis stand for contributions at $m = 0$. Using (4.8) we obtain to $\mathcal{O}(\lambda)$

$$\Delta_{(m)}(T_{\mu\nu}) = \Delta_{(m)}(\partial_\lambda \phi \partial_\rho \phi) \left(\delta_\mu^\lambda \delta_\nu^\rho - \frac{1}{2} \delta^{\lambda\rho} \delta_{\mu\nu} \right) - \lambda \Delta_{(m)}(\partial_\mu \phi \partial_\nu \phi (\partial\phi)^2) \quad (4.10)$$

$$- \frac{1}{2} m^2 \delta_{\mu\nu} \Delta(\phi^2) - \lambda \delta_{\mu\nu} \Delta_{(m)}(T\bar{T}_0), \quad (4.11)$$

where renormalization $\Delta_{(m)}$ of various operators on the r.h.s. is to be taken up to the appropriate order in λ , so that $\Delta_{(m)}(T_{\mu\nu})$ is of the linear order in λ . Notice that $\Delta(\phi^2)$ in (4.10) includes both the zero-mass and (possible) finite-mass contributions.

From the action (4.1), (4.4) we can derive the renormalized composite operator $[T\bar{T}_0]$ to $\mathcal{O}(\lambda)$,

$$[T\bar{T}_0] = \mu^{-\epsilon} \frac{\partial L}{\partial \lambda} = T\bar{T}_0 - \frac{m^2}{4\pi\epsilon} \left(2(\partial\phi)^2 + 3m^2 \phi^2 \right) \Rightarrow \Delta_{(m)}(T\bar{T}_0) = \frac{m^2}{4\pi\epsilon} \left(2(\partial\phi)^2 + 3m^2 \phi^2 \right). \quad (4.12)$$

As a consistency check, the relation (4.12) can alternatively be obtained by directly renormalizing (4.2) at $\mathcal{O}(\lambda^0)$ order using (4.7) and (here we skip for now renormalization terms proportional to the identity operator $\mathbb{I}$, these will be restored in the final expression)

$$\begin{aligned} \phi^4 &= [\phi^4] + 6\phi^2 \langle \phi^2 \rangle + \dots = [\phi^4] + \frac{3}{\pi\epsilon} \phi^2 + \dots, \\ ((\partial\phi)^2)^2 &= [((\partial\phi)^2)^2] + 2\langle (\partial\phi)^2 \rangle (\partial\phi)^2 + 4\langle \partial_\alpha \phi \partial_\beta \phi \rangle \partial^\alpha \phi \partial^\beta \phi + \dots \\ &= [((\partial\phi)^2)^2] - \frac{2m^2}{\pi\epsilon} (\partial\phi)^2 + \dots \end{aligned} \quad (4.13)$$

Similarly, to $\mathcal{O}(\lambda^0)$ we derive

$$\begin{aligned}\Delta_{(m)}(\partial_\mu\phi\partial_\nu\phi(\partial\phi)^2) &= 2\langle\partial_\mu\phi\partial_\alpha\phi\rangle\partial_\nu\phi\partial^\alpha\phi + 2\langle\partial_\nu\phi\partial_\alpha\phi\rangle\partial_\mu\phi\partial^\alpha\phi \\ &\quad + \langle\partial_\mu\phi\partial_\nu\phi\rangle(\partial\phi)^2 + \partial_\mu\phi\partial_\nu\phi\langle(\partial\phi)^2\rangle \\ &= -\frac{3m^2}{2\pi\epsilon}\partial_\mu\phi\partial_\nu\phi - \frac{m^2}{4\pi\epsilon}\delta_{\mu\nu}(\partial\phi)^2.\end{aligned}\quad (4.14)$$

The mass contribution to renormalization of ϕ^2 at $\mathcal{O}(\lambda)$, which needs to be added to the expression (2.7), evaluated for $n = 2$, is given by

$$\Delta_{(m)}(\phi^2) = -\frac{\lambda\mu^\epsilon}{4}m^4\phi^2\int d^dx\phi^4 = -3\lambda m^4\int d^dx\langle\phi(0)\phi(x)\rangle^2\phi^2, \quad (4.15)$$

where we recall that for the free massive scalar the propagator is given by

$$\langle\phi(0)\phi(x)\rangle = \frac{1}{(2\pi)^{\frac{d}{2}}}\left(\frac{|m|}{|x|}\right)^{\frac{d-2}{2}}K_{\frac{d-2}{2}}(|m||x|). \quad (4.16)$$

It can be seen that (4.15) does not contain any logarithmic divergencies (is regular in $d \rightarrow 2$ limit),

$$\Delta_{(m)}(\phi^2) = 0. \quad (4.17)$$

Next, extending the calculation of subsection 2.2, we consider mass contributions to renormalization of $\partial_\mu\phi\partial_\nu\phi$,

$$\begin{aligned}\Delta_{(m)}(\partial_\mu\phi\partial_\nu\phi) &= \frac{\lambda}{4}\int d^dx\left(4\partial_\mu\partial_\alpha\langle\phi(0)\phi(x)\rangle\partial_\nu\partial^\alpha\langle\phi(0)\phi(x)\rangle(\partial\phi(x))^2\right. \\ &\quad \left.+ 8\partial_\mu\partial_\alpha\langle\phi(0)\phi(x)\rangle\partial_\nu\partial_\beta\langle\phi(0)\phi(x)\rangle\partial^\alpha\phi(x)\partial^\beta\phi(x)\right)\Big|_m \\ &\quad - 3m^4\int d^dx\partial_\mu\langle\phi(0)\phi(x)\rangle\partial_\nu\langle\phi(0)\phi(x)\rangle\phi^2.\end{aligned}\quad (4.18)$$

Here the first two terms on the r.h.s. are recognized as having already appeared in the analogous calculation (2.12) in the massless case. Therefore now in these terms we only retain contributions due to the mass m , as we indicated with the subscript m . The last term in (4.18) is due to the m^4 term in the $T\bar{T}$ operator (4.2). We denote these two groups of contributions correspondingly as $\Delta_{(m)}^{(1,2)}(\partial_\mu\phi\partial_\nu\phi)$,

$$\Delta_{(m)}(\partial_\mu\phi\partial_\nu\phi) = \Delta_{(m)}^{(1)}(\partial_\mu\phi\partial_\nu\phi) + \Delta_{(m)}^{(2)}(\partial_\mu\phi\partial_\nu\phi). \quad (4.19)$$

Using the expansion around zero mass,

$$\partial_\mu\partial_\alpha\langle\phi(0)\phi(x)\rangle = \frac{\Gamma\left(\frac{d+2}{2}\right)}{\pi^{\frac{d}{2}}}\frac{1}{|x|^{d+2}}\left(x_\alpha x_\mu - \frac{\delta_{\alpha\mu}}{d}|x|^2\right)\left(1 - \frac{m^2|x|^2}{2d}\right) \quad (4.20)$$

$$+ \frac{\delta_{\mu\alpha}}{4d\pi^{\frac{d}{2}}}\frac{m^2|x|^2}{|x|^d}\left(\Gamma\left(\frac{d-2}{2}\right) + \Gamma\left(\frac{2-d}{2}\right)\frac{(m|x|)^{d-2}}{2^{d-2}}\right), \quad (4.21)$$

we obtain

$$\Delta_{(m)}^{(1)}(\partial_\mu\phi\partial_\nu\phi) = \Delta_{(m)}^{(1,1)}(\partial_\mu\phi\partial_\nu\phi) + \Delta_{(m)}^{(1,2)}(\partial_\mu\phi\partial_\nu\phi), \quad (4.22)$$

where $\Delta_{(m)}^{(1,1)}(\partial_\mu\phi\partial_\nu\phi)$ is due to the first line in (4.21) and $\Delta_{(m)}^{(1,2)}(\partial_\mu\phi\partial_\nu\phi)$ is due to the second line in (4.21). Specifically,

$$\begin{aligned} \Delta_{(m)}^{(1,1)}(\partial_\mu\phi\partial_\nu\phi) &= -\frac{\lambda m^2}{2\pi^2} ((\partial\phi)^2\delta^{\alpha\beta} + 2\partial^\alpha\phi\partial^\beta\phi) \left(\mathcal{I}_{\mu\nu\alpha\beta} - \frac{1}{2}\mathcal{I}_{\mu\alpha}\delta_{\nu\beta} - \frac{1}{2}\mathcal{I}_{\nu\alpha}\delta_{\mu\beta} + \frac{1}{4}\delta_{\mu\alpha}\delta_{\nu\beta} \right) \\ &= -\frac{\lambda m^2}{2\pi\epsilon} (\partial\phi)^2\delta_{\mu\nu}, \end{aligned} \quad (4.23)$$

and

$$\begin{aligned} \Delta_{(m)}^{(1,2)}(\partial_\mu\phi\partial_\nu\phi) &= \frac{\lambda m^2}{4\pi^2} ((\partial\phi)^2\delta_\nu^\alpha + 2\partial^\alpha\phi\partial_\nu\phi) \\ &\quad \times \left(\left(I_{\mu\alpha} - \frac{\delta_{\mu\alpha}}{d} I \right) \Gamma\left(\frac{d-2}{2}\right) + \frac{m^{d-2}}{2^{d-2}} \Gamma\left(\frac{2-d}{2}\right) \int d^d x \frac{1}{|x|^{d+2}} \left(x_\mu x_\alpha - \frac{\delta_{\mu\alpha}}{d} |x|^2 \right) \right) \\ &= \frac{\lambda m^2}{4\pi^2} ((\partial\phi)^2\delta_\nu^\alpha + 2\partial^\alpha\phi\partial_\nu\phi) \int \frac{d^d x}{|x|^{2d}} \left(x_\mu x_\alpha - \frac{\delta_{\mu\alpha}}{d} |x|^2 \right) \left(\Gamma\left(\frac{d-2}{2}\right) + \frac{(m|x|)^{d-2}}{2^{d-2}} \Gamma\left(\frac{2-d}{2}\right) \right). \end{aligned} \quad (4.24)$$

Here gamma-functions cancel each other's singularities at $d = 2$, and as a result $\Delta_{(m)}^{(1,2)}$ does not have a singularity at $d = 2$. Finally, for the calculation of $\Delta_{(m)}^{(2)}(\partial_\mu\phi\partial_\nu\phi)$, corresponding to the last term on the r.h.s. of (4.18), it is sufficient to use the massless propagator (2.8), which gives

$$\Delta_{(m)}^{(2)}(\partial_\mu\phi\partial_\nu\phi) = -\frac{3\lambda m^4}{4\pi^2} \mathcal{I}_{\mu\nu} \phi^2 = -\frac{3\lambda m^4}{4\pi\epsilon} \delta_{\mu\nu} \phi^2. \quad (4.25)$$

Combining everything together we obtain

$$\Delta_{(m)}(\partial_\mu\phi\partial_\nu\phi) = -\frac{\lambda m^2}{4\pi\epsilon} (3m^2\phi^2 + 2(\partial\phi)^2) \delta_{\mu\nu}. \quad (4.26)$$

Using (2.10) for $n = 2$, (4.12), (4.14), (4.17), (4.26) in (4.10) we obtain

$$\Delta_{(m)}(T_{\mu\nu}) = \frac{3\lambda m^2}{2\pi\epsilon} T_{\mu\nu} + \mathcal{O}(\lambda^2), \quad (4.27)$$

which completes our derivation of (4.3).

5 Discussion

In this paper we considered perturbative renormalization of the composite operators in the $T\bar{T}$ -deformed two-dimensional free field theories. In the massless case renormalization of various operators satisfies (1.2). This universal relation holds true regardless of whether the operator is spinless or not, and whether it is a primary or a descendant. Our explicit calculations agree with [1], where a generic $T\bar{T}$ deformed CFT was studied. However, in

the massive case there is no universal formula to compare to. Hence, it would be interesting to derive such a formula for a generic $T\bar{T}$ -deformed gapped quantum field theory.

Conserved Noether currents correspond to a particularly interesting class of composite operators whose renormalization one can study. Due to the Ward identities an allowed divergent structures of the Noether current — the so-called improvement counterterms [31] — are separately conserved. The stress-energy tensor is an example of the Noether current where improvement terms are directly related to the gravitational counterterms induced in a theory coupled to a curved background [31, 32].

To understand the structure of improvement counterterms in the $T\bar{T}$ -deformed field theories, we considered renormalization of the canonical stress-energy tensor and the $U(1)$ current in the case of scalar and Dirac fields. We found that, to linear order in the $T\bar{T}$ coupling, neither stress-energy tensor nor the $U(1)$ current are renormalized in the massless case. However, this is not true if the undeformed theory is gapped. Since the very definition of the $T\bar{T}$ deformation in the presence of curvature is obscure, these observations partially unravel the way a $T\bar{T}$ theory couples to gravity and add an extra incentive to the program of generalizing the $T\bar{T}$ deformation to the theories living on a curved background.

It would be interesting to extend the analysis of this paper to other quantum field theories. Of particular interest is the case of interacting CFTs. While we leave this problem for future research, we would like to sketch a possible route towards this direction.

As an example of the $T\bar{T}$ -deformed interacting CFT, let us consider the Wess-Zumino-Witten (WZW) model on the group manifold G in the presence of $T\bar{T}$ deformation. In conformal gauge for the two-dimensional metric, the (anti-)holomorphic components of the stress-energy tensor can be written in terms of the Kac-Moody currents, $T(z) = \frac{1}{\kappa} j^a j^a(z)$, $\tilde{T}(\bar{z}) = \frac{1}{\kappa} \tilde{j}^a \tilde{j}^a(\bar{z})$. Here the constant κ is determined by the level of the WZW model, the index a is summed over n values, where n is the rank of the group G , and $j^a(z)$, $\tilde{j}^a(\bar{z})$ represent the (anti-)holomorphic Kac-Moody currents. These currents exhibit simple correlation functions $\langle j^a(z) j^b(0) \rangle = \delta^{ab}/z^2$, $\langle \tilde{j}^a(\bar{z}) \tilde{j}^b(0) \rangle = \delta^{ab}/\bar{z}^2$, which can be thought of as correlators of the Noether currents $\partial\phi^a$, $\bar{\partial}\phi^a$ associated with translation invariance in the space of n free massless scalar fields ϕ^a .

Now notice that the calculation in section 2.4 for a single free massless scalar is applicable to the case of n decoupled scalars ϕ^a . In particular, the non-renormalization of the stress-energy tensor, in this case, rests on manipulating the integrals over the correlation functions of the currents $\partial\phi^a$, $\bar{\partial}\phi^a$. Furthermore, since the Kac-Moody currents have the same correlation functions as $\partial\phi^a$ and $\bar{\partial}\phi^a$, we can literally repeat the same arguments to conclude that the stress-energy tensor for an arbitrary WZW model is not renormalized in the presence of $T\bar{T}$ deformation.

Finally, we would like to point out a possible application of our results in the context of entanglement entropy calculations for a $T\bar{T}$ -deformed field theory. In general, the flow of entanglement entropy along the one-parameter family of theories defined by (1.1) can be formulated in terms of the correlation function of the renormalized $T\bar{T}$ operator and a modular Hamiltonian associated with the entangling region of interest [33]. This correlation function can be calculated perturbatively in λ , see [34–38], provided that the energy cut off is sufficiently low. Our findings might be useful to explicitly carry out this sort of

calculations. In particular, a comparison with the proposed holographic duals can be done to understand better strongly coupled $T\bar{T}$ theories, see e.g., [39, 40] and references therein for a related discussion.

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A Master integrals

In this appendix we calculate the divergent parts of the master integrals

$$\mathcal{I}_{\alpha_1\beta_1\cdots\alpha_k\beta_k} = \int d^d x f(|x|) \frac{x_{\alpha_1} x_{\beta_1} \cdots x_{\alpha_k} x_{\beta_k}}{|x|^{2(d-1+k)}}, \quad (\text{A.1})$$

where $k = 0, 1, 2, \dots$, and $f(|x|)$ is a test function introduced to regularize spurious IR divergence (which is an artifact of expanding our integrands around $x = 0$). Imposing the condition $f(0) = 1$ we ensure that presence of the test function $f(|x|)$ does not affect the form of the UV divergencies of the master integrals, which we are interested in. In this paper we define

$$d = 2 - \epsilon \quad (\text{A.2})$$

and for the purpose of regularization we are interested in terms which are singular in the $\epsilon \rightarrow 0$ limit.

For instance, in the case $k = 0$ in (A.1), integrating by parts, expanding around $\epsilon = 0$, and keeping only divergent term, we find

$$\begin{aligned} \mathcal{I} &= \int d^d x f(|x|) \frac{1}{|x|^{2d-2}} \\ &= \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \frac{1}{\epsilon} \int d|x| \left(\frac{d}{d|x|} (f(|x|) |x|^\epsilon) - \frac{df}{d|x|} (1 + \mathcal{O}(\epsilon)) \right) \\ &\rightarrow \frac{2\pi}{\epsilon}. \end{aligned} \quad (\text{A.3})$$

Similarly, we calculate

$$\mathcal{I}_{\alpha\beta} = \frac{\pi}{\epsilon} \delta_{\alpha\beta}, \quad (\text{A.4})$$

$$\mathcal{I}_{\alpha\beta\mu\nu} = \frac{\pi}{4\epsilon} (\delta_{\alpha\beta}\delta_{\mu\nu} + (\beta \leftrightarrow \mu) + (\beta \leftrightarrow \nu)), \quad (\text{A.5})$$

$$I_{\alpha\beta\mu\nu\rho\sigma} = \frac{\pi}{24\epsilon} (\delta_{\alpha\beta}(\delta_{\mu\nu}\delta_{\rho\sigma} + (\nu \leftrightarrow \rho) + (\nu \leftrightarrow \sigma)) + (\beta \leftrightarrow \mu) + (\beta \leftrightarrow \nu) + (\beta \leftrightarrow \rho) + (\beta \leftrightarrow \sigma)). \quad (\text{A.6})$$

B Details of $[(\bar{\psi}\psi)^n]$ calculation

Expanding around $d = 2$ and keeping only singular terms in (3.26) we obtain (where c_2 and c_3 are real-valued, hence the coefficient in front of them is doubled, while c_1 is defined to explicitly contain its value plus the complex conjugate)

$$\Delta_{(2)}^{(2)}(\bar{\psi}\psi)^n = \frac{\lambda n(n-1)}{64\pi^2} (\bar{\psi}\psi)^{n-2} (c_1 + 2c_2 + 2c_3), \quad (\text{B.1})$$

$$\begin{aligned} c_1 = & -2\partial_\alpha(\bar{\psi}\gamma^\lambda\gamma^\rho\psi(0)\bar{\psi}(0)\gamma_\nu\gamma_\lambda\partial_\rho\psi)\mathcal{I}^{\alpha\nu} + 4\partial_\alpha(\bar{\psi}\gamma^\lambda\gamma_\mu\psi(0)\bar{\psi}(0)\gamma_\nu\gamma_\lambda\partial_\rho\psi)\mathcal{I}^{\alpha\mu\nu\rho} \\ & - 2\partial_\alpha(\bar{\psi}\gamma^\lambda\gamma^\rho\psi(0)\bar{\psi}(0)\gamma_\nu\gamma_\rho\partial_\lambda\psi)\mathcal{I}^{\alpha\nu} + 4\partial_\alpha(\bar{\psi}\gamma^\lambda\gamma_\mu\psi(0)\bar{\psi}(0)\gamma_\nu\gamma_\rho\partial_\lambda\psi)\mathcal{I}^{\alpha\mu\nu\rho} \\ & + \text{c.c.}, \end{aligned} \quad (\text{B.2})$$

$$c_2 = \partial^\lambda\bar{\psi}\gamma^\rho\gamma_\mu\psi(0)\bar{\psi}(0)\gamma_\nu\gamma_\lambda\partial_\rho\psi\mathcal{I}^{\mu\nu} + \partial^\rho\bar{\psi}\gamma^\lambda\gamma_\mu\psi(0)\bar{\psi}(0)\gamma_\nu\gamma_\lambda\partial_\rho\psi\mathcal{I}^{\mu\nu}, \quad (\text{B.3})$$

$$\begin{aligned} c_3 = & \frac{1}{2}\partial_\alpha\partial_\beta(\bar{\psi}\gamma_\lambda\gamma_\rho\psi(0)\bar{\psi}(0)\gamma^\lambda\gamma^\rho\psi)\mathcal{I}^{\alpha\beta} - \partial_\alpha\partial_\beta(\bar{\psi}\gamma_\lambda\gamma_\rho\psi(0)\bar{\psi}(0)\gamma_\nu\gamma^\rho\psi)\mathcal{I}^{\alpha\beta\lambda\nu} \\ & - \partial_\alpha\partial_\beta(\bar{\psi}\gamma_\lambda\gamma_\nu\psi(0)\bar{\psi}(0)\gamma^\lambda\gamma_\rho\psi)\mathcal{I}^{\alpha\beta\nu\rho} + 2\partial_\alpha\partial_\beta(\bar{\psi}\gamma_\lambda\gamma_\mu\psi(0)\bar{\psi}(0)\gamma_\nu\gamma_\rho\psi)\mathcal{I}^{\alpha\beta\lambda\rho\mu\nu} \\ & + \frac{1}{2}\partial_\alpha\partial_\beta(\bar{\psi}\gamma_\lambda\gamma_\rho\psi(0)\bar{\psi}(0)\gamma^\rho\gamma^\lambda\psi)\mathcal{I}^{\alpha\beta} - \partial_\alpha\partial_\beta(\bar{\psi}\gamma_\lambda\gamma_\rho\psi(0)\bar{\psi}(0)\gamma_\nu\gamma^\lambda\psi)\mathcal{I}^{\alpha\beta\rho\nu} \\ & - \partial_\alpha\partial_\beta(\bar{\psi}\gamma_\lambda\gamma_\nu\psi(0)\bar{\psi}(0)\gamma_\rho\gamma^\lambda\psi)\mathcal{I}^{\alpha\beta\nu\rho} + 2\partial_\alpha\partial_\beta(\bar{\psi}\gamma_\lambda\gamma_\mu\psi(0)\bar{\psi}(0)\gamma_\nu\gamma^\lambda\psi)\mathcal{I}^{\alpha\beta\mu\nu}. \end{aligned} \quad (\text{B.4})$$

Using $\mathcal{O}(\lambda^0)$ e.o.m. one can show that

$$c_1 = 4c_2 = 4c_3 = \frac{32\pi}{\epsilon} \partial_\mu(\bar{\psi}\psi(0))\partial^\mu(\bar{\psi}(0)\psi). \quad (\text{B.5})$$

Using this expression in (B.1) we obtain (3.28).

Finally, expanding around $d = 2$ and keeping only singular terms in (3.27) we obtain (this expression is real-valued, so adding complex conjugate to it simply doubles its value)

$$\begin{aligned} \Delta_{(2)}^{(3)}(\bar{\psi}\psi)^n = & -\frac{\lambda n(n-1)}{64\pi^2} (\bar{\psi}\psi)^{n-2} \\ & \times \left(\bar{\psi}(0)\gamma_\mu\gamma_\lambda\partial_\rho\psi(0)\bar{\psi}(0)\gamma_\nu\gamma^\lambda\partial^\rho\psi(0)\mathcal{I}^{\mu\nu} + \bar{\psi}(0)\gamma_\mu\gamma_\rho\partial_\lambda\psi(0)\bar{\psi}(0)\gamma_\nu\gamma^\lambda\partial^\rho\psi(0)\mathcal{I}^{\mu\nu} \right. \\ & - \partial_\alpha(\bar{\psi}(0)\gamma_\mu\gamma_\lambda\partial_\rho\psi\bar{\psi}(0)\gamma^\lambda\gamma^\rho\psi)\mathcal{I}^{\mu\alpha} + 2\partial_\alpha(\bar{\psi}(0)\gamma_\mu\gamma_\lambda\partial_\rho\psi\bar{\psi}(0)\gamma_\nu\gamma^\rho\psi)\mathcal{I}^{\alpha\mu\nu\lambda} \\ & - \partial_\alpha(\bar{\psi}(0)\gamma_\mu\gamma_\lambda\partial_\rho\psi\bar{\psi}(0)\gamma^\rho\gamma^\lambda\psi)\mathcal{I}^{\mu\alpha} + 2\partial_\alpha(\bar{\psi}(0)\gamma_\mu\gamma_\lambda\partial_\rho\psi\bar{\psi}(0)\gamma_\nu\gamma^\lambda\psi)\mathcal{I}^{\alpha\mu\nu\rho} \\ & + \frac{1}{2}\partial_\alpha\partial_\beta(\bar{\psi}\gamma_\lambda\gamma_\rho\psi(0)\bar{\psi}\gamma^\lambda\gamma^\rho\psi(0))\mathcal{I}^{\alpha\beta} - \partial_\alpha\partial_\beta(\bar{\psi}\gamma_\lambda\gamma_\rho\psi(0)\bar{\psi}\gamma^\lambda\gamma_\nu\psi(0))\mathcal{I}^{\alpha\beta\rho\nu} \\ & - \partial_\alpha\partial_\beta(\bar{\psi}\gamma_\lambda\gamma_\mu\psi(0)\bar{\psi}\gamma^\lambda\gamma^\rho\psi(0))\mathcal{I}^{\alpha\beta\mu}_\rho + 2\partial_\alpha\partial_\beta(\bar{\psi}\gamma_\lambda\gamma_\mu\psi(0)\bar{\psi}\gamma^\lambda\gamma_\nu\psi(0))\mathcal{I}^{\alpha\beta\mu\nu} \\ & + \frac{1}{2}\partial_\alpha\partial_\beta(\bar{\psi}\gamma_\rho\gamma_\lambda\psi(0)\bar{\psi}\gamma^\lambda\gamma^\rho\psi(0))\mathcal{I}^{\alpha\beta} - \partial_\alpha\partial_\beta(\bar{\psi}\gamma_\rho\gamma_\lambda\psi(0)\bar{\psi}\gamma^\lambda\gamma_\nu\psi(0))\mathcal{I}^{\alpha\beta\rho\nu} \\ & - \partial_\alpha\partial_\beta(\bar{\psi}\gamma_\rho\gamma_\mu\psi(0)\bar{\psi}\gamma^\lambda\gamma^\rho\psi(0))\mathcal{I}^{\alpha\beta\mu}_\lambda + 2\partial_\alpha\partial_\beta(\bar{\psi}\gamma_\rho\gamma_\mu\psi(0)\bar{\psi}\gamma^\lambda\gamma_\nu\psi(0))\mathcal{I}^{\alpha\beta\mu\nu\rho}_\lambda \\ & - \partial_\alpha(\bar{\psi}\gamma_\lambda\gamma_\rho\psi(0)\partial^\lambda\bar{\psi}\gamma^\rho\gamma_\nu\psi(0))\mathcal{I}^{\alpha\nu} + 2\partial_\alpha(\bar{\psi}\gamma_\lambda\gamma_\mu\psi(0)\partial^\lambda\bar{\psi}\gamma^\rho\gamma_\nu\psi(0))\mathcal{I}^{\alpha\mu\nu}_\rho \\ & \left. - \partial_\alpha(\bar{\psi}\gamma_\lambda\gamma_\rho\psi(0)\partial^\rho\bar{\psi}\gamma^\lambda\gamma_\nu\psi(0))\mathcal{I}^{\alpha\nu} + 2\partial_\alpha(\bar{\psi}\gamma_\lambda\gamma_\mu\psi(0)\partial^\rho\bar{\psi}\gamma^\lambda\gamma_\nu\psi(0))\mathcal{I}^{\alpha\mu\nu}_\rho \right) + \text{c.c.} \end{aligned} \quad (\text{B.6})$$

Simplifying this expression using $\mathcal{O}(\lambda^0)$ e.o.m. one obtains (3.29).

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