The large charge expansion and AdS/CFT

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ABSTRACT: The scaling dimensions of charged operators in conformal field theory were recently computed in a large charge expansion. We verify this expansion in a dual AdS model. Specifically, we numerically construct solitonic boson star solutions of Einstein-Maxwell-Scalar theory in AdS₄ and find that its mass at large charge reproduces the universal form of the lowest operator dimension in the large U(1) charge sector of the dual 2+1 dimensional CFT.

KEYWORDS: AdS-CFT Correspondence, Conformal Field Theory, Gauge-gravity correspondence, Holography and condensed matter physics (AdS/CMT)

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Contents

1	Introduction	1
2	Setup	2
3	Results	3
4	Discussion	4

1 Introduction

The most successful methods in quantum field theory involve a perturbation expansion in a small parameter. Recently [1–3], a new such parameter was found in conformal field theory (CFT). When two of the operators in a correlation function have a large global charge, the correlator may be computable in a large charge expansion.

Consider a 2+1 dimensional CFT with a U(1) global symmetry. Let Δ be the scaling dimension of the lowest dimension operator of charge Q . The large charge expansion predicts

$$\Delta = Q^{3/2} \sum_{n=0}^{\infty} \frac{c_n}{Q^n} + \text{calculable}, \quad (1.1)$$

where the c_n are theory dependent Wilson coefficients and “calculable” can be viewed as universal quantum corrections (giving rise to precise numerical predictions [4, 5]). In this note, we verify the form of (1.1) in a simple model using AdS/CFT.

The large Q expansion makes use of the state-operator correspondence. Let $|Q\rangle$ be the state corresponding to the operator under consideration. By construction, $|Q\rangle$ has a non-vanishing U(1) charge density. Furthermore, it is assumed that in the infinite volume limit at fixed charge density, the U(1) symmetry is spontaneously broken. Such a state is known as a superfluid [6]. We will construct a simple model in AdS₄ that is dual to a superfluid in CFT₃.

By the AdS/CFT correspondence:

1. The stress tensor of the CFT is dual to a dynamical metric with asymptotically AdS boundary conditions.
2. The global U(1) current J of the CFT is dual to a U(1) gauge field A in AdS.
3. The non-vanishing charge density imposes the boundary condition $A^0 \rightarrow \mu + z\langle Q|J^0|Q\rangle$, where z is the radial coordinate and μ is the chemical potential.

Physically, the boundary condition on A imposes a non-vanishing radial electric field at the boundary. Any bulk solution with such an electric field will be dual to a CFT state of large charge. However, in order to describe a superfluid, we will restrict ourselves to states that spontaneously break the global $U(1)$ symmetry. This means that in the CFT, there is an operator $\mathcal{O}$ with $\langle \mathcal{O} \rangle \neq 0$. In AdS,

4. $\mathcal{O}$ is dual to a field ϕ with the boundary condition $\phi \rightarrow z^\Delta \langle \mathcal{O} \rangle + z^{d-\Delta} \times \text{source}$. Since the symmetry is spontaneously broken, the source term vanishes.

A minimal model with these properties is Einstein-Maxwell-Scalar theory on asymptotically global AdS. We will numerically construct solitonic boson star solutions of this model and find that its mass at large charge reproduces (1.1).

Many aspects of Einstein-Maxwell-Scalar theory were previously studied in [7–12]. Numerics were also employed using the Poincaré patch counterpart to this model in the study of holographic superconductors [13–16].

2 Setup

We consider

$$S = \frac{1}{16\pi G_4} \int d^4x \sqrt{-G} \left[\frac{6}{L^2} + \mathcal{R} - \frac{1}{4} F_{ab} F^{ab} - (D_a \phi)^* D^a \phi + \frac{2}{L^2} |\phi|^2 \right], \quad (2.1)$$

where G_4 is the gravitational constant, $\mathcal{R}$ is the Ricci scalar, and L denotes the AdS radius.¹ The covariant derivative is defined as $D_a \phi = \partial_a \phi - iq A_a \phi$, where q is the charge of the scalar field.

We are looking for static and spherically symmetric solutions and therefore choose the following ansatz for the bulk metric,

$$ds^2 = -g(r)b(r)^2 dt^2 + \frac{dr^2}{g(r)} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2, \quad (2.2)$$

where $b \rightarrow 1$ and $g \rightarrow \frac{r^2}{L^2} + 1$ as $r \rightarrow \infty$ for asymptotically AdS boundary conditions.

Spherical symmetry, time-independence, radial gauge, and the restriction to electrically charged solutions allows the following ansatz for the fields,

$$\phi = \phi(r) \quad A = a(r) dt. \quad (2.3)$$

Moreover, the r -component of Maxwell equations implies that the phase of ϕ must be constant, allowing us to take ϕ to be real without loss of generality.

The equations of motion result in four coupled differential equations and the following asymptotic behaviour of the fields,

$$\phi(r \rightarrow \infty) = \frac{\phi_1}{r} + \frac{\phi_2}{r^2} + O(1/r^3) \quad (2.4)$$

$$a(r \rightarrow \infty) = \mu - \frac{\rho}{r} + O(1/r^2) \quad (2.5)$$

$$g(r \rightarrow \infty) = r^2 + 1 + \frac{\phi_1^2}{2} - \frac{m}{r} + O(1/r^2). \quad (2.6)$$

¹The equations of motion are invariant under the scaling $r \rightarrow \lambda r$, $t \rightarrow \lambda t$, $L \rightarrow \lambda L$, $q \rightarrow q/\lambda$ which we shall use to set $L = 1$.

Figure 1. The numerical result of $M(Q_{\text{AdS}}) = \Delta_Q(Q)$ along the soliton branch $q^2 = 1.4$.

To impose that the dual CFT state is a superfluid with zero entropy, the solutions we consider have no horizon, vanishing ϕ_1 , and non-vanishing μ , ρ , and ϕ_2 .

The parameter ρ is then related to the total charge Q_{AdS} of the solution through Gauss' law, which in our convention is

$$Q_{\text{AdS}} = \frac{L^2}{16\pi G_4} \lim_{r \rightarrow \infty} \int_{S^2} \star F = \frac{L^4}{4G_4} \rho, \quad (2.7)$$

where S^2 is a two-sphere at constant t and r and $\star F$ is the Hodge star dual of the field strength. Similarly, the AdS invariant boundary conditions allow us to make the link between the physical mass M of the solution and the coefficient m in the fall-off of the lapse function (2.6)

$$M = \frac{L^4}{2G_4} m. \quad (2.8)$$

The AdS/CFT dictionary relates the boundary values m and ρ to the dimension Δ_Q/l and charge Q of the CFT respectively. Following [12] we will choose units in which $2G_4 = 1$, such that²

$$M = m = \Delta_Q \quad \text{and} \quad Q_{\text{AdS}} = \frac{1}{2} \rho = Q. \quad (2.9)$$

3 Results

We numerically integrated the EOMs using a shooting method and constructed horizonless solutions with non-vanishing scalar, electric, and gravitational fields. Such solutions are often referred as “boson stars” [7–11] and consists of a one parameter family for each value of the scalar charge q . For concreteness, we take $q^2 = 1.4$ and plot Δ_Q in figure 1. Assuming the expansion (1.1), a fit of the numerical data gives the following,

$$\Delta_Q^{\text{fit}} = 1.56369 Q^{3/2} + 0.68411 Q^{1/2} - 0.03174 \frac{1}{Q^{1/2}}. \quad (3.1)$$

Next, we plot the difference between the numerical values of Δ_Q (figure 1) and the terms of Δ_Q^{fit} (3.1) in figures 2a and 2b. Fitting the curves 2a and 2b with a functions

²A change to Fefferman-Graham coordinates shows that one can choose the AdS radius to coincide with the radius of the cylinder $L = l = 1$.

Figure 2. (a) Subtracting the first term of the fit (3.1) from the data reveals a $\sim Q^{1/2}$ behavior, confirming the subleading term in (1.1). (b) Subtracting also the β -term of (3.1) reveals the $\sim Q^{-1/2}$ behavior.

of the form bQ^a yields $a_b \simeq 0.510$ and $a_c \simeq -0.506$ respectively, clearly showing the subleading and the second order behaviours $\sim Q^{1/2}$ and $\sim Q^{-1/2}$. Figure 2b indicates that the precision of our code is starting to reach its limits at the second order. However, the agreement with (1.1) is still clearly visible. We therefore confirm the predicted universal form of the scaling dimension Δ_Q (1.1) of the operator of lowest dimension with charge Q .

4 Discussion

We numerically constructed boson star solutions in a simple AdS model and computed their mass as a function of their charge. We showed that the result is consistent with the prediction (1.1). Our computation was purely classical and therefore did not reproduce the “calculable” terms in (1.1). To reproduce those, we would need to include quantum fluctuations about the boson star background. Such fluctuations would be dual to the fluctuations of the goldstone boson in the CFT [17]. We note that higher order terms in the action (2.1) will presumably only change the numerical values of the coefficients c_n in the expansion (1.1).

Apart from boson stars, the theory (2.1) also admits other charged solutions, such as black holes with scalar hair. See [18] for a phase diagram. However, such black holes do not correspond to superfluids because they have a large zero-temperature entropy. Furthermore, the Reisner-Nordstrom black hole considered in [19] does not break the $U(1)$ symmetry and therefore does not have the correct symmetry breaking pattern of a superfluid.

In future work, we would like to compute the spectrum of small fluctuations as well as boundary correlation functions. This would allow us to further check the predictions of [1, 3] and to make contact with the bootstrap analysis of [20].

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