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Effective gravitational couplings of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories

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Abstract: The low energy effective couplings of a four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory to topological invariants of the background gravitational field are described by two functions A and B. These two functions play an important role in the study of topologically twisted four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories and in the computation of central charges of $\mathcal{N} = 2$ superconformal theories. In this paper, we compute A and B from the partition function in the Ω-background for SU(2) gauge theories. Our results not only confirm the predicted expressions of the effective gravitational couplings, but also give the previously undetermined overall multiplicative factors. We also analyze A and B for the SU(N) super-Yang-Mills theory, and confirm all the previous predictions.

Keywords: Extended Supersymmetry, Supersymmetric Gauge Theory, Topological Field Theories, Nonperturbative Effects

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Contents

1 Introduction

The work of Seiberg and Witten [1, 2] on four-dimensional $\mathcal{N} = 2$ SU(2) supersymmetric gauge theories using holomorphy and electric-magnetic duality has revolutionized our understanding of non-perturbative dynamics in quantum field theory. After a quarter century of hard work, the Seiberg-Witten solution has been generalized to a large class of $\mathcal{N} = 2$ theories.

The Coulomb moduli space $\mathcal{M}$, parameterized by a set of gauge-invariant order parameters $u = \{u_1, \cdots, u_r\}$, is a complex manifold whose dimension is the rank r of the gauge group. At a generic point in $\mathcal{M}$, the gauge group is broken to a maximal torus $\mathrm{U}(1)^r$. We can choose a duality frame with local special coordinates $a = \{a_1, \cdots, a_r\}$, and the low energy effective theory is described in terms of r abelian vector multiplets. The perturbative corrections to the low energy effective prepotential $\mathcal{F}$ arise only at the one loop order, while non-perturbative corrections are entirely from instantons. It is remarkable that $\mathcal{F}$ can be solved exactly, and the solution is elegantly encoded in the Seiberg-Witten geometry. At singular loci $\mathcal{D} = \{\mathcal{D}_s\}$ in $\mathcal{M}$ extra massless particles appear.

Meanwhile, the achievement of Seiberg and Witten has also led to enormous advances in the theory of four-manifolds. Following the earlier development of topological field theory pioneered by Witten [3], the famous Donaldson invariants of four-manifolds [4] can be interpreted physically as correlation functions in the topologically twisted $\mathcal{N} = 2$ SU(2) super-Yang-Mills theory. With the understanding of the low energy effective dynamics of the theory, an alternative formulation of the Donaldson invariants was conjectured in terms of the Seiberg-Witten invariants [5]. Subsequently, a physical derivation of the conjecture was given in [6] and later extended and clarified in [7–20].

The path integral of the topologically twisted low energy effective theory on a curved four-manifold X receives two different contributions, one from an integral over the Coulomb branch (often called the u-plane integral), and the other from Seiberg-Witten invariants associated to extra massless particles. Hence, the Donaldson-Witten partition function $Z_{\rm DW}$, which is a generating function of the Donaldson invariants, takes the form

$$Z_{\rm DW} = Z_u + \sum_s Z_{{\rm SW},s}, \tag{1.1}$$

where Z_u is the contribution from the u-plane, and $Z_{{\rm SW},s}$ is the Seiberg-Witten contribution from the singular locus $\mathcal{D}_s$. When $b_1(X) = 0$ and $b_2^+(X) = 1$, the expression of Z_u is given by

$$Z_u = K_u \int [da d\bar{a}]\, A(u)^\chi B(u)^\sigma \Psi\,[\mathcal{K}]\,. \tag{1.2}$$

The normalization factor K_u is chosen so that Z_u is dimensionless. The measure factor $A(u)^\chi B(u)^\sigma$ is holomorphic in u, and encodes the couplings of the low energy effective theory to topological invariants of the background gravitational field, where χ and σ are the Euler characteristic and the signature of the four-manifold, respectively,

$$\chi = \frac{1}{32\pi^2}\int {\rm tr} R\wedge\tilde{R}, \quad \sigma = \frac{1}{24\pi^2}\int {\rm tr} R\wedge R. \tag{1.3}$$

The term $\Psi\,[\mathcal{K}]$ comes essentially from the evaluation of the photon partition function of the low energy effective abelian gauge theory, and takes the form of a Siegel-Narain theta function with kernel $\mathcal{K}$ depending on the inserted observables [19, 20].

It was found by Shapere and Tachikawa [21] that the functions A and B appearing in the topologically twisted theory can be used to compute the central charges of the physical $\mathcal{N} = 2$ superconformal theory that corresponds to a superconformal point in the Coulomb moduli space of an $\mathcal{N} = 2$ supersymmetric gauge theory. By definition, the $\mathtt{c}$ and $\mathtt{a}$ central charges are coefficients of the Weyl tensor and the Euler density associated with the curvature of the background gravitational field in the conformal anomaly,

$$\langle T^\mu{}_\mu\rangle = \frac{\mathtt{c}}{16\pi^2}\,({\rm Weyl})^2 - \frac{\mathtt{a}}{16\pi^2}\,({\rm Euler})\,, \tag{1.4}$$

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where the Weyl tensor and the Euler density associated with the curvature of the background gravitational field are given by

$$(\mathrm{Weyl})^2 = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2, \quad (\mathrm{Euler}) = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2. \tag{1.5}$$

We introduce a background $\mathrm{SU}(2)_R$ gauge connection with field strength $W^a_{\mu\nu}$. We can get the anomaly for the $\mathrm{U}(1)_R$-current $\mathcal{R}^\mu$ from the conformal anomaly using the superconformal algebra [22–24],

$$\partial_\mu \mathcal{R}^\mu = \frac{\mathtt{c}-\mathtt{a}}{8\pi^2} R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} + \frac{2\mathtt{a}-\mathtt{c}}{8\pi^2} W^a_{\mu\nu}\tilde{W}^{\mu\nu}_a. \tag{1.6}$$

We perform a topological twist by setting the $\mathrm{SU}(2)_R$ gauge connection equal to the self-dual part of the spin connection. Integrating the anomaly equation (1.6) over the four-manifold, we obtain the $\mathrm{U}(1)_R$ anomaly of the vacuum

$$\Delta\mathcal{R} = 2\,(2\mathtt{a}-\mathtt{c})\,\chi + 3\mathtt{c}\sigma. \tag{1.7}$$

On the other hand, if there are r free vector multiplets and h free neutral hypermultiplets in the low energy effective theory, we can also read the $\mathrm{U}(1)_R$ anomaly from the low energy effective action on the curved manifold. The $\mathrm{U}(1)_R$ anomalies of a free vector multiplet and a free hypermultiplet are $\frac{1}{2}(\chi+\sigma)$ and $\frac{1}{4}\sigma$, respectively. If the $\mathrm{U}(1)_R$-charges of A and B are $\mathcal{R}(A)$ and $\mathcal{R}(B)$, respectively, then the $\mathrm{U}(1)_R$ anomaly of the vacuum is also given by

$$\Delta\mathcal{R} = \mathcal{R}(A)\chi + \mathcal{R}(B)\sigma + \frac{r}{2}(\chi+\sigma) + \frac{h}{4}\sigma. \tag{1.8}$$

Combining (1.7) and (1.8), we obtain the central charges

$$\mathtt{a} = \frac{1}{4}\mathcal{R}(A) + \frac{1}{6}\mathcal{R}(B) + \frac{5}{24}r + \frac{1}{24}h, \quad \mathtt{c} = \frac{1}{3}\mathcal{R}(B) + \frac{1}{6}r + \frac{1}{12}h. \tag{1.9}$$

Our interest in the u-plane integral also comes from the study of the non-trivial six-dimensional $\mathcal{N}=(2,0)$ superconformal theories, whose existence is one of the most striking predictions of string theory [25–27]. We can realize the six-dimensional $\mathcal{N}=(2,0)$ theory of type A_{N-1} using a stack of N parallel M5-branes [28]. After compactifying on a Riemann surface $\mathcal{C}$ with punctures, we can obtain a four-dimensional $\mathcal{N}=2$ supersymmetric field theory $\mathcal{T}^{\mathrm{4d}}_N[\mathcal{C}]$ [29–32]. Such theories are called the $\mathcal{N}=2$ theories of class $\mathcal{S}$. In particular, for $\mathcal{N}=2$ superconformal theories of class $\mathcal{S}$, the space of coupling constants can be identified with Teichmüller space, the universal covering space of the moduli space of complex structures of punctured Riemann surfaces. Moreover, the ultraviolet S-duality group is identified with the group of large diffeomorphisms acting on $\mathcal{C}$ that leave its complex structure fixed.

The low energy effective theory of $\mathcal{T}^{\mathrm{4d}}_N[\mathcal{C}]$ on the Coulomb branch is governed by a single smooth M5-brane wrapped on the Seiberg-Witten curve $\Sigma \subset T^*\mathcal{C}$ [30], which is an algebraic curve depending on the Coulomb branch order parameters u, the masses m_f of the hypermultiplets, as well as the cutoff Λ for asymptotically free theories or the ultraviolet coupling τ_{UV} for superconformal theories. The low energy dynamics of a single M5-brane is governed by a six-dimensional $\mathcal{N}=(2,0)$ abelian tensor multiplet, which can be described

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using an action principle [33–40]. Now we put the six-dimensional $\mathcal{N}=(2,0)$ abelian theory $\mathcal{T}^{\text{6d}}$ on $X\times\Sigma$. The R-symmetry group of $\mathcal{T}^{\text{6d}}$ is $\text{Spin}(5)_R$, which has a subgroup $\text{Spin}(3)_R\times\text{Spin}(2)_R\cong\text{SU}(2)_R\times\text{U}(1)_R$. Let $\text{SU}(2)'_+$ and $\text{U}(1)'_\Sigma$ be the diagonal subgroups of $\text{SU}(2)_+\times\text{SU}(2)_R$ and $\text{U}(1)_\Sigma\times\text{U}(1)_R$, respectively. We can apply the standard procedure of topological twisting[1] and replace the holonomy group $\text{SU}(2)_-\times\text{SU}(2)_+$ of X and the holonomy group $\text{U}(1)_\Sigma$ of Σ with $\text{SU}(2)_-\times\text{SU}(2)'_+$ and $\text{U}(1)'_\Sigma$, respectively. In order to compute the partition function of $\mathcal{T}^{\text{6d}}$ on $X\times\Sigma$, we can either first compactify $\mathcal{T}^{\text{6d}}$ on Σ to obtain the low energy effective theory $\mathcal{T}^{\text{4d}}_{\text{IR}}[\Sigma]$ of the ultraviolet theory $\mathcal{T}^{\text{4d}}_N[\mathcal{C}]$ on X with Donaldson-Witten twist, or first compactify $\mathcal{T}^{\text{6d}}$ on X to get a two-dimensional $\mathcal{N}=(0,2)$ theory $\mathcal{T}^{\text{2d}}[X]$ on Σ with half-twist [41, 42]. Because of the topological nature of the setup, the integrand of the u-plane integral of $\mathcal{T}^{\text{4d}}_{\text{IR}}[\Sigma]$ on X should coincide with a correlation function in $\mathcal{T}^{\text{2d}}[X]$ on Σ. Therefore, we can deduce A and B using this correspondence by changing the topology of X. The relation of four-manifold invariants with two-dimensional $\mathcal{N}=(0,2)$ has been discussed in [43–46]. In spite of this work, the full derivation of the Coulomb branch integrals for topologically twisted class $\mathcal{S}$ theories remains to be completed. We will leave a discussion of this interesting topic for another occasion.

Based on the requirements of holomorphy, the $\text{U}(1)_R$ R-symmetry, and the single-valuedness of the integrand of Z_u, the general forms of A and B were predicted to be [6, 7, 9, 21, 47]

$$A=\alpha\left(\det\frac{du_i}{da_j}\right)^{\frac{1}{2}},\quad B=\beta\Delta^{\frac{1}{8}}. \tag{1.10}$$

Here Δ is the physical discriminant, which is a holomorphic function with first order zeroes at the locus $\{u_s\}$ where extra particles becomes massless. For SU(2) gauge theories, we normalize Δ as

$$\Delta=\prod_s(u-u_s)\,. \tag{1.11}$$

The physical discriminant can be different from the mathematical discriminant of the Seiberg-Witten curve for two reasons [21]. First, the Seiberg-Witten curve is not unique for a given $\mathcal{N}=2$ gauge theory [48–50]. Different forms give the same solution to the low energy effective theory and the same BPS spectrum, but may give different mathematical discriminants. Second, it is not guaranteed that all the cycles of the Seiberg-Witten curve correspond to physical states, and if some zeroes of the mathematical discriminant do not indicate the appearance of extra massless particles, we should not include them when we compute Δ.

[1] We would like to emphasize that we use Lie groups rather than Lie algebras in the procedure of topological twisting: this procedure requires the introduction of a bundle with connection associated with the R-symmetry group, and together with an isomorphism of bundles such that relevant connections are mapped to each other under the isomorphism. In the study of the Donaldson-Witten theory, the required $\text{SU}(2)_R$ bundle might not exist, but the $\text{SO}(3)_R$ bundle associated to the adjoint representation always exists. One can choose an isomorphism of this adjoint bundle with the bundle of self-dual two-forms. Then one puts a connection on the adjoint $\text{SO}(3)_R$ bundle so that under this isomorphism we get the Levi-Civita connection on the self-dual two-forms. In our case, however, one must choose a $\text{Spin}(5)_R$ bundle together with a reduction of the structure group to $\text{Spin}(3)_R\times\text{Spin}(2)_R$.

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The overall multiplicative factors α and β in (1.10) are constants on the Coulomb branch that have not been determined yet. In principle, they can depend on the theory, the masses of hypermultiplets, and also on the cutoff Λ for asymptotically free theories or the ultraviolet coupling $\tau_{\rm UV}$ for superconformal theories. For the SU(2) super-Yang-Mills theory, we need to choose (K_u, α, β) so that the partition function (1.1) matches precisely with known results of Donaldson invariants from the mathematical literature. The choice made in [20] is that[2]

$$K_u = 2^{-\frac{5}{2}}\Lambda^{-3}, \quad \alpha = 2^{\frac{1}{8}} e^{-\frac{\pi \mathrm{i}}{8}} \pi^{-\frac{1}{2}}, \quad \beta = 2^{\frac{5}{8}} e^{-\frac{\pi \mathrm{i}}{8}} \pi^{-\frac{1}{2}}. \tag{1.13}$$

For other theories, there is no mathematical result to compare with. It was predicted in [9] that the N-dependence of α and β in the SU(N) super-Yang-Mills theory should be

$$\alpha(N) = e^{\kappa_1^{(\alpha)} N + \kappa_2^{(\alpha)} N^2}, \quad \beta(N) = e^{\kappa_1^{(\beta)} N + \kappa_2^{(\beta)} N^2}, \tag{1.14}$$

where $\kappa_1^{(\alpha,\beta)}$ and $\kappa_2^{(\alpha,\beta)}$ are N-independent constants that can depend on Λ. It was also argued in [14] that α and β are independent of masses for asymptotically free theories.

Up to now, almost nothing has been known about α and β for superconformal theories. It is certainly interesting to figure out how α and β depend on the parameters of the theory, especially on the conformal manifold for superconformal theories. It was proposed by Labastida and Lozano [12] that for the SU(2) $\mathcal{N} = 2^*$ theory[3]

$$K_u \alpha^\chi \beta^\sigma = -\frac{4\mathrm{i}}{\pi} 2^{\frac{3}{8}\chi + \frac{21}{16}\sigma} \mu^{2\chi + 3\sigma} \eta\left(\tau_{\rm UV}\right)^{-3\chi - \frac{3}{2}\sigma} m^{\frac{1}{8}\sigma}, \tag{1.17}$$

so that the Donaldson-Witten partition function $Z_{\rm DW}$ in the massless limit coincides with the Vafa-Witten partition function [51] on $K3$ manifolds. The function μ was not determined since $2\chi + 3\sigma = 0$ for $K3$ manifolds. Clearly, at least one of K_u, α and β must depend nontrivially on $\tau_{\rm UV}$. We expect that for general superconformal theories of class $\mathcal{S}$, α and β are automorphic forms on the Teichmüller space.

Given the importance of A and B, it is definitely beneficial to cross-check the prediction (1.10) using other approaches. In this paper, we shall specify the gravitational background to be the Ω-background of $\mathbb{R}^4 \cong \mathbb{C}^2$ and apply the powerful instanton counting

[2]We have rescaled here the Λ and a of [20] to compare their α and β to ours,

$$\Lambda_{\rm KMMN} = \sqrt{2}\Lambda, \quad a_{\rm KMMN} = \frac{a}{\sqrt{2}}, \tag{1.12}$$

while $u_{\rm KMMN} = u$.

[3]In order to compare the α and β of [12] to ours, we need to rescale the m, u and a of [12] by

$$m_{\rm LL} = \sqrt{2}m, \quad u_{\rm LL} = 2u, \quad a_{\rm LL} = 2a. \tag{1.15}$$

We should also notice that the discriminant used in [12] is the mathematical discriminant of the Seiberg-Witten curve, which is related to the physical discriminant Δ used in this paper by

$$\Delta_{\rm LL} = 8\eta\left(\tau_{\rm UV}\right)^{12}\Delta. \tag{1.16}$$

techniques [52] to compute A and B. Our strategy is to expand the exact partition function $\mathcal{Z}$ in the Ω-background around the flat space limit $\varepsilon_1, \varepsilon_2 \to 0$,

$$\varepsilon_1\varepsilon_2 \log \mathcal{Z} = -\mathcal{F} + (\varepsilon_1 + \varepsilon_2)\,\mathcal{H} + \varepsilon_1\varepsilon_2 \log A + \frac{\varepsilon_1^2+\varepsilon_2^2}{3} \log B + \cdots\,, \tag{1.18}$$

where ε_1 and ε_2 are two deformation parameters of the Ω-background, and $\cdots$ includes higher order terms in $\varepsilon_1, \varepsilon_2$ that are irrelevant to our problem. The leading term coincides with the low energy effective prepotential $\mathcal{F}$ [52]. This gives us an opportunity to derive rigorously the Seiberg-Witten geometry for a large class of $\mathcal{N}=2$ theories. In fact, by the saddle point analysis, the partition function $\mathcal{Z}$ in the limit $\varepsilon_1, \varepsilon_2 \to 0$ is dominated by a particular instanton configuration determined by the limit shape equations, whose solution leads to the Seiberg-Witten curve [53–57]. A priori, we cannot rule out the next-to-leading order term $\mathcal{H}$, but it vanishes in every example we will be dealing with. The identification of the next two terms follows from the equivariant Euler characteristic and the equivariant signature of $\mathbb{C}^2$ [58],

$$\chi\left(\mathbb{C}^2\right) = \varepsilon_1\varepsilon_2, \quad \sigma\left(\mathbb{C}^2\right) = \frac{\varepsilon_1^2+\varepsilon_2^2}{3}. \tag{1.19}$$

Hence from the partition function $\mathcal{Z}$ we can directly compute A and B, and determine α and β from first principles.[4] Similar expansions were performed in [60, 61], leading to a modular anomaly equation. However, they simply disregarded the a-independent terms and the important U(1) factors in their analysis. These terms can be ignored when we are only interested in the dynamics of the theory, but they are crucial to our problem.

There is an important subtlety regarding the normalization involved in our analysis. Since the partition function $\mathcal{Z}$ is naturally normalized to have vanishing mass dimension, we see that the mass dimensions of A and B are zero. On the other hand, in the standard normalization of the u-plane integral, A and B have nonzero mass dimensions. In order to resolve this problem, we notice that we only consider the situation $b_1(X) = 0$ and $b_2^+(X) = 1$. Therefore, we have $\chi + \sigma = 4$, and there is a normalization ambiguity [17]

$$(K_u, \alpha, \beta) \sim \left(\kappa^{-4} K_u, \kappa\alpha, \kappa\beta\right). \tag{1.20}$$

We can use this ambiguity to relate the results computed from $\mathcal{Z}$ with those appearing in Z_u. Notice that the ratio β/α is unambiguous.

To illustrate our method, we shall mainly focus on the simple examples of SU(2) gauge theories. We can write down the partition function $\mathcal{Z}$ explicitly up to an arbitrary order of the instanton number. We then compare our results with those computed using the Seiberg-Witten curve. In this way, we successfully confirm the prediction (1.10), and obtain the overall factors α and β. We also confirm (1.10) and (1.14) for the SU(N) super-Yang-Mills theory.

The rest of the paper is organized as follows. In section 2 we summarize the useful results of the partition function of the four-dimensional $\mathcal{N}=2$ supersymmetric gauge

[4] In fact, the information of A and B can be extracted from $\mathcal{Z}$ using two linearly independent limits for the ε_1 and ε_2. For example, we can use the topological string limit $\varepsilon_+ \to 0$ and the Nekrasov-Shatashvili limit $\varepsilon_2 \to 0$ [59].

theory in the Ω-background. In section 3 we consider the SU(2) super-Yang-Mills theory. In section 4 we deal with the SU(2) theory with an adjoint hypermultiplet. In section 5 we study the SU(2) gauge theory with four fundamental hypermultiplets. In section 6 we analyze A and B in the SU(N) super-Yang-Mills theory. We conclude in section 7 with a discussion of some subtleties of our results as well as an outlook of future directions. In appendix A we discuss the definition and the expansion of the special function $\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda)$. In appendix B we review how to compute the period integrals on an elliptic curve. In appendix C we collect a few essential aspects of the theory of modular forms and Jacobi theta functions. In appendix D we review Weierstrass's elliptic function.

2 Partition function in the Ω-background

Let us consider the four-dimensional $\mathcal{N}=2$ supersymmetric gauge theory with gauge group $G=\mathrm{SU}(N)$ and massive hypermultiplets[5] in a representation $\mathfrak{R}$ of G. We can decompose $\mathfrak{R}$ into irreducible representations of G,

$$\mathfrak{R}=\bigoplus_f \mathfrak{R}_f. \tag{2.1}$$

We require the beta function of the gauge coupling constant g to be non-positive so that we can have a well-defined microscopic theory,

$$\Lambda\frac{\partial g}{\partial\Lambda}=-\frac{g^3}{16\pi^2}\left(2N-2T\left(\mathfrak{R}\right)\right)\leq 0, \tag{2.2}$$

where Λ is the cutoff scale, and $T(\mathfrak{R})$ is the quadratic Casimir of the representation $\mathfrak{R}$ satisfying $T(\mathfrak{R}_1\oplus\mathfrak{R}_2)=T(\mathfrak{R}_1)+T(\mathfrak{R}_2)$. In this paper, we are mainly interested in the adjoint and fundamental representations,

$$T\left(\mathrm{adj}\right)=N,\quad T\left(\mathrm{fund}\right)=\frac{1}{2}. \tag{2.3}$$

For asymptotically free theories, we define the instanton counting parameter $\mathtt{q}$ to be

$$\mathtt{q}=\Lambda^{2N-2T(\mathfrak{R})}. \tag{2.4}$$

For superconformal theories we have the ultraviolet complexified coupling

$$\tau_{\mathrm{UV}}=\frac{\vartheta_{\mathrm{UV}}}{2\pi}+\frac{4\pi\mathrm{i}}{g_{\mathrm{UV}}^2}, \tag{2.5}$$

where ϑ_{UV} and g_{UV} are the ultraviolet theta angle and gauge coupling constant, respectively, and we define

$$\mathtt{q}=e^{2\pi\mathrm{i}\tau_{\mathrm{UV}}}. \tag{2.6}$$

We choose the vacuum expectation value of the scalar field ϕ in the vector multiplet to be the local special coordinates a on the Coulomb branch,

$$a=\langle\phi\rangle=\mathrm{diag}\left(a_1,\cdots,a_N\right). \tag{2.7}$$

[5]In this paper, we always consider full hypermultiplets. See [57, 62, 63] for work on half-hypermultiplets in the Ω-background.

It is useful to introduce the Ω-deformation of the theory [52, 64], so that the Poincaré symmetry of $\mathbb{R}^4 \cong \mathbb{C}^2$ is broken in a rotationally covariant way, while still preserving a particular linear combination of supercharges

$$\mathcal{Q} = \bar{Q} + \Omega^\mu Q_\mu. \tag{2.8}$$

Here $\bar{Q}$ and Q_μ are the scalar and vector supercharges in the topologically twisted $\mathcal{N}=2$ theories [3], and $\Omega^\mu \partial_\mu$ is the Killing vector generating the $\mathrm{U}(1)^2$ isometry of $\mathbb{C}^2$,

$$\Omega^\mu \partial_\mu = \mathrm{i}\varepsilon_1 \left(z_1 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} \right) + \mathrm{i}\varepsilon_2 \left(z_2 \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right). \tag{2.9}$$

The supersymmetric action in the Ω-background can be constructed from the flat space action by replacing ϕ by an operator [65]

$$\phi \mapsto \phi + \Omega^\mu D_\mu. \tag{2.10}$$

We also define

$$\varepsilon_\pm = \frac{\varepsilon_1 \pm \varepsilon_2}{2}. \tag{2.11}$$

Clearly, the Ω-background is closely related to the topological twist, since $\mathcal{Q}$ will become the usual scalar supercharge $\bar{Q}$ used in the topologically twisted theory when we take the limit $\varepsilon_1, \varepsilon_2 \to 0$.

Using the powerful localization techniques, the partition function $\mathcal{Z}$ in the Ω-background can be calculated exactly and is given by a product of the classical, one-loop and instanton contributions [52],

$$\mathcal{Z} = \mathcal{Z}^{\mathrm{cl}} \mathcal{Z}^{1-\mathrm{loop}} \mathcal{Z}^{\mathrm{inst}}. \tag{2.12}$$

It is convenient to start with the gauge group $\mathrm{U}(N)$. The classical contribution is given by

$$\mathcal{Z}^{\mathrm{cl}}(a, \mathsf{q}; \varepsilon_1, \varepsilon_2) = \mathsf{q}^{-\frac{1}{2\varepsilon_1\varepsilon_2}\sum_{i=1}^N a_i^2}. \tag{2.13}$$

The one-loop contributions of the vector multiplet and the hypermultiplet in the fundamental or adjoint representation are given by

$$\mathcal{Z}^{1-\mathrm{loop,vec}} = \prod_{i<j} \exp\left[-\gamma_{\varepsilon_1,\varepsilon_2}(a_i - a_j; \Lambda) - \gamma_{\varepsilon_1,\varepsilon_2}(a_i - a_j - 2\varepsilon_+; \Lambda)\right], \tag{2.14}$$

$$\mathcal{Z}^{1-\mathrm{loop,fund}} = \prod_{i=1}^N \exp\left[\gamma_{\varepsilon_1,\varepsilon_2}(a_i + m - \varepsilon_+; \Lambda)\right], \tag{2.15}$$

$$\mathcal{Z}^{1-\mathrm{loop,adj}} = \prod_{i,j=1}^N \exp\left[\gamma_{\varepsilon_1,\varepsilon_2}(a_i - a_j + m - \varepsilon_+; \Lambda)\right], \tag{2.16}$$

where the definition and basic properties of the special function $\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda)$ are given in appendix A. For asymptotically free theories, it is convenient to absorb the classical contribution into the one-loop contribution by redefining Λ. Notice that we do not include

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the contributions from $i = j$ for the vector multiplet but we should include them for the adjoint hypermultiplet.

The instanton partition function is given by

$$\mathcal{Z}^{\text{inst}} = \sum_{k=0}^{\infty} \mathfrak{q}^k \int_{\mathfrak{M}_k} \mathrm{e}\left(\mathcal{E}_{\text{matter}} \to \mathfrak{M}_k\right), \tag{2.17}$$

where $\mathfrak{M}_k$ is the moduli space of framed noncommutative $\mathrm{U}(N)$ instantons on $\mathbb{C}^2$ with instanton charge k,[6] and $\mathrm{e}\left(\mathcal{E}_{\text{matter}} \to \mathfrak{M}_k\right)$ is the equivariant Euler class of the matter bundle whose fiber is the space of the virtual zero modes for the Dirac operator associated with the hypermultiplet in the instanton background. $\mathcal{Z}^{\text{inst}}$ can be evaluated using the equivariant localization formula. The fixed points in $\mathfrak{M} = \cup_k \mathfrak{M}_k$ are labeled by N-tuple of Young diagrams $\vec{Y} = \left\{Y^{(1)}, \cdots Y^{(N)}\right\}$, and the equivariant Euler class at fixed points can be computed from the equivariant Chern characters. Then $\mathcal{Z}^{\text{inst}}$ is reduced to a statistical sum over Young diagrams,

$$\mathcal{Z}^{\text{inst}} = \sum_{\vec{Y}} \mathfrak{q}^{\left|\vec{Y}\right|} z_{\text{vec}}\left(a, \vec{Y}\right) \prod_f z_{\text{hyper}}^{\mathfrak{R}_f}\left(a, m_f, \vec{Y}\right), \tag{2.18}$$

where $\left|\vec{Y}\right|$ is the total number of boxes in N Young diagrams. We introduce the conversion operator ϵ which maps characters into weights,

$$\epsilon\left\{\sum_i n_i e^{x_i}\right\} = \prod_i x_i^{n_i}, \tag{2.19}$$

and the dual operator $\vee$,

$$\left(\sum_i n_i e^{x_i}\right)^{\vee} = \left(\sum_i n_i e^{-x_i}\right). \tag{2.20}$$

The contributions from the vector multiplet and the hypermultiplet in the fundamental or adjoint representation can be written compactly as [52, 56, 66]

$$z_{\text{vec}}\left(a, \vec{Y}\right) = \epsilon\left\{-\mathcal{N}\mathcal{K}_{\vec{Y}}^{\vee} - e^{2\varepsilon_+}\mathcal{K}_{\vec{Y}}\mathcal{N}^{\vee} + \mathcal{P}\mathcal{K}_{\vec{Y}}\mathcal{K}_{\vec{Y}}^{\vee}\right\}, \tag{2.21}$$

$$z_{\text{hyper}}^{\text{fund}}\left(a, m, \vec{Y}\right) = \epsilon\left\{e^{m+\varepsilon_+}\mathcal{K}_{\vec{Y}}\right\}, \tag{2.22}$$

$$z_{\text{hyper}}^{\text{adj}}\left(a, m, \vec{Y}\right) = \epsilon\left\{e^{m-\varepsilon_+}\left(\mathcal{N}\mathcal{K}_{\vec{Y}}^{\vee} + e^{2\varepsilon_+}\mathcal{K}_{\vec{Y}}\mathcal{N}^{\vee} - \mathcal{P}\mathcal{K}_{\vec{Y}}\mathcal{K}_{\vec{Y}}^{\vee}\right)\right\}, \tag{2.23}$$

where

$$\mathcal{N} = \sum_{i=1}^{N} e^{a_i}, \tag{2.24}$$

$$\mathcal{K}_{\vec{Y}} = \sum_{i=1}^{N} \sum_{(x,y)\in Y^{(i)}} e^{a_i + \varepsilon_1(x-1) + \varepsilon_2(y-1)}, \tag{2.25}$$

$$\mathcal{P} = \left(1 - e^{\varepsilon_1}\right)\left(1 - e^{\varepsilon_2}\right). \tag{2.26}$$

[6] Here the noncommutative deformation is introduced to resolve the singularities of the moduli space due to point-like instantons.

This expression of $\mathcal{Z}^{\text{inst}}$ can reproduce the standard expression in terms of arm and leg lengths using combinatorial formulas [67, 68].

It is worth emphasizing that the masses m_f appearing in (2.15), (2.16), (2.22), (2.23) differ from the masses m'_f in the original paper [52, 53] by a constant shift of ε_+ [69–71],

$$m_f = m'_f + \varepsilon_+. \tag{2.27}$$

This shift is due to the fact that the scalars in a hypermultiplet become spinors in the Donaldson-Witten twist, and the Dirac complex is the Dolbeault complex twisted by the square-root of the canonical bundle of the four-manifold. This shift can often be ignored in many applications of the Ω-background, since it will not modify the dynamics of the theory on flat space where $\varepsilon_+ = 0$. However, the functions A and B are defined in the Donaldson-Witten twist, and it is necessary to use m_f as the mass parameters.

When we move from the gauge group $\mathrm{U}(N)$ to $\mathrm{SU}(N)$, we have to modify carefully the partition function in the Ω-background. First of all, we need to set

$$\sum_{i=1}^{N} a_i = 0. \tag{2.28}$$

In particular, for $G = \mathrm{SU}(2)$, we take

$$\langle\phi\rangle = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}. \tag{2.29}$$

Second, while the tensor product $\mathrm{fund}\otimes\overline{\mathrm{fund}}$ of the fundamental and the anti-fundamental representations gives the adjoint representation for the group $\mathrm{U}(N)$, we have to subtract the trivial representation to get the adjoint representation for the group $\mathrm{SU}(N)$. Therefore, the one-loop contribution of the $\mathrm{SU}(N)$ adjoint hypermultiplet is given by (2.16) divided by $\exp\left[\gamma_{\varepsilon_1,\varepsilon_2}(m-\varepsilon_+;\Lambda)\right]$. Finally, we need to factor out the residual contribution of the $\mathrm{U}(1) \subset \mathrm{U}(N)$ gauge field from the instanton partition function,

$$\mathcal{Z}^{\text{inst}}_{\mathrm{U}(N)} = \mathcal{Z}^{\text{inst}}_{\mathrm{SU}(N)} \mathcal{Z}^{\text{inst}}_{\text{extra}}. \tag{2.30}$$

The explicit expression of $\mathcal{Z}^{\text{inst}}_{\text{extra}}$ was first proposed in [72], and later derived from the non-perturbative Dyson-Schwinger equations [66, 73, 74]. For the SU(2) gauge theory with one adjoint hypermultiplet of mass m, we have

$$\mathcal{Z}^{\text{inst}}_{\text{extra}} = \left[\prod_{n=1}^{\infty}(1-\mathsf{q}^n)\right]^{-\frac{2}{\varepsilon_1\varepsilon_2}(m+\varepsilon_-)(m-\varepsilon_-)}, \tag{2.31}$$

and for the SU(2) gauge theory with four fundamental hypermultiplets of masses m_1, m_2, m_3, m_4, we have[7]

$$\mathcal{Z}^{\text{inst}}_{\text{extra}} = (1-\mathsf{q})^{\frac{2}{\varepsilon_1\varepsilon_2}\left(\frac{m_1+m_2}{2}+\varepsilon_+\right)\left(\frac{m_3+m_4}{2}+\varepsilon_+\right)}. \tag{2.32}$$

[7]We consider here four fundamental hypermultiplets rather than two fundamental and two anti-fundamental hypermultiplets as in [72]. The factor $\mathcal{Z}^{\text{inst}}_{\text{extra}}$ breaks the Spin(8) symmetry of the masses. However, the breaking only affects the low energy effective prepotential $\mathcal{F}$ in the expansion (1.18), and leads to a constant shift in $\mathcal{F}$.

In the following three sections, we will focus on SU(2) gauge theories. For this gauge group, we can also take the advantage of the equivalence $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$ and directly perform the computation using the Sp(1) gauge theory [54, 75–77]. It is known that the Sp(1) instanton moduli space looks rather different from the SU(2) instanton moduli space [78]. Nevertheless, it was shown in [62] that these two partition functions agree, possibly up to an a-independent factor and after a nontrivial mapping of parameters. Therefore, if (1.10) can be demonstrated using the SU(2) partition function, it automatically holds if we use the Sp(1) partition function. We should also not worry about the a-independent factor, since α and β depend on the precise microscopic definition of the theory. The choice made in this paper is a natural choice of ultraviolet regularizations, and it turns out to be consistent with all the previous results.

3 The SU(2) super-Yang-Mills theory

The simplest but most important example is the SU(2) super-Yang-Mills theory.

3.1 Expansion of the partition function

The partition function of the theory in the Ω-background is given in section 2 with $\mathcal{Z}^{\text{inst}}_{\text{extra}} = 1$.

It is straightforward to compute the expansion (1.18). The leading term gives the low energy effective prepotential $\mathcal{F}$,

$$\mathcal{F} = -4a^2\left(\log\left(\frac{2a}{\Lambda}\right) - \frac{3}{2}\right) + \frac{\Lambda^4}{2a^2} + \frac{5\Lambda^8}{64a^6} + \frac{3\Lambda^{12}}{64a^{10}} + \frac{1469\Lambda^{16}}{32768a^{14}} + \mathcal{O}\left(\frac{\Lambda^{20}}{a^{18}}\right), \tag{3.1}$$

from which we can compute the Coulomb moduli order parameter u [79, 80],

$$\begin{aligned} u &= \frac{1}{2}\left\langle \mathrm{Tr}\phi^2 \right\rangle = \frac{1}{4}\Lambda\frac{\partial \mathcal{F}}{\partial \Lambda} \\ &= a^2 + \frac{\Lambda^4}{2a^2} + \frac{5\Lambda^8}{32a^6} + \frac{9\Lambda^{12}}{64a^{10}} + \frac{1469\Lambda^{16}}{8192a^{14}} + \mathcal{O}\left(\frac{\Lambda^{20}}{a^{18}}\right). \end{aligned} \tag{3.2}$$

The next-to-leading order term $\mathcal{H} = 0$. In fact, the perturbative contribution vanishes because of the expansion (A.6) of the function $\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda)$, and the instanton contribution also vanishes since $\mathcal{Z}^{\text{inst}}$ is invariant under $(\varepsilon_1, \varepsilon_2) \to (-\varepsilon_1, -\varepsilon_2)$. The second order terms in the expansion (1.18) are

$$\log A = \frac{1}{2}\log\left(\frac{2a}{\Lambda}\right) - \frac{\Lambda^4}{4a^4} - \frac{19\Lambda^8}{64a^8} - \frac{47\Lambda^{12}}{96a^{12}} - \frac{15151\Lambda^{16}}{16384a^{16}} + \mathcal{O}\left(\frac{\Lambda^{20}}{a^{20}}\right), \tag{3.3}$$

$$\log B = \frac{1}{2}\log\left(\frac{2a}{\Lambda}\right) - \frac{3\Lambda^4}{8a^4} - \frac{63\Lambda^8}{128a^8} - \frac{55\Lambda^{12}}{64a^{12}} - \frac{55335\Lambda^{16}}{32768a^{16}} + \mathcal{O}\left(\frac{\Lambda^{20}}{a^{20}}\right). \tag{3.4}$$

3.2 Comparison to the prediction

In order to compare our results computed from the partition function in the Ω-background to the prediction (1.10), we consider the Seiberg-Witten curve

$$y^2 = \left(x^2 - u\right)^2 - 4\Lambda^4, \tag{3.5}$$

with the Seiberg-Witten differential λ determined by

$$\frac{\partial \lambda}{\partial u} = \frac{1}{2\pi \mathtt{i}} \frac{dx}{y}. \tag{3.6}$$

Using the result of the period integral (B.6), we have

$$\begin{aligned}
\frac{da}{du} &= \frac{1}{2\pi \mathtt{i}} \oint_A \frac{dx}{y} \\
&= \left(\sqrt{u-2\Lambda^2}+\sqrt{u+2\Lambda^2}\right)^{-1} {}_2F_1\left(\frac{1}{2},\frac{1}{2},1,\left(\frac{\sqrt{u-2\Lambda^2}-\sqrt{u+2\Lambda^2}}{\sqrt{u-2\Lambda^2}+\sqrt{u+2\Lambda^2}}\right)^2\right) \\
&= \frac{1}{2\sqrt{u}} + \frac{3\Lambda^4}{8u^{5/2}} + \frac{105\Lambda^8}{128u^{9/2}} + \frac{1155\Lambda^{12}}{512u^{13/2}} + \frac{225225\Lambda^{16}}{32768u^{17/2}} + \mathcal{O}\left(\frac{\Lambda^{20}}{u^{21/2}}\right).
\end{aligned} \tag{3.7}$$

Using techniques from the theory of elliptic curves, one can express the observable da/du as a function of the complex structure τ of the curve (3.5) in closed form,[8]

$$\Lambda \frac{da}{du} = \frac{1}{4}\theta_2(\tau)^2, \tag{3.8}$$

where θ_2 is one of the Jacobi theta functions defined in (C.10).

Integrating with respect to u, we get

$$a(u) = \sqrt{u} - \frac{\Lambda^4}{4u^{3/2}} - \frac{15\Lambda^8}{64u^{7/2}} - \frac{105\Lambda^{12}}{256u^{11/2}} - \frac{15015\Lambda^{16}}{16384u^{15/2}} + \mathcal{O}\left(\frac{\Lambda^{20}}{u^{19/2}}\right), \tag{3.9}$$

and its inverse function is

$$u(a) = a^2 + \frac{\Lambda^4}{2a^2} + \frac{5\Lambda^8}{32a^6} + \frac{9\Lambda^{12}}{64a^{10}} + \frac{1469\Lambda^{16}}{8192a^{14}} + \mathcal{O}\left(\frac{\Lambda^{20}}{a^{18}}\right). \tag{3.10}$$

As a function of τ, u reads

$$\frac{u(\tau)}{\Lambda^2} = 4\frac{\theta_3(\tau)^4}{\theta_2(\tau)^4} - 2. \tag{3.11}$$

Returning to the results for the instanton partition function in the Ω-background, we recognize that the expansion in (3.10) matches with the result (3.2). From (3.7), we determine

$$\log\left(\frac{du}{da}\right) = \log(2a) - \frac{\Lambda^4}{2a^4} - \frac{19\Lambda^8}{32a^8} - \frac{47\Lambda^{12}}{48a^{12}} - \frac{15151\Lambda^{16}}{8192a^{16}} + \mathcal{O}\left(\frac{\Lambda^{20}}{a^{20}}\right). \tag{3.12}$$

On the other hand, the singularities of the Coulomb branch are at $u = \pm 2\Lambda^2$ where we have extra massless BPS states. Therefore the physical discriminant is given by $\Delta = u^2 - 4\Lambda^4$, whose logarithm is given by

$$\log \Delta = 4\log(a) - \frac{3\Lambda^4}{a^4} - \frac{63\Lambda^8}{16a^8} - \frac{55\Lambda^{12}}{8a^{12}} - \frac{55335\Lambda^{16}}{4096a^{16}} + \mathcal{O}\left(\frac{\Lambda^{20}}{a^{20}}\right). \tag{3.13}$$

[8]This equation and (3.11) appear different from those in [20] due to the different normalizations of a, Λ and τ.

By comparing (3.12) and (3.13) with A and B given in (3.3) and (3.4), respectively, we find

$$A = \Lambda^{-\frac{1}{2}} \left(\frac{du}{da}\right)^{\frac{1}{2}}, \quad B = \sqrt{2}\Lambda^{-\frac{1}{2}}\Delta^{\frac{1}{8}}, \tag{3.14}$$

which reproduce (1.10). We also match the unambiguous ratio with (1.13),

$$\frac{\beta}{\alpha} = \sqrt{2} = \frac{2^{\frac{5}{8}} e^{-\frac{\pi i}{8}} \pi^{-\frac{1}{2}}}{2^{\frac{1}{8}} e^{-\frac{\pi i}{8}} \pi^{-\frac{1}{2}}}. \tag{3.15}$$

Finally, we note that we can express $\tau = \frac{1}{4\pi i}\frac{\partial^2 \mathcal{F}}{\partial a^2}$ as an expansion in Λ/a using (3.1). Substitution of this expansion in (3.8) and (3.11) reproduces the expansions in (3.7) and (3.11).

4 The SU(2) $\mathcal{N} = 2^*$ theory

The simplest $\mathcal{N} = 2$ superconformal theory is the $\mathcal{N} = 4$ super-Yang-Mills theory, which is the $\mathcal{N} = 2$ gauge theory with one adjoint hypermultiplet. We turn on the $\mathcal{N} = 2$ invariant bare mass term and the resulting theory is often called the $\mathcal{N} = 2^*$ theory. In this section, we take the gauge group $G = \mathrm{SU}(2)$, and denote the mass by m. In the class $\mathcal{S}$ construction, this theory arises by compactifying the six-dimensional $(2, 0)$ theory of type A_1 on a torus with one puncture.

4.1 Expansion of the partition function

We can compute the expansion (1.18) of the partition function in the Ω-background. The leading term is the low energy effective prepotential $\mathcal{F}$. Up to $\mathcal{O}\left(\mathsf{q}^5\right)$, it is given explicitly by

$$\begin{aligned}
\mathcal{F} = \; & a^2 \log \mathsf{q} + m^2 \left(\log \frac{2a}{\Lambda} + \frac{1}{2}\log\frac{m}{\Lambda} - \frac{3}{4}\right) \\
& + \frac{m^4}{a^2}\left(-\frac{1}{48} + \frac{1}{2}\mathsf{q} + \frac{3}{2}\mathsf{q}^2 + 2\mathsf{q}^3 + \frac{7}{2}\mathsf{q}^4 + \mathcal{O}\left(\mathsf{q}^5\right)\right) \\
& + \frac{m^6}{a^4}\left(-\frac{1}{960} - \frac{3}{4}\mathsf{q}^2 - 4\mathsf{q}^3 - \frac{45}{4}\mathsf{q}^4 + \mathcal{O}\left(\mathsf{q}^5\right)\right) \\
& + \frac{m^8}{a^6}\left(-\frac{1}{10752} + \frac{5}{64}\mathsf{q}^2 + \frac{5}{2}\mathsf{q}^3 + \frac{1095}{64}\mathsf{q}^4 + \mathcal{O}\left(\mathsf{q}^5\right)\right) + \mathcal{O}\left(\frac{m^{10}}{a^8}\right),
\end{aligned} \tag{4.1}$$

where we organize $\mathcal{F}$ as a series in inverse powers of a^2. Because of the S-duality of the ultraviolet theory, we expect that the q-series in each parentheses is the first few terms of a quasi-modular form, which can be written in terms of the Eisenstein series E_2, E_4 and E_6. Indeed, we can complete the q-series to get

$$\begin{aligned}
\mathcal{F} = \; & a^2 \log \mathsf{q} + m^2 \left(\log \frac{2a}{\Lambda} + \frac{1}{2}\log\frac{m}{\Lambda} - \frac{3}{4}\right) - \frac{m^4 E_2}{48a^2} \\
& - \frac{m^6\left(5E_2^2 + E_4\right)}{5760a^4} - \frac{m^8\left(175E_2^3 + 84E_2E_4 + 11E_6\right)}{2903040a^6} + \mathcal{O}\left(\frac{m^{10}}{a^8}\right).
\end{aligned} \tag{4.2}$$

The appearance of the quasi-modular form E_2 is unavoidable in order for $\mathcal{F}$ to transform properly under S-duality [81]. The Λ dependent part of $\mathcal{F}$ is

$$\mathcal{F} \sim -\frac{3}{2} m^2 \log \Lambda. \tag{4.3}$$

If we weakly gauge the U(1) flavor symmetry, then m can be viewed as the vacuum expectation value of the corresponding vector multiplet. The hypermultiplet transforms in the adjoint representation of the gauge group SU(2). From the $\mathcal{N} = 2$ preserving superpotential

$$W = \sqrt{2}\mathrm{Tr}\tilde{Q}\Phi Q + m\mathrm{Tr}\tilde{Q}Q, \tag{4.4}$$

we know that the hypermultiplet has charge ± 1 under this U(1). We can get the coefficient of the one-loop beta function for the U(1) coupling constant from

$$\Lambda \frac{\partial^3 \mathcal{F}}{\partial \Lambda \partial m^2} = -3, \tag{4.5}$$

where the sign is opposite to that of an asymptotically free theory, and 3 is the dimension of the adjoint representation of SU(2).

Using the derivatives of the Eisenstein series (C.5), we obtain the Coulomb branch order parameter

$$\begin{aligned} u &= \frac{1}{2}\left\langle \mathrm{Tr}\phi^2 \right\rangle = \mathsf{q}\frac{\partial \mathcal{F}}{\partial \mathsf{q}} \\ &= a^2 + \frac{m^4\left(-E_2^2 + E_4\right)}{576a^2} + \frac{m^6\left(-5E_2^3 + 3E_2E_4 + 2E_6\right)}{34560a^4} \\ &\quad + \frac{m^8\left(-35E_2^4 + 7E_2^2E_4 + 10E_4^2 + 18E_2E_6\right)}{2322432a^6} + \mathcal{O}\left(\frac{m^{10}}{a^8}\right), \end{aligned} \tag{4.6}$$

which is independent of Λ.

We then go beyond the leading order in the expansion (1.18). It is interesting that we still have $\mathcal{H} = 0$ in the presence of the adjoint hypermultiplet. At the second order, we have two terms which are our main interest,

$$\begin{aligned} \log A = \frac{1}{2}\log\frac{2a}{\Lambda} &+ \frac{m^4}{a^4}\left(-\frac{1}{4}\mathsf{q} - \frac{3}{2}\mathsf{q}^2 - 3\mathsf{q}^3 - 7\mathsf{q}^4 + \mathcal{O}\left(\mathsf{q}^5\right)\right) \\ &+ \frac{m^6}{a^6}\left(\frac{3}{2}\mathsf{q}^2 + 12\mathsf{q}^3 + 45\mathsf{q}^4 + \mathcal{O}\left(\mathsf{q}^5\right)\right) \\ &+ \frac{m^8}{a^8}\left(-\frac{19}{64}\mathsf{q}^2 - 12\mathsf{q}^3 - \frac{3405}{32}\mathsf{q}^4 + \mathcal{O}\left(\mathsf{q}^5\right)\right) + \mathcal{O}\left(\frac{m^{10}}{a^{10}}\right), \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \log B = \frac{3}{4}\log\frac{2a}{\Lambda} + \frac{1}{8}\log\frac{m}{\Lambda} &+ \frac{m^2}{a^2}\left(-\frac{1}{32} + \frac{3}{4}\mathsf{q} + \frac{9}{4}\mathsf{q}^2 + 3\mathsf{q}^3 + \frac{21}{4}\mathsf{q}^4 + \mathcal{O}\left(\mathsf{q}^5\right)\right) \\ + \frac{m^4}{a^4}&\left(-\frac{1}{256} - \frac{3}{8}\mathsf{q} - \frac{81}{16}\mathsf{q}^2 - \frac{39}{2}\mathsf{q}^3 - \frac{843}{16}\mathsf{q}^4 + \mathcal{O}\left(\mathsf{q}^5\right)\right) \\ + \frac{m^6}{a^6}&\left(-\frac{1}{1536} + \frac{195}{64}\mathsf{q}^2 + \frac{75}{2}\mathsf{q}^3 + \frac{12465}{64}\mathsf{q}^4 + \mathcal{O}\left(\mathsf{q}^5\right)\right) \\ + \frac{m^8}{a^8}&\left(-\frac{1}{8192} - \frac{63}{128}\mathsf{q}^2 - \frac{441}{16}\mathsf{q}^3 - \frac{83097}{256}\mathsf{q}^4 + \mathcal{O}\left(\mathsf{q}^5\right)\right) + \mathcal{O}\left(\frac{m^{10}}{a^{10}}\right). \end{aligned} \tag{4.8}$$

Similar to the treatment of $\mathcal{F}$, we complete each $\mathtt{q}$-series into a quasi-modular form,

$$\begin{aligned}\log A &= \frac{1}{2}\log\frac{2a}{\Lambda}+\frac{m^4\left(E_2^2-E_4\right)}{1152a^4}+\frac{m^6\left(5E_2^3-3E_2E_4-2E_6\right)}{34560a^6}\\&\quad+\frac{m^8\left(203E_2^4-28E_2^2E_4-67E_4^2-108E_2E_6\right)}{9289728a^8}+\mathcal{O}\left(\frac{m^{10}}{a^{10}}\right),\end{aligned} \tag{4.9}$$

$$\begin{aligned}\log B &= \frac{3}{4}\log\frac{2a}{\Lambda}+\frac{1}{8}\log\frac{m}{\Lambda}-\frac{m^2E_2}{32a^2}-\frac{m^4\left(E_2^2+E_4\right)}{512a^4}-\frac{m^6\left(25E_2^3+48E_2E_4+17E_6\right)}{138240a^6}\\&\quad-\frac{m^8\left(1225E_2^4+3332E_2^2E_4+1055E_4^2+1948E_2E_6\right)}{61931520a^8}+\mathcal{O}\left(\frac{m^{10}}{a^{10}}\right).\end{aligned} \tag{4.10}$$

We can get the pure $\mathcal{N}=2$ super-Yang-Mills theory from the $\mathcal{N}=2^*$ theory by taking a certain decoupling limit. This limit is not manifest in (4.2) since it is written in the limit $m/a\to 0$. The expression of $\mathcal{F}$ in the limit $m/a\to\infty$ is given by

$$\begin{aligned}\mathcal{F} &= m^2\left(\frac{3}{2}\log\left(\frac{m}{\Lambda}\right)-\frac{9}{4}\right)+a^2\log\frac{\mathtt{q}m^4}{\Lambda^4}-4a^2\left(\log\left(\frac{2a}{\Lambda}\right)-\frac{3}{2}\right)\\&\quad+\mathtt{q}\frac{m^4}{2a^2}+\mathtt{q}^2\left(\frac{5m^8}{64a^6}-\frac{3m^6}{4a^4}+\frac{3m^4}{2a^2}\right)+\mathcal{O}\left(\mathtt{q}^3,\frac{a}{m}\right).\end{aligned} \tag{4.11}$$

Therefore, if we take the limit

$$m\to\infty,\quad \mathtt{q}\to 0,\quad \mathtt{q}m^4=\Lambda^4, \tag{4.12}$$

the effective prepotential (4.11) becomes (3.1) up to a constant,

$$\mathcal{F}_{\mathcal{N}=2^*}\to\mathcal{F}_{\mathrm{SYM}}+m^2\left(\frac{3}{2}\log\left(\frac{m}{\Lambda}\right)-\frac{9}{4}\right), \tag{4.13}$$

and the relation between the order parameters $u_{\mathcal{N}=2^*}$ and u_{SYM} is given by

$$u_{\mathcal{N}=2^*}\to u_{\mathrm{SYM}}-\frac{3m^2}{8}. \tag{4.14}$$

Similarly, we can consider the limit (4.12) for $\log A$ and $\log B$,

$$\begin{aligned}\log A &= \frac{1}{2}\log\left(\frac{2a}{\Lambda}\right)-\mathtt{q}\frac{m^4}{4a^4}-\mathtt{q}^2\left(\frac{19m^8}{64a^8}-\frac{3m^6}{2a^6}+\frac{3m^4}{2a^4}\right)+\mathcal{O}\left(\mathtt{q}^3,\frac{a}{m}\right)\\&\to\frac{1}{2}\log\left(\frac{2a}{\Lambda}\right)-\frac{\Lambda^4}{4a^4}-\frac{19\Lambda^8}{64a^8}+\mathcal{O}\left(\frac{\Lambda^{12}}{a^{12}}\right),\end{aligned} \tag{4.15}$$

$$\begin{aligned}\log B &= \frac{1}{2}\log\left(\frac{2a}{\Lambda}\right)+\frac{3}{8}\log\left(\frac{m}{\Lambda}\right)-\mathtt{q}\left(\frac{3m^4}{8a^4}-\frac{3m^2}{4a^2}\right)\\&\quad-\mathtt{q}^2\left(\frac{63m^8}{128a^8}-\frac{195m^6}{64a^6}+\frac{81m^4}{16a^4}-\frac{9m^2}{4a^2}\right)+\mathcal{O}\left(\mathtt{q}^3,\frac{a}{m}\right)\\&\to\frac{1}{2}\log\left(\frac{2a}{\Lambda}\right)+\frac{3}{8}\log\left(\frac{m}{\Lambda}\right)-\frac{3\Lambda^4}{8a^4}-\frac{63\Lambda^8}{128a^8}+\mathcal{O}\left(\frac{\Lambda^{12}}{a^{12}}\right).\end{aligned} \tag{4.16}$$

Hence we have

$$A_{\mathcal{N}=2^*}\to A_{\mathrm{SYM}},\quad B_{\mathcal{N}=2^*}\to\left(\frac{m}{\Lambda}\right)^{\frac{3}{8}}B_{\mathrm{SYM}}. \tag{4.17}$$

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4.2 Comparison to the prediction

In order to compare our results with the prediction (1.10), we take the Seiberg-Witten curve and the Seiberg-Witten differential to be [30–32]

$$t^2 = \tilde{u} - \nu m^2 \wp\left(z;\tau_{\rm UV}\right), \quad \lambda = t dz, \tag{4.18}$$

where the parameter $\tilde{u}$ in the curve is the same as u up to an additive constant,

$$\tilde{u} = u + m^2 h\left(\tau_{\rm UV}\right), \tag{4.19}$$

and $\wp\left(z;\tau_{\rm UV}\right)$ is Weierstrass's elliptic function (see appendix D for its basic properties). We see from the curve (4.18) that $\tilde{u}$ is modular under the ultraviolet S-duality transformation. The adjustable numerical constant ν depends on the normalization and will be fixed later. In fact, (4.18) is a special example of the Seiberg-Witten geometry constructed using the elliptic Calogero-Moser integrable system [82, 83].

We can extract $a\left(\tilde{u}\right)$ in the usual way from the period integral,

$$\begin{aligned} a\left(\tilde{u}\right) &= \frac{1}{\pi}\oint_A \sqrt{\tilde{u} - \nu m^2 \wp\left(z;\tau_{\rm UV}\right)} dz \\ &= \sqrt{\tilde{u}}\left(1 - \frac{\nu m^2}{2\tilde{u}}\mathcal{P}_1 - \frac{\nu^2 m^4}{8\tilde{u}^2}\mathcal{P}_2 - \frac{\nu^3 m^6}{16\tilde{u}^3}\mathcal{P}_3 - \frac{5\nu^4 m^8}{128\tilde{u}^4}\mathcal{P}_4 + \mathcal{O}\left(\frac{m^{10}}{\tilde{u}^5}\right)\right), \end{aligned} \tag{4.20}$$

where we define

$$\mathcal{P}_n = \frac{1}{\pi}\oint_A \wp^n\left(z;\tau_{\rm UV}\right) dz, \tag{4.21}$$

whose explicit expressions are given in appendix D. We can solve $\tilde{u}$ in terms of a by inverting (4.20),

$$\begin{aligned} \tilde{u} &= a^2 + \nu m^2 \mathcal{P}_1 - \frac{\nu^2 m^4\left(\mathcal{P}_1^2 - \mathcal{P}_2\right)}{4a^2} + \frac{\nu^3 m^6\left(2\mathcal{P}_1^3 - 3\mathcal{P}_1\mathcal{P}_2 + \mathcal{P}_3\right)}{8a^4} \\ &\quad - \frac{5\nu^4 m^8\left(4\mathcal{P}_1^4 - 8\mathcal{P}_1^2\mathcal{P}_2 + \mathcal{P}_2^2 + 4\mathcal{P}_1\mathcal{P}_3 - \mathcal{P}_4\right)}{64a^6} + \mathcal{O}\left(\frac{m^{10}}{a^8}\right) \\ &= a^2 - \frac{\nu m^2 E_2}{3} + \frac{\nu^2 m^4\left(-E_2^2 + E_4\right)}{36a^2} + \frac{\nu^3 m^6\left(-5E_2^3 + 3E_2E_4 + 2E_6\right)}{540a^4} \\ &\quad + \frac{\nu^4 m^8\left(-35E_2^4 + 7E_2^2E_4 + 10E_4^2 + 18E_2E_6\right)}{9072a^6} + \mathcal{O}\left(\frac{m^{10}}{a^8}\right), \end{aligned} \tag{4.22}$$

which matches (4.6) if we choose

$$\tilde{u} = u - \frac{m^2 E_2}{12}, \quad \nu = \frac{1}{4}. \tag{4.23}$$

In fact, one can give a closed form expression for $\tilde{u}$ as function of τ and $\tau_{\rm UV}$ [71],

$$\tilde{u}(\tau,\tau_{\rm UV}) = -\frac{m^2}{4}\frac{e_1(\tau_{\rm UV})^2(e_2(\tau) - e_3(\tau)) + \text{cyclic}}{e_1(\tau_{\rm UV})(e_2(\tau) - e_3(\tau)) + \text{cyclic}}, \tag{4.24}$$

where the e_j are defined in (D.5). Note $\tilde{u}$ is a modular form of weight 0 in τ and weight 2 in $\tau_{\rm UV}$.

Using (4.22), we can compute

$$\begin{aligned}\log\left(\frac{du}{da}\right) &= \log\left(\frac{d\tilde{u}}{da}\right)\\ &= \log 2a + \frac{m^4\left(E_2^2-E_4\right)}{576a^4} + \frac{m^6\left(5E_2^3-3E_2E_4-2E_6\right)}{17280a^6}\\ &\quad + \frac{m^8\left(203E_2^4-28E_2^2E_4-67E_4^2-108E_2E_6\right)}{4644864a^8} + \mathcal{O}\left(\frac{m^{10}}{a^{10}}\right). \end{aligned} \tag{4.25}$$

From the relation (4.14), we know that in the limit (4.12),

$$\left(\frac{du}{da}\right)_{\mathcal{N}=2^*} \to \left(\frac{du}{da}\right)_{\mathrm{SYM}}. \tag{4.26}$$

As function of τ and τ_{UV}, da/du can be expressed as

$$\frac{da}{du} = \frac{1}{4\,m\,\eta(\tau_{\mathrm{UV}})^6}\left(\theta_4(\tau)^4\,\theta_3(\tau_{\mathrm{UV}})^4 - \theta_3(\tau)^4\,\theta_4(\tau_{\mathrm{UV}})^4\right)^{\frac{1}{2}}, \tag{4.27}$$

where η is the Dedekind eta function given in (C.6).

There are three singularities on the Coulomb branch where we have extra massless particles. From (4.18) we know that the singularities are at points

$$\tilde{u} = \frac{m^2}{4}e_i, \quad i=1,2,3, \tag{4.28}$$

where e_i are defined in (D.3). Therefore, the physical discriminant Δ is given by

$$\begin{aligned}\Delta &= \prod_{i=1}^{3}\left(\tilde{u}-\frac{m^2}{4}e_i\right)\\ &= \tilde{u}^3 - \frac{m^2}{4}\left(e_1+e_2+e_3\right)\tilde{u}^2 + \frac{m^4}{16}\left(e_1e_2+e_1e_3+e_2e_3\right)\tilde{u} - \frac{m^6}{64}e_1e_2e_3\\ &= \tilde{u}^3 - \frac{m^4E_4}{48}\tilde{u} - \frac{m^6E_6}{864}, \end{aligned} \tag{4.29}$$

which using (4.24) can be written as

$$\Delta = (2m)^6\,\eta(\tau_{\mathrm{UV}})^{24}\frac{\eta(\tau)^{12}}{\left(\theta_4(\tau)^4\theta_3(\tau_{\mathrm{UV}})^4-\theta_3(\tau)^4\theta_4(\tau_{\mathrm{UV}})^4\right)^3}. \tag{4.30}$$

Substituting (4.22) into (4.29), we obtain

$$\begin{aligned}\log\Delta &= 6\log a - \frac{m^2E_2}{4a^2} - \frac{m^4\left(E_2^2+E_4\right)}{64a^4} - \frac{m^6\left(25E_2^3+48E_2E_4+17E_6\right)}{17280a^6}\\ &\quad - \frac{m^8\left(1225E_2^4+3332E_2^2E_4+1055E_4^2+1948E_2E_6\right)}{7741440a^8} + \mathcal{O}\left(\frac{m^{10}}{a^{10}}\right). \end{aligned} \tag{4.31}$$

In the decoupling limit (4.12),

$$\Delta_{\mathcal{N}=2^*} \to \left(u-\frac{m^2}{4}\right)\left(u-2\Lambda^2\right)\left(u+2\Lambda^2\right) \to -\frac{m^2}{4}\Delta_{\mathrm{SYM}}. \tag{4.32}$$

Comparing (4.25), (4.31) with (4.9), (4.10), we find that

$$A = \Lambda^{-\frac{1}{2}} \left(\frac{du}{da}\right)^{\frac{1}{2}}, \quad B = 2^{\frac{3}{4}} m^{\frac{1}{8}} \Lambda^{-\frac{7}{8}} \Delta^{\frac{1}{8}}. \tag{4.33}$$

We can get the unambiguous ratio

$$\frac{\beta}{\alpha} = 2^{\frac{3}{4}} m^{\frac{1}{8}} \Lambda^{-\frac{3}{8}}. \tag{4.34}$$

Similar to the Λ dependence of the prepotential (4.3), the strange Λ dependence of β/α can be understood as a remnant of gravitational couplings of the weakly gauged U(1) flavor symmetry. In fact, we can see from (A.6) that the remnant contribution of the adjoint hypermultiplet to $\alpha^\chi \beta^\sigma$ is $\Lambda^{-\frac{3}{8}}$, which precisely gives the Λ dependence in β/α.

On the other hand, from (1.17) we have

$$\frac{\beta}{\alpha} = 2^{\frac{15}{16}} \mu \eta \left(\tau_{\mathrm{UV}}\right)^{\frac{3}{2}} m^{\frac{1}{8}}. \tag{4.35}$$

Combining (4.34) and (4.35), we get

$$\mu = 2^{-\frac{3}{16}} \Lambda^{-\frac{3}{8}} \eta \left(\tau_{\mathrm{UV}}\right)^{-\frac{3}{2}}. \tag{4.36}$$

Therefore,

$$K_u \alpha^\chi \beta^\sigma = -\frac{4\mathrm{i}}{\pi} 2^{\frac{3}{4}\sigma} \Lambda^{-\frac{3}{4}\chi - \frac{9}{8}\sigma} \eta \left(\tau_{\mathrm{UV}}\right)^{-6\chi - 6\sigma} m^{\frac{1}{8}\sigma}. \tag{4.37}$$

Finally, we can express $\tau = \frac{1}{4\pi i} \frac{\partial^2 \mathcal{F}}{\partial a^2}$ as a series in a using (4.1). Substitution of this series in the closed expressions (4.24), (4.27) and (4.32) matches with the expansions (4.22), (4.25) and (4.31).

4.3 Mass parameter

As stressed in section 2, we need to be very careful about the masses (2.27). We would like to show explicitly in this theory that we need the mass m rather than m' to get the sensible result from the point of view of the u-plane integral.

It is interesting to notice that the instanton partition function $\mathcal{Z}^{\text{inst}}$ is a-independent if we take either the limit $m \to 0$ or $m' \to 0$. In fact, this is what we expect, since in the massless limit we recover the $\mathcal{N} = 4$ super-Yang-Mills theory, and there is no instanton corrections when we study the dynamics of the theory.

However, if we identify m' rather than m as the mass of the hypermultiplet, we can again naively compute the expansion (1.18). The leading term will not change, and is still given by the prepotential. The next-to-leading order term $\mathcal{H}$ no longer vanishes,

$$\begin{aligned} \mathcal{H} &= -m' \log \frac{2a}{\Lambda} - \frac{1}{2} m' \log \frac{m'}{\Lambda} + \frac{m'}{2} \\ &\quad + \frac{m'^3}{a^2} \left(\frac{1}{24} - \mathsf{q} - 3\mathsf{q}^2 - 4\mathsf{q}^3 + \mathcal{O}\left(\mathsf{q}^4\right) \right) + \mathcal{O}\left(\frac{m'^5}{a^4} \right). \end{aligned} \tag{4.38}$$

Moreover, we have

$$\log A = -\frac{1}{8}\log\left(\frac{m'^2}{\Lambda^2}\right) + \frac{m'^2}{a^2}\left(\frac{1}{16} - \frac{3}{2}\mathsf{q} - \frac{9}{2}\mathsf{q}^2 - 6\mathsf{q}^3 + \mathcal{O}\left(\mathsf{q}^4\right)\right)$$
$$+\frac{m'^4}{a^4}\left(\frac{1}{128} - \frac{1}{4}\mathsf{q} + \frac{33}{8}\mathsf{q}^2 + 27\mathsf{q}^3 + \mathcal{O}\left(\mathsf{q}^4\right)\right) + \mathcal{O}\left(\frac{m'^6}{a^6}\right), \tag{4.39}$$

and

$$\log B = -\frac{1}{8}\log\left(\frac{m'^2}{\Lambda^2}\right) + \frac{m'^2}{a^2}\left(\frac{1}{16} - \frac{3}{2}\mathsf{q} - \frac{9}{2}\mathsf{q}^2 - 6\mathsf{q}^3 + \mathcal{O}\left(\mathsf{q}^4\right)\right)$$
$$+\frac{m'^2}{a^4}\left(\frac{1}{128} - \frac{3}{8}\mathsf{q} + \frac{27}{8}\mathsf{q}^2 + \frac{51}{2}\mathsf{q}^3 + \mathcal{O}\left(\mathsf{q}^4\right)\right) + \mathcal{O}\left(\frac{m'^6}{a^6}\right). \tag{4.40}$$

Clearly, A and B will violate the general forms (1.10).

On the other hand, we can compute the $\mathrm{U}(N)$ partition function of the $\mathcal{N} = 2^*$ theory with the limit $m' \to 0$ on a compact manifold, and the partition function is known to be the generating function of the Euler characteristic of the moduli space of unframed semi-stable equivariant torsion-free sheaves [51]. An explicit example of the $\mathrm{U}(2)$ partition function on $\mathbb{CP}^2$ was given in [84, 85], and the result was given in terms of mock modular forms.

For $K3$ manifolds we have $2\chi + 3\sigma = 0$. If we express χ and σ of the Ω-background of $\mathbb{C}^2$ in terms of ε_1 and ε_2 using (1.19), we get

$$0 = 2\chi + 3\sigma = 2\left(\varepsilon_1\varepsilon_2\right) + 3\left(\frac{\varepsilon_1^2 + \varepsilon_2^2}{3}\right) = \left(2\varepsilon_+\right)^2. \tag{4.41}$$

Hence, $m' = m$ and we no longer need to distinguish between the Donaldson-Witten twist and the Vafa-Witten twist. This is the reason why (1.17) can make sense.

5 The SU(2) theory with fundamental hypermultiplets

Now we consider the $\mathrm{SU}(2)$ gauge theory with $N_f \leq 4$ fundamental hypermultiplets. We will mainly focus on the $N_f = 4$ case which is superconformal, and we turn on mass deformations with four masses m_1, m_2, m_3, m_4. In the class $\mathcal{S}$ construction [31, 32], the $\mathrm{SU}(2)$ gauge theory with four fundamental hypermultiplets arises by compactifying the six-dimensional $(2,0)$ theory of type A_1 on a sphere with four punctures. There are three cusps in the moduli space where we have weakly coupled descriptions of the theory. For each cusp we can define a cross ratio of the four punctures. This cross ratio is identified with the instanton counting parameter $\mathsf{q} = e^{2\pi\mathrm{i}\tau_{\mathrm{UV}}}$ for that weak-coupling description.

5.1 Expansion of the partition function

Similar to the previous cases, we can compute the expansion (1.18). Here we introduce the shorthand notation $[\![\ ]\!]$ to indicate the sum of all terms that make m_1, m_2, m_3, m_4 totally symmetric. For example,

$$[\![m_1^2]\!] = \sum_{i=1}^{4} m_i^2, \quad [\![m_1^2 m_2^2]\!] = \sum_{i<j} m_i^2 m_j^2. \tag{5.1}$$

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We also define

$$\mathrm{Pf}m = m_1 m_2 m_3 m_4. \tag{5.2}$$

The Spin(8) flavor symmetry is broken by the masses down to a Weyl group of the Spin(8) symmetry, and the above combinations of masses are Weyl-group invariant. The explicit expression of the low energy effective prepotential $\mathcal{F}$ is given by

$$\begin{aligned}\mathcal{F} =& a^2\left(\log\frac{\mathsf{q}}{16}+\frac{\mathsf{q}}{2}+\frac{13\mathsf{q}^2}{64}+\frac{23\mathsf{q}^3}{192}\right)+[\![m_1^2]\!]\log\left(\frac{a}{\Lambda}\right)\\ &+\left((m_1m_2+m_3m_4)\left(\frac{1}{2}\mathsf{q}+\frac{1}{4}\mathsf{q}^2+\frac{1}{6}\mathsf{q}^3\right)+\frac{1}{64}[\![m_1^2]\!]\left(\mathsf{q}^2+\mathsf{q}^3\right)\right)\\ &+\frac{1}{a^2}\left(-\frac{1}{12}[\![m_1^4]\!]+\frac{1}{64}[\![m_1^2m_2^2]\!]\left(\mathsf{q}^2+\mathsf{q}^3\right)\right.\\ &\qquad\left.+\mathrm{Pf}m\left(\frac{1}{2}\mathsf{q}+\frac{1}{4}\mathsf{q}^2+\frac{11}{64}\mathsf{q}^3\right)\right)+\mathcal{O}\left(\mathsf{q}^4,\frac{m_i^6}{a^4}\right).\end{aligned} \tag{5.3}$$

Note that the Weyl group symmetry is broken by the a-independent expression in the second line above. This is not surprising since we broke the symmetry by moving to a weak-coupling cusp.[9] Similar to the previous example, it is interesting to analyze the Λ dependence of $\mathcal{F}$

$$\mathcal{F}\sim-[\![m_1^2]\!]\log\Lambda. \tag{5.4}$$

Now we should weakly gauge the Spin(8) flavor symmetry group, which has a subgroup $\mathrm{U}(1)^4$. We view m_i as the vacuum expectation value of the ith U(1) vector multiplet. The hypermultiplet transforms in the fundamental representation of the gauge group SU(2) and has charge ±1 under this U(1). We can get the coefficient of the one-loop beta function for the ith U(1) coupling constant from

$$\Lambda\frac{\partial^3\mathcal{F}}{\partial\Lambda\partial m_i^2}=-2. \tag{5.5}$$

Again the sign is opposite to that of an asymptotically free theory, and 2 is the dimension of the fundamental representation of SU(2).

The Coulomb branch order parameter u is

$$\begin{aligned}u =& \frac{1}{2}\left\langle\mathrm{Tr}\phi^2\right\rangle=\mathsf{q}\frac{\partial\mathcal{F}}{\partial\mathsf{q}}\\ =& a^2\left(1+\frac{\mathsf{q}}{2}+\frac{13\mathsf{q}^2}{32}+\frac{23\mathsf{q}^3}{64}\right)\\ &+\frac{1}{2}(m_1m_2+m_3m_4)\left(\mathsf{q}+\mathsf{q}^2+\mathsf{q}^3\right)+[\![m_1^2]\!]\left(\frac{1}{32}\mathsf{q}^2+\frac{3}{64}\mathsf{q}^3\right)\\ &+\frac{1}{a^2}\left([\![m_1^2m_2^2]\!]\left(\frac{1}{32}\mathsf{q}^2+\frac{3}{64}\mathsf{q}^3\right)\right.\\ &\qquad\left.+\mathrm{Pf}m\left(\frac{1}{2}\mathsf{q}+\frac{1}{2}\mathsf{q}^2+\frac{33}{64}\mathsf{q}^3\right)\right)+\mathcal{O}\left(\mathsf{q}^4,\frac{m_i^6}{a^4}\right).\end{aligned} \tag{5.6}$$

[9] In the Sp(1) gauge theory description, the Weyl group of the Spin(8) symmetry is preserved in the weak-coupling limit. This does not lead to a contradiction because the a-independent part of $\mathcal{F}$ has no effect on the low energy effective action and is therefore not physical.

Notice that this definition of u breaks the Weyl group symmetry acting on the masses. We can define a new Coulomb branch order parameter u' which is invariant under the Weyl group symmetry by subtracting an a-independent constant from u,

$$u' = u - \frac{\mathsf{q}\left(m_1 m_2 + m_3 m_4\right)}{2(1-\mathsf{q})}. \tag{5.7}$$

In this example, the vanishing of $\mathcal{H}$ is a little nontrivial. It is crucial that we factor out the residual contribution $\mathcal{Z}^{\text{inst}}_{\text{extra}}$.

We have two interesting terms at the second order,

$$\begin{aligned}\log A =& \frac{1}{2}\log\left(\frac{2a}{\Lambda}\right) + \frac{\mathsf{q}}{4} + \frac{9\mathsf{q}^2}{64} + \frac{19\mathsf{q}^3}{192} \\ & -\frac{1}{a^4}\left(\frac{1}{64}\left[\!\left[m_1^2 m_2^2\right]\!\right]\left(\mathsf{q}^2+\mathsf{q}^3\right)\right. \\ & \left. +\mathrm{Pf}m\left(\frac{1}{4}\mathsf{q} + \frac{1}{8}\mathsf{q}^2 + \frac{3}{32}\mathsf{q}^3\right)\right) + \mathcal{O}\left(\mathsf{q}^4, \frac{m_i^6}{a^6}\right),\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}\log B =& \frac{3}{2}\log\left(\frac{a}{\Lambda}\right) + \frac{1}{2}\log 2 + \frac{3\mathsf{q}}{8} + \frac{27\mathsf{q}^2}{128} + \frac{19\mathsf{q}^3}{128} \\ & +\frac{1}{a^2}\left(\left[\!\left[m_1^2\right]\!\right]\left(-\frac{1}{8} + \frac{3}{256}\mathsf{q}^2 + \frac{3}{256}\mathsf{q}^3\right)\right) \\ & +\frac{1}{a^4}\left(-\frac{3}{64}\left[\!\left[m_1^2 m_2^2\right]\!\right]\left(\mathsf{q}^2+\mathsf{q}^3\right)\right. \\ & \left. -\mathrm{Pf}m\left(\frac{1}{16} + \frac{3}{8}\mathsf{q} + \frac{3}{16}\mathsf{q}^2 + \frac{3}{16}\mathsf{q}^3\right)\right) + \mathcal{O}\left(\mathsf{q}^4, \frac{m_i^6}{a^6}\right).\end{aligned} \tag{5.9}$$

5.2 Comparison to the prediction

Now we would like to compare our explicit results of A and B with the prediction (1.10).

If all the hypermultiplets are massless, the Seiberg-Witten curve is given by [2]

$$y^2 = \prod_{i=1}^{3}\left(x - e_i\left(\tau_{\text{SW}}\right)\hat{u}\right), \tag{5.10}$$

which describes the double cover of a sphere with four punctures. The argument τ_{SW} of e_i coincides with the complex structure of the curve, and is the same as the low energy effective coupling τ_{eff}. It takes value in the upper half plane which is the universal cover of the punctured sphere parameterized by q. The coupling τ_{SW} is related to the coupling τ_{UV} by [86],

$$e^{2\pi\mathrm{i}\tau_{\text{UV}}} = \frac{\theta_2\left(\tau_{\text{SW}}\right)^4}{\theta_3\left(\tau_{\text{SW}}\right)^4} = 16q_{\text{SW}}^{\frac{1}{2}} - 128q_{\text{SW}} + 704q_{\text{SW}}^{\frac{3}{2}} - 3072q_{\text{SW}}^2 + 11488q_{\text{SW}}^{\frac{5}{2}} + \mathcal{O}\left(q_{\text{SW}}^3\right), \tag{5.11}$$

where $q_{\text{SW}} = \exp\left(2\pi\mathrm{i}\tau_{\text{SW}}\right)$.

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When we turn on masses, the curve proposed by Seiberg and Witten [2] is

$$\begin{aligned}
y^2 &= W_1W_2W_3 + A\left(W_1T_1\left(e_2-e_3\right)+W_2T_2\left(e_3-e_1\right)+W_3T_3\left(e_1-e_2\right)\right) - A^2N,\\
W_i &= x - e_i\hat{u} - e_i^2R,\\
A &= \left(e_1-e_2\right)\left(e_2-e_3\right)\left(e_3-e_1\right),\\
R &= \frac{1}{2}\left[\!\left[\hat{m}_1^2\right]\!\right],\\
T_1 &= \frac{1}{12}\left[\!\left[\hat{m}_1^2\hat{m}_2^2\right]\!\right] - \frac{1}{24}\left[\!\left[\hat{m}_1^4\right]\!\right],\\
T_{2,3} &= \pm\frac{1}{2}\mathrm{Pf}\hat{m} - \frac{1}{24}\left[\!\left[\hat{m}_1^2\hat{m}_2^2\right]\!\right] + \frac{1}{48}\left[\!\left[\hat{m}_1^4\right]\!\right],\\
N &= \frac{3}{16}\left[\!\left[\hat{m}_1^2\hat{m}_2^2\hat{m}_3^2\right]\!\right] - \frac{1}{96}\left[\!\left[\hat{m}_1^4\hat{m}_2^2\right]\!\right] + \frac{1}{96}\left[\!\left[\hat{m}_1^6\right]\!\right].
\end{aligned} \tag{5.12}$$

Here the argument of e_i is still τ_{SW}, but it is no longer the complex structure of the curve, and therefore is different from the low energy effective coupling τ_{eff}. In principle, we can compare our results computed from the partition function $\mathcal{Z}$ with the curve (5.12). However, it turns out that the parameter $\hat{u}$ and the masses $\hat{m}_i$ in the curve (5.12) are related to u and m_i used in $\mathcal{Z}$ in a complicated way [87, 88],

$$\hat{u} = h_u\left(u,\mathfrak{q},m_i\right), \quad \hat{m}_i = h_i\left(\mathfrak{q},m_i\right). \tag{5.13}$$

Due to this problem, it is complicated to compare our results directly with this form of the Seiberg-Witten curve.

A more conceptual reason why the Seiberg-Witten curve (5.12) is not suitable for the instanton counting is the following. Since the Seiberg-Witten curve (5.12) is obtained from a mass deformation of (5.10), the parameters appearing in (5.12) are measured in the limit $\hat{m}_i \to 0$, or equivalently $a \to \infty$, and τ_{SW} is defined as

$$\tau_{\mathrm{SW}} = \frac{1}{2\pi\mathrm{i}}\left.\frac{\partial^2\mathcal{F}}{\partial a^2}\right|_{\hat{m}_i=0}. \tag{5.14}$$

On the other hand, the partition function $\mathcal{Z}$ is computed as a series expansion in $\mathfrak{q}$, and the convergence of the series requires that $\mathfrak{q}$ is small. All the parameters appearing in $\mathcal{Z}$ are naturally measured in the limit $\mathfrak{q} \to 0$, which is also the degenerate limit of the punctured sphere in the class $\mathcal{S}$ construction. Two limits $\mathfrak{q} \to 0$ and $a \to \infty$ are the same for asymptotically free theories, but in general are different for superconformal theories.

To make life simple, we would like to work with another Seiberg-Witten curve. Our choice of the Seiberg-Witten curve is constructed from the qq-characters of the theory. The fundamental qq-character of the SU(2) gauge theory with $N_f \leq 4$ fundamental hypermultiplets is given by [66, 89]

$$\mathcal{X}(x) = \mathcal{Y}\left(x+2\varepsilon_+\right) + \mathfrak{q}\mathcal{Y}(x)^{-1}\prod_{i=1}^{N_f}\left(x+m_i+\varepsilon_+\right), \tag{5.15}$$

where the observable $\mathcal{Y}(x)$ is the quantum corrected characteristic polynomial of ϕ in the Ω-background,

$$\mathcal{Y}(x)=x^2\exp\left(-\sum_{n=1}^{\infty}\frac{1}{nx^n}\mathrm{Tr}\phi^n\right)=x^2-\frac{1}{2}\mathrm{Tr}\phi^2+\mathcal{O}\left(x^{-1}\right). \tag{5.16}$$

Although the expectation value of $\mathcal{Y}(x)$ contains singularities in x, $\mathcal{X}(x)$ satisfies the non-perturbative Dyson-Schwinger equation [66],

$$\langle\mathcal{X}(x)\rangle=\mathcal{T}(x), \tag{5.17}$$

where $\mathcal{T}(x)$ is a quadratic polynomial in the variable x and can be fixed by comparing the large x expansions of both sides. For example, for $N_f=4$ we have

$$\begin{aligned}\mathcal{T}(x)&=\left\langle(\mathcal{X}(x))_+\right\rangle\\&=(x+\varepsilon_1+\varepsilon_2)^2-\tilde{u}+\mathtt{q}\left(x^2+\left(\sum_{i=1}^{4}m_i\right)x+\tilde{u}\right),\end{aligned} \tag{5.18}$$

where $(\cdot)_+$ means the polynomial part of the Laurent series, and $\tilde{u}$ is identified with u up to an additive constant,

$$\tilde{u}=u+\sum_{n=1}^{\infty}\mathtt{q}^n f_n\left(m_i\right). \tag{5.19}$$

It is not difficult to work out f_n explicitly. However, we should notice that in the proof of the non-perturbative Dyson-Schwinger equation we consider $\mathrm{U}(N)$ gauge theories, and the relation between $\tilde{u}$ and u will be modified when we restrict ourselves to gauge group $\mathrm{SU}(N)$. The Seiberg-Witten curve is given by taking the flat space limit $\varepsilon_1,\varepsilon_2\to 0$,

$$Y+\frac{\mathtt{q}\prod_{i=1}^{N_f}(x+m_i)}{Y}=T(x), \tag{5.20}$$

where

$$Y=\langle\mathcal{Y}(x)\rangle\,,\quad T(x)=\lim_{\varepsilon_1,\varepsilon_2\to 0}\mathcal{T}(x)=(1+\mathtt{q})\,x^2+\mathtt{q}\left(\sum_{i=1}^{4}m_i\right)x-(1-\mathtt{q})\,\tilde{u}, \tag{5.21}$$

and the canonical Seiberg-Witten differential is given by

$$\lambda=x\frac{dY}{Y}. \tag{5.22}$$

It is convenient to perform a change of variables,

$$y=\frac{2Y-T(x)}{1-\mathtt{q}}, \tag{5.23}$$

so that the Seiberg-Witten curve becomes

$$y^2=\left(\frac{T(x)}{1-\mathtt{q}}\right)^2-\frac{4\mathtt{q}}{(1-\mathtt{q})^2}\prod_{i=1}^{N_f}(x+m_i)\,. \tag{5.24}$$

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The right hand side of (5.24) is now a monic polynomial in x of degree four. The curve (5.24) can be viewed as a hybrid of the Seiberg-Witten curve (5.12) and the class S curve [31, 32]. It describes a torus rather than a punctured sphere, but the parameters are measured in the same way as those in the class $\mathcal{S}$ curve. The Seiberg-Witten differential λ is determined by

$$\frac{\partial\lambda}{\partial\tilde{u}} = \frac{1}{2\pi\mathrm{i}}\frac{dx}{y}, \tag{5.25}$$

whose period integral gives

$$\frac{\partial a}{\partial\tilde{u}} = \frac{1}{2\pi\mathrm{i}}\oint_A\frac{dx}{y}. \tag{5.26}$$

Using the result reviewed in appendix B, we can write down the exact result of the period integral (5.26) in terms of the hypergeometric function. We then expand it as

$$\begin{aligned}\frac{\partial a}{\partial\tilde{u}} =& \frac{1}{\tilde{u}^{1/2}}\left(\frac{1}{2}-\frac{\mathsf{q}}{8}-\frac{7\mathsf{q}^2}{128}-\frac{17\mathsf{q}^3}{512}\right)\\ &+\frac{1}{\tilde{u}^{3/2}}\left(\llbracket m_1^2\rrbracket\left(\frac{1}{128}\mathsf{q}^2+\frac{5}{512}\mathsf{q}^3\right)-\llbracket m_1m_2\rrbracket\left(\frac{1}{8}\mathsf{q}+\frac{3}{32}\mathsf{q}^2+\frac{41}{512}\mathsf{q}^3\right)\right)\\ &+\frac{1}{\tilde{u}^{5/2}}\left(\llbracket m_1^2m_2^2\rrbracket\left(\frac{9}{128}\mathsf{q}^2+\frac{63}{512}\mathsf{q}^3\right)+\mathrm{Pf}m\left(\frac{3}{8}\mathsf{q}+\frac{3}{4}\mathsf{q}^2+\frac{531}{512}\mathsf{q}^3\right)\right.\\ &\qquad\left.+\llbracket m_1^2m_2m_3\rrbracket\left(\frac{3}{32}\mathsf{q}^2+\frac{81}{512}\mathsf{q}^3\right)-\llbracket m_1^3m_2\rrbracket\left(\frac{3}{512}\mathsf{q}^3\right)\right)+\mathcal{O}\left(\mathsf{q}^4,\frac{m_i^6}{\tilde{u}^{7/2}}\right).\end{aligned} \tag{5.27}$$

We integrate (5.27) over $\tilde{u}$ to get $a(\tilde{u})$, and then solve the inversion $\tilde{u}(a)$,

$$\begin{aligned}\tilde{u} =& a^2\left(1+\frac{\mathsf{q}}{2}+\frac{13\mathsf{q}^2}{32}+\frac{23\mathsf{q}^3}{64}\right)\\ &+\left(\llbracket m_1^2\rrbracket\left(\frac{1}{32}\mathsf{q}^2+\frac{3}{64}\mathsf{q}^3\right)-\frac{1}{2}\llbracket m_1m_2\rrbracket\left(\mathsf{q}+\mathsf{q}^2+\mathsf{q}^3\right)\right)\\ &+\frac{1}{a^2}\left(\llbracket m_1^2m_2^2\rrbracket\left(\frac{1}{32}\mathsf{q}^2+\frac{3}{64}\mathsf{q}^3\right)\right.\\ &\qquad\left.+\mathrm{Pf}m\left(\frac{1}{2}\mathsf{q}+\frac{1}{2}\mathsf{q}^2+\frac{33}{64}\mathsf{q}^3\right)\right)+\mathcal{O}\left(\mathsf{q}^4,\frac{m_i^6}{a^4}\right),\end{aligned} \tag{5.28}$$

which matches u computed in (5.6) up to a-independent terms. It is easy to compute

$$\begin{aligned}\log\left(\frac{du}{da}\right) =& \log\left(\frac{d\tilde{u}}{da}\right)\\ =& \log(2a)+\frac{\mathsf{q}}{2}+\frac{9\mathsf{q}^2}{32}+\frac{19\mathsf{q}^3}{96}\\ &-\frac{1}{a^4}\left(\frac{1}{32}\llbracket m_1^2m_2^2\rrbracket\left(\mathsf{q}^2+\mathsf{q}^3\right)\right.\\ &\qquad\left.+\mathrm{Pf}m\left(\frac{1}{2}\mathsf{q}+\frac{1}{4}\mathsf{q}^2+\frac{3}{16}\mathsf{q}^3\right)\right)+\mathcal{O}\left(\mathsf{q}^4,\frac{m_i^6}{a^6}\right).\end{aligned} \tag{5.29}$$

There are 6 singularities on the Coulomb moduli space for the SU(2) gauge theory with $N_f = 4$ fundamental hypermultiplets. Unlike the previous case, it is complicated to write

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down the explicit expressions of the discriminant loci where we have extra massless BPS states. What we can do is to compute the physical discriminant Δ from the mathematical discriminant $\hat{\Delta}$ of the Seiberg-Witten curve (5.24) by dividing the $\tilde{u}^6$ coefficient of $\hat{\Delta}$,

$$\Delta = \frac{\hat{\Delta}}{\mathrm{Coeff}_{\tilde{u}^6}\left(\hat{\Delta}\right)}. \tag{5.30}$$

Then Δ is indeed a monic polynomial in $\tilde{u}$ of degree 6. We can compute

$$\begin{aligned}\log\Delta =\ & 12\log(a) + 3\mathsf{q} + \frac{27\mathsf{q}^2}{16} + \frac{19\mathsf{q}^3}{16} \\ & + \frac{1}{a^2}\left(\llbracket m_1^2 \rrbracket \left(-1 + \frac{3}{32}\mathsf{q}^2 + \frac{3}{32}\mathsf{q}^3\right)\right) \\ & + \frac{1}{a^4}\left(-\frac{3}{8}\llbracket m_1^2 m_2^2 \rrbracket \left(\mathsf{q}^2 + \mathsf{q}^3\right)\right. \\ & \qquad \left. - \mathrm{Pf}m\left(\frac{1}{2} + 3\mathsf{q} + \frac{3}{2}\mathsf{q}^2 + \frac{3}{2}\mathsf{q}^3\right)\right) + \mathcal{O}\left(\mathsf{q}^4, \frac{m_i^6}{a^6}\right).\end{aligned} \tag{5.31}$$

By comparing (5.29), (5.31) with the explicit calculation in the Ω-background (5.8), (5.9), we find that

$$A = \Lambda^{-\frac{1}{2}}\left(\frac{du}{da}\right)^{\frac{1}{2}}, \quad B = \sqrt{2}\Lambda^{-\frac{3}{2}}\Delta^{\frac{1}{8}}. \tag{5.32}$$

We see that the explicit dependence on q disappears. Therefore, we confirm (1.10), and we find the unambiguous ratio

$$\frac{\beta}{\alpha} = \sqrt{2}\Lambda^{-1}. \tag{5.33}$$

Similar to the previous case, it is easy to check that we cannot get (1.10) if we use m'_f rather than m_f as the mass parameters. The strange Λ dependence of β/α can be again understood as a remnant of gravitational couplings of the weakly gauged Spin(8) flavor symmetry. Each fundamental hypermultiplet contributes a factor of $\Lambda^{-\frac{1}{4}}$ to β/α, and we have four fundamental hypermultiplets.

It is straightforward to perform the same computation in asymptotically free theories with $N_f \leq 3$ fundamental hypermultiplets. The Seiberg-Witten curve can be constructed in the same way from the fundamental qq-character. The expansion (1.18) matches (1.10) for each case, and the overall factors α and β depend on Λ but not on masses.

6 Perturbative analysis in SU(N) super-Yang-Mills theory

In this section, we will study A and B in the SU(N) super-Yang-Mills theory. In particular, we would like to check the prediction (1.14). For the purpose of determining α and β, it is sufficient to neglect the complicated instanton contributions and use only the perturbative part of the partition function.

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The one-loop partition function is given by

$$\mathcal{Z}^{1-\text{loop}} = \prod_{i<j} \exp\left[-\gamma_{\varepsilon_1,\varepsilon_2}\left(a_i - a_j;\Lambda\right) - \gamma_{\varepsilon_1,\varepsilon_2}\left(a_i - a_j - 2\varepsilon_+;\Lambda\right)\right], \tag{6.1}$$

with the constraint

$$\sum_{i=1}^{N} a_i = 0. \tag{6.2}$$

We can expand (6.1) around the flat space limit (1.18) using (A.6),

$$\begin{aligned}
\mathcal{F}^{1-\text{loop}} &= \sum_{i<j}\left((a_i-a_j)^2\log\left(\frac{a_i-a_j}{\Lambda}\right) - \frac{3}{2}(a_i-a_j)^2\right),\\
\mathcal{H}^{1-\text{loop}} &= 0,\\
\log A^{1-\text{loop}} &= \frac{1}{2}\sum_{i<j}\log\left(\frac{a_i-a_j}{\Lambda}\right),\\
\log B^{1-\text{loop}} &= \frac{1}{2}\sum_{i<j}\log\left(\frac{a_i-a_j}{\Lambda}\right).
\end{aligned} \tag{6.3}$$

We take the Seiberg-Witten curve to be [90, 91]

$$y^2 = \left(\langle\det\left(x-\phi\right)\rangle\right)^2 - 4\Lambda^{2N}, \tag{6.4}$$

with the Coulomb branch order parameters

$$u_n = \left\langle \frac{1}{n}\mathrm{Tr}\phi^n \right\rangle, \quad n = 2,\cdots,N. \tag{6.5}$$

Ignoring the instanton corrections, the Seiberg-Witten curve degenerates to $y^2 = \left(\langle\det\left(x-\phi\right)\rangle\right)^2$, and u_n are simply given by the classical result,

$$u_n = \frac{1}{n}\sum_{i=1}^{N} a_i^n. \tag{6.6}$$

We take $a_2,\cdots,a_N$ as independent parameters. Then we have

$$\left(\frac{du_i}{da_j}\right)^{1-\text{loop}} = a_j^{i-1} - a_1^{i-1}, \tag{6.7}$$

and consequently

$$
\begin{aligned}
\det\left(\frac{du_i}{da_j}\right)^{1-\text{loop}} &= \begin{vmatrix} a_2-a_1 & \cdots & a_N-a_1 \\ \vdots & \ddots & \vdots \\ a_2^{N-1}-a_1^{N-1} & \cdots & a_N^{N-1}-a_1^{N-1} \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 & \cdots & 0 \\ a_1 & a_2-a_1 & \cdots & a_N-a_1 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{N-1} & a_2^{N-1}-a_1^{N-1} & \cdots & a_N^{N-1}-a_1^{N-1} \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_N \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{N-1} & a_2^{N-1} & \cdots & a_N^{N-1} \end{vmatrix} \\
&= \prod_{i<j}(a_i-a_j)\,. \qquad (6.8)
\end{aligned}
$$

Meanwhile, the perturbative discriminant is given by

$$\Delta^{1-\text{loop}} = \left[\prod_{i<j}(a_i-a_j)^2\right]^2 . \qquad (6.9)$$

By comparing with (6.3), we reproduce the expression (1.10),

$$A = \Lambda^{-\frac{N(N-1)}{4}} \det\left(\frac{du_i}{da_j}\right)^{\frac{1}{2}}, \quad B = \Lambda^{-\frac{N(N-1)}{4}}\Delta^{\frac{1}{8}}. \qquad (6.10)$$

Hence, we obtain

$$\alpha = \beta = \Lambda^{-\frac{N(N-1)}{4}}, \qquad (6.11)$$

which confirms the prediction (1.14). This also matches our result (3.14) when $N = 2$. The overall numerical constants of β are different due to the different normalization of the discriminant Δ.

7 Discussions and outlook

In this paper, we use the partition function in the Ω-background to compute explicitly the low energy effective couplings A and B to topological invariants of the background gravitational field in four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories with gauge group SU(2). We also study the SU(N) super-Yang-Mills theory at the perturbative level. Our results confirm the previous predictions. We also determine the ratio of the overall factors β/α. For SU(2) theory with either an adjoint hypermultiplet or four fundamental hypermultiplets, we find that β/α is independent of τ_{UV}. Nevertheless, $K_u\alpha^\chi\beta^\sigma$ can still be a nontrivial function of τ_{UV}. It would be interesting to have a better understanding of this fact. Since β/α naturally shows up in the blowup formula [6, 7], it may be useful to analyze carefully the behavior of the u-plane integral under blowups for superconformal theories.

There is no conceptual problem in extending our computation to any other $\mathcal{N}=2$ theory whose partition function in the Ω-background can be calculated. Technically, our brute force expansion in $\mathtt{q}$ can be rather complicated. It would be very interesting to see whether one could directly obtain the all-instanton results using methods of topological recursion [92, 93]. A possible strategy is to use the theory of qq-characters [66, 74, 94–96], and generalize the derivation presented in [73, 89]. This will be discussed in the future.

We should also point out that A and B were exactly computed for the SU(2) super-Yang-Mills theory [58, 68] and the SU(2) gauge theory with one fundamental hypermultiplet [69] using the partition function in the Ω-background of the blowup $\widehat{\mathbb{C}^2}$. This blowup approach is also powerful enough to determine the contact terms in the u-plane integral. We shall discuss the generalization of this approach to other gauge theories in a separate paper. Unfortunately, this blowup approach is not always useful for superconformal theories due to the lack of an important vanishing theorem.

The supersymmetric localization method allows us to provide a contour integral formula for the exact partition function of $\mathcal{N}=2$ supersymmetric gauge theories on compact toric four-manifolds [97–99], generalizing the pioneering work of Pestun [100]. It was shown in [84, 85] that the equivariant Donaldson invariants can be calculated by explicitly evaluating the contour integral for U(2) super-Yang-Mills theory on $\mathbb{CP}^2$. These equivariant Donaldson polynomials correctly reproduce ordinary Donaldson invariants in the limit $\varepsilon_1, \varepsilon_2 \to 0$. It would be interesting to have a better understanding of these computations from the point of view of the u-plane integral, and to perform similar computations with hypermultiplets.

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A Special function $\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda)$

The special function $\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda)$ is defined through the zeta function regularization,

$$\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda) = \frac{d}{ds}\bigg|_{s=0} \frac{\Lambda^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s \frac{e^{-xt}}{(e^{\varepsilon_1 t}-1)(e^{\varepsilon_2 t}-1)}. \tag{A.1}$$

It is related to Barnes' double Gamma function $\Gamma_2(x|\varepsilon_1,\varepsilon_2)$ by

$$\gamma_{\varepsilon_1,\varepsilon_2}(x;1) = \log\Gamma_2(x+\varepsilon_1+\varepsilon_2|\varepsilon_1,\varepsilon_2). \tag{A.2}$$

Let us define $\{c_n, n\in\mathbb{N}\}$ by

$$\frac{1}{(e^{\varepsilon_1 t}-1)(e^{\varepsilon_2 t}-1)} = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^{n-2}, \tag{A.3}$$

where

$$c_0 = \frac{1}{\varepsilon_1\varepsilon_2}, \quad c_1 = -\frac{\varepsilon_1+\varepsilon_2}{2\varepsilon_1\varepsilon_2}, \quad c_2 = \frac{\varepsilon_1^2+3\varepsilon_1\varepsilon_2+\varepsilon_2^2}{6\varepsilon_1\varepsilon_2}. \tag{A.4}$$

Then the expansion of $\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda)$ around the flat space limit $\varepsilon_1,\varepsilon_2\to 0$ can be computed using analytic continuation,

$$\begin{aligned}
\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda) &= \frac{d}{ds}\bigg|_{s=0} \frac{\Lambda^s}{\Gamma(s)}\int_0^\infty dt \sum_{n=0}^{\infty}\frac{c_n}{n!}t^{s+n-3}e^{-xt} \\
&= \frac{d}{ds}\bigg|_{s=0} \frac{\Lambda^s}{\Gamma(s)}\sum_{n=0}^{\infty}\frac{c_n}{n!}\Gamma(s+n-2)x^{2-s-n} \\
&= \frac{1}{\varepsilon_1\varepsilon_2}\left(-\frac{1}{2}x^2\log\left(\frac{x}{\Lambda}\right)+\frac{3}{4}x^2\right) - \frac{\varepsilon_1+\varepsilon_2}{2\varepsilon_1\varepsilon_2}\left(x\log\left(\frac{x}{\Lambda}\right)-x\right) \\
&\quad -\frac{\varepsilon_1^2+3\varepsilon_1\varepsilon_2+\varepsilon_2^2}{12\varepsilon_1\varepsilon_2}\log\left(\frac{x}{\Lambda}\right) + \sum_{n=2}^{\infty}\frac{c_n}{n(n-1)(n-2)}x^{2-n}.
\end{aligned} \tag{A.5}$$

In this paper, we need the expansions of the following two combinations

$$\begin{aligned}
&\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda) + \gamma_{\varepsilon_1,\varepsilon_2}(x-2\varepsilon_+;\Lambda) \\
&\quad = \frac{1}{\varepsilon_1\varepsilon_2}\left(-x^2\log\left(\frac{x}{\Lambda}\right)+\frac{3}{2}x^2\right) - \frac{\varepsilon_1^2+3\varepsilon_1\varepsilon_2+\varepsilon_2^2}{6\varepsilon_1\varepsilon_2}\log\left(\frac{x}{\Lambda}\right)+\cdots, \\
&\quad \times\gamma_{\varepsilon_1,\varepsilon_2}(x-\varepsilon_+;\Lambda) \\
&\quad = \frac{1}{\varepsilon_1\varepsilon_2}\left(-\frac{1}{2}x^2\log\left(\frac{x}{\Lambda}\right)+\frac{3}{4}x^2\right) + \frac{\varepsilon_1^2+\varepsilon_2^2}{24\varepsilon_1\varepsilon_2}\log\left(\frac{x}{\Lambda}\right)+\cdots.
\end{aligned} \tag{A.6}$$

Notice that there is no $\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1\varepsilon_2}$-term in the expansion of both combinations.

B Period integrals on elliptic curves

A general elliptic curve can be written as

$$y^2 = x^4 - c_1x^3 + c_2x^2 - c_3x + c_4 = (x-r_1)(x-r_2)(x-r_3)(x-r_4), \tag{B.1}$$

where

$$c_n = \sum_{1\le i_1<\cdots<i_n\le 4} r_{i_1}\cdots r_{i_n}. \tag{B.2}$$

We assume that $r_1 < r_2 < r_3 < r_4$ are all real. The general case can be obtained by analytic continuation. We define the A-cycle and the B-cycle to enclose the cut $[r_1, r_2]$ and $[r_2, r_3]$, respectively. The period integrals of the holomorphic one-form are

$$\Pi_\gamma = \frac{1}{2\pi \mathrm{i}}\oint_\gamma \frac{dx}{y} = \frac{1}{2\pi \mathrm{i}}\oint_\gamma \frac{dx}{\sqrt{(x-r_1)(x-r_2)(x-r_3)(x-r_4)}}, \quad \gamma = A, B. \tag{B.3}$$

In this paper, we only need the period integral Π_A over the A-cycle. In order to compute the integral, we consider a useful variable change

$$x = \frac{(r_2-r_1)r_4t + (r_4-r_2)r_1}{(r_2-r_1)t+(r_4-r_2)}, \tag{B.4}$$

so that $x = r_1, r_2, r_3, r_4$ are mapped to $t = 0, 1, \frac{1}{\kappa}, \infty$, with

$$\kappa = \frac{(r_1 - r_2)(r_3 - r_4)}{(r_1 - r_3)(r_2 - r_4)}. \tag{B.5}$$

Then we have

$$\begin{aligned} \Pi_A &= [(r_1 - r_3)(r_2 - r_4)]^{-\frac{1}{2}} \frac{1}{\pi} \int_0^1 \frac{dt}{\sqrt{t(1-t)(1-zt)}} \\ &= [(r_1 - r_3)(r_2 - r_4)]^{-\frac{1}{2}} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, \kappa\right), \end{aligned} \tag{B.6}$$

where we used the integral representation of hypergeometric function

$${}_2F_1(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx\, x^{\beta-1} (1-x)^{\gamma-\beta-1} (1-xz)^{-\alpha}. \tag{B.7}$$

We also define the discriminant Δ of the elliptic curve to be

$$\begin{aligned} \Delta &= \prod_{i<j} (r_i - r_j)^2 \\ &= -27c_4^2c_1^4 - 4c_3^3c_1^3 + 18c_2c_3c_4c_1^3 + c_2^2c_3^2c_1^2 + 144c_2c_4^2c_1^2 \\ &\quad -4c_2^3c_4c_1^2 - 6c_3^2c_4c_1^2 + 18c_2c_3^3c_1 - 192c_3c_4^2c_1 - 80c_2^2c_3c_4c_1 \\ &\quad -27c_3^4 + 256c_4^3 - 4c_2^3c_3^2 - 128c_2^2c_4^2 + 16c_2^4c_4 + 144c_2c_3^2c_4. \end{aligned} \tag{B.8}$$

In general, the expression for the roots can be rather complicated. Moreover, in the formula (B.6), the four roots r_1, r_2, r_3, r_4 are not on equal footing. Using the quadratic transformation identity of the hypergeometric function [101],

$${}_2F_1(\alpha, \beta, 2\beta; z) = (1-z)^{-\frac{1}{2}\alpha} {}_2F_1\left(\frac{1}{2}\alpha, \beta - \frac{1}{2}\alpha, \beta + \frac{1}{2}; -\frac{z^2}{4(1-z)}\right), \tag{B.9}$$

with $\alpha = \beta = \frac{1}{2}$, we get

$$\Pi_A = \xi^{-\frac{1}{4}} {}_2F_1\left(\frac{1}{4}, \frac{1}{4}, 1, \tilde{\kappa}\right), \tag{B.10}$$

where

$$\begin{aligned} \xi &= (r_1 - r_3)(r_2 - r_3)(r_1 - r_4)(r_2 - r_4), \\ \tilde{\kappa} &= -\frac{\kappa^2}{4(1-\kappa)} = -\frac{\Delta}{4\xi^3}. \end{aligned} \tag{B.11}$$

We see that the formula (B.10) is now symmetric in r_1, r_2 and r_3, r_4, but not in all of them. We can further apply the cubic transformation identities of the hypergeometric function [101],

$${}_2F_1\left(3\alpha, \frac{1}{3} - \alpha, 2\alpha + \frac{5}{6}; z\right) = (1-4z)^{-3\alpha} {}_2F_1\left(\alpha, \alpha + \frac{1}{3}, 2\alpha + \frac{5}{6}; \frac{27z}{(4z-1)^3}\right), \tag{B.12}$$

with $\alpha = \frac{1}{12}$ to obtain

$$\Pi_A = \rho^{-\frac{1}{4}} {}_2F_1\left(\frac{1}{12}, \frac{5}{12}, 1, \frac{27\Delta}{4\rho^3}\right), \tag{B.13}$$

where

$$\rho = \xi + \frac{\Delta}{\xi^2} = \frac{1}{4}\left(\sum_{i<j}(r_i - r_j)^2\right)^2 - \frac{3}{4}\sum_{i<j}(r_i - r_j)^4 = c_2^2 - 3c_1c_3 + 12c_4. \tag{B.14}$$

This formula makes all the roots completely symmetric. Furthermore, we no longer need to solve the roots for a given elliptic curve in order to obtain the period Π_A, thereby making the computation much simpler.

C Modular forms and theta functions

Eisenstein series. Let $\tau \in \mathbb{H}$ and $q = e^{2\pi i\tau}$. The Eisenstein series E_{2k} is defined by

$$\begin{aligned} E_{2k} &= \frac{1}{2\zeta(2k)} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq. (0,0)}} \frac{1}{(m+n\tau)^{2k}} \\ &= 1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \frac{n^{2k-1}q^n}{1-q^n} \\ &= 1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \end{aligned} \tag{C.1}$$

where $\sigma_p(n)$ is the divisor sum, the sum of the pth powers of the divisors of n. The following explicit expansions of the Eisenstein series are useful,

$$\begin{aligned} E_2 &= 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n = 1 - 24q - 72q^2 - 96q^3 - 168q^4 + \mathcal{O}\left(q^5\right), \\ E_4 &= 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \mathcal{O}\left(q^5\right), \\ E_6 &= 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n = 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 + \mathcal{O}\left(q^5\right). \end{aligned} \tag{C.2}$$

The Eisenstein series E_{2k} is a modular form of weight $2k$ under the $\mathrm{SL}(2,\mathbb{Z})$ modular transformation for $k \geq 2$,

$$E_{2k}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{2k} E_{2k}(\tau), \quad a,b,c,d \in \mathbb{Z}, \quad ad - bc = 1. \tag{C.3}$$

The space of modular forms of $\mathrm{SL}(2,\mathbb{Z})$ forms a ring that is generated by $E_4(\tau)$ and $E_6(\tau)$. For $k=1$, E_2 is quasi-modular,

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) + \frac{6}{\pi i} c(c\tau+d). \tag{C.4}$$

All quasi-modular forms can be expressed as polynomials of E_2, E_4 and E_6. The derivatives of the Eisenstein series are given by

$$\begin{aligned} q\frac{dE_2}{dq} &= \frac{E_2^2 - E_4}{12}, \\ q\frac{dE_4}{dq} &= \frac{E_2E_4 - E_6}{3}, \\ q\frac{dE_6}{dq} &= \frac{E_2E_6 - E_4^2}{2}. \end{aligned} \tag{C.5}$$

Dedekind eta function. The Dedekind eta function is defined by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) = q^{\frac{1}{24}} \phi(q), \tag{C.6}$$

where $\phi(q)$ is called the Euler function. Under the generators of $\mathrm{SL}(2,\mathbb{Z})$, $\eta(\tau)$ transforms as

$$\eta(\tau+1) = e^{\frac{\pi \mathrm{i}}{12}} \eta(\tau), \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-\mathrm{i}\tau}\,\eta(\tau). \tag{C.7}$$

The derivative of $\eta(\tau)$ is related to E_2 by

$$q\frac{d}{dq} \log \eta(\tau) = \frac{E_2}{24}. \tag{C.8}$$

We also use the expansion

$$\log \phi(q) = \sum_{n=1}^{\infty} \log(1 - q^n) = -q - \frac{3}{2}q^2 - \frac{4}{3}q^3 - \frac{7}{4}q^4 + \mathcal{O}\left(q^5\right). \tag{C.9}$$

Jacobi theta functions. The Jacobi theta functions are defined for two complex variables $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$ as

$$\begin{aligned} \theta_1(z;\tau) &= \mathrm{i} \sum_{n\in\mathbb{Z}} (-1)^n\, w^{n+\frac{1}{2}} q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}, \\ \theta_2(z;\tau) &= \sum_{n\in\mathbb{Z}} w^{n+\frac{1}{2}} q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}, \\ \theta_3(z;\tau) &= \sum_{n\in\mathbb{Z}} w^n q^{\frac{1}{2}n^2}, \\ \theta_4(z;\tau) &= \sum_{n\in\mathbb{Z}} (-1)^n\, w^n q^{\frac{1}{2}n^2}, \end{aligned} \tag{C.10}$$

where $w = e^{2\pi \mathrm{i} z}$ and $q = e^{2\pi \mathrm{i}\tau}$. When evaluated at $z = 0$, $\theta_1(0;\tau) = 0$ and $\theta_j(\tau) = \theta_j(0;\tau)$ for $j = 2, 3, 4$ satisfy

$$\theta_2(\tau)^4 + \theta_4(\tau)^4 = \theta_3(\tau)^4, \quad \theta_2(\tau)\theta_3(\tau)\theta_4(\tau) = 2\eta^3(\tau). \tag{C.11}$$

They are also related to the Eisenstein series E_4 and E_6 by

$$E_4 = \frac{1}{2}\left(\theta_2^8 + \theta_3^8 + \theta_4^8\right), \quad E_6 = \frac{1}{2}\left(\theta_2^4 + \theta_3^4\right)\left(\theta_3^4 + \theta_4^4\right)\left(\theta_4^4 - \theta_2^4\right), \tag{C.12}$$

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The transformation of $\theta_j(\tau)$ under the generators of $\mathrm{SL}(2,\mathbb{Z})$ are

$$\begin{aligned}
\theta_2\left(-\frac{1}{\tau}\right) &= \sqrt{-\mathrm{i}\tau}\theta_4(\tau), \quad \theta_2(\tau+1) = e^{\frac{\pi\mathrm{i}}{4}}\theta_2(\tau)\\
\theta_3\left(-\frac{1}{\tau}\right) &= \sqrt{-\mathrm{i}\tau}\theta_3(\tau), \quad \theta_3(\tau+1) = \theta_4(\tau),\\
\theta_4\left(-\frac{1}{\tau}\right) &= \sqrt{-\mathrm{i}\tau}\theta_2(\tau), \quad \theta_4(\tau+1) = \theta_3(\tau).
\end{aligned} \tag{C.13}$$

D Weierstrass's elliptic function

Let z be a coordinate of the torus, which can be viewed as the complex plane with the identification $z \sim z+\pi \sim z+\pi\tau$. We define Weierstrass's elliptic function $\wp(z;\tau)$ to be a meromorphic function in the complex plane with a double pole at each lattice point,

$$\begin{aligned}
\wp(z;\tau) &= \wp(z;\pi,\pi\tau)\\
&= \frac{1}{z^2} + \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \left[\frac{1}{(z+m\pi+n\pi\tau)^2} - \frac{1}{(m\pi+n\pi\tau)^2}\right],
\end{aligned} \tag{D.1}$$

satisfying the doubly periodic condition,

$$\wp(z;\tau) = \wp(z+\pi;\tau) = \wp(z+\pi\tau;\tau). \tag{D.2}$$

The function $\wp(z;\tau)$ satisfies the differential equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3 = 4(\wp-e_1)(\wp-e_2)(\wp-e_3), \tag{D.3}$$

where

$$g_2 = \frac{4}{3}E_4(\tau), \quad g_3 = \frac{8}{27}E_6(\tau), \tag{D.4}$$

and the roots e_1, e_2, e_3 can be expressed in terms of Jacobi theta functions as

$$\begin{aligned}
e_1 &= \frac{1}{3}\left(\theta_3^4+\theta_4^4\right),\\
e_2 &= -\frac{1}{3}\left(\theta_2^4+\theta_3^4\right),\\
e_3 &= \frac{1}{3}\left(\theta_2^4-\theta_4^4\right).
\end{aligned} \tag{D.5}$$

The modular discriminant Δ is defined as

$$\Delta = g_2^3 - 27g_3^2 = (2\pi)^{12}\eta^{24}(\tau). \tag{D.6}$$

When $\Delta > 0$, all three are real and it is conventional to choose $e_1 > e_2 > e_3$.

The function $\wp(z;\tau)$ is related to the Jacobi theta function by

$$\wp(z;\tau) = -\frac{d^2}{dz^2}\log\theta_1(z;\tau) - \frac{1}{3}E_2, \tag{D.7}$$

where the constant term is fixed by comparing the Laurent expansion of $\wp(z;\tau)$ at $z=0$,

$$\wp(z;\tau) = \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}g_3 z^4 + \mathcal{O}\left(z^6\right), \tag{D.8}$$

and the Laurent expansion of $\log\theta_1(z;\tau)$.

We are interested in calculating the integrals

$$\mathcal{P}_n = \frac{1}{\pi}\oint_A \wp^n dz, \tag{D.9}$$

where we define the A-cycle to be $0 \le z \le \pi$. By definition, we have

$$\mathcal{P}_0 = \frac{1}{\pi}\oint_A dz = 1, \tag{D.10}$$

$$\mathcal{P}_1 = \frac{1}{\pi}\oint_A \wp dz = \frac{1}{\pi}\oint_A \left(-\frac{d^2}{dz^2}\log\theta_1(z;\tau) - \frac{1}{3}E_2\right)dz = -\frac{1}{3}E_2. \tag{D.11}$$

For $n=2$, we can obtain from the derivative of (D.3) that

$$2\wp'' = 12\wp^2 - g_2, \tag{D.12}$$

from which we get

$$\mathcal{P}_2 = \frac{1}{\pi}\oint_A \wp^2 dz = \frac{1}{\pi}\oint_A \left(\frac{1}{6}\wp'' + \frac{1}{12}g_2\right)dz = \frac{1}{12}g_2. \tag{D.13}$$

The period integrals $\mathcal{P}_n$ for $n \ge 3$ can be derived recursively [102]. In fact, using (D.12) we have

$$\begin{aligned}
\mathcal{P}_n &= \frac{1}{\pi}\oint_A \wp^n dz \\
&= \frac{1}{\pi}\oint_A \wp^{n-2}\left(\frac{1}{6}\wp'' + \frac{1}{12}g_2\right)dz \\
&= \frac{1}{6\pi}\oint_A \wp^{n-2}\wp'' dz + \frac{1}{12}g_2\mathcal{P}_{n-2}.
\end{aligned} \tag{D.14}$$

Integrating by parts the first term and substituting (D.3) gives

$$\begin{aligned}
\oint_A \wp^{n-2}\wp'' dz &= -(n-2)\oint_A \wp^{n-3}\left(4\wp^3 - g_2\wp - g_3\right)dz \\
&= -(n-2)\oint_A \left(4\wp^n - g_2\wp^{n-2} - g_3\wp^{n-3}\right)dz.
\end{aligned} \tag{D.15}$$

Therefore, we obtain the following recurrence relation

$$\mathcal{P}_n = \frac{2n-3}{8n-4}g_2\mathcal{P}_{n-2} + \frac{n-2}{4n-2}g_3\mathcal{P}_{n-3}, \quad n \ge 3. \tag{D.16}$$

Here we list the first few explicit expressions for $\mathcal{P}_n$, $n \ge 2$, as polynomials in E_2, E_4 and E_6,

$$\begin{aligned}
\mathcal{P}_2 &= \frac{1}{9}E_4, \\
\mathcal{P}_3 &= -\frac{1}{15}E_2E_4 + \frac{4}{135}E_6, \\
\mathcal{P}_4 &= \frac{5}{189}E_4^2 - \frac{8}{567}E_2E_6, \\
\mathcal{P}_5 &= -\frac{7}{405}E_2E_4^2 + \frac{16}{1215}E_4E_6.
\end{aligned} \tag{D.17}$$

References

[1] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, Nucl. Phys. **B 426** (1994) 19 [*Erratum ibid.* **B 430** (1994) 485] [hep-th/9407087] [INSPIRE].

[2] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, Nucl. Phys. **B 431** (1994) 484 [hep-th/9408099] [INSPIRE].

[3] E. Witten, *Topological quantum field theory*, Commun. Math. Phys. **117** (1988) 353 [INSPIRE].

[4] S.K. Donaldson and P.B. Kronheimer, *The geometry of four-manifolds*, Oxford mathematical monographs, Oxford University Press, Oxford, U.K. (1990).

[5] E. Witten, *Monopoles and four manifolds*, Math. Res. Lett. **1** (1994) 769 [hep-th/9411102] [INSPIRE].

[6] G.W. Moore and E. Witten, *Integration over the u-plane in Donaldson theory*, Adv. Theor. Math. Phys. **1** (1997) 298 [hep-th/9709193] [INSPIRE].

[7] A. Losev, N. Nekrasov and S.L. Shatashvili, *Issues in topological gauge theory*, Nucl. Phys. **B 534** (1998) 549 [hep-th/9711108] [INSPIRE].

[8] A. Lossev, N. Nekrasov and S.L. Shatashvili, *Testing Seiberg-Witten solution*, *NATO Sci. Ser. C* **520** (1999) 359 [hep-th/9801061] [INSPIRE].

[9] M. Mariño and G.W. Moore, *The Donaldson-Witten function for gauge groups of rank larger than one*, Commun. Math. Phys. **199** (1998) 25 [hep-th/9802185] [INSPIRE].

[10] K. Takasaki, *Integrable hierarchies and contact terms in u-plane integrals of topologically twisted supersymmetric gauge theories*, Int. J. Mod. Phys. **A 14** (1999) 1001 [hep-th/9803217] [INSPIRE].

[11] M. Mariño and G.W. Moore, *Donaldson invariants for nonsimply connected manifolds*, Commun. Math. Phys. **203** (1999) 249 [hep-th/9804104] [INSPIRE].

[12] J.M.F. Labastida and C. Lozano, *Duality in twisted $N = 4$ supersymmetric gauge theories in four-dimensions*, Nucl. Phys. **B 537** (1999) 203 [hep-th/9806032] [INSPIRE].

[13] M. Mariño and G.W. Moore, *Three manifold topology and the Donaldson-Witten partition function*, Nucl. Phys. **B 547** (1999) 569 [hep-th/9811214] [INSPIRE].

[14] M. Mariño, G.W. Moore and G. Peradze, *Superconformal invariance and the geography of four manifolds*, Commun. Math. Phys. **205** (1999) 691 [hep-th/9812055] [INSPIRE].

[15] M.-C. Tan, *Integration over the u-plane in Donaldson theory with surface operators*, JHEP **05** (2011) 007 [arXiv:0912.4261] [INSPIRE].

[16] G. Korpas and J. Manschot, *Donaldson-Witten theory and indefinite theta functions*, JHEP **11** (2017) 083 [arXiv:1707.06235] [INSPIRE].

[17] G.W. Moore and I. Nidaiev, *The partition function of Argyres-Douglas theory on a four-manifold*, arXiv:1711.09257 [INSPIRE].

[18] G. Korpas, *Donaldson-Witten theory, surface operators and mock modular forms*, arXiv:1810.07057 [INSPIRE].

[19] G. Korpas, J. Manschot, G. Moore and I. Nidaiev, *Renormalization and BRST symmetry in Donaldson-Witten theory*, Annales Henri Poincaré **20** (2019) 3229 [arXiv:1901.03540] [INSPIRE].

[20] G. Korpas, J. Manschot, G.W. Moore and I. Nidaiev, *Mocking the u-plane integral*, arXiv:1910.13410 [INSPIRE].

[21] A.D. Shapere and Y. Tachikawa, *Central charges of $N = 2$ superconformal field theories in four dimensions*, JHEP **09** (2008) 109 [arXiv:0804.1957] [INSPIRE].

[22] D. Anselmi, D.Z. Freedman, M.T. Grisaru and A.A. Johansen, *Nonperturbative formulas for central functions of supersymmetric gauge theories*, Nucl. Phys. **B 526** (1998) 543 [hep-th/9708042] [INSPIRE].

[23] D. Anselmi, J. Erlich, D.Z. Freedman and A.A. Johansen, *Positivity constraints on anomalies in supersymmetric gauge theories*, Phys. Rev. **D 57** (1998) 7570 [hep-th/9711035] [INSPIRE].

[24] S.M. Kuzenko and S. Theisen, *Correlation functions of conserved currents in $N = 2$ superconformal theory*, Class. Quant. Grav. **17** (2000) 665 [hep-th/9907107] [INSPIRE].

[25] E. Witten, *Some comments on string dynamics*, in *Future perspectives in string theory. Proceedings, Conference, Strings '95*, Los Angeles, CA, U.S.A., 13–18 March 1995, pg. 501 [hep-th/9507121] [INSPIRE].

[26] N. Seiberg and E. Witten, *Comments on string dynamics in six-dimensions*, Nucl. Phys. **B 471** (1996) 121 [hep-th/9603003] [INSPIRE].

[27] N. Seiberg, *Nontrivial fixed points of the renormalization group in six-dimensions*, Phys. Lett. **B 390** (1997) 169 [hep-th/9609161] [INSPIRE].

[28] A. Strominger, *Open p-branes*, Phys. Lett. **B 383** (1996) 44 [hep-th/9512059] [INSPIRE].

[29] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N.P. Warner, *Selfdual strings and $N = 2$ supersymmetric field theory*, Nucl. Phys. **B 477** (1996) 746 [hep-th/9604034] [INSPIRE].

[30] E. Witten, *Solutions of four-dimensional field theories via M-theory*, Nucl. Phys. **B 500** (1997) 3 [hep-th/9703166] [INSPIRE].

[31] D. Gaiotto, *$N = 2$ dualities*, JHEP **08** (2012) 034 [arXiv:0904.2715] [INSPIRE].

[32] D. Gaiotto, G.W. Moore and A. Neitzke, *Wall-crossing, Hitchin systems and the WKB approximation*, arXiv:0907.3987 [INSPIRE].

[33] M. Henneaux and C. Teitelboim, *Dynamics of chiral (selfdual) p-forms*, Phys. Lett. **B 206** (1988) 650 [INSPIRE].

[34] M. Perry and J.H. Schwarz, *Interacting chiral gauge fields in six-dimensions and Born-Infeld theory*, Nucl. Phys. **B 489** (1997) 47 [hep-th/9611065] [INSPIRE].

[35] P. Pasti, D.P. Sorokin and M. Tonin, *On Lorentz invariant actions for chiral p-forms*, Phys. Rev. **D 55** (1997) 6292 [hep-th/9611100] [INSPIRE].

[36] P. Pasti, D.P. Sorokin and M. Tonin, *Covariant action for a $D = 11$ five-brane with the chiral field*, Phys. Lett. **B 398** (1997) 41 [hep-th/9701037] [INSPIRE].

[37] I.A. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D.P. Sorokin and M. Tonin, *Covariant action for the superfive-brane of M-theory*, *Phys. Rev. Lett.* **78** (1997) 4332 [hep-th/9701149] [INSPIRE].

[38] M. Aganagic, J. Park, C. Popescu and J.H. Schwarz, *World volume action of the M-theory five-brane*, *Nucl. Phys.* **B 496** (1997) 191 [hep-th/9701166] [INSPIRE].

[39] I.A. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D.P. Sorokin and M. Tonin, *On the equivalence of different formulations of the M-theory five-brane*, *Phys. Lett.* **B 408** (1997) 135 [hep-th/9703127] [INSPIRE].

[40] D. Belov and G.W. Moore, *Holographic action for the self-dual field*, hep-th/0605038 [INSPIRE].

[41] E. Witten, *Mirror manifolds and topological field theory*, hep-th/9112056 [INSPIRE].

[42] E. Witten, *Phases of $N = 2$ theories in two-dimensions*, *Nucl. Phys.* **B 403** (1993) 159 [hep-th/9301042] [INSPIRE].

[43] G.W. Moore and I. Nidaiev, unpublished.

[44] A. Gadde, S. Gukov and P. Putrov, *Fivebranes and 4-manifolds*, *Prog. Math.* **319** (2016) 155 [arXiv:1306.4320] [INSPIRE].

[45] M. Dedushenko, S. Gukov and P. Putrov, *Vertex algebras and 4-manifold invariants*, in *Proceedings, Nigel Hitchin's* 70^{th} *birthday conference. Geometry and physics: a festschrift in honour of Nigel Hitchin*, volume 1, Oxford University Press, Oxford, U.K. (2018), pg. 249 [arXiv:1705.01645] [INSPIRE].

[46] I. Nidaiev, *Cohomological field theories and four-manifold invariants*, Ph.D. thesis, Rutgers University, Piscataway, NJ, U.S.A. (2019).

[47] E. Witten, *On S duality in Abelian gauge theory*, *Selecta Math.* **1** (1995) 383 [hep-th/9505186] [INSPIRE].

[48] E.J. Martinec and N.P. Warner, *Integrable systems and supersymmetric gauge theory*, *Nucl. Phys.* **B 459** (1996) 97 [hep-th/9509161] [INSPIRE].

[49] J. Schulze and N.P. Warner, *BPS geodesics in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 498** (1997) 101 [hep-th/9702012] [INSPIRE].

[50] E. Caceres and P. Pasanen, *M theory five-brane wrapped on curves for exceptional groups*, *Nucl. Phys.* **B 543** (1999) 572 [hep-th/9806224] [INSPIRE].

[51] C. Vafa and E. Witten, *A strong coupling test of S duality*, *Nucl. Phys.* **B 431** (1994) 3 [hep-th/9408074] [INSPIRE].

[52] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2003) 831 [hep-th/0206161] [INSPIRE].

[53] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, *Prog. Math.* **244** (2006) 525 [hep-th/0306238] [INSPIRE].

[54] S. Shadchin, *Saddle point equations in Seiberg-Witten theory*, *JHEP* **10** (2004) 033 [hep-th/0408066] [INSPIRE].

[55] S. Shadchin, *Cubic curves from instanton counting*, *JHEP* **03** (2006) 046 [hep-th/0511132] [INSPIRE].

[56] N. Nekrasov and V. Pestun, *Seiberg-Witten geometry of four dimensional $N = 2$ quiver gauge theories*, arXiv:1211.2240 [INSPIRE].

[57] X. Zhang, *Seiberg-Witten geometry of four-dimensional $N = 2$ SO-USp quiver gauge theories*, *Phys. Rev.* **D 100** (2019) 125015 [arXiv:1910.10104] [INSPIRE].

[58] H. Nakajima and K. Yoshioka, *Lectures on instanton counting*, in *CRM workshop on algebraic structures and moduli spaces*, Montreal, Canada, 14–20 July 2003 [math.AG/0311058] [INSPIRE].

[59] N.A. Nekrasov and S.L. Shatashvili, *Quantization of integrable systems and four dimensional gauge theories*, in *Proceedings*, 16^{th} *International Congress on Mathematical Physics (ICMP09)*, Prague, Czech Republic, 3–8 August 2009, World Scientific, Singapore (2009), pg. 265 [arXiv:0908.4052] [INSPIRE].

[60] M. Billó, M. Frau, L. Gallot, A. Lerda and I. Pesando, *Deformed $N = 2$ theories, generalized recursion relations and S-duality*, *JHEP* **04** (2013) 039 [arXiv:1302.0686] [INSPIRE].

[61] M. Billó, M. Frau, L. Gallot, A. Lerda and I. Pesando, *Modular anomaly equation, heat kernel and S-duality in $N = 2$ theories*, *JHEP* **11** (2013) 123 [arXiv:1307.6648] [INSPIRE].

[62] L. Hollands, C.A. Keller and J. Song, *From SO/Sp instantons to W-algebra blocks*, *JHEP* **03** (2011) 053 [arXiv:1012.4468] [INSPIRE].

[63] L. Hollands, C.A. Keller and J. Song, *Towards a 4d/2d correspondence for Sicilian quivers*, *JHEP* **10** (2011) 100 [arXiv:1107.0973] [INSPIRE].

[64] G.W. Moore, N. Nekrasov and S. Shatashvili, *Integrating over Higgs branches*, *Commun. Math. Phys.* **209** (2000) 97 [hep-th/9712241] [INSPIRE].

[65] N. Nekrasov and E. Witten, *The omega deformation, branes, integrability and Liouville theory*, *JHEP* **09** (2010) 092 [arXiv:1002.0888] [INSPIRE].

[66] N. Nekrasov, *BPS/CFT correspondence: non-perturbative Dyson-Schwinger equations and qq-characters*, *JHEP* **03** (2016) 181 [arXiv:1512.05388] [INSPIRE].

[67] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, American Mathematical Society, U.S.A. (1999).

[68] H. Nakajima and K. Yoshioka, *Instanton counting on blowup.* 1. *4-dimensional pure gauge theory*, *Invent. Math.* **162** (2005) 313 [math.AG/0306198].

[69] L. Gottsche, H. Nakajima and K. Yoshioka, *Donaldson = Seiberg-Witten from Mochizuki's formula and instanton counting*, *Publ. Res. Inst. Math. Sci. Kyoto* **47** (2011) 307 [arXiv:1001.5024] [INSPIRE].

[70] T. Okuda and V. Pestun, *On the instantons and the hypermultiplet mass of $N = 2^*$ super Yang-Mills on S^4*, *JHEP* **03** (2012) 017 [arXiv:1004.1222] [INSPIRE].

[71] M.-X. Huang, A.-K. Kashani-Poor and A. Klemm, *The Ω deformed B-model for rigid $N = 2$ theories*, *Annales Henri Poincaré* **14** (2013) 425 [arXiv:1109.5728] [INSPIRE].

[72] L.F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, *Lett. Math. Phys.* **91** (2010) 167 [arXiv:0906.3219] [INSPIRE].

[73] S. Jeong and X. Zhang, *BPZ equations for higher degenerate fields and non-perturbative Dyson-Schwinger equations*, arXiv:1710.06970 [INSPIRE].

[74] N. Nekrasov, *BPS/CFT correspondence V: BPZ and KZ equations from qq-characters*, arXiv:1711.11582 [INSPIRE].

[75] M. Mariño and N. Wyllard, *A note on instanton counting for $N = 2$ gauge theories with classical gauge groups*, JHEP **05** (2004) 021 [hep-th/0404125] [INSPIRE].

[76] N. Nekrasov and S. Shadchin, *ABCD of instantons*, Commun. Math. Phys. **252** (2004) 359 [hep-th/0404225] [INSPIRE].

[77] F. Fucito, J.F. Morales and R. Poghossian, *Instantons on quivers and orientifolds*, JHEP **10** (2004) 037 [hep-th/0408090] [INSPIRE].

[78] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Yu. I. Manin, *Construction of instantons*, Phys. Lett. **A 65** (1978) 185 [INSPIRE].

[79] M. Matone, *Instantons and recursion relations in $N = 2$ SUSY gauge theory*, Phys. Lett. **B 357** (1995) 342 [hep-th/9506102] [INSPIRE].

[80] R. Flume, F. Fucito, J.F. Morales and R. Poghossian, *Matone's relation in the presence of gravitational couplings*, JHEP **04** (2004) 008 [hep-th/0403057] [INSPIRE].

[81] J.A. Minahan, D. Nemeschansky and N.P. Warner, *Instanton expansions for mass deformed $N = 4$ super Yang-Mills theories*, Nucl. Phys. **B 528** (1998) 109 [hep-th/9710146] [INSPIRE].

[82] E. D'Hoker and D.H. Phong, *Calogero-Moser systems in $\mathrm{SU}(N)$ Seiberg-Witten theory*, Nucl. Phys. **B 513** (1998) 405 [hep-th/9709053] [INSPIRE].

[83] E. D'Hoker and D.H. Phong, *Spectral curves for super Yang-Mills with adjoint hypermultiplet for general Lie algebras*, Nucl. Phys. **B 534** (1998) 697 [hep-th/9804126] [INSPIRE].

[84] M. Bershtein, G. Bonelli, M. Ronzani and A. Tanzini, *Exact results for $N = 2$ supersymmetric gauge theories on compact toric manifolds and equivariant Donaldson invariants*, JHEP **07** (2016) 023 [arXiv:1509.00267] [INSPIRE].

[85] M. Bershtein, G. Bonelli, M. Ronzani and A. Tanzini, *Gauge theories on compact toric surfaces, conformal field theories and equivariant Donaldson invariants*, J. Geom. Phys. **118** (2017) 40 [arXiv:1606.07148] [INSPIRE].

[86] T.W. Grimm, A. Klemm, M. Mariño and M. Weiss, *Direct integration of the topological string*, JHEP **08** (2007) 058 [hep-th/0702187] [INSPIRE].

[87] N. Dorey, V.V. Khoze and M.P. Mattis, *On $N = 2$ supersymmetric QCD with four flavors*, Nucl. Phys. **B 492** (1997) 607 [hep-th/9611016] [INSPIRE].

[88] P.C. Argyres and S. Pelland, *Comparing instanton contributions with exact results in $N = 2$ supersymmetric scale invariant theories*, JHEP **03** (2000) 014 [hep-th/9911255] [INSPIRE].

[89] S. Jeong and X. Zhang, *A note on chiral trace relations from qq-characters*, JHEP **04** (2020) 026 [arXiv:1910.10864] [INSPIRE].

[90] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, *Simple singularities and $N = 2$ supersymmetric Yang-Mills theory*, Phys. Lett. **B 344** (1995) 169 [hep-th/9411048] [INSPIRE].

[91] P.C. Argyres and A.E. Faraggi, *The vacuum structure and spectrum of $N = 2$ supersymmetric $\mathrm{SU}(N)$ gauge theory*, Phys. Rev. Lett. **74** (1995) 3931 [hep-th/9411057] [INSPIRE].

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[92] B. Eynard, *Topological expansion for the 1-Hermitian matrix model correlation functions*, *JHEP* **11** (2004) 031 [hep-th/0407261] [INSPIRE].

[93] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion*, *Commun. Num. Theor. Phys.* **1** (2007) 347 [math-ph/0702045] [INSPIRE].

[94] N. Nekrasov, *BPS/CFT correspondence II: instantons at crossroads, moduli and compactness theorem*, *Adv. Theor. Math. Phys.* **21** (2017) 503 [arXiv:1608.07272] [INSPIRE].

[95] N. Nekrasov, *BPS/CFT correspondence III: gauge origami partition function and qq-characters*, *Commun. Math. Phys.* **358** (2018) 863 [arXiv:1701.00189] [INSPIRE].

[96] N. Nekrasov, *BPS/CFT correspondence IV: σ-models and defects in gauge theory*, *Lett. Math. Phys.* **109** (2019) 579 [arXiv:1711.11011] [INSPIRE].

[97] N. Hama and K. Hosomichi, *Seiberg-Witten theories on ellipsoids*, *JHEP* **09** (2012) 033 [arXiv:1206.6359] [INSPIRE].

[98] D. Rodriguez-Gomez and J. Schmude, *Partition functions for equivariantly twisted $N = 2$ gauge theories on toric Kähler manifolds*, *JHEP* **05** (2015) 111 [arXiv:1412.4407] [INSPIRE].

[99] G. Festuccia, J. Qiu, J. Winding and M. Zabzine, *Twisting with a flip (the art of pestunization)*, *Commun. Math. Phys.* **377** (2020) 341 [arXiv:1812.06473] [INSPIRE].

[100] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, *Commun. Math. Phys.* **313** (2012) 71 [arXiv:0712.2824] [INSPIRE].

[101] H. Bateman and A. Erdélyi, *Higher transcendental functions*, Calif. Inst. Technol. Bateman Manuscr. Project, McGraw-Hill, New York, NY, U.S.A. (1955).

[102] M.P. Grosset and A.P. Veselov, *Elliptic Faulhaber polynomials and Lamé densities of states*, math-ph/0508066.

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