$N = 3$ SCFTs in 4 dimensions and non-simply laced groups

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ABSTRACT: In this paper we discuss various $N = 3$ SCFTs in 4 dimensions and in particular those which can be obtained as a discrete gauging of an $N = 4$ SYM theories with non-simply laced groups. The main goal of the project was to compute the Coulomb branch superconformal index and moduli space Hilbert series for the $N = 3$ SCFTs that are obtained from gauging a discrete subgroup of the global symmetry group of $N = 4$ Super Yang-Mills theory. The discrete subgroup contains elements of both $SU(4)$ R-symmetry group and the S-duality group of $N = 4$ SYM. This computation was done for the simply laced groups (where the S-duality groups is $SL(2, \mathbb{Z})$ and Langlands dual of the algebra ${}^L\mathfrak{g}$ is simply $\mathfrak{g}$) by Bourton et al. [1], and we extended it to the non-simply laced groups. We also considered the orbifolding groups of the Coulomb branch for the cases when Coulomb branch is relatively simple; in particular, we compared them with the results of Argyres et al. [2], who classified all $N \geq 3$ moduli space orbifold geometries at rank 2 and with the results of Bonetti et al. [3], who listed all possible orbifolding groups for the freely generated Coulomb branches of $N \geq 3$ SCFTs. Finally, we have considered sporadic complex crystallographic reflection groups with rank greater than 2 and analyzed, which of them can correspond to an $N = 3$ SCFT with a principal Dirac pairing.

KEYWORDS: Extended Supersymmetry, Conformal Field Theory, Supersymmetric Gauge Theory

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1 Introduction

In the last few years there have been many advances in the studies of superconformal field theories with extended supersymmetry in 4 dimensions. A particularly fruitful area of research was research of $N = 3$ SCFTs; for example, in [4] it was shown, that the relation between dimensions of Coulomb branch operators and $2a - c$ [5] is only true if theory doesn't possess a discrete gauge group, and in [6] it was shown, that, contrary to a long-standing belief, Coulomb branch doesn't have to be freely generated or even to be a complete intersection manifold and, in fact, may even be not a complete intersection manifold (see also [7] for the further discussion). Moreover, in [1] Bourton et al. have shown that a “generic” $N = 3$ SCFT Coulomb branch is not a complete intersection manifold.

Large progress has also been made in classifying superconformal field theories. In the series of papers [2, 8–11] by Argyres et al. authors have studied the Coulomb branches of $N \geq 2$ SCFTs and have employed various methods to probe the relations in the holomorphic polynomial ring and the orbifolding structure of the Coulomb branch manifold. In particular, in [8] Argyres and Martone have suggested a method to refine the Coulomb limit of the superconformal index, that simplifies tracking the relations in the coordinate ring. In [2] they have classified Coulomb branches for rank-2 SCFTs with $N > 2$ supersymmetry. It is also possible to study Coulomb branch manifold in a “bottom-up” approach; in [3] Bonetti et al. have considered a class of $N = 2$ vertex operator algebras W_G labeled by crystallographic complex reflection groups, that are extensions of the $N = 2$ super Virasoro algebra obtained by introducing additional generators; they have also found a way to recover the Macdonald limit of the superconformal index of the parent $4d$ theory from the corresponding vertex operator algebra, when such a theory exists. Their construction

is also interesting because for every rank- r $N > 2$ SCFT with a freely generated Coulomb branch its complex structure can be written as $\mathbb{C}^r/\Gamma$, where Γ is a crystallographic complex reflection group acting irreducibly on $\mathbb{C}^r$. However, not every crystallographic complex reflection group Γ will correspond to an $N = 3$ SCFT with a principal Dirac pairing. For example, none of the rank-2 crystallographic complex reflection groups G_4, G_5, G_8 ¹ can be an orbifolding group for $N > 2$ SCFT with a principal Dirac pairing [2, 12].

Another way to analyze the landscape of superconformal field theories is by constructing them and then computing their index. A particular example of such study can be found in a paper by Bourton et al. [1], where authors have computed Coulomb limit of the superconformal index and Higgs branch Hilbert series for various $N = 3$ and $N = 4$ SCFTs and analyzed the Coulomb branches of these theories. To construct new theories, they have considered $N = 4$ SCFTs with simply laced gauge groups and noticed, that these theories have an enhanced discrete global symmetry at certain values of the gauge coupling. They then refined obtained superconformal index by a fugacity for the enhanced discrete symmetry. The index of the discretely gauged daughter theory is then obtained by “integrating” over the additional fugacity, which takes values in the discrete group. The enhanced discrete symmetry is constructed from a subgroup of S-duality group,² so authors restricted their studies to the case of simply laced gauge groups, leaving the non-simply laced groups for the further studies.

In this paper we continued studying the landscape of $N > 2$ SCFTs in 4 dimensions, found more new $N = 3$ theories in the spirit of [1] and bridged some of the gaps between the research of Argyres et al. [2, 8], Bonetti et al. [3], and Bourton et al. [1]. In order to do that, we extended the results of [1] to the non-simply laced groups. We also analyzed the geometry of the moduli space of the theories we obtained; for the cases when the Coulomb branch is freely generated the orbifolding group is a complex reflection group in agreement with [3]. The Coulomb branches of the rank-2 theories we found are in agreement with the results of [2]. Finally, we considered sporadic crystallographic complex reflection groups³ and, using the methods of [12], checked, which of them can correspond to $N = 3$ SCFTs with principal Dirac pairing; we compute moduli space Hilbert series for $N = 3$ SCFTs that can originate from these groups.

2 $N = 3$ SCFTs from gauging $N = 4$ SCFTs with non-simply laced gauge groups

2.1 Coulomb branch index computation

The first part of the computation is to compute Coulomb branch limit index. It is more instructing to do these computations along the lines of [8], since the results are similar to

¹Notation for the complex crystallographic reflection groups is in agreement with Shephard and Todd [13].

²It is important to notice that analysis of Bourton et al. doesn’t account for the line operators, so the discrete global symmetry isn’t always present and not every $N = 3$ theory they list actually exists [14], see appendix A for more details.

³The non-sporadic groups were considered in [12].

the computations done as in [1], but the method from [8] gives more refined version of the index that simplifies analysis of the Coulomb branch manifold. The computation method is as follows:

1. We start with an $N = 4$ SYM theory with a non-simply laced gauge group (B_n, C_n, F_4 or G_2); the simply laced cases have been discussed in [1]. The non-simply laced case is more complicated than the simply laced one, so writing the S-duality group and finding the discrete symmetry group is slightly trickier. For the simply laced groups the S-duality group is simply $\mathrm{SL}(2, \mathbb{Z})$, while for the non-simply laced groups it is rather $\Gamma_0(q) \subset \mathrm{SL}(2, \mathbb{Z})$ ⁴ where q is the square of length ratio of the short roots to the long ones; $q = 2$ for B_n, C_n, F_4 and $q = 3$ for G_2 . $\mathrm{SL}(2, \mathbb{Z})$ is generated by the three elements $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $C = -1$, while $\Gamma_0(q)$ is generated by C , T , $A = ST^q S$. $\mathrm{SL}(2, \mathbb{Z})$ generators obey the relations $C^2 = 1$, $S^2 = (ST)^3 = C$, while for $\Gamma_0(q)$ $(AT)^q = C$. It is important to notice that for almost every algebra (with the exception of A_n) C is a part of Weyl group and so acts on the theory trivially; only the quotient of the S-duality group by its center acts faithfully on the theory. For B_n, C_n the only global discrete symmetry subgroup of S-duality group (apart from the trivial transformation generated by C) is generated by AT . As B_n, C_n theories have the same Coulomb and Higgs branches, superconformal indices we consider in this paper cannot distinguish between these theories, so we will restrict ourselves to considering B_n theories from now on.
2. For G_2 and F_4 the S-duality group is extended to the Hecke group $H(2q)$ (see [15] for more details) generated by $C, T, \tilde{S}$, where $\tilde{S}$ is such that $\tilde{S}T\tilde{S} = ST^q S$. $C, T, \tilde{S}$ obey the relations $C^2 = 1$, $\tilde{S}^2 = (\tilde{S}T)^{2q} = C$. This extension happens because for G_2 and F_4 $\tilde{S}$ takes us back to the same group, while for B_n it moves us to C_n . $\tilde{S}$ is not a part of the Weyl group, so it acts on the moduli space non-trivially; in [15] it was shown that for G_2 the Coulomb branch operators transform as $(U_2, U_6) \xrightarrow{\tilde{S}} (U_2, -U_6)$, and for F_4 the rule is $(U_2, U_6, U_8, U_{12}) \xrightarrow{\tilde{S}} (U_2, -U_6, U_8, -U_{12})$.
3. Now we can find, for which values of the coupling τ the discrete subgroups of the S-duality group for the various theories we have considered above will leave τ unchanged. As C belongs to the Weyl group and is therefore a trivial operation, only the quotient of the S-duality group subgroup by its center acts faithfully on the $N = 4$ theory, and from now on we will consider only the quotients. For G_2 we have three different options [15]: we can consider either $\tau = \frac{i}{\sqrt{3}}$, which is fixed under $\mathbb{Z}_2$ generated by $\tilde{S}$, or $\tau = -\frac{1}{2} \pm \frac{i}{2\sqrt{3}}$,⁵ for which we have two options for the symmetry group: $\mathbb{Z}_6$, generated by $(\tilde{S}T)$, and $\mathbb{Z}_3 \subset \mathbb{Z}_6$ discrete symmetry with $\mathbb{Z}_3$ generated by $(\tilde{S}T)^2$. For F_4 we also have three different options: we have enhanced $\mathbb{Z}_2$ symmetry at $\tau = \frac{i}{\sqrt{2}}$ (generated by $\tilde{S}$), $\mathbb{Z}_4$ symmetry at $\tau = -\frac{1}{2} \pm \frac{i}{2}$ (generated by $(\tilde{S}T)$), and a $\mathbb{Z}_2 \subset \mathbb{Z}_4$ discrete symmetry with the generator $(\tilde{S}T)^2$, it is realized differently

⁴With a caveat for G_2, F_4 we mention in the next item.

⁵Apparently there is a typo in [15].

comparing to the $\mathbb{Z}_2$ generated by $\tilde{S}$. For B_n groups we only have $\mathbb{Z}_2$ at $\tau = -\frac{1}{2} \pm \frac{i}{2}$, generated by (AT) .

4. The S-duality transformations transform the chiral supercharges by a phase [16]. Namely, if an element of the S-duality group transforms the SYM coupling as

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad \tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad (2.1)$$

then the chiral supercharges transform as

$$Q_\alpha^i \rightarrow e^{i\chi} Q_\alpha^i; \quad e^{i\chi} = \left(\frac{|c\tau + d|}{c\tau + d} \right)^{1/2}. \quad (2.2)$$

In particular, the transformations we have found above transform the chiral supercharges as following:

$\mathfrak{g} \backslash \sigma$	$\tilde{S}$	$\tilde{S}T$	$(\tilde{S}T)^2 = AT$
G_2	$Q_\alpha^i \rightarrow e^{-i\pi/4} Q_\alpha^i$	$Q_\alpha^i \rightarrow e^{-i\pi/12} Q_\alpha^i$	$Q_\alpha^i \rightarrow e^{-i\pi/6} Q_\alpha^i$
F_4	$Q_\alpha^i \rightarrow e^{-i\pi/4} Q_\alpha^i$	$Q_\alpha^i \rightarrow e^{-i\pi/8} Q_\alpha^i$	$Q_\alpha^i \rightarrow e^{-i\pi/4} Q_\alpha^i$
B_n	—	—	$Q_\alpha^i \rightarrow e^{-i\pi/4} Q_\alpha^i$

Now we need to offset the action of the S-duality transformation; to do that, we will use elements of the R-symmetry group $SU(4)_R = SO(6)_R$. We can organize six real adjoint scalar fields $\phi^I, I \in \mathbf{6}$ of $SU(4)_R$ into a triplet of complex scalars $\varphi^a, a \in \mathbf{3}$ of $U(3)$; $\varphi^a = \phi^{2a-1} + i\phi^{2a}$. The R-symmetry group element ρ can be represented by a simultaneous rotation in three orthogonal planes in $\mathbb{R}^6 \simeq \mathbb{C}^3$:

$$\rho = \begin{pmatrix} e^{i\psi_1} & & \\ & e^{i\psi_2} & \\ & & e^{i\psi_3} \end{pmatrix} \in U(3) \subset SU(4)_R. \quad (2.3)$$

Then ρ rotates the complex scalars by a phase $\varphi_a \xrightarrow{\rho} e^{i\psi_a} \varphi_a$, and the four chiral supercharges transform as

$$Q_\alpha^1 \xrightarrow{\rho} e^{i(\psi_1+\psi_2+\psi_3)/2} Q_\alpha^1 \quad (2.4)$$

$$Q_\alpha^2 \xrightarrow{\rho} e^{i(\psi_1-\psi_2-\psi_3)/2} Q_\alpha^2 \quad (2.5)$$

$$Q_\alpha^3 \xrightarrow{\rho} e^{i(-\psi_1+\psi_2-\psi_3)/2} Q_\alpha^3 \quad (2.6)$$

$$Q_\alpha^4 \xrightarrow{\rho} e^{i(-\psi_1-\psi_2+\psi_3)/2} Q_\alpha^4 \quad (2.7)$$

Now we can choose ψ_a in such a fashion that the combination of ρ, σ will leave Q^1, Q^2, Q^3 invariant; this means that for a given σ , ρ should be equal to $\rho =$

$\text{diag}(e^{2i\pi/n}, e^{2i\pi/n}, e^{-2i\pi/n})$, where n is the value of the denominator in the exponent at the corresponding cell of the (2.1). All in all, resulting Coulomb branch will be described by $\mathbb{C}^r/(\Gamma \rtimes \Gamma_k)$, where Γ_k can be $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6$ for $G_2, \mathbb{Z}_2, \mathbb{Z}_2'$ (different from $\mathbb{Z}_2$), and $\mathbb{Z}_4$ for F_4 , and $\mathbb{Z}_2$ for B_n theories, with Γ denoting the orbifolding group of the original theory.

5. Using this knowledge, we can now compute the refined Molien series as in [8] (see equation (4.13) and derivation around it):

$$P_{J_{\Gamma_k}}(t_1, \dots, t_l) = \frac{1}{|\Gamma_k|} \sum_{g \in \Gamma_k} \frac{1}{\det(1 - g \text{diag}(t_1, \dots, t_l))}, \quad (2.8)$$

where t_i are coordinates that correspond to the Coulomb branch operators of the original theory. Applying the plethystic logarithm⁶ to $P_{J_{\Gamma_k}}$ one obtains the generators of the Coulomb branch of the resulting theory, as well as the relations between them in the form

$$\mathcal{F}_{\Gamma}(t) = \sum_k c_k^+ t^k - \sum_{k'} c_{k'}^- t^{k'}, \quad (2.9)$$

where the positive coefficients count the number of generators of degree k and the negative ones count the number of relations at degree k' . If the Coulomb branch turns out to be not a complete intersection manifold, then $\mathcal{F}_{\Gamma}(t)$ should be not a polynomial, but rather an infinite power series.⁷

6. We can now do a cross-check of the results we have obtained by considering the orbifolding group of the Coulomb branch directly. The orbifolding group Γ of the Coulomb branch in the original $N = 4$ theory is a Weyl group of a Lie algebra and, therefore, a crystallographic Coxeter group. If the Coulomb branch is freely generated, $\Gamma \rtimes \Gamma_k$ should be a crystallographic complex reflection group (see [3] table 1 for the table of the irreducible crystallographic complex reflection groups, divided into non-Coxeter and Coxeter groups; from it one can also read the dimensions of the Coulomb branch generators). Thus the cross-check is done by computing $\Gamma \rtimes \Gamma_k$ and checking, if the resulting group is a crystallographic complex reflection group; if it is, then the degrees of its fundamental invariants should match the dimensions of the Coulomb branch operators obtained from the Molien series computations results.

2.2 Moduli space Hilbert series computation

Next we can move on to the computation of moduli space Hilbert series. The algorithm for the computation is as follows:

⁶Plethystic logarithm PLog is $\text{PLog}(f(t)) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log f(t^n)$, where $\mu(n)$ is the Möbius function.

⁷This is not always true: there can be “unexpected” cancellations between factors in the numerator and denominator of the Molien series (2.8) [8]. This can happen when the degree of a relation happens to be the same as that of an affine parameter in the coordinate ring, or if the degree of a syzygy happens to coincide with that of a relation, etc. As the Coulomb branch rank increases, such accidental cancellations become more likely, but, at least for the low-rank examples, one might expect that the plethystic logarithm will accurately capture the degrees and counting of generators and relations.

1. The Higgs branch Hilbert series can be constructed in a fashion similar to the Coulomb branch index. We can use the fact that $N = 2$ Higgs branch and $N = 2$ Coulomb branch are parts of the moduli space that is defined as

$$\mathcal{M}_\Gamma = \mathbb{C}^{3r} / \rho_\tau(\Gamma) = \mathbb{C}^{3r} / (\mathbf{1}_3 \otimes \mu_\tau(\Gamma)), \quad (2.10)$$

where $\rho_\tau : \Gamma \rightarrow \mathrm{GL}(3r, \mathbb{C})$, $\mu_\tau : \Gamma \rightarrow \mathrm{GL}(r, \mathbb{C})$. The complex structure of $\mathcal{M}_\Gamma$ is determined by picking one left-handed supercharge in the $N = 3$ algebra and calling the complex scalars which are taken to left-handed Weyl spinors by the action of that supercharge the holomorphic coordinates on $\mathcal{M}_\Gamma$. The special coordinates on $\mathcal{M}_\Gamma$ are not holomorphic; from every $\mathrm{SU}(3)_R$ triplet two can be taken to be holomorphic and the third anti-holomorphic. Thus, for example,

$$(z_i^1, z_i^2, z_{3i}) \equiv (a_i^1, a_i^2, \bar{a}_{3i}); 1 \leq i \leq r, \quad (2.11)$$

can be taken as the holomorphic coordinates (see discussion near eq. (2.7)). When we choose a $N = 2$ subalgebra of $N = 3$, we choose a minimally embedded $\mathrm{SU}(2)_R \subset \mathrm{SU}(3)_R$. Then the subspace fixed by the $\mathrm{SU}(2)_R$ is the $N = 2$ Coulomb branch. If we now assume that $\mathcal{M}_\Gamma$ is an orbifold, and $\mu_\tau : \Gamma \rightarrow \mathrm{GL}(r, \mathbb{C})$ then it can be written as

$$\mathcal{M}_\Gamma \equiv \mathbb{C}^{3r} / \mu_\tau(\Gamma) \oplus \mu_\tau(\Gamma) \oplus \bar{\mu}_\tau(\Gamma), \quad (2.12)$$

with the Coulomb branch $\mathcal{C}_\Gamma \equiv \mathbb{C}^r / \mu_\tau(\Gamma)$ and Higgs branch

$$\mathcal{H}_\Gamma \equiv \mathbb{C}^{2r} / \mu_\tau(\Gamma) \oplus \bar{\mu}_\tau(\Gamma). \quad (2.13)$$

Therefore, we can construct the Higgs branch Hilbert series in a fashion similar to the Coulomb branch (see [8] for more details).

2. The usual Higgs branch Hilbert series (the unrefined version) has only one fugacity that tracks scaling dimensions of the operators. Since $\mathcal{M}_\Gamma$ carries a non-holomorphic $\mathrm{U}(3)_R$ isometry, we can refine the Hilbert series as

$$\begin{aligned} H_{\mathcal{M}_\Gamma}(t, v, u_1, u_2) &= \\ &= \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \frac{1}{\det(\mathbf{1} - tvu_1\mu_\tau(g))} \frac{1}{\det(\mathbf{1} - tv\frac{u_2}{u_1}\mu_\tau(g))} \frac{1}{\det(\mathbf{1} - t\frac{u_2}{v}\bar{\mu}_\tau(g))} \end{aligned} \quad (2.14)$$

The fact that the Hilbert series factorizes in three pieces is an immediate consequence of the fact that the group action on $\mathbb{C}^{3r}$ is chosen to be a direct sum of three factors $\rho = \mu_\tau \oplus \mu_\tau \oplus \bar{\mu}_\tau$, each of which acts independently on $\mathbb{C}^r$. The choice of fugacities is in agreement with the $\mathrm{U}(3)_R$ weights of the holomorphic coordinates ($[1; 0]_1$ for z_i^1 , $[-1; 1]_1$ for z_i^2 , $[0; 1]_{-1}$ for $\bar{z}_i^3$); u_1, u_2 fugacities powers are the $\mathrm{SU}(3)_R$ weights, t corresponds to the scaling dimension and v tracks the $\mathrm{U}(1)_R$ charge. We can now reduce (2.14) to obtain Molien formula for Higgs and Coulomb branches:

$$H_{\mathcal{C}_\Gamma}(t, v, u_1, u_2) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \frac{1}{\det(\mathbf{1} - tvu_1\mu_\tau(g))} \quad (2.15)$$

$$H_{\mathcal{H}_\Gamma}(t, v, u_1, u_2) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \frac{1}{\det(\mathbf{1} - tv\frac{u_2}{u_1}\mu_\tau(g))} \frac{1}{\det(\mathbf{1} - t\frac{u_2}{v}\bar{\mu}_\tau(g))} \quad (2.16)$$

In [2] authors consider the whole moduli space Hilbert series (2.14), while in [1] the authors restrict to the smaller Higgs branch series (2.16); this can be seen by comparing eq. (6.14) of [1] to eq. (4.10) of [2]. We will stick to the definition chosen in [2], since it is formulated in the $N = 3$ language.

3. Now let us compute the moduli space Hilbert series. In order to do that, one has to find the embedding of the group Γ in $\text{GL}(r, \mathbb{C})$. In our case $\Gamma = \mathcal{W}(\mathfrak{g}) \rtimes \mathbb{Z}_n$, so we only need to find how to embed $\mathbb{Z}_n$ properly. To check whether the $\mathbb{Z}_n$ embedding we've chosen is correct we can plug in Γ into (2.15) and compare the results with the ones we obtained with the other method in subsection 2.1. One more sanity check is to consider terms up to t^3 in the (2.14) expansion; according to [2] the expansion for $N = 3$ SCFTs should go as

$$\mathcal{I}_H = t^2 \left(u_1 u_2 + \frac{u_2^2}{u_1} \right) + O(t^3). \quad (2.17)$$

2.3 Results

2.3.1 Gauging $N = 4$ G_2 SYM

The action of the generators $\mathcal{C}_l$ of the $\mathbb{Z}_l$ groups on the Coulomb branch coordinates is

$$\mathcal{C}_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{C}_3 = \begin{pmatrix} \exp(\frac{4i\pi}{3}) & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{C}_6 = \begin{pmatrix} \exp(\frac{2i\pi}{3}) & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.18)$$

and the dimensions of the Coulomb branch operators are $\Delta = 4, 6$ for Γ_2 gauging of the theory, $\Delta = 6, 6$ for Γ_3 gauging and $\Delta = 6, 12$ for Γ_6 gauging. In every case Coulomb branch is freely generated.

The orbifolding group for the original manifold is $\text{Weyl}(\mathfrak{g}_2) = G(6, 6, 2)$ (we use Shepard and Todd notation for the complex reflection groups), and it is easy to check directly that $G(6, 6, 2) \rtimes \mathbb{Z}_2 = G(6, 3, 2)$, $G(6, 6, 2) \rtimes \mathbb{Z}_3 = G(6, 2, 2)$, $G(6, 6, 2) \rtimes \mathbb{Z}_6 = G(6, 1, 2)$. These groups are also present in the table 1 of [2],⁸ so our result match theirs and fill some of the gaps in the classification of the $N \geq 3$ SCFTs with rank-2 moduli spaces.

The moduli space Hilbert series for $\mathbb{Z}_2$ gauging is

$$t^2 \left(\frac{u_2^2}{u_1} + u_1 u_2 \right) + t^4 \left(u_1^4 v^4 + \frac{u_2^4 v^4}{u_1^4} + \frac{u_2^3 v^4}{u_1^2} + \right. \\ \left. + 2u_2^2 v^4 + u_1^2 u_2 v^4 + \frac{u_2^4}{v^4} + \frac{u_2^4}{u_1^2} + u_2^3 + u_1^2 u_2^2 \right) + O(t^6), \quad (2.19)$$

for $\mathbb{Z}_3$ gauging it is given by

$$t^2 \left(\frac{u_2^2}{u_1} + u_1 u_2 \right) + t^4 \left(\frac{u_2^4}{u_1^2} + u_2^3 + u_1^2 u_2^2 \right) + O(t^6) \quad (2.20)$$

and for $\mathbb{Z}_6$ gauging it is given by

$$t^2 \left(\frac{u_2^2}{u_1} + u_1 u_2 \right) + t^4 \left(\frac{u_2^4}{u_1^2} + u_2^3 + u_1^2 u_2^2 \right) + O(t^6); \quad (2.21)$$

we can use $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_6$ as the generators of $\mathbb{Z}_n$; indices for $\mathbb{Z}_3, \mathbb{Z}_6$ gaugings differ at the t^6 order.

⁸ $G(6, 2, 2)$ is written in [2] as $\text{Weyl}(\mathfrak{su}_3) \rtimes \mathbb{Z}_6$.

2.3.2 Gauging $N = 4$ F_4 SYM

The action of the generators $\mathcal{C}_l$ of the $\mathbb{Z}_l$ groups on the Coulomb branch coordinates is

$$\mathcal{C}_2 = \text{diag}(-1, 1, 1, -1), \quad \mathcal{C}_4 = \text{diag}(i, i, 1, 1), \quad \mathcal{C}'_2 = \text{diag}(-1, -1, 1, 1), \quad (2.22)$$

where $\mathcal{C}_2$ corresponds to the discrete symmetry related to $\tilde{S}$ and $\mathcal{C}'_2$ corresponds to the discrete symmetry related to $(\tilde{S}T)^2$. The Coulomb branch is not freely generated in any of these three cases, for $\mathbb{Z}_4$ it is not a complete intersection:

$$\mathbb{Z}_4 : \quad \text{PLog} \left\{ \frac{1 + U_2 U_6 (U_2^2 + U_2 U_6 + U_6^2)}{(1 - U_2^4)(1 - U_6^4)(1 - U_8)(1 - U_{12})} \right\}, \quad (2.23)$$

and in the two other cases the generators obey the relations

$$\mathbb{Z}_2 : \quad \tilde{u}_1 = U_2^2, \quad \tilde{u}_2 = U_6, \quad \tilde{u}_3 = U_8, \quad \tilde{u}_4 = U_{12}^2, \quad \tilde{u}_5 = U_2 U_{12}; \quad \tilde{u}_5^2 = \tilde{u}_1 \tilde{u}_4, \quad (2.24)$$

$$\mathbb{Z}'_2 : \quad \tilde{u}_1 = U_2^2, \quad \tilde{u}_2 = U_6^2, \quad \tilde{u}_3 = U_8, \quad \tilde{u}_4 = U_{12}, \quad \tilde{u}_5 = U_2 U_6; \quad \tilde{u}_5^2 = \tilde{u}_1 \tilde{u}_2. \quad (2.25)$$

The orbifolding group for the original manifold is $\text{Weyl}(\mathfrak{f}_4) = G_{28}$, and its semidirect product with $\mathbb{Z}_2, \mathbb{Z}_4$ doesn't yield a complex reflection group.

The action of the generators $\mathcal{A}_l$ of the $\mathbb{Z}_l$ groups on the fields ϕ can be chosen to be

$$\mathcal{A}_2 = i \cdot R, \quad \mathcal{A}_4 = e^{i\pi/4} R, \quad \mathcal{A}'_2 = i\mathbb{1}; \quad R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad (2.26)$$

and the moduli space Hilbert series is given by

$$\begin{aligned} \mathcal{I}_H^{F_4, \mathbb{Z}_2} = t^2 \left(\frac{u_2^2}{u_1} + u_1 u_2 \right) + t^4 \left(u_1^4 v^4 + \frac{u_2^4 v^4}{u_1^4} + \frac{u_2^3 v^4}{u_1^2} + \right. \\ \left. + 2u_2^2 v^4 + u_1^2 u_2 v^4 + \frac{u_2^4}{v^4} + \frac{u_2^4}{u_1^2} + u_2^3 + u_1^2 u_2^2 \right) + O(t^6) \end{aligned} \quad (2.27)$$

$$\begin{aligned} \mathcal{I}_H^{F_4, \mathbb{Z}'_2} = t^2 \left(\frac{u_2^2}{u_1} + u_1 u_2 \right) + t^4 \left(u_1^4 v^4 + \frac{u_2^4 v^4}{u_1^4} + \frac{u_2^3 v^4}{u_1^2} + \right. \\ \left. + 2u_2^2 v^4 + u_1^2 u_2 v^4 + \frac{u_2^4}{v^4} + \frac{u_2^4}{u_1^2} + u_2^3 + u_1^2 u_2^2 \right) + O(t^6) \end{aligned} \quad (2.28)$$

$$\mathcal{I}_H^{F_4, \mathbb{Z}_4} = t^2 \left(\frac{u_2^2}{u_1} + u_1 u_2 \right) + t^4 \left(\frac{u_2^4}{u_1^2} + u_2^3 + u_1^2 u_2^2 \right) + O(t^6) \quad (2.29)$$

$\mathcal{I}_H^{F_4, \mathbb{Z}_2}$ and $\mathcal{I}_H^{F_4, \mathbb{Z}'_2}$ differ at the t^6 order.

2.3.3 Gauging $N = 4$ B_n SYM

These gauge groups haven't been analyzed in [15]. As for the theories with B_n (or C_n) gauge groups S transformation takes us to another theory [14], from the S-duality side we should only consider $\mathbb{Z}_2$ generated by AT . This transformation leaves the Coulomb branch

invariant, so when we mix it with the R-symmetry part, we get that the action of the generator $\mathcal{C}_2$ of the Γ_2 groups on the Coulomb branch operators is given by

$$\mathcal{C}_2 = \text{diag}(-1, 1, \dots, (-1)^n), \quad (2.30)$$

where n corresponds to the B_n gauge group.

For B_2 gauge group we get that the Coulomb branch is freely generated, the dimensions of Coulomb branch operators are $\Delta = 4, 4$, and the orbifolding group is $G(4, 2, 2)$, which is in agreement with [2].

For B_3, B_4 gauge groups we found that the Coulomb branch is not freely generated, and the generators obey the relations

$$B_3 : \quad \tilde{u}_2 = U_2^2, \quad \tilde{u}_4 = U_4, \quad \tilde{u}_6 = U_6^2, \quad \tilde{u}_c = U_2 U_6; \quad \tilde{u}_c^2 = \tilde{u}_2 \tilde{u}_6 \quad (2.31)$$

$$B_4 : \quad \tilde{u}_2 = U_2^2, \quad \tilde{u}_4 = U_4, \quad \tilde{u}_6 = U_6^2, \quad \tilde{u}_8 = U_8, \quad \tilde{u}_c = U_2 U_6; \quad \tilde{u}_c^2 = \tilde{u}_2 \tilde{u}_6 \quad (2.32)$$

For $n \geq 5$ gauging Γ_4 yields a theory with Coulomb branch that is not a complete intersection manifold, the Molien series for e.g. B_5 is given by

$$B_5 : \quad \frac{1 + U_2 U_6 + U_2 U_{10} + U_6 U_{10}}{(1 - U_2^2)(1 - U_4)(1 - U_6^2)(1 - U_8)(1 - U_{10}^2)} \quad (2.33)$$

The action of the generators $\mathcal{A}_2$ of the $\mathbb{Z}_2$ groups on the fields ϕ can be chosen to be

$$\mathcal{A}_2 = i\mathbb{1}. \quad (2.34)$$

The Hilbert series for $B_2 - B_5$ $\mathcal{N} = 4$ theories with Γ_2 gauged is

$$\begin{aligned} \mathcal{I}_H^B = t^2 \left(\frac{u_2^2}{u_1} + u_1 u_2 \right) + t^4 \left(2u_1^4 v^4 + \frac{2u_2^4 v^4}{u_1^4} + \frac{2u_3^3 v^4}{u_1^2} + \right. \\ \left. + 3u_2^2 v^4 + 2u_1^2 u_2 v^4 + \frac{2u_2^4}{v^4} + \frac{2u_2^4}{u_1^2} + 2u_2^3 + 2u_1^2 u_2^2 \right) + O(t^6). \end{aligned} \quad (2.35)$$

The difference between B_2 and B_3 indices appears at t^6 order, between B_3 and B_4 — at t^8 order and between B_4 and B_5 — at t^{10} order.

3 $\mathcal{N} = 3$ SCFTs from complex crystallographic reflection groups

Another area of interest in $\mathcal{N} = 3$ SCFT studies is theories that have freely generated Coulomb branch. Until a few years ago it was commonly believed that every $\mathcal{N} \geq 2$ theory possesses a freely generated Coulomb branch, but in [1, 8] it was shown that there exist many $\mathcal{N} = 3$ SCFTs that possess a non-freely generated Coulomb branch. In [12] Caorsi and Cecotti argued that for every $\mathcal{N} = 3$ SCFT with a freely generated Coulomb branch the Coulomb branch is given by $\mathbb{C}^r/G$, where G is complex crystallographic reflection group (CCRG).⁹ However, these CCRGs split into two different classes: the ones that can correspond to an $\mathcal{N} = 3$ SCFT with a principal Dirac pairing and those that can correspond

⁹If G is real, SUSY is enhanced to $\mathcal{N} = 4$.

only to an $N = 3$ SCFT with a non-principal Dirac pairing. The distinction is important, because theories with non-principal Dirac pairing have a non-maximal spectrum of line operators and are very peculiar.¹⁰ In [14] it was shown that Lagrangian $N = 2$ SCFTs cannot have a non-maximal spectrum of line operators, while in [12] authors claim that an $N = 2$ SCFT with non-principal Dirac pairing cannot come from a stringy construction and its special geometry cannot be described by a Seiberg-Witten curve. All in all, it is unclear how to generate a theory with non-principal Dirac pairing in 4 dimensions, and it might be that such a theory can only be a relative field theory [2].

If a rank- k CCRG G corresponds to an $N = 3$ SCFT with a principal Dirac pairing, then consistency with Dirac quantization requires that $G \subset \mathrm{Sp}(2k, \mathbb{Z})$.¹¹ Caorsi and Cecotti argued that non-sporadic groups complex crystallographic reflection groups can always be embedded into $\mathrm{Sp}(2k, \mathbb{Z})$, and in [2] the issue of rank-2 sporadic groups has been addressed, so the groups left to analyze are G_{24} , G_{25} , G_{26} , G_{29} , G_{31} , G_{32} , G_{33} , G_{34} . In order to do that, we considered embedding of these groups into $\mathrm{GL}(k, \mathbb{Z}[\zeta])$, where k is the rank of the group and ζ is primitive third root of 1 for G_{25} , G_{26} , G_{32} , G_{33} , G_{34} ; $\zeta = i$ for G_{29} , G_{31} and $\zeta = \sqrt{-7}$ for G_{24} .¹² Then we computed an invariant $(k \times k)$ Hermitian form H for each of these groups. Afterwards we constructed $2k \times 2k$ skew-symmetric form Ω from H by considering

$$\frac{1}{\zeta - \bar{\zeta}} H_{ij} \psi^i \wedge \bar{\psi}^j, \quad (3.1)$$

where $\psi^i = x^i + \zeta y^i \in \mathbb{Z}[\zeta]^k$. Ω is then obtained by clearing denominators and dividing by a non-trivial common factor for all the entries of the matrix in consideration if needed. Then the necessary and sufficient condition on whether the embedding $G \hookrightarrow \mathrm{GL}(k, \mathbb{Z}[\zeta])$ induces an embedding $G \hookrightarrow \mathrm{Sp}(2k, \mathbb{Z})$ is simply $\det \Omega = 1$; otherwise $\mathrm{Sp}(2k, \mathbb{Z})$ should be replaced by the arithmetic group:

$$S(\Omega)_{\mathbb{Z}} = \{M \in \mathrm{GL}(2k, \mathbb{Z}) : M^T \Omega M = \Omega\} \quad (3.2)$$

The direct computation shows that for $G = G_{24}$, G_{25} , G_{26} , G_{32} , G_{33} $G \not\subset \mathrm{Sp}(2k, \mathbb{Z})$. Let us list lowest terms of the moduli space Hilbert series expansion for G_{29} , G_{31} :¹³

$$G_{29} : \quad \mathcal{I}_H = t^2 \left(\frac{u_2^2}{u_1} + u_1 u_2 \right) + t^4 \left(u_1^4 v^4 + \frac{u_2^4 v^4}{u_1^4} + \frac{u_2^3 v^4}{u_1^2} + u_2^2 v^4 + u_1^2 u_2 v^4 + \frac{u_2^4}{v^4} \right) + O(t^6) \quad (3.3)$$

$$G_{31} : \quad \mathcal{I}_H = t^2 \left(\frac{u_2^2}{u_1} + u_1 u_2 \right) + O(t^8). \quad (3.4)$$

We can identify that at the t^2 order the only contribution to the index comes from the $N = 3$ stress-tensor multiplet. The index expansion for G_{31} has contributions at t^2 order and then only at t^8 .

¹⁰I am grateful to Yuji Tachikawa and Gabi Zafrir for clarifying this.

¹¹If this requirement is satisfied, it still doesn't mean there exist a corresponding $N = 3$ SCFT.

¹²One should also take into account that G_{25} , G_{26} , G_{33} have two inequivalent embeddings into $\mathrm{Sp}(2k, \mathbb{Z})$, both of which should be considered [17].

¹³ G_{34} has about 3.9×10^7 elements, so it is computationally unfeasible to calculate moduli space Hilbert series for it.

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A Existence of theories listed in Bourton et al.

In [1] Bourton et al. classify various $N = 3$ SCFTs obtained from $N = 4$ SCFTs with ADE or $U(n)$ gauge groups. In particular, they have mixed finite cyclic subgroups of $SL(2, \mathbb{Z})$ self-duality groups with the $\mathbb{Z}_n \subset SU(R)_4$ of the R-symmetry group and then gauged the resulting group; for $n = 3, 4, 6$ they have obtained $N = 3$ SCFTs, while for $n = 2$ they have got $N = 4$ SCFTs. However, there is a fine point first observed in [14] related to the fact that there might be more than one theory for a given gauge algebra $\mathfrak{g}$, depending on the line operators present in the theory. S, T transformations then may transform a theory with one set of line operators to a physically distinct theory with another set of line operators; an $N = 4$ SCFT with gauge algebra $\mathfrak{g}$ (listed in [1]) will have a $Z_k \subset SL(2, \mathbb{Z})$ iff there is a theory which is self-dual under the corresponding S -duality transformation. Therefore, it turns out that not every $N = 3$ SCFT listed in [1] exists; using [14] and [8], one can find that the following $N = 3$ SCFTs exist:

	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_6$
$SU(2)$	+	−	+	−
$SU(3)$	+	+	−	+
$SU(4)$	+	+	+	+
$SU(5)$	+	−	+	−
$SO(2d), d > 1$	+	+	+	+
$U(d)$	+	+	+	+
E_6	+	+	−	+
E_7	+	−	+	−
E_8	+	+	+	+

These results are in agreement with [2], with the superficial exception of the $U(2)$ and $SO(4)$ $\mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ gaugings. The classification of Argyres et al. contains the Coulomb branch orbifold geometries for these theories (table 3, entry 32 for $\mathbb{Z}_3$ gauging of $U(2)$, table 3, entry 33 for $\mathbb{Z}_4$ gauging of $U(2)$, $SO(4)$ and table 4, entry 52 for $\mathbb{Z}_6$ gauging of $U(2)$ and $\mathbb{Z}_3, \mathbb{Z}_6$ gauging of $SO(4)$). These entries were ruled out by Argyres et al. because the corresponding theories have two stress tensors (this can be seen from the Hilbert series analysis). However, as the mother $N = 4$ theory had a non-simple gauge group, getting

theory with two stress tensors after gauging a discrete subgroup is the expected outcome and does not mean the geometries in question should be discarded.

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