Conformal symmetry and composite operators in the $O(N)^3$ tensor field theory

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ABSTRACT: We continue the study of the bosonic $O(N)^3$ model with quartic interactions and long-range propagator. The symmetry group allows for three distinct invariant ϕ^4 composite operators, known as tetrahedron, pillow and double-trace. As shown in [1, 2], the tetrahedron operator is exactly marginal in the large- N limit and for a purely imaginary tetrahedron coupling a line of real infrared fixed points (parametrized by the absolute value of the tetrahedron coupling) is found for the other two couplings. These fixed points have real critical exponents and a real spectrum of bilinear operators, satisfying unitarity constraints. This raises the question whether at large- N the model is unitary, despite the tetrahedron coupling being imaginary.

In this paper, we first rederive the above results by a different regularization and renormalization scheme. We then discuss the operator mixing for composite operators and we give a perturbative proof of conformal invariance of the model at the infrared fixed points by adapting a similar proof from the long-range Ising model. At last, we identify the scaling operators at the fixed point and compute the two- and three-point functions of ϕ^4 and ϕ^2 composite operators. The correlations have the expected conformal behavior and the OPE coefficients are all real, reinforcing the claim that the large- N CFT is unitary.

KEYWORDS: $1/N$ Expansion, Conformal Field Theory, Renormalization Group, Field Theories in Lower Dimensions

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1 Introduction

Finding and studying non-supersymmetric interacting conformal field theories (CFT) in $d > 2$ dimensions is a very challenging task with a long history. A young and successful approach is the conformal bootstrap [3], working completely within the framework of CFT and seeking to identify CFT's by imposing consistency conditions. The most standard and historical approach is however based on the renormalization group: here the challenge is to find non-trivial (interacting) fixed points. They correspond by construction to scale-invariant theories, but very commonly invariance under the full conformal group arises as

well [4]. The great hurdle faced by the renormalization group is that our main tool of investigation, perturbation theory, demands that we approach the interacting fixed point by perturbing the free theory. Clearly, the task becomes more daunting the further the two theories are, and therefore it is extremely important from the theoretical point of view to have adjustable parameters that allow one to bring the interacting fixed point closer to the free theory or to tame the perturbative series. There are two widely exploited parameters of this sort [5]: one is the analytically continued spacetime dimension, as typically the interacting fixed point collapses into the non-interacting one at some critical dimension; the other is the number of field components $\mathcal{N}$, when their interactions are constrained by a (global or local) symmetry group, as the large- $\mathcal{N}$ limit can lead to drastic simplifications of the perturbative expansion.

Tensor models, in which $\mathcal{N} = N^r$ with r the rank of the tensor, are a recent entry in the menu of field theories admitting an interesting large- N limit. They typically admit a large N limit dominated by *melonic* diagrams [6–10] different from both the large N limit of vector models ($r = 1$, dominated by bubble diagrams [11]) and the one of matrix models ($r = 2$, dominated by planar diagrams [12]). The melonic dominance translates into a perturbative expansion which is richer than the one of vectors and more manageable than that of matrices. It is therefore interesting to construct models of tensor field theories in d dimensions and look for their fixed points at large N [1, 2, 13–18].¹ We call the resulting conformal field theories *melonic*.

In this paper we study the bosonic tensor field theory introduced in [1], that is the Carrozza-Tanasa-Klebanov-Tarnopolsky [26, 44] (CTKT) model with long-range propagator. The model has the following features:

1. The global symmetry group is $O(N)^3$ with fields transforming in the tri-fundamental representation, that is the fields are rank-3 tensors. The $O(N)^3$ models were introduced in zero dimensions in [44], and studied in dimension one and higher in [26] and later in [13, 16]. The existence of a large- N expansion (with melonic dominance at leading order for the case of a tetrahedron interaction) was proved in [44] by adapting the methods of [6, 45].
2. The action contains only quartic interactions, and there are three of them which are allowed by the symmetry; they are known as tetrahedron, pillow, and double-trace invariants. The same set of interactions were considered in ref. [13], but the two models are distinguished by the following two features.
3. The propagator is long-range, $C(p) = 1/p^{d/2}$ in $d < 4$ dimensions, such that the quartic couplings are dimensionless in any dimension d . From this point of view the model can be thought of as a generalization of the long-range Ising model [46, 47], which has been studied extensively with various methods, including constructive

¹Tensor models were initially studied in zero dimension in the context of quantum gravity and random geometry [7, 19–23]. They were then studied in one dimension [24–36] (see also [8, 37] for reviews) as a generalization of the Sachdev-Ye-Kitaev model [38–43] without quenched disorder.

methods [48, 49], large- N expansion [50], functional renormalization group [51], and CFT methods [52, 53].

4. The tetrahedron coupling is purely imaginary. As the tetrahedron invariant is unbounded (from above and below) the choice of imaginary coupling is reminiscent of the Lee-Yang model with an $i\lambda\phi^3$ interaction [54, 55].

The main result of ref. [1] is that in the large- N limit the tetrahedron coupling g is exactly marginal (its beta function is identically zero) while the beta functions of the other two couplings are quadratic in the couplings themselves with g -dependent coefficients. The model has four g -dependent fixed points, which are real for g purely imaginary and below some critical value. One of them is IR attractive for both the pillow and the double-trace couplings. Notice that from the perspective of the long-range Ising model ($N = 1$, only one quartic interaction) this infrared fixed point is surprising. In fact, our kinetic term corresponds to the transition point between the long-range behavior and the mean field theory one, in which the infrared fixed point disappears in the Ising case. The existence of a non-trivial IR fixed point in our $O(N)^3$ model is exclusively due to the tensor structure and the large- N limit. The critical exponents at the IR fixed point, that is the scaling dimensions of the pillow and double-trace operators, are real and above the unitarity bounds. The spectrum of dimensions of bilinear operators with arbitrary spin, as well as their OPE coefficients with two fundamental fields, has been computed in ref. [2], where it was found again to be real and above the unitarity bounds.

Given the fact that the model has an imaginary coupling, it is to be expected that it is non-unitary, but the results of [1, 2] raise the tantalizing possibility that in the large- N limit we could find a unitary theory.² For example, non-unitarity could manifest itself in some dimensions or OPE coefficients having imaginary parts which are suppressed in $1/N$.

In this paper we address the following two questions: *(i)* does the large- N infrared fixed point of ref. [1] define a conformal field theory? *(ii)* Does it define a unitary theory?

In order to tackle the first question, we will adapt to our model the methods of ref. [52], which gives a proof of conformal invariance to all orders in perturbation theory for the infrared fixed-point of the long-range Ising model with propagator $C(p) = 1/p^{(d+\epsilon)/2}$. Most of that proof is built on standard ideas (e.g. from ref. [56]), except that the non-local propagator of the long-range model implies the absence of a local energy-momentum tensor. The main point is to use the Caffarelli-Silvestre trick [57] of localizing the kinetic term by means of an embedding of the theory in $d + p$ dimensions, with $p = 2 - (d + \epsilon)/2$.

The main differences in our case are that: first, we are interested in $\epsilon = 0$, and second, we must deal with multiple quartic interactions which mix under renormalization. We thus need to revisit the results of ref. [1] and [2] with an analytic regularization (rather than using a momentum cutoff), and construct renormalized composite operators. Once this is done we conclude along the lines of [52] that the fixed-point theory of our $O(N)^3$ model is indeed conformally invariant.

²Here and occasionally in the rest of the paper, by an abuse of terminology, we talk about unitarity rather than reflection positivity despite working exclusively in Euclidean signature. It should be clear that we have in mind unitarity of the Wick-rotated theory.

We then compute two and three-point functions among the renormalized composite operators, which we use to further test conformal invariance and to address our second question. Conformal invariance greatly constraints these correlators [58]: the two-point functions between operators of different scaling dimension are zero, the three-point functions among three operators are completely fixed by their dimensions (up to an overall OPE coefficient), and so on. Such constraints are respected by all the correlators we have computed.

For special values of the dimensions of operators some subtle issues appear [59]. For instance one should be careful to distinguish between the d -dimensional Dirac delta $\delta(x)$, which is a homogeneous distribution under conformal transformations, and $|x|^{-d}$ which has a singular Fourier transform and upon regularization does not transform homogeneously [60]. We need to deal with this issue as in our model we have an exactly marginal operator, the tetrahedron, which by definition has dimension d . We find contact terms (i.e. terms including a delta function) in its three-point function with itself eq. (6.26), as expected [61, 62]. These contact terms do not lead to an anomaly as their coefficients are finite. However, an anomaly can arise from its two-point function [63], which has the functional form $1/|x|^{2d}$, and which in two dimensions has a singular distributional limit: $1/|x|^{2d-\epsilon} \sim \frac{1}{\epsilon}(\partial^2)\delta(x)$. We therefore expect a conformal anomaly in $d = 2$. Such a conformal anomaly in two dimensions would not be a big surprise, but in the absence of a local energy-momentum tensor it is not obvious how it should be interpreted. It should also be noted that in ref. [2] a puzzling discontinuity was found in the spectrum of bilinear operators: the computation at $d = 2$ differs from that at $d = 2 + \epsilon$ in the limit of vanishing ϵ . We hope to come back in future work to the two-dimensional case to clarify these issues.

The computation of two- and three-point functions allows us to address also the question about unitarity. Unitarity constrains the correlators [3]: we can check whether our two-point functions satisfy reflection positivity, whether the OPE coefficients appearing in our three-point functions are real, and so on. It turns out that these constraints are satisfied.

Melons vs. fishnets. We conclude this overview of results by a remark. There are a number of intriguing similarities between our model and the conformal fishnet theory introduced in ref. [64], or more precisely with its generalization to dimension $d < 4$ [65], which requires a long-range propagator like ours. The conformal fishnet theory is a model of two complex matrices, with a single-trace chiral quartic interaction, without its hermitian conjugate. The interaction is therefore complex, as in our model. Furthermore this interaction is exactly marginal in the large- N limit, as in our model. Moreover, renormalizability requires the introduction of double-trace interactions, and the four-point function renormalizing them is built out of ladders and bubbles in a similar fashion to what was found in ref. [1] for our model (see also section 3.2.1 below). The resulting beta functions for the running couplings are therefore quadratic in both models, with coefficients parametrically depending on the exactly marginal coupling (compare eq. (13) of ref. [66], with our beta functions (3.44) below).

There are of course also important differences. The different names that have been attached to the two conformal theories are not an accident: whereas at large- N our model is dominated by melonic diagrams, the model of Gürdogan and Kazakov is dominated by

fishnet diagrams. What is accidental is the fact that for the four-point functions of fundamental fields both types of diagrams reduce to ladders: indeed a ladder can be thought either as melonic graph which has been open on two edges (and with resummed propagators), or as a fishnet with periodicity of length two in one direction. However, all the other n -point functions of fundamental fields are different. In particular, while the two-point function in a melonic CFT is given by a sum over melonic two-point diagrams, in the fishnet CFT there is no correction to the bare propagator (and hence no mass or wave function renormalization) at leading order. Such a difference is not very important for our long-range model in $d < 4$, which has no wave function renormalization anyway, but it becomes relevant for models with a standard short-range propagator, including ours at $d = 4$. Last but not least, the conformal fishnet theory is a logarithmic CFT [67], while so far we have not found logarithmic correlators in our model. We will study in future work whether this property survives in other correlators or at sub-leading order in $1/N$.

Plan of the paper. In section 2, we introduce and review the model. In section 3, we discuss renormalization and fixed points of the model, both in the Wilsonian picture and in the minimal subtraction scheme. Then, in section 4, we discuss the mixing of the ϕ^4 composite operators under the renormalization group flow. In section 5, we give a proof of the conformal symmetry at the infrared fixed point of a class of correlations, based on the $D = d + p$ dimensional embedding method of ref. [52]. Lastly, in section 6, we use the perturbative expansion at the fixed-point (our small parameter being the exactly marginal tetrahedron coupling) in order to compute the two- and three-point functions (and hence the OPE coefficients) among the ϕ^4 and ϕ^2 composite operators.

In the appendices we include some detailed computations and additional remarks. Appendix A contains detailed computations for several integrals we use in the main text. The beta functions for the spin zero bilinear operators are presented in appendix B. In appendix C, we discuss the large- N scaling of the maximally single-trace (MST) and maximally multi-trace (MMT) operators, and in appendix D, we discuss correlators containing the pillow operator, which is neither of MST nor MMT.

2 Overview of the model

We study the tensor model of [1, 2], that is, the $O(N)^3$ tensor model of Klebanov and Tarnopolsky [26] and Carrozza and Tanasa [44] (CTKT model) with a long-range covariance. The fundamental field is a real tensor field of rank 3, $\phi_{a^1 a^2 a^3}(x)$, transforming under $O(N)^3$ with indices distinguished by the position, and we denote $\mathbf{a} = (a^1, a^2, a^3)$. The action of the model is:

$$\begin{aligned}
 S[\phi] &= \frac{1}{2} \int d^d x \phi_{\mathbf{a}}(x) (-\partial^2)^\zeta \phi_{\mathbf{a}}(x) + S^{\text{int}}[\phi], \\
 S^{\text{int}}[\phi] &= \frac{m^{2\zeta}}{2} \int d^d x \phi_{\mathbf{a}}(x) \phi_{\mathbf{a}}(x) \\
 &\quad + \frac{1}{4} \int d^d x \left[i \lambda \hat{\delta}_{\mathbf{abcd}}^t + \lambda_1 \hat{P}_{\mathbf{ab};\mathbf{cd}}^{(1)} + \lambda_2 \hat{P}_{\mathbf{ab};\mathbf{cd}}^{(2)} \right] \phi_{\mathbf{a}}(x) \phi_{\mathbf{b}}(x) \phi_{\mathbf{c}}(x) \phi_{\mathbf{d}}(x),
 \end{aligned}
 \tag{2.1}$$

where repeated tensor indices are summed over $a^i = 1, \dots, N$ and we introduced the projectors:

$$\hat{P}_{\mathbf{ab};\mathbf{cd}}^{(1)} = 3(\hat{\delta}_{\mathbf{ab};\mathbf{cd}}^p - \hat{\delta}_{\mathbf{ab};\mathbf{cd}}^d), \quad \hat{P}_{\mathbf{ab};\mathbf{cd}}^{(2)} = \hat{\delta}_{\mathbf{ab};\mathbf{cd}}^d. \quad (2.2)$$

and the rescaled operators:

$$\hat{\delta}_{\mathbf{abcd}}^t = \frac{1}{N^{3/2}} \delta_{\mathbf{abcd}}^t, \quad \hat{\delta}_{\mathbf{ab};\mathbf{cd}}^p = \frac{1}{N^2} \delta_{\mathbf{ab};\mathbf{cd}}^p, \quad \hat{\delta}_{\mathbf{ab};\mathbf{cd}}^d = \frac{1}{N^3} \delta_{\mathbf{ab};\mathbf{cd}}^d, \quad (2.3)$$

with

$$\begin{aligned} \delta_{\mathbf{abcd}}^t &= \delta_{a^1 b^1} \delta_{c^1 d^1} \delta_{a^2 c^2} \delta_{b^2 d^2} \delta_{a^3 d^3} \delta_{b^3 c^3}, \\ \delta_{\mathbf{ab};\mathbf{cd}}^p &= \frac{1}{3} \sum_{i=1}^3 \delta_{a^i c^i} \delta_{b^i d^i} \prod_{j \neq i} \delta_{a^j b^j} \delta_{c^j d^j}, \quad \delta_{\mathbf{ab};\mathbf{cd}}^d = \delta_{\mathbf{ab}} \delta_{\mathbf{cd}}. \end{aligned} \quad (2.4)$$

Here t stands for *tetrahedron*, p for *pillow*, and d for *double-trace*. Such names refer to the graphical representation of the respective pattern of contraction of indices, as recalled below.

We use the following shorthand notations for the quadratic invariant:

$$\phi^2(x) \equiv \phi_{\mathbf{a}}(x) \phi_{\mathbf{a}}(x), \quad (2.5)$$

and for the quartic invariants:

$$\begin{aligned} \phi_t^4(x) &\equiv i \hat{\delta}_{\mathbf{abcd}}^t \phi_{\mathbf{a}}(x) \phi_{\mathbf{b}}(x) \phi_{\mathbf{c}}(x) \phi_{\mathbf{d}}(x), \\ \phi_i^4(x) &\equiv \hat{P}_{\mathbf{ab};\mathbf{cd}}^{(i)} \phi_{\mathbf{a}}(x) \phi_{\mathbf{b}}(x) \phi_{\mathbf{c}}(x) \phi_{\mathbf{d}}(x), \quad (i = 1, 2). \end{aligned} \quad (2.6)$$

The difference between this model and the CTKT model is that the Laplacian is allowed to have a non integer power $0 < \zeta \leq 1$. This modification preserves the reflection positivity of the propagator: the free theory is unitary for any $\zeta \leq 1$. The choice $\zeta = d/4$ renders the quartic invariants marginal in any d [1]. It is this value of ζ that interests us in this paper. Moreover, we will restrict to $d < 4$ in the following, in order to avoid a wave function renormalization (see section 3.1, and footnote 3 in particular).

We have not assigned any subscript to the coupling of the tetrahedral invariant, as it plays a special role in the model. Observe also that the infrared fixed point found in [1, 2], and that we aim to study, occurs for a purely imaginary tetrahedral coupling, hence we have chosen here to make that explicit from the onset, by writing the coupling as $i\lambda$, with $\lambda \in \mathbb{R}$.

As usual, it is convenient to introduce a graphical representation of the $O(N)^3$ invariants, which also justifies their names. We represent every tensor ($\phi_{\mathbf{a}}$, $\phi_{\mathbf{b}}$ and so on) as a three-valent node and every contraction of two indices (a^i and b^i for instance) as an edge with a color $i = 1, 2$, or 3 (red, green, or blue) corresponding to the position i of the indices. As a result, $O(N)^3$ invariants are represented by 3-colored graphs. The graphs corresponding to the quartic invariants of eq. (2.1) are depicted in figure 1.

We can expand the free energy and the connected n -point functions perturbatively around the Gaussian theory. We introduce two graphical representations for the terms in

Figure 1. Graphical representation of the quartic $O(N)^3$ invariants. From left to right: the tetrahedron, the pillow, and the double-trace (there are three pillow contractions, distinguished by the color of the vertical edge).

Figure 2. Two 4-colored graphs, with external tensor contractions equivalent to the pillow (left) and double-trace (right) invariants.

Figure 3. Two Feynman diagrams obtained from the two 4-colored graphs of figure 2 by shrinking the colored edges. Half-edges are also added to keep track of the external fields.

the perturbative expansion. In the first, each interaction invariant is a 3-colored graph, and the Feynman propagators are represented as edges with a new color, which we call 0 (pictured in black), connecting the tensors. This leads to a representation of the perturbative expansion in terms of 4-colored graphs, as for example in figure 2. In the second representation, we simplify the graphs by shrinking each interaction invariant (i.e. all its edges with colors from 1 to 3) to a point. We call the resulting object Feynman diagrams, as they represent in a more straightforward way the spacetime integrals associated to the amplitude. An important class of graphs and diagrams are the melonic ones, which however have very different features depending on whether it is the 4-colored graph which is melonic, or the (single color) diagram. The pillow and double-trace interactions are examples of melonic 3-colored graphs, and models based on such type of interactions are known to be dominated by melonic 4-colored graphs at leading-order in $1/N$ [45]. The corresponding Feynman diagrams are cactus diagrams, as in vector models. On the contrary, the tetrahedron interaction is not a melonic 3-colored graph, but it leads to melonic Feynman diagrams in the large- N limit [26, 44].

As a result of the combination of pillow, double-trace, and tetrahedron interactions, our model has a $1/N$ expansion dominated by *melon-tadpole* diagrams [1] with melons based on couples of tetrahedral vertices and tadpoles based on either pillow or double-trace vertices (see figure 4).

Figure 4. A vacuum melon-tadpole Feynman diagram, where all the invariants have been shrunk to point-like vertices.

3 Renormalization and fixed points

We consider the d -dimensional theory in Euclidean signature, and our aim is to study the case $\zeta = d/4$.

We introduce an infrared regulator μ by modifying the free covariance of the theory to:

$$C_\mu(p) = \frac{1}{(p^2 + \mu^2)^\zeta} = \frac{1}{\Gamma(\zeta)} \int_0^\infty da \, a^{\zeta-1} e^{-(p^2 + \mu^2)a}, \quad (3.1)$$

and we regulate the UV divergences by setting:

$$\zeta = \frac{d + \epsilon}{4}, \quad (0 \leq \epsilon \ll 1). \quad (3.2)$$

This implies that the ultraviolet dimension of the field is

$$\Delta_\phi = (d - 2\zeta)/2 = (d - \epsilon)/4. \quad (3.3)$$

The reader should keep in mind that for us ϵ is just a regulator and we always intend to send $\epsilon \rightarrow 0$ in the end. In particular, we are not looking for Wilson-Fisher type of fixed points at small but finite ϵ .

We will need the following Fourier transform, which holds for $d/2 > \gamma > 0$:

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{|p|^{2\gamma}} e^{ip(x-y)} = \frac{c(\gamma)}{|x-y|^{d-2\gamma}}, \quad c(\gamma) = \frac{\Gamma(\frac{d}{2} - \gamma)}{2^{2\gamma} \pi^{d/2} \Gamma(\gamma)}. \quad (3.4)$$

We sometimes rescale the couplings to $\lambda = (4\pi)^{d/2} \Gamma(\zeta)^2 \tilde{\lambda}$ and so on, and we denote the dimensionless running couplings at scale μ by g, g_1, g_2 (respectively $\tilde{g}, \tilde{g}_1, \tilde{g}_2$).

3.1 The two-point function

We now discuss the bare and the full two-point functions of the model.

The bare propagator. The bare propagator in the direct space, with infrared regulator set to zero, is obtained from eq. (3.4) by simply setting $\gamma = \zeta$:

$$C(x-y) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip(x-y)}}{|p|^{2\zeta}} = \frac{c(\zeta)}{|x-y|^{d-2\zeta}}. \quad (3.5)$$

We will encounter below the convolution of the cube of the bare propagators with another propagator. Using repeatedly the Fourier transform in eq. (3.4), we obtain the formal result:

$$\int d^d z \, C(x-z) C(z-y)^3 = \frac{c(\zeta)^3}{c(3\zeta-d)} \frac{c(4\zeta-d)}{|x-y|^{3d-8\zeta}}. \quad (3.6)$$

The problem with this formula is that we used eq. (3.4) for $\gamma = 3\zeta - d = -d/4 + 3\epsilon/4 < 0$, hence the result is only formal. In fact this convolution hides an ultraviolet divergence which needs to be subtracted. Taking this into account, we obtain:³

$$\begin{aligned} \int d^d z C(x-z) \left(C(z-y)^3 - \delta(z-y) \int d^d u C(u)^3 \right) &= \frac{c(\zeta)^3}{c(3\zeta-d)} \frac{c(4\zeta-d)}{|x-y|^{8\Delta_\phi-d}} \\ &= -\frac{4\Gamma(1-\frac{d}{4})}{d(4\pi)^d \Gamma(\frac{3d}{4})} \delta(x-y) + O(\epsilon), \end{aligned} \quad (3.8)$$

where this time the integral is convergent, and we have used the distributional limit $\lim_{\epsilon \rightarrow 0} c(\epsilon)/|x-y|^{d-2\epsilon} = \delta(x-y)$ (alternatively one can take the limit in Fourier space to reach the same conclusion).

The full two-point function. We now discuss the full two-point function of the model. In the absence of spontaneous symmetry breaking,⁴ the full two-point function is diagonal in the tensor indices. That is, it can be written as:

$$G_{\mathbf{ab}}(x, y) = \langle \phi_{\mathbf{a}}(x) \phi_{\mathbf{b}}(y) \rangle = \delta_{\mathbf{ab}} G(x-y). \quad (3.9)$$

Due to the infrared regulator, the full two-point function also acquires a μ -dependence, hence we write its regulated diagonal component as $G_\mu(x-y)$.

Following [1], we observe that $G_\mu(p)$ at leading order in N respects the melonic Schwinger-Dyson (SD) equation:

$$\begin{aligned} G_\mu(p)^{-1} &= (p^2 + \mu^2)^\zeta + m^{2\zeta} + \lambda_2 \int \frac{d^d q}{(2\pi)^d} G_\mu(q) \\ &+ \lambda^2 \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} G_\mu(q_1) G_\mu(q_2) G_\mu(p + q_1 + q_2), \end{aligned} \quad (3.10)$$

where the sign of the last term is changed with respect to [1] due to the explicit i factor in our action. Similar to eq. (3.6), this equation exhibits a power divergence in the ultraviolet. The critical theory is obtained by tuning the bare mass to exactly cancel this divergence such that the renormalized mass is zero. The melonic SD equation in the limit $\mu \rightarrow 0$ for

³In Fourier space the subtracted melon contribution is:

$$\begin{aligned} \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} C(q_1) C(q_2) [C(p + q_1 + q_2) - C(q_1 + q_2)] &= -\frac{p^{2d-6\zeta}}{(4\pi)^d \Gamma(\zeta)^3} \int_0^\infty da_1 da_2 da_3 \\ &\frac{(a_1 a_2 a_3)^{\zeta-1}}{(a_1 a_2 + a_1 a_3 + a_2 a_3)^{d/2}} \left(1 - e^{-\frac{a_1 a_2 a_3}{a_1 a_2 + a_1 a_3 + a_2 a_3}} \right) = \frac{p^{2d-6\zeta}}{(4\pi)^d \Gamma(\zeta)^3} \frac{\Gamma(1+3\zeta-d)}{3\zeta-d} \frac{\Gamma(\frac{d}{2}-\zeta)^3}{\Gamma(\frac{3}{2}d-3\zeta)}, \end{aligned} \quad (3.7)$$

where the integral over the a_i parameters is convergent for $d < 4$ and $\epsilon < d/3$. We computed it in appendix A. Multiplying eq. (3.7) by $p^{-2\zeta}$, Fourier transforming back to the direct space (which is allowed for $\epsilon \geq 0$) and using the analytic continuation of the Γ function we obtain eq. (3.8)

⁴Spontaneous symmetry breaking in tensor field theories has been so far not much explored, but see refs. [17, 68, 69].

the critical theory is:

$$G(p)^{-1} = p^{2\zeta} + \lambda^2 \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} G(q_1) G(q_2) [G(p + q_1 + q_2) - G(q_1 + q_2)], \quad (3.11)$$

where $G(p)$ denotes the full two-point function with no cutoff. It is solved self consistently at $\epsilon = 0$, for any $d < 4$ by the ansatz $G(p)^{-1} = Z p^{2\zeta}$:

$$Z p^{2\zeta} = p^{2\zeta} - \frac{p^{2d-6\zeta}}{Z^3} \lambda^2 \left(-\frac{c(\zeta)^3}{c(3\zeta - d)} \right), \quad \zeta = \frac{d}{4}, \quad (3.12)$$

provided that the constant Z is chosen as:

$$Z = 1 - \frac{\lambda^2}{Z^3} \left(-\frac{c(d/4)^3}{c(-d/4)} \right). \quad (3.13)$$

This in turn fixes the bare mass to $m_c^{2\zeta} = m_c^{2\zeta}$, where:

$$m_c^{2\zeta} = -\frac{\lambda_2}{Z} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^{2\zeta}} - \frac{\lambda^2}{Z^2} \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{q_1^{2\zeta}} \frac{1}{q_2^{2\zeta}} \frac{1}{(q_1 + q_2)^{2\zeta}}. \quad (3.14)$$

Summarizing, in the critical theory, the net effect of all the Feynman diagrams on the two-point function is a just multiplicative factor: $G(p) = Z^{-1} C(p)$. The constant Z can be seen as a finite wave function renormalization which resums all the melonic insertions. By resumming an infinite series of diagrams, Z carries non-perturbative information on the radius of convergence of the perturbative series [1] at large N .

The fact that we can perform such a resummation is an important property of our tensor model at $\epsilon = 0$. Nevertheless, in order to regulate the UV divergences, below we will consider $\epsilon > 0$. In this case, the resummation of the leading-order two-point function can not be performed explicitly. Furthermore, for theories with a non-local propagator, due to the locality of counterterms, there is no need to renormalize the fields. Therefore, it is common to set the wave function renormalization to 1 (e.g. [52]), and we will do the same here. We thus seem to have a discontinuous Z at $\epsilon = 0$, but in fact this is not the case. It is all only a matter of repackaging of diagrams: the two-point melonic insertions contribute also at $\epsilon > 0$, but as we let $\epsilon \rightarrow 0$, we can conveniently resum them by working with melon-free diagrams and propagators divided by Z . Differences appear if we decide to rescale or not the fields by $\sqrt{Z}$: rescaling is a natural choice at $\epsilon = 0$, as it brings the two-point back function to its original form, but it is not justified at $\epsilon > 0$ as $G(p)$ and $C(p)$ have different functional forms. In any case, rescaling by a finite constant is just a choice of renormalization scheme like others, and as such it will not affect the universal part of the beta functions.

For $\epsilon = 0$ the melonic SD equation in direct space is:

$$(G^{-1})_{xy} = (C^{-1})_{xy} + \lambda^2 \left[G_{xy}^3 - \delta_{xy} \int d^d u G(u)^3 \right], \quad G(x - y) = \frac{c(d/4)}{Z |x - y|^{d/2}}, \quad (3.15)$$

which can also be written as:

$$\begin{aligned} \int d^d z G(x - z) \left(G(z - y)^3 - \delta_{zy} \int d^d u G(u)^3 \right) &= \frac{Z - 1}{Z \lambda^2} \delta(x - y), \\ \frac{Z - 1}{Z \lambda^2} &= \frac{1}{Z^4} \frac{c(d/4)^3}{c(-d/4)}. \end{aligned} \quad (3.16)$$

The SD equation simplifies if the integral over z is understood in the sense of dimensional regularization. In this case the local part of the melonic correction and the tadpoles are set to zero. In the Wilsonian picture the melon integral combines with the explicit mass counterterms which provide the subtraction. This works as long as the position y is an internal position. If y is an external argument, that is when this is a contribution to a correlator $\langle \dots \phi_t^4(y) \rangle$ one needs to replace the tetrahedral operator by the renormalized tetrahedral operator:

$$\langle \dots [\phi_t^4(y)] \rangle, \quad [\phi_t^4(y)] = \phi_t^4(y) - 2\phi^2(y) \int d^d u G^3(u) + \dots \quad (3.17)$$

3.2 The beta functions

We review the β functions of the theory defined by the action (2.1) tuned to criticality.

3.2.1 Quartic couplings

We denote $i\Gamma_t^R, \Gamma_1^R$, and Γ_2^R the appropriately normalized one-particle irreducible four-point functions at zero external momentum. They are computed using the bare expansion in terms of connected amputated one-particle irreducible four-point diagrams $\mathcal{G}$ with amplitude:

$$A(\mathcal{G}) = \mu^{(d-4\Delta_\phi)[1-n(\mathcal{G})]} \hat{A}(\mathcal{G}) = \mu^{\epsilon-n(\mathcal{G})\epsilon} \hat{A}(\mathcal{G}), \quad (3.18)$$

$$\hat{A}(\mathcal{G}) = \int_0^\infty \left(\prod_{e \in \mathcal{G}} d\alpha_e \alpha_e^{\zeta-1} \right) \frac{1}{[\sum_{\mathcal{T} \subset \mathcal{G}} \prod_{e \notin \mathcal{T}} \alpha_e]^{d/2}} e^{-\sum_{e \in \mathcal{G}} \alpha_e}, \quad (3.19)$$

where $n(\mathcal{G})$ denotes the number of vertices of $\mathcal{G}$, $e \in \mathcal{G}$ denotes the edges of $\mathcal{G}$, and $\mathcal{T}$ runs over the spanning trees in $\mathcal{G}$ (each having $n(\mathcal{G}) - 1$ edges). The tetrahedral four-point function is trivial, as it receives no radiative corrections at large N :

$$\Gamma_t^R = \tilde{\lambda}. \quad (3.20)$$

At leading order in $1/N$ the remaining four-point functions Γ_1^R and Γ_2^R are identical up to replacing $\tilde{\lambda}^2$ by $3\tilde{\lambda}^2$ and $\tilde{\lambda}_1$ by $\tilde{\lambda}_2$. We discuss Γ_1^R . Only chain diagrams [1] contribute to Γ_1^R at leading order in $1/N$. A chain diagram $\mathcal{G}$ is a sequence of irreducible pieces connected by pairs of parallel horizontal edges. The irreducible pieces are either vertical ladder rungs with two tetrahedral couplings, or bare vertices λ_1 . There are 2^n chain diagrams with n irreducible parts (that is vertical rungs or bare vertices). The edges of a chain diagram are decorated by arbitrary melonic insertions, but we do not include tadpoles, as they are assumed to have been taken care of (by dimensional regularization or mass subtraction).

The chain diagram consisting in a bare vertex has amplitude 1. We denote $n_t(\mathcal{G})$ the number of tetrahedral vertices of $\mathcal{G}$ (which is always even), $n_1(\mathcal{G})$ the numbers of vertices λ_1 of $\mathcal{G}$, and $\mathfrak{G}$ the set of connected chain diagrams with at least two internal vertices. We have:

$$\Gamma_1^R = \tilde{\lambda}_1 + \mu^\epsilon \sum_{\mathcal{G} \in \mathfrak{G}} (-1)^{1+n_1(\mathcal{G})+n_t(\mathcal{G})} \left(\mu^{-\epsilon} \tilde{\lambda} \right)^{n_t(\mathcal{G})} \left(\mu^{-\epsilon} \tilde{\lambda}_1 \right)^{n_1(\mathcal{G})} \hat{A}(\mathcal{G}). \quad (3.21)$$

Figure 5. The pure ladders U_1 , U_2 , and U_3 .

Figure 6. The caps S_1 , S_2 , and S_3 . The blue vertex represents λ_1 .

The case $\epsilon = 0$ is special. The series in (3.21) can be further simplified: all the melonic insertions can be analytically resummed at the price of multiplying the bare propagator by Z^{-1} from eq. (3.13). Furthermore, in this case the four-point function itself should be divided by Z^2 . The overall effect is that eq. (3.21) can be rewritten at $\epsilon = 0$ by dividing all the couplings by Z^2 and reducing the sum to chain diagrams with no melonic insertions.

The chain diagrams can be analyzed in terms of their one-vertex irreducible components. Adapting the notation of [1] to the case $\epsilon > 0$, we denote:

- $\hat{U}_r$ the sum of dimensionless amplitudes of the ladders with $r \geq 1$ rungs, and with melonic insertions; we include in $\hat{U}_r$ the $\tilde{\lambda}$ -dependence due to melonic insertions, but not that due to the pairs of vertices in a rung. Therefore, we write the sum over the ladders of arbitrary length as:

$$U(x) = \sum_{r \geq 1} (-1)^r x^{2r} \hat{U}_r(x), \quad (3.22)$$

with $x = \mu^{-\epsilon} \tilde{\lambda}$. Notice that unlike in [1], $U(x)$ is not the generating function of the amplitudes $\hat{U}_r(x)$, due to the x -dependence of the latter. We also define $U_r \equiv \hat{U}_r(0)$, for the amplitudes without melonic insertions (figure 5).

- $\hat{S}_r$ the sum of dimensionless amplitudes of the caps with $r \geq 1$ rungs, i.e. ladders with r rungs closed on a λ_1 vertex on one side, with melonic insertions; we write the sum over caps as:

$$S(x) = \sum_{r \geq 1} (-1)^r x^{2r} \hat{S}_r(x), \quad (3.23)$$

with $x = \mu^{-\epsilon} \tilde{\lambda}$. We also define $S_r \equiv \hat{S}_r(0)$, for the amplitudes without melonic insertions (figure 6).

- $\hat{T}_r$ the sum of dimensionless amplitudes of the double-caps with $r \geq 0$ rungs, i.e. ladders with r rungs closed on a λ_1 vertex on each side, with melonic insertions; we write the sum over double-caps as:

$$T(x) = \sum_{r \geq 0} (-1)^r x^{2r} \hat{T}_r(x), \quad (3.24)$$

with $x = \mu^{-\epsilon} \tilde{\lambda}$. We also define $T_r \equiv \hat{T}_r(0)$, for the amplitudes without melonic insertions (figure 7).

Figure 7. The caps T_0 , T_1 , T_2 , and T_3 . The blue vertices represent λ_1 .

Observe that for the caps and double-caps exactly one and two vertices, respectively, correspond to couplings λ_1 . The leading $1/\epsilon$ behavior of these amplitudes is $U_r \sim \epsilon^{-(2r-1)}$, $S_r \sim \epsilon^{-2r}$, $T_r \sim \epsilon^{-(2r+1)}$. In terms of the resummed amplitudes, the bare expansion writes [1]:

$$\mu^{-\epsilon} \Gamma_1^R = -U \left(\mu^{-\epsilon} \tilde{\lambda} \right) + \mu^{-\epsilon} \tilde{\lambda}_1 \frac{[1 + S(\mu^{-\epsilon} \tilde{\lambda})]^2}{1 + \mu^{-\epsilon} \tilde{\lambda}_1 T(\mu^{-\epsilon} \tilde{\lambda})}, \quad (3.25)$$

which exhibits an ascending series of poles in $1/\epsilon$. In the limit $\epsilon \rightarrow 0$, as discussed above, we can restrict to ladders, caps, and double-caps with no melonic insertions (that is, we can remove hats from $\hat{U}_r$, $\hat{S}_r$, and $\hat{T}_r$), provided that the couplings and the amplitude are rescaled by Z^{-2} .

Wilsonian beta functions. In the Wilsonian picture we identify the four-point functions with the running couplings:

$$\tilde{g} = \mu^{-\epsilon} \Gamma_t^R, \quad \tilde{g}_1 = \mu^{-\epsilon} \Gamma_1^R, \quad \tilde{g}_2 = \mu^{-\epsilon} \Gamma_2^R, \quad (3.26)$$

and for the theory at $\epsilon = 0$ we would further rescale the right-hand side by Z^{-2} . The beta functions are the scale derivative of the running couplings at fixed bare couplings $\tilde{\lambda}, \tilde{\lambda}_1, \tilde{\lambda}_2$. The beta functions of $\tilde{g}_1$ and $\tilde{g}_2$ decouple. In order to compute them, we invert the bare series:

$$\tilde{\lambda} = \mu^\epsilon \tilde{g}, \quad \tilde{\lambda}_1 = \mu^\epsilon \frac{\tilde{g}_1 + U(\tilde{g})}{[1 + S(\tilde{g})]^2 - [\tilde{g}_1 + U(\tilde{g})]T(\tilde{g})}. \quad (3.27)$$

We first observe that the β function of the tetrahedral coupling is trivial:

$$\beta_t(\tilde{g}) = \mu \partial_\mu \tilde{g} = -\epsilon \tilde{g}. \quad (3.28)$$

In particular β_t is identically zero at $\epsilon = 0$ indicating a line of fixed points in that case [1]. For the other couplings, the beta functions are obtained by taking $\mu \frac{d}{d\mu}$ in equation (3.27), leading to:

$$\beta_1(\tilde{g}_1, \tilde{g}) = \mu \partial_\mu \tilde{g}_1 = \epsilon \left[\tilde{g} U' - \left(\frac{2\tilde{g} S'}{1+S} + 1 \right) (\tilde{g}_1 + U) + \frac{\tilde{g} T' + T}{(1+S)^2} (\tilde{g}_1 + U)^2 \right], \quad (3.29)$$

where $U = -\tilde{g}^2 \hat{U}_1 + \dots$, $S = -\tilde{g}^2 \hat{S}_1 + \dots$ and $T = \hat{T}_0 - \tilde{g}^2 \hat{T}_1 + \dots$ and their derivatives are evaluated at the effective tetrahedral coupling $\tilde{g}$. Comparing with the results of [1], we see that the beta functions remain a quadratic polynomial in $\tilde{g}_1$ and $\tilde{g}_2$ also at $\epsilon > 0$:

$$\beta_1(\tilde{g}_1, \tilde{g}) = -\epsilon \tilde{g}_1 + \beta^{(0)}(\tilde{g}^2) - 2\beta^{(1)}(\tilde{g}^2) \tilde{g}_1 + \beta^{(2)}(\tilde{g}^2) \tilde{g}_1^2, \quad (3.30)$$

$$\beta_2(\tilde{g}_2, \tilde{g}) = -\epsilon \tilde{g}_2 + \beta^{(0)}(3\tilde{g}^2) - 2\beta^{(1)}(3\tilde{g}^2) \tilde{g}_2 + \beta^{(2)}(3\tilde{g}^2) \tilde{g}_2^2, \quad (3.31)$$

where the coefficients are:

$$\begin{aligned}\beta^{(0)}(\tilde{g}^2) &= \epsilon \left[\tilde{g}U' - U - 2\frac{U}{1+S}\tilde{g}S' + \frac{U^2}{(1+S)^2}(\tilde{g}T' + T) \right], \\ \beta^{(1)}(\tilde{g}^2) &= \epsilon \left[\frac{1}{1+S}\tilde{g}S' - \frac{U}{(1+S)^2}(\tilde{g}T' + T) \right], \quad \beta^{(2)}(\tilde{g}^2) = \epsilon \frac{\tilde{g}T' + T}{(1+S)^2}.\end{aligned}\quad (3.32)$$

Up to cubic order in the couplings (that is, two loops), and using the integrals $D \equiv T_0(0) = U_1(0)$ and $S_1(0)$ discussed in appendix A, we find:

$$\beta_1(\tilde{g}_1, \tilde{g}) = -\epsilon\tilde{g}_1 - \epsilon D\tilde{g}^2 + 2\epsilon(2S_1 - D^2)\tilde{g}^2\tilde{g}_1 + \epsilon D\tilde{g}_1^2 + O(g^4, g_1^2g^2), \quad (3.33)$$

and at first order in ϵ the coefficients $\beta^{(0)}$, $\beta^{(1)}$ and $\beta^{(2)}$ are:

$$\begin{aligned}\beta^{(0)} &= -\epsilon\tilde{g}^2D = -2\frac{\Gamma(d/4)^2}{\Gamma(d/2)}\tilde{g}^2 + O(\epsilon, g^4, g_1^2g^2), \\ \beta^{(1)} &= -\epsilon(2S_1 - D^2)\tilde{g}^2 = -\frac{2\Gamma(d/4)^4}{\Gamma(d/2)^2} \left[\psi(1) + \psi(d/2) - 2\psi(d/4) \right] \tilde{g}^2 + O(\epsilon, g^4, g_1^2g^2), \\ \beta^{(2)} &= \epsilon D = 2\frac{\Gamma(d/4)^2}{\Gamma(d/2)} + O(\epsilon, g^4, g_1^2g^2),\end{aligned}\quad (3.34)$$

which reproduce the results of [1] upon dividing all the couplings by Z^2 .

Minimal subtraction. The minimal subtraction consists in fixing a series of counterterms for each coupling, having ascending series of poles in $1/\epsilon$, with residues chosen such that the four-point functions expressed in terms of the renormalized couplings $\tilde{g}, \tilde{g}_1, \tilde{g}_2$ do not have any poles in $1/\epsilon$. The tetrahedron coupling is still trivial, as before. For the others we have:

$$\tilde{\lambda}_1 = \mu^\epsilon \left(\tilde{g}_1 + \sum_{k \geq 1} \frac{B^{(k)}(\tilde{g}_1, \tilde{g})}{\epsilon^k} \right), \quad (3.35)$$

where the $B^{(k)}(\tilde{g}_1, \tilde{g})$ are ϵ -independent. A standard manipulation [70] then leads to:

$$\beta_1(\tilde{g}_1, \tilde{g}^2) = -\epsilon\tilde{g}_1 + (\tilde{g}\partial_{\tilde{g}} + \tilde{g}_1\partial_{\tilde{g}_1} - 1)B^{(1)}(\tilde{g}_1, \tilde{g}). \quad (3.36)$$

The difference between minimal subtraction and Wilsonian scheme amounts to a mapping between renormalized couplings. In fact, in minimal subtraction we find:

$$\mu^{-\epsilon}\Gamma_1^R = -U^{(0)}(\tilde{g}) + \tilde{g}_1 \frac{[1 + S^{(0)}(\tilde{g})]^2}{1 + \tilde{g}_1 T^{(0)}(\tilde{g})} \equiv F(\tilde{g}_1, \tilde{g}), \quad (3.37)$$

where we have used:

$$U(\tilde{g}) = \sum_{k \geq 0} \frac{U^{(k)}(\tilde{g})}{\epsilon^k}, \quad (3.38)$$

and similar expansions for $S^{(0)}(\tilde{g})$ and $T^{(0)}(\tilde{g})$. The superscript here denotes the order of the pole in ϵ , and each amplitude U_r and so on has an expansion of the form (3.38). Therefore, we have:

$$\tilde{g}_1^{\text{Wilson}} = F(\tilde{g}_1^{\text{MS}}, \tilde{g}). \quad (3.39)$$

We could in principle plug this transformation in eq. (3.27), find the expansion in eq. (3.35), and write the beta function in terms of the one-vertex irreducible amplitudes. The result is rather cumbersome.

Let us instead compute directly the β functions in minimal subtraction up to two loops (order λ^3). At this order, the beta functions of marginal couplings are scheme independent up to $O(\epsilon)$ terms because $F(\tilde{g}_1, \tilde{g}) = \tilde{g}_1 + O(\tilde{g}^2)$. The bare series of Γ_1^R at cubic order is:

$$\begin{aligned} \mu^{-\epsilon} \Gamma_1^R &= \mu^{-\epsilon} \tilde{\lambda}_1 + (\mu^{-\epsilon} \tilde{\lambda})^2 D - (\mu^{-\epsilon} \tilde{\lambda}_1)^2 D - 2S_1 (\mu^{-\epsilon} \tilde{\lambda}_1) (\mu^{-\epsilon} \tilde{\lambda})^2 \\ &\quad + (\mu^{-\epsilon} \tilde{\lambda}_1)^3 D^2 + O(\tilde{\lambda}^4, \tilde{\lambda}^2 \tilde{\lambda}_1^2, \tilde{\lambda}_1^4). \end{aligned} \quad (3.40)$$

Its pole singularities arise from $D = \frac{D^{(1)}}{\epsilon} + D^{(0)} + O(\epsilon)$ and $S_1 = \frac{S_1^{(2)}}{\epsilon^2} + \frac{S_1^{(1)}}{\epsilon} + S_1^{(0)} + O(\epsilon)$, with $2S_1^{(2)} = (D^{(1)})^2$. They are cancelled by choosing:

$$\begin{aligned} \mu^{-\epsilon} \tilde{\lambda} &= \tilde{g}, \\ \mu^{-\epsilon} \tilde{\lambda}_1 &= \tilde{g}_1 - \frac{D^{(1)}}{\epsilon} \tilde{g}^2 + \frac{D^{(1)}}{\epsilon} \tilde{g}_1^2 - \left(\frac{2S_1^{(2)}}{\epsilon^2} - \frac{2S_1^{(1)} - 2D^{(1)}D^{(0)}}{\epsilon} \right) \tilde{g}^2 \tilde{g}_1 \\ &\quad + \frac{(D^{(1)})^2}{\epsilon^2} \tilde{g}_1^3 + O(\tilde{g}^4), \end{aligned} \quad (3.41)$$

which in the limit $\epsilon \rightarrow 0$ leads to:

$$F(\tilde{g}_1, \tilde{g}) = \tilde{g}_1 + D^{(0)}(\tilde{g}^2 - \tilde{g}_1^2) - 2S_1^{(0)} \tilde{g}^2 \tilde{g}_1 + D^{(0)2} \tilde{g}_1^3 + O(\tilde{g}^4, \tilde{g}^2 \tilde{g}_1^2, \tilde{g}_1^4). \quad (3.42)$$

From eq. (3.41) we can read off $B^{(1)}(\tilde{g}_1, \tilde{g})$, and plugging it into eq. (3.36), we find up to two loops:

$$\beta_t(\tilde{g}^2) = -\epsilon \tilde{g}, \quad \beta_1(\tilde{g}_1, \tilde{g}^2) = -\epsilon \tilde{g}_1 - D^{(1)} \tilde{g}^2 + D^{(1)} \tilde{g}_1^2 + 2(2S_1^{(1)} - 2D^{(1)}D^{(0)}) \tilde{g}^2 \tilde{g}_1. \quad (3.43)$$

The reader can check that these functions coincide with the Wilsonian beta functions in eq. (3.28) and (3.33), up to terms that vanish for $\epsilon \rightarrow 0$. At higher perturbative orders, even at $\epsilon = 0$, the two schemes will differ. However, also in minimal subtraction, the β functions at all orders have the form:

$$\beta_1(\tilde{g}_1, \tilde{g}) = -\epsilon \tilde{g}_1 + \bar{\beta}^{(0)}(\tilde{g}^2) - 2\bar{\beta}^{(1)}(\tilde{g}^2) \tilde{g}_1 + \bar{\beta}^{(2)}(\tilde{g}^2) \tilde{g}_1^2, \quad (3.44)$$

$$\beta_2(\tilde{g}_2, \tilde{g}) = -\epsilon \tilde{g}_2 + \bar{\beta}^{(0)}(3\tilde{g}^2) - 2\bar{\beta}^{(1)}(3\tilde{g}^2) \tilde{g}_2 + \bar{\beta}^{(2)}(3\tilde{g}^2) \tilde{g}_2^2, \quad (3.45)$$

with $\bar{\beta}^{(i)}(\tilde{g}^2)$ the minimal subtraction versions of eq. (3.32).

Fixed points and critical exponents. Without loss of generality we consider $\tilde{g} > 0$. While the fixed point below exists at all orders in perturbation theory, we will restrict to the first non trivial order. The beta functions at one loop:

$$\beta_t = -\epsilon \tilde{g}, \quad \beta_1 = -\epsilon \tilde{g}_1 + 2 \frac{\Gamma(d/4)^2}{\Gamma(d/2)} (\tilde{g}_1^2 - \tilde{g}^2), \quad \beta_2 = -\epsilon \tilde{g}_2 + 2 \frac{\Gamma(d/4)^2}{\Gamma(d/2)} (\tilde{g}_2^2 - 3\tilde{g}^2), \quad (3.46)$$

admit at $\epsilon = 0$, for any $\tilde{g} = \tilde{g}^*$, a fixed point $\tilde{g}_1^* = \tilde{g}^*, \tilde{g}_2^* = \sqrt{3}\tilde{g}^*$ which is infrared attractive:

$$\partial_{\tilde{g}_1}\beta_1|_* = 4\frac{\Gamma(d/4)^2}{\Gamma(d/2)}\tilde{g}^* > 0, \quad \partial_{\tilde{g}_2}\beta_2|_* = 4\frac{\Gamma(d/4)^2}{\Gamma(d/2)}\sqrt{3}\tilde{g}^* > 0. \quad (3.47)$$

The quartic operators acquire anomalous scaling $\Delta_a^* = 4\Delta_\phi^* + \delta h_a$ (with $a = t, 1, 2$). Here $\Delta_\phi^* = d/4$ is the dimension of the field at the fixed point. As the stability matrix $\partial_{\tilde{g}_a}\beta_b$ is triangular, $\delta h_a = \partial_{\tilde{g}_a}\beta_a|_*$ hence:

$$\delta h_t = 0, \quad \delta h_1 = 4\frac{\Gamma(d/4)^2}{\Gamma(d/2)}\tilde{g}^*, \quad \delta h_2 = 4\frac{\Gamma(d/4)^2}{\Gamma(d/2)}\sqrt{3}\tilde{g}^*. \quad (3.48)$$

We will come back to the corresponding scaling operators in section 4.2, after having discussed composite operators and their mixing.

3.2.2 Quadratic couplings

Let us consider a ϕ^2 perturbation of the critical theory. This comes to considering from the onset a mass parameter $m^{2\zeta} = m_c^{2\zeta} + \lambda_{\phi^2}$, with $m_c^{2\zeta}$ given by (3.14). With respect to our previous discussion we now need to add bi-valent vertices in the theory, corresponding to the insertion of λ_{ϕ^2} . The back reaction of λ_{ϕ^2} on the flow of g_1 and g_2 can be neglected, as the ultraviolet behavior of the massive propagator $[(p^2 + \mu^2)^\zeta + \lambda_{\phi^2}]^{-1}$ is identical to that of the massless one.

Let us consider the one-particle irreducible two-point function at zero external momentum $\Gamma_{(2)}^R(0) = G_\mu^{-1}(0)$. At one loop, only the tadpole with an insertion of a bi-valent vertex contributes:

$$\Gamma_{(2)}^R(0) = \lambda_{\phi^2} - \lambda_2 \lambda_{\phi^2} \int \frac{d^d p}{(2\pi)^d} \left(\frac{1}{(p^2 + \mu^2)^\zeta} \right)^2 = \lambda_{\phi^2} - \frac{\lambda_2 \lambda_{\phi^2}}{(4\pi)^{d/2} \Gamma(\zeta)^2} \mu^{-\epsilon} D, \quad (3.49)$$

with D the integral of appendix A. For both the Wilsonian prescription and in minimal subtraction, we get:

$$\tilde{\lambda}_{\phi^2} = \mu^{d-2\Delta_\phi} \left(\tilde{g}_{\phi^2} + \frac{D^{(1)}}{\epsilon} \tilde{g}_2 \tilde{g}_{\phi^2} \right), \quad \beta_{\phi^2} = -(d - 2\Delta_\phi) \tilde{g}_{\phi^2} + D^{(1)} \tilde{g}_{\phi^2} \tilde{g}_2, \quad (3.50)$$

which is valid for $\epsilon \geq 0$, and where $2\Delta_\phi = \frac{d-\epsilon}{2}$ is the classical dimension of the ϕ^2 operator. Observe that $\tilde{g}_{\phi^2} = 0$ is always a fixed point of this equation. For $\epsilon = 0$ (when the dimension of the field becomes $\Delta_\phi^* = d/4$), the beta function becomes:

$$\beta_{\phi^2} = -(d - 2\Delta_\phi^*) \tilde{g}_{\phi^2} + 2\frac{\Gamma(d/4)^2}{\Gamma(d/2)} \tilde{g}_{\phi^2} \tilde{g}_2, \quad (3.51)$$

and therefore, close to the fixed point $\tilde{g}_2^* = \sqrt{3}\tilde{g}^*$ the operator ϕ^2 acquires the anomalous scaling

$$\delta h_{\phi^2} = 2\frac{\Gamma(d/4)^2}{\Gamma(d/2)} \tilde{g}_2^* = 2\frac{\Gamma(d/4)^2}{\Gamma(d/2)} \sqrt{3} \tilde{g}^*, \quad \Delta_{\phi^2}^* = 2\Delta_\phi^* + \delta h_{\phi^2}, \quad (3.52)$$

reproducing the anomalous dimension found in [1] by diagonalizing the four-point kernel.⁵

The spin-zero bilinear operators of the type: $\phi(-\partial^2)^n\phi$ can be treated similarly. We present a detail computation of the corresponding beta functions in appendix B. The result again reproduces the anomalous dimensions of this type of operators derived in [1] by diagonalizing the four-point kernel.

4 Composite operators

Consider a field theory described by the action:

$$S[\phi] = S_0[\phi] + S^{\text{int}}[\phi], \quad S^{\text{int}}[\phi] = \sum_a \lambda_a \int d^d x \mathcal{O}_a(x), \quad (4.1)$$

where S_0 is some free quadratic action and the bare perturbation $S^{\text{int}}[\phi]$ is a sum over local operators $\mathcal{O}_a$ with associated bare couplings λ_a . To simplify the discussion we assume that we do not have a wave function renormalization, which is in particular the case for our model. We regulate the logarithmic ultraviolet divergences by some dimensional continuation and we assume that the power ultraviolet divergences have already been taken care of. We denote μ the infrared regulator (which could be a cutoff on the covariance) and Δ_a the canonical dimension of the operator $\mathcal{O}_a(x)$.

In order to eliminate the logarithmic ultraviolet divergences we replace the bare couplings by renormalized couplings plus counterterms:

$$\lambda_a = \mu^{d-\Delta_a} \left(g_a + \sum_{k \geq 1} \frac{B_a^{(k)}(\{g_b\})}{\epsilon^k} \right), \quad (4.2)$$

where the counterterms of λ_a can depend on all the renormalized couplings $\{g_b\}$. The counterterms can have finite parts in ϵ , but do not have if we use minimal subtraction. The renormalized action is obtained by substituting the bare couplings in terms of the renormalized ones and the infrared scale μ :

$$S[\phi] = S_0[\phi] + S^{\text{int}}[\phi], \quad S^{\text{int}}[\phi] = \sum_a \mu^{d-\Delta_a} \left(g_a + \sum_{k \geq 1} \frac{B_a^{(k)}(\{g_b\})}{\epsilon^k} \right) \int d^d x \mathcal{O}_a(x), \quad (4.3)$$

and the counterterms are chosen such that the connected correlations:

$$\langle \phi(x_1) \dots \phi(x_n) \rangle_c = \frac{1}{\int [d\phi] e^{-S(\phi)}} \int [d\phi] \phi(x_1) \dots \phi(x_n) e^{-S[\phi]} - (\text{disconnected}), \quad (4.4)$$

have no poles in $1/\epsilon$ when expressed in terms of the renormalized couplings.

⁵For the comparison, one should note that in this paper we explicitly separated the imaginary unit i from the tetrahedral coupling, see eq. (2.1). This factor was instead included in the coupling in [1].

4.1 Operator mixing

The (integrated) renormalized operator $[\mathcal{O}_a]$ is the derivative of the renormalized action with respect to the dimensionless renormalized coupling g_a :

$$\int d^d x [\mathcal{O}_a](x) \equiv \mu^{-(d-\Delta_a)} \frac{\partial S^{\text{int}}}{\partial g_a}, \quad (4.5)$$

and it is easy to check that when acting on a connected correlation the derivative with respect to g_a brings down an $[\mathcal{O}_a]$ operator:

$$-\frac{\partial}{\partial g_a} \langle \phi(x_1) \dots \phi(x_n) \rangle_c = \mu^{d-\Delta_a} \left\langle \phi(x_1) \dots \phi(x_n) \int d^d x [\mathcal{O}_a](x) \right\rangle_c. \quad (4.6)$$

As $\langle \phi(x_1) \dots \phi(x_n) \rangle_c$ is finite, i.e. it has no poles in $1/\epsilon$, its derivative is also finite, with the possible exception of some critical points.

Observe that the renormalized operator $[\mathcal{O}_a] = \mathcal{O}_a + \dots$ has the same dimension Δ_a as its bare counterpart $\mathcal{O}_a$. The bare operators are linear combinations of the renormalized ones:

$$\mathcal{O}_a = \sum_b \mu^{\Delta_a} M_{ab} \mu^{-\Delta_b} [\mathcal{O}_b], \quad [\mathcal{O}_a] = \sum_b \mu^{\Delta_a} Z_{ab} \mu^{-\Delta_b} \mathcal{O}_b, \quad (4.7)$$

where M is the mixing matrix and $Z = M^{-1}$. The matrix elements M_{ab} and Z_{ab} are dimensionless. For a free theory, that is, neglecting all the radiative corrections, the mixing matrix is $M_{ab} = \delta_{ab} = Z_{ab}$. In the interacting case M is determined by observing that, as insertions in correlation functions, we have:

$$[\mathcal{O}_a] = \mu^{-(d-\Delta_a)} \sum_b \frac{\partial \lambda_b}{\partial g_a} \mathcal{O}_b \Rightarrow Z_{ab} = \frac{\partial \lambda_b}{\partial g_a} \mu^{\Delta_b-d} = (M^{-1})_{ab}. \quad (4.8)$$

Denoting Δ the diagonal matrix with entries Δ_a , and organizing the couplings λ and the beta functions in (row) vectors, we can obtain the β functions in terms of the mixing matrix as:

$$0 = \mu \frac{d}{d\mu} \lambda = \lambda (d - \Delta) + \beta Z \mu^{d-\Delta} \Rightarrow \beta = -\lambda (d - \Delta) \mu^{\Delta-d} M. \quad (4.9)$$

Below we will deal with the energy-momentum tensor. The contribution of the interaction to the trace of the energy-momentum tensor is:

$$\sum_b \lambda_b (d - \Delta_b) \mathcal{O}_b = \lambda (d - \Delta) \mu^\Delta M \mu^{-\Delta} [\mathcal{O}] = -\beta \mu^{d-\Delta} [\mathcal{O}]. \quad (4.10)$$

4.2 Stability matrix and scaling operators

The renormalized n -point functions (4.4) of our model satisfy, in the limit $\mu \rightarrow 0$, the usual Callan-Symanzik (CS) equation with zero anomalous dimension:

$$\left(\mu \partial_\mu + \sum_a \beta_a \partial_{g_a} \right) \langle \phi(x_1) \dots \phi(x_n) \rangle_c = 0. \quad (4.11)$$

Although this is a very familiar equation, some remarks are due. In the renormalized expansion any correlation is expressed as an explicit function of μ and g with no poles in $1/\epsilon$. The left-hand side of the CS equation is just the total derivative with respect to μ of the correlation.

The right-hand side of the CS equation is exactly zero only if the fields acquire no anomalous dimension and if one uses a massless renormalization scheme in which the bare expansion is μ independent. This is *not* the case for our regularization scheme, where μ is an explicit infrared cutoff in the covariance of the bare action. When expressed in terms of bare constants, the correlations will depend on μ and the right-hand side of eq. (4.11), rather than being zero, gives

$$\langle -(\mu \partial_\mu S_0[\phi]) \phi(x_1) \dots \phi(x_n) \rangle_c = -\zeta \mu^2 \int d^d x \langle (\phi(\partial^2 + \mu^2)^{\zeta-1} \phi)(x) \phi(x_1) \dots \phi(x_n) \rangle_c. \quad (4.12)$$

This is an n -point function with a two-valent vertex insertion, and it goes to zero faster than the left-hand side of (4.11) when the infrared regulator μ descends below the scale of the external momenta of the correlation.⁶ In other words, the CS equation with zero on the right-hand side holds in any schemes in the deep infrared regime. This is due to the fact that the external momenta become the true infrared regulators and the correlation stops depending on μ . The reader will note that this is in fact what we did in section 3.1 in order to solve self-consistently for the full two point function in the case $\epsilon = 0$.

Notice that there exists a well established renormalization scheme in which the bare functions are truly μ independent: the subtraction at momentum scale μ , *à la* Gell-Mann and Low. In this case one does not introduce any infrared regulator in the bare action, but notices that the correlation functions are infrared convergent provided that the external momenta are non exceptional (i.e. no subset of external momenta adds up to zero). One eliminates the poles in $1/\epsilon$ by using a renormalization condition at scale μ , such as $\Gamma^{(4)}(p)|_{p=\bar{p}} = \mu^\epsilon g$, with $\bar{p}_i \cdot \bar{p}_j = \frac{\mu^2}{4}(4\delta_{ij} - 1)$, and solving for the bare coupling $\lambda = \mu^\epsilon(g + \dots)$. As the β functions at two loops are scheme independent, one can safely use the CS equation (4.11) up to this order in any scheme.

We now use the CS equation as stated in eq. (4.11) to study the scaling operators close to an infrared fixed point. Acting on the CS equation with a derivative $-\partial_{g_b}$, we obtain:

$$0 = \sum_a \left[\delta_{ab} \mu \partial_\mu + (\partial_{g_b} \beta_a) + \beta_a \partial_{g_b} \right] \mu^{d-\Delta_a} \int d^d x \langle \phi(x_1) \dots \phi(x_n) [\mathcal{O}_a](x) \rangle_c. \quad (4.13)$$

At a fixed point we have $\beta_a|_\star = 0$ hence the third term drops out. Defining the stability matrix $Y_{ba} = \partial_{g_b} \beta_a|_\star$, the CS equation becomes, in matrix notation (arranging the operators $[\mathcal{O}]$ in a column vector) and dropping the integral over x :

$$0 = \left[\mu \partial_\mu + d + Y \right] \mu^{-\Delta} \langle [\mathcal{O}](x) \phi(x_1) \dots \phi(x_n) \rangle_c. \quad (4.14)$$

⁶Such situation is similar to that concerning the CS equation for massive theories, with μ replaced by the physical mass (see for example ref. [71]).

Assuming that the stability matrix is diagonalizable, $Y = U\nu U^{-1}$, with ν being by definition the (diagonal matrix of) critical exponents, we find:

$$0 = (\mu\partial_\mu + d + \nu)U^{-1}\mu^{-\Delta}[\mathcal{O}](x), \quad (4.15)$$

in the sense of operators inserted in correlation functions. This allows us to identify scaling operators at the fixed point, that is the operators which satisfy:

$$\mu \frac{d}{d\mu} \left[\{\mathcal{O}_b\}(x) \right] = 0. \quad (4.16)$$

In matrix notation, they write:

$$\{\mathcal{O}\}(x) = \mu^{d+\nu} U^{-1} \mu^{-\Delta} [\mathcal{O}](x) = \mu^{d+\nu} U^{-1} Z \mu^{-\Delta} \mathcal{O}(x) \equiv \mu^{\Delta^*} \{\mathcal{O}\}'(\mu x), \quad (4.17)$$

where the (inverse) mixing matrix Z is evaluated at the fixed point, and we denoted by prime the dimensionless version of the operator. The scaling operators are the eigenvectors of the dilatation, with eigenvalue $\Delta_b^* = d + \nu_b$, which is the scaling dimension of $\{\mathcal{O}_b\}$. We write the fixed-point scaling dimensions in terms of canonical and anomalous dimensions as $\Delta_b^* = \Delta_b + \delta h_b$. In the case of our quartic operators we have $\Delta_b = d$ at $\epsilon = 0$, hence the critical exponent and the anomalous dimension coincide.

The stability matrix can be computed further from eq. (4.8) and (4.9). In matrix notation we get:

$$Y = \partial_g \beta = -Z \mu^{d-\Delta} (d - \Delta) \mu^{\Delta-d} Z^{-1} + \lambda (d - \Delta) \mu^{\Delta-d} Z^{-1} (\partial_g Z) Z^{-1}. \quad (4.18)$$

Observing that, as Z does not depend explicitly on μ , the matrix of anomalous dimensions can be written as:

$$\gamma = -\mu \left(\frac{d}{d\mu} Z \right) Z^{-1} = -\beta_a (\partial_{g_a} Z) Z^{-1}, \quad (4.19)$$

and using $\partial_{g_a} Z_{bc} = \partial_{g_b} Z_{ac}$, we conclude that:

$$d + Y = Z \Delta Z^{-1} + \gamma. \quad (4.20)$$

4.3 Action, mixing, and scaling operators at first order

Gathering the first order results in eq. (3.41) and (3.50), the minimally-subtracted action, with bare mass $m^{2\zeta} = m_c^{2\zeta} + \lambda_{\phi^2}$ and $m_c^{2\zeta}$ given by (3.14), and up to quadratic order in the couplings, is:

$$\begin{aligned} S = & \frac{1}{2} \int d^d x d^d y \phi_{\mathbf{a}}(x) C_\mu^{-1}(x, y) \phi_{\mathbf{a}}(y) + \frac{1}{2} \left(\lambda_{\phi^2} - \lambda^2 \int d^d u C_\mu(u)^3 \right) \int d^d x \phi^2(x) \\ & + \frac{1}{4} \int d^d x \left[\lambda \phi_t^4(x) + \lambda_1 \phi_1^4(x) + \lambda_2 \left(\phi_2^4(x) - 2C_\mu(0) \phi^2(x) \right) \right], \end{aligned} \quad (4.21)$$

where $\Delta_\phi = \frac{d-\epsilon}{4}$ and the bare couplings at first order are:

$$\lambda_{\phi^2} = \mu^{d-2\Delta_\phi} g_{\phi^2} \left(1 + Q \frac{g_2}{\epsilon} \right), \quad \lambda = \mu^{d-4\Delta_\phi} g, \quad (4.22)$$

$$\lambda_1 = \mu^{d-4\Delta_\phi} \left(g_1 - Q \frac{g^2}{\epsilon} + Q \frac{g_1^2}{\epsilon} \right), \quad \lambda_2 = \mu^{d-4\Delta_\phi} \left(g_2 - 3Q \frac{g^2}{\epsilon} + Q \frac{g_2^2}{\epsilon} \right),$$

with:

$$Q \equiv \frac{2}{(4\pi)^{d/2} \Gamma(d/2)}. \quad (4.23)$$

Observe that in the second line of eq. (4.21) we can recognize the Wick ordered double-trace interaction at leading order in N and up to a vacuum term⁷: $\phi_2^4 : \equiv \phi_2^4(x) - 2C_\mu(0) \phi^2(x)$. At this order in $1/N$ it is superfluous to Wick order the other quartic interactions.

Contrary to the previous discussion, we have explicitly taken into account the counterterms necessary for eliminating the mass (power law) ultraviolet divergence⁸. The renormalized operators are computed by taking the derivatives of the renormalized action with respect to the renormalized couplings:

$$\begin{aligned} [\phi^2](x) &= Z_{\phi^2} \phi^2(x), & [\phi_1^4](x) &= Z_{\phi_1^4} \phi_1^4(x), \\ [\phi_2^4](x) &= Z_{\phi_2^4} : \phi_2^4(x) : + Z_{\phi_2^4; \phi^2} \phi^2(x), \\ [\phi_t^4](x) &= \phi_t^4(x) + Z_{\phi_t^4; \phi_1^4} \phi_1^4(x) + Z_{\phi_t^4; \phi_2^4} : \phi_2^4(x) : + Z_{\phi_t^4; \phi^2} \phi^2(x), \end{aligned} \quad (4.24)$$

where, contrary to the previous section, we included the dimensions directly in the (inverse) mixing matrix elements:

$$\begin{aligned} Z_{\phi^2} &= 1 + Q \frac{g_2}{\epsilon}, & Z_{\phi_1^4} &= 1 + 2Q \frac{g_1}{\epsilon}, \\ Z_{\phi_2^4} &= 1 + 2Q \frac{g_2}{\epsilon}, & Z_{\phi_2^4; \phi^2} &= \mu^{2\Delta_\phi} 2Q \frac{g \phi^2}{\epsilon}, \\ Z_{\phi_t^4; \phi_1^4} &= -2Q \frac{g}{\epsilon}, & Z_{\phi_t^4; \phi_2^4} &= -6Q \frac{g}{\epsilon}, \\ Z_{\phi_t^4; \phi^2} &= -\mu^{d-4\Delta_\phi} 4g \int d^d u C(u)^3. \end{aligned} \quad (4.25)$$

Going back to our beta functions (3.46) and (3.51), we find the following diagonalizing transformation for the stability matrix, corresponding to a vector ordered as $\{g_{\phi^2}, g, g_1, g_2\}$:

$$U^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \sqrt{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (4.26)$$

and using (4.17) and (4.24), the scaling operators are:

$$\{\phi^2\}(x) = \mu^{\delta h_{\phi^2}} Z_{\phi^2} \phi^2(x),$$

⁷The standard definition of Wick ordering [72] gives:

$$\begin{aligned} : \phi_2^4(x) : &:= \exp \left\{ -\frac{1}{2} \int d^d x d^d y \partial_{\phi_{\mathbf{a}}(x)} C_\mu(x, y) \partial_{\phi_{\mathbf{a}}(y)} \right\} \phi_2^4(x) \\ &= \phi_2^4(x) - 2 \left(1 + \frac{2}{N^3} \right) \phi^2(x) C_\mu(0) + 2(N^3 + 2) C_\mu(0)^2. \end{aligned}$$

⁸Using $C_\mu(x) = \mu^{2\Delta_\phi} C_1(\mu x)$, we have $C_\mu(0) = \mu^{2\Delta_\phi} C_1(0)$, $\int d^d u C_\mu(u)^3 = \mu^{6\Delta_\phi - d} \int d^d u' C_1(u')$.

$$\begin{aligned}
 \{\phi_t^4\}(x) &= \phi_t^4(x) + \left(Z_{\phi_1^4} + Z_{\phi_t^4; \phi_1^4}\right) \phi_1^4(x) \\
 &\quad + \left(\sqrt{3}Z_{\phi_2^4} + Z_{\phi_t^4; \phi_2^4}\right) : \phi_2^4(x) : + \left(\sqrt{3}Z_{\phi_2^4; \phi^2} + Z_{\phi_t^4; \phi^2}\right) \phi^2(x), \\
 \{\phi_1^4\}(x) &= \mu^{\delta h_1} Z_{\phi_1^4} \phi_1^4(x), \\
 \{\phi_2^4\}(x) &= \mu^{\delta h_2} Z_{\phi_2^4} : \phi_2^4(x) : .
 \end{aligned} \tag{4.27}$$

5 $D = d + p$ dimensional embedding

In this section, we give a proof of the conformal symmetry of the model (2.1) at the infrared fixed point, following the discussion of ref. [52]. The main idea is that the model (2.1) can be written with a standard short-range kinetic term by embedding it in $D = d + p$ (where $p = 2 - 2\zeta$) dimensional space. In this enlarged space, the action becomes:

$$\begin{aligned}
 S[\Phi] &= \frac{1}{2} \int d^D X \left(\partial_M \Phi_{\mathbf{a}}(X) \right)^2 \\
 &\quad + \frac{1}{4} \int_{y=0} d^d x \left[i \lambda \hat{\delta}_{\mathbf{abcd}}^t + \lambda_1 \hat{P}_{\mathbf{ab;cd}}^{(1)} + \lambda_2 \hat{P}_{\mathbf{ab;cd}}^{(2)} \right] \Phi_{\mathbf{a}}(x) \Phi_{\mathbf{b}}(x) \Phi_{\mathbf{c}}(x) \Phi_{\mathbf{d}}(x), \quad (5.1)
 \end{aligned}$$

where the D -dimensional coordinates are labeled by $X^M = (x^\mu, y^m)$ and the original field is obtained by $\Phi|_{y \rightarrow 0} = \phi$. In this D -dimensional space, one can write down a local energy-momentum tensor:

$$\begin{aligned}
 T_{MN} &= \sum_{\mathbf{a}} \left[\partial_M \Phi_{\mathbf{a}} \partial_N \Phi_{\mathbf{a}} - \frac{1}{2} \delta_{MN} (\partial_K \Phi_{\mathbf{a}})^2 \right] - \frac{i \lambda}{4} \delta_{MN}^{\parallel} \delta^p(y) \sum_{\mathbf{a,b,c,d}} \hat{\delta}_{\mathbf{abcd}}^t \Phi_{\mathbf{a}} \Phi_{\mathbf{b}} \Phi_{\mathbf{c}} \Phi_{\mathbf{d}} \\
 &\quad - \frac{\delta_{MN}^{\parallel} \delta^p(y)}{4} \sum_{\mathbf{a,b,c,d}} \left(\lambda_1 \hat{P}_{\mathbf{ab;cd}}^{(1)} + \lambda_2 \hat{P}_{\mathbf{ab;cd}}^{(2)} \right) \Phi_{\mathbf{a}} \Phi_{\mathbf{b}} \Phi_{\mathbf{c}} \Phi_{\mathbf{d}}, \quad (5.2)
 \end{aligned}$$

where $\delta_{MN}^{\parallel} = \delta_{\mu\nu}$ if both indices are in the d -dimensional space, and zero otherwise. We also introduce the orthogonal projector $\delta_{MN}^{\perp} = \delta_{MN} - \delta_{MN}^{\parallel}$. Now we can write the divergence and trace of the energy-momentum tensor as:

$$\partial^M T_{MN} = -E_N + \delta^p(y) \delta_{MN}^{\perp} D^M, \tag{5.3}$$

$$\begin{aligned}
 T^M_M &= -\Delta_{\phi} E + \left(\frac{1}{2} - \frac{D}{4} \right) \sum_{\mathbf{a}} \partial_K^2 \Phi_{\mathbf{a}}^2 \\
 &\quad - \frac{\epsilon \delta^p(y)}{4} \sum_{\mathbf{a,b,c,d}} \left(i \lambda \hat{\delta}_{\mathbf{abcd}}^t + \lambda_1 \hat{P}_{\mathbf{ab;cd}}^{(1)} + \lambda_2 \hat{P}_{\mathbf{ab;cd}}^{(2)} \right) \Phi_{\mathbf{a}} \Phi_{\mathbf{b}} \Phi_{\mathbf{c}} \Phi_{\mathbf{d}}, \quad (5.4)
 \end{aligned}$$

with

$$E = \sum_{\mathbf{a}} \Phi_{\mathbf{a}} \left[-\partial_K^2 \Phi_{\mathbf{a}} + \delta^p(y) \sum_{\mathbf{b,c,d}} \left(i \lambda \hat{\delta}_{\mathbf{abcd}}^t + \lambda_1 \hat{P}_{\mathbf{ab;cd}}^{(1)} + \lambda_2 \hat{P}_{\mathbf{ab;cd}}^{(2)} \right) \Phi_{\mathbf{b}} \Phi_{\mathbf{c}} \Phi_{\mathbf{d}} \right], \tag{5.5}$$

$$E_N = \sum_{\mathbf{a}} \partial_N \Phi_{\mathbf{a}} \left[-\partial_K^2 \Phi_{\mathbf{a}} + \delta^p(y) \sum_{\mathbf{b,c,d}} \left(i \lambda \hat{\delta}_{\mathbf{abcd}}^t + \lambda_1 \hat{P}_{\mathbf{ab;cd}}^{(1)} + \lambda_2 \hat{P}_{\mathbf{ab;cd}}^{(2)} \right) \Phi_{\mathbf{b}} \Phi_{\mathbf{c}} \Phi_{\mathbf{d}} \right], \tag{5.6}$$

$$D^N = \sum_{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}} \left(i \lambda \hat{\delta}_{\mathbf{abcd}}^t + \lambda_1 \hat{P}_{\mathbf{ab}; \mathbf{cd}}^{(1)} + \lambda_2 \hat{P}_{\mathbf{ab}; \mathbf{cd}}^{(2)} \right) (\partial_N \Phi_{\mathbf{a}}) \Phi_{\mathbf{b}} \Phi_{\mathbf{c}} \Phi_{\mathbf{d}}. \quad (5.7)$$

We note that E and E_N are proportional to the equation of motion, so they vanish on-shell. Therefore, on-shell the trace of the energy momentum tensor, eq. (5.4) is equal to a double divergence, $\partial_K^2 \phi_{\mathbf{a}}^2$, plus a term proportional to ϵ , and thus the theory is classically conformal invariant at $\epsilon = 0$. However, due to radiative corrections, the $\Phi^4|_{y=0} = \phi^4$ operators lead to infinities, hence they need to be renormalized as we discussed in the previous section. Using eq. 4.10 we arrive at the following expression for the trace of the energy-momentum tensor:

$$T^M_M = -\Delta_\phi E + \left(\frac{1}{2} - \frac{D}{4} \right) \sum_{\mathbf{a}} \partial_K^2 \Phi_{\mathbf{a}}^2 + \frac{\mu^\epsilon \delta^p(y)}{4} \left[\beta_t(g) [\phi_t^4](x) + \beta_1(g, g_1) [\phi_1^4](x) + \beta_2(g, g_2) [\phi_2^4](x) \right], \quad (5.8)$$

where we observe that in terms of the renormalized operators, the last terms survive at $\epsilon = 0$, leading in general to a breaking of conformal invariance. In order to simplify the notation we have omitted the bare mass term. We can include this term (restricted to $y = 0$) in eq. (5.1) and tune the renormalized mass to zero. The trace of the energy momentum tensor still writes in terms of beta functions using eq. (4.10). To keep things simple, we will continue ignoring this mass term and its mixing with the quartic operators.

The dilatation and special conformal transformation currents are constructed from the energy-momentum tensor as:

$$\mathcal{D}_M = T_{MN} X^N, \quad \mathcal{C}_M^N = T_{MK} (2X^K X^N - \delta^{KN} X^2). \quad (5.9)$$

Their divergences are found as:

$$\partial^M \mathcal{D}_M = -X^M E_M + T^M_M, \quad (5.10)$$

$$\partial^M \mathcal{C}_M^N = -(2X^K X^N - \delta^{KN} X^2) (E_K - \delta^p(y) \delta_{MK}^\perp D^M) + 2X^N T^M_M. \quad (5.11)$$

Inserting these operators in a renormalized n -point function and integrating over the insertion point, up to a possible boundary term we obtain:

$$\int d^D X \left\langle \left(-X^M E_M(X) + T^M_M(X) \right) \Phi(X_1) \cdots \Phi(X_n) \right\rangle = 0, \quad (5.12)$$

$$\int d^D X \left\langle \left[(2X^K X^N - \delta^{KN} X^2) (E_K(X) - \delta^p(y) \delta_{MK}^\perp D^M(X)) - 2X^N T^M_M \right] \Phi(X_1) \cdots \Phi(X_n) \right\rangle = 0. \quad (5.13)$$

The Schwinger-Dyson equations involving E and E_M are obtained from the path integral expression for the n -point function by the field redefinitions $\Phi \rightarrow \Phi(1 + \delta\Phi)$ and $\Phi \rightarrow \Phi + (\partial_M \Phi) \delta\Phi$, respectively. The result is [56]:

$$\langle E(X) \Phi(X_1) \cdots \Phi(X_n) \rangle = \sum_{i=1}^n \delta(X - X_i) \langle \Phi(X_1) \cdots \Phi(X_n) \rangle, \quad (5.14)$$

$$\langle E_M(X) \Phi(X_1) \cdots \Phi(X_n) \rangle = \sum_{i=1}^n \delta(X - X_i) \frac{\partial}{\partial X_i^M} \langle \Phi(X_1) \cdots \Phi(X_n) \rangle. \quad (5.15)$$

Using these equations, together with the expression of the trace of the energy-momentum tensor (5.8), we obtain the Ward identities for the dilatation and special conformal transformation currents. Since the n -point function behaves continuously in the limit $y \rightarrow 0$ [52], we can write down the Ward identities in the original d -dimensional space as:

$$\begin{aligned} & \sum_{i=1}^n \left[x_i \cdot \partial_{x_i} + \Delta_\phi \right] \langle \phi(x_1) \cdots \phi(x_n) \rangle \\ &= \frac{\mu^\epsilon}{4} \int d^d x \left\langle \left[\beta_t(g) [\phi_t^4](x) + \beta_1(g, g_1) [\phi_1^4](x) + \beta_2(g, g_2) [\phi_2^4](x) \right] \phi(x_1) \cdots \phi(x_n) \right\rangle, \end{aligned} \quad (5.16)$$

and

$$\begin{aligned} & \sum_{i=1}^n \left[(2x_i^\mu x_i^\nu - \delta^{\mu\nu} x_i^2) \frac{\partial}{\partial x_i^\mu} + 2\Delta_\phi x_i^\nu \right] \langle \phi(x_1) \cdots \phi(x_n) \rangle \\ &= \frac{\mu^\epsilon}{2} \int d^d x x^\nu \left\langle \left[\beta_t(g) [\phi_t^4](x) + \beta_1(g, g_1) [\phi_1^4](x) + \beta_2(g, g_2) [\phi_2^4](x) \right] \phi(x_1) \cdots \phi(x_n) \right\rangle. \end{aligned} \quad (5.17)$$

These Ward identities are well defined even at $\epsilon = 0$ and the right-hand sides survives in this limit. We thus get in general a breaking of scale and special conformal invariance due to renormalization. At a fixed point, as $\beta_t(g) = \beta_1(g, g_1) = \beta_2(g, g_2) = 0$, the invariance is restored. In fact, one should check that the integral on the right-hand side of the Ward identities does not blow up when we approach the fixed point as the inverse of the beta functions or worse. This subtlety is discussed in detail in ref. [52] and that discussion goes through in our case.⁹ The main point of repeating the derivation of the Ward identities here is to check that the structure of right-hand side (beta function times operator insertion) generalizes to our model with multiple interaction terms.

We conclude that the n -point functions of fundamental fields are conformal invariant at the fixed point. The next step would be to generalize the above result to correlators of composite operators. A possible strategy, starting from correlators of fundamental fields and using an operator-product expansion to generate the composite fields, is sketched in ref. [52]. In the next section we will follow a different route, and compute perturbatively three-point functions of (quartic and quadratic) composite operators.

⁹In our case, we can actually understand the absence of such singularity on the right-hand side of the scale Ward identity (5.16) thanks to the presence of a small parameter, the tetrahedron coupling g . First, we can express the right-hand side of (5.16) by means of (4.6), thus reducing the scale Ward identity to a Callan-Symanzik equation. Then, for infinitesimal g , we notice that n -point functions near the fixed point are essentially polynomials in g , hence their derivative at the fixed point is finite. A singularity might instead arise at finite g , and in particular at the critical value determined by the invertibility of the relation $g = \lambda Z(\lambda)^{-2}$, with $Z(\lambda)$ defined by the solution of eq. (3.13) (see ref. [1]).

Figure 8. Would-be leading order diagram for $\langle \{\phi_2^4\} \{\phi_2^4\} \rangle$.

6 Correlation functions

In this section we will explicitly compute large- N three-point functions among the ϕ^4 and ϕ^2 composite operators we discussed in the previous sections at the interacting IR fixed point. Since this fixed point depends parametrically on the exactly marginal coupling g , we will work at small g , at lowest perturbative order in the couplings.

Large- N and conformal limits. Before proceeding, we stress one important aspect about the interplay between large- N limit and conformal limit. Namely, it should be kept in mind that in principle we should take the conformal limit (i.e. tune the theory to the fixed point) before or at least together with the large- N limit, in order to keep non-trivial two-point functions in the limit. For example, if we were to take the large- N limit first, away from criticality, the two-point function of double-trace operators would be dominated by the diagram in figure 8, rather than that of figure 11. Away from criticality, such diagram leads to a finite contribution (after renormalization) determining the two-point function at leading order of the large- N expansion. However, at criticality its renormalized amplitude vanishes, and we would be left with a zero two-point function. The way out is to assume from the beginning that diagrams such as that in figure 8 are zero because we are in the massless limit, and to rescale the operators in such a way that the first non-vanishing contribution to a two-point function at criticality is always normalized to order N^0 . Making sure that such a scaling exists in the case of tensor models is non-trivial, and it is what we want to address now.

Let us first review how correlators of invariants are treated in the case of matrix field theories. The analogue of our interacting action (2.1) is:

$$S^{\text{int}}[\phi] = \int d^d x \left(\frac{m^{2\zeta}}{2} \text{Tr}[\phi(x)^2] + \frac{\lambda_s}{4N} \text{Tr}[\phi(x)^4] + \frac{\lambda_d}{4N^2} (\text{Tr}[\phi(x)^2])^2 \right). \quad (6.1)$$

It is convenient to rescale $\phi \rightarrow \sqrt{N}\phi$, so that the action becomes:

$$S[\phi] = N \int d^d x \left(\frac{1}{2} \phi_{ab}(x) (-\partial^2)^\zeta \phi_{ba}(x) + \frac{m^{2\zeta}}{2} \text{Tr}[\phi(x)^2] + \frac{\lambda_s}{4} \text{Tr}[\phi(x)^4] + \frac{\lambda_d}{4N} (\text{Tr}[\phi(x)^2])^2 \right). \quad (6.2)$$

In such rescaled variables, the functional:

$$W[J] = \frac{1}{N^2} \log \int [d\phi] e^{-S[\phi] + \sum_{t \geq 1} N^{2-t} \sum_i \int J_i^{(t)} \mathcal{O}_i^{(t)}}, \quad (6.3)$$

admits a large- N limit, in the perturbative sense. Here, $\mathcal{O}_i^{(t)}$ stands for an invariant built from t traces of products of ϕ fields and contains *no explicit factors* of N . The functional

derivatives of $W[J]$ with respect to the sources $J_i^{(t)}$ are proportional to the connected correlators of $\mathcal{O}_i^{(t)}$:

$$\frac{\delta^n W}{\delta J_{i_1}^{(t_1)}(x_1) \cdots \delta J_{i_n}^{(t_n)}(x_n)} = N^{\sum_j (2-t_j)} \left\langle \mathcal{O}_1^{(t_1)}(x_1) \cdots \mathcal{O}_1^{(t_n)}(x_n) \right\rangle_{\text{connected}}, \quad (6.4)$$

and have an expansion in $1/N$ starting at order N^0 . Restricting to single-trace operators, the leading order graphs are connected planar ribbon graphs (or 3-colored graphs in our notation). Including multi-trace operators, the leading order graphs are nodal surfaces, i.e. trees of bubbles (or cacti) diagrams with bubbles made of planar graphs. Disconnected correlators, obtained from $Z[J] = e^{N^2 W[J]}$, have leading-order contributions of order N^{2c} , with c the number of connected components.

In applications to conformal field theories it is customary to consider instead correlators of $\mathcal{O}_i^{(t)}$, without explicit factors of N . They are obtained by rescaling appropriately the derivatives of W , or equivalently, by taking derivatives with respect to $\tilde{J}_i^{(t)} = N^{2-t} J_i^{(t)}$ of the functional $\tilde{W}[\{\tilde{J}_i^{(t)}\}] = N^2 W[\{N^{t-2} \tilde{J}_i^{(t)}\}]$. In this normalization, the connected two-point functions of single-trace operators at leading-order are of order N^0 , while their higher-point functions are suppressed in $1/N$. For multi-trace operators, a subtlety rarely emphasized appears, namely, their leading-order two-point functions naively seem to be of higher order than N^0 , and higher-point functions seem to have even higher powers of N . In fact, we have:

$$\frac{\delta^n \tilde{W}}{\delta \tilde{J}_{i_1}^{(t_1)}(x_1) \cdots \delta \tilde{J}_{i_n}^{(t_n)}(x_n)} = N^{2+\sum_j (t_j-2)} \frac{\delta^n W}{\delta J_{i_1}^{(t_1)}(x_1) \cdots \delta J_{i_n}^{(t_n)}(x_n)}, \quad (6.5)$$

with a leading order contribution scaling as $N^{2+\sum_j (t_j-2)}$. For $n = 2$ and $t_j = 1$, we get a correlator of order N^0 , as anticipated. On the other hand, if for example all the t_i are greater than two, we naively seem to get an arbitrarily large power of N . However, the (naive) leading-order contributions for such multi-trace operators come from cactus diagrams, as the one of figure 8, which always contain a factor corresponding to the one-point function of a single-trace component of the multi-trace invariant (the end leaves of the cactus). In the conformal limit such contributions vanish, and a factorization into two-point functions of single-trace components dominates.¹⁰ Notice that since two-point functions of single-trace operators are normalized to one, by factorization also the two-point-functions of multi-trace operators are normalized to one. Summarizing, one is typically interested in:

$$\tilde{W}[\tilde{J}] = \log \int [d\phi] e^{-S_{\text{CFT}}[\phi] + \sum_{t \geq 1} \sum_i \int \tilde{J}_i^{(t)} \mathcal{O}_i^{(t)}} \quad (6.6)$$

¹⁰The factorization property is often stated in a slightly different fashion in the literature [73]. Namely, considering only single trace operators, one observes that

$$\left\langle \mathcal{O}_1^{(1)}(x_1) \cdots \mathcal{O}_n^{(1)}(x_n) \right\rangle_{\text{conn.}} = \sum_{g \geq 0} N^{2-2g-n} F_g(\lambda),$$

from which it is deduced that higher-point functions are suppressed with respect to two-point functions. However, as one-point functions are more favored than two-point functions, in order to have a factorization in two-point functions it must be assumed that one-point functions are zero, as guaranteed by scale invariance.

which gives two-point functions normalized to one, and higher-point functions suppressed in $1/N$.

We now go back to our tensor field theory. In analogy to the matrix case, we rescale the field as:

$$\phi_{\mathbf{a}} \rightarrow N^{3/4} \phi_{\mathbf{a}}, \quad (6.7)$$

so that the action becomes:

$$\begin{aligned} S[\phi] &= N^{3/2} \frac{1}{2} \int d^d x \phi_{\mathbf{a}}(x) (-\partial^2)^\zeta \phi_{\mathbf{a}}(x) + S^{\text{int}}[\phi], \\ S^{\text{int}}[\phi] &= N^{3/2} \left[\frac{m^{2\zeta}}{2} \int d^d x \phi_{\mathbf{a}}(x) \phi_{\mathbf{a}}(x) \right. \\ &\quad \left. + \frac{1}{4} \int d^d x \left(i \lambda \delta_{\mathbf{abcd}}^t + \lambda_p N^{-1/2} \delta_{\mathbf{ab};\mathbf{cd}}^p + \lambda_d N^{-3/2} \delta_{\mathbf{ab};\mathbf{cd}}^d \right) \right. \\ &\quad \left. \times \phi_{\mathbf{a}}(x) \phi_{\mathbf{b}}(x) \phi_{\mathbf{c}}(x) \phi_{\mathbf{d}}(x) \right], \end{aligned} \quad (6.8)$$

where we reinstated the pillow and double trace couplings. Observe that the operators δ^t, δ^p and δ^d , defined in eq. (2.4), merely contract indices and do not contain explicit factors of N .

According to appendix C, the generating functional of connected correlators admitting a large- N limit in the perturbative sense is now:

$$W[J] = \frac{1}{N^3} \log \int [d\phi] e^{-S[\phi] + \sum_b N^{\frac{3}{2}-\rho_b} \int J_b \mathcal{O}_b}, \quad (6.9)$$

with $\rho_b \geq 0$ chosen according to the optimal scaling defined in ref. [44]:

$$\rho_b = \frac{F_b - 3}{2}, \quad (6.10)$$

where F_b counts the total number of cycles of alternating colors i and j with $i, j \in \{1, 2, 3\}$. Invariants with $\rho_b = 0$ are called *maximally single-trace* (MST) [9].

Defining again correlators as derivatives with respect to non-rescaled sources $\tilde{J}_b = N^{\frac{3}{2}-\rho_b} J_b$ of the functional $\tilde{W}[\{\tilde{J}_b\}] = N^3 W[\{N^{\rho_b-\frac{3}{2}} \tilde{J}_b\}]$, we have the following expansion:

$$\frac{\delta^n \tilde{W}}{\delta \tilde{J}_{b_1}(x_1) \cdots \delta \tilde{J}_{b_n}(x_n)} = \sum_{\omega \geq 0} N^{3-\omega-\sum_i (\frac{3}{2}-\rho_{b_i})} F_\omega(\lambda). \quad (6.11)$$

For MST operators we thus have an analogous result as for single-trace matrix operators, that is, their two-point functions are of order one, and higher-point functions are suppressed. Assuming again that one-point functions are zero, for operators which are products of MST operators, which we will call *maximally multi-trace* (MMT), we obtain a factorization property as for the matrix multi-trace operators (see appendix C for more details).

For non-MMT operators with $\rho_b > 0$ we seem to have again two-point functions with a higher scaling than N^0 , and for those with $\rho_b > 3/2$ we seem to have again a possible

arbitrarily large power of N . However, we conjecture that, as for multi-trace matrix operators, the leading order diagrams (those with $\omega = 0$) vanish in the conformal case, and the first non-vanishing order has $\omega \geq 3 - \sum_i (\frac{3}{2} - \rho_{b_i})$. We hope to return to this conjecture in full generality in a future publication. For the pillow invariant which is contained in ϕ_1^4 , and which has $\rho_p = 1/2$, we can explicitly check that for example the two-point function at leading order is of order N^0 (see appendix D).

Now we proceed to the perturbative computation of the two and three-point connected correlations of the MST operators ϕ_t^4 and ϕ^2 . The correlations involving the double trace ϕ_2^4 are obtained at leading order using the large N factorization. We do not study here the correlations involving ϕ_1^4 , which due to the pillow operator is neither MST nor MMT, but we will briefly discuss them in appendix D. As explained above, a connected correlation function of n MST operators has an a priori scaling $N^{3-\frac{3}{2}n}$, that is N^0 for two-point correlations and $N^{-3/2}$ for the three-point ones. Whenever a correlation scales less than that, we consider it suppressed in the large N limit.

Rescaled action and renormalized operators. In order to regularize the UV divergences we set $\epsilon > 0$. We compute the correlations up to first order in the couplings in the critical theory, $\lambda_{\phi^2} = 0$. At the relevant order the action in eq. (4.21) becomes:

$$S = N^{3/2} \int \left[\frac{1}{2} \phi_{\mathbf{a}} C_{\mu}^{-1} \phi_{\mathbf{a}} + \frac{\lambda}{4} \phi_t^4 + \frac{\lambda_1}{4N^{1/2}} \phi_1^4 + \frac{\lambda_2}{4N^{3/2}} : \phi_2^4 : \right], \quad (6.12)$$

where the operators are defined as before, except for the factors of N due to the rescaling (6.7) (remember the distinction between hatted and un-hatted deltas in eq. 2.3):

$$\begin{aligned} \phi_t^4(x) &\equiv i \delta_{\mathbf{abcd}}^t \phi_{\mathbf{a}}(x) \phi_{\mathbf{b}}(x) \phi_{\mathbf{c}}(x) \phi_{\mathbf{d}}(x), \\ \phi_1^4(x) &\equiv 3 \left(\delta_{\mathbf{ab};\mathbf{cd}}^p - \frac{1}{N} \delta_{\mathbf{ab};\mathbf{cd}}^d \right) \phi_{\mathbf{a}}(x) \phi_{\mathbf{b}}(x) \phi_{\mathbf{c}}(x) \phi_{\mathbf{d}}(x), \\ \phi_2^4(x) &\equiv \delta_{\mathbf{ab};\mathbf{cd}}^d \phi_{\mathbf{a}}(x) \phi_{\mathbf{b}}(x) \phi_{\mathbf{c}}(x) \phi_{\mathbf{d}}(x), \end{aligned} \quad (6.13)$$

where, in spite of the new scaling in N we maintain the same notation for the quartic invariants.

At this order we have $\lambda = \mu^{d-4\Delta_\phi} g$, $\lambda_1 = \mu^{d-4\Delta_\phi} g_1$, $\lambda_2 = \mu^{d-4\Delta_\phi} g_2$ with g, g_1, g_2 the dimensionless couplings. Below we will use λ 's for the perturbative expressions at $\epsilon > 0$, and g 's for the final expressions in dimensionless variables. At linear order, the renormalized operators are:¹¹

$$\begin{aligned} [\phi^2] &= Z_{\phi^2} \phi^2, & [\phi_1^4] &= Z_{\phi_1^4} \phi_1^4, & [\phi_2^4] &= Z_{\phi_2^4} : \phi_2^4 :, \\ Z_{\phi^2} &= 1 + Q \frac{g_2}{\epsilon}, & Z_{\phi_1^4} &= 1 + 2Q \frac{g_1}{\epsilon}, & Z_{\phi_2^4} &= 1 + 2Q \frac{g_2}{\epsilon}, \\ [\phi_t^4] &= \phi_t^4 + N^{-1/2} Z_{\phi_t^4, \phi_1^4} \phi_1^4 + N^{-3/2} Z_{\phi_t^4, \phi_2^4} : \phi_2^4 : + Z_{\phi_t^4, \phi^2} \phi^2, \end{aligned} \quad (6.14)$$

¹¹Notice that to obtain from eq. (4.8) the renormalized operators at linear order, we need to use the couplings at quadratic order in eq. (4.22).

$$Z_{\phi_t^4; \phi_1^4} = -2Q \frac{g}{\epsilon}, \quad Z_{\phi_t^4; \phi_2^4} = -6Q \frac{g}{\epsilon}, \quad Z_{\phi_t^4; \phi^2} = -4\mu^{d-4\Delta_\phi} g \int d^d u C(u)^3,$$

with $Q = 2(4\pi)^{-d/2} \Gamma(d/2)^{-1}$. All our correlations will be written up to terms of order g^2 which from now we omit. We will fix the pillow and double-trace couplings at their IR fixed point, which at the same order in N and g reads $g_1^* = g^*, g_2^* = \sqrt{3}g^*$.

The scaling operators are obtained as in section 4.2, but with some different factors of N . In particular, from the above discussion, we have the following modification of eq. (4.13):

$$\begin{aligned} 0 &= -\partial_{g_b} (\mu \partial_\mu + \beta_a \partial_{g_a}) \langle \phi(x_1) \dots \phi(x_n) \rangle_c \\ &= \sum_a (\delta_{ab} \mu \partial_\mu + (\partial_{g_b} \beta_a)) \mu^{d-\Delta_a} N^{\frac{3}{2}-\rho_a} \int d^d x \langle \phi(x_1) \dots \phi(x_n) [\mathcal{O}_a](x) \rangle_c \\ &\quad - \sum_a \mu^{2d-\Delta_a-\Delta_b} N^{3-\rho_a-\rho_b} \beta_a \int d^d x d^d y \langle \phi(x_1) \dots \phi(x_n) [\mathcal{O}_a](x) [\mathcal{O}_b](y) \rangle_c, \end{aligned} \quad (6.15)$$

from which, going at the fixed point and multiplying by $N^{-\frac{3}{2}+\rho_b}$, we arrive at the same expression for the scaling operators as in eq. (4.17), but with the replacement $(U^{-1})_{ba} \rightarrow N^{\rho_b} (U^{-1})_{ba} N^{-\rho_a}$. The scaling operators are then:

$$\begin{aligned} \{\phi^2\}(x) &= \mu^{\delta h_{\phi^2}} Z_{\phi^2} \phi^2(x), \\ \{\phi_t^4\}(x) &= \phi_t^4(x) + N^{-\frac{1}{2}} \left(Z_{\phi_1^4} + Z_{\phi_t^4; \phi_1^4} \right) \phi_1^4(x) \\ &\quad + N^{-\frac{3}{2}} \left(\sqrt{3} Z_{\phi_2^4} + Z_{\phi_t^4; \phi_2^4} \right) : \phi_2^4(x) : + \left(N^{-\frac{3}{2}} \sqrt{3} Z_{\phi_2^4; \phi^2} + Z_{\phi_t^4; \phi^2} \right) \phi^2(x), \\ \{\phi_1^4\}(x) &= \mu^{\delta h_1} Z_{\phi_1^4} \phi_1^4(x), \\ \{\phi_2^4\}(x) &= \mu^{\delta h_2} Z_{\phi_2^4} : \phi_2^4(x) : . \end{aligned} \quad (6.16)$$

Contact terms. Because of the melonic convolution in eq. (3.8), we will encounter “contact” terms proportional to $(\partial^2)^n \delta(x_{ij})$. This should not come as a surprise, as such terms are for example expected in the three-point function of exactly marginal operators [61]. One should distinguish them from terms like $\lim_{\epsilon \rightarrow 0} \frac{1}{|x|^{d-\epsilon}} \sim \frac{1}{\epsilon} \delta(x)$, which require regularization and lead to an anomaly [59, 74]. Contact terms with finite coefficients are compatible with conformal transformations. The difference can be reformulated by the observation [60] that in d dimensions the only homogeneous distribution of dimension d is the Dirac delta, while any generalized function which coincides with $\frac{1}{|x|^d}$ for $|x| \neq 0$ is an associate homogeneous distribution [75], that is, under scale transformations it transforms with an inhomogeneous contact term.

6.1 Two-point functions

We denote by prime dimensionless positions, $x' = \mu x$ and so on. We first compute the relevant two-point functions.

$$\langle \{\phi^2\}(x) \{\phi^2\}(y) \rangle = \text{Diagram 1} + \text{Diagram 2}$$

Figure 9. Leading order diagrams for $\langle \{\phi^2\} \{\phi^2\} \rangle$.

$$\langle \{\phi_t^4\}(x) \{\phi_t^4\}(y) \rangle = \text{Diagram 1}$$

Figure 10. Leading order diagram for $\langle \{\phi_t^4\} \{\phi_t^4\} \rangle$.

The $\langle \{\phi^2\} \{\phi^2\} \rangle$ correlation. Up to the first order in the coupling constant, at leading order in N only the two diagrams represented in figure 9 contribute. They give:

$$\langle \{\phi^2\}(x) \{\phi^2\}(y) \rangle = \mu^{2\delta h_{\phi^2}} 2 \left[Z_{\phi^2}^2 C(x-y)^2 - \lambda_2 \int d^d z C(x-z)^2 C(z-y)^2 \right], \quad (6.17)$$

where Z_{ϕ^2} is the renormalization constant of the ϕ^2 operator in eq. (6.14).

Computing the integral at first order in ϵ (see appendix A.1) we obtain:

$$\begin{aligned} \langle \{\phi^2\}(x) \{\phi^2\}(y) \rangle &= \mu^{4\Delta_\phi + 2\delta h_{\phi^2}} \frac{2c(\zeta)^2}{|x' - y'|^{4\Delta_\phi}} \\ &\times \left[Z_{\phi^2}^2 - 2Qg_2 \left(\frac{1}{\epsilon} + \log|x' - y'| - \log 2 - \psi\left(\frac{d}{4}\right) + \mathcal{O}(\epsilon) \right) \right]. \end{aligned} \quad (6.18)$$

As expected, the $1/\epsilon$ pole cancels as, according to eq. (6.14), $Z_{\phi^2} = 1 + Q\frac{g_2}{\epsilon}$. We now take ϵ to 0. The $g_2 \log|x' - y'|$ term combines with the constant term to give a correction to the scaling law, hence at first order in g the two-point function is:

$$\langle \{\phi^2\}(x) \{\phi^2\}(y) \rangle = \frac{2c(d/4)^2}{|x - y|^{4\Delta_\phi^* + 2\delta h_{\phi^2}}} \left[1 + 2Qg_2^* \left(\log 2 + \psi(d/4) \right) \right], \quad (6.19)$$

where $\Delta_\phi^* = d/4$ is the dimension of the field at $\epsilon = 0$. The dimension of ϕ^2 is $\Delta_{\phi^2}^* = 2\Delta_\phi^* + \delta h_{\phi^2}$ with $\delta h_{\phi^2} = Qg_2^*$ which reproduces the anomalous dimension found in eq. (3.52).

The $\langle \{\phi_t^4\} \{\phi_t^4\} \rangle$ correlation. Next, we consider the two-point functions of ϕ_t^4 operators. As the operator is complex we take the two-point function of ϕ_t^4 with the hermitian conjugate $(\phi_t^4)^\dagger = -\phi_t^4$.

At first order in the couplings and at leading order in N only one diagram, depicted in figure 10, contributes, yielding:

$$\langle \{\phi_t^4\}(x) \{(\phi_t^4)^\dagger\}(y) \rangle = 4C(x-y)^4 = \frac{4c(\zeta)^4}{|x - y|^{8\Delta_\phi}}. \quad (6.20)$$

The $\epsilon \rightarrow 0$ limit is trivial. This is the standard form for the two-point function of an exactly marginal operator, leading to a conformal anomaly in even dimensions [63]. The appearance of an anomaly can be understood in the spirit of our earlier comment on contact terms by noticing that for $d = 2n$ we have $1/|x|^{2d-\epsilon} \sim \frac{1}{\epsilon} (\partial^2)^n \delta(x)$. Since in our model we have assumed $d < 4$ from the beginning, the anomaly only concerns $d = 2$.

Figure 11. Leading order diagrams for $\langle \{\phi_2^4\} \{\phi_2^4\} \rangle$.

Figure 12. Leading order diagram for $\langle \{\phi^2\} \{\phi_t^4\} \rangle$.

The $\langle \{\phi_2^4\} \{\phi_2^4\} \rangle$ correlation. At the appropriate order we have two graphs (see figure 11), where the quartic vertex is $:\phi_2^4:$, yielding:

$$\langle \{\phi_2^4\}(x) \{\phi_2^4\}(y) \rangle = \mu^{2\delta h_2} 8 \left[Z_{\phi_2^4}^2 C(x-y)^4 - 2 \lambda_2 C(x-y)^2 \int d^d z C(x-z)^2 C(z-y)^2 \right]. \quad (6.21)$$

Following the same steps as before we get, after cancellation of the pole and taking the limit $\epsilon \rightarrow 0$:

$$\langle \{\phi_2^4\}(x) \{\phi_2^4\}(y) \rangle = \frac{8 c(d/4)^4}{|x-y|^{8\Delta_\phi^*+2\delta h_2}} \left[1 + 4 Q g_2^* \left(\log 2 + \psi\left(\frac{d}{4}\right) \right) \right], \quad (6.22)$$

with $\delta h_2 = 2 Q g_2^*$ reproducing the perturbative result eq. (3.48). As expected for a multi-trace operator we have

$$\langle \{\phi_2^4\}(x) \{\phi_2^4\}(y) \rangle = 2 \langle \{\phi^2\}(x) \{\phi^2\}(y) \rangle \langle \{\phi^2\}(x) \{\phi^2\}(y) \rangle. \quad (6.23)$$

The $\langle \{\phi^2\} \{\phi_t^4\} \rangle$ correlation. For a conformal field theory we expect that the two-point functions of operators with different dimensions are zero. We check this for $\langle \{\phi^2\} \{\phi_t^4\} \rangle$. The leading-order contribution to this correlator is depicted in figure 12 (and we get the appropriate counterterm subtraction from the mixing of ϕ_t^4 with ϕ^2) which yields:

$$\langle \{\phi^2\}(x) \{\phi_t^4\}(y) \rangle \sim \lambda C(x-y) \int d^d z C(x-z) \left(C(z-y)^3 - \delta_{yz} \int d^d u C^3(u) \right). \quad (6.24)$$

Using the convolution (3.8), one can evaluate the z -integral and obtain:

$$\langle \{\phi^2\}(x) \{\phi_t^4\}(y) \rangle \sim 4 \lambda \frac{c(\zeta)^4 c(4\zeta - d)}{c(3\zeta - d)} \frac{1}{|x-y|^{10\Delta_\phi - d}}, \quad (6.25)$$

which is zero in the $\epsilon \rightarrow 0$ limit. We conclude that $\langle \{\phi^2\}(x) \{\phi_t^4\}(y) \rangle \lesssim N^{-1}$, hence it is suppressed at large N .

6.2 Three-point functions

We now compute the three-point functions at first order in the couplings and leading order in large N .

$$\langle \{\phi_t^4\} \{\phi_t^4\} \{\phi_t^4\} \rangle = \text{Diagram}$$

Figure 13. Leading order diagram for $\langle \{\phi_t^4\} \{\phi_t^4\} \{\phi_t^4\} \rangle$.

$$\langle \{\phi^2\} \{\phi^2\} \{\phi^2\} \rangle = \text{Diagram 1} + \text{Diagram 2}$$

Figure 14. Leading order diagrams for $\langle \{\phi^2\} \{\phi^2\} \{\phi^2\} \rangle$.

The $\langle \{\phi_t^4\} \{\phi_t^4\} \{\phi_t^4\} \rangle$ correlation. We consider the three-point function $\langle \{\phi_t^4\}(x) \{\phi_t^4\}(y) \{\phi_t^4\}(z) \rangle$. Up to the first order in the coupling constant, the only relevant Feynman diagram at order $N^{-3/2}$ is depicted in figure 13 (and we get the appropriate counterterm subtraction from the mixing of ϕ_t^4 with ϕ^2). Its contribution is given by:

$$\begin{aligned} \langle \{\phi_t^4\}(x) \{\phi_t^4\}(y) \{\phi_t^4\}(z) \rangle &= N^{-\frac{3}{2}} 4^2 \lambda C(x-y)^3 C(y-z) \\ &\times \int d^d u \left[C(z-u)^3 - \delta_{zu} \int d^d v C(v)^3 \right] C(u-x) \\ &+ (\text{permutations}) \\ &= N^{-\frac{3}{2}} 4^2 \lambda \frac{c(d/4)^7}{c(-d/4)} \frac{1}{|x-y|^{2d}} \delta(x-z) + (\text{permutations}) , \end{aligned} \quad (6.26)$$

where we used again eq. (3.8). This is of the expected form for the three-point function of exactly marginal operators [61], and up to contact terms with finite coefficients, it is consistent with the conclusion of ref. [59], according to which a marginal operator should have vanishing three-point function with itself.

The $\langle \{\phi^2\} \{\phi^2\} \{\phi^2\} \rangle$ correlation. Up to the first order in the coupling we have the diagrams depicted in figure 14.

They yield:

$$\begin{aligned} \langle \{\phi^2\}(x) \{\phi^2\}(y) \{\phi^2\}(z) \rangle &= \mu^{3\delta h_{\phi^2}} 8N^{-\frac{3}{2}} \left[Z_{\phi^2}^3 C(x-y) C(y-z) C(z-x) \right. \\ &\quad - \lambda_2 C(x-y) \int d^d u C(x-u) C(y-u) C(z-u)^2 \\ &\quad \left. - (x \leftrightarrow z) - (y \leftrightarrow z) \right] . \end{aligned} \quad (6.27)$$

Figure 15. Leading order diagrams for $\langle \{\phi_t^4\}\{\phi_t^4\}\{\phi^2\} \rangle$.

Using appendix A.1, the right and side of the above equation becomes:

$$8N^{-\frac{3}{2}}c(\zeta)^3\mu^{6\Delta_\phi+3\delta h_{\phi^2}}\frac{Z_{\phi^2}^3-Qg_2\left[\frac{3}{\epsilon}+\log(|x'-y'||y'-z'||z'-x'|)\right]}{|x'-y'|^{2\Delta_\phi}|y'-z'|^{2\Delta_\phi}|z'-x'|^{2\Delta_\phi}}, \quad (6.28)$$

where the dots denote some finite part. As expected, the pole cancels and in the $\epsilon \rightarrow 0$ limit we get:

$$\langle \{\phi^2\}(x)\{\phi^2\}(y)\{\phi^2\}(z) \rangle = N^{-\frac{3}{2}}\frac{8c(d/4)^3(1-Qg_2^*)}{|x-y|^{2\Delta_\phi^*+\delta h_{\phi^2}}|y-z|^{2\Delta_\phi^*+\delta h_{\phi^2}}|z-x|^{2\Delta_\phi^*+\delta h_{\phi^2}}}, \quad (6.29)$$

which is the right form of the conformal three-point function with the correct anomalous dimension given in eq. (3.52).

The $\langle \{\phi_t^4\}\{\phi_t^4\}\{\phi^2\} \rangle$ correlation. For the three-point function $\langle \{\phi_t^4\}\{\phi_t^4\}\{\phi^2\} \rangle$ two graphs, represented in figure 15 contribute.

We have:

$$\begin{aligned} \langle \{\phi_t^4\}(x)\{\phi_t^4\}(y)\{\phi^2\}(z) \rangle &= \mu^{\delta h_{\phi^2}}32N^{-\frac{3}{2}}C(x-y)^3\left[Z_{\phi^2}C(x-z)C(y-z) \right. \\ &\quad \left.-\lambda_2\int d^duC(x-u)C(y-u)C(u-z)^2\right]. \end{aligned} \quad (6.30)$$

Using the conformal integral given in appendix A.1, the right hand side above writes:

$$32N^{-\frac{3}{2}}c(\zeta)^5\mu^{10\Delta_\phi+\delta h_{\phi^2}}\frac{Z_{\phi^2}-Qg_2\left[\frac{1}{\epsilon}+\log\frac{|y'-z'||z'-x'|}{|x'-y'|}\right]}{|x'-y'|^{6\Delta_\phi}|y'-z'|^{2\Delta_\phi}|z'-x'|^{2\Delta_\phi}}. \quad (6.31)$$

Again the pole cancels and in the $\epsilon \rightarrow 0$ limit we get:

$$\langle \{\phi_t^4\}(x)\{\phi_t^4\}(y)\{\phi^2\}(z) \rangle = N^{-\frac{3}{2}}\frac{32c(d/4)^5(1-Qg_2^*)}{|x-y|^{6\Delta_\phi^*-\delta h_{\phi^2}}|y-z|^{2\Delta_\phi^*+\delta h_{\phi^2}}|z-x|^{2\Delta_\phi^*+\delta h_{\phi^2}}}, \quad (6.32)$$

which is of the correct conformal form.

The $\langle \{\phi_t^4\}\{\phi^2\}\{\phi^2\} \rangle$ correlation. The leading order contributions to this correlation are depicted in eq. (6.16).

Taking into account the mixing and the scaling operators we get:

$$\langle \{\phi_t^4\}(x)\{\phi^2\}(y)\{\phi^2\}(z) \rangle$$

Figure 16. Leading diagrams for $\langle \{\phi_t^4\}\{\phi^2\}\{\phi^2\} \rangle$.

$$\begin{aligned}
 &= \mu^{2\delta h_{\phi^2}} \left(\langle \phi_t^4(x) \phi^2(y) \phi^2(z) \rangle + N^{-3/2} (\sqrt{3} Z_{\phi_2^4} + Z_{\phi_t^4; \phi_2^4} Z_{\phi^2}^2 \langle \phi_2^4(x) \phi^2(y) \phi^2(z) \rangle) \right) \\
 &= \mu^{2\delta h_{\phi^2}} N^{-\frac{3}{2}} \left[16 \lambda C(x-y) C(y-z) \int d^d u C(x-u)^3 C(u-z) + (y \leftrightarrow z) \right. \\
 &\quad + 48 \lambda C(x-y) C(x-z) \int d^d u C(x-u)^2 C(y-u) C(z-u) \\
 &\quad + 8 \left(\sqrt{3} \left(1 + 4Q \frac{g_2}{\epsilon} \right) - 6Q \frac{g}{\epsilon} \right) C(x-y)^2 C(x-z)^2 \\
 &\quad \left. - 8\sqrt{3} \lambda_2 C(x-y)^2 \int d^d u C(x-u)^2 C(u-z)^2 + (y \leftrightarrow z) \right], \quad (6.33)
 \end{aligned}$$

where the terms correspond to the diagrams in figure 16. In order to obtain this equation we note that at order $N^{-3/2}$ and up to linear order in the couplings the term coming from the mixing with ϕ_1^4 vanishes and $\langle \phi_t^4 \phi^2 \phi^2 \rangle$ starts at order g , hence we can ignore the renormalization of ϕ^2 in this correlation. However in $\langle \phi_2^4 \phi^2 \phi^2 \rangle$ we need to take it into account. We also omit the subtraction term coming from the mixing of ϕ_t^4 with ϕ^2 . We use once more eq. (3.8) and obtaining contact terms proportional to $\delta(x-z)/|x-y|^d$. In conjunction with appendix. A.1, we find at first order in the coupling constant:

$$\begin{aligned}
 \langle \{\phi_t^4\}(x) \{\phi^2\}(y) \{\phi^2\}(z) \rangle &= N^{-\frac{3}{2}} \mu^{2\delta h_{\phi^2}} \frac{16 c(d/4)^5 g^*}{c(-d/4)} \left[\frac{\delta(x-z)}{|x-y|^{4\Delta_\phi^*}} + \frac{\delta(x-y)}{|x-z|^{4\Delta_\phi^*}} \right] \\
 &\quad + N^{-\frac{3}{2}} \frac{8\sqrt{3} c(d/4)^4 [1 + 2\sqrt{3} Q g_*(\kappa - 2 \log 2 - 2\psi(d/4))]}{|x-y|^{4\Delta_\phi^*} |x-z|^{4\Delta_\phi^*} |y-z|^{2\delta h_{\phi^2}}}, \quad (6.34)
 \end{aligned}$$

where $\delta h_{\phi^2} = Q g_2^* = \sqrt{3} Q g^*$ and κ is defined in eq. (A.25). Notice that in the first term of the above correlator there is no correction to scaling associated to δh_{ϕ^2} , because such term is proportional to g^* , hence the scaling correction will appear at next order.

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A Integrals

We compute in this appendix several integrals we encountered in this paper.

The subtracted melon integral. We first compute the subtracted melon integral:

$$\int_0^\infty da_1 da_2 da_3 \frac{(a_1 a_2 a_3)^{\zeta-1}}{(a_1 a_2 + a_1 a_3 + a_2 a_3)^{d/2}} \left(1 - e^{-\frac{a_1 a_2 a_3}{a_1 a_2 + a_1 a_3 + a_2 a_3}}\right). \quad (\text{A.1})$$

Using a Taylor expansion with integral rest we have:

$$\begin{aligned} & \int_0^1 dt \int_0^\infty da \frac{(a_1 a_2 a_3)^{\zeta-1}}{(a_1 a_2 + a_1 a_3 + a_2 a_3)^{d/2}} \left(-\frac{d}{dt} e^{-t \frac{a_1 a_2 a_3}{a_1 a_2 + a_1 a_3 + a_2 a_3}}\right) \\ &= \int_0^1 dt t^{d-3\zeta-1} \int_0^\infty da \frac{(a_1 a_2 a_3)^\zeta}{(a_1 a_2 + a_1 a_3 + a_2 a_3)^{d/2+1}} e^{-\frac{a_1 a_2 a_3}{a_1 a_2 + a_1 a_3 + a_2 a_3}}. \end{aligned} \quad (\text{A.2})$$

The integral over t converges for $d > 3\zeta$. Let us compute a slight generalization of this integral to q parameters α . Changing variables to $\beta = a^{-1}$ and integrating out t yields:

$$\frac{1}{\frac{d}{2}(q-1) - \zeta q} \int_0^\infty d\beta \frac{\prod_{i=1}^q \beta_i^{d/2-\zeta-1}}{(\sum_{i=1}^q \beta_i)^{d/2+1}} e^{-\frac{1}{\sum_{i=1}^q \beta_i}}. \quad (\text{A.3})$$

Introducing $x = \sum_i \beta_i$ and $\beta_i = s_i x$ the integral becomes:

$$\begin{aligned} & \frac{1}{\frac{d}{2}(q-1) - \zeta q} \int_0^\infty dx x^{q(d/2-\zeta)-d/2-1} e^{-\frac{1}{x}} \int_0^1 \frac{ds_1}{s_1} s_1^{\frac{d}{2}-\zeta} \int_0^{1-s_1} \frac{ds_2}{s_2} s_2^{\frac{d}{2}-\zeta} \dots \\ & \times \int_0^{1-s_1-\dots-s_{q-2}} ds_{q-1} s_{q-1}^{\frac{d}{2}-\zeta-1} (1-s_1-\dots-s_{q-1})^{\frac{d}{2}-\zeta-1}. \end{aligned} \quad (\text{A.4})$$

We now use:

$$\int_0^{1-x} ds s^{u-1} (1-x-s)^{v-1} = (1-x)^{u+v-1} \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \quad (\text{A.5})$$

and we finally obtain the subtracted integral:

$$\frac{\Gamma[1 - [\frac{d}{2}(q-1) - \zeta q]]}{\frac{d}{2}(q-1) - \zeta q} \frac{\Gamma(\frac{d}{2} - \zeta)^q}{\Gamma[(\frac{d}{2} - \zeta) q]}. \quad (\text{A.6})$$

The D integral. We will repeatedly use below the integral:¹²

$$\int_0^\infty [da] \frac{(a_1 a_2)^{u-1}}{(a_1 + a_2)^\gamma} e^{-(a_1 + a_2)} = \frac{\Gamma(u)^2 \Gamma(2u - \gamma)}{\Gamma(2u)}, \quad (\text{A.7})$$

which is convergent for $2\text{Re}(u) > \text{Re}(\gamma)$ and $\text{Re}(u) > 0$. In the particular case $u = \zeta, \gamma = d/2$ we get:

$$D = \frac{\Gamma(\zeta)^2}{\Gamma(2\zeta)} \Gamma(\epsilon/2). \quad (\text{A.8})$$

Denoting ψ the digamma function (the logarithmic derivative of Γ) we have:

$$\begin{aligned} D &= \frac{2}{\epsilon} \frac{\Gamma(d/4 + \epsilon/4)^2}{\Gamma(d/2 + \epsilon/2)} \Gamma(1 + \epsilon/2) \\ &= \frac{2}{\epsilon} \frac{\Gamma(d/4)^2}{\Gamma(d/2)} + \frac{\Gamma(d/4)^2}{\Gamma(d/2)} \left[\psi(d/4) - \psi(d/2) + \psi(1) \right] + O(\epsilon) \end{aligned} \quad (\text{A.9})$$

The S_1 integral. The next integral we want to compute is:

$$S_1 = \int_0^\infty [dad b] \frac{(a_1 a_2 b_1 b_2)^{\zeta-1}}{[(a_1 + a_2)(b_1 + b_2) + b_1 b_2]^{d/2}} e^{-(a_1 + a_2 + b_1 + b_2)}. \quad (\text{A.10})$$

We use Mellin parameters to write:

$$\begin{aligned} &\frac{1}{[(a_1 + a_2)(b_1 + b_2) + b_1 b_2]^{d/2}} \\ &= \int_{0-i\infty}^{0+i\infty} \frac{dz}{2\pi i} \frac{\Gamma(d/2 + z)}{\Gamma(d/2)} \Gamma(-z) \frac{(b_1 b_2)^z}{(b_1 + b_2)^{z+d/2}} \frac{1}{(a_1 + a_2)^{z+d/2}}, \end{aligned} \quad (\text{A.11})$$

and eq. (A.7) allows us to integrate a and b . We thus obtain:

$$S_1 = \frac{\Gamma(\zeta)^2}{\Gamma(2\zeta)\Gamma(d/2)} \int_{0-i\infty}^{0+i\infty} \frac{dz}{2\pi i} \Gamma(d/2 + z) \frac{\Gamma(\zeta + z)^2}{\Gamma(2\zeta + 2z)} \Gamma\left(\frac{\epsilon}{2} + z\right) \Gamma(-z) \Gamma\left(\frac{\epsilon}{2} - z\right). \quad (\text{A.12})$$

In the right half complex plane the integrand has poles at $z = n, n + \epsilon/2$. Only the poles in $0, \epsilon/2$ have large residues at $\epsilon \rightarrow 0$, hence:

$$\begin{aligned} S_1 &= \frac{\Gamma(\zeta)^4}{\Gamma(2\zeta)^2} \Gamma(\epsilon/2) \Gamma(\epsilon/2) + \frac{\Gamma(\zeta)^2}{\Gamma(2\zeta)\Gamma(d/2)} \Gamma(d/2 + \epsilon/2) \frac{\Gamma(\zeta + \epsilon/2)^2}{\Gamma(2\zeta + \epsilon)} \Gamma(\epsilon) \Gamma(-\epsilon/2) \\ &\quad + \frac{\Gamma(\zeta)^2}{\Gamma(2\zeta)\Gamma(d/2)} \int_{1-i\infty}^{1+i\infty} \frac{dz}{2\pi i} \Gamma(d/2 + z) \frac{\Gamma(\zeta + z)^2}{\Gamma(2\zeta + 2z)} \Gamma\left(\frac{\epsilon}{2} + z\right) \Gamma(-z) \Gamma\left(\frac{\epsilon}{2} - z\right), \end{aligned} \quad (\text{A.13})$$

¹²We have:

$$\begin{aligned} \int_0^\infty [da] \frac{(a_1 a_2)^{u-1}}{(a_1 + a_2)^\gamma} e^{-(a_1 + a_2)} &= \int_0^\infty ds s^{-\gamma} e^{-s} \int_0^s da_1 a_1^{u-1} (s - a_1)^{u-1} \\ &= \int_0^\infty ds s^{2u-\gamma-1} e^{-s} \int_0^1 dx x^{u-1} (1-x)^{u-1}. \end{aligned}$$

and the last line is finite in the $\epsilon \rightarrow 0$ limit. At small ϵ we get:

$$S_1 = \frac{2}{\epsilon^2} \frac{\Gamma(d/4)^4}{\Gamma(d/2)^2} + \frac{1}{\epsilon} \frac{\Gamma(d/4)^4}{\Gamma(d/2)^2} [3\psi(1) - \psi(d/2)] + O(\epsilon^0), \quad (\text{A.14})$$

In particular, we have

$$2S_1 - D^2 = \frac{2}{\epsilon} \frac{\Gamma(d/4)^4}{\Gamma(d/2)^2} [\psi(1) + \psi(d/2) - 2\psi(d/4)] + O(\epsilon^0). \quad (\text{A.15})$$

The melon integral with momentum insertion. We are interested in evaluating the coefficient of p^{2n} in the Taylor expansion of the integral:

$$\int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{[(p + q_1 + q_2)^2 + \mu^2]^\zeta} \frac{1}{(q_2^2 + \mu^2)^\zeta} \frac{1}{(q_1^2 + \mu^2)^\zeta} (q_1^2)^n \frac{1}{(q_1^2 + \mu^2)^\zeta}. \quad (\text{A.16})$$

Using the parametric representation, and observing that $(q_1^2)^n e^{-a_1 q_1^2} = (-\partial_{a_1})^n e^{-a_1 q_1^2}$, the momentum integrals can be computed, yielding:

$$\frac{1}{(4\pi)^d \Gamma(\zeta)^4} \int_{a,b} (a_1 a_2 b_1 b_2)^{\zeta-1} e^{-\mu^2(a_1+a_2+b_1+b_2)} \times \left(-\frac{\partial}{\partial a_1} \right)^n \left[\frac{e^{-p^2 \frac{b_1 b_2 (a_1+a_2)}{(a_1+a_2)(b_1+b_2)+b_1 b_2}}}{[(a_1+a_2)(b_1+b_2)+b_1 b_2]^{d/2}} \right]. \quad (\text{A.17})$$

The coefficient of p^{2n} in the Taylor expansion of this integral is $\frac{\mu^{-2\epsilon}}{(4\pi)^d \Gamma(\zeta)^4} S_1^{(n)}$ with:

$$S_1^{(n)} = \frac{1}{n!} \int_{a,b} (a_1 a_2 b_1 b_2)^{\zeta-1} e^{-(a_1+a_2+b_1+b_2)} \times \left(\frac{\partial}{\partial a_1} \right)^n \left[\frac{[b_1 b_2 (a_1+a_2)]^n}{[(a_1+a_2)(b_1+b_2)+b_1 b_2]^{d/2+n}} \right]. \quad (\text{A.18})$$

In order to compute the leading divergence of $S_1^{(n)}$, we note that:

$$\frac{1}{[(a_1+a_2)(b_1+b_2)+b_1 b_2]^{d/2+n}} = \int_{0^- - i\infty}^{0^- + i\infty} \frac{dz}{2\pi i} \frac{\Gamma(d/2+n+z)}{\Gamma(d/2+n)} \Gamma(-z) \frac{(a_1+a_2)^z (b_1+b_2)^z}{(b_1 b_2)^{d/2+n+z}}, \quad (\text{A.19})$$

and, as we encounter no singularities, we move the integration contour to $z = -d/2 + i\mathbb{R}$. We thus get:

$$S_1^{(n)} = \frac{1}{n!} \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \frac{\Gamma(n+z)}{\Gamma(d/2+n)} \Gamma(d/2-z) \int_b \frac{(b_1 b_2)^{\zeta-z-1}}{(b_1+b_2)^{d/2-z}} e^{-(b_1+b_2)} \times \int_a \frac{(a_1 a_2)^{\zeta-1}}{(a_1+a_2)^{d/2-z}} e^{-(a_1+a_2)} \frac{\Gamma(n+1-d/2+z)}{\Gamma(n-d/2+z)}, \quad (\text{A.20})$$

and using eq. (A.7) to integrate out the a 's and b 's we obtain:

$$S_1^{(n)} = \frac{1}{n!} \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \frac{\Gamma(n+z)}{\Gamma(d/2+n)} \Gamma(d/2-z) \times \frac{\Gamma(\zeta-z)^2}{\Gamma(2\zeta-2z)} \Gamma\left(\frac{\epsilon}{2}-z\right) \frac{\Gamma(n+1-d/2+z)}{\Gamma(n-d/2+z)} \frac{\Gamma(\zeta)^2}{\Gamma(2\zeta)} \Gamma\left(\frac{\epsilon}{2}+z\right). \quad (\text{A.21})$$

The only pole of the integrand in the right-half complex plane with residue of order $1/\epsilon$ is located at $z = \epsilon/2$. Moving the contour across the pole we get:

$$S_1^{(n)} = \frac{1}{\epsilon} \frac{\Gamma(d/4)^4}{\Gamma(d/2)} \frac{\Gamma(n+1-d/2)}{n\Gamma(d/2+n)\Gamma(1-d/2)} + O(\epsilon^0). \quad (\text{A.22})$$

A.1 The conformal integrals

We work at $\epsilon > 0$, that is $\Delta_\phi = (d-\epsilon)/4$. In the main text we encounter the following integrals involving bare propagators:

$$\int d^d u C(x-u)^2 C(y-u)^2 = \frac{2Q c(\zeta)^2}{|x-y|^{4\Delta_\phi}} \left[\frac{1}{\epsilon} + \log|x-y| - \log 2 - \psi\left(\frac{d}{4}\right) + \mathcal{O}(\epsilon) \right], \quad (\text{A.23})$$

and

$$\begin{aligned} \int d^d u C(x-u) C(y-u) C(z-u)^2 \\ = \frac{Q c(\zeta)^2}{|y-z|^{2\Delta_\phi} |x-z|^{2\Delta_\phi}} \left[\frac{1}{\epsilon} + \log\left(\frac{|z-x||z-y|}{|x-y|}\right) + \kappa + \mathcal{O}(\epsilon) \right], \end{aligned} \quad (\text{A.24})$$

with

$$\kappa = \frac{1}{2} \left[\psi\left(\frac{d}{2}\right) - 4\psi\left(\frac{d}{4}\right) - \gamma - 2\log 2 \right]. \quad (\text{A.25})$$

These integrals are computed using two conformal integrals. First we have:

$$\begin{aligned} I_d(\nu_1, \nu_2) &= \int \frac{d^d u}{|x-u|^{2\nu_1} |y-u|^{2\nu_2}} \\ &= \frac{\pi^{\frac{d}{2}} \Gamma(d/2 - \nu_1) \Gamma(d/2 - \nu_2) \Gamma(\nu_1 + \nu_2 - d/2)}{\Gamma(\nu_1) \Gamma(\nu_2) \Gamma(d - \nu_1 - \nu_2)} \frac{1}{|x-y|^{2\nu_1+2\nu_2-d}}, \end{aligned} \quad (\text{A.26})$$

which follows from the Fourier transform (3.5). In particular for $\nu_1 = \nu_2 = 2\Delta_\phi = \frac{d-\epsilon}{2}$ we get:

$$I_d(2\Delta_\phi, 2\Delta_\phi) = \frac{2(2\pi)^d Q}{|x-y|^d} \left[\frac{1}{\epsilon} + 2\log|x-y| + \mathcal{O}(\epsilon) \right], \quad (\text{A.27})$$

and multiplying $c(\zeta)^4$ and rearranging the coefficient we obtain eq. (A.23).

In order to prove Eq. (A.24), we start from:

$$I_d(\nu_1, \nu_2, \nu_3) = \int \frac{d^d u}{|x-u|^{2\nu_1} |y-u|^{2\nu_2} |z-u|^{2\nu_3}}. \quad (\text{A.28})$$

In particular, we are interested in the case $\nu_1 = \nu_2 = \Delta_\phi = (d-\epsilon)/4$, $\nu_3 = 2\Delta_\phi = (d-\epsilon)/2$. Using the Mellin-Barnes representation [76], we rewrite the integral as:

$$\begin{aligned} I_d(\nu_1, \nu_2, \nu_3) &= \frac{\pi^{\frac{d}{2}} |x-y|^{d-2\sum \nu_i}}{\Gamma(d - \sum \nu_i) \prod \Gamma(\nu_i)} \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} \alpha^s \beta^t \Gamma(-s) \Gamma(-t) \\ &\quad \times \Gamma\left(\frac{d}{2} - \nu_2 - \nu_3 - s\right) \Gamma\left(\frac{d}{2} - \nu_1 - \nu_3 - t\right) \Gamma(\nu_3 + s + t) \\ &\quad \times \Gamma\left(\sum \nu_i - \frac{d}{2} + s + t\right), \end{aligned} \quad (\text{A.29})$$

where we defined $\alpha = (y - z)^2/(x - y)^2$, $\beta = (z - x)^2/(x - y)^2$. The integration contour (i.e. the constant c) is chosen to separate all poles of the first four Gamma functions from the poles of the last two Gamma functions. We close the contour to the right so that we pick up all poles of the first four Gamma functions, but none of the poles of the last two Gamma functions. The relevant poles are located at:

$$\begin{aligned} s &= n_1, & s &= \frac{d}{2} - \nu_2 - \nu_3 + n_2, \\ t &= m_1, & t &= \frac{d}{2} - \nu_1 - \nu_3 + m_2, \end{aligned} \quad (\text{A.30})$$

with $n_{1,2}, m_{1,2} = 0, 1, 2, \dots$. The complete answer for the integral is given by the sum of all of these pole contributions. This is a daunting task to complete, so we look at the singular contribution in the limit $\epsilon \rightarrow 0$ with the choice $\nu_1 = \nu_2 = \Delta_\phi$, $\nu_3 = 2\Delta_\phi$. We note that for this choice one of the Gamma functions in the overall coefficient becomes:

$$\Gamma\left(d - \sum_i \nu_i\right) = \Gamma(\epsilon) = \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0). \quad (\text{A.31})$$

The leading behavior of the integral is $\mathcal{O}(\epsilon^{-2})$, coming from the poles at $n_2 = m_2 = 0$. Namely, the poles of the third and fourth gamma functions at $s = t = (3\epsilon - d)/4$ lead to a $\mathcal{O}(\epsilon^{-1})$ contribution from each of the fifth and sixth gamma functions. Overall we get:

$$I_d(\Delta_\phi, \Delta_\phi, 2\Delta_\phi) = \frac{(2\pi)^d Q}{|y - z|^{\frac{d}{2}} |z - x|^{\frac{d}{2}}} \left[\frac{1}{\epsilon} + \log \frac{|y - z|^{3/2} |z - x|^{3/2}}{|x - y|} + \kappa' + \mathcal{O}(\epsilon) \right], \quad (\text{A.32})$$

where

$$\kappa' = \frac{1}{2} \left[\psi\left(\frac{d}{2}\right) - 2\psi\left(\frac{d}{4}\right) - \gamma \right]. \quad (\text{A.33})$$

Multiplying by $c(\zeta)^4$, using $\Delta_\phi = (d - \epsilon)/4$, and rearranging the coefficient we obtain eq. (A.24).

B The bilinear operators

The spin-zero bilinear operators of the type $\phi(-\partial^2)^n \phi$ can be treated similarly to the ϕ^2 perturbation. We start by including a bare perturbation:

$$-\frac{1}{2} \lambda_{(n)} \int d^d x \phi(-\partial^2)^n \phi, \quad (\text{B.1})$$

and we evaluate the Taylor coefficient of p^{2n} in the one-particle irreducible two-point function, which we denote $\Gamma_{(n)}^R$. As the tadpole is local, only the melon with one bi-valent vertex $\lambda_{(n)}$ inserted on one of its edges contributes:

$$\begin{aligned} \Gamma_{(n)}^R &= \lambda_{(n)} - 3\lambda^2 \lambda_{(n)} \left[\int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \right. \\ &\quad \left. + \frac{1}{[(p + q_1 + q_2)^2 + \mu^2]^\zeta} \frac{1}{(q_2^2 + \mu^2)^\zeta} \frac{1}{(q_1^2 + \mu^2)^\zeta} (q_1^2)^n \frac{1}{(q_1^2 + \mu^2)^\zeta} \right]_{p^{2n}}, \end{aligned} \quad (\text{B.2})$$

where the subscript p^{2n} signifies that we are only interested in the coefficient of p^{2n} in the Taylor expansion of the integral. Using appendix A, the bare expansion becomes:

$$\begin{aligned}\Gamma_{(n)}^R &= \lambda_{(n)} - 3\lambda_{(n)} \frac{\lambda^2}{(4\pi)^d \Gamma(\zeta)^4} \mu^{-2\epsilon} S_1^{(n)}, \\ S_1^{(n)} &= \frac{1}{\epsilon} \frac{\Gamma(d/4)^4}{\Gamma(d/2)} \frac{\Gamma(n+1-d/2)}{n\Gamma(d/2+n)\Gamma(1-d/2)} + O(\epsilon^0).\end{aligned}\quad (\text{B.3})$$

Similar to the mass parameter, we obtain:

$$\tilde{\lambda}_{(n)} = \mu^{d-\Delta_{(n)}} \left(\tilde{g}_{(n)} + 3\tilde{g}^2 \tilde{g}_{(n)} S_1^{(n)} \right), \quad \beta_{(n)} = -(d - \Delta_{(n)}) \tilde{g}_{(n)} + 6\tilde{g}_{(n)} \tilde{g}^2 \epsilon S_1^{(n)}, \quad (\text{B.4})$$

where $\Delta_{(n)} = 2\Delta_\phi + 2n$ is the classical dimension of the operator $\phi(-\partial^2)^n \phi$. We note that $\tilde{g}_{(n)} = 0$ is always a fixed point of this equation, and that at $\epsilon = 0$ the beta function simplifies to:

$$\beta_{(n)} = -\left[d - \left(2\Delta_\phi^* + 2n \right) \right] \tilde{g}_{(n)} + 6 \frac{\Gamma(d/4)^4}{\Gamma(d/2)} \frac{\Gamma(n+1-d/2)}{n\Gamma(d/2+n)\Gamma(1-d/2)} \tilde{g}_{(n)} \tilde{g}^2. \quad (\text{B.5})$$

At the fixed point $\tilde{g}^*$ the operator $\phi(-\partial^2)^n \phi$ acquires an anomalous dimension

$$\delta h_{(n)} = 6 \frac{\Gamma(d/4)^4}{\Gamma(d/2)} \frac{\Gamma(n+1-d/2)}{n\Gamma(d/2+n)\Gamma(1-d/2)} \tilde{g}^{*2}, \quad \Delta_{(n)}^* = 2\Delta_\phi^* + 2n + \delta h_{(n)}, \quad (\text{B.6})$$

reproducing the results derived in [1] by diagonalizing the four-point kernel.

C The $1/N$ expansion revisited

In the main body of the paper we are interested in tensors of rank $D = 3$. However, the discussion below applies to any rank D .

A D -colored graph [7, 21, 22] is a graph such that:

- all the vertices are D -valent
- the edges have a color $1, \dots, D$ and at any vertex we have exactly one incident edge for each color

A D -colored graph is connected if any two vertices are joined by a path of (colored) edges such that two consecutive edges in the path share a vertex. For the D -colored graph h we denote $V(h)$, $E(h)$, $C(h)$ and $F(h)$ the numbers of vertices, edges, connected components and faces (i.e. bi colored cycles) of h . We also denote $E^c(h)$ the number of edges of color c and $F^c(h)$ the number of faces which contain the color c .

Invariants and Feynman graphs. The $O(N)^D$ invariants $\text{Tr}_b(T)$ are D -colored graphs [7] b . The vertices of b are associated to the tensors T and the edges (colored $1, \dots, D$) are associated to the contractions of indices:

$$\text{Tr}_b(T) = \prod_{v \in b} T_{a_v^1 a_v^2 a_v^3} \prod_{e^c = (v, w)} \delta_{a_v^c a_w^c}, \quad (\text{C.1})$$

where v runs over the vertices of b and e^c over its edges (c denotes the color of the edge e^c). The tetrahedral graph in $D = 3$ corresponds to $\delta_{\mathbf{abcd}}^t T_{\mathbf{a}} T_{\mathbf{b}} T_{\mathbf{c}} T_{\mathbf{d}}$. We call b the *bubbles*.

We are interested in the partition function:

$$W = \frac{1}{N^D} \ln \left(\int [dT] e^{-N^{D/2} (T_{\mathbf{a}} T_{\mathbf{a}} + \sum_b \lambda_b N^{-\rho_b} \text{Tr}_b(T))} \right), \quad (\text{C.2})$$

where $\rho_b \geq 0$ are scalings chosen such that the large N limit of W exists. Observe that, contrary to [44], we allow the bubbles b to have several connected components. This is for instance the case of the double trace interaction bubble $\delta_{\mathbf{ab};\mathbf{cd}}^d T_{\mathbf{a}} T_{\mathbf{b}} T_{\mathbf{c}} T_{\mathbf{d}}$. Somewhat abusively, we some times call a bubble with several connected components a “multi-trace” interaction.

The generating function W is a sum over connect Feynman graphs G which have a new color 0 for the Wick contractions (propagators). As the propagators represent pairings of tensors, they connect vertices and G is a $(D+1)$ -colored graph. Denoting $G^{\hat{0}}$ the graph obtained from G by erasing the edges of color 0 we have:

$$W = \sum_{\text{i-connected } G}^{G^{\hat{0}} = \cup b} \left(\prod_b -\lambda_b \right) N^{-D - \frac{D}{2} E^0(G) + \sum_b \left(\frac{D}{2} - \rho_b \right) + F^0(G)}. \quad (\text{C.3})$$

Due to the disconnected bubbles (multi trace interactions), the notion of connectivity in equation (C.3) is subtle, hence the notation “i-connected” in the sum. The graph G is i-connected if any two interaction bubbles are joined by a path of edges of color 0 such that any two consecutive edge in the path are incident to the same interaction bubble. However, the graph G can be disconnected as a colored graph, $C(G) > 1$, because the edges in this path can be incident to different connected components in the bubbles. An example of an i-connected graph G which has $C(G) > 1$ is a double trace interaction decorated by two tadpole edges $\delta_{\mathbf{ab};\mathbf{cd}}^d \langle T_{\mathbf{a}} T_{\mathbf{b}} \rangle \langle T_{\mathbf{c}} T_{\mathbf{d}} \rangle$.

It is a standard result [7, 22] that the total number of faces of a $(D+1)$ -colored graph G is:

$$F(G) = D C(G) + \frac{D(D-1)}{4} V(G) - \bar{\omega}(G), \quad \bar{\omega}(G) = \frac{1}{2(D-1)!} \sum_{\pi} k(\pi) \geq 0, \quad (\text{C.4})$$

where π runs over the $D!$ jackets of G (that is the embedding of G corresponding to cycles over the colors) and $k(\pi)$ is the non orientable genus of the jacket π . The non negative half integer $\bar{\omega}(G)$ is the *degree* of G . The degree of a disconnected graph is the sum of the degrees of its connect components. The bubbles b have only D colors therefore:

$$F(b) = (D-1) C(b) + \frac{(D-1)(D-2)}{4} V(b) - \bar{\omega}(b). \quad (\text{C.5})$$

The crucial property of the degree is that for any $(D+1)$ -colored graph G :

$$\bar{\omega}(G) \geq \frac{D}{D-1} \bar{\omega}(G^{\hat{0}}). \quad (\text{C.6})$$

This is a bit subtle. As G has $D + 1$ colors, $G^{\hat{0}}$ has only D colors. There is a D to 1 correspondence between the jackets π of G and the jackets $\pi^{\hat{0}}$ of $G^{\hat{0}}$ consisting in deleting the edges of color 0 in the jacket. As the non orientable genus can not increase by deleting edges we have $k(\pi) \geq k(\pi^{\hat{0}})$ and consequently $\sum_{\pi} k(\pi) \geq D \sum_{\pi^{\hat{0}}} k(\pi^{\hat{0}})$.

A $(D + 1)$ -colored graph G has at most $\sum_{b \in G^{\hat{0}}} C(b)$ connected components. If G is i-connected, then it posses a tree of edges of color 0 connecting all the bubbles. Each edge in this tree joins two connected components on two different bubbles, hence decrease the maximal number of connect components of G by 1. Overall we get an upper bound on the number of connected components of G :

$$1 + \sum_{b \in G^{\hat{0}}} [C(b) - 1] \geq C(G). \quad (\text{C.7})$$

Among the invariants (i.e. bubbles), an interesting subclass consists in the *maximally single trace* (MST) ones. They are those bubbles with only one face for each couple of colors. They are obviously connected and have exactly $D(D - 1)/2$ faces hence maximal possible degree:

$$\bar{\omega}(b) = -\frac{(D - 1)(D - 2)}{2} + \frac{(D - 1)(D - 2)}{4} V(b), \quad (\text{C.8})$$

at fixed number of vertices.

$D = 3$. Let us fix the ideas for $D = 3$. The bubbles b are 3 colored graphs. As such they are embedded graphs (ribbon graph, combinatorial map) with $V(b) - E(b) + F(b) = 2C(b) - k(b)$, where $k(b)$ is the non orientable genus of b . Every b admits two jackets, (123) and (132), which are identical up to orientation and have non orientable genus $k(b)$. The degree of b is its non orientable genus $\bar{\omega}(b) = k(b)$. The MST invariants have three faces and non orientable genus $k(b) = -1 + V(b)/2$. For instance the tetrahedron is MST and has non orientable genus 1. The wheel sextic interaction [77] is also MST and has non orientable genus 2. The Feynman graphs G have 4 color and 6 jackets, which are in $3 \rightarrow 1$ correspondence with the jackets of $G^{\hat{0}}$:

$$\left\{ \begin{array}{l} (0123) \\ (0231) \\ (0312) \end{array} \right\} \rightarrow (123) \quad \left\{ \begin{array}{l} (0132) \\ (0213) \\ (0321) \end{array} \right\} \rightarrow (132), \quad \omega(G) = \frac{1}{4} \sum_{6 \text{ cycles } \pi} k(\pi) \geq \frac{3}{2} k(G^{\hat{0}}). \quad (\text{C.9})$$

The $1/N$ series. As $E^0(G) = \sum_b V(b)/2$ and $F^0(G) = F(G) - \sum_b F(b)$ the scaling with N of a Feynman graph G :

$$-D - \frac{D}{4} \sum_b V(b) + \sum_b \left(\frac{D}{2} - \rho_b \right) + F(G) - \sum_b F(b). \quad (\text{C.10})$$

We now chose to scale all the invariants by the “optimal scaling” introduced in [9, 44]:

$$\rho_b = \frac{F(b)}{D - 1} - \frac{D}{2} \geq 0. \quad (\text{C.11})$$

With this optimal scaling eq. (C.3) becomes:

$$W = \sum_{i \text{--connected } G}^{G^0 = \cup b} \left(\prod_b -\lambda_b \right) N^{-D \left(1 + \sum_b [C(b) - 1] - C(G) \right) - \left(\bar{\omega}(G) - \frac{D}{D-1} \sum_b \bar{\omega}(b) \right)}, \quad (\text{C.12})$$

which, due to the inequalities (C.6) (C.7), is a series in $1/N$ indexed by:

$$\omega(G) = D \left(1 + \sum_b [C(b) - 1] - C(G) \right) + \left(\bar{\omega}(G) - \frac{D}{D-1} \sum_b \bar{\omega}(b) \right) \geq 0. \quad (\text{C.13})$$

The optimal scaling leads to a good large N limit. For some classes of interaction bubbles (like the MST or the melonic ones) the optimal scaling is the minimal scaling which still leads to a large N limit. It should be stressed however that this is not true in general: finding the minimal ρ_b which still leads to a large N limit for an arbitrary interaction is a difficult open question [78].

Correlations. The connected correlation of p bubbles $b_1, \dots, b_p$ is

$$\begin{aligned} \langle \text{Tr}_{b_1}(T) \dots \text{Tr}_{b_p}(T) \rangle_{\text{connected}} &= N^{D - \sum_i \left(\frac{D}{2} - \rho_{b_i} \right)} \frac{\delta^p W}{\delta \lambda_{b_1} \dots \delta \lambda_{b_p}} \\ &= \sum_{i \text{--connected } G}^{G^0 \supset \cup b_i} N^{D - \sum_i \left(\frac{D}{2} - \rho_{b_i} \right) - \omega(G)} \left((-1)^p \prod_{b \subset G^0}^{b \neq b_i} -\lambda_b \right). \end{aligned} \quad (\text{C.14})$$

For MST invariants we have:

$$\langle \text{Tr}_{b_1}(T) \dots \text{Tr}_{b_p}(T) \rangle_{\text{connected}} \lesssim N^{D - \frac{D}{2}p}, \quad (\text{C.15})$$

which leads to a large N factorization of the expectations. Any correlation factors into connected correlations:

$$\langle \text{Tr}_{b_1}(T) \dots \text{Tr}_{b_p}(T) \rangle = \sum_P \prod_{B \in P} \left\langle \prod_{j \in B} \text{Tr}_{b_j}(T) \right\rangle_{\text{connected}}, \quad (\text{C.16})$$

where the sum runs over the partitions P of $\{1, \dots, p\}$, B runs over the blocks in the partition P and j over the elements in the block B . In this decomposition the partitions with the largest number of blocks will dominate. As the one point functions are zero in a CFT, the dominant partition will have either only connected two-point functions or at most one connected three-point function. Furthermore, if O is a product of single trace operators and $[\text{Tr}_{b_1}(T) \text{Tr}_{b_1}(T)]$ is a double trace one we have:

$$\begin{aligned} \langle O[\text{Tr}_{b_1}(T) \text{Tr}_{b_1}(T)] \rangle_{\text{connected}} &= \langle O \text{Tr}_{b_1}(T) \text{Tr}_{b_2}(T) \rangle_{\text{connected}} \\ &\quad + \langle O \text{Tr}_{b_1}(T) \rangle_{\text{connected}} \langle \text{Tr}_{b_2}(T) \rangle_{\text{connected}} \\ &\quad + \langle O \text{Tr}_{b_2}(T) \rangle_{\text{connected}} \langle \text{Tr}_{b_1}(T) \rangle_{\text{connected}}, \end{aligned} \quad (\text{C.17})$$

and the last two terms are zero in a CFT.

D Comments on the correlation functions of ϕ_1^4

The ϕ_1^4 operator is neither MST nor MMT, as the pillow is a connected invariant with $\rho_b = 1/2$. In order to get a general idea of how it contributes to n -point functions we will consider first a simplified with only pillow operators of a single type, and then we will explicitly compute the two-point function of ϕ_1^4 at leading order.

Let us first consider a general correlator of n pillow operators with an arbitrary number of perturbative pillow vertices. We restrict to a single type of pillow operator, with single lines of color one: $\mathcal{O}_p = \phi_{a_1 b_1 c_1} \phi_{a_2 b_1 c_1} \phi_{a_1 b_2 c_2} \phi_{a_2 b_2 c_2}$. The intermediate field representation, known also as Hubbard-Stratonovich transformation, amounts to replacing it in the path integral by the integral over an auxiliary real symmetric $N \times N$ matrix field (the intermediate field), with ultralocal free covariance proportional to N (from the scaling of the pillow in the original action) and which couples to the composite matrix $\phi_{a_1 b_1 c_1} \phi_{a_2 b_1 c_1}$ (see for example [15, 79]). The original field appears then only quadratically in the new action, and thus it forms V_q loop-vertices of valency q , for $q \geq 1$, each containing two faces of the tensor model. Denoting by E the number of intermediate field propagators, and by F the number of faces that the intermediate field forms, we thus have that the connected n -point function of pillows scales as

$$N^{\sum_{q \geq 1} (2 - \frac{3}{2}q) V_q + F + E - n} = N^{\sum_{q \geq 1} (1 - \frac{q}{2}) V_q + 2 - 2g - n}, \quad (\text{D.1})$$

where the factor n is due to the fact that the inserted operators, unlike the perturbative vertices, carry no factor N . The amplitude would therefore be dominated by an intermediate field graph which is planar and which maximizes the number of univalent loop-vertices, that is, a usual cactus diagram. However, assuming that univalent loop-vertices (tadpoles in the original representation) have zero amplitude, we are left with dominant graphs being made of two-valent loop-vertices, joined in a planar way. Their amplitude scales like N^{2-n} . Comparing with eq. (6.11), this means that such dominant graphs have $\omega = 1$. Therefore, we conclude that also pillows at criticality have two-point functions of order N^0 , and higher-point functions suppressed in $1/N$.

Two-point function of ϕ_1^4 . We now study the two-point function $\langle \{\phi_1^4\} \{\phi_1^4\} \rangle$. The same two graphs as for the correlation $\langle \{\phi_2^4\} \{\phi_2^4\} \rangle$ contribute with replacing λ_2 by λ_1 . The global scaling is N^0 (as for the other two-point functions) and we get:

$$\langle \{\phi_1^4\}(x) \{\phi_1^4\}(y) \rangle = \mu^{2\delta h_1} Z_{\phi_1^4}^2 \left(4C(x-y)^4 - 8\lambda_1 C(x-y)^2 \int d^d z C(x-z)^2 C(z-y)^2 \right). \quad (\text{D.2})$$

Recalling that $Z_{\phi_1^4} = 1 + 2Q \frac{g_1}{\epsilon}$, the $1/\epsilon$ pole cancels and in the $\epsilon \rightarrow 0$ limit we obtain:

$$\langle \{\phi_1^4\}(x) \{\phi_1^4\}(y) \rangle = \mu^{8\Delta_\phi^* + 2\delta h_1} \frac{4c(d/4)^4}{|x' - y'|^{8\Delta_\phi^* + 2\delta h_1}} \left[1 + 4Q g_1^* \left(\log 2 + \psi(d/4) \right) \right], \quad (\text{D.3})$$

with $\delta h_1 = 2Q g_1^* = 2Q g^*$ reproducing the perturbative computation in eq. (3.48).

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