Two-loop renormalization of the matter superfields and finiteness of $\mathcal{N} = 1$ supersymmetric gauge theories regularized by higher derivatives

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ABSTRACT: The two-loop anomalous dimension of the chiral matter superfields is calculated for a general $\mathcal{N} = 1$ supersymmetric gauge theory regularized by higher covariant derivatives. We obtain both the anomalous dimension defined in terms of the bare couplings, and the one defined in terms of the renormalized couplings for an arbitrary renormalization prescription. For the one-loop finite theories we find a simple relation between the higher derivative regulators under which the anomalous dimension defined in terms of the bare couplings vanishes in the considered approximation. In this case the one-loop finite theory is also two-loop finite in the HD+MSL scheme. Using the assumption that with the higher covariant derivative regularization the NSVZ equation is satisfied for RGFs defined in terms of the bare couplings, we construct the expression for the three-loop β -function. Again, the result is written both for the β -function defined in terms of the bare couplings and for the one defined in terms of the renormalized couplings for an arbitrary renormalization prescription.

KEYWORDS: Renormalization Group, Renormalization Regularization and Renormalons, Supersymmetric Gauge Theory

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1 Introduction

Quantum properties of supersymmetric theories have a lot of very interesting features. The maximally extended $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) is finite in all loops [1–4], while $\mathcal{N} = 2$ supersymmetric gauge theories are finite beyond the one-loop approximation [1, 4, 5]. This implies that $\mathcal{N} = 2$ theories finite in the one-loop approximation are also finite in all orders [6]. The one-loop finiteness can be achieved by a special choice of a gauge group and a representation for the matter superfields. In $\mathcal{N} = 1$ supersymmetric theories the divergent quantum corrections can exist in all orders. However, supersymmetry leads to some interesting relations between various renormalization constants. For example, due to the finiteness of the superpotential [7] the renormalizations of masses and Yukawa couplings are related to the renormalization of the chiral matter superfields. Similarly, the non-renormalization of the triple gauge-ghost vertices [8] allows choosing a renormalization prescription for which

$$Z_\alpha^{-1/2} Z_c Z_V = 1, \tag{1.1}$$

where Z_α , Z_c , and Z_V are the renormalization constants for the gauge coupling constant, the Faddeev-Popov ghosts, and the quantum gauge superfield, respectively. However, the most interesting quantum feature of $\mathcal{N} = 1$ supersymmetric gauge theories is the existence

of a relation between the β -function and the anomalous dimension of the matter superfields called “the exact NSVZ β -function” [9–12]. Usually it is written as¹

$$\beta(\alpha, \lambda) = -\frac{\alpha^2(3C_2 - T(R) + C(R)_i^j(\gamma_\phi)_j^i(\alpha, \lambda)/r)}{2\pi(1 - C_2\alpha/(2\pi))}, \quad (1.2)$$

where r is the dimension of the gauge group G with the structure constants f^{ABC} , and the group Casimirs are defined as

$$f^{ABC}f^{ABD} = \delta^{CD}C_2; \quad \text{tr}(T^AT^B) = T(R)\delta^{AB}; \quad C(R)_i^j = (T^AT^A)_i^j. \quad (1.3)$$

In our notation T^A are the generators of the representation to which the matter superfields belong. They should be distinguished from the generators of the fundamental representation denoted by t^A , which are assumed to be normalized by the condition $\text{tr}(t^At^B) = \delta^{AB}/2$.

If $\mathcal{N} = 2$ supersymmetric gauge theories are considered as a particular case of $\mathcal{N} = 1$ theories, then the NSVZ relation leads to the finiteness beyond the one-loop approximation, provided the quantization is manifestly $\mathcal{N} = 2$ supersymmetric [13, 14]. A natural way to provide this is to use harmonic superspace [15, 16] and an invariant regularization [17]. In this case we will also obtain the all-loop finiteness of $\mathcal{N} = 4$ SYM theory as a consequence of the NSVZ equation.

It is important to recall that the NSVZ equation is valid only for certain renormalization prescriptions which are usually called “the NSVZ schemes”. In particular, the most popular $\overline{\text{DR}}$ scheme (which is obtained in the case of using dimensional reduction [18] supplemented by modified minimal subtraction [19]) is not NSVZ [20–24]. The MOM scheme is also not NSVZ [25]. Nevertheless, the $\overline{\text{DR}}$ calculations implicitly confirm the NSVZ equation, because they demonstrate the validity of scheme-independent consequences following from the NSVZ equation [25, 26]. These scheme independent relations appear because some terms in the renormalization group functions (RGFs) remain invariant under finite renormalizations, see, e.g., [27]. The fact that these relations are satisfied indicates the existence of NSVZ schemes. In fact, there are an infinite number of the NSVZ schemes which constitute a continuous set. In the Abelian case this set has been described in ref. [28] and, in particular, includes the on-shell [29] and HD+MSL [30] schemes. The latter scheme is obtained if a theory is regularized by higher covariant derivatives [31, 32] (see refs. [33, 34] for supersymmetric versions of this regularization) and the divergences are removed by minimal subtractions of logarithms, when only powers of $\ln \Lambda/\mu$ are included into renormalization constants [35, 36]. Equivalently, this scheme can be introduced by imposing certain boundary conditions on the renormalization constants [30]. Presumably, the HD+MSL scheme is also NSVZ in the non-Abelian case [8]. This is confirmed by some explicit calculations made in such an approximation where the scheme dependence is essential [37–39].

The reason why the HD+MSL scheme turns out to be NSVZ is that the NSVZ equation is satisfied by RGFs defined in terms of the bare couplings for theories regularized by higher

¹In this section we do not specify the definition of the renormalization group functions. This will be done below.

covariant derivatives. Such RGFs are independent of a renormalization prescription for a fixed regularization, so that eq. (1.2) holds for them in an arbitrary subtraction scheme. However, the calculation of ref. [40] indicates that with dimensional reduction these RGFs do not satisfy the NSVZ equation starting from the three-loop approximation for the β -function. For Abelian theories regularized by higher derivatives the validity of the NSVZ equation for RGFs defined in terms of the bare couplings has been proved in all loops in refs. [41, 42]. This proof is based on the fact that the loop integrals which give the β -function defined in terms of the bare coupling constant in supersymmetric theories are integrals of double total derivatives with respect to the loop momenta. The factorization into total and double total derivatives has first been noted in calculating the lowest-order quantum corrections for $\mathcal{N} = 1$ supersymmetric electrodynamics (SQED) in refs. [43] and [44], respectively. The subsequent generalizations of the proof made in ref. [41] allowed deriving the all-loop NSVZ-like equations for the Adler D -function [45] in $\mathcal{N} = 1$ supersymmetric chromodynamics [46, 47] and for the renormalization of the photino mass in softly broken $\mathcal{N} = 1$ SQED [48]. (In softly broken supersymmetric theories the NSVZ-like equation for the renormalization of the gaugino mass has first been found in [49–51].) In both cases the HD+MSL scheme is NSVZ [52, 53], because in this scheme RGFs defined in terms of the renormalized couplings coincide with RGFs defined in terms of the bare couplings up to the renaming of arguments, see eq. (2.23) below.

In the non-Abelian case the all-loop proof of the NSVZ equation by a similar method has not yet been completed, although its main steps are at present quite clear. First, the NSVZ equation should be rewritten in the equivalent form [8]

$$\frac{\beta(\alpha, \lambda)}{\alpha^2} = -\frac{1}{2\pi} \left(3C_2 - T(R) - 2C_2\gamma_c(\alpha, \lambda) - 2C_2\gamma_V(\alpha, \lambda) + C(R)_i{}^j (\gamma_\phi)_j{}^i(\alpha, \lambda)/r \right) \quad (1.4)$$

with the help of eq. (1.1), where γ_c and γ_V are the anomalous dimensions of the Faddeev-Popov ghosts and the quantum gauge superfield, respectively. According to [54] the β -function is given by integrals of double total derivatives with respect to the loop momenta in all loops if the higher derivatives are used for regularization. (Again, in the case of using dimensional reduction it is not so [55].) Certainly, this is confirmed by a large number of explicit calculations [37, 38, 56–58]. Due to this structure of the loop integrals the β -function beyond the one-loop approximation is given by a sum of δ -singularities, which appear when double total derivatives act on an inverse squared momentum. The all-loop sums of singularities which occur when the double total derivatives act on inverse squared momenta of the matter superfields and of the Faddeev-Popov ghosts give the corresponding anomalous dimensions in eq. (1.4) [59]. For remaining singularities produced by momenta of the quantum gauge superfield the corresponding paper is in preparation. Thus, there are strong indications that the NSVZ equation is satisfied by RGFs defined in terms of the bare couplings for theories regularized by higher covariant derivatives independently of a way of renormalization.

The above discussion demonstrates that the higher covariant derivative regularization helps to reveal the underlying structure of quantum corrections which is responsible for the appearance of the NSVZ equation in perturbation theory. That is why it is especially

interesting to use it for investigating $\mathcal{N} = 1$ finite theories (see ref. [60] for a recent review of the theoretical aspects and the phenomenological applications). The direct calculation of ref. [61] made in the $\overline{\text{DR}}$ scheme demonstrated that if an $\mathcal{N} = 1$ supersymmetric gauge theory is finite in the one-loop approximation, then it is also finite in the two-loop approximation. The same result was obtained from arguments based on anomalies [62, 63]. According to [64], for theories finite in the L -th loop the β -function vanishes in the $(L+1)$ -th loop. The same statement immediately follows from the NSVZ equation. We know that the NSVZ equation naturally appears with the higher covariant derivative regularization and is not valid in the $\overline{\text{DR}}$ scheme. Therefore, we are tempted to suggest that the higher derivative regularization could reveal some features of one-loop finite $\mathcal{N} = 1$ supersymmetric theories leading to their possible all-loop finiteness. For this purpose one should investigate the anomalous dimension of the matter superfields. The calculation made in ref. [65] demonstrated that in the $\overline{\text{DR}}$ scheme the three-loop anomalous dimension does not vanish. However, it has explicitly been verified [66] that it is possible to tune a subtraction scheme in such a way that a one-loop finite theory will also be finite in the three-loop approximation. According to the general argumentation of refs. [67–70], a scheme in which a one-loop finite theory is finite in all loops should exist, but at present there is no simple prescription for constructing it. Possibly, the use of the higher covariant derivative regularization could help to solve this problem.

One more interesting subject is the possible existence of the exact expression for the anomalous dimension of the matter superfields for theories obeying the $P = \frac{1}{3}Q$ constraint proposed by Jack and Jones in ref. [71]. In our notation it can be written as

$$\lambda_{imn}^* \lambda^{jmn} - 4\pi\alpha C(R)_i^j = \frac{2\pi\alpha}{3} Q \delta_i^j, \quad \text{where} \quad Q \equiv T(R) - 3C_2. \quad (1.5)$$

According to ref. [66], for theories which satisfy the condition (1.5) the anomalous dimension of the matter superfields can possibly be written in the Jack, Jones, North (JJN) form

$$(\gamma_\phi)_i^j(\alpha, \lambda) \rightarrow (\gamma_\phi)_i^j(\alpha) = \frac{\alpha Q}{6\pi(1 + \alpha Q/6\pi)} \delta_i^j, \quad (1.6)$$

while the β -function does not depend on the Yukawa couplings and is given by the geometric series

$$\beta(\alpha, \lambda) \rightarrow \beta(\alpha) = \frac{\alpha^2 Q}{2\pi(1 + \alpha Q/6\pi)}. \quad (1.7)$$

(Note that here we again do not specify a definition of RGFs, a regularization, and a renormalization prescription.) Although the three-loop calculation of ref. [66] presumably excludes this possibility, this particular case seems to be very interesting and deserving a further investigation. As a justification, it is possible to suggest that eq. (1.6) can be valid in higher loops if some more constraints are imposed on the theory together with eq. (1.5).

In this paper we consider a general renormalizable $\mathcal{N} = 1$ supersymmetric theory regularized by higher covariant derivatives, which is described in section 2. In section 3 for this theory we calculate the two-loop anomalous dimension of the matter superfields defined in terms of the bare couplings. It is demonstrated that there is a simple relation between

the higher derivative regulators and the Pauli-Villars masses for which it vanishes for the one-loop finite theories. Also there is a regularization for which eq. (1.6) is valid for the anomalous dimension defined in terms of the bare couplings under the $P = \frac{1}{3}Q$ constraint. Certainly, the same statements are valid for the anomalous dimension defined in terms of the renormalized couplings in the HD+MSL scheme. For a general renormalization prescription the expression for the anomalous dimension defined in terms of the renormalized couplings is found in section 4. Using the statement that the NSVZ equation is presumably valid for RGFs defined in terms of the bare couplings with the higher derivative regularization, the expression for the three-loop β -function is written in section 5. Again this is done for the β -functions defined both in terms of the bare couplings and in terms of the renormalized couplings. The particular cases of the one-loop finite theories and theories satisfying the constraint (1.5) are investigated. Also in section 5 we demonstrate that for the one-loop finite theories the NSVZ equation in the considered approximation is valid in an arbitrary subtraction scheme.

2 The theory under consideration

We consider the $\mathcal{N} = 1$ SYM theory with a simple gauge group G interacting with the chiral matter superfields ϕ_i in a representation R , which can in general be reducible. At the classical level this theory in the massless limit is described by the superfield action [72–74]

$$S = \frac{1}{2e_0^2} \text{tr} \text{Re} \int d^4x d^2\theta W^a W_a + \frac{1}{4} \int d^4x d^4\theta \phi^{*i} (e^{2V})_i{}^j \phi_j + \left(\frac{1}{6} \int d^4x d^2\theta \lambda_0^{ijk} \phi_i \phi_j \phi_k + \text{c.c.} \right), \quad (2.1)$$

where the supersymmetric gauge field strength is given by the chiral superfield $W_a = \bar{D}^2(e^{-2V} D_a e^{2V})/8$. Note that $V = e_0 V^A t^A$ in the first term of the action (2.1), while $V = e_0 V^A T^A$ in the second one.

To quantize the theory, one should take into account that the quantum gauge superfield is renormalized in a nonlinear way [75–77]. Also it is convenient to use the background field method [78–80] in the superfield formulation [1, 72], because it produces a manifestly gauge invariant effective action. All this can be achieved by making the replacement

$$e^{2V} \rightarrow e^{2\mathcal{F}(V)} e^{2\mathbf{V}}, \quad (2.2)$$

where $\mathbf{V}$ and V are the background and quantum superfields, respectively. Note that in our notation the latter superfield satisfies the constraint $V^+ = e^{-2\mathbf{V}} V e^{2\mathbf{V}}$. The parameters of the nonlinear renormalization are included into the function

$$\mathcal{F}(V) = V + O(V^3). \quad (2.3)$$

The first nonlinear term in this function has been calculated in refs. [81, 82]. Subsequently it was demonstrated that the presence of the nonlinear renormalization is very essential for the renormalization group equations to be satisfied [83].

After the replacement (2.2), the action regularized by higher covariant derivatives in the massless limit can be written as

$$\begin{aligned}
 S_{\text{reg}} = & \frac{1}{2e_0^2} \text{Re tr} \int d^4x d^2\theta W^a \left[e^{-2\mathbf{V}} e^{-2\mathcal{F}(V)} R \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right) e^{2\mathcal{F}(V)} e^{2\mathbf{V}} \right]_{\text{Adj}} W_a \\
 & + \frac{1}{4} \int d^4x d^4\theta \phi^{*i} \left[F \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right) e^{2\mathcal{F}(V)} e^{2\mathbf{V}} \right]_i^j \phi_j \\
 & + \left(\frac{1}{6} \lambda_0^{ijk} \int d^4x d^2\theta \phi_i \phi_j \phi_k + \text{c.c.} \right),
 \end{aligned} \tag{2.4}$$

where the left and right covariant spinor derivatives are given by the equations

$$\bar{\nabla}_{\dot{a}} \equiv e^{2\mathcal{F}(V)} e^{2\mathbf{V}} \bar{D}_{\dot{a}} e^{-2\mathbf{V}} e^{-2\mathcal{F}(V)}; \quad \nabla_a \equiv D_a, \tag{2.5}$$

respectively. The regulator functions $F(x)$ and $R(x)$ are infinite at infinity and approach 1 at $x = 0$. Note that in eq. (2.4) the gauge superfield strength is defined as

$$W_a \equiv \frac{1}{8} \bar{D}^2 \left[e^{-2\mathbf{V}} e^{-2\mathcal{F}(V)} D_a \left(e^{2\mathcal{F}(V)} e^{2\mathbf{V}} \right) \right]. \tag{2.6}$$

In this paper we will use the gauge fixing term

$$S_{\text{gf}} = -\frac{1}{16\xi_0 e_0^2} \text{tr} \int d^4x d^4\theta \nabla^2 V R \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right)_{\text{Adj}} \bar{\nabla}^2 V \tag{2.7}$$

containing the parameter ξ_0 . Due to the presence of the background covariant derivatives

$$\bar{\nabla}_{\dot{a}} \equiv e^{2\mathbf{V}} \bar{D}_{\dot{a}} e^{-2\mathbf{V}}; \quad \nabla_a \equiv D_a \tag{2.8}$$

it is invariant under the background gauge symmetry. According to [58] the one-loop renormalization of the gauge parameter is described by the equation

$$\frac{1}{\xi_0 e_0^2} = \frac{1}{\xi e^2} + \frac{C_2(1-\xi)}{12\pi^2 \xi} \left(\ln \frac{\Lambda}{\mu} + a_1 \right) + O(e^2), \tag{2.9}$$

where a_1 is a finite constant which originates from the arbitrariness in choosing a renormalization prescription. In this paper we will use the Feynman gauge $\xi = 1$ in which

$$\xi_0 e_0^2 = e^2 + O(e^6). \tag{2.10}$$

The quantization procedure (see, e.g., [73]) also requires introducing the Faddeev-Popov and Nielsen-Kallosh ghosts. Their actions (S_{FP} and S_{NK} , respectively) can be found, e.g., in ref. [87].

Due to the presence of the higher derivative regulators $R(x)$ and $F(x)$ in the actions (2.4) and (2.7) all divergences disappear beyond the one-loop approximation. This is a general feature of the higher covariant derivative regularization, see, e.g., [84]. For regularizing the remaining one-loop divergences one has to introduce the Pauli-Villars determinants [85]. Following refs. [58, 86], we define the generating functional as

$$\begin{aligned}
 Z[\text{Sources}] = & \int D\mu \text{Det}(PV, M_\varphi)^{-1} (\text{Det}(PV, M))^c \\
 & \times \exp \left\{ i \left(S_{\text{reg}} + S_{\text{gf}} + S_{\text{FP}} + S_{\text{NK}} + S_{\text{sources}} \right) \right\},
 \end{aligned} \tag{2.11}$$

where the Pauli-Villars determinants are given by the functional integrals

$$\text{Det}(PV, M_\varphi)^{-1} = \int D\varphi_1 D\varphi_2 D\varphi_3 e^{iS_\varphi}; \quad \text{Det}(PV, M)^{-1} = \int D\Phi e^{iS_\Phi}. \quad (2.12)$$

Here φ_a is a set of three commuting chiral superfields in the adjoint representation with the action

$$S_\varphi = \frac{1}{2e_0^2} \text{tr} \int d^4x d^4\theta \left\{ \varphi_1^+ \left[R \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right) e^{2\mathcal{F}(V)} e^{2V} \right]_{\text{Adj}} \varphi_1 + \varphi_2^+ [e^{2\mathcal{F}(V)} e^{2V}]_{\text{Adj}} \varphi_2 \right. \\ \left. + \varphi_3^+ [e^{2\mathcal{F}(V)} e^{2V}]_{\text{Adj}} \varphi_3 \right\} + \frac{1}{2e_0^2} \left(\text{tr} \int d^4x d^2\theta M_\varphi (\varphi_1^2 + \varphi_2^2 + \varphi_3^2) + \text{c.c.} \right), \quad (2.13)$$

and Φ_i is a multiplet of commuting chiral superfields in a representation R_{PV} that admits a gauge invariant mass term with the action

$$S_\Phi = \frac{1}{4} \int d^4x d^4\theta \Phi^{*i} \left[F \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right) e^{2\mathcal{F}(V)} e^{2V} \right]_i^j \Phi_j + \frac{1}{4} \left(\int d^4x d^2\theta M^{ij} \Phi_i \Phi_j + \text{c.c.} \right). \quad (2.14)$$

We assume that the invariant tensor M^{ij} satisfies the condition

$$M^{ik} M_{kj}^* = M^2 \delta_j^i, \quad (2.15)$$

and the Pauli-Villars masses are proportional to the constant Λ in the higher derivative terms,

$$M_\varphi = a_\varphi \Lambda; \quad M = a \Lambda, \quad (2.16)$$

where a_φ and a are independent of couplings. Then the one-loop divergences and subdivergences will be regularized if the constant c in the generating functional (2.11) is equal to $T(R)/T(R_{PV})$.

In this paper we will calculate the anomalous dimension of the matter superfields. To construct it, first, one should consider the part of the effective action which corresponds to self-energy diagrams for the chiral matter superfields. It can be written in the form

$$\Gamma_\phi^{(2)} = \frac{1}{4} \int \frac{d^4q}{(2\pi)^4} d^4\theta \phi^{*i}(-q, \theta) \phi_j(q, \theta) (G_\phi)_i^j(\alpha_0, \lambda_0, q^2/\Lambda^2), \quad (2.17)$$

where $(G_\phi)_i^j = \delta_i^j + (\Delta G_\phi)_i^j$, and $(\Delta G_\phi)_i^j$ is of order α_0 or λ_0^2 .

Let $\alpha \equiv e^2/4\pi$ and λ^{ijk} denote the renormalized gauge and Yukawa coupling constants, respectively. The bare and renormalized chiral matter superfields are related by the equation

$$\phi_i = (\sqrt{Z_\phi})_i^j (\phi_R)_j, \quad (2.18)$$

where the renormalization constant $(Z_\phi)_i^j$ is determined by requiring the finiteness of the product $(Z_\phi)_i^k (G_\phi)_k^j$ expressed in terms of the renormalized couplings in the limit $\Lambda \rightarrow \infty$. Note that this requirement does not allow fixing this constant uniquely. The remaining arbitrariness is removed by choosing a renormalization prescription.

Divergences of a theory are encoded in RGFs. According to [30], it is necessary to distinguish between RGFs defined in terms of the bare couplings and the ones (standardly)

defined in terms of the renormalized couplings. In terms of the bare couplings, the definitions of the β -function and the anomalous dimension of the chiral matter superfields read as

$$\beta(\alpha_0, \lambda_0) = \left. \frac{d\alpha_0}{d \ln \Lambda} \right|_{\alpha, \lambda = \text{const}}; \quad (2.19)$$

$$(\gamma_\phi)_i^j(\alpha_0, \lambda_0) = - \left. \frac{d(\ln Z_\phi)_i^j}{d \ln \Lambda} \right|_{\alpha, \lambda = \text{const}}. \quad (2.20)$$

For a fixed regularization these RGFs are independent of a renormalization procedure. That is why they (presumably) satisfy the NSVZ relation in an arbitrary subtraction scheme supplementing the higher covariant derivative regularization. In terms of the renormalized couplings RGFs are defined by the equations

$$\tilde{\beta}(\alpha, \lambda) = \left. \frac{d\alpha}{d \ln \mu} \right|_{\alpha_0, \lambda_0 = \text{const}}; \quad (2.21)$$

$$(\tilde{\gamma}_\phi)_i^j(\alpha, \lambda) = \left. \frac{d(\ln Z_\phi)_i^j}{d \ln \mu} \right|_{\alpha_0, \lambda_0 = \text{const}}, \quad (2.22)$$

where the derivatives are taken with respect to the logarithm of the renormalization point μ at fixed values of the bare couplings. The β -function (2.21) and the anomalous dimension (2.22) are scheme-dependent starting from the three- and two-loop approximation, respectively. Up to the renaming of arguments both definitions of RGFs coincide in the HD+MSL scheme, in which the renormalization constants contain only powers of $\ln \Lambda/\mu$ and all finite constants fixing a subtraction scheme are set to 0,

$$\beta(\alpha_0 \rightarrow \alpha, \lambda_0 \rightarrow \lambda) = \tilde{\beta}(\alpha, \lambda) \Big|_{\text{HD+MSL}}; \quad (\gamma_\phi)_i^j(\alpha_0 \rightarrow \alpha, \lambda_0 \rightarrow \lambda) = (\tilde{\gamma}_\phi)_i^j(\alpha, \lambda) \Big|_{\text{HD+MSL}}. \quad (2.23)$$

RGFs defined in terms of the bare couplings are related to the corresponding Green functions. For example, using the finiteness of the expression $(Z_\phi G_\phi)_i^j$ the anomalous dimension (2.20), which will be calculated in this paper, can be presented in the equivalent form

$$(\gamma_\phi)_i^j(\alpha_0, \lambda_0) = \left. \frac{d(\ln G_\phi)_i^j}{d \ln \Lambda} \right|_{\alpha, \lambda = \text{const}; q \rightarrow 0}, \quad (2.24)$$

where the limit $q \rightarrow 0$ is necessary here to get rid of the terms vanishing in the limit $\Lambda \rightarrow \infty$.

To use eq. (2.24), one needs to know how the renormalized and bare couplings are related at lower orders. In particular, for calculating the two-loop anomalous dimension of the matter superfields, one needs the one-loop relation between the bare and renormalized couplings. Moreover, we will also need the two-loop relation between the bare and renormalized gauge coupling constants for investigating the three-loop β -function defined in terms of the renormalized couplings in section 5.2. It can be found by integrating eq. (2.19) in which we substitute the two-loop β -function. With the regularization considered in this paper it has been calculated in ref. [87]. The result contains some arbitrary integration

constants b_i , which reflect the arbitrariness in choosing a subtraction scheme,

$$\begin{aligned} \frac{1}{\alpha} - \frac{1}{\alpha_0} = & -\frac{3}{2\pi}C_2\left(\ln\frac{\Lambda}{\mu} + b_{11}\right) + \frac{1}{2\pi}T(R)\left(\ln\frac{\Lambda}{\mu} + b_{12}\right) - \frac{3\alpha}{4\pi^2}C_2^2\left(\ln\frac{\Lambda}{\mu} + b_{21}\right) \\ & + \frac{\alpha}{4\pi^2r}C_2\text{tr}C(R)\left(\ln\frac{\Lambda}{\mu} + b_{22}\right) + \frac{\alpha}{2\pi^2r}\text{tr}[C(R)^2]\left(\ln\frac{\Lambda}{\mu} + b_{23}\right) \\ & - \frac{1}{8\pi^3r}C(R)_j{}^i\lambda_{imn}^*\lambda^{jmn}\left(\ln\frac{\Lambda}{\mu} + b_{24}\right) + O(\alpha^2, \alpha\lambda^2, \lambda^4). \end{aligned} \quad (2.25)$$

Due to the non-renormalization theorem of ref. [7], the superpotential does not receive divergent radiative corrections. This implies that the renormalization of the Yukawa couplings is related to the renormalization of the matter superfields by the equation

$$\lambda^{ijk} = (\sqrt{Z_\phi})_l{}^i (\sqrt{Z_\phi})_m{}^j (\sqrt{Z_\phi})_n{}^k \lambda_0^{lmn}. \quad (2.26)$$

In this paper we will consider only such renormalization schemes for which it is valid (although, in general, it is possible to construct subtraction schemes breaking this equation). The one-loop expression for the renormalization constant $(Z_\phi)_i{}^j$ can be written as

$$(Z_\phi)_i{}^j(\alpha, \lambda) = \delta_i{}^j + \frac{\alpha}{\pi}C(R)_i{}^j\left(\ln\frac{\Lambda}{\mu} + g_{11}\right) - \frac{1}{4\pi^2}\lambda_{imn}^*\lambda^{jmn}\left(\ln\frac{\Lambda}{\mu} + g_{12}\right) + O(\alpha^2, \alpha\lambda^2, \lambda^4), \quad (2.27)$$

where g_{11} and g_{12} are arbitrary constants similar to b_i in eq. (2.25). Eqs. (2.26) and (2.27) determine the one-loop renormalization of the Yukawa couplings,

$$\begin{aligned} \lambda_0^{ijk} = & \lambda^{ijk} - \frac{\alpha}{2\pi}\left(C(R)_m{}^i\lambda^{mjk} + C(R)_m{}^j\lambda^{imk} + C(R)_m{}^k\lambda^{ijm}\right)\left(\ln\frac{\Lambda}{\mu} + g_{11}\right) \\ & + \frac{1}{8\pi^2}\left(\lambda^{mjk}\lambda_{mab}^*\lambda^{iab} + \lambda^{imk}\lambda_{mab}^*\lambda^{jab} + \lambda^{ijm}\lambda_{mab}^*\lambda^{kab}\right)\left(\ln\frac{\Lambda}{\mu} + g_{12}\right) \\ & + O(\alpha^2\lambda, \alpha\lambda^3, \lambda^5). \end{aligned} \quad (2.28)$$

By definition, in the HD+MSL renormalization scheme all constants b_i and g_i are set to 0.

3 Two-loop anomalous dimension

Superdiagrams contributing to the two-loop anomalous dimension of the matter superfields can be divided into two parts. The first part contains the superdiagrams without Yukawa vertices. They are presented in figure 1. The gray circles in the superdiagrams (17) and (18) denote insertions of the one-loop polarization operator of the quantum gauge superfield, which is equal to the sum of the supergraphs presented in figure 2. They have been calculated in ref. [86]. The superdiagrams of the second part presented in figure 3 contain the Yukawa vertices. They have already been calculated in refs. [37, 38] for the higher derivative regulators $R(x) = 1 + x^m$ and $F(x) = 1 + x^n$. In this paper the integrals written in refs. [37, 38] are calculated for an arbitrary form of the functions $R(x)$ and $F(x)$. The details of this calculation are presented in appendix A.

Figure 1. Superdiagrams without Yukawa vertices contributing to the two-loop anomalous dimension of the matter superfields. In the diagrams (17) and (18) the gray circles denote insertions of the one-loop polarization operator of the quantum gauge superfield.

$$\text{Diagram with gray circle} = \left\{ \begin{array}{ccc} \text{Wavy loop} & \text{Dashed loop} & \text{Solid loop} \\ \text{Wavy loop with external lines} & \text{Dashed loop with external lines} & \text{Solid loop with external lines} \end{array} \right.$$

Figure 2. The one-loop polarization operator of the quantum gauge superfield. The second column contains diagrams with a loop of the Faddeev-Popov ghosts, and the third one contains diagrams with a loop of the matter and Pauli-Villars superfields.

Figure 3. Superdiagrams contributing to the two-loop anomalous dimension of the matter superfields which contain Yukawa vertices.

The superdiagrams presented in figure 1 have been calculated in ref. [52] for $\mathcal{N} = 1$ SQCD. It is essential that the action of $\mathcal{N} = 1$ SQCD does not contain the Yukawa term cubic in the matter superfields, so that there is no need for the regulator $F(x)$. That is why the calculation of ref. [52] has been done for $F(x) = 1$. However, for theories containing the Yukawa interaction the higher covariant derivative regularization should include $F(x) \neq 1$. In this paper we consider the general case. The presence of the function F generates new vertices, which have to be taken into account. For example, the triple gauge-matter vertex is written as

$$\frac{1}{2} \int d^4x d^4\theta \phi^+ \left\{ VF \left(\frac{\partial^2}{\Lambda^2} \right) + \sum_{\alpha=1}^{\infty} f_{\alpha} \sum_{\beta=0}^{\alpha-1} \left(\frac{\partial^2}{\Lambda^2} \right)^{\beta} \frac{\bar{D}^2[V, D^2]}{16\Lambda^2} \left(\frac{\partial^2}{\Lambda^2} \right)^{\alpha-1-\beta} \right\} \phi, \quad (3.1)$$

where the coefficients f_{α} are determined by the equation $F(x) = 1 + \sum_{\alpha=1}^{\infty} f_{\alpha} x^{\alpha}$.

Remarkably, the sum of the supergraphs (1) — (16) in figure 1 turns out to be independent of F . This regulator is present only in the superdiagrams (17) and (18) inside the expression for the polarization operator. Explicitly writing the sum of the superdiagrams depicted in figure 1, the two-loop anomalous dimension of the matter superfields can be presented as

$$\begin{aligned} (\gamma_{\phi})_i^j &= \frac{d}{d \ln \Lambda} \left\{ -C(R)_i^j \int \frac{d^4k}{(2\pi)^4} \frac{2e_0^2}{k^4 R_k} \left[1 - \frac{2e_0^2}{R_k} \left(C_2 f(k/\Lambda) + T(R) h(k/\Lambda) \right) \right] \right. \\ &\quad \left. + 4 [C(R)^2]_i^j e_0^4 \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \left(\frac{1}{(l+k)^2 R_{l+k} k^4 R_k l^2} - \frac{1}{2l^4 R_l k^4 R_k} \right) \right\}_{\alpha, \lambda = \text{const}} \\ &\quad + \lambda\text{-dependent terms} + \text{higher orders}. \end{aligned} \quad (3.2)$$

The explicit expression for the λ -dependent terms can be found in ref. [38], see also appendix A. The functions $f(k/\Lambda)$ and $h(k/\Lambda)$ are related to the one-loop polarization operator of the quantum gauge superfield in the Feynman gauge,

$$\Pi(\alpha_0, \lambda_0, k^2/\Lambda^2) = -8\pi\alpha_0 \left(C_2 f(k/\Lambda) + T(R) h(k/\Lambda) \right) + O(\alpha_0^2, \alpha_0 \lambda_0^2). \quad (3.3)$$

In our notation the polarization operator $\Pi(\alpha_0, \lambda_0, k^2/\Lambda^2)$ is defined by the equation

$$d_q^{-1}(\alpha_0, \lambda_0, k^2/\Lambda^2) - \alpha_0^{-1} R(k^2/\Lambda^2) \equiv -\alpha_0^{-1} \Pi(\alpha_0, \lambda_0, k^2/\Lambda^2), \quad (3.4)$$

where

$$\Gamma_V^{(2)} - S_{\text{gf}}^{(2)} \equiv -\frac{1}{8\pi} \text{tr} \int \frac{d^4 k}{(2\pi)^4} d^4 \theta V(-k, \theta) \partial^2 \Pi_{1/2} V(k, \theta) d_q^{-1}(\alpha_0, \lambda_0, k^2/\Lambda^2). \quad (3.5)$$

The explicit expressions for the functions $f(k/\Lambda)$ and $h(k/\Lambda)$ are rather large. They can be found in ref. [86]. However, in this paper we need only their asymptotic behavior at small k ,

$$f(k/\Lambda) = -\frac{3}{16\pi^2} \left(\ln \frac{\Lambda}{k} + \ln a_\varphi + 1 + o(1) \right); \quad (3.6)$$

$$h(k/\Lambda) = \frac{1}{16\pi^2} \left(\ln \frac{\Lambda}{k} + \ln a + 1 + o(1) \right), \quad (3.7)$$

where $o(1)$ denotes terms that vanish in the limit $k \rightarrow 0$.

In eq. (3.2) the differentiation with respect to $\ln \Lambda$ is to be performed at fixed values of the renormalized couplings α and λ before the momentum integration. This makes the integrals well-defined and finite in the infrared region.

The integrals giving the two-loop anomalous dimension are calculated in appendix A. We calculate both the integrals explicitly written in eq. (3.2) and the integrals present in the λ -dependent terms for an arbitrary form of the functions $R(x)$ and $F(x)$. The result for the two-loop anomalous dimension defined in terms of the bare couplings obtained in appendix A is written as

$$\begin{aligned} (\gamma_\phi)_i^j(\alpha_0, \lambda_0) = & -\frac{\alpha_0}{\pi} C(R)_i^j + \frac{1}{4\pi^2} \lambda_{0imn}^* \lambda_0^{jmn} + \frac{\alpha_0^2}{2\pi^2} [C(R)^2]_i^j - \frac{1}{16\pi^4} \lambda_{0iac}^* \lambda_0^{jab} \lambda_{0bde}^* \lambda_0^{cde} \\ & - \frac{3\alpha_0^2}{2\pi^2} C_2 C(R)_i^j \left(\ln a_\varphi + 1 + \frac{A}{2} \right) + \frac{\alpha_0^2}{2\pi^2} T(R) C(R)_i^j \left(\ln a + 1 + \frac{A}{2} \right) \\ & - \frac{\alpha_0}{8\pi^3} \lambda_{0lmn}^* \lambda_0^{jmn} C(R)_i^l (1 - B + A) \\ & + \frac{\alpha_0}{4\pi^3} \lambda_{0imn}^* \lambda_0^{jml} C(R)_l^n (1 - A + B) + O(\alpha_0^3, \alpha_0^2 \lambda_0^2, \alpha_0 \lambda_0^4, \lambda_0^6), \end{aligned} \quad (3.8)$$

where the constants A and B are given by the integrals

$$A = \int_0^\infty dx \ln x \frac{d}{dx} \frac{1}{R(x)}; \quad B = \int_0^\infty dx \ln x \frac{d}{dx} \frac{1}{F^2(x)}. \quad (3.9)$$

For the regulators $R(x) = 1 + x^m$ and $F(x) = 1 + x^n$ (which were used in refs. [38, 88]) these integrals can be taken,

$$A = 0; \quad B = \frac{1}{n}. \quad (3.10)$$

For the particular case of $\mathcal{N} = 1$ SQED with N_f flavors eq. (3.8) gives

$$\gamma(\alpha_0) = -\frac{\alpha_0}{\pi} + \frac{\alpha_0^2}{\pi^2} \left(N_f \ln a + N_f + \frac{N_f A}{2} + \frac{1}{2} \right) + O(\alpha_0^3). \quad (3.11)$$

This expression agrees with the results of refs. [26, 88, 89] obtained for $R(x) = 1 + x^m$ (and, therefore, $A = 0$) and generalizes them to the case of an arbitrary regulator function. Note that according to ref. [89] eq. (3.11) is valid for an arbitrary ξ -gauge.

The anomalous dimension (3.8) is considerably simplified for theories finite in the one-loop approximation, which are obtained if

$$T(R) = 3C_2; \quad \lambda_{0imn}^* \lambda_0^{jmn} = 4\pi\alpha_0 C(R)_i^j. \quad (3.12)$$

The first equation (which is an analog of the Banks-Zaks prescription [90]) leads to the vanishing one-loop β -function, while the second one follows from the vanishing of the one-loop anomalous dimension of the matter superfields. In this case we obtain

$$\begin{aligned} (\gamma_\phi)_i^j(\alpha_0, \lambda_0) = & -\frac{3\alpha_0^2}{2\pi^2} C_2 C(R)_i^j \ln \frac{a_\varphi}{a} \\ & -\frac{\alpha_0}{4\pi^2} \left(\frac{1}{\pi} \lambda_{0imn}^* \lambda_0^{jml} C(R)_l^n + 2\alpha_0 [C(R)^2]_i^j \right) (A - B) \\ & + O(\alpha_0^3, \alpha_0^2 \lambda_0^2, \alpha_0 \lambda_0^4, \lambda_0^6). \end{aligned} \quad (3.13)$$

This implies that the one-loop finite theories remain finite in the two-loop approximation if the higher derivative regularization is chosen in such a way that $A = B$ and $a = a_\varphi$. The first condition is automatically satisfied if $R(x) = F^2(x)$. That is why a version of the higher covariant derivative regularization leading to the two-loop finiteness is obtained by imposing the conditions

$$R(x) = F^2(x) \quad \text{and} \quad a = a_\varphi. \quad (3.14)$$

Possibly, this could help to reveal if the finiteness of the considered class of supersymmetric theories takes place in all loops and what is its underlying reason.

Another interesting particular case corresponds to the theories which satisfy the $P = \frac{1}{3}Q$ condition. In terms of the bare couplings it is written as

$$\lambda_{0imn}^* \lambda_0^{jmn} - 4\pi\alpha_0 C(R)_i^j = \frac{2\pi\alpha_0}{3} Q \delta_i^j. \quad (3.15)$$

In this case the anomalous dimension (3.8) takes the form

$$\begin{aligned} (\gamma_\phi)_i^j(\alpha_0, \lambda_0) = & \left(\frac{\alpha_0}{6\pi} Q - \frac{\alpha_0^2}{36\pi^2} Q^2 \right) \delta_i^j \\ & -\frac{\alpha_0}{4\pi^2} \left(\frac{1}{\pi} \lambda_{0imn}^* \lambda_0^{jml} C(R)_l^n + 2\alpha_0 [C(R)^2]_i^j \right) (A - B) \\ & -\frac{\alpha_0^2}{12\pi^2} Q C(R)_i^j (A - B) - \frac{3\alpha_0^2}{2\pi^2} C_2 C(R)_i^j \left(\ln a_\varphi + \frac{1}{2} + \frac{A}{2} \right) \\ & + \frac{\alpha_0^2}{2\pi^2} T(R) C(R)_i^j \left(\ln a + \frac{1}{2} + \frac{A}{2} \right) + O(\alpha_0^3, \alpha_0^2 \lambda_0^2, \alpha_0 \lambda_0^4, \lambda_0^6). \end{aligned} \quad (3.16)$$

From this result we see that in the considered approximation the anomalous dimension satisfies the exact JJN equation (1.6) if the parameters of the higher covariant derivative regularization satisfy the conditions

$$A = B; \quad a = a_\varphi = \exp \left(-\frac{1}{2}(A + 1) \right). \quad (3.17)$$

In particular, the first equality is valid for $R(x) = F^2(x)$, while the second one can be obtained by a special choice of the Pauli-Villars masses. In this case

$$\begin{aligned}
 (\gamma_\phi)_i^j(\alpha_0, \lambda_0) &= \left(\frac{\alpha_0}{6\pi} Q - \frac{\alpha_0^2}{36\pi^2} Q^2 \right) \delta_i^j + O\left(\alpha_0^3, \alpha_0^2 \lambda_0^2, \alpha_0 \lambda_0^4, \lambda_0^6 \right) \\
 &= \frac{\alpha_0 Q}{6\pi(1 + \alpha_0 Q/6\pi)} \delta_i^j + O\left(\alpha_0^3, \alpha_0^2 \lambda_0^2, \alpha_0 \lambda_0^4, \lambda_0^6 \right). \quad (3.18)
 \end{aligned}$$

4 Two-loop anomalous dimension defined in terms of the renormalized couplings

To obtain the anomalous dimension (standardly) defined in terms of the renormalized couplings, it is necessary to rewrite the left hand side of the equation (2.20) in terms of the renormalized couplings with the help of eqs. (2.25) and (2.28) and integrate the result with respect to $\ln \Lambda$. Then we obtain the function $(\ln Z_\phi)_i^j(\alpha, \lambda, \ln \Lambda/\mu)$, which should be rewritten in terms of the bare couplings using eqs. (2.25) and (2.28). According to eq. (2.22), the result should be differentiated with respect to $\ln \mu$. The final expression for the anomalous dimension is obtained after expressing the bare couplings in terms of the renormalized ones again using eqs. (2.25) and (2.28). As a correctness check, one can verify cancellation of all $\ln \Lambda/\mu$, which follows from the general theory of the renormalization group (see, e.g., [91]). The result of the above described calculation is written as

$$\begin{aligned}
 (\tilde{\gamma}_\phi)_i^j(\alpha, \lambda) &= -\frac{\alpha}{\pi} C(R)_i^j + \frac{1}{4\pi^2} \lambda_{imn}^* \lambda^{jmn} + \frac{\alpha^2}{2\pi^2} [C(R)^2]_i^j - \frac{1}{16\pi^4} \lambda_{iac}^* \lambda^{jab} \lambda_{bde}^* \lambda^{cde} \\
 &\quad - \frac{3\alpha^2}{2\pi^2} C_2 C(R)_i^j \left(\ln a_\varphi + 1 + \frac{A}{2} - b_{11} + g_{11} \right) \\
 &\quad + \frac{\alpha^2}{2\pi^2} T(R) C(R)_i^j \left(\ln a + 1 + \frac{A}{2} - b_{12} + g_{11} \right) \\
 &\quad - \frac{\alpha}{8\pi^3} \lambda_{imn}^* \lambda^{jmn} C(R)_i^l \left(1 - B + A - 2g_{12} + 2g_{11} \right) \\
 &\quad + \frac{\alpha}{4\pi^3} \lambda_{imn}^* \lambda^{jml} C(R)_l^n \left(1 - A + B + 2g_{12} - 2g_{11} \right) + O\left(\alpha^3, \alpha^2 \lambda^2, \alpha \lambda^4, \lambda^6 \right). \quad (4.1)
 \end{aligned}$$

It depends on the finite constants b_i and g_i , which determine the renormalization prescription. Certainly, this agrees with the statement that the anomalous dimension defined in terms of the renormalized couplings is scheme dependent starting from the two-loop order. However, the two-loop terms proportional to $[C(R)^2]_i^j$ and to $\lambda_{iac}^* \lambda^{jab} \lambda_{bde}^* \lambda^{cde}$ are scheme independent in agreement with refs. [25] and [37], respectively.

Now, let us consider particular cases of the general expression (4.1).

First, we write the expression for the anomalous dimension in the HD+MSL scheme, in which all finite constants (namely, g_{11} , g_{12} , b_{11} , and b_{12} in the considered approximation) should be set to 0, so that

$$\begin{aligned}
 (\tilde{\gamma}_\phi)_i^j(\alpha, \lambda)_{\text{HD+MSL}} &= -\frac{\alpha}{\pi} C(R)_i^j + \frac{1}{4\pi^2} \lambda_{imn}^* \lambda^{jmn} + \frac{\alpha^2}{2\pi^2} [C(R)^2]_i^j \\
 &\quad - \frac{1}{16\pi^4} \lambda_{iac}^* \lambda^{jab} \lambda_{bde}^* \lambda^{cde} - \frac{3\alpha^2}{2\pi^2} C_2 C(R)_i^j \left(\ln a_\varphi + 1 + \frac{A}{2} \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{\alpha^2}{2\pi^2} T(R) C(R)_i^j \left(\ln a + 1 + \frac{A}{2} \right) \\
 & - \frac{\alpha}{8\pi^3} \lambda_{lmn}^* \lambda^{jmn} C(R)_i^l \left(1 - B + A \right) \\
 & + \frac{\alpha}{4\pi^3} \lambda_{imn}^* \lambda^{jml} C(R)_l^n \left(1 - A + B \right) + O\left(\alpha^3, \alpha^2 \lambda^2, \alpha \lambda^4, \lambda^6\right). \quad (4.2)
 \end{aligned}$$

In agreement with the general statement [30], this expression coincides with eq. (3.8) up to the renaming of arguments $\alpha \rightarrow \alpha_0$, $\lambda \rightarrow \lambda_0$.

Eq. (4.1) should also agree with the result in the $\overline{\text{DR}}$ -scheme, which can be found, e.g., in ref. [20]. This means that there must be such finite constants that the anomalous dimension takes the form

$$\begin{aligned}
 (\tilde{\gamma}_\phi)_i^j(\alpha, \lambda)_{\overline{\text{DR}}} = & -\frac{\alpha}{\pi} C(R)_i^j + \frac{1}{4\pi^2} \lambda_{imn}^* \lambda^{jmn} + \frac{\alpha^2}{2\pi^2} [C(R)^2]_i^j - \frac{1}{16\pi^4} \lambda_{iac}^* \lambda^{jab} \lambda_{bde}^* \lambda^{cde} \\
 & - \frac{\alpha^2}{4\pi^2} (3C_2 - T(R)) C(R)_i^j - \frac{\alpha}{8\pi^3} \lambda_{lmn}^* \lambda^{jmn} C(R)_i^l \\
 & + \frac{\alpha}{4\pi^3} \lambda_{imn}^* \lambda^{jml} C(R)_l^n + O\left(\alpha^3, \alpha^2 \lambda^2, \alpha \lambda^4, \lambda^6\right). \quad (4.3)
 \end{aligned}$$

This is really true for the finite constants satisfying the equations

$$b_{11} - g_{11} = \ln a_\varphi + \frac{1}{2}(1+A); \quad b_{12} - g_{11} = \ln a + \frac{1}{2}(1+A); \quad g_{12} - g_{11} = \frac{1}{2}(A-B). \quad (4.4)$$

The existence of these values can be considered as a non-trivial check of the calculation correctness. Note that the constant g_{11} remains unfixed due to the arbitrariness of choosing the renormalization point μ . Its value can be found by comparing the renormalized one-loop two-point Green functions for the matter superfields calculated with the higher covariant derivative regularization and in the $\overline{\text{DR}}$ scheme, see appendix A and ref. [92] for details. This gives

$$g_{11} = -\frac{1}{2} - \frac{A}{2}. \quad (4.5)$$

For $\mathcal{N} = 1$ SQED with N_f flavors from eq. (4.1) we obtain the expression

$$\tilde{\gamma}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2} \left(N_f \ln a + N_f + \frac{N_f A}{2} + \frac{1}{2} - N_f b_{12} + N_f g_{11} \right) + O(\alpha^3), \quad (4.6)$$

which agrees with ref. [26].

The $\mathcal{N} = 1$ supersymmetric theories finite in the one-loop approximation are of a special interest. Written in terms of the renormalized couplings the finiteness conditions are

$$T(R) = 3C_2; \quad \lambda_{imn}^* \lambda^{jmn} = 4\pi \alpha C(R)_i^j. \quad (4.7)$$

In this case the expression (4.1) takes the form

$$\begin{aligned}
 (\tilde{\gamma}_\phi)_i^j(\alpha, \lambda) = & -\frac{3\alpha^2}{2\pi^2} C_2 C(R)_i^j \left(\ln \frac{a_\varphi}{a} - b_{11} + b_{12} \right) \\
 & - \frac{\alpha}{4\pi^2} \left(\frac{1}{\pi} \lambda_{imn}^* \lambda^{jml} C(R)_l^n + 2\alpha [C(R)^2]_i^j \right) \\
 & \times (A - B - 2g_{12} + 2g_{11}) + O\left(\alpha^3, \alpha^2 \lambda^2, \alpha \lambda^4, \lambda^6\right). \quad (4.8)
 \end{aligned}$$

We see that in general the one-loop finiteness does not lead to the two-loop finiteness. At first sight, this result contradicts the anomaly based consideration of refs. [62, 63]. However, it seems that in refs. [62, 63] the use of the $\overline{\text{DR}}$ scheme is implicitly assumed [93]. For this scheme the finite constants are given by eq. (4.4), and the anomalous dimension (4.8) really vanishes. Also it is possible to find other subtraction schemes in which it is so. In particular, a more interesting example is the HD+MSL scheme obtained for the higher covariant derivative regularization (3.14). The reason for this is that the HD+MSL scheme seems to be NSVZ in all loops. Moreover, the restrictions on the choice of the regularization can presumably reveal a deep structure of a theory needed for the finiteness, say, a certain symmetry underlying it.

One more interesting special case is the theories which satisfy the $P = \frac{1}{3}Q$ condition (1.5), under which the anomalous dimension is given by the expression

$$\begin{aligned}
 (\tilde{\gamma}_\phi)_i{}^j(\alpha, \lambda) = & \left(\frac{\alpha}{6\pi} Q - \frac{\alpha^2}{36\pi^2} Q^2 \right) \delta_i^j \\
 & + \left(\frac{\alpha^2}{2\pi^2} [C(R)^2]_i{}^j + \frac{\alpha}{4\pi^3} \lambda_{imn}^* \lambda^{jml} C(R)_l{}^n \right) (-A + B - 2g_{11} + 2g_{12}) \\
 & - \frac{\alpha^2}{12\pi^2} Q C(R)_i{}^j (A - B - 2g_{12} + 2g_{11}) \\
 & - \frac{3\alpha^2}{2\pi^2} C_2 C(R)_i{}^j \left(\ln a_\phi + \frac{1}{2}(1 + A) - b_{11} + g_{11} \right) \\
 & + \frac{\alpha^2}{2\pi^2} T(R) C(R)_i{}^j \left(\ln a + \frac{1}{2}(1 + A) - b_{12} + g_{11} \right) \\
 & + O(\alpha^3, \alpha^2 \lambda^2, \alpha \lambda^4, \lambda^6).
 \end{aligned} \tag{4.9}$$

Again we see that there are two ways to obtain the JJN equation (1.6) in the considered approximation. Namely, it is possible to choose finite constants for which eq. (4.4) is valid. The corresponding subtraction scheme is equivalent to the $\overline{\text{DR}}$ scheme. Another way is to use the HD+MSL scheme with the regularization satisfying eq. (3.17).

5 Three-loop β -function

5.1 The β -function defined in terms of the bare couplings

As we have already mentioned, there are strong indications that the NSVZ equation is valid in all loops for RGFs defined in terms of the bare couplings in the case of using the higher covariant derivative regularization. If we believe that this is really so, then it is possible to construct the expression for the three-loop β -function,

$$\begin{aligned}
 \frac{\beta(\alpha_0, \lambda_0)}{\alpha_0^2} = & -\frac{1}{2\pi} \left(3C_2 - T(R) \right) \left(1 + \frac{\alpha_0 C_2}{2\pi} + \frac{\alpha_0^2 C_2^2}{4\pi^2} \right) - \frac{1}{2\pi r} C(R)_j{}^i (\gamma_{\phi, 1\text{-loop}})_i{}^j \\
 & - \frac{\alpha_0 C_2}{4\pi^2 r} C(R)_j{}^i (\gamma_{\phi, 1\text{-loop}})_i{}^j - \frac{1}{2\pi r} C(R)_j{}^i (\gamma_{\phi, 2\text{-loop}})_i{}^j \\
 & + O(\alpha_0^3, \alpha_0^2 \lambda_0^2, \alpha_0 \lambda_0^4, \lambda_0^6),
 \end{aligned} \tag{5.1}$$

where $(\gamma_{\phi,1\text{-loop}})_i^j$ and $(\gamma_{\phi,2\text{-loop}})_i^j$ are the one- and two-loop parts of the anomalous dimension, respectively. Substituting these contributions we obtain

$$\begin{aligned} \frac{\beta(\alpha_0, \lambda_0)}{\alpha_0^2} = & -\frac{1}{2\pi} \left(3C_2 - T(R) \right) + \frac{\alpha_0}{4\pi^2} \left\{ -3C_2^2 + \frac{1}{r} C_2 \text{tr} C(R) + \frac{2}{r} \text{tr} [C(R)^2] \right\} \\ & - \frac{1}{8\pi^3 r} C(R)_j^i \lambda_{0imn}^* \lambda_0^{jmn} + \frac{\alpha_0^2}{8\pi^3} \left\{ -3C_2^3 + \frac{1}{r} C_2^2 \text{tr} C(R) - \frac{2}{r} \text{tr} [C(R)^3] \right. \\ & + \frac{2}{r} C_2 \text{tr} [C(R)^2] \left(3 \ln a_\varphi + 4 + \frac{3A}{2} \right) - \frac{2}{r^2} \text{tr} C(R) \text{tr} [C(R)^2] \left(\ln a + 1 + \frac{A}{2} \right) \Big\} \\ & - \frac{\alpha_0 C_2}{16\pi^4 r} C(R)_j^i \lambda_{0imn}^* \lambda_0^{jmn} + \frac{\alpha_0}{16\pi^4 r} [C(R)^2]_j^i \lambda_{0imn}^* \lambda_0^{jmn} (1 + A - B) \\ & - \frac{\alpha_0}{8\pi^4 r} C(R)_j^i C(R)_l^n \lambda_{0imn}^* \lambda_0^{jml} (1 - A + B) \\ & + \frac{1}{32\pi^5 r} C(R)_j^i \lambda_{0iac}^* \lambda_0^{jab} \lambda_{0bde}^* \lambda_0^{cde} + O\left(\alpha_0^3, \alpha_0^2 \lambda_0^2, \alpha_0 \lambda_0^4, \lambda_0^6\right). \end{aligned} \quad (5.2)$$

Certainly, this expression is independent of a renormalization prescription (for a fixed regularization) as any RGF defined in terms of the bare couplings. In the case of $\mathcal{N} = 1$ SQED with N_f flavors it gives

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{N_f}{\pi} + \frac{\alpha_0 N_f}{\pi^2} + \frac{\alpha_0^2 N_f}{\pi^3} \left[-\frac{1}{2} - N_f \left(\ln a + 1 + \frac{A}{2} \right) \right] + O(\alpha_0^3) = \frac{N_f}{\pi} \left(1 - \gamma(\alpha_0) \right) + O(\alpha_0^3). \quad (5.3)$$

For the one-loop finite theories (which satisfy eq. (3.12)) the expression (5.2) is reduced to

$$\begin{aligned} \frac{\beta(\alpha_0, \lambda_0)}{\alpha_0^2} = & \frac{\alpha_0}{8\pi^3 r} \left(\frac{1}{\pi} C(R)_j^i C(R)_l^n \lambda_{0imn}^* \lambda_0^{jml} + 2\alpha_0 \text{tr} [C(R)^3] \right) (A - B) \\ & + \frac{3\alpha_0^2}{4\pi^3 r} C_2 \text{tr} [C(R)^2] \ln \frac{a_\varphi}{a} + O\left(\alpha_0^3, \alpha_0^2 \lambda_0^2, \alpha_0 \lambda_0^4, \lambda_0^6\right). \end{aligned} \quad (5.4)$$

We see that the three-loop β -function vanishes for the higher covariant derivative regularization with $A = B$ and $a_\varphi = a$, and, in particular, for the regularization (3.14).

For theories satisfying the $P = \frac{1}{3}Q$ constraint (3.15) the three-loop β -function defined in terms of the bare couplings takes the form

$$\begin{aligned} \frac{\beta(\alpha_0, \lambda_0)}{\alpha_0^2} = & \frac{1}{2\pi} Q - \frac{\alpha_0}{12\pi^2} Q^2 + \frac{\alpha_0^2}{72\pi^3} Q^3 \\ & + \frac{\alpha_0}{8\pi^3 r} \left(\frac{1}{\pi} C(R)_j^i C(R)_l^n \lambda_{0imn}^* \lambda_0^{jml} + 2\alpha_0 \text{tr} [C(R)^3] \right) (A - B) \\ & + \frac{3\alpha_0^2}{4\pi^3 r} C_2 \text{tr} [C(R)^2] \ln \frac{a_\varphi}{a} + \frac{\alpha_0^2}{24\pi^3 r} Q \text{tr} [C(R)^2] \left(-6 \ln a - 3 - 2A - B \right) \\ & + O\left(\alpha_0^3, \alpha_0^2 \lambda_0^2, \alpha_0 \lambda_0^4, \lambda_0^6\right). \end{aligned} \quad (5.5)$$

From this equation we see that in the considered approximation the JJN equation (1.7) is satisfied by the β -function defined in terms of the bare couplings for the higher derivative regularization with the parameters obeying the conditions (3.17).

5.2 The β -function defined in terms of the renormalized couplings

To find the β -function defined in terms of the renormalized couplings, first, it is necessary to rewrite the right hand side of eq. (5.2) in terms of the renormalized couplings and, then, integrate it with respect to $\ln \Lambda$. The relations between the bare and renormalized gauge and Yukawa couplings are given by eqs. (2.25) and (2.28), respectively. After integrating eq. (5.2) with respect to $\ln \Lambda$ at fixed values of the renormalized couplings we find the three-loop expression for the renormalized gauge coupling constant as a function of the bare couplings. Next, the result is differentiated with respect to $\ln \mu$ (at fixed values of the bare couplings). Finally, the bare couplings are expressed in terms of the renormalized ones. Note that, according to the general theory of the renormalization group, all $\ln \Lambda/\mu$ should disappear. This fact can be used as a correctness check.

The procedure described above gives the three-loop β -function defined in terms of the renormalized couplings for a general renormalization prescription supplementing the higher covariant derivative regularization,

$$\begin{aligned}
 \frac{\tilde{\beta}(\alpha, \lambda)}{\alpha^2} = & -\frac{1}{2\pi} \left(3C_2 - T(R) \right) + \frac{\alpha}{4\pi^2} \left\{ -3C_2^2 + \frac{1}{r} C_2 \operatorname{tr} C(R) + \frac{2}{r} \operatorname{tr} [C(R)^2] \right\} \\
 & -\frac{1}{8\pi^3 r} C(R)_j^i \lambda_{imn}^* \lambda^{jmn} + \frac{\alpha^2}{8\pi^3} \left\{ -3C_2^3 \left(1 + 3b_{21} - 3b_{11} \right) \right. \\
 & + \frac{1}{r} C_2^2 \operatorname{tr} C(R) \left(1 + 3b_{21} - 3b_{12} + 3b_{22} - 3b_{11} \right) \\
 & - \frac{2}{r} \operatorname{tr} [C(R)^3] - \frac{1}{r^2} C_2 [\operatorname{tr} C(R)]^2 (b_{22} - b_{12}) \\
 & + \frac{1}{r} C_2 \operatorname{tr} [C(R)^2] \left(6 \ln a_\varphi + 8 + 3A + 6b_{23} - 6b_{11} \right) \\
 & \left. + \frac{1}{r^2} \operatorname{tr} C(R) \operatorname{tr} [C(R)^2] \left(-2 \ln a - 2 - A - 2b_{23} + 2b_{12} \right) \right\} \\
 & - \frac{\alpha C_2}{16\pi^4 r} C(R)_j^i \lambda_{imn}^* \lambda^{jmn} \\
 & + \frac{\alpha}{16\pi^4 r} [C(R)^2]_j^i \lambda_{imn}^* \lambda^{jmn} \left(1 + A - B - 2b_{24} + 2g_{11} \right) \\
 & - \frac{\alpha}{8\pi^4 r} C(R)_j^i C(R)_l^n \lambda_{imn}^* \lambda^{jml} \left(1 - A + B + 2b_{24} - 2g_{11} \right) \\
 & + \frac{1}{16\pi^5 r} C(R)_j^i \lambda_{iac}^* \lambda^{jab} \lambda_{bde}^* \lambda^{cde} \left(\frac{1}{2} + b_{24} - g_{12} \right) \\
 & + \frac{1}{32\pi^5 r} C(R)_j^i \lambda_{imn}^* \lambda^{kmn} \lambda_{kpq}^* \lambda^{jpq} \left(b_{24} - g_{12} \right) \\
 & + O(\alpha^3, \alpha^2 \lambda^2, \alpha \lambda^4, \lambda^6). \tag{5.6}
 \end{aligned}$$

We see that the terms corresponding to the three-loop contribution depend on the constants b_i and g_i . Certainly, this follows from the fact that the β -function defined in terms of the renormalized couplings is scheme dependent starting from the three-loop approximation. The terms containing the Yukawa couplings exactly coincide with the ones obtained by the direct calculations of refs. [37, 38].

By definition, in the HD+MSL scheme all finite constants b_i and g_i are set to 0, so that the expression for the β -function takes the form

$$\begin{aligned}
 \left(\frac{\tilde{\beta}(\alpha, \lambda)}{\alpha^2} \right)_{\text{HD+MSL}} &= -\frac{1}{2\pi} \left(3C_2 - T(R) \right) \\
 &+ \frac{\alpha}{4\pi^2} \left\{ -3C_2^2 + \frac{1}{r} C_2 \text{tr} C(R) + \frac{2}{r} \text{tr} [C(R)^2] \right\} \\
 &- \frac{1}{8\pi^3 r} C(R)_j^i \lambda_{imn}^* \lambda^{jmn} + \frac{\alpha^2}{8\pi^3} \left\{ -3C_2^3 + \frac{1}{r} C_2^2 \text{tr} C(R) \right. \\
 &- \frac{2}{r} \text{tr} [C(R)^3] + \frac{1}{r} C_2 \text{tr} [C(R)^2] \left(6 \ln a_\varphi + 8 + 3A \right) \\
 &+ \left. \frac{1}{r^2} \text{tr} C(R) \text{tr} [C(R)^2] \left(-2 \ln a - 2 - A \right) \right\} \\
 &- \frac{\alpha C_2}{16\pi^4 r} C(R)_j^i \lambda_{imn}^* \lambda^{jmn} + \frac{\alpha}{16\pi^4 r} [C(R)^2]_j^i \lambda_{imn}^* \lambda^{jmn} (1 + A - B) \\
 &- \frac{\alpha}{8\pi^4 r} C(R)_j^i C(R)_l^n \lambda_{imn}^* \lambda^{jml} (1 - A + B) \\
 &+ \frac{1}{32\pi^5 r} C(R)_j^i \lambda_{iac}^* \lambda^{jab} \lambda_{bde}^* \lambda^{cde} + O(\alpha^3, \alpha^2 \lambda^2, \alpha \lambda^4, \lambda^6) \quad (5.7)
 \end{aligned}$$

and up to the renaming of arguments $\alpha \rightarrow \alpha_0$, $\lambda \rightarrow \lambda_0$ coincides with the β -function defined in terms of the bare couplings given by eq. (5.2).

For certain values of the finite constants b_i and g_i the expression (5.6) should also reproduce the $\overline{\text{DR}}$ -result first found in refs. [20, 21], which in our notation takes the form

$$\begin{aligned}
 \left(\frac{\tilde{\beta}(\alpha, \lambda)}{\alpha^2} \right)_{\overline{\text{DR}}} &= -\frac{1}{2\pi} \left(3C_2 - T(R) \right) + \frac{\alpha}{4\pi^2} \left\{ -3C_2^2 + \frac{1}{r} C_2 \text{tr} C(R) + \frac{2}{r} \text{tr} [C(R)^2] \right\} \\
 &- \frac{1}{8\pi^3 r} C(R)_j^i \lambda_{imn}^* \lambda^{jmn} + \frac{\alpha^2}{8\pi^3} \left\{ -\frac{21}{4} C_2^3 + \frac{5}{2r} C_2^2 \text{tr} C(R) \right. \\
 &- \frac{1}{4r^2} C_2 [\text{tr} C(R)]^2 - \frac{2}{r} \text{tr} [C(R)^3] + \frac{13}{2r} C_2 \text{tr} [C(R)^2] \\
 &- \left. \frac{3}{2r^2} \text{tr} C(R) \text{tr} [C(R)^2] \right\} - \frac{\alpha C_2}{16\pi^4 r} C(R)_j^i \lambda_{imn}^* \lambda^{jmn} \\
 &+ \frac{\alpha}{32\pi^4 r} [C(R)^2]_j^i \lambda_{imn}^* \lambda^{jmn} - \frac{3\alpha}{16\pi^4 r} C(R)_j^i C(R)_l^n \lambda_{imn}^* \lambda^{jml} \\
 &+ \frac{3}{64\pi^4 r} C(R)_j^i \lambda_{iac}^* \lambda^{jab} \lambda_{bde}^* \lambda^{cde} + \frac{1}{128\pi^5 r} C(R)_j^i \lambda_{imn}^* \lambda^{kmn} \lambda_{kpq}^* \lambda^{jpq} \\
 &+ O(\alpha^3, \alpha^2 \lambda^2, \alpha \lambda^4, \lambda^6). \quad (5.8)
 \end{aligned}$$

In particular, this implies that all scheme independent terms in eq. (5.6) should coincide with the corresponding terms in eq. (5.8). We see that it is really so. The other terms coincide if

$$\begin{aligned}
 b_{21} - b_{11} &= \frac{1}{4}; & b_{21} - b_{12} + b_{22} - b_{11} &= \frac{1}{2}; & b_{22} - b_{12} &= \frac{1}{4}; \\
 b_{23} - b_{12} &= -\ln a - \frac{A}{2} - \frac{1}{4}; & b_{23} - b_{11} &= -\ln a_\varphi - \frac{A}{2} - \frac{1}{4}; \\
 b_{24} - g_{11} &= \frac{A}{2} - \frac{B}{2} + \frac{1}{4}; & b_{24} - g_{12} &= \frac{1}{4}. \quad (5.9)
 \end{aligned}$$

From these equations and eq. (4.4) one can express all finite constants in terms of g_{11} given by eq. (4.5),

$$\begin{aligned}
 b_{11} &= g_{11} + \ln a_\varphi + \frac{1}{2}(1 + A) = \ln a_\varphi; & b_{12} &= g_{11} + \ln a + \frac{1}{2}(1 + A) = \ln a; \\
 b_{21} &= g_{11} + \ln a_\varphi + \frac{3}{4} + \frac{A}{2} = \ln a_\varphi + \frac{1}{4}; & b_{22} &= g_{11} + \ln a + \frac{3}{4} + \frac{A}{2} = \ln a + \frac{1}{4}; \\
 b_{23} &= g_{11} + \frac{1}{4} = -\frac{1}{4} - \frac{A}{2}; & b_{24} &= g_{11} + \frac{A}{2} - \frac{B}{2} + \frac{1}{4} = -\frac{1}{4} - \frac{B}{2}; \\
 g_{12} &= g_{11} + \frac{1}{2}(A - B) = -\frac{1}{2} - \frac{B}{2}; & g_{11} &= -\frac{1}{2} - \frac{A}{2}.
 \end{aligned} \tag{5.10}$$

The values (5.10) of the finite constants b_i and g_i correspond to the $\overline{\text{DR}}$ renormalization scheme.

Comparing eqs. (5.6) and (4.1) we see that for a general renormalization prescription the NSVZ relation does not take place. However, if the finite constants b_i and g_i satisfy the equations

$$b_{21} = b_{11}; \quad b_{22} = b_{12}; \quad b_{23} = g_{11}; \quad b_{24} = g_{12}, \tag{5.11}$$

then the NSVZ β -function in the considered approximation is valid for RGFs defined in terms of the renormalized couplings. Evidently, eq. (5.11) does not unambiguously fix the values of all finite constants. This implies that there is a class of NSVZ schemes similar to the one in the Abelian case which was described in ref. [28].

From eq. (5.6) we see that the β -function of $\mathcal{N} = 1$ SQED with N_f flavors is given by the expression

$$\frac{\tilde{\beta}(\alpha)}{\alpha^2} = \frac{N_f}{\pi} + \frac{\alpha N_f}{\pi^2} + \frac{\alpha^2 N_f}{\pi^3} \left[-\frac{1}{2} + N_f \left(-\ln a - 1 - \frac{A}{2} - b_{23} + b_{12} \right) \right] + O(\alpha^3), \tag{5.12}$$

which agrees with ref. [26].

For the one-loop finite theories (which satisfy the conditions (4.7)) the three-loop β -function defined in terms of the renormalized couplings takes the form

$$\begin{aligned}
 \frac{\tilde{\beta}(\alpha, \lambda)}{\alpha^2} &= \frac{3\alpha^2}{4\pi^3 r} C_2 \text{tr} [C(R)^2] \left(\ln \frac{a_\varphi}{a} + b_{12} - b_{11} \right) \\
 &+ \frac{\alpha}{8\pi^3 r} \left(\frac{1}{\pi} C(R)_j{}^i C(R)_l{}^n \lambda_{imn}^* \lambda^{jml} + 2\alpha \text{tr} [C(R)^3] \right) (A - B - 2g_{12} + 2g_{11}) \\
 &+ O(\alpha^3, \alpha^2 \lambda^2, \alpha \lambda^4, \lambda^6).
 \end{aligned} \tag{5.13}$$

Comparing this expression with eq. (4.8) we see that in this case the NSVZ equation

$$\frac{\tilde{\beta}(\alpha, \lambda)}{\alpha^2} = -\frac{C(R)_j{}^i (\tilde{\gamma}_\phi)_i{}^j(\alpha, \lambda)}{2\pi r (1 - \alpha C_2/2\pi)} \tag{5.14}$$

is valid in the considered approximation for a general renormalization prescription in agreement with the general statement that for a theory finite in a certain loop the β -function

always vanishes in the next loop [64]. Note that in ref. [64] the $\overline{\text{DR}}$ scheme was essentially used. However, in general, the NSVZ equation is not valid in the $\overline{\text{DR}}$ scheme (see, e.g., [20]). Now we see how this seeming contradiction is solved in the considered approximation.

Also we see that for theories satisfying eq. (4.7) the three-loop β -function vanishes if

$$\ln \frac{a_\varphi}{a} + b_{12} - b_{11} = 0; \quad A - B - 2g_{12} + 2g_{11} = 0. \quad (5.15)$$

In particular, these equations are valid in the $\overline{\text{DR}}$ scheme and in the HD+MSL scheme supplementing the regularization with $R(x) = F^2(x)$ and $a_\varphi = a$.

For theories which satisfy the $P = \frac{1}{3}Q$ condition (1.5) the β -function defined in terms of the renormalized couplings is

$$\begin{aligned} \frac{\tilde{\beta}(\alpha, \lambda)}{\alpha^2} = & \frac{1}{2\pi}Q - \frac{\alpha}{12\pi^2}Q^2 + \frac{\alpha^2}{72\pi^3}Q^3 \left(1 + 3b_{24} - 3g_{12}\right) \\ & + \frac{\alpha^2}{8\pi^3}C_2Q^2 \left(b_{24} - b_{22} + b_{12} - g_{12}\right) + \frac{3\alpha^2}{8\pi^3}C_2^2Q \left(b_{21} - b_{11} - b_{22} + b_{12}\right) \\ & + \frac{\alpha}{8\pi^3r} \left(\frac{1}{\pi}C(R)_j{}^i C(R)_l{}^n \lambda_{imn}^* \lambda^{jml} + 2\alpha \text{tr} [C(R)^3] \right) (A - B - 2g_{12} + 2g_{11}) \\ & + \frac{\alpha^2}{24\pi^3r}Q \text{tr} [C(R)^2] \left(-6 \ln a - 3 - 2A - B - 6b_{23} + 6b_{12} + 6b_{24} + 2g_{11} - 8g_{12} \right) \\ & + \frac{3\alpha^2}{4\pi^3r}C_2 \text{tr} [C(R)^2] \left(\ln \frac{a_\varphi}{a} + b_{12} - b_{11} \right) \\ & + O(\alpha^3, \alpha^2\lambda^2, \alpha\lambda^4, \lambda^6). \end{aligned} \quad (5.16)$$

Therefore, eq. (1.7) in the considered approximation is valid, for example, in the HD+MSL scheme with the regularization satisfying the conditions (3.17). Another possibility is the NSVZ scheme constructed from the $\overline{\text{DR}}$ scheme by a special redefinition of the gauge coupling constant in refs. [20–22]. This scheme corresponds to the finite constants

$$\begin{aligned} b_{21} = b_{11} &= \ln a_\varphi; & b_{22} = b_{12} &= \ln a; \\ b_{23} = g_{11} &= -\frac{1}{2} - \frac{A}{2}; & b_{24} = g_{12} &= -\frac{1}{2} - \frac{B}{2}. \end{aligned} \quad (5.17)$$

6 Conclusion

In this paper the two-loop anomalous dimension of the matter superfields has been calculated for a general $\mathcal{N} = 1$ supersymmetric gauge theory regularized by higher covariant derivatives. The result has been found both for the anomalous dimension defined in terms of the bare couplings and for the one defined in terms of the renormalized couplings for an arbitrary renormalization prescription. For theories finite in the one-loop approximation the two-loop anomalous dimension defined in terms of the bare couplings does not in general vanish. However, for a version of the higher derivative regularization with the parameters $R(x) = F^2(x)$ and $a = a_\varphi$ this is so. Possibly, this could help to understand deeper reasons responsible for the finiteness of a theory. Moreover, we analysed a possibility of satisfying the exact equation (1.6) for the anomalous dimension which was proposed

by Jack, Jones, and North for theories obeying the $P = \frac{1}{3}Q$ condition. It appears that in the considered approximation for this equation to be valid the regularization parameters should satisfy the constraints (3.17).

Using the statement that for RGFs defined in terms of the bare couplings the NSVZ equation is valid in the case of using the higher covariant derivative regularization we also construct the expression for the three-loop β -function. Again, we present the results for the β -function defined in terms of the bare couplings and for the one defined in terms of the renormalized couplings. For the one-loop finite theories the three-loop β -function defined in terms of the bare couplings vanishes under the same conditions (3.14) as the two-loop anomalous dimension. For theories satisfying eq. (1.5) the exact expression for the β -function in the form of the geometric series (1.7) is valid with the regularization (3.17).

For RGFs defined in terms of the renormalized couplings the above results are valid in the HD+MSL scheme. Also we have demonstrated that for the one-loop finite theories the NSVZ equation in the considered approximation is valid for an arbitrary renormalization prescription.

RGFs obtained in this paper with the help of the higher covariant derivative regularization are in agreement with the ones in the $\overline{\text{DR}}$ scheme in a sense that there are certain values of finite constants fixing the renormalization prescription for which the results of this paper reproduce the $\overline{\text{DR}}$ results.

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A Calculation of the loop integrals giving the two-loop anomalous dimension

The two-loop anomalous dimension can be presented in the form

$$\begin{aligned}
 (\gamma_\phi)_i^j(\alpha_0, \lambda_0) = & -8\pi\alpha C(R)_i^j I_1 + 2\lambda_{imn}^* \lambda^{jmn} I_2 - 64\pi^2 \alpha^2 [C(R)^2]_i^j I_{10} \\
 & + 64\pi^2 \alpha^2 C_2 C(R)_i^j \left[I_{11} + \frac{3}{16\pi^2} \left(\ln \frac{\Lambda}{\mu} + b_{11} \right) I_1 - \frac{3}{2} I_8 \right] \\
 & + 64\pi^2 \alpha^2 C(R)_i^j T(R) \left[I_{12} - \frac{1}{16\pi^2} \left(\ln \frac{\Lambda}{\mu} + b_{12} \right) I_1 + \frac{1}{2} I_8 \right] \\
 & + 16\pi\alpha \lambda_{lmn}^* \lambda^{jmn} C(R)_i^l \left[I_7 - I_4 - I_8 + \frac{1}{8\pi^2} \left(\ln \frac{\Lambda}{\mu} + g_{11} \right) I_1 \right] \\
 & + 32\pi\alpha \lambda_{imn}^* \lambda^{jml} C(R)_l^n \left[I_3 + I_9 - \frac{1}{8\pi^2} \left(\ln \frac{\Lambda}{\mu} + g_{11} \right) I_2 \right]
 \end{aligned}$$

$$\begin{aligned}
 & -2\lambda_{iab}^* \lambda^{kab} \lambda_{kcd}^* \lambda^{jcd} I_6 - 8\lambda_{iac}^* \lambda^{jab} \lambda_{bde}^* \lambda^{cde} \left[I_5 + I_9 - \frac{1}{8\pi^2} \left(\ln \frac{\Lambda}{\mu} + g_{12} \right) I_2 \right] \\
 & + O(\alpha^3, \alpha^2 \lambda^2, \alpha \lambda^4, \lambda^6),
 \end{aligned} \tag{A.1}$$

where the Euclidean integrals $I_1 - I_{12}$ are given by the expressions

$$I_1 \equiv \frac{d}{d \ln \Lambda} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^4 R_k} = \frac{1}{8\pi^2}; \tag{A.2}$$

$$I_2 \equiv \frac{d}{d \ln \Lambda} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^4 F_k^2} = \frac{1}{8\pi^2}; \tag{A.3}$$

$$I_3 \equiv \frac{d}{d \ln \Lambda} \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^4 F_l^2} \left(\int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 R_k (l+k)^2} - \frac{1}{8\pi^2} \ln \frac{\Lambda}{l} \right); \tag{A.4}$$

$$I_4 \equiv \frac{d}{d \ln \Lambda} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^4 R_k} \left(\int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 F_l (l+k)^2 F_{l+k}} - \frac{1}{8\pi^2} \ln \frac{\Lambda}{k} \right); \tag{A.5}$$

$$I_5 \equiv \frac{d}{d \ln \Lambda} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^4 F_k^3} \left(\int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 F_l (l+k)^2 F_{l+k}} - \frac{1}{8\pi^2} F_k \ln \frac{\Lambda}{k} \right); \tag{A.6}$$

$$I_6 \equiv \frac{d}{d \ln \Lambda} \left[\int \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{1}{k^4 F_k^2 l^4 F_l^2} - \frac{1}{4\pi^2} \left(\ln \frac{\Lambda}{\mu} + g_{12} \right) \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^4 F_k^2} \right]; \tag{A.7}$$

$$\begin{aligned}
 I_7 \equiv \frac{d}{d \ln \Lambda} \left[\int \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{1}{k^4 R_k l^4 F_l^2} - \frac{1}{8\pi^2} \left(\ln \frac{\Lambda}{\mu} + g_{11} \right) \int \frac{d^4 k}{(2\pi)^4} \left(\frac{1}{k^4 F_k^2} + \frac{1}{k^4 R_k} \right) \right]; \\
 \tag{A.8}
 \end{aligned}$$

$$I_8 \equiv \frac{1}{8\pi^2} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^4} \ln \frac{\Lambda}{k} \frac{d}{d \ln \Lambda} \frac{1}{R_k}; \tag{A.9}$$

$$I_9 \equiv \frac{1}{8\pi^2} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^4} \ln \frac{\Lambda}{k} \frac{d}{d \ln \Lambda} \frac{1}{F_k^2}; \tag{A.10}$$

$$I_{10} \equiv \frac{d}{d \ln \Lambda} \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{k_\mu l^\mu}{l^4 R_l k^4 R_k (l+k)^2}; \tag{A.11}$$

$$I_{11} \equiv \frac{d}{d \ln \Lambda} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^4 R_k^2} \left[f(k/\Lambda) + \frac{3R_k}{16\pi^2} \ln \frac{\Lambda}{k} \right]; \tag{A.12}$$

$$I_{12} \equiv \frac{d}{d \ln \Lambda} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^4 R_k^2} \left[h(k/\Lambda) - \frac{R_k}{16\pi^2} \ln \frac{\Lambda}{k} \right]. \tag{A.13}$$

The (rather large) explicit expressions for the functions $f(k/\Lambda)$ and $h(k/\Lambda)$ inside the integrals I_{11} and I_{12} can be found in ref. [86].

The integrals I_6 , I_7 , and I_{10} have been calculated in refs. [37, 38], and [29], respectively. The results are written as

$$I_6 = -\frac{1}{32\pi^4} \left(\ln \frac{\Lambda}{\mu} + g_{12} \right); \quad I_7 = -\frac{1}{32\pi^4} \left(\ln \frac{\Lambda}{\mu} + g_{11} \right); \quad I_{10} = -\frac{1}{128\pi^4}. \tag{A.14}$$

Calculating the integrals I_8 and I_9 in the four-dimensional spherical coordinates one can relate them to the constants A and B defined by eq. (3.9),

$$I_8 = \frac{A}{128\pi^4}; \quad I_9 = \frac{B}{128\pi^4}. \tag{A.15}$$

Earlier the integrals I_3 , I_4 , I_5 and the integrals I_{11} , I_{12} have been found only for the higher derivative regulators $R(x) = 1 + x^m$ and $F(x) = 1 + x^n$, see refs. [38] and [83], respectively. However, it is possible to generalize the results to the case of arbitrary functions $R(x)$ and $F(x)$. For this purpose we will use the equation

$$\frac{d}{d \ln \Lambda} \int \frac{d^4 k}{(2\pi)^4} \frac{a(k/\Lambda)}{k^4} = \frac{1}{8\pi^2} a(0) \quad (\text{A.16})$$

valid for a nonsingular function $a(k/\Lambda)$ rapidly decreasing at infinity. Using this equation the integrals under consideration can be presented in the form

$$I_3 = \lim_{l \rightarrow 0} \frac{1}{8\pi^2} \left(\int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 R_k(l+k)^2} - \frac{1}{8\pi^2} \ln \frac{\Lambda}{l} \right); \quad (\text{A.17})$$

$$I_4 = I_5 = \lim_{k \rightarrow 0} \frac{1}{8\pi^2} \left(\int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 F_l(l+k)^2 F_{l+k}} - \frac{1}{8\pi^2} \ln \frac{\Lambda}{k} \right); \quad (\text{A.18})$$

$$I_{11} = \lim_{k \rightarrow 0} \frac{1}{8\pi^2} \left(f(k/\Lambda) + \frac{3}{16\pi^2} \ln \frac{\Lambda}{k} \right) = -\frac{3}{128\pi^4} \left(\ln a_\varphi + 1 \right); \quad (\text{A.19})$$

$$I_{12} = \lim_{k \rightarrow 0} \frac{1}{8\pi^2} \left(h(k/\Lambda) - \frac{1}{16\pi^2} \ln \frac{\Lambda}{k} \right) = \frac{1}{128\pi^4} \left(\ln a + 1 \right), \quad (\text{A.20})$$

where the last equalities in eqs. (A.19) and (A.20) follow from eqs. (3.6) and (3.7), respectively.

Now, let us calculate the integral I_3 for a general higher derivative regulator $R(x)$. Introducing the Feynman parameter x and making the substitution $k_\mu \rightarrow k_\mu - x l_\mu$ the integral in eq. (A.17) can be rewritten as

$$\begin{aligned} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 R_k(l+k)^2} &= \int_0^1 dx \int \frac{d^4 k}{(2\pi)^4} \frac{1}{R_{k-xl}(k^2 + x(1-x)l^2)^2} \\ &= \int_0^1 dx \int \frac{d^4 k}{(2\pi)^4} \frac{1}{R_k(k^2 + x(1-x)l^2)^2} + O(\Lambda^{-2}), \end{aligned} \quad (\text{A.21})$$

where we took into account that

$$R_{k-xl} \equiv R\left((k_\mu - x l_\mu)^2 / \Lambda^2\right) = R(k^2 / \Lambda^2) + O(\Lambda^{-2}) = R_k + O(\Lambda^{-2}). \quad (\text{A.22})$$

Next, it is convenient to change the integration variable to $z = k^2 / \Lambda^2$ and introduce the notation $\epsilon \equiv x(1-x)l^2 / \Lambda^2$. Taking into account that ϵ is small, we obtain

$$\begin{aligned} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{R_k(k^2 + x(1-x)l^2)^2} &= \frac{1}{(4\pi)^2} \int_0^\infty \frac{k^2 dk^2}{R_k(k^2 + x(1-x)l^2)^2} \\ &= \frac{1}{(4\pi)^2} \int_0^\infty \frac{z dz}{R(z)(z + \epsilon)^2} \\ &= \frac{1}{(4\pi)^2} \frac{\partial}{\partial \epsilon} \int_0^\infty \frac{dz \epsilon}{R(z)(z + \epsilon)} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{(4\pi)^2} \frac{\partial}{\partial \epsilon} \left(\epsilon \ln(z + \epsilon) \frac{1}{R(z)} \Big|_0^{+\infty} - \int_0^\infty dz \epsilon \ln(z + \epsilon) \frac{d}{dz} \frac{1}{R(z)} \right) \\
 &= \frac{1}{(4\pi)^2} \frac{\partial}{\partial \epsilon} \left(-\epsilon \ln \epsilon - \int_0^\infty dz \ln z \frac{d}{dz} \frac{\epsilon}{R(z)} + O(\epsilon^2 \ln \epsilon) \right) \\
 &= \frac{1}{(4\pi)^2} (-\ln \epsilon - 1 - A) + O(\epsilon \ln \epsilon),
 \end{aligned} \tag{A.23}$$

where the constant A is defined by eq. (3.9). Integrating this expression with respect to x and omitting terms vanishing in the limit $\Lambda \rightarrow \infty$, the considered integral can be written as

$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 R_k(l+k)^2} = \frac{1}{8\pi^2} \left(\ln \frac{\Lambda}{l} + \frac{1}{2} - \frac{A}{2} \right). \tag{A.24}$$

The result for the integral I_3 is calculated by substituting this expression into eq. (A.17),

$$I_3 = \frac{1}{128\pi^4} (1 - A). \tag{A.25}$$

According to ref. [38],

$$\begin{aligned}
 &\lim_{k \rightarrow 0} \int \frac{d^4 l}{(2\pi)^4} \frac{1}{k^2 (k+l)^2 F_l} \left(\frac{1}{F_{k+l}} - \frac{1}{F_l} \right) \\
 &= \lim_{k \rightarrow 0} \left(\int \frac{d^4 l}{(2\pi)^4} \frac{F_l - F_{k+l}}{(l^2 - (k+l)^2)} \frac{1}{F_l^2 F_{k+l} (k+l)^2} \right. \\
 &\quad \left. - \int \frac{d^4 l}{(2\pi)^4} \frac{F_l - F_{k+l}}{(l^2 - (k+l)^2)} \frac{1}{F_l^2 F_{k+l} l^2} \right) = 0,
 \end{aligned} \tag{A.26}$$

because the last two integrals are well defined in the limit $k \rightarrow 0$. Therefore,

$$I_4 = I_5 = \lim_{k \rightarrow 0} \frac{1}{8\pi^2} \left(\int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 F_l^2 (l+k)^2} - \frac{1}{8\pi^2} \ln \frac{\Lambda}{k} \right). \tag{A.27}$$

This expression is completely analogous to the integral I_3 . Repeating the calculation described above, we obtain

$$I_4 = I_5 = \frac{1}{128\pi^4} (1 - B), \tag{A.28}$$

where the constant B is defined by eq. (3.9).

Thus, the integrals (A.2) — (A.13) are given by the following expressions:

$$\begin{aligned}
 I_1 = I_2 &= \frac{1}{8\pi^2}; & I_3 &= \frac{1}{128\pi^4} (1 - A); & I_4 &= \frac{1}{128\pi^4} (1 - B); \\
 I_5 &= \frac{1}{128\pi^4} (1 - B); & I_6 &= -\frac{1}{32\pi^4} \left(\ln \frac{\Lambda}{\mu} + g_{12} \right); & I_7 &= -\frac{1}{32\pi^4} \left(\ln \frac{\Lambda}{\mu} + g_{11} \right); \\
 I_8 &= \frac{A}{128\pi^4}; & I_9 &= \frac{B}{128\pi^4}; & I_{10} &= -\frac{1}{128\pi^4}; \\
 I_{11} &= -\frac{3}{128\pi^4} (\ln a_\varphi + 1); & I_{12} &= \frac{1}{128\pi^4} (\ln a + 1).
 \end{aligned} \tag{A.29}$$

Substituting them into eq. (A.1) the anomalous dimension can be written as

$$\begin{aligned}
(\gamma_\phi)_i{}^j = & -\frac{\alpha}{\pi}C(R)_i{}^j + \frac{1}{4\pi^2}\lambda_{imn}^*\lambda^{jmn} + \frac{\alpha^2}{2\pi^2}[C(R)^2]_i{}^j \\
& + \frac{3\alpha^2}{2\pi^2}C_2C(R)_i{}^j\left[\ln\frac{\Lambda}{\mu} + b_{11} - \ln a_\varphi - 1 - \frac{A}{2}\right] \\
& - \frac{\alpha^2}{2\pi^2}T(R)C(R)_i{}^j\left[\ln\frac{\Lambda}{\mu} + b_{12} - \ln a - 1 - \frac{A}{2}\right] \\
& - \frac{\alpha}{4\pi^3}\lambda_{lmn}^*\lambda^{jmn}C(R)_i{}^l\left[\ln\frac{\Lambda}{\mu} + g_{11} + \frac{1}{2} - \frac{B}{2} + \frac{A}{2}\right] \\
& - \frac{\alpha}{2\pi^3}\lambda_{imn}^*\lambda^{jml}C(R)_l{}^n\left[\ln\frac{\Lambda}{\mu} + g_{11} - \frac{1}{2} - \frac{B}{2} + \frac{A}{2}\right] \\
& + \frac{1}{16\pi^4}\lambda_{iab}^*\lambda^{kab}\lambda_{kcd}^*\lambda^{jcd}\left[\ln\frac{\Lambda}{\mu} + g_{12}\right] + \frac{1}{8\pi^4}\lambda_{iac}^*\lambda^{jab}\lambda_{bde}^*\lambda^{cde}\left[\ln\frac{\Lambda}{\mu} + g_{12} - \frac{1}{2}\right] \\
& + O\left(\alpha^3, \alpha^2\lambda^2, \alpha\lambda^4, \lambda^6\right). \tag{A.30}
\end{aligned}$$

Rewriting the result in terms of the bare couplings using eqs. (2.25) and (2.28) we obtain the anomalous dimension (3.8). Note that all $\ln\Lambda/\mu$ and all finite constants cancel each other in the resulting expression. Certainly, this can be considered as a correctness check.

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References

- [1] M.T. Grisaru and W. Siegel, *Supergraphity. 2. Manifestly covariant rules and higher loop finiteness*, *Nucl. Phys. B* **201** (1982) 292 [Erratum *ibid.* **B 206** (1982) 496] [[INSPIRE](#)].
- [2] S. Mandelstam, *Light cone superspace and the ultraviolet finiteness of the $N = 4$ model*, *Nucl. Phys. B* **213** (1983) 149 [[INSPIRE](#)].
- [3] L. Brink, O. Lindgren and B.E.W. Nilsson, *$N = 4$ Yang-Mills theory on the light cone*, *Nucl. Phys. B* **212** (1983) 401 [[INSPIRE](#)].
- [4] P.S. Howe, K.S. Stelle and P.K. Townsend, *Miraculous ultraviolet cancellations in supersymmetry made manifest*, *Nucl. Phys. B* **236** (1984) 125 [[INSPIRE](#)].
- [5] I.L. Buchbinder, S.M. Kuzenko and B.A. Ovrut, *On the $D = 4$, $N = 2$ nonrenormalization theorem*, *Phys. Lett. B* **433** (1998) 335 [[hep-th/9710142](#)] [[INSPIRE](#)].
- [6] P.S. Howe, K.S. Stelle and P.C. West, *A class of finite four-dimensional supersymmetric field theories*, *Phys. Lett.* **124B** (1983) 55 [[INSPIRE](#)].
- [7] M.T. Grisaru, W. Siegel and M. Roček, *Improved methods for supergraphs*, *Nucl. Phys. B* **159** (1979) 429 [[INSPIRE](#)].
- [8] K.V. Stepanyantz, *Non-renormalization of the $V\bar{c}c$ -vertices in $\mathcal{N} = 1$ supersymmetric theories*, *Nucl. Phys. B* **909** (2016) 316 [[arXiv:1603.04801](#)] [[INSPIRE](#)].
- [9] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Exact Gell-Mann-Low function of supersymmetric Yang-Mills theories from instanton calculus*, *Nucl. Phys. B* **229** (1983) 381 [[INSPIRE](#)].

- [10] D.R.T. Jones, *More on the axial anomaly in supersymmetric Yang-Mills theory*, *Phys. Lett.* **123B** (1983) 45 [[INSPIRE](#)].
- [11] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *The β -function in supersymmetric gauge theories. Instantons versus traditional approach*, *Phys. Lett.* **166B** (1986) 329 [[INSPIRE](#)].
- [12] M.A. Shifman and A.I. Vainshtein, *Solution of the anomaly puzzle in SUSY gauge theories and the wilson operator expansion*, *Nucl. Phys. B* **277** (1986) 456 [*Sov. Phys. JETP* **64** (1986) 428] [*Zh. Eksp. Teor. Fiz.* **91** (1986) 723] [[INSPIRE](#)].
- [13] M.A. Shifman and A.I. Vainshtein, *Instantons versus supersymmetry: fifteen years later*, in *ITEP lectures on particle physics and field theory*, M.A. Shifman ed., World Scientific, Singapore (1989), [hep-th/9902018](#) [[INSPIRE](#)].
- [14] I.L. Buchbinder and K.V. Stepanyantz, *The higher derivative regularization and quantum corrections in $N = 2$ supersymmetric theories*, *Nucl. Phys. B* **883** (2014) 20 [[arXiv:1402.5309](#)] [[INSPIRE](#)].
- [15] A. Galperin et al., *Unconstrained $N = 2$ matter, Yang-Mills and supergravity theories in Harmonic superspace*, *Class. Quant. Grav.* **1** (1984) 469 [Erratum *ibid.* **2** (1985) 127] [[INSPIRE](#)].
- [16] A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky and E.S. Sokatchev, *Harmonic superspace*, Cambridge University Press, Cambridge U.K. (2001).
- [17] I.L. Buchbinder, N.G. Pletnev and K.V. Stepanyantz, *Manifestly $N = 2$ supersymmetric regularization for $N = 2$ supersymmetric field theories*, *Phys. Lett. B* **751** (2015) 434 [[arXiv:1509.08055](#)] [[INSPIRE](#)].
- [18] W. Siegel, *Supersymmetric dimensional regularization via dimensional reduction*, *Phys. Lett.* **84B** (1979) 193 [[INSPIRE](#)].
- [19] W.A. Bardeen, A.J. Buras, D.W. Duke and T. Muta, *Deep inelastic scattering beyond the leading order in asymptotically free gauge theories*, *Phys. Rev. D* **18** (1978) 3998 [[INSPIRE](#)].
- [20] I. Jack, D.R.T. Jones and C.G. North, *$N = 1$ supersymmetry and the three loop gauge β -function*, *Phys. Lett. B* **386** (1996) 138 [[hep-ph/9606323](#)] [[INSPIRE](#)].
- [21] I. Jack, D.R.T. Jones and C.G. North, *Scheme dependence and the NSVZ β -function*, *Nucl. Phys. B* **486** (1997) 479 [[hep-ph/9609325](#)] [[INSPIRE](#)].
- [22] I. Jack, D.R.T. Jones and A. Pickering, *The connection between DRED and NSVZ*, *Phys. Lett. B* **435** (1998) 61 [[hep-ph/9805482](#)] [[INSPIRE](#)].
- [23] R.V. Harlander et al., *Four-loop β -function and mass anomalous dimension in dimensional reduction*, *JHEP* **12** (2006) 024 [[hep-ph/0610206](#)] [[INSPIRE](#)].
- [24] L. Mihaila, *Precision calculations in supersymmetric theories*, *Adv. High Energy Phys.* **2013** (2013) 607807 [[arXiv:1310.6178](#)] [[INSPIRE](#)].
- [25] A.L. Kataev and K.V. Stepanyantz, *The NSVZ β -function in supersymmetric theories with different regularizations and renormalization prescriptions*, *Theor. Math. Phys.* **181** (2014) 1531 [[arXiv:1405.7598](#)] [[INSPIRE](#)].
- [26] A.L. Kataev and K.V. Stepanyantz, *Scheme independent consequence of the NSVZ relation for $N = 1$ SQED with N_f flavors*, *Phys. Lett. B* **730** (2014) 184 [[arXiv:1311.0589](#)] [[INSPIRE](#)].

- [27] A.L. Kataev, *Conformal symmetry limit of QED and QCD and identities between perturbative contributions to deep-inelastic scattering sum rules*, *JHEP* **02** (2014) 092 [[arXiv:1305.4605](#)] [[INSPIRE](#)].
- [28] I.O. Goriachuk, A.L. Kataev and K.V. Stepanyantz, *A class of the NSVZ renormalization schemes for $\mathcal{N} = 1$ SQED*, *Phys. Lett. B* **785** (2018) 561 [[arXiv:1808.02050](#)] [[INSPIRE](#)].
- [29] A.L. Kataev, A.E. Kazantsev and K.V. Stepanyantz, *On-shell renormalization scheme for $\mathcal{N} = 1$ SQED and the NSVZ relation*, *Eur. Phys. J. C* **79** (2019) 477 [[arXiv:1905.02222](#)] [[INSPIRE](#)].
- [30] A.L. Kataev and K.V. Stepanyantz, *NSVZ scheme with the higher derivative regularization for $\mathcal{N} = 1$ SQED*, *Nucl. Phys. B* **875** (2013) 459 [[arXiv:1305.7094](#)] [[INSPIRE](#)].
- [31] A.A. Slavnov, *Invariant regularization of nonlinear chiral theories*, *Nucl. Phys. B* **31** (1971) 301 [[INSPIRE](#)].
- [32] A.A. Slavnov, *Invariant regularization of gauge theories*, *Theor. Math. Phys.* **13** (1972) 1064 [*Teor. Mat. Fiz.* **13** (1972) 174].
- [33] V.K. Krivoshchekov, *Invariant regularizations for supersymmetric gauge theories*, *Theor. Math. Phys.* **36** (1978) 745 [*Teor. Mat. Fiz.* **36** (1978) 291].
- [34] P.C. West, *Higher derivative regulation of supersymmetric theories*, *Nucl. Phys. B* **268** (1986) 113 [[INSPIRE](#)].
- [35] V. Yu. Shakhmanov and K.V. Stepanyantz, *New form of the NSVZ relation at the two-loop level*, *Phys. Lett. B* **776** (2018) 417 [[arXiv:1711.03899](#)] [[INSPIRE](#)].
- [36] K.V. Stepanyantz, *Structure of quantum corrections in $\mathcal{N} = 1$ supersymmetric gauge theories*, *Bled Workshops Phys.* **18** (2017) 197 [[arXiv:1711.09194](#)].
- [37] V.Yu. Shakhmanov and K.V. Stepanyantz, *Three-loop NSVZ relation for terms quartic in the Yukawa couplings with the higher covariant derivative regularization*, *Nucl. Phys. B* **920** (2017) 345 [[arXiv:1703.10569](#)] [[INSPIRE](#)].
- [38] A.E. Kazantsev, V.Y. Shakhmanov and K.V. Stepanyantz, *New form of the exact NSVZ β -function: the three-loop verification for terms containing Yukawa couplings*, *JHEP* **04** (2018) 130 [[arXiv:1803.06612](#)] [[INSPIRE](#)].
- [39] M.D. Kuzmichev et al., *Three-loop contribution of the Faddeev–Popov ghosts to the β -function of $\mathcal{N} = 1$ supersymmetric gauge theories and the NSVZ relation*, *Eur. Phys. J. C* **79** (2019) 809 [[arXiv:1908.10586](#)] [[INSPIRE](#)].
- [40] S.S. Aleshin, I.O. Goriachuk, A.L. Kataev and K.V. Stepanyantz, *The NSVZ scheme for $\mathcal{N} = 1$ SQED with N_f flavors, regularized by the dimensional reduction, in the three-loop approximation*, *Phys. Lett. B* **764** (2017) 222 [[arXiv:1610.08034](#)] [[INSPIRE](#)].
- [41] K.V. Stepanyantz, *Derivation of the exact NSVZ β -function in $\mathcal{N} = 1$ SQED, regularized by higher derivatives, by direct summation of Feynman diagrams*, *Nucl. Phys. B* **852** (2011) 71 [[arXiv:1102.3772](#)] [[INSPIRE](#)].
- [42] K.V. Stepanyantz, *The NSVZ β -function and the Schwinger-Dyson equations for $\mathcal{N} = 1$ SQED with N_f flavors, regularized by higher derivatives*, *JHEP* **08** (2014) 096 [[arXiv:1404.6717](#)] [[INSPIRE](#)].
- [43] A.A. Soloshenko and K.V. Stepanyantz, *Three loop β -function for $\mathcal{N} = 1$ supersymmetric electrodynamics, regularized by higher derivatives*, *Theor. Math. Phys.* **140** (2004) 1264 [*Teor. Mat. Fiz.* **140** (2004) 437] [[hep-th/0304083](#)] [[INSPIRE](#)].

- [44] A.V. Smilga and A. Vainshtein, *Background field calculations and nonrenormalization theorems in 4D supersymmetric gauge theories and their low-dimensional descendants*, *Nucl. Phys. B* **704** (2005) 445 [[hep-th/0405142](#)] [[INSPIRE](#)].
- [45] S.L. Adler, *Some Simple vacuum polarization phenomenology: $e^+e^- \rightarrow$ hadrons: the μ -mesic atom X-ray discrepancy and g_μ^{-2}* , *Phys. Rev. D* **10** (1974) 3714 [[INSPIRE](#)].
- [46] M. Shifman and K. Stepanyantz, *Exact Adler function in supersymmetric QCD*, *Phys. Rev. Lett.* **114** (2015) 051601 [[arXiv:1412.3382](#)] [[INSPIRE](#)].
- [47] M. Shifman and K.V. Stepanyantz, *Derivation of the exact expression for the D function in $N = 1$ SQCD*, *Phys. Rev. D* **91** (2015) 105008 [[arXiv:1502.06655](#)] [[INSPIRE](#)].
- [48] I.V. Nartsev and K.V. Stepanyantz, *Exact renormalization of the photino mass in softly broken $\mathcal{N} = 1$ SQED with N_f flavors regularized by higher derivatives*, *JHEP* **04** (2017) 047 [[arXiv:1610.01280](#)] [[INSPIRE](#)].
- [49] J. Hisano and M.A. Shifman, *Exact results for soft supersymmetry breaking parameters in supersymmetric gauge theories*, *Phys. Rev. D* **56** (1997) 5475 [[hep-ph/9705417](#)] [[INSPIRE](#)].
- [50] I. Jack and D.R.T. Jones, *The gaugino β -function*, *Phys. Lett. B* **415** (1997) 383 [[hep-ph/9709364](#)] [[INSPIRE](#)].
- [51] L.V. Avdeev, D.I. Kazakov and I.N. Kondrashuk, *Renormalizations in softly broken SUSY gauge theories*, *Nucl. Phys. B* **510** (1998) 289 [[hep-ph/9709397](#)] [[INSPIRE](#)].
- [52] A.L. Kataev, A.E. Kazantsev and K.V. Stepanyantz, *The Adler D -function for $\mathcal{N} = 1$ SQCD regularized by higher covariant derivatives in the three-loop approximation*, *Nucl. Phys. B* **926** (2018) 295 [[arXiv:1710.03941](#)] [[INSPIRE](#)].
- [53] I.V. Nartsev and K.V. Stepanyantz, *NSVZ-like scheme for the photino mass in softly broken $\mathcal{N} = 1$ SQED regularized by higher derivatives*, *JETP Lett.* **105** (2017) 69 [[arXiv:1611.09091](#)] [[INSPIRE](#)].
- [54] K.V. Stepanyantz, *The β -function of $\mathcal{N} = 1$ supersymmetric gauge theories regularized by higher covariant derivatives as an integral of double total derivatives*, *JHEP* **10** (2019) 011 [[arXiv:1908.04108](#)] [[INSPIRE](#)].
- [55] S.S. Aleshin, A.L. Kataev and K.V. Stepanyantz, *Structure of three-loop contributions to the β -function of $\mathcal{N} = 1$ supersymmetric QED with N_f flavors regularized by the dimensional reduction*, *JETP Lett.* **103** (2016) 77 [[arXiv:1511.05675](#)] [[INSPIRE](#)].
- [56] A.B. Pimenov, E.S. Shevtsova and K.V. Stepanyantz, *Calculation of two-loop β -function for general $N = 1$ supersymmetric Yang–Mills theory with the higher covariant derivative regularization*, *Phys. Lett. B* **686** (2010) 293 [[arXiv:0912.5191](#)] [[INSPIRE](#)].
- [57] K.V. Stepanyantz, *Factorization of integrals defining the two-loop β -function for the general renormalizable $N = 1$ SYM theory, regularized by the higher covariant derivatives, into integrals of double total derivatives*, [arXiv:1108.1491](#) [[INSPIRE](#)].
- [58] S.S. Aleshin, A.E. Kazantsev, M.B. Skoptsov and K.V. Stepanyantz, *One-loop divergences in non-Abelian supersymmetric theories regularized by BRST-invariant version of the higher derivative regularization*, *JHEP* **05** (2016) 014 [[arXiv:1603.04347](#)] [[INSPIRE](#)].
- [59] K.V. Stepanyantz, *The NSVZ β -function for theories regularized by higher covariant derivatives: the all-loop sum of matter and ghost singularities*, *JHEP* **01** (2020) 192 [[arXiv:1912.12589](#)] [[INSPIRE](#)].
- [60] S. Heinemeyer, M. Mondragón, N. Tracas and G. Zoupanos, *Reduction of Couplings and its application in particle physics*, *Phys. Rept.* **814** (2019) 1 [[arXiv:1904.00410](#)] [[INSPIRE](#)].

- [61] A. Parkes and P.C. West, *Finiteness in rigid supersymmetric theories*, *Phys. Lett.* **138B** (1984) 99 [[INSPIRE](#)].
- [62] D.R.T. Jones and L. Mezincescu, *The β -function in supersymmetric Yang-Mills theory*, *Phys. Lett.* **136B** (1984) 242 [[INSPIRE](#)].
- [63] D.R.T. Jones and L. Mezincescu, *The chiral anomaly and a class of two loop finite supersymmetric gauge theories*, *Phys. Lett.* **138B** (1984) 293 [[INSPIRE](#)].
- [64] M.T. Grisaru, B. Milewski and D. Zanon, *The structure of UV divergences in SSYM theories*, *Phys. Lett.* **155B** (1985) 357 [[INSPIRE](#)].
- [65] A.J. Parkes, *Three loop finiteness conditions in $N = 1$ super-Yang-Mills*, *Phys. Lett.* **156B** (1985) 73 [[INSPIRE](#)].
- [66] I. Jack, D.R.T. Jones and C.G. North, *$N = 1$ supersymmetry and the three loop anomalous dimension for the chiral superfield*, *Nucl. Phys. B* **473** (1996) 308 [[hep-ph/9603386](#)] [[INSPIRE](#)].
- [67] D.I. Kazakov, *Finite $N = 1$ SUSY field theories and dimensional regularization*, *Phys. Lett. B* **179** (1986) 352 [[INSPIRE](#)].
- [68] A.V. Ermushev, D.I. Kazakov and O.V. Tarasov, *Finite $N = 1$ supersymmetric grand unified theories*, *Nucl. Phys. B* **281** (1987) 72 [[INSPIRE](#)].
- [69] C. Lucchesi, O. Piguet and K. Sibold, *Vanishing β -functions in $N = 1$ Supersymmetric gauge theories*, *Helv. Phys. Acta* **61** (1988) 321 [[INSPIRE](#)].
- [70] C. Lucchesi, O. Piguet and K. Sibold, *Necessary and sufficient conditions for all order vanishing β -functions in supersymmetric Yang-Mills Theories*, *Phys. Lett. B* **201** (1988) 241 [[INSPIRE](#)].
- [71] I. Jack and D.R.T. Jones, *Renormalization group invariance and universal soft supersymmetry breaking*, *Phys. Lett. B* **349** (1995) 294 [[hep-ph/9501395](#)] [[INSPIRE](#)].
- [72] S.J. Gates, M.T. Grisaru, M. Roček and W. Siegel, *Superspace or One Thousand and one lessons in supersymmetry*, *Front. Phys.* **58** (1983) 1 [[hep-th/0108200](#)] [[INSPIRE](#)].
- [73] P.C. West, *Introduction to supersymmetry and supergravity*, World Scientific, Singapore (1990).
- [74] I.L. Buchbinder and S.M. Kuzenko, *Ideas and methods of supersymmetry and supergravity: or a walk through superspace*, IOP, Bristol, U.K. (1998).
- [75] O. Piguet and K. Sibold, *Renormalization of $N = 1$ supersymmetrical Yang-Mills theories. 1. The classical theory*, *Nucl. Phys. B* **197** (1982) 257 [[INSPIRE](#)].
- [76] O. Piguet and K. Sibold, *Renormalization of $N = 1$ supersymmetrical Yang-Mills theories. 2. The radiative corrections*, *Nucl. Phys. B* **197** (1982) 272 [[INSPIRE](#)].
- [77] I.V. Tyutin, *Renormalization of supergauge theories with nonextended supersymmetry* (in russian), *Yad. Fiz.* **37** (1983) 761 [[INSPIRE](#)].
- [78] B.S. DeWitt, *Dynamical theory of groups and fields*, *Conf. Proc. C* **630701** (1964) 585 [Les Houches Lect. Notes **13** (1964) 585].
- [79] L.F. Abbott, *The background field method beyond one loop*, *Nucl. Phys. B* **185** (1981) 189 [[INSPIRE](#)].
- [80] L.F. Abbott, *Introduction to the background field method*, *Acta Phys. Polon. B* **13** (1982) 33 [[INSPIRE](#)].

- [81] J.W. Juer and D. Storey, *Nonlinear renormalization in superfield gauge Theories*, *Phys. Lett.* **119B** (1982) 125 [[INSPIRE](#)].
- [82] J.W. Juer and D. Storey, *One loop renormalization of superfield Yang-Mills Theories*, *Nucl. Phys. B* **216** (1983) 185 [[INSPIRE](#)].
- [83] A.E. Kazantsev et al., *Two-loop renormalization of the Faddeev-Popov ghosts in $\mathcal{N} = 1$ supersymmetric gauge theories regularized by higher derivatives*, *JHEP* **06** (2018) 020 [[arXiv:1805.03686](#)] [[INSPIRE](#)].
- [84] L.D. Faddeev and A.A. Slavnov, *Gauge fields. Introduction to quantum theory*, *Front. Phys.* **50** (1980) 1 [[INSPIRE](#)].
- [85] A.A. Slavnov, *The Pauli-Villars regularization for nonabelian gauge theories*, *Theor. Math. Phys.* **33** (1977) 977 [*Teor. Mat. Fiz.* **33** (1977) 210].
- [86] A.E. Kazantsev, M.B. Skoptsov and K.V. Stepanyantz, *One-loop polarization operator of the quantum gauge superfield for $\mathcal{N} = 1$ SYM regularized by higher derivatives*, *Mod. Phys. Lett. A* **32** (2017) 1750194 [[arXiv:1709.08575](#)] [[INSPIRE](#)].
- [87] K. Stepanyantz, *The higher covariant derivative regularization as a tool for revealing the structure of quantum corrections in supersymmetric gauge theories*, [arXiv:1910.03242](#) [[INSPIRE](#)].
- [88] A.A. Soloshenko and K.V. Stepanyants, *Two-loop anomalous dimension of $N = 1$ supersymmetric quantum electrodynamics regularized using higher covariant derivatives*, *Theor. Math. Phys.* **134** (2003) 377 [[INSPIRE](#)].
- [89] S.S. Aleshin et al., *Three-loop verification of a new algorithm for the calculation of a β -function in supersymmetric theories regularized by higher derivatives for the case of $\mathcal{N} = 1$ SQED*, *Nucl. Phys. B* **956** (2020) 115020 [[arXiv:2003.06851](#)] [[INSPIRE](#)].
- [90] T. Banks and A. Zaks, *On the phase structure of vector-Like gauge Theories with massless fermions*, *Nucl. Phys. B* **196** (1982) 189 [[INSPIRE](#)].
- [91] A.A. Vladimirov and D.V. Shirkov, *The renormalization group and ultraviolet asymptotics*, *Sov. Phys. Usp.* **22** (1979) 860 [*Usp. Fiz. Nauk* **129** (1979) 407] [[INSPIRE](#)].
- [92] S.S. Aleshin, A.L. Kataev and K.V. Stepanyantz, *The three-loop adler D -function for $\mathcal{N} = 1$ SQCD regularized by dimensional reduction*, *JHEP* **03** (2019) 196 [[arXiv:1902.08602](#)] [[INSPIRE](#)].
- [93] D.R.T. Jones, *Coupling constant reparametrization and finite field theories*, *Nucl. Phys. B* **277** (1986) 153 [[INSPIRE](#)].