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On the supersymmetric solutions of the Heterotic Superstring effective action

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ABSTRACT: We consider the effective action of the Heterotic Superstring to first order in α' and derive the necessary and sufficient conditions that a field configuration has to satisfy in order to admit at least one Killing spinor using the spinor bilinear method in an arbitrary spinorial basis and corresponding arbitrary gamma matrices. As a previous step in this derivation, we compute the complete spinor bilinear algebra using the Fierz identities, obtaining as a by-product the algebra satisfied by the $\text{Spin}(7)$ structure contained in the bilinears in an arbitrary basis. We find the off-shell relations existing between the bosonic equations of motion evaluated on supersymmetric field configurations using the Killing Spinor Identities instead of the (far more complicated) integrability conditions of the Killing Spinor Equations as it is common in the literature. We show how to include the Kalb-Ramond's Bianchi identity in the Killing Spinor Identities.

KEYWORDS: Superstring Vacua, Superstrings and Heterotic Strings

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1 Introduction

The construction and study of the classical solutions of a theory always provides a great deal of information about its properties and predictions. The fundamental rôle played by the Schwarzschild solution in the conceptual development of General Relativity, as well as in more mundane computations of testable predictions of this theory, is a very clear example that cannot be overstated. For these reasons, the construction and study of solutions of the Superstring Theory effective action (compactification backgrounds, pp -wave backgrounds, black holes, cosmological models) has been a very active area of research for almost 30 years and it is not surprising that some of the solutions found, such as the 3-charge black hole [1, 2] used by Strominger and Vafa to compare the Bekenstein-Hawking entropy with that obtained by the first microstate counting in ref. [3], have also had a huge impact in the development of Superstring Theory. Near-horizon geometries, pp -waves and other Penrose limits of solutions provide further examples.

The methods used to construct new solutions of Superstring Theory based on dualities and supersymmetry have probably been the most fruitful ones. Dualities transform solutions into solutions, sometimes with very different properties. The original solutions are required to satisfy only a minimal number of conditions such as the existence of isometries for T-duality. In contrast, supersymmetry methods can only be used to construct supersymmetric solutions, but, in general, they provide very general recipes that permit the construction of very general families of supersymmetric solutions such as all supersymmetric black holes of a given Superstring effective field theory (in the end, a supergravity theory). In Superstring Theory, supersymmetric solutions often describe the fields generated by non-perturbative extended objects such as D p -branes and provide a way to learn more about them. Supersymmetric compactification backgrounds are an essential ingredient of many superstring phenomenological models as well. But supersymmetric solutions are also interesting in their own right because, often, they involve structures and enjoy properties of great physical and mathematical relevance. All this justifies the great deal of effort employed in the characterization and classification of all supersymmetric solutions of Superstring Theory via the characterization and classification of all supersymmetric solutions of all supergravity theories.

This effort started with the pioneering work of Gibbons, Hull and Tod [4, 5] in pure $\mathcal{N} = 2$, $d = 4$ supergravity. This theory is just the simplest of the very rich family of $\mathcal{N} = 2$, $d = 4$ supergravities which have different matter contents (vector multiplets and hypermultiplets) and couplings (some of them associated to gaugings of their global symmetries). All of them have been studied from this point of view in a long series of papers [6–13] of increasing complexity using the “spinor bilinear” method of ref. [14], which we will also use and explain in this paper. The most general case, considered in ref. [13] has only been solved for the “timelike” case and the “null” supersymmetric solutions of theories with non-Abelian gaugings still have to be characterized. The supersymmetric solutions (both timelike and null) of the pure $\mathcal{N} = 4$, $d = 4$ theory have also been characterized in refs. [15, 16], but neither the matter-coupled nor gauged theories have been studied.¹

¹It is believed that the supersymmetric solutions of these, and other supergravity theories are in one to one correspondence to those of their $\mathcal{N} = 2$ truncations, although this has not been formally proven.

Finally, since it is possible to treat all 4-dimensional supergravities with vector multiplets in a unified form, all their timelike supersymmetric solutions were characterized in a unified form in [17]. The null case and the gauged theories remain to be studied.

In $d = 5$ dimensions the situation in $\mathcal{N} = 1$ theories² is better because all timelike and null solutions have been characterized with the most general matter content and couplings in refs. [14, 18–24]. The $\mathcal{N} > 1, d = 5$ theories have not been studied systematically.

In the case of $\mathcal{N} = (1, 0), d = 6$ supergravity, all supersymmetric solutions have been characterized systematically in refs. [25–30], but those of the rest of the 6-dimensional supergravity theories have not.

For the sake of making this short review of what has been accomplished in this field of research in $d \leq 6$ complete, let us also mention the work done in maximal and half-maximal $d = 3$ supergravities in refs. [31–34] and also $\mathcal{N} = 1, d = 4$ supergravity in refs. [35, 36]. For a review on this topic and additional references on related work, see refs. [37, 38].

In dimensions higher than six there are only supergravities with 16 or 32 supercharges. Many of them can be obtained via dimensional reduction from the 10- and 11-dimensional theories and, therefore, most of the work has been focused directly on these. Pure $\mathcal{N} = 1, d = 10$ supergravity can be viewed as the effective field theory of the Heterotic or the Type I Superstrings (depending on the stringy interpretation of the supergravity fields) at lowest order in an expansion in terms of the Regge slope parameter α' . One of the most important features of these theories is the presence of massless gauge vectors in their spectra. These occur at first order in α' in the effective action, but there is no problem to accommodate them in $\mathcal{N} = 1, d = 10$ supergravity as vector supermultiplets [39, 40]. However, at this order in α' , the Heterotic Superstring effective action contains more terms which can only be accommodated in $\mathcal{N} = 1, d = 10$ supergravity if more terms are also included to preserve invariance under supersymmetry transformations at a given order [41, 42]. The additional terms are of higher order in derivatives as well, which leads to very complicated equations of motion.

From the heterotic superstring effective action point of view, the neglect of the α' corrections has to be justified in each particular solution. In general, this imposes constraints on the charges of the solutions or signals (typically high-curvature) regions of the solutions which cannot be trusted as good string theory solutions because they are bound to be modified once the neglected terms in the theory are reconsidered. For this reason, from the Superstring Theory point of view, it is important to take into account these possible complicated α' corrections in the analysis from the onset.

However, even though the corrections to the equations of motion are very involved, it so happens that the Killing Spinor Equations (KSEs) only get “implicit” first-order α' corrections and can be analyzed at zeroth or first order at the same time. For these reasons, in the first systematic studies of supersymmetric solutions refs. [43–45] the KSEs were solved at zeroth/first order but involving only the zeroth-order equations of motion.³

²This means 8 supercharges. Sometimes they are referred to as $\mathcal{N} = 2, d = 5$ supergravities in the literature although the minimal spinor in 5 dimensions has 8 components because of their relation with $\mathcal{N} = 2, d = 4$ theories.

³Eqs. (2.1) of ref. [43]. The first α' corrections to the equations of motion have been considered more recently in [38]. See section 4.2 for a discussion on their rôle in the KSEs’ integrability conditions.

In all these works, the so-called *spinorial geometry* method was used [46]. In this method many computations are streamlined by the use of a privileged basis of spinors, and, correspondingly, of gamma matrices. The downside of this approach is that the results obtained concerning spinors are written in that particular basis and, sometimes, a basis-independent form could be more desirable for some purposes and could also give further insights into the structure and interpretation of the supersymmetric solutions, as we will discuss below.

Our goal in this paper is to repeat part of the analysis carried out in refs. [43–45] using the spinor bilinear formalism without making any choice of spinor basis, to first order in α' . Thus, we expect to obtain

1. The algebra of spinor bilinears in an arbitrary basis (see section A.4). This algebra includes the relations satisfied by the Spin(7) structure 4-form Ω_{abcd} .
2. The form of the Killing spinors in a general spinorial basis. In particular, we will obtain the supersymmetry projectors using arbitrary gamma matrices (section A.5). The form of the projector eq. (A.45b) suggests that the solutions with only one Killing spinor can be viewed as a multiple intersection of S5-branes, with volume form of the 4-dimensional transverse space of each of them entering the Spin(7) structure 4-form Ω_{abcd} . This projector can be compared with eq. (3.10) of ref. [43]. They are equivalent, but the later is expressed in a particular basis and its form will change under a general change of basis while the form of the projector eq. (A.45b) will not. Furthermore, the intersection interpretation is lost in the simplified form.
3. The necessary and sufficient conditions that a supersymmetric field configuration has to satisfy, originally found in refs. [43–45]. These will be obtained in essentially the same form.
4. The relations that hold between the first-order in α' -corrected equations of motion when they are evaluated on supersymmetric configurations, via the Killing Spinor Identities (see below). Relations of this kind have been derived recently as integrability conditions of the Killing Spinor Equations in ref. [38]. We discuss the relation with ours in section 4.2.

We are just interested in the general characterization of the supersymmetric solutions (those admitting (at least) one Killing spinor), and we will not try to study case by case, what happens when the solution admits 2 or more Killing spinors, as it has been done in full detail in refs. [43–45].

There are several important difficulties which prevent us from making this analysis at higher orders in α' :

1. Not all α' corrections to the action and supersymmetry transformation rules are known. In ref. [42], which we will use here, they were determined to cubic order in α' (quartic order in curvatures) by supersymmetrizing the first-order terms (specially the Chern-Simons terms in the Kalb-Ramond 3-form field strength) which had been

found by other means. This supersymmetrization leads to a recursive procedure for introducing the terms of next order in α' in the Kalb-Ramond 3-form field strength H . H occurs in the action and in the supersymmetry transformation rules, and part of the α' corrections in them are introduced implicitly through H . However, apart from these, there are other α' corrections of increasing complexity both in the action and in the supersymmetry transformation rules and this will force us to work only at first order in α' and analyze the KSEs only up to that order. It has been suggested that higher order corrections to the KSEs could be absorbed into redefinitions of the torsionful spin connection or the Kalb-Ramond 3-form field strength which would preserve their first-order form, though, [42, 47]. If this was proven, the first-order analysis would be sufficient.

2. As we have just said, many (probably most) of the higher-order terms in the action cannot be constructed by recursion and they are only known explicitly up to cubic order. The equations of motion⁴ have a very large number of complicated terms. At first order, though, it is known that many of them can be ignored because they are proportional to the zeroth-order equations of motion. It is not known if similar simplifications take place at higher orders and, if one wants to find explicit solutions at higher orders one will have to deal with very complicated equations.
3. In most of the literature on the characterization or classification of supersymmetric solutions the following fact is used: the integrability conditions of the KSEs (adequately treated) are combinations of the off-shell equations of motion. This leads to non-trivial relations between them which simplify the problem of finding supersymmetric solutions, because only a small number of equations of motion are independent and have to be solved. At higher orders in α' , computing the integrability conditions of the KSEs and recognizing in them the complicated higher-order equations of motion can be very difficult.

The Killing Spinor Identities (KSIs) derived in ref. [48] offer an alternative path, because, based on the invariance of the theory under local supersymmetry transformations, they yield the same relations between equations of motion⁵ even if the form of the latter is not known explicitly [49]. The main ingredients are the supersymmetry transformations of the bosonic fields of the theory, which, usually, are much simpler than those of the fermions.

4. However, even though, in most cases, the supersymmetry transformations of the bosonic fields are not modified when the supergravity theory is gauged or deformed in other ways, in this case the Kalb-Ramond 3-form field strength is modified at first order in α' by the addition of the Lorentz- and Yang-Mills-Chern-Simons terms that

⁴We only consider purely bosonic configurations and, therefore, we always refer, implicitly, to the equations of motion of the bosonic fields of the theory.

⁵As we will explain in section 4.1, the equations of motion usually considered in the literature are combinations of those one obtains by direct variation of the action where terms proportional to the lowest-order equations of motion are eliminated. In order to use the KSIs, we need to take into account these facts.

modify the gauge transformations of the Kalb-Ramond 2-form. The supersymmetry transformations of this field acquire additional terms related to these gauge transformations which results in a non-trivial modification of the KSIs at first order in α' we have to deal with.

5. The main drawback of the KSIs is that, by construction, they never include the Bianchi identities satisfied by the field strengths in the relations obtained. The Bianchi identity of the Kalb-Ramond 3-form is typically one of the key equations to be solved, though. This deficiency of the KSI approach can be overcome by including the 6-form dual of the Kalb-Ramond 2-form in the derivation of the KSIs, as we will show in section 4.2. The equation of motion of the 6-form is, by definition, the Bianchi identity of the Kalb-Ramond 3-form field strength. We are not including the Bianchi identity of the Yang-Mills gauge field strength because in order to write it one needs to know the gauge connection, which completely determines and trivializes the Bianchi identity.

After considering all these difficulties and the solutions found for some of them, it is clear that, being conservative, we will have to content ourselves with carrying our program to first order in α' only: we will determine necessary and sufficient conditions for unbroken supersymmetry valid to $\mathcal{O}(\alpha')$ and relations between equations of motion evaluated on supersymmetric configurations valid to the same order in α' . We will not be able to simplify the independent equations of motion beyond $\mathcal{O}(\alpha')$, either, because it is not clear if they can be simplified.

This paper is organized as follows: we start by reviewing section 2 the bosonic Heterotic Superstring effective action, equations of motion and supersymmetry transformation rules. Appendix A contains our conventions for gamma matrices and spinors, the definitions of the spinor bilinears used in the main text and the computation of the algebra satisfied by these bilinears using the Fierz identities in an arbitrary basis which we are going to use in section 3.⁶ As a byproduct we will derive the algebra of the Spin(7) structure 4-form which is always present in the bilinear algebra. In section 3 we will determine the necessary conditions for a field configuration to admit one Killing spinor (summarized in 3.4) and we will show that they are also sufficient by solving explicitly the KSEs. We will make use of the spinor projector given explicitly in terms of the Spin(7) structure in eq. (A.45b). In section 4 we determine which independent equations have to be imposed on the supersymmetric field configurations determined in the previous section to ensure that they are solutions of all equations of motion of the theory. We explain how the equations of motion are obtained and simplified at first order in α' and the derivation of the KSIs involving also the Bianchi identity of the Kalb-Ramond 3-form field strength. Finally, section 5 contains a brief discussion of the results obtained and their spinoffs.

⁶We have learned, after doing this calculation, that there is a much more efficient way of computing the bilinear algebra [50].

2 The Heterotic Superstring effective action

In this section we are going to review the Heterotic Superstring effective action was given in ref. [42], where it was constructed up to cubic order in α' (quartic in derivatives) by demanding invariance of the action under supersymmetry (to that order). Here we will use the conventions of ref. [37].⁷

We start by defining recursively the 3-form field strength of the Kalb-Ramond 2-form B . The zeroth-order, it is given by

$$H^{(0)} \equiv dB. \quad (2.1)$$

Using it as torsion, one can define the zeroth-order torsionful spin (or Lorentz) connections

$$\Omega_{(\pm)}^{(0) a}{}_b = \omega^a{}_b \pm \frac{1}{2} H_{\mu}^{(0) a}{}_b dx^{\mu}, \quad (2.2)$$

where $\omega^a{}_b$ is the (torsionless, metric-compatible) Levi-Civita spin connection 1-form.

The curvature 2-forms of these connections and the (Lorentz-) Chern-Simons 3-forms are defined as⁸

$$R_{(\pm)}^{(0) a}{}_b = d\Omega_{(\pm)}^{(0) a}{}_b - \Omega_{(\pm)}^{(0) a}{}_c \wedge \Omega_{(\pm)}^{(0) c}{}_b, \quad (2.3)$$

$$\omega_{(\pm)}^{L(0)} = d\Omega_{(\pm)}^{(0) a}{}_b \wedge \Omega_{(\pm)}^{(0) b}{}_a - \frac{2}{3} \Omega_{(\pm)}^{(0) a}{}_b \wedge \Omega_{(\pm)}^{(0) b}{}_c \wedge \Omega_{(\pm)}^{(0) c}{}_a. \quad (2.4)$$

We will denote the gauge field 1-form by A^A , where the indices $A, B, C, \dots$ take values in the Lie algebra of the gauge group. The (Yang-Mills) gauge field strength and Chern-Simons 3-form are defined by

$$F^A = dA^A + \frac{1}{2} f_{BC}{}^A A^B \wedge A^C, \quad (2.5)$$

$$\omega^{\text{YM}} = dA_A \wedge A^A + \frac{1}{3} f_{ABC} A^A \wedge A^B \wedge A^C, \quad (2.6)$$

where the Killing metric of the gauge group's Lie algebra in the relevant representation, K_{AB} , assumed to be invertible and positive definite, has been used to lower the index of the structure constants $f_{ABC} \equiv f_{AB}{}^D K_{DB}$ and of the indices of the gauge fields $A_A \equiv K_{AB} A^B$.

Then, using the Yang-Mills- and zeroth-order Lorentz-Chern-Simons 3-forms,⁹ the first-order the Kalb-Ramond 3-form field strength is defined to be

$$H^{(1)} = dB + \frac{\alpha'}{4} \left(\omega^{\text{YM}} + \omega_{(-)}^{L(0)} \right), \quad (2.7)$$

⁷The relation with the fields in ref. [42] is as follows: the metric and gauged fields can be identified; the Kalb-Ramond fields are related by $B_{\text{BdR} \mu\nu} = \frac{1}{\sqrt{2}} B_{\mu\nu}$ and their field strengths by $H_{\text{BdR} \mu\nu\rho} = \frac{1}{3\sqrt{2}} H_{\mu\nu\rho}$. The dilaton fields are related by $\phi_{\text{BdR}} = e^{2\phi/3}$. The gravitino and dilatino are related by $\psi_{\text{BdR} \mu} = \sqrt{2} \psi_{\mu}$, $\lambda_{\text{BdR}} = -\frac{1}{2} \lambda$ while the gaugini are related by $\chi_{\text{BdR}}^A = \sqrt{2} \chi^A$. The relation between the 6-form dual of the Kalb-Ramond 2-form in refs. [51, 52], $A_{\alpha_1 \dots \alpha_6}$ by the same authors and our 6-form $\tilde{B}_{\alpha_1 \dots \alpha_6}$ is $A_{\alpha_1 \dots \alpha_6} = \frac{1}{2\sqrt{2} \cdot 6!} \tilde{B}_{\alpha_1 \dots \alpha_6}$. Finally, the supersymmetry parameters are related by $\epsilon_{\text{BdR}} = \sqrt{2} \epsilon$ and the parameters α and β are both equal to $\alpha'/4$.

⁸The same definition applies to other spin connections with the same indices; in particular, for the curvature 2-form of the Levi-Civita connection (the Riemann tensor).

⁹Only $\Omega_{(-)}$ occurs in H .

and using it as torsion, we obtain the first-order torsionful spin connections

$$\Omega_{(\pm)}^{(1) a}{}_b = \omega^a{}_b \pm \frac{1}{2} H_{\mu}^{(1) a}{}_b dx^{\mu}, \quad (2.8)$$

and their curvatures and Lorentz-Chern-Simons terms $R_{(\pm)}^{(1) a}{}_b, \omega_{(\pm)}^{L(1)}$ are obtained by plugging them into the above definitions. Then, the second-order Kalb-Ramond field strength is defined as

$$H^{(2)} = dB + \frac{\alpha'}{4} \left(\omega^{\text{YM}} + \omega_{(-)}^{L(1)} \right), \quad \dots \quad H^{(n)} = dB + \frac{\alpha'}{4} \left(\omega^{\text{YM}} + \omega_{(-)}^{L(n-1)} \right), \quad (2.9)$$

etc.

For many practical purposes it is advantageous to work with general H and $\Omega_{(\pm)}$ and then restrict them to a given order when needed. This will allow us to work with the Killing spinor equations at an arbitrary order in α' , for instance, because the only α' corrections are contained in the definitions of H and $\Omega_{(+)}$. In the action, apart from the α' corrections implicit in the definitions of H and $\Omega_{(-)}$, there are additional terms of higher order in curvatures that have to be included explicitly and which are known only to cubic order in α' [42].¹⁰ It is understood that all terms above a certain order in α' have to be ignored. Thus, with this understanding, we will omit the upper indices (n) from now on.

It is convenient to define an affine torsionful connection $\Gamma^{(\pm)}_{\mu\nu}{}^{\rho}$ via the Vielbein postulate

$$\nabla^{(+)}_{\mu} e^a{}_{\nu} = \partial_{\mu} e^a{}_{\nu} - \Omega^{(\pm)}_{\mu}{}^a{}_b e^b{}_{\nu} - \Gamma^{(\pm)}_{\mu\nu}{}^{\rho} e^a{}_{\rho} = 0. \quad (2.10)$$

Solving the above equation one finds that it is given by

$$\Gamma^{(\pm)}_{\mu\nu}{}^{\rho} = \left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\} \pm \frac{1}{2} H_{\mu\nu}{}^{\rho}, \quad (2.11)$$

where $\left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\}$ stands for the Christoffel symbols of the metric $g_{\mu\nu} = \eta_{ab} e^a{}_{\mu} e^b{}_{\nu}$.

It is also convenient to use the so-called “ T -tensors” associated to the α' corrections in the equations of motion and in the Bianchi identity of the Kalb-Ramond 3-form field strength. They are defined by

$$\begin{aligned} T^{(4)} &\equiv \frac{\alpha'}{4} \left[F_A \wedge F^A + R_{(-)}^a{}_b \wedge R_{(-)}^b{}_a \right], \\ T^{(2)}_{\mu\nu} &\equiv \frac{\alpha'}{4} \left[F_{A\mu\rho} F^A{}_{\nu}{}^{\rho} + R_{(-)\mu\rho}^a{}_b R_{(-)\nu}{}^{\rho b}{}_a \right], \\ T^{(0)} &\equiv T^{(2)}{}^{\mu}{}_{\mu}. \end{aligned} \quad (2.12)$$

The Heterotic Superstring effective action, written in the string frame to cubic order in α' in terms of the objects we have just defined takes the form

$$S = \frac{g_s^2}{16\pi G_N^{(10)}} \int d^{10}x \sqrt{|g|} e^{-2\phi} \left\{ R - 4(\partial\phi)^2 + \frac{1}{12} H^2 - \frac{1}{2} T^{(0)} - \frac{\alpha'}{48} (T^{(4)})^2 - \frac{\alpha'}{4} (T^{(2)})^2 \right\}, \quad (2.13)$$

¹⁰And not completely: only the quartic terms that follow from the supersymmetrization of the Chern-Simons terms in H were determined, but there may be more [53–55].

where R is the Ricci scalar of the string-frame metric $g_{\mu\nu}$, ϕ is the dilaton field and its vacuum expectation value of e^ϕ is the Heterotic Superstring coupling constant g_s and where $G_N^{(10)}$ is the 10-dimensional Newton constant.

Observe that, to have all $\mathcal{O}(\alpha'^3)$ terms in the action, we have to use $H^{(3)}$ and $\Omega_{(-)}^{(3)}$, disregarding all terms of higher order that arise from H^2 etc. in the above action. As explained in the Introduction, though, we will only work at first order in α' .

Finally, let us consider the supersymmetry transformation rules of the bosons and of the fermionic fields (gravitino ψ_μ , dilatino λ and gaugini χ^A)¹¹ for vanishing fermions and to first order in α' . They can be written, respectively, as follows:

$$\delta_\epsilon e^a{}_\mu = \bar{\epsilon} \Gamma^a \psi_\mu, \quad (2.14)$$

$$\delta_\epsilon B_{\mu\nu} = 2\bar{\epsilon} \Gamma_{[\mu} \psi_{\nu]} + \frac{\alpha'}{2} \left\{ A_{A[\mu} \delta_\epsilon A^A{}_{\nu]} + \Omega_{(-)[\mu}{}^a{}_b \delta_\epsilon \Omega_{(-)\nu]}{}^b{}_a \right\}, \quad (2.15)$$

$$\delta_\epsilon \tilde{B}_{\mu_1 \dots \mu_6} = 6e^{-2\phi} \bar{\epsilon} \Gamma_{[\mu_1 \dots \mu_5} \left[\psi_{\mu_6]} - \frac{1}{6} \Gamma_{\mu_6]} \lambda \right], \quad (2.16)$$

$$\delta_\epsilon \phi = \frac{1}{2} \bar{\epsilon} \lambda, \quad (2.17)$$

$$\alpha' \delta_\epsilon A^A{}_\mu = \alpha' \bar{\epsilon} \Gamma_\mu \chi^A, \quad (2.18)$$

where, at this order,

$$\delta_\epsilon \Omega_{(-)\mu}{}^{ab} = \bar{\epsilon} \Gamma_\mu \psi^{ab}, \quad (2.19)$$

with the gravitino field strength $\psi_{\mu\nu}$ defined as

$$\psi_{\mu\nu} \equiv 2\mathcal{D}_{[\mu}^{(+)} \psi_{\nu]} \equiv 2\partial_{[\mu} \psi_{\nu]} - \frac{1}{2} \Omega_{(+)[\mu}{}^{ab} \Gamma_{ab} \psi_{\nu]}, \quad (2.20)$$

and

$$\delta_\epsilon \psi_a = \nabla^{(+)}{}_a \epsilon \equiv \left(\partial_a - \frac{1}{4} \Omega^{(+)}{}_{abc} \Gamma^{bc} \right) \epsilon, \quad (2.21)$$

$$\delta_\epsilon \lambda = \left(\partial_a \phi \Gamma^a - \frac{1}{12} H_{abc} \Gamma^{abc} \right) \epsilon, \quad (2.22)$$

$$\alpha' \delta_\epsilon \chi^A = -\frac{1}{4} \alpha' F^A{}_{ab} \Gamma^{ab} \epsilon. \quad (2.23)$$

3 Supersymmetric configurations

The (purely bosonic) supersymmetric field configurations of this theory are those for which the Killing Spinor Equations (KSEs) $\delta_\epsilon \psi_a = \delta_\epsilon \lambda = \delta_\epsilon \chi^A = 0$ admit at least one solution called Killing spinor that we will denote by ϵ . Thus, if $e^a{}_\mu, B_{\mu\nu}, \phi$ describe a supersymmetric field configuration of this theory, there is an ϵ satisfying the KSEs

$$\nabla^{(+)}{}_a \epsilon = 0, \quad (3.1)$$

$$\left(\partial_a \phi \Gamma^a - \frac{1}{12} H_{abc} \Gamma^{abc} \right) \epsilon = 0, \quad (3.2)$$

$$F^A{}_{ab} \Gamma^{ab} \epsilon = 0. \quad (3.3)$$

¹¹All fermions and the supersymmetry parameter ϵ are Majorana-Weyl spinors and ϵ has positive chirality in the conventions given in appendix A.

When the spinor bilinears ℓ_a and $W_{a_1 \dots a_5}$ defined in appendix A.2, are constructed with the Killing spinor ϵ that satisfies the above equations, they must satisfy certain some other equations apart from the algebraic relations found in appendix A.4. In what follows, we are going to determine those equations and their immediate consequences, for each KSE.

3.1 The gravitino KSE

Using the torsionful affine connection defined in eq. (2.11), the gravitino KSE eq. (3.1) immediately leads to these two differential equations:

$$\nabla_a^{(+)} \ell_b = 0, \quad (3.4)$$

$$\nabla_a^{(+)} W_{b_1 \dots b_5} = 0. \quad (3.5)$$

Using eq. (A.28), these two equations lead to another equation for the 4-form $\Omega_{a_1 \dots a_4}$

$$\ell_{b_1} \nabla_a^{(+)} \Omega_{b_2 \dots b_4} = 0, \quad (3.6)$$

where we are using the same convention as in the appendix: all indices with the same Latin letter (here $b_1 \dots b_5$) are assumed to be fully antisymmetrized.

The symmetric and antisymmetric parts of eq. (3.4) indicate that the null vector ℓ_a is a Killing vector

$$\nabla_{(a} \ell_{b)} = 0, \quad (3.7)$$

(∇ is the standard Levi-Civita connection) and that

$$2\nabla_{[a} \ell_{b]} = \ell^c H_{cab}, \quad (3.8)$$

or, in the language of differential forms

$$i_\ell H = d\ell. \quad (3.9)$$

It is customary to introduce an auxiliary null vector n , dual to ℓ

$$n^2 = 0, \quad n^a \ell_a = 1. \quad (3.10)$$

Then, we can use the 1-forms

$$\ell_\mu dx^\mu \equiv e^+, \quad n_\mu dx^\mu \equiv e^- \quad (3.11)$$

as the first two elements of a Vielbein basis $\{e^+, e^-, e^m\}$ $m = 1, \dots, 8$ in which the metric takes the form

$$ds^2 = e^+ \otimes e^- + e^- \otimes e^+ - \delta_{mn} e^m \otimes e^n. \quad (3.12)$$

Eq. (3.9) can be interpreted as the $+$ component of the first Cartan structure equations ($de^a = \omega^a_b \wedge e^b$ in our conventions) and from it we find that

$$\Omega^{(+)}_{[ab]-} = 0. \quad (3.13)$$

On the other hand, from eq. (3.6) we get these two equations in the above basis:

$$\nabla^{(+)}_a \Omega_{m_1 \dots m_4} = 0, \quad (3.14a)$$

$$\nabla^{(+)}_a \Omega_{m_1 m_2 m_3 -} = 0. \quad (3.14b)$$

Since, in this basis, the 4-form Ω 's only non-vanishing components are those with transverse indices $m, n, p, \dots$, eq. (3.14b) implies that

$$\Omega^{(+)}_{am-} = 0, \quad (3.15)$$

which, together with eq. (3.13) gives

$$\Omega^{(+)}_{ab-} = 0, \quad \Rightarrow \quad \omega_{ab-} = \omega_{[ab]-} = -\frac{1}{2}H_{ab-}, \quad \forall a, b. \quad (3.16)$$

It is not difficult to see that, if one uses coordinates adapted to the null Killing vector $\ell^\mu \partial_\mu = e_-^\mu \partial_\mu$ and, therefore, $\partial_- e_{m\mu} = 0$, we have the property $\omega_{-mn} = -\omega_{mn-}$ which, combined with eq. (3.16) leads to

$$\Omega^{(+)}_{-mn} = 0, \quad (3.17)$$

which, in its turn, used in eq. (3.14a), sheds the result

$$\partial_- \Omega_{m_1 \dots m_4} = 0, \quad (3.18)$$

in agreement with the use of adapted coordinates.

The $a = +, m$ components of eq. (3.14a) still need to be analyzed, but this requires a more detailed choice of coordinates that we are going to make next.

3.1.1 The metric

All metrics characterized by the existence of a null, generically not covariantly-constant nor hypersurface-orthogonal, Killing vector can be written in a common way (see, e.g. ref. [30]: first of all, we introduce null coordinates v, u through

$$\partial_- \equiv \partial_v, \quad (3.19a)$$

$$e^+ \equiv f(du + \beta), \quad (3.19b)$$

where $\beta = \beta_m dx^m$, $m = 1, \dots, 8$ is a 1-form in the 8-dimensional space orthogonal to e^+ , f is a scalar function and both f and β are independent of v (but, in general, not of u nor of the remaining coordinates x^m).

Next, we write e^- in the form

$$e^- \equiv dv + Kdu + \omega, \quad (3.20)$$

where $\omega = \omega_m dx^m$ is another 1-form in the transverse 8-dimensional space and K is a scalar function which are also independent of v .

Choosing the Vielbeins e^m to only have non-vanishing v -independent components in the transverse directions, $e^m = e^m_{\underline{n}} dx^{\underline{n}}$, the metric takes the form

$$ds^2 = 2f(du + \beta)(dv + Kdu + \omega) - h_{\underline{mn}} dx^{\underline{m}} dx^{\underline{n}}, \quad (3.21)$$

where the metric in the transverse space is given by

$$h_{\underline{mn}} = \delta_{mn} e^m_{\underline{m}} e^n_{\underline{n}}. \quad (3.22)$$

It is clear, then, that the transverse components of the spin connection ω_{mnp} only depend on the transverse Vielbeins e^m and the $a = m$ components of eq. (3.14a)

$$\nabla^{(+)}_m \Omega_{n_1 \dots n_4} = 0, \quad (3.23)$$

are those of an equation in transverse space. We can rewrite it as

$$\nabla_m \Omega_{n_1 \dots n_4} = 2H_{mn_1}{}^p \Omega_{pn_2 n_3 n_4}, \quad (3.24)$$

and multiplying it by $\Omega_q{}^{n_2 n_3 n_4}$ and using eqs. (A.40g) and (A.40f) we find

$$H_{mnp} = -\frac{7}{2} H_{mnp}^{(-)} + \frac{1}{8} \nabla_{[m} \Omega_n{}^{s_1 s_2 s_3} \Omega_{p] s_1 s_2 s_3}, \quad (3.25)$$

where we have used the projector acting on 3-forms defined in appendix A.5.

The the $a = +$ components of eq. (3.14a), on the other hand, can be written in the form

$$\nabla_+ \Omega_{m_1 \dots m_4} = 2H_{+m_1}{}^n \Omega_{nm_2 m_3 m_4}, \quad (3.26)$$

and using the same properties, we find

$$H_{+mn}^{(-)} = \frac{1}{48} \Omega_m{}^{s_1 s_2 s_3} \nabla_+ \Omega_{ns_1 s_2 s_3}. \quad (3.27)$$

Finally, observe that the components of the spin connection are determined by the objects that occur in the metric: the scalar functions f, K , the transverse 1-forms ω, β and the transverse metric h . Via eqs. (3.16) they also determine the H_{ab-} components of the Kalb-Ramond field strength. These components are constrained by the dilatino KSE and the constraints become constraints on the objects that occur in the metric.

3.2 The dilatino KSE

Multiplying the dilatino KSE by $\bar{\epsilon}$ from the left, we get

$$\ell^\mu \partial_\mu \phi = 0, \quad \Rightarrow \quad \partial_v \phi = 0. \quad (3.28)$$

If we multiply by $\bar{\epsilon} \Gamma_{ab}$ from the left, we get

$$2\ell_{[a} \partial_{b]} \phi - \frac{1}{12} W_{ab}{}^{cde} H_{cde} + \frac{1}{2} \ell^c H_{cab} = 0. \quad (3.29)$$

In terms of the 4-form Ω we arrive to

$$e^+_{[a} \left[2\partial_{b]} \phi - \frac{1}{6} \Omega_{b]}{}^{cde} H_{cde} \right] - \frac{1}{4} \Omega_{ab}{}^{cd} H_{cd-} + \frac{1}{2} H_{ab-} = 0. \quad (3.30)$$

The $a = +, m$ components of this equation give a pair of non-trivial equations in transverse space¹²

$$\frac{1}{6}\Omega_m{}^{npq}H_{npq} + H_{+-m} - 2\partial_m\phi = 0, \quad (3.31a)$$

$$H_{qr-}^{(-)} = 0. \quad (3.31b)$$

If we multiply the dilatino KSE by $\bar{\epsilon}\Gamma_{a_1\cdots a_4}$ from the left, we get

$$W_{a_1\cdots a_4}{}^c\partial_c\phi - W_{a_1a_2a_3}{}^{bc}H_{a_4bc} + 2\ell_{a_1}H_{a_2a_3a_4} = 0. \quad (3.32)$$

Using again eq. (A.28) we get another pair of equations in transverse space

$$(2\partial_n\phi - H_{+-n})\Omega_{m_1m_2m_3}^n + \frac{3}{2}H_{m_1}{}^{np}\Omega_{m_2m_3np} - H_{m_1m_2m_3} = 0, \quad (3.33a)$$

$$\Omega_{m_1m_2m_3}{}^n H_{m_4n-} = 0. \quad (3.33b)$$

It turns out that these two equations are just combinations of eqs. (3.31a) and (3.31b): if we multiply eq. (3.33b) by the 4-form Ω , contracting just 3 of the 4 free indices and using eqs. (A.40g) and (A.40d) we obtain eq. (3.31b). The same happens with eq. (3.33a). Therefore, the only three independent equations one obtains from the dilatino KSE are eq. (3.28), (3.31a) and (3.31b).

Eq. (3.33a) can be rewritten in the form

$$H_{mnp}^{(-)} = \frac{1}{7}(2\partial_q\phi - H_{+-q})\Omega^q{}_{mnp}, \quad (3.34)$$

and, combining this result with eq. (3.25) we can solve for the components H_{mnp} :

$$H_{mnp} = -\frac{1}{2}(2\partial_q\phi - H_{+-q})\Omega^q{}_{mnp} + \frac{1}{8}\Omega_{[m}{}^{s_1s_2s_3}\nabla_n\Omega_{p]s_1s_2s_3}. \quad (3.35)$$

3.3 The gaugino KSE

Multiplying the gaugino KSE eq. (3.3) by $\bar{\epsilon}\Gamma_a$, $\bar{\epsilon}\Gamma_{abc}$ and $\bar{\epsilon}\Gamma_{a_1\cdots a_5}$ from the left, we get

$$\ell^b F^A{}_{ba} = 0, \quad \Rightarrow \quad F^A{}_{a-} = 0, \quad (3.36a)$$

$$F^A{}_{c_1c_2}W_{c_1c_2a_1a_2a_3} - 6\ell_{a_1}F^A{}_{a_2a_3} = 0, \quad (3.36b)$$

$$F^A{}_{a_1}{}^c W_{ca_2\cdots a_4} = 0. \quad (3.36c)$$

respectively.

Using eq. (3.36) and the decomposition of W in terms of ℓ and Ω eqs. (3.36b) and (3.36c) lead to

$$F^A{}_{pq}\Pi^{(-)pq}{}_{mn} = 0, \quad (3.37a)$$

$$\Omega_{m_1m_2m_3}{}^n F^A{}_{m_4n} = 0. \quad (3.37b)$$

¹²Observe that we could write, using eq. (3.28) $d\phi - i_n d\phi\ell = \tilde{d}\phi$.

Observe that these two equations for F^A_{mn} have exactly the same form as eqs. (3.31b) and (3.33b) for H_{mn-} and, therefore, they are equivalent by virtue of the properties of the 4-form Ω . The components F^A_{m+} remain undetermined.

Eq. (3.37a) is the natural generalization of the standard self-duality condition of Yang-Mills instantons in 8 dimensions. As a matter of fact, eq. (3.37a) is the defining relation of the “octonionic instanton” constructed in ref. [56] and which was used as source for the “octonionic superstring soliton” solution of the Heterotic Superstring of ref. [57]. Since this equation is just a necessary condition to have at least one supersymmetry, we notice that all supersymmetric solutions of the Heterotic Superstring effective action must satisfy it. In particular, the gauge field of the “gauge 5-brane” solution of ref. [58] (a $SU(2)$ BPST instanton [59]) must satisfy it and, indeed, the self-duality condition on the gauge field strength of the BPST instanton as just the result of imposing the condition eq. (3.37a) on a gauge field that lives on a 4-dimensional subspace. For gauge fields that live in subspaces of dimensions larger than 4 and smaller than 8, eq. (3.37a) defines Yang-Mills instantons of gauge groups related to the holonomy of the Killing spinors. An intermediate example between the octonionic ($Spin(7)$) one and the BPST one is provided by the G_2 instanton and its associated heterotic string solution [60].

3.4 Summary of the necessary conditions for unbroken supersymmetry

1. The metric has to admit a null Killing vector ℓ^a . If v is the null coordinate adapted to this isometry, this means that the metric can be written in the form eq. (3.21) which we rewrite here (as we will do with other formulae) for the sake of convenience

$$ds^2 = 2f(du + \beta)(dv + Kdu + \omega) - h_{mn}dx^m dx^n, \quad \beta = \beta_{\underline{m}}dx^{\underline{m}}, \quad \omega = \omega_{\underline{m}}dx^{\underline{m}}. \quad (3.38)$$

All objects in the metric are v -independent.

2. There exists a v -independent 4-form Ω satisfying the properties eqs. (A.40b)–(A.40h) and which satisfies

$$\nabla_+^{(+)}\Omega_{m_1\dots m_4} = 0, \quad (3.39a)$$

$$\nabla_n^{(+)}\Omega_{m_1\dots m_4} = 0. \quad (3.39b)$$

3. The following relations between certain components of the matter fields must be satisfied:

$$\partial_- \phi = 0, \quad (3.40a)$$

$$F^A_{a-} = 0, \quad (3.40b)$$

$$F^A_{mn} = F^{A(+)}_{mn}, \quad (3.40c)$$

$$H_{-mn} = H_{-mn}^{(+)}, \quad (3.40d)$$

$$H_{mnp}^{(-)} = \frac{1}{7}(2\partial_q \phi - H_{+-q})\Omega^q_{mnp}, \quad (3.40e)$$

4. The torsionful spin connection $\Omega^{(+)}$ satisfies the following conditions:

$$\Omega^{(+)}_{ab-} = 0, \quad (3.41a)$$

$$\Omega^{(+)}_{-mn} = 0. \quad (3.41b)$$

These conditions relate certain components of the Levi-Civita spin connection (and, hence, some of the objects that occur in the metric) to certain components of the Kalb-Ramond 3-form.

3.5 Sufficiency

Let us now check that the necessary conditions for having the minimal amount of unbroken supersymmetry previously identified are also sufficient.

3.5.1 Gaugino KSE

Let us start with the gaugino KSE eq. (3.3). The necessary conditions that the gauge field strength has to satisfy are eq. (3.40b) (3.40c). Then,

$$F^A_{ab}\Gamma^{ab}\epsilon = F^{A(+)}_{mn}\Gamma^{mn}\Pi^{(-)}\epsilon + 2F^A_{m+}\Gamma^m\Gamma^+\epsilon, \quad (3.42)$$

where we have used the property eq. (A.46a) and the spinor projector eq. (A.45b). This equation is solved by demanding¹³

$$\Gamma^+\epsilon = 0, \quad (3.43a)$$

$$\Pi^-\epsilon = 0. \quad (3.43b)$$

Observe that, when $F^A_{m+} = 0$ the first condition seems to be unnecessary. However, Π^- is only idempotent when that condition is satisfied.

3.5.2 Dilatino KSE

Using eq. (3.40a) and the spinor projector eq. (3.43a), this equation reduces to

$$\left\{ \left(\partial_m\phi - \frac{1}{2}H_{+-m} \right) \Gamma^m - \frac{1}{12}H_{mnp}\Gamma^{mnp} - \frac{1}{4}H_{-mn}\Gamma^-\Gamma^{mn} \right\} \epsilon = 0. \quad (3.44)$$

Now we do two things:

1. First use eq. (3.40e) into eq. (A.46b) to eliminate Π^+H_{mnp} from the latter, solve the resulting equation for H_{mnp} and substitute the result in the above equation.
2. Use eq. (3.40d) and then eq. (A.46a).

¹³The constraint eq. (3.43a) is associated to the projector

$$\frac{1}{2}\Gamma^-\Gamma^+ = \frac{1}{2}(1 + \Gamma^{++}).$$

Eq. (3.44) takes the form

$$\left\{ \frac{7}{8} \left(\partial_m \phi - \frac{1}{2} H_{+-m} \right) \left(\Gamma^m - \frac{1}{42} \Omega^m_{npq} \Gamma^{npq} \right) - \frac{1}{12} (H_{mnp} \Gamma^{mnp} + 3H_{-mn} \Gamma^- \Gamma^{mn}) \Pi^{(-)} \right\} \epsilon = 0. \quad (3.45)$$

Then, eq. (A.47) allows us to rewrite the whole equation as

$$\left\{ \partial_m \phi - \frac{1}{12} (H_{mnp} \Gamma^{mnp} + 3H_{-mn} \Gamma^- \Gamma^{mn} + 6H_{+-m} \Gamma^m) \right\} \Pi^{(-)} \epsilon = 0, \quad (3.46)$$

which is solved by demanding eq. (3.43b).

3.5.3 Gravitino KSE

The projection eq. (3.43a) and the supersymmetry condition eq. (3.41a) bring the gravitino KSE to the form

$$\left(\partial_a - \frac{1}{4} \Omega^{(+)}_{a mn} \Gamma^{mn} \right) \epsilon = 0. \quad (3.47)$$

Due to the condition eq. (3.41b), the $a = -$ component is solved by v -independent spinors

$$\partial_- \epsilon = 0. \quad (3.48)$$

The $a = +, m$ components are guaranteed to be satisfied because of the supersymmetry conditions eqs. (3.39a) and (3.39b). Observe that these two equations lead to

$$\Omega^{(+)}_{a mn} = \left(\Pi^{(+)} \Omega^{(+)}_a \right)_{mn} + \frac{1}{96} \partial_b \Omega_{mp_1 p_2 p_3} \Omega_n^{p_1 p_2 p_3}. \quad (3.49)$$

If we use a frame in which the components of the 4-form are constant, then eq. (3.47) takes the form

$$\left(\partial_a - \frac{1}{4} \left(\Pi^{(+)} \Omega^{(+)}_a \right)_{mn} \Gamma^{mn} \right) \epsilon = \partial_a \epsilon = 0, \quad (3.50)$$

by virtue of the property eq. (A.46a) and of the projection eq. (3.43b). Then, in that frame, the Killing spinors are just constant spinors satisfying the two conditions eqs. (3.43a) and (3.43b).

4 Supersymmetric solutions

In this section we are going to study under which conditions, the supersymmetric field configurations that we have identified in the previous section are also solutions of the equations of motion of the theory. We start by reviewing the equations of motion that follow from the action eq. (2.13) at first order in α' and finding the relations with the simplified equations which are usually solved. Of course, nothing but the sheer difficulty prevents us from deriving higher-order equations of motion from eq. (2.13) because it is not known if the simplifications that occur at first order in α' have an analogue at higher orders.

4.1 Equations of motion

The equations of motion that follow from the action eq. (2.13) are very complicated. If we stay at first order in α' , though, there are important simplifications. Following ref. [61], we can separate the variations with respect to each field in the action (except for the dilaton and Yang-Mills fields) into those corresponding to occurrences via $\Omega_{(-)}^{a_b}$, that we will call *implicit*,¹⁴ and the rest, that we will call *explicit*, as follows:¹⁵

$$\begin{aligned} \delta S &= \frac{\delta S}{\delta e^a_\mu} \delta e^a_\mu + \frac{\delta S}{\delta B_{\mu\nu}} \delta B_{\mu\nu} + \frac{\delta S}{\delta A^A_\mu} \delta A^A_\mu + \frac{\delta S}{\delta \phi} \delta \phi \\ &= \left. \frac{\delta S}{\delta g_{\mu\nu}} \right|_{\text{exp.}} \delta g_{\mu\nu} + \left. \frac{\delta S}{\delta B_{\mu\nu}} \right|_{\text{exp.}} \delta B_{\mu\nu} + \frac{\delta S}{\delta A^A_\mu} \delta A^A_\mu + \frac{\delta S}{\delta \phi} \delta \phi \\ &\quad + \frac{\delta S}{\delta \Omega_{(-)\mu}^{a_b}} \left(\frac{\delta \Omega_{(-)\mu}^{a_b}}{\delta e^c_\rho} \delta e^c_\rho + \frac{\delta \Omega_{(-)\mu}^{a_b}}{\delta B_{\nu\rho}} \delta B_{\nu\rho} \right). \end{aligned} \quad (4.1)$$

A lemma proven in ref. [42] states that $\delta S / \delta \Omega_{(-)\mu}^{a_b}$ is proportional to α' multiplied by combinations of zeroth-order equations of motion of the fields e^a_μ , $B_{\mu\nu}$ and ϕ plus terms of higher orders in α' .

This lemma has important consequences: if we consider field configurations which solve the zeroth-order equations of motion up to terms of order α' or higher, then $\delta S / \delta \Omega_{(-)\mu}^{a_b} = X^\mu_{a_b}$ will automatically vanish up to terms of second order in α' or higher and can be ignored to the order at which we are working. Then all the terms involving X^μ_{ab} in the complete equations of motion eqs. (B.1)–(B.4) can be ignored.

Further simplifications are possible combining the equations of motion:

$$-\frac{e^{2\phi}}{2e} \left[\frac{\delta S}{\delta e^a_\mu} + \frac{1}{2} e^a_\mu \frac{\delta S}{\delta \phi} \right]_{\text{exp.}} = R^\mu_a - 2\nabla^\mu \partial_a \phi + \frac{1}{4} H^{\mu\rho\sigma} H_{a\rho\sigma} - T^{(2)\mu}_a, \quad (4.2)$$

$$\frac{e^{2\phi}}{4e} \left[e^a_\mu \frac{\delta S}{\delta e^a_\mu} + \frac{d-2}{2} \frac{\delta S}{\delta \phi} \right]_{\text{exp.}} = (\partial\phi)^2 - \frac{1}{2} \nabla^2 \phi - \frac{1}{4 \cdot 3!} H^2 + \frac{1}{8} T^{(0)}, \quad (4.3)$$

$$-\frac{2}{e} \left. \frac{\delta S}{\delta B_{\nu\rho}} \right|_{\text{exp.}} = \nabla_\mu \left(e^{-2\phi} H^{\mu\nu\rho} \right), \quad (4.4)$$

$$\frac{2}{e} \left[\frac{\delta S}{\delta A^A_\nu} + A^A_\mu \left. \frac{\delta S}{\delta B_{\mu\nu}} \right|_{\text{exp.}} \right] = \alpha' e^{2\phi} \nabla_{(+)\mu} \left(e^{-2\phi} F^{A\mu\nu} \right). \quad (4.5)$$

These are the equations of motion that are usually solved in the literature. Observe that, although we have arrived at them by assuming that the zeroth-order equations of motion are satisfied, since these can be obtained by setting $\alpha' = 0$ in eqs. (4.2)–(4.5), if we find a solution of eqs. (4.2)–(4.5), then it is automatically a solution of the zeroth-order equations of motion up to terms of first order in α' , the lemma can be applied, the complete equations of motion reduce to eqs. (4.2)–(4.5) and the solution is a solution of the complete equations of motion.

¹⁴The dilaton does not occur in the torsionful spin connection, neither do the Yang-Mills fields to the order in α' we are considering.

¹⁵The complete equations of motion of all the fields and $\delta S / \delta \Omega_{(-)\mu}^{a_b}$ can be found in appendix B to first order in α' . The proof of the lemma of ref. [61] mentioned below can also be found there.

The non-trivial relation between the complete equations of motion that one obtains from the action eqs. (B.1)–(B.4) and the standard equations of motion considered in the literature eqs. (4.2)–(4.5) that we have just explained has to be kept in mind when using the Killing Spinor Identities (KSIs) [48, 49] because, in principle, they involve the complete ones eqs. (B.1)–(B.4) (see section 4.2).

If a solution is given in terms of the 3-form field strength, we also need to solve the Bianchi identity

$$dH - T^{(4)} = 0, \quad (4.6)$$

as well.

This identity can be rewritten as the equation of motion of the 6-form $\tilde{B}_{\mu_1 \dots \mu_6}$ dual to the Kalb-Ramond 2-form:

$$\nabla_\nu \left(\tilde{H}^{\nu \mu_1 \dots \mu_6} \right) - \star T^{(4) \mu_1 \dots \mu_6} = 0, \quad (4.7)$$

where $\tilde{H} = d\tilde{B} = e^{-2\phi} \star H$ is its field strength. Then, we can introduce it in the Killing Spinor Identities (see section 4.2) which only involve equations of motion if we use the dual formulation of the 10-dimensional supergravity in refs. [51, 52] if we take into account that the above equation corresponds to

$$\frac{6!}{\sqrt{|g|}} \frac{\delta S}{\delta \tilde{B}_{\mu_1 \dots \mu_6}} = 0. \quad (4.8)$$

Apart from the Kalb-Ramond 2-form, there are no other fields in the action which only occur through their field strengths. Thus, we cannot formulate the equations of motion only in terms of those field strengths and imposing a Bianchi identity on them is utterly unnecessary. Observe that, for instance, in order to write the Bianchi identity for the Yang-Mills field strength (or the Riemann curvature tensor) it is necessary to know the gauge field (the connection). Therefore, we will not need any more equations.

4.2 Killing Spinor Identities

The equations of motion of theories with local symmetries are related off-shell by the so-called Noether (or gauge) identities. In a supergravity theory the Noether identities relate the bosonic and fermionic equations of motion.¹⁶ These identities are valid for any field configurations but, if we restrict ourselves to purely bosonic field configurations admitting Killing spinors, such as those we have characterized in the previous section, it can be shown that the equations of motion of the bosonic fields are related by the so-called *Killing Spinor Identities*, first derived in ref. [48].

As shown in ref. [49], the KSIs are essentially equivalent to the relations obtained from the integrability conditions of the Killing spinor equations, but, in general, they are much easier to derive because, usually, only algebraic operations are required. These relations between the bosonic equations of motion of supersymmetric configurations can be used to reduce the number of independent equations that need to be checked in order to prove that

¹⁶The invariance of the action of $N = 1, d = 4$ supergravity was first proven analytically in ref. [62] precisely by checking that the supersymmetric Noether identity was satisfied off-shell.

a given supersymmetric field configuration is also a solution of all equations of motion. Our goal in this section is to find the KSIs and determine the independent equations of motion that need to be checked in the case of the Heterotic Superstring effective action.

An important point to be stressed is that the generic form of the KSIs only depends on the supersymmetry transformation laws of the bosonic fields. The equations of motion of a supergravity theory change when we gauge it, deform it with mass terms or, as it is the case here, when we add the α' corrections, but, in many cases, the relations that hold between them when they are evaluated on supersymmetric configurations, the KSIs, do not, because they only depend on the supersymmetry transformation rules of the bosonic fields, which do not change.¹⁷

In the present case, however, the supersymmetry transformation rules of the bosons do get α' corrections and this generic property will not hold true. In particular, as explained in the Introduction, the Kalb-Ramond 2-form supersymmetry transformation rules acquire two new terms of first order in α' (see eq. (2.15)) associated to the Nicolai-Townsend (“anomalous”) gauge transformations that arise when the Lorentz- and Yang-Mills-Chern-Simons terms are included in the field strength. At higher orders, new terms have to be taken into account and, although it has been suggested that they may be absorbed in redefinitions of the fields, this has not been explored systematically and we will restrict our analysis to the first order in α' at which the only modification of the bosons’ supersymmetry transformation rules is the one we have just discussed.

A disadvantage of this approach is that the proof of the KSIs in ref. [48] assumes the existence of the potentials in the field strengths (the equations of motion are the first variations of the action with respect to them) or, equivalently, that the Bianchi identities are satisfied and, sometimes, we would like not to assume this and solve a different set of equations. The Bianchi identities appear explicitly in the integrability conditions, but it is usually very hard to compute them.

There is, however, a simple way to make the Bianchi identities appear in the KSIs: we just have to view them as the equations of motion of the dual potentials (as long as their supersymmetry transformation laws are known). In the case at hand, this means that, if we want to find KSIs including the Bianchi identity of the Kalb-Ramond 3-form field strength, we must view it as the equation of motion of the dual 6-form potential $\tilde{B}_{\mu_1 \dots \mu_6}$ and use the supersymmetry transformation law of this field, given in refs. [51, 52] and which we have rewritten in our conventions in eq. (2.16). Observe that, if denote by $\mathcal{E}_{\tilde{B}}^{\mu_1 \dots \mu_6}$ the equation of motion of the dual 6-form $\tilde{B}_{\mu_1 \dots \mu_6}$ (see below), and we denote by $\mathcal{B}_{H \mu_1 \dots \mu_4}$ the Bianchi identity of the Kalb-Ramond 3-form,¹⁸ they are each other’s Hodge dual:

$$\mathcal{E}_{\tilde{B}}^{\mu_1 \dots \mu_6} = \frac{1}{4! \sqrt{|g|}} \epsilon^{\mu_1 \dots \mu_6 \nu_1 \dots \nu_4} \mathcal{B}_{H \nu_1 \dots \nu_4}. \quad (4.9)$$

¹⁷This observation was used in ref. [63] to prove the exactness of the maximally supersymmetric solutions of 5-dimensional supergravity when higher-order corrections are included.

¹⁸Up to global factors, the components of the Bianchi identity eq. (4.6) are

$$\mathcal{B}_{H \mu_1 \dots \mu_4} = 4 \partial_{[\mu_1} H_{\mu_2 \mu_3 \mu_4]} - T^{(4)}_{\mu_1 \dots \mu_4}.$$

Taking into account this last point, using the definitions

$$\begin{aligned}\mathcal{E}_e^\mu{}_a &\equiv \left. \frac{\delta S}{\delta e^a{}_\mu} \right|_{\text{exp}}, & \mathcal{E}_B^{\mu\nu} &\equiv \left. \frac{\delta S}{\delta B_{\mu\nu}} \right|_{\text{exp}}, & \mathcal{E}_{\tilde{B}}^{\mu_1 \dots \mu_6} &\equiv \frac{\delta S}{\delta \tilde{B}_{\mu_1 \dots \mu_6}}, \\ \mathcal{E}_\phi &\equiv \frac{\delta S}{\delta \phi}, & \mathcal{E}_A^\mu &\equiv \frac{\delta S}{\delta A^A{}_\mu}, & X^\mu{}_{ab} &\equiv \frac{1}{\sqrt{|e|}} \frac{\delta S}{\delta \Omega_{(-)\mu}{}^{ab}},\end{aligned}\quad (4.10)$$

and splitting the variations with respect to the fields into variations with respect to explicit occurrences and variations with respect to implicit occurrences via the torsionful spin connection,¹⁹ the general recipe in ref. [48] takes the form

$$\mathcal{E}_{ea}{}^\nu \frac{\delta(\delta_\epsilon e^a{}_\nu)}{\delta \psi_\mu} + \mathcal{E}_B{}^{\nu\rho} \frac{\delta(\delta_\epsilon B_{\nu\rho})}{\delta \psi_\mu} + \mathcal{E}_{\tilde{B}}^{\mu_1 \dots \mu_6} \frac{\delta(\delta_\epsilon \tilde{B}_{\mu_1 \dots \mu_6})}{\delta \psi_\mu} + \sqrt{|g|} X^\nu{}_{ab} \frac{\delta(\delta_\epsilon \Omega_{(-)\nu}{}^{ab})}{\delta \psi_\mu} = 0, \quad (4.11a)$$

$$\mathcal{E}_\phi \frac{\delta(\delta_\epsilon \phi)}{\delta \lambda} + \mathcal{E}_{\tilde{B}}^{\mu_1 \dots \mu_6} \frac{\delta(\delta_\epsilon \tilde{B}_{\mu_1 \dots \mu_6})}{\delta \lambda} = 0, \quad (4.11b)$$

$$\mathcal{E}_C{}^\mu \frac{\delta(\delta_\epsilon A^C{}_\mu)}{\delta \chi^A} + \mathcal{E}_B{}^{\mu\nu} \frac{\delta(\delta_\epsilon B_{\mu\nu})}{\delta \chi^A} = 0. \quad (4.11c)$$

Using the supersymmetry transformations of the bosons in eqs. (2.14)–(2.18), we find that the KSIs of the theory at hands are the spinorial equations

$$\bar{\epsilon} \left\{ \Gamma^a \mathcal{E}_{ea}{}^\mu + 2\Gamma_\alpha \mathcal{E}_B{}^{\alpha\mu} + 6\Gamma_{\alpha_1 \dots \alpha_5} e^{-2\phi} \mathcal{E}_{\tilde{B}}^{\alpha_1 \dots \alpha_5 \mu} \right\} - \mathcal{D}_{(+)\lambda} \left(\bar{\epsilon} \Gamma_a Y^{a\lambda\mu} \right) = 0, \quad (4.12a)$$

$$\bar{\epsilon} \left\{ \mathcal{E}_\phi - 2\Gamma_{\alpha_1 \dots \alpha_6} e^{-2\phi} \mathcal{E}_{\tilde{B}}^{\alpha_1 \dots \alpha_6} \right\} = 0, \quad (4.12b)$$

$$\bar{\epsilon} \Gamma_\alpha \left(\mathcal{E}_A{}^\alpha + \frac{\alpha'}{2} A_{A\mu} \mathcal{E}_B{}^{\mu\alpha} \right) = 0, \quad (4.12c)$$

where we have defined the combination

$$Y^\mu{}_{ab} \equiv \sqrt{|g|} X^\mu{}_{ab} - \frac{\alpha'}{2} \mathcal{E}_B{}^{\rho\mu} \Omega_{(-)\rho ab}. \quad (4.13)$$

Using eq. (B.12),

$$Y^\nu{}_{ab} \equiv -\frac{\alpha'}{2} e^{-2\phi} \nabla_{(+)[a}^{(0)} \left[e^{2\phi} \left(\mathcal{E}_e^{(0)\nu}{}_{|b]} + 2\mathcal{E}_B^{(0)\nu}{}_{|b]} + \frac{1}{2} e_{|b]}{}^\nu \mathcal{E}_\phi^{(0)} \right) \right], \quad (4.14)$$

and, taking into account that the Killing spinor ϵ satisfies $\nabla_{(+)\lambda}^{(0)} \epsilon = \mathcal{O}(\alpha')$, we find that

$$\bar{\epsilon} \Gamma_a Y^{a\lambda\mu} = -\frac{\alpha'}{2} e^{-2\phi} \nabla_{(+)}^{(0)[\lambda} \left[e^{2\phi} \bar{\epsilon} \Gamma_a \left(\mathcal{E}_e^{(0)|\mu]a} + 2\mathcal{E}_B^{(0)|\mu]a} + \frac{1}{2} e^{a|\mu]a} \mathcal{E}_\phi^{(0)} \right) \right], \quad (4.15)$$

up to terms of order $\mathcal{O}(\alpha'^2)$. Now we can use the KSI eq. (4.12a) at zeroth order in α' to show that

$$e^{2\phi} \bar{\epsilon} \Gamma_a \left(\mathcal{E}_e^{(0)a\mu} + 2\mathcal{E}_B^{(0)a\mu} + \frac{1}{2} e^{a\mu} \mathcal{E}_\phi^{(0)} \right) = \bar{\epsilon} \Gamma_{\alpha_1 \dots \alpha_6}{}^\mu \mathcal{E}_{\tilde{B}}^{(0)\alpha_1 \dots \alpha_6}, \quad (4.16)$$

so that

$$\bar{\epsilon} \Gamma_a Y^{a\lambda\mu} = -\frac{\alpha'}{2} e^{-2\phi} \nabla_{(+)}^{(0)[\lambda} \left(\bar{\epsilon} \Gamma_{\alpha_1 \dots \alpha_6}{}^{\mu]} \mathcal{E}_{\tilde{B}}^{(0)\alpha_1 \dots \alpha_6} \right). \quad (4.17)$$

¹⁹This step is fundamental to recover the standard equations of motion eqs. (4.2)–(4.5).

The upshot is that, to $\mathcal{O}(\alpha'^2)$, the KSE eq. (4.12a), can be replaced by

$$\bar{\epsilon} \left\{ \Gamma^a \mathcal{E}_a{}^\mu + 2\Gamma_\alpha \mathcal{E}_B{}^{\alpha\mu} + 6\Gamma_{\alpha_1 \dots \alpha_5} e^{-2\phi} \mathcal{E}_{\tilde{B}}{}^{\alpha_1 \dots \alpha_5 \mu} \right\} + \bar{\epsilon} \left(\Gamma^{(7)} C \right)^\mu = 0, \quad (4.18)$$

where C is a complicated tensor that contains $\mathcal{E}_{\tilde{B}}^{(0)}$ and $\Gamma^{(7)}$ is the antisymmetrized product of 7 gamma matrices.

Furthermore, observe that the combination of equations of motion that occurs in eq. (4.12c)

$$\mathcal{E}_A{}^\alpha + \frac{\alpha'}{2} A_{A\mu} \mathcal{E}_B{}^{\mu\alpha} = \frac{\alpha'}{2} \sqrt{|g|} \nabla_{(+)\nu} \left(e^{-2\phi} F^{A\nu\mu} \right) \equiv \hat{\mathcal{E}}_A{}^\alpha, \quad (4.19)$$

and it is convenient to rewrite the KSI in the form

$$\bar{\epsilon} \Gamma_\alpha \hat{\mathcal{E}}_A{}^\alpha = 0. \quad (4.20)$$

Eqs. (4.12b), (4.18) and (4.20) should be identical to the relations between the bosonic equations of motion obtained from the KSEs' integrability conditions. These have been recently presented in ref. [38] (eqs. (205)) and should be compared to eqs. (4.12b), (4.18) and (4.20). Apart from some minor differences (the equation of motion of the dual 6-form is replaced by the Bianchi identity of the 3-form and the Einstein and Maxwell equations used in ref. [38] are the combinations of our equations of motion described in eqs. (4.2)–(4.5)) the authors of ref. [38] have used an expansion of the equations of motion and Bianchi identities in powers of α' that, for instance, does not take into account the definition of the Kalb-Ramond field strength H in terms of potentials eq. (2.7) and the corresponding expansion in α' . Instead, they just assume the existence of field strengths which are not expanded in powers of α' , satisfying certain (anomaly-corrected) Bianchi identities. Using the explicit form of the Bianchi identity for H it is possible to rewrite their eqs. (205) in exactly the same form we have obtained.

4.3 Tensorial KSIs

Eqs. (4.12b), (4.18) and (4.20) are the off-shell relations between the bosonic equations of motion we were after but, in order to make use of them, we must transform them into tensorial equations. Let us start with the simplest of them, eq. (4.20): if we hit it with ϵ and $\Gamma^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon$ from the right we obtain, respectively,

$$\ell_\mu \hat{\mathcal{E}}_A{}^\mu = 0, \quad (4.21a)$$

$$W_{\mu_1 \mu_2 \mu_3 \mu_4 \mu} \hat{\mathcal{E}}_A{}^\mu = 0. \quad (4.21b)$$

Using eq. (A.28) and the first equation in the second, and contracting it with the null vector n , it takes the simpler form

$$\Omega_{\mu_1 \mu_2 \mu_3 \mu} \hat{\mathcal{E}}_A{}^\mu = 0. \quad (4.22)$$

Contracting this equation with $\Omega^{\mu_1 \mu_2 \mu_3 \nu}$ and using eq. (A.35b), we get

$$\tilde{g}_{\mu\nu} \hat{\mathcal{E}}_A{}^\nu = 0, \quad (4.23)$$

where $\tilde{g}_{\mu\nu} = g_{\mu\nu} - 2\ell_{(\mu}n_{\nu)}$ is the (curved indices) induced metric in the 8-dimensional Euclidean transverse space defined in eq. (A.31). An equivalent way of writing this equation is

$$\hat{\mathcal{E}}_A{}^m = 0. \quad (4.24)$$

We conclude, that all components of the Yang-Mills equations $\hat{\mathcal{E}}_A{}^\mu$, except for $n_\mu \hat{\mathcal{E}}_A{}^\mu$, are automatically satisfied by supersymmetric field configurations.

Hitting now eq. (4.12b) with $\Gamma^\mu \epsilon$ from the right and contracting the result with n_μ , we arrive at

$$\mathcal{E}_\phi = 60\ell_\mu n_\nu \mathcal{E}_{\tilde{B}}{}^{\mu\nu}, \quad (4.25)$$

where we have defined

$$\mathcal{E}_{\tilde{B}}{}^{\mu\nu} \equiv \Omega_{\alpha_1\alpha_2\alpha_3\alpha_4} e^{-2\phi} \mathcal{E}_{\tilde{B}}{}^{\alpha_1\alpha_2\alpha_3\alpha_4\mu\nu}, \quad (4.26)$$

because this combination appears very often and plays an interesting rôle. Often, in the literature, the Bianchi identity is assumed to be solved from the beginning. In that case, the dilaton equation is automatically solved on supersymmetric field configurations, but the above KSI allows us to assume that the dilaton equation is solved from the beginning, which would imply that the component $\ell_\mu n_\nu \mathcal{E}_{\tilde{B}}{}^{\mu\nu}$ of the Bianchi identity is automatically solved.

Observe that the relation eq. (4.9) implies that

$$\ell_\mu n_\nu \mathcal{E}_{\tilde{B}}{}^{\mu\nu} = -e^{-2\phi} \Omega^{\alpha_1\cdots\alpha_4} \mathcal{B}_{H\alpha_1\cdots\alpha_4}, \quad (4.27)$$

so the dilaton equation is related to a single combination of the transverse components of the Bianchi identity.

Next, let us consider eq. (4.18). Hitting it with ϵ from the right, and taking into account that the last term will not contribute, we get

$$\ell_\mu \left(\mathcal{E}_e{}^{\mu\nu} + 2\hat{\mathcal{E}}_B{}^{\mu\nu} \right) = 0, \quad (4.28)$$

where we have defined

$$\hat{\mathcal{E}}_B{}^{\mu\nu} \equiv \mathcal{E}_B{}^{\mu\nu} + 15\mathcal{E}_{\tilde{B}}{}^{\mu\nu}, \quad (4.29)$$

from which we get

$$\ell_\mu \ell_\nu \mathcal{E}_e{}^{\mu\nu} = 0. \quad (4.30)$$

Observe that, while the complete Einstein equation, $e^{a\nu} \delta S / \delta e^a{}_\mu$, is not symmetric in the pair of indices $\mu\nu$, $\mathcal{E}_e{}^{\mu\nu} = e^{a\mu} \mathcal{E}_{e a}{}^\nu$, which only contains the variation of the action with respect to explicit occurrences of the Vielbein, is.²⁰

In what follows, in order to get rid of the complicated contributions of the last term in eq. (4.18) we will assume that the Bianchi identity is satisfied at zeroth order in α' ,

$$\mathcal{E}_{\tilde{B}}^{(0)} = 0, \quad \mathcal{E}_{\tilde{B}} = \mathcal{O}(\alpha'). \quad (4.31)$$

²⁰This fact follows from the Noether identities, see eq. (B.20).

Using this assumption and hitting eq. (4.18) with $\Gamma^\rho \Gamma^\sigma \epsilon$ from the right and using eq. (4.28) we get

$$\left(\mathcal{E}_e^{[\rho|\mu} + 2\hat{\mathcal{E}}_B^{[\rho|\mu} \right) \ell^{\sigma]} + 120\ell_\nu e^{-2\phi} \mathcal{E}_{\tilde{B}}^{\alpha_1\alpha_2\alpha_3[\rho|\nu\mu} \Omega_{\alpha_1\alpha_2\alpha_3}^{\sigma]} = 0. \quad (4.32)$$

If the antisymmetrized indices ρ and σ are transverse indices m and n the first term vanishes identically and we get

$$\ell_\nu \mathcal{E}_{\tilde{B}}^{\alpha_1\alpha_2\alpha_3[m|\nu\mu} \Omega_{\alpha_1\alpha_2\alpha_3}^{n]} = 0. \quad (4.33)$$

Furthermore, hitting eq. (4.32) equation with n_σ and using the fact that the metric $\tilde{g}_{\mu\nu}$ projects onto the transverse components, we get

$$\mathcal{E}_e^{m\mu} + 2\hat{\mathcal{E}}_B^{m\mu} + 120\ell_\nu n_\rho e^{-2\phi} \mathcal{E}_{\tilde{B}}^{\alpha_1\alpha_2\alpha_3\mu\nu\rho} \Omega_{\alpha_1\alpha_2\alpha_3}^m = 0. \quad (4.34)$$

Contracting this identity with $e_{m\mu}$, n_μ and ℓ_μ and using eq. (4.33) we get

$$\mathcal{E}_e^{mn} + 120\ell_\nu n_\rho e^{-2\phi} \mathcal{E}_{\tilde{B}}^{\alpha_1\alpha_2\alpha_3\mu\nu(m} \Omega_{\alpha_1\alpha_2\alpha_3}^{n)} = 0, \quad (4.35)$$

$$\hat{\mathcal{E}}_B^{mn} = 0. \quad (4.36)$$

$$n_\mu \left(\mathcal{E}_e^{m\mu} + 2\hat{\mathcal{E}}_B^{m\mu} \right) = 0, \quad (4.37)$$

$$\ell_\mu \left(\mathcal{E}_e^{m\mu} + 2\hat{\mathcal{E}}_B^{m\mu} \right) = 0. \quad (4.38)$$

Combining the last equation with eq. (4.28) we get

$$\ell_\mu \mathcal{E}_e^{\mu m} = \ell_\mu \hat{\mathcal{E}}_B^{\mu m} = 0. \quad (4.39)$$

Observe that eq. (4.35) leads to a relation between the trace of the spatial components of the Einstein equation and, yet again, the component $\Omega^{\alpha_1\cdots\alpha_4} \mathcal{B}_{H\alpha_1\cdots\alpha_4}$ of the Bianchi identity. On account of eq. (4.25) we can write

$$\mathcal{E}_\phi = -\frac{1}{2} \mathcal{E}_e^m{}_m = -60e^{-2\phi} \Omega^{\alpha_1\cdots\alpha_4} \mathcal{B}_{H\alpha_1\cdots\alpha_4}. \quad (4.40)$$

If, instead, we hit eq. (4.18) with $\Gamma_\mu \Gamma^\sigma \epsilon$ from the right and use eq. (4.28) to eliminate the terms containing $\mathcal{E}_B$, we get $\ell_\mu \mathcal{E}_e^{\mu m} = 60\ell_\mu \mathcal{E}_{\tilde{B}}^{\mu m}$, from which it follows that

$$\ell_\mu \mathcal{E}_{\tilde{B}}^{\mu m} = 0. \quad (4.41)$$

Finally, eq. (4.18) with $\Gamma_{\alpha_1\cdots\alpha_4} \epsilon$ from the right we get

$$\begin{aligned} & (e^{a\mu} \mathcal{E}_{ea}{}^\nu + 2\mathcal{E}_B^{\mu\nu}) W_{\mu\alpha_1\cdots\alpha_4} + 6e^{-2\phi} \mathcal{E}_{\tilde{B}}^{\beta_1\cdots\beta_5\nu} \epsilon_{\beta_1\cdots\beta_5\alpha_1\cdots\alpha_4\mu} \ell^\mu \\ & - 720e^{-2\phi} \mathcal{E}_{\tilde{B}\beta_1\beta_2\beta_3[\alpha_1}{}^\nu W_{\beta_1\beta_2\beta_3\alpha_3\alpha_4]} + 720\ell^\mu e^{-2\phi} \mathcal{E}_{\tilde{B}\alpha_1\cdots\alpha_4\mu}{}^\nu = 0. \end{aligned} \quad (4.42)$$

Decomposing W and contracting the resulting expression with n^{α_4} ,²¹

$$\begin{aligned} & (e^{a\mu} \mathcal{E}_{ea}{}^\nu + 2\mathcal{E}_B^{\mu\nu}) \Omega_{\mu\alpha_1\alpha_2\alpha_3} - 720\ell_\mu n_\rho e^{-2\phi} \mathcal{E}_{\tilde{B}\alpha_1\alpha_2\alpha_3}{}^{\nu\mu\rho} \\ & + 6e^{-2\phi} \mathcal{E}_{\tilde{B}}^{\beta_1\cdots\beta_5\nu} \tilde{\epsilon}_{\beta_1\cdots\beta_5\alpha_1\alpha_2\alpha_3} + 1080\ell_\mu n_\rho e^{-2\phi} \mathcal{E}_{\tilde{B}\beta_1\beta_2[\alpha_1}{}^{\nu\mu\rho} \Omega_{\beta_1\beta_2\alpha_2\alpha_3]} \\ & + 720n_\rho e^{-2\phi} \mathcal{E}_{\tilde{B}\beta_1\beta_2\beta_3[\alpha_1}{}^{\nu\rho} \Omega_{\beta_1\beta_2\beta_3\alpha_2} \ell_{\alpha_3]} - 360e^{-2\phi} \mathcal{E}_{\tilde{B}\beta_1\beta_2\beta_3[\alpha_1\alpha_2}{}^\nu \Omega_{\beta_1\beta_2\beta_3\alpha_3]} = 0. \end{aligned} \quad (4.43)$$

²¹Contracting this expression with $\Omega^{\alpha_1\cdots\alpha_4}$ and ℓ^{α_4} we obtain $0 = 0$ and contracting it with ℓ_ν we get an identity that has already been derived.

This is a complicated identity that we can simplify by hitting it with ℓ, n, Ω in different ways.

Hitting eq. (4.43) with $g^{\alpha_1}{}_\nu$ and using several of the identities derived above, we get another constraint on the Bianchi identity

$$20 \left(\tilde{g}^{\mu\nu}{}_{\rho\sigma} + \frac{1}{4} \Omega^{\mu\nu}{}_{\rho\sigma} \right) \mathcal{E}_{\tilde{B}}^{\rho\sigma} - e^{-2\phi} \mathcal{E}_{\tilde{B}}^{\beta_1 \dots \beta_6} \tilde{\epsilon}_{\beta_1 \dots \beta_6}{}^{\mu\nu} = 0. \quad (4.44)$$

To summarize, the components of the equations of motion of the Vierbein, the Kalb-Ramond 2-form and its dual implied by supersymmetry are:

$$\begin{aligned} \mathcal{E}_e^{++} &= 0, \\ \mathcal{E}_e^{+m} &= 0, \\ \mathcal{E}_e^{mn} &= -120 \ell_\nu n_\rho e^{-2\phi} \mathcal{E}_{\tilde{B}}^{\alpha_1 \alpha_2 \alpha_3 \mu \nu (m} \Omega_{\alpha_1 \alpha_2 \alpha_3}{}^{n)}, \end{aligned} \quad \begin{aligned} \mathcal{E}_e^{+-} + 2\hat{\mathcal{E}}_B^{+-} &= 0, \\ \mathcal{E}_B^{+m} &= 0, \quad \mathcal{E}_{\tilde{B}}^{+m} = 0, \\ \mathcal{E}_e^{m-} + 2\hat{\mathcal{E}}_{\tilde{B}}^{m-} &= 0, \\ \hat{\mathcal{E}}_B^{mn} &= 0, \end{aligned} \quad (4.45)$$

and those of the gauge fields and dilaton are

$$\hat{\mathcal{E}}_A^+ = 0, \quad \hat{\mathcal{E}}_A^m = 0, \quad \mathcal{E}_\phi = 60 \mathcal{E}_{\tilde{B}}^{+-}. \quad (4.46)$$

4.4 Remaining equations for supersymmetric solutions

A possible choice of independent equations of motion to be checked is

$$\begin{aligned} \mathcal{E}_e^{--} &= 0, \\ \mathcal{E}_B^{+-} &= 0, \quad \mathcal{E}_B^{-m} = 0, \\ \mathcal{B}_{H\mu\nu\rho\sigma} &= 0, \\ \hat{\mathcal{E}}_A^- &= 0, \end{aligned} \quad (4.47)$$

although some combinations of the components of the Bianchi identity are automatically satisfied for supersymmetric field configurations. It should be remembered that, in order to derive some of the relations, we had to assume that the Bianchi identity is satisfied at lowest order in α' .

5 Discussion

In this paper we have re-analyzed the problem of characterizing all the supersymmetric solutions of the Heterotic Superstring effective action to first order in α' working in a general spinorial basis, instead of working on a particular or privileged basis as in it is done in refs. [43–45].

Thus, we have first computed the algebra of bilinears in an arbitrary basis in section A.4. It is this computation that allowed us to obtain the conditions necessary for unbroken supersymmetry and to construct the supersymmetry projectors necessary to prove the sufficiency of the conditions in an arbitrary basis. As explained below eq. (A.45b), the

explicit form of the supersymmetry projector for the Killing spinors suggests a physical interpretation of minimally supersymmetric field configurations which is entirely lost if one works in a privileged spinor basis. The form of the projector eq. (A.45b), for instance, is general, basis-independent. One can always evaluate it in a particular basis such as the one used in refs. [43–45] if that is needed in a specific calculation.

Another important result is the derivation of the set of relations existing between the equations of motion evaluated on supersymmetric configurations to first order in α' obtained in section 4.2 using the Killing Spinor Identities (KSIs). One of the novelties in this result is the procedure through which we have obtained it, using the KSIs in this, more complicated, context. We have also shown how to include the Bianchi identity of the 3-form Kalb-Ramond field strengths in the KSIs.

Finally, we have also re-derived the set of conditions necessary for unbroken supersymmetry, summarized in section 3.4. They are, of course, completely equivalent to those obtained in refs. [43–45].

The computation of the bilinear algebra in an arbitrary basis is a very useful result because the algebras of bilinears of half-maximal supergravities²² in lower dimensions are exactly the same, up to relabeling of the components of the bilinear forms [64], which implies the existence of a Spin(7) structure hidden in any supersymmetric solution of any half-maximal theory. This observation deserves further discussion.²³

Let us consider the supersymmetry condition of the Yang-Mills fields eq. (3.40c) which we rewrite here for the sake of convenience:

$$F^{A mn} = \frac{1}{2} \Omega^{mnpq} F^A_{pq}, \quad m, n, p, q = 1, \dots, 8. \quad (5.1)$$

This equation can be seen as an 8-dimensional generalization of the 4-dimensional self-duality condition

$$F^{A mn} = \frac{1}{2} \epsilon^{mnpq} F^A_{pq}, \quad m, n, p, q = 1, \dots, 4, \quad (5.2)$$

that characterizes 4-dimensional Yang-Mills instantons such as the BPST instanton with gauge group $SU(2) \subset SO(4)$ [59]. This instanton is part of the *gauge 5-brane* solution of Heterotic Supergravity [58], sourcing the gravitational and dilaton field. Combined with the *solitonic (or NS) 5-brane* of ref. [66] as in ref. [67], in which the source is a magnetic Kalb-Ramond field, one can obtain the so-called *symmetric 5-brane* [67], which is considered an exact solution of the Heterotic Superstring effective action to all orders in α' . It is clear that this solution should fit into our general result and that the gauge field satisfying eq. (5.2) should obey eq. (5.1) for some Spin(7) structure 4-form. As a matter of fact, one can view the Spin(7) structure 4-form as a collection of volume forms in 4-dimensional manifolds (hyper-planes in 8-dimensional Euclidean space, [68]) and, if we simply restrict

²²Actually, of any half-maximal supersymmetric theory, including global supersymmetry.

²³Observe that, ultimately, this is a property of the Clifford algebra itself that will hold whenever (1,9)-dimensional Majorana-Weyl spinors are at play. For instance, in ref. [65] it has been shown that any 10-dimensional Lorentzian manifold admits a real chiral spinor if and only if it admits a null vector such that the associated metric in the corresponding “screen bundle” is Spin(7).

eq. (5.1) to the 4-dimensional subspace in which the gauge field is defined to live, we just get (up to a sign) eq. (5.2). Therefore, eq. (5.2) is included as a particular case in eq. (5.1).

A solution to eq. (5.1) that does not assume that the gauge field lives in less than 8 dimensions is the so-called *octonionic instanton* of Nicolai and Fubini [56], whose gauge group is $\text{Spin}(7) \subset \text{SO}(8)$. Observe that the use of a generic basis for the $\text{Spin}(7)$ structure 4-form and the knowledge of the algebra it satisfies plays an important rôle in the construction of the solution. The octonionic instanton has been used to construct the *octonionic superstring soliton* of ref. [57], which is, actually, a $\mathcal{O}(\alpha')$ solution of the Heterotic Superstring effective action preserving exactly one supersymmetry and, therefore, a very good example of the characterization discussed in this paper. It is, on the other hand, a solution closely related to that of the symmetric 5-brane mentioned above: both of them are sourced by Yang-Mills instanton fields satisfying eq. (5.1), the main difference being the number of transverse directions the gauge fields do not depend upon and the number of isometries of the metric (8 to 4) and the absence of a (known) solution of the same kind with no gauge fields, sourced only by the Kalb-Ramond field, in analogy with the solitonic 5-brane.²⁴

It should be possible to consider solutions to eq. (5.1) for cases in between the full 8-dimensional dependence of the octonionic instanton and the 4-dimensional dependence of the BPST instanton, with gauge groups that can be embedded in $\text{SO}(n)$, $4 < n < 8$ and which should coincide with (some of) the special holonomy groups found in refs. [43–45]. Let us consider, for instance, the $n = 7$ case, in which the gauge field lives in a 7-dimensional space or, alternatively, does not depend on one of the original 8 transverse coordinates, x^8 , say. It is not difficult to see that only the components Ω_{mnpq} with $m, n, p, q = 1, \dots, 7$ occur in eq. (5.1) and, due to the 8-dimensional selfduality of the $\text{Spin}(7)$ structure, they can be rewritten in terms of the 3-form $\Sigma_{mnp} \equiv \Omega_{mnp8}$ which satisfies the algebra of a G_2 structure. Thus, it should not be surprising that one can construct G_2 instantons in $\mathbb{R}^7$ using an ansatz similar to Nicolai and Fubini’s (or ’t Hooft’s for the BPST instanton) and a complete solution of the Heterotic Superstring effective action sourced by such an instanton. As a matter of fact, both the instanton and the solution were constructed in ref. [60]. It should be possible to find more intermediate cases, with a number of isometries ranging between 2 (the *string*, $\text{Spin}(7)$ case) and 6 (the *fivebrane*, $\text{SU}(2)$ case), so they can be interpreted as $(n + 1)$ -brane solitons. Their existence would greatly enhance the spectrum of non-perturbative extended solitons of the Heterotic Superstring. Work in this direction is in progress.

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²⁴The torsionful spin connection $\Omega_{(+)}$ of that solution is also that of an “octonionic instanton”.

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A $d = 10$ gamma matrices, spinors and the algebra of bilinears

A.1 $d = 10$ gamma matrices and spinors

In this appendix, Γ^a , $a, b, c, \dots = 0, \dots, 9$ are the 10-dimensional gamma matrices, satisfying the Clifford algebra

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}, \quad (\text{A.1})$$

where $(\eta_{ab}) = \text{diag}(+ - \dots -)$ is the 10-dimensional Minkowski metric.

The chirality matrix Γ_{11} is defined to satisfy the relations

$$\Gamma_{11}\Gamma^{a_1\dots a_n} = \frac{(-1)^{[(10-n)/2]+1}}{(10-n)!} \epsilon^{a_1\dots a_n b_1\dots b_{10-n}} \Gamma_{b_1\dots b_{10-n}}, \quad (\text{A.2})$$

so that, in particular,

$$\Gamma_{11} = \frac{1}{10!} \epsilon_{b_1\dots b_{10}} \Gamma^{b_1\dots b_{10}} = -\Gamma^0 \dots \Gamma^9, \quad (\text{A.3a})$$

$$\Gamma^{a_1\dots a_5} \Gamma_{11} = \frac{1}{5!} \epsilon^{a_1\dots a_5 b_1\dots b_5} \Gamma_{b_1\dots b_5}, \quad (\text{A.3b})$$

where $\epsilon^{0\dots 9} = -\epsilon_{0\dots 9} = +1$. Furthermore,

$$\Gamma_{11}^\dagger = \Gamma_{11}^T = \Gamma_{11}. \quad (\text{A.4})$$

The charge-conjugation and Dirac-conjugation matrices $\mathcal{C}$ and $\mathcal{D}$ are defined by the properties

$$\mathcal{D}\Gamma^a = \Gamma^{a\dagger}\mathcal{D}, \quad \Rightarrow \quad \mathcal{D}\Gamma^{a_1\dots a_n} = (-1)^{[n/2]} (\Gamma^{a_1\dots a_n})^\dagger \mathcal{D}, \quad (\text{A.5a})$$

$$\mathcal{C}\Gamma^a = \Gamma^{aT}\mathcal{C}, \quad \Rightarrow \quad \mathcal{C}\Gamma^{a_1\dots a_n} = (-1)^{[n/2]} (\Gamma^{a_1\dots a_n})^T \mathcal{C}. \quad (\text{A.5b})$$

The particular matrices we have chosen are

$$\mathcal{C} = -i\Gamma^0\Gamma^3\Gamma^4\Gamma^6\Gamma^8, \quad \mathcal{D} = \Gamma^0, \quad (\text{A.6})$$

and satisfy

$$\mathcal{C}^T = \mathcal{C} = -\mathcal{C}^{-1} = -\mathcal{C}^\dagger, \quad (\text{A.7a})$$

$$\mathcal{D}^\dagger = \mathcal{D}, \quad (\text{A.7b})$$

$$\mathcal{D}\Gamma_{11} = -\Gamma_{11}\mathcal{D}, \quad (\text{A.7c})$$

$$\mathcal{C}\Gamma_{11} = -\Gamma_{11}\mathcal{C}. \quad (\text{A.7d})$$

Given a 10-dimensional spinor ψ , using these matrices, we define its Dirac and Majorana conjugates, respectively $\bar{\psi}$ and ψ^c , by

$$\bar{\psi} \equiv \psi^\dagger \mathcal{D}, \quad (\text{A.8a})$$

$$\psi^c \equiv \psi^T \mathcal{C}. \quad (\text{A.8b})$$

Majorana spinors are defined by the property

$$\bar{\psi} = \psi^c. \quad (\text{A.9})$$

With the particular choices of $\mathcal{C}$ and $\mathcal{D}$ that we have made, they are neither purely real nor purely imaginary, but this is the most convenient choice for reducing them to symplectic-Majorana spinors in $d = 6$ dimensions (which will be useful in a forthcoming work [64]).

The supersymmetry parameter of Heterotic Supergravity, ϵ , is a Majorana-Weyl spinor. We choose the convention

$$\Gamma_{11}\epsilon = +\epsilon, \quad \Rightarrow \quad \bar{\epsilon}\Gamma_{11} = -\bar{\epsilon}. \quad (\text{A.10})$$

A.2 $d = 10$ spinor bilinears

Let us consider the bilinears of these spinors (taken as commuting) $\bar{\epsilon}\Gamma^{a_1 \dots a_n}\epsilon$. Using the above properties we find

$$\bar{\epsilon}\Gamma^{a_1 \dots a_n}\epsilon = 0, \quad \forall n \text{ even}. \quad (\text{A.11a})$$

$$(\bar{\epsilon}\Gamma^{a_1 \dots a_n}\epsilon)^T = (-1)^{[n/2]} \bar{\epsilon}\Gamma^{a_1 \dots a_n}\epsilon, \quad (\text{A.11b})$$

$$(\bar{\epsilon}\Gamma^{a_1 \dots a_n}\epsilon)^\dagger = (-1)^{[n/2]} \bar{\epsilon}\Gamma^{a_1 \dots a_n}\epsilon, \quad (\text{A.11c})$$

from which it follows that only for $n = 1(\text{mod } 4)$ the bilinear is generically non-vanishing and that, in those cases, it is real. Since the n and $\tilde{n} = 10 - n$ bilinears are related by the duality eq. (A.2) we end up with just two independent, real bilinears: a 1-form that we denote by ℓ_a and a selfdual 5-form that we denote by $W_{a_1 \dots a_5}$

$$\ell_a \equiv \bar{\epsilon}\Gamma_a\epsilon, \quad (\text{A.12a})$$

$$W_{a_1 \dots a_5} \equiv \bar{\epsilon}\Gamma_{a_1 \dots a_5}\epsilon = \frac{1}{5!}\epsilon^{a_1 \dots a_5 b_1 \dots b_5} W_{b_1 \dots b_5}. \quad (\text{A.12b})$$

A.3 $d = 10$ Fierz identities

If $\{\mathcal{O}^I\}$ and $\{\mathcal{O}_I\}$ are dual bases of 32×32 matrices with the trace as inner product

$$\text{Tr}(\mathcal{O}^I \mathcal{O}_J) = 32 \delta^I_J, \quad (\text{A.13})$$

M and N are two arbitrary 32×32 matrices and $\lambda, \chi, \psi, \varphi$ are four 32-component commuting spinors, the Fierz identities take the form

$$(\bar{\lambda} M \chi)(\bar{\psi} N \varphi) = \frac{1}{32} \sum_I (\bar{\lambda} M \mathcal{O}^I N \varphi)(\bar{\psi} \mathcal{O}_I \chi). \quad (\text{A.14})$$

Choosing the dual bases in the space of 32×32 matrices

$$\{\mathcal{O}^I\} \equiv \{1, \Gamma^a, i\Gamma^{ab}, i\Gamma^{abc}, \Gamma^{a_1 \dots a_4}, \Gamma^{a_1 \dots a_5}, \Gamma^{a_1 \dots a_4} \Gamma_{11}, \Gamma^{abc} \Gamma_{11}, i\Gamma^{ab} \Gamma_{11}, i\Gamma^a \Gamma_{11}, \Gamma_{11}\}, \quad (\text{A.15a})$$

$$\{\mathcal{O}_I\} \equiv \{1, \Gamma_a, i\Gamma_{ab}, i\Gamma_{abc}, \Gamma_{a_1 \dots a_4}, \Gamma_{a_1 \dots a_5}, \Gamma_{a_1 \dots a_4} \Gamma_{11}, \Gamma_{abc} \Gamma_{11}, i\Gamma_{ab} \Gamma_{11}, i\Gamma_a \Gamma_{11}, \Gamma_{11}\}, \quad (\text{A.15b})$$

and assuming that the spinors $\lambda, \chi, \psi, \varphi$ have positive chirality (so M and N have to be products of odd numbers of gammas and, therefore, anticommute with Γ_{11}), the Fierz identity takes the explicit form

$$\begin{aligned} (\bar{\lambda}M\chi)(\bar{\psi}N\varphi) &= \frac{1}{16}(\bar{\lambda}M\Gamma^aN\varphi)(\bar{\psi}\Gamma_a\chi) - \frac{1}{16 \cdot 3!}(\bar{\lambda}M\Gamma^{abc}N\varphi)(\bar{\psi}\Gamma_{abc}\chi) \\ &\quad + \frac{1}{32 \cdot 5!}(\bar{\lambda}M\Gamma^{a_1 \dots a_5}N\varphi)(\bar{\psi}\Gamma_{a_1 \dots a_5}\chi). \end{aligned} \quad (\text{A.16})$$

Finally, if $\lambda = \chi = \psi = \varphi = \epsilon$, the 3-form vanishes and, using the above definitions for the bilinears, we are left with just

$$(\bar{\epsilon}M\epsilon)(\bar{\epsilon}N\epsilon) = \frac{1}{16}(\bar{\epsilon}M\Gamma^aN\epsilon)\ell_a + \frac{1}{32 \cdot 5!}(\bar{\epsilon}M\Gamma^{a_1 \dots a_5}N\epsilon)W_{a_1 \dots a_5}, \quad (\text{A.17})$$

A.4 $d = 10$ bilinear algebra

It is now straightforward to compute the products of the bilinears ℓ_a and $W_{a_1 \dots a_5}$ using the Fierz identities we just derived in the previous section. Each of the identities obtained has been checked to be consistent with the self-duality of the 5-form, which imposes strong constraints on the possible combinations that can occur in the right-hand side.

To start with, we find for the product of 1-forms

$$\ell_a \ell_b = \frac{1}{14 \cdot 4!} W_a{}^{c_1 \dots c_4} W_{bc_1 \dots c_4}, \quad (\text{A.18})$$

and we observe that the right-hand-side is equal to itself when we replace W by its dual, in agreement with the invariance under duality of the left-hand side.

For the product of a 1-form and one self-dual 5-form we have obtained²⁵

$$\ell^a W_{b_1 \dots b_5} = \frac{5}{7} \ell_{b_1} W_{b_2 \dots b_5}{}^a - \frac{5}{14} W_{b_1 b_2 b_3}{}^{c_1 c_2} W_{b_4 b_5 c_1 c_2}{}^a, \quad (\text{A.19})$$

where we have used the fact that the term

$$\eta^a{}_{b_1} W_{b_2 \dots b_5 d} \ell^d, \quad (\text{A.20})$$

which occurs in the right-hand side vanishes identically due to the self-duality of W . The expression in the right-hand side is self-dual in the five b_i indices, just as the left-hand side.

In order to express the product of two self-dual 5-forms we have defined, first, for the sake of convenience, the following products and contractions of the self-dual 5-form with itself:

$$A \equiv W_{a_1 \dots a_4 b_5} W_{b_1 \dots b_4 a_5}, \quad (\text{A.21a})$$

$$B \equiv W_{a_1 a_2 a_3 b_4 b_5} W_{b_1 b_2 b_3 a_4 a_5}, \quad (\text{A.21b})$$

$$C \equiv W_{a_1 a_2 a_3}{}^{c_1 c_2} W_{b_1 b_2 b_3 c_1 c_2} \eta_{a_4 a_5, b_4 b_5}, \quad (\text{A.21c})$$

$$D \equiv W_{a_1 a_2 b_3}{}^{c_1 c_2} W_{b_1 b_2 a_3 c_1 c_2} \eta_{a_4 a_5, b_4 b_5}, \quad (\text{A.21d})$$

$$E \equiv W_{a_1}{}^{c_1 \dots c_4} W_{b_1 c_1 \dots c_4} \eta_{a_2 \dots a_5, b_2 \dots b_5}, \quad (\text{A.21e})$$

$$F \equiv W_{a_1 \dots a_5} W_{b_1 \dots b_5}. \quad (\text{A.21f})$$

²⁵Here and in what follows, it is assumed that all indices that have the same letter ($b_1, \dots, b_5$ in this case) are always antisymmetrized. The brackets of the antisymmetrizers have been suppressed to avoid the cluttering of the formulae.

The rest of the terms quadratic in W that can occur in the right-hand side are linear combinations of them, as can be seen by replacing W by its dual. These relations are

$$W_{a_1 a_2}{}^{c_1 c_2 c_3} W_{b_1 b_2 c_1 c_2 c_3} \eta_{a_3 a_4 a_5, b_3 b_4 b_5} = \frac{1}{2} E, \quad (\text{A.22a})$$

$$W_{a_1 b_2}{}^{c_1 c_2 c_3} W_{b_1 a_2 c_1 c_2 c_3} \eta_{a_3 a_4 a_5, b_3 b_4 b_5} = -\frac{1}{4} E, \quad (\text{A.22b})$$

$$W_{a_1 \dots a_4}{}^c W_{b_1 \dots b_4 c} \eta_{a_5, b_5} = 4C - E, \quad (\text{A.22c})$$

$$W_{a_1 a_2 a_3 b_4}{}^c W_{b_1 b_2 b_3 a_4 c} \eta_{a_5, b_5} = \frac{1}{4} C + \frac{9}{4} D - \frac{1}{4} E, \quad (\text{A.22d})$$

$$W_{a_1 a_2 b_3 b_4}{}^c W_{b_1 b_2 a_3 a_4 c} \eta_{a_5, b_5} = 2D - \frac{1}{6} E, \quad (\text{A.22e})$$

and they allow us to use A, B, C, D and E as a basis for these products.

Then, using this notation, we find

$$\begin{aligned} W_{a_1 \dots a_5} W_{b_1 \dots b_5} &= \frac{1}{3} \left(\ell_{a_1} \epsilon_{a_2 \dots a_5 b_1 \dots b_5} \ell^d + a \leftrightarrow b \right) \\ &\quad - 80 \left(\ell_{a_1} W_{a_2 a_3 b_1 b_2 b_3} \eta_{a_4 a_5, b_4 b_5} + a \leftrightarrow b \right) \\ &\quad + 80 \ell_{a_1} \ell_{b_1} \eta_{a_2 \dots a_5, b_2 \dots b_5} \\ &\quad + \frac{5}{3} A + \frac{20}{3} B - \frac{20}{3} C - 60D + \frac{25}{3} E. \end{aligned} \quad (\text{A.23})$$

All terms in the right-hand side of this expression are $a \leftrightarrow b$ symmetric, as the left-hand side.

Furthermore, the left-hand side is self-dual in the a and b indices separately. It can be checked that the right-hand side has the same property: the combination of the first three terms is self-dual and the combination of terms in the fourth line is also self-dual,²⁶ as can be seen by using the properties

$$\star A = \frac{1}{5} F - 4C + E, \quad (\text{A.24a})$$

$$\star B = \frac{1}{10} F - 3C + E, \quad (\text{A.24b})$$

$$\star C = \frac{1}{10} F - A + B, \quad (\text{A.24c})$$

$$\star D = \frac{1}{30} F - \frac{1}{3} A + \frac{1}{3} B - \frac{1}{3} C + D, \quad (\text{A.24d})$$

$$\star E = \frac{1}{5} F - 3A + 4B. \quad (\text{A.24e})$$

A.4.1 Consequences

The selfduality of W implies

$$W^2 \equiv W^{a_1 \dots a_5} W_{a_1 \dots a_5} = 0. \quad (\text{A.25})$$

Then, eq. (A.18) implies that ℓ is null:

$$\ell^2 = \ell^a \ell_a = 0. \quad (\text{A.26})$$

²⁶Actually, up to a global factor, it is the only self-dual combination.

Lowering the upper index of eq. (A.19) and antisymmetrizing it with the rest leads to

$$\ell_{b_1} W_{b_2 \dots b_6} = 0, \quad (\text{A.27})$$

which implies that

$$W_{a_1 \dots a_5} = 5\ell_{a_1} \Omega_{a_2 \dots a_5}, \quad (\text{A.28})$$

for a certain 4-form Ω . We will see that this 4-form, which was first found in the supergravity context in ref. [69], satisfies the relations of a Spin(7) structure. Plugging this result back into eq. (A.19) and contracting now the upper index with one of the lower ones we find that

$$\ell^b W_{a_1 \dots a_4 b} = 0, \quad \Rightarrow \quad \ell^b \Omega_{a_1 a_2 a_3 b} = 0, \quad (\text{A.29})$$

so Ω lives in the 8-dimensional space transverse to the null vector ℓ . It is useful to introduce a null vector n dual to ℓ :

$$n^2 = 0, \quad n^a \ell_a = 1, \quad (\text{A.30})$$

and define the metric induced in the 8-dimensional space transverse to ℓ as

$$\tilde{\eta}_{ab} \equiv \eta_{ab} - 2\ell_{(a} n_{b)}, \quad (\text{A.31})$$

and a fully antisymmetric tensor in that space as well

$$\tilde{\epsilon}^{c_1 \dots c_8} \equiv \epsilon^{abc_1 \dots c_8} n_a \ell_b, \quad \tilde{\epsilon}_{c_1 \dots c_8} \equiv \epsilon_{abc_1 \dots c_8} n^a \ell^b. \quad (\text{A.32})$$

Ω satisfies

$$\Omega_{a_1 \dots a_4} = \tilde{\eta}_{a_1 \dots a_4}{}^{b_1 \dots b_4} \Omega_{b_1 \dots b_4}, \quad (\text{A.33a})$$

$$\Omega^{a_1 \dots a_4} = \frac{1}{4!} \tilde{\epsilon}^{a_1 \dots a_4 b_1 \dots b_4} \Omega_{b_1 \dots b_4}. \quad (\text{A.33b})$$

Furthermore, using eq. (A.28) in eqs. (A.18) and eq. (A.19), we find

$$\Omega_{a_1 \dots a_4} \Omega^{a_1 \dots a_4} \equiv \Omega^2 = 14 \cdot 4!, \quad (\text{A.34a})$$

$$\Omega_{a_1 a_2}{}^{b_1 b_2} \Omega_{a_3 a_4 b_1 b_2} = -4\Omega_{a_1 \dots a_4}. \quad (\text{A.34b})$$

The selfduality of Ω in the 8-dimensional transverse space eq. (A.33a) together with eq. (A.34a) implies

$$\Omega_{a_1 a_2 a_3}{}^c \Omega_{b_1 b_2 b_3 c} = -21 \tilde{\eta}_{a_1 a_2 a_3, b_1 b_2 b_3} + \frac{9}{4} \Omega_{a_1 a_2}{}^{c_1 c_2} \Omega_{b_1 b_2 c_1 c_2} \tilde{\eta}_{a_3 b_3}, \quad (\text{A.35a})$$

$$\Omega_{ac_1 c_2 c_3} \Omega_b{}^{c_1 c_2 c_3} = 42 \tilde{\eta}_{ab}. \quad (\text{A.35b})$$

The last product, eq. (A.23), gives the following expression for the product of two 4-forms:

$$\begin{aligned} \Omega_{a_1 \dots a_4} \Omega_{b_1 \dots b_4} = & \frac{1}{7} \tilde{\epsilon}_{a_1 \dots a_4 b_1 \dots b_4} + \frac{864}{7} \tilde{\eta}_{a_1 \dots a_4, b_1 \dots b_4} - \frac{144}{7} \Omega_{a_1 a_2 b_1 b_2} \tilde{\eta}_{a_3 a_4, b_3 b_4} \\ & + \frac{16}{7} \Omega_{a_1 a_2 a_3 b_4} \Omega_{b_1 b_2 b_3 a_4} + \frac{18}{7} \Omega_{a_1 a_2 b_3 b_4} \Omega_{b_1 b_2 a_3 a_4} \\ & - \frac{36}{7} \Omega_{a_1 a_2}{}^{c_1 c_2} \Omega_{b_1 b_2 c_1 c_2} \tilde{\eta}_{a_3 a_4, b_3 b_4} + \frac{72}{7} \Omega_{a_1 b_2}{}^{c_1 c_2} \Omega_{b_1 a_2 c_1 c_2} \tilde{\eta}_{a_3 a_4, b_3 b_4}. \end{aligned} \quad (\text{A.36})$$

This result is $a \leftrightarrow b$ symmetric and fully consistent with eqs. (A.34a) and (A.34b). It immediately leads to these two identities,

$$\Omega_{a_1 \dots a_4} \Omega_{a_5 \dots a_8} = \frac{1}{5} \tilde{\epsilon}_{a_1 \dots a_8}, \quad (\text{A.37a})$$

$$\Omega_{a_1 a_2}{}^{c_1 c_2} \Omega_{b_1 b_2 c_1 c_2} = -\frac{2}{5} \Omega_{a_1 b_2}{}^{c_1 c_2} \Omega_{b_1 a_2 c_1 c_2} - \frac{12}{5} \Omega_{a_1 a_2 b_1 b_2} - \frac{72}{5} \tilde{\eta}_{a_1 a_2, b_1 b_2}. \quad (\text{A.37b})$$

Eqs. (A.34a) and (A.34b) can be obtained from the last of these equations.

Antisymmetrizing eq. (A.37b), we find

$$\Omega_{a_1 a_2}{}^{c_1 c_2} \Omega_{b_1 b_2 c_1 c_2} = -4 \Omega_{a_1 a_2 b_1 b_2} + 12 \tilde{\eta}_{a_1 a_2, b_1 b_2}, \quad (\text{A.38a})$$

$$\Omega_{a_1 b_2}{}^{c_1 c_2} \Omega_{b_1 a_2 c_1 c_2} = +4 \Omega_{a_1 a_2 b_1 b_2} + 6 \tilde{\eta}_{a_1 a_2, b_1 b_2}. \quad (\text{A.38b})$$

Substituting these two identities back into eq. (A.36) we get

$$\begin{aligned} \Omega_{a_1 \dots a_4} \Omega_{b_1 \dots b_4} &= \frac{1}{7} \tilde{\epsilon}_{a_1 \dots a_4 b_1 \dots b_4} - \frac{188}{7} \Omega_{a_1 a_2 b_1 b_2} \tilde{\eta}_{a_3 a_4, b_3 b_4} \\ &\quad + \frac{16}{7} \Omega_{a_1 a_2 a_3 b_4} \Omega_{b_1 b_2 b_3 a_4} + \frac{18}{7} \Omega_{a_1 a_2 b_3 b_4} \Omega_{b_1 b_2 a_3 a_4}. \end{aligned} \quad (\text{A.39})$$

Further simplifications of this general formula are possible, but we will not try to obtain them here.

Summarizing, the main relations involving the product and contractions of two 4-forms that we will use are²⁷

$$\Omega_{a_1 \dots a_4} = \frac{1}{4!} \tilde{\epsilon}_{a_1 \dots a_4 b_1 \dots b_4} \Omega^{b_1 \dots b_4}, \quad (\text{A.40a})$$

$$\Omega_{a_1 \dots a_4} \Omega_{a_5 \dots a_8} = \frac{1}{5} \tilde{\epsilon}_{a_1 \dots a_8}, \quad (\text{A.40b})$$

$$\Omega_{a_1 a_2 a_3}{}^c \Omega_{b_1 b_2 b_3 c} = -9 \Omega_{a_1 a_2 b_1 b_2} \tilde{\eta}_{a_3 b_3} + 6 \tilde{\eta}_{a_1 a_2 a_3, b_1 b_2 b_3}, \quad (\text{A.40c})$$

$$\Omega_{a_1 a_2}{}^{c_1 c_2} \Omega_{b_1 b_2 c_1 c_2} = -4 \Omega_{a_1 a_2 b_1 b_2} + 12 \tilde{\eta}_{a_1 a_2, b_1 b_2}, \quad (\text{A.40d})$$

$$\Omega_{a_1 b_2}{}^{c_1 c_2} \Omega_{b_1 a_2 c_1 c_2} = +4 \Omega_{a_1 a_2 b_1 b_2} + 6 \tilde{\eta}_{a_1 a_2, b_1 b_2}. \quad (\text{A.40e})$$

$$\Omega_{a_1 a_2}{}^{b_1 b_2} \Omega_{a_3 a_4 b_1 b_2} = -4 \Omega_{a_1 \dots a_4}, \quad (\text{A.40f})$$

$$\Omega_{a c_1 c_2 c_3} \Omega_b{}^{c_1 c_2 c_3} = 42 \tilde{\eta}_{ab}, \quad (\text{A.40g})$$

$$\Omega_{a_1 \dots a_4} \Omega^{a_1 \dots a_4} \equiv \Omega^2 = 14 \cdot 4!. \quad (\text{A.40h})$$

A.5 Projectors

The 4-form $\Omega_{m_1 \dots m_4}$ can be used to construct projectors acting on 2- and 3-forms in the 8-dimensional transverse space and on spinors.

²⁷Not all these equations are independent. We quote all of them for their usefulness.

For 2-forms, and 3-forms, respectively we have these two complementary pairs of projectors²⁸

$$\Pi^{(+)}{}_{m_1 m_2}{}_{n_1 n_2} = \frac{3}{4} \left(\tilde{\eta}^{m_1 m_2}{}_{n_1 n_2} + \frac{1}{6} \Omega^{m_1 m_2}{}_{n_1 n_2} \right), \quad (\text{A.41a})$$

$$\Pi^{(-)}{}_{m_1 m_2}{}_{n_1 n_2} \equiv \frac{1}{4} \left(\tilde{\eta}^{m_1 m_2}{}_{n_1 n_2} - \frac{1}{2} \Omega^{m_1 m_2}{}_{n_1 n_2} \right), \quad (\text{A.41b})$$

$$\Pi^{(+)}{}_{m_1 m_2 m_3}{}_{n_1 n_2 n_3} \equiv \frac{6}{7} \left(\tilde{\eta}^{m_1 m_2 m_3}{}_{n_1 n_2 n_3} + \frac{1}{4} \Omega^{m_1 m_2}{}_{n_1 n_2} \tilde{\eta}^{m_3}{}_{n_3} \right), \quad (\text{A.41c})$$

$$\Pi^{(-)}{}_{m_1 m_2 m_3}{}_{n_1 n_2 n_3} \equiv \frac{1}{7} \left(\tilde{\eta}^{m_1 m_2 m_3}{}_{n_1 n_2 n_3} - \frac{3}{2} \Omega^{m_1 m_2}{}_{n_1 n_2} \tilde{\eta}^{m_3}{}_{n_3} \right). \quad (\text{A.41d})$$

With them we can decompose 2-forms and 3-forms as follows:

$$F_{mn} = F_{mn}^{(+)} + F_{mn}^{(-)}, \quad (\text{A.42a})$$

$$H_{mnp} = H_{mnp}^{(+)} + H_{mnp}^{(-)}, \quad (\text{A.42b})$$

where the components satisfy

$$\begin{aligned} \Pi^{(\pm)} F^{(\pm)} &= F^{(\pm)}, & \Pi^{(\pm)} F^{(\mp)} &= 0, \\ \Pi^{(\pm)} H^{(\pm)} &= H^{(\pm)}, & \Pi^{(\pm)} H^{(\mp)} &= 0. \end{aligned} \quad (\text{A.43})$$

On positive chirality spinors satisfying the constraint

$$\ell_a \Gamma^a \epsilon \equiv \Gamma^+ \epsilon = 0, \quad (\text{A.44})$$

it is consistent to define the projectors

$$\Pi^{(+)} \equiv \frac{1}{8} \left(1 + \frac{1}{48} \Omega_{m_1 \dots m_4} \Gamma^{m_1 \dots m_4} \right), \quad (\text{A.45a})$$

$$\Pi^{(-)} \equiv \frac{7}{8} \left(1 - \frac{1}{336} \Omega_{m_1 \dots m_4} \Gamma^{m_1 \dots m_4} \right). \quad (\text{A.45b})$$

Observe that these projectors can be interpreted as supersymmetry projectors associated to a multiple intersection of S5-branes: if ω_{6789} is the volume 4-form of the space transverse to a S5-brane whose worldvolume occupies the directions $0, 1, \dots, 5$, its associate supersymmetry projector is $\sim (1 + \omega_{6789} \Gamma^{6789})$. There is a basis [68] in which the Spin(7) structure 4-form is a linear combination of 4-forms associated to the transverse space of intersecting S5-branes and, therefore, the above projector can be understood as a superposition of the associated supersymmetric projectors of those S5-branes. This suggests that the supersymmetric configurations with minimal supersymmetry can be understood as configurations describing multiple intersection S5-branes.

²⁸They are mutually orthogonal and their sum is the identity in the space of 2- and 3-forms in 8 dimensions. They are properly normalized to be idempotent.

The projectors acting on spinors and forms satisfy several relations. The two relations that we will use are

$$F_{mn}^{(+)}\Gamma^{mn}\epsilon = F_{mn}^{(+)}\Gamma^{mn}\Pi^{(-)}\epsilon, \quad (\text{A.46a})$$

$$H_{mnp}\Gamma^{mnp}\Pi^{(-)}\epsilon = \frac{1}{8}\left[H_{mnp}\Gamma^{mnp}\epsilon + 7(\Pi^{(+)}H)_{mnp}\Gamma^{mnp} - \Omega_m{}^{npq}H_{npq}\Gamma^m\right]\epsilon. \quad (\text{A.46b})$$

Finally, we notice the relation

$$\Gamma^m\Pi^{(-)}\epsilon = \frac{7}{8}\left(\Gamma^m - \frac{1}{42}\Omega_m{}^{npq}\Gamma^{npq}\right). \quad (\text{A.47})$$

B Equations of motion at first order in α'

In this appendix we collect the explicit form of the complete equations of motion that follow from the action eq. (2.13) at first order in α' and ignoring the factor $g_s^2/(16\pi G_N^{(10)})$.

$$\frac{e^{2\phi}}{\sqrt{|g|}}\frac{\delta S}{\delta\phi} = 8\left[\nabla^2\phi - (\partial\phi)^2\right] - 2R - \frac{1}{6}H^2 + \frac{\alpha'}{4}\left(F^2 + R_{(-)}^2\right), \quad (\text{B.1})$$

$$\frac{1}{\sqrt{|g|}}\frac{\delta S}{\delta A_{A\mu}} = \frac{\alpha'}{2}\left[\nabla_{(+)\nu}\left(e^{-2\phi}F^{A\nu\mu}\right) + \frac{1}{2}A^A{}_\nu\nabla_\rho\left(e^{-2\phi}H^{\rho\nu\mu}\right)\right], \quad (\text{B.2})$$

$$\frac{1}{\sqrt{|g|}}\frac{\delta S}{\delta B_{\mu\nu}} = -\frac{1}{2}\nabla_\rho\left(e^{-2\phi}H^{\rho\mu\nu}\right) - 2\nabla_\rho X^{[\rho\mu\nu]}, \quad (\text{B.3})$$

$$\begin{aligned} -\frac{e^{2\phi}}{2\sqrt{|g|}}\frac{\delta S}{\delta e^a{}_\mu} &= G_a{}^\mu - 2e^\mu{}_a(\partial\phi)^2 - 2\left[\nabla^\mu\partial_a\phi - e^\mu{}_a\nabla^2\phi\right] \\ &\quad + \frac{1}{4}\left[H^{\mu\nu\rho}H_{a\nu\rho} - \frac{1}{6}e^\mu{}_aH^2\right] - \frac{\alpha'}{4}\left[F_A{}^{\mu\nu}F^A{}_{a\nu} - \frac{1}{4}e^\mu{}_aF^2\right] \\ &\quad - \frac{\alpha'}{4}\left[R_{(-)}{}^{\mu\nu b}{}_cR_{(-)a\nu}{}^c{}_b - \frac{1}{4}e^\mu{}_aR_{(-)}^2\right] \\ &\quad - \frac{1}{2}e^{2\phi}X^{\rho\nu\mu}H_{\rho\nu a} + \frac{1}{2}e^{2\phi}\nabla_\rho\left(X^\rho{}_a{}^\mu - X^\mu{}_a{}^\rho + X_a{}^{\rho\mu}\right), \end{aligned} \quad (\text{B.4})$$

where $\nabla_{(+)\mu}$ stands for the total (gauge, Lorentz, general coordinate transformations) covariant derivative with torsionful connection $\Omega_{(+)}$:

$$e^{2\phi}\nabla_{(+)\nu}\left(e^{-2\phi}F^{A\nu\mu}\right) = e^{2\phi}\nabla_\nu\left(e^{-2\phi}F^{A\nu\mu}\right) + f_{BC}{}^A A^B{}_\nu F^{C\nu\mu} + \frac{1}{2}H_{\nu\rho}{}^\mu \wedge F^{A\nu\rho} = 0. \quad (\text{B.5})$$

where we have used the shorthand notations

$$\begin{aligned} F^2 &\equiv F_{A\mu\nu}F^{A\mu\nu}, \\ R_{(-)}^2 &\equiv R_{(-)\mu\nu}{}^a{}_b R_{(-)}{}^{\mu\nu b}{}_a, \\ X^\nu{}_{ab} &\equiv \frac{1}{\sqrt{|g|}}\frac{\delta S}{\delta\Omega_{(-)\nu}{}^{ab}} = -\frac{\alpha'}{2}\left[\mathcal{D}_{(+,-)\mu}\left(e^{-2\phi}R_{(-)}{}^{\mu\nu}{}_{ab}\right) + \frac{1}{2}\Omega_{(-)\mu ab}\nabla_\rho\left(e^{-2\phi}H^{\rho\mu\nu}\right)\right], \end{aligned} \quad (\text{B.6})$$

where the last covariant derivative uses the $\Omega_{(+)}$ torsionful spin connection with respect to the index ν and $\Omega_{(-)}$ torsionful spin connection with respect to the indices ab and the Levi-Civita connection with respect to the index μ .

Using the identities

$$R_{(+)}{}_{abcd} - R_{(-)}{}_{abcd} = 2\nabla_{[a}H_{bcd]}, \quad (\text{B.7})$$

$$\mathcal{D}_{(\pm)[\mu]}R_{(\pm)[\nu\rho]}{}^{ab} = 0, \quad (\text{B.8})$$

where the covariant derivative $\mathcal{D}_{(\pm)\mu}$ with respect to the torsionful spin connections $\Omega_{(\pm)}$ only acts on the Lorentz indices, and the Bianchi identity of H , one can show that, up to terms of higher order in α' (produced by the Bianchi identity of H)

$$\mathcal{D}_{(+,-)\mu} \left(e^{-2\phi} R_{(-)}{}^{\mu c}{}_{ab} \right) = 2e^{-2\phi} \nabla_{(+)[a]} \left(R_{(+)}{}_{|b]}{}^c - 2\nabla_{(+)|b]} \partial^c \phi \right). \quad (\text{B.9})$$

The expression in parenthesis is a combination of the zeroth-order equations of motion:

$$R_{(+)}{}_{b}{}^c - 2\nabla_{(+)}{}_b \partial^c \phi = R_b{}^c - 2\nabla_b \partial^c \phi + \frac{1}{4} H^{(0)}{}_{bde} H^{(0)}{}^{cde} - \frac{1}{2} e^{2\phi} \nabla_d \left(e^{-2\phi} H^{(0)}{}^d{}_b{}^c \right). \quad (\text{B.10})$$

This is the proof of the lemma of ref. [42] mentioned in section 4.1.

The particular combination of the zeroth-order equations of motion that appears in the above expression is, in the notation introduced in section 4.1 of the main text,

$$R_{(+)}{}_{b}{}^c - 2\nabla_{(+)}{}_b \partial^c \phi = \frac{e^{2\phi}}{2\sqrt{|g|}} \left[\mathcal{E}_e{}^{(0)}{}_b{}^c + 2\mathcal{E}_B{}^{(0)}{}_b{}^c + \frac{1}{2} g_b{}^c \mathcal{E}_\phi^{(0)} \right], \quad (\text{B.11})$$

and, therefore,

$$X^\nu{}_{ab} = -\frac{\alpha'}{2\sqrt{|g|}} \left\{ e^{-2\phi} \nabla_{(+)[a]} \left[e^{2\phi} \left(\mathcal{E}_e{}^{(0)}{}^\nu{}_{|b]} + 2\mathcal{E}_B{}^{(0)}{}^\nu{}_{|b]} + \frac{1}{2} e_{|b]}{}^\nu \mathcal{E}_\phi^{(0)} \right) \right] - \Omega_{(-)\mu ab} \mathcal{E}_B^{(0)\mu\nu} \right\}, \quad (\text{B.12})$$

where the upper (0) indices indicate zeroth order in α' .

B.1 Noether identities

In order to understand better the structure of the equations of motion, it is convenient to study the Noether identities that relate them as a consequence of the gauge symmetries of the theory. Associated to the standard gauge transformations of the Kalb-Ramond 2-form with parameter Λ_μ

$$\delta B_{\mu\nu} = 2\partial_{[\mu} \Lambda_{\nu]}, \quad (\text{B.13})$$

we find

$$\partial_\nu \frac{\delta S}{\delta B_{\nu\rho}} = 0. \quad (\text{B.14})$$

Associated to the invariance under Yang-Mills gauge transformations with gauge parameters ξ^A

$$\begin{aligned} \delta A^A{}_\mu &= \mathcal{D}_\mu \xi^A, \\ \delta B_{\mu\nu} &= -\frac{\alpha'}{2} A_{A[\mu} \partial_{\nu]} \xi^A, \end{aligned} \quad (\text{B.15})$$

we find

$$\mathcal{D}_\mu \frac{\delta S}{\delta A^A{}_\mu} - \frac{\alpha'}{2} \partial_\nu \left(A_{A\mu} \frac{\delta S}{\delta B_{\mu\nu}} \right) = 0. \quad (\text{B.16})$$

If this identity is true, then, this one is also true as well:

$$\mathcal{D}_{(-)\mu} \frac{\delta S}{\delta \Omega_{(-)\mu}{}^{ab}} + \frac{\alpha'}{2} \partial_\nu \left(\Omega_{(-)\mu}{}^{ab} \frac{\delta S}{\delta B_{\mu\nu}} \right) = 0. \quad (\text{B.17})$$

Associated to the invariance under local Lorentz transformations with gauge parameters σ^{ab}

$$\begin{aligned} \delta e^a{}_\mu &= \sigma^a{}_b e^b{}_\mu, \\ \delta \Omega_{(-)\mu}{}^{ab} &= \mathcal{D}_{(-)\mu} \sigma^{ab}, \\ \delta B_{\mu\nu} &= -\frac{\alpha'}{2} \Omega_{(-)[\mu}{}^a{}_b \partial_{\nu]} \sigma^b{}_a, \end{aligned} \quad (\text{B.18})$$

we find

$$\left. \frac{\delta S}{\delta e^a{}_\mu} \right|_{\text{exp}} e_{b] \mu} - \mathcal{D}_{(-)\mu} \frac{\delta S}{\delta \Omega_{(-)\mu}{}^{ab}} - \frac{\alpha'}{2} \partial_\nu \left(\Omega_{(-)\mu}{}^{ab} \frac{\delta S}{\delta B_{\mu\nu}} \right) = 0, \quad (\text{B.19})$$

which can be simplified with eq. (B.17) to

$$\left. \frac{\delta S}{\delta e^a{}_\mu} \right|_{\text{exp}} e_{b] \mu} = 0. \quad (\text{B.20})$$

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References

- [1] A.A. Tseytlin, *Extreme dyonic black holes in string theory*, *Mod. Phys. Lett. A* **11** (1996) 689 [[hep-th/9601177](#)] [[INSPIRE](#)].
- [2] M. Cvetič and A.A. Tseytlin, *Solitonic strings and BPS saturated dyonic black holes*, *Phys. Rev. D* **53** (1996) 5619 [*Erratum ibid.* **D 55** (1997) 3907] [[hep-th/9512031](#)] [[INSPIRE](#)].
- [3] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett. B* **379** (1996) 99 [[hep-th/9601029](#)] [[INSPIRE](#)].
- [4] G.W. Gibbons and C.M. Hull, *A Bogomolny Bound for General Relativity and Solitons in $N = 2$ Supergravity*, *Phys. Lett.* **109B** (1982) 190 [[INSPIRE](#)].
- [5] K.p. Tod, *All Metrics Admitting Supercovariantly Constant Spinors*, *Phys. Lett.* **121B** (1983) 241 [[INSPIRE](#)].
- [6] M.M. Caldarelli and D. Klemm, *All supersymmetric solutions of $N = 2$, $D = 4$ gauged supergravity*, *JHEP* **09** (2003) 019 [[hep-th/0307022](#)] [[INSPIRE](#)].
- [7] P. Meessen and T. Ortín, *The Supersymmetric configurations of $N = 2$, $D = 4$ supergravity coupled to vector supermultiplets*, *Nucl. Phys. B* **749** (2006) 291 [[hep-th/0603099](#)] [[INSPIRE](#)].
- [8] M. Huebscher, P. Meessen and T. Ortín, *Supersymmetric solutions of $N = 2$ $D = 4$ SUGRA: The Whole ungauged shebang*, *Nucl. Phys. B* **759** (2006) 228 [[hep-th/0606281](#)] [[INSPIRE](#)].
- [9] S.L. Cacciatori, D. Klemm, D.S. Mansi and E. Zorzan, *All timelike supersymmetric solutions of $N = 2$, $D = 4$ gauged supergravity coupled to abelian vector multiplets*, *JHEP* **05** (2008) 097 [[arXiv:0804.0009](#)] [[INSPIRE](#)].

- [10] M. Huebscher, P. Meessen, T. Ortín and S. Vaula, $N = 2$ Einstein-Yang-Mills's BPS solutions, *JHEP* **09** (2008) 099 [[arXiv:0806.1477](#)] [[INSPIRE](#)].
- [11] D. Klemm and E. Zorzan, All null supersymmetric backgrounds of $N = 2$, $D = 4$ gauged supergravity coupled to abelian vector multiplets, *Class. Quant. Grav.* **26** (2009) 145018 [[arXiv:0902.4186](#)] [[INSPIRE](#)].
- [12] D. Klemm and E. Zorzan, The timelike half-supersymmetric backgrounds of $N = 2$, $D = 4$ supergravity with Fayet-Iliopoulos gauging, *Phys. Rev. D* **82** (2010) 045012 [[arXiv:1003.2974](#)] [[INSPIRE](#)].
- [13] P. Meessen and T. Ortín, Supersymmetric solutions to gauged $N = 2$ $d = 4$ SUGRA: the full timelike shebang, *Nucl. Phys. B* **863** (2012) 65 [[arXiv:1204.0493](#)] [[INSPIRE](#)].
- [14] J.P. Gauntlett, J.B. Gutowski, C.M. Hull, S. Pakis and H.S. Reall, All supersymmetric solutions of minimal supergravity in five- dimensions, *Class. Quant. Grav.* **20** (2003) 4587 [[hep-th/0209114](#)] [[INSPIRE](#)].
- [15] K.P. Tod, More on supercovariantly constant spinors, *Class. Quant. Grav.* **12** (1995) 1801 [[INSPIRE](#)].
- [16] J. Bellowin and T. Ortín, All the supersymmetric configurations of $N = 4$, $d = 4$ supergravity, *Nucl. Phys. B* **726** (2005) 171 [[hep-th/0506056](#)] [[INSPIRE](#)].
- [17] P. Meessen, T. Ortín and S. Vaula, All the timelike supersymmetric solutions of all ungauged $d = 4$ supergravities, *JHEP* **11** (2010) 072 [[arXiv:1006.0239](#)] [[INSPIRE](#)].
- [18] J.P. Gauntlett and J.B. Gutowski, All supersymmetric solutions of minimal gauged supergravity in five-dimensions, *Phys. Rev. D* **68** (2003) 105009 [Erratum *ibid.* **D 70** (2004) 089901] [[hep-th/0304064](#)] [[INSPIRE](#)].
- [19] J.B. Gutowski and H.S. Reall, General supersymmetric AdS_5 black holes, *JHEP* **04** (2004) 048 [[hep-th/0401129](#)] [[INSPIRE](#)].
- [20] J.P. Gauntlett and J.B. Gutowski, General concentric black rings, *Phys. Rev. D* **71** (2005) 045002 [[hep-th/0408122](#)] [[INSPIRE](#)].
- [21] J.B. Gutowski and W. Sabra, General supersymmetric solutions of five-dimensional supergravity, *JHEP* **10** (2005) 039 [[hep-th/0505185](#)] [[INSPIRE](#)].
- [22] J. Bellowin, P. Meessen and T. Ortín, All the supersymmetric solutions of $N = 1$, $d = 5$ ungauged supergravity, *JHEP* **01** (2007) 020 [[hep-th/0610196](#)] [[INSPIRE](#)].
- [23] J. Bellowin and T. Ortín, Characterization of all the supersymmetric solutions of gauged $N = 1$, $d = 5$ supergravity, *JHEP* **08** (2007) 096 [[arXiv:0705.2567](#)] [[INSPIRE](#)].
- [24] J. Bellowin, Supersymmetric solutions of gauged five-dimensional supergravity with general matter couplings, *Class. Quant. Grav.* **26** (2009) 195012 [[arXiv:0810.0527](#)] [[INSPIRE](#)].
- [25] J.B. Gutowski, D. Martelli and H.S. Reall, All Supersymmetric solutions of minimal supergravity in six-dimensions, *Class. Quant. Grav.* **20** (2003) 5049 [[hep-th/0306235](#)] [[INSPIRE](#)].
- [26] M. Cariglia and O.A.P. Mac Conamhna, The General form of supersymmetric solutions of $N = (1, 0)$ $U(1)$ and $SU(2)$ gauged supergravities in six-dimensions, *Class. Quant. Grav.* **21** (2004) 3171 [[hep-th/0402055](#)] [[INSPIRE](#)].
- [27] D.C. Jong, A. Kaya and E. Sezgin, 6D Dyonie String With Active Hyperscalars, *JHEP* **11** (2006) 047 [[hep-th/0608034](#)] [[INSPIRE](#)].
- [28] M. Akyol and G. Papadopoulos, Spinorial geometry and Killing spinor equations of 6-D supergravity, *Class. Quant. Grav.* **28** (2011) 105001 [[arXiv:1010.2632](#)] [[INSPIRE](#)].

- [29] H. Het Lam and S. Vandoren, *BPS solutions of six-dimensional $(1,0)$ supergravity coupled to tensor multiplets*, *JHEP* **06** (2018) 021 [[arXiv:1804.04681](#)] [[INSPIRE](#)].
- [30] P.A. Cano and T. Ortín, *The structure of all the supersymmetric solutions of ungauged $\mathcal{N} = (1,0)$, $d = 6$ supergravity*, *Class. Quant. Grav.* **36** (2019) 125007 [[arXiv:1804.04945](#)] [[INSPIRE](#)].
- [31] N.S. Deger, H. Samtleben and O. Sarioglu, *On The Supersymmetric Solutions of $D = 3$ Half-maximal Supergravities*, *Nucl. Phys. B* **840** (2010) 29 [[arXiv:1003.3119](#)] [[INSPIRE](#)].
- [32] J. de Boer, D.R. Mayerson and M. Shigemori, *Classifying Supersymmetric Solutions in 3D Maximal Supergravity*, *Class. Quant. Grav.* **31** (2014) 235004 [[arXiv:1403.4600](#)] [[INSPIRE](#)].
- [33] E. Ó Colgáin, *All supersymmetric solutions of 3D $U(1)^3$ gauged supergravity*, *JHEP* **11** (2015) 116 [[arXiv:1502.04668](#)] [[INSPIRE](#)].
- [34] N.S. Deger, G. Moutsopoulos, H. Samtleben and Ö. Sarioglu, *All timelike supersymmetric solutions of three-dimensional half-maximal supergravity*, *JHEP* **06** (2015) 147 [[arXiv:1503.09146](#)] [[INSPIRE](#)].
- [35] U. Gran, J. Gutowski and G. Papadopoulos, *Geometry of all supersymmetric four-dimensional $N = 1$ supergravity backgrounds*, *JHEP* **06** (2008) 102 [[arXiv:0802.1779](#)] [[INSPIRE](#)].
- [36] T. Ortín, *The Supersymmetric solutions and extensions of ungauged matter-coupled $N = 1$, $d = 4$ supergravity*, *JHEP* **05** (2008) 034 [[arXiv:0802.1799](#)] [[INSPIRE](#)].
- [37] T. Ortín, *Gravity and Strings*, 2nd edition, Cambridge University Press (2015) [[INSPIRE](#)].
- [38] U. Gran, J. Gutowski and G. Papadopoulos, *Classification, geometry and applications of supersymmetric backgrounds*, *Phys. Rept.* **794** (2019) 1 [[arXiv:1808.07879](#)] [[INSPIRE](#)].
- [39] E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, *Ten-Dimensional Maxwell-Einstein Supergravity, Its Currents and the Issue of Its Auxiliary Fields*, *Nucl. Phys. B* **195** (1982) 97 [[INSPIRE](#)].
- [40] G.F. Chapline and N.S. Manton, *Unification of Yang-Mills Theory and Supergravity in Ten-Dimensions*, *Phys. Lett.* **120B** (1983) 105 [[INSPIRE](#)].
- [41] E. Bergshoeff and M. de Roo, *Supersymmetric Chern-Simons Terms in Ten-dimensions*, *Phys. Lett. B* **218** (1989) 210 [[INSPIRE](#)].
- [42] E.A. Bergshoeff and M. de Roo, *The Quartic Effective Action of the Heterotic String and Supersymmetry*, *Nucl. Phys. B* **328** (1989) 439 [[INSPIRE](#)].
- [43] U. Gran, P. Lohrmann and G. Papadopoulos, *The Spinorial geometry of supersymmetric heterotic string backgrounds*, *JHEP* **02** (2006) 063 [[hep-th/0510176](#)] [[INSPIRE](#)].
- [44] U. Gran, G. Papadopoulos, D. Roest and P. Sloane, *Geometry of all supersymmetric type-I backgrounds*, *JHEP* **08** (2007) 074 [[hep-th/0703143](#)] [[INSPIRE](#)].
- [45] U. Gran, G. Papadopoulos and D. Roest, *Supersymmetric heterotic string backgrounds*, *Phys. Lett. B* **656** (2007) 119 [[arXiv:0706.4407](#)] [[INSPIRE](#)].
- [46] J. Gillard, U. Gran and G. Papadopoulos, *The Spinorial geometry of supersymmetric backgrounds*, *Class. Quant. Grav.* **22** (2005) 1033 [[hep-th/0410155](#)] [[INSPIRE](#)].
- [47] R.R. Metsaev and A.A. Tseytlin, *Order alpha-prime (Two Loop) Equivalence of the String Equations of Motion and the σ -model Weyl Invariance Conditions: Dependence on the Dilaton and the Antisymmetric Tensor*, *Nucl. Phys. B* **293** (1987) 385 [[INSPIRE](#)].
- [48] R. Kallosh and T. Ortín, *Killing spinor identities*, [hep-th/9306085](#) [[INSPIRE](#)].

- [49] J. Bellorín and T. Ortín, *A Note on simple applications of the Killing Spinor Identities*, *Phys. Lett. B* **616** (2005) 118 [[hep-th/0501246](#)] [[INSPIRE](#)].
- [50] V. Cortés, C. Lazaroiu and C.S. Shahbazi, *Spinors of real type as polyforms and the generalized Killing equation*, [arXiv:1911.08658](#) [[INSPIRE](#)].
- [51] E.A. Bergshoeff and M. de Roo, *The string effective action in the dual formulation of $D = 10$ supergravity*, *Phys. Lett. B* **247** (1990) 530 [[INSPIRE](#)].
- [52] E.A. Bergshoeff and M. de Roo, *Duality transformations of string effective actions*, *Phys. Lett. B* **249** (1990) 27 [[INSPIRE](#)].
- [53] D.J. Gross and E. Witten, *Superstring Modifications of Einstein's Equations*, *Nucl. Phys. B* **277** (1986) 1 [[INSPIRE](#)].
- [54] M.T. Grisaru, A.E.M. van de Ven and D. Zanon, *Two-Dimensional Supersymmetric σ -models on Ricci Flat Kähler Manifolds Are Not Finite*, *Nucl. Phys. B* **277** (1986) 388 [[INSPIRE](#)].
- [55] M.T. Grisaru, A.E.M. van de Ven and D. Zanon, *Four Loop Divergences for the $N = 1$ Supersymmetric Nonlinear σ -model in Two-Dimensions*, *Nucl. Phys. B* **277** (1986) 409 [[INSPIRE](#)].
- [56] S. Fubini and H. Nicolai, *The Octonionic Instanton*, *Phys. Lett.* **155B** (1985) 369 [[INSPIRE](#)].
- [57] J.A. Harvey and A. Strominger, *Octonionic superstring solitons*, *Phys. Rev. Lett.* **66** (1991) 549 [[INSPIRE](#)].
- [58] A. Strominger, *Heterotic solitons*, *Nucl. Phys. B* **343** (1990) 167 [*Erratum ibid.* **B 353** (1991) 565] [[INSPIRE](#)].
- [59] A.A. Belavin, A.M. Polyakov, A.S. Schwartz and Y.S. Tyupkin, *Pseudoparticle Solutions of the Yang-Mills Equations*, *Phys. Lett.* **59B** (1975) 85 [[INSPIRE](#)].
- [60] M. Günaydin and H. Nicolai, *Seven-dimensional octonionic Yang-Mills instanton and its extension to an heterotic string soliton*, *Phys. Lett. B* **351** (1995) 169 [[hep-th/9502009](#)] [[INSPIRE](#)].
- [61] E.A. Bergshoeff, R. Kallosh and T. Ortín, *Supersymmetric string waves*, *Phys. Rev. D* **47** (1993) 5444 [[hep-th/9212030](#)] [[INSPIRE](#)].
- [62] S. Deser and B. Zumino, *Consistent Supergravity*, *Phys. Lett.* **62B** (1976) 335 [[INSPIRE](#)].
- [63] P. Meessen, *All-order consistency of 5d SUGRA vacua*, *Phys. Rev. D* **76** (2007) 046006 [*Erratum ibid.* **D 85** (2012) 129902] [[arXiv:0705.1966](#)] [[INSPIRE](#)].
- [64] A. Fontanella and T. Ortín, work in progress.
- [65] C.S. Shahbazi, *Spin(7) structures from chiral spinors on ten-dimensional Lorentzian spin manifolds*, to appear.
- [66] M.J. Duff and J.X. Lu, *Elementary five-brane solutions of $D = 10$ supergravity*, *Nucl. Phys. B* **354** (1991) 141 [[INSPIRE](#)].
- [67] C.G. Callan Jr., J.A. Harvey and A. Strominger, *World sheet approach to heterotic instantons and solitons*, *Nucl. Phys. B* **359** (1991) 611 [[INSPIRE](#)].
- [68] D.D. Joyce, *Compact Manifolds with Special Holonomy*, Oxford Mathematical Monographs, Oxford University Press, Oxford, U.K. (2000).
- [69] B. de Wit and H. Nicolai, *The Parallelizing S_7 Torsion in Gauged $N = 8$ Supergravity*, *Nucl. Phys. B* **231** (1984) 506 [[INSPIRE](#)].