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On even spin $\mathcal{W}_\infty$

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ABSTRACT: We study the even spin $\mathcal{W}_\infty$ which is a universal $\mathcal{W}$ -algebra for orthosymplectic series of $\mathcal{W}$ -algebras. We use the results of Fateev and Lukyanov to embed the algebra into $\mathcal{W}_{1+\infty}$. Choosing the generators to be quadratic in those of $\mathcal{W}_{1+\infty}$, we find that the algebra has quadratic operator product expansions. Truncations of the universal algebra include principal Drinfeld-Sokolov reductions of BCD series of simple Lie algebras, orthogonal and symplectic cosets as well as orthosymplectic Y -algebras of Gaiotto and Rapčák. Based on explicit calculations we conjecture a complete list of co-dimension 1 truncations of the algebra.

KEYWORDS: Conformal and W Symmetry, Conformal Field Theory, Higher Spin Symmetry, Supersymmetric Gauge Theory

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1 Introduction

$\mathcal{W}$ -algebras and their incarnations as affine Yangians [1, 2], degenerate double affine Hecke algebras [3] or cohomological Hall algebras [4] play an important role in various areas of mathematical physics. They were originally introduced in the context of integrable hierarchies of partial differential equations, and soon after in conformal field theory. Some more recent applications include four-dimensional $\mathcal{N} = 2$ gauge theories [5–7], M -theory [8, 9] or higher spin AdS_3/CFT_2 dualities [10].

One of the most important $\mathcal{W}$ -algebra is the universal two-parametric family of algebras called $\mathcal{W}_{1+\infty}$, which has one generating field of every integer spin [11, 12]. It interpolates

between algebras of the $\mathcal{W}_N$ family which are among the most well studied examples of $\mathcal{W}$ -algebras. The very distinct property of $\mathcal{W}_{1+\infty}$ is that it has quantum triality symmetry [13]. The vacuum character of this algebra is given by the MacMahon function which connects the representation theory of $\mathcal{W}_{1+\infty}$ to combinatorics of plane partition, plane tilings and dimer models.

A less studied example of universal interpolating algebra is the even spin $\mathcal{W}_\infty$ which is freely generated by fields of every even spin [14–17]. Here we want to study this algebra in more detail. We first review and slightly extend the primary bootstrap approach to this algebra [16]. A different choice of independent structure constants with respect to [16] eliminates ambiguities arising from square root factors and spurious duality symmetries. Using three elementary minimal representations of the algebra, we introduce a convenient triality covariant parametrization, analogous to parametrization of $\mathcal{W}_\infty$ used in [18]. We also introduce another parametrization which is closer to the parametrization of Gaiotto and Rapčák [7] featuring the Kapustin-Witten parameter. Next we identify our parameters with parameters of well-known truncations of the algebra, including orthogonal and symplectic quotients and principal Drinfel'd-Sokolov reductions of BCD families of simple Lie algebras. We also compare even spin $\mathcal{W}_\infty$ with orthosymplectic Y -algebras introduced in [7].

In the following section we use the results of Fateev and Lukyanov [19] to embed the even spin $\mathcal{W}_\infty$ into $\mathcal{W}_{1+\infty}$. Although in retrospect this is not very surprising, the possibility of doing this at the level of non-linear quantum algebras is not at all obvious. Actually the principal Drinfel'd-Sokolov reductions of all simple Lie algebras are subalgebras of truncations of $\mathcal{W}_{1+\infty}$ (except for F_4 where this is not known) so in this sense $\mathcal{W}_{1+\infty}$ can be thought of as the interpolating algebra for all $\mathcal{W}$ -algebras associated to simple Lie algebras via Drinfel'd-Sokolov reduction (except for F_4). One would like to start with Miura operator for $GL(N)$ $\mathcal{W}$ -algebras and fold it to obtain a Miura operator for BCD -type algebras [20]. Unfortunately it is not clear how to do this at the quantum level. The trick used by Fateev and Lukyanov is instead to consider D_n algebras which have an additional Pfaffian generator of dimension n and study its operator product expansion with itself. From there we can identify the generators of even spin $\mathcal{W}_\infty$ as quadratic composites of the generators of $\mathcal{W}_{1+\infty}$. We verify by explicit calculations using `OPEdefs` [21] that the resulting subalgebra quadratically closes (up to sum of spins 20) and we find a map between parameters of $\mathcal{W}_{1+\infty}$ and those of even spin $\mathcal{W}_\infty$. The operator product expansions in the quadratic basis share many nice properties with $\mathcal{W}_{1+\infty}$ or even with the matrix valued $\mathcal{W}_{1+\infty}$ [22, 23], but unlike in those cases it is not clear at the moment how to sum the derivative terms.

In the following section we list the known truncation curves and study co-dimension 1 truncations of vacuum representation up to level 12. The structure seems to be more complicated than in the case of $\mathcal{W}_\infty$, but all truncations found agree nicely with a simple formula for truncation curves and also with Gaiotto-Rapčák Y -algebras. We conjecture a general formula for the level of the first singular vector in all of these truncations. In the last section we show on an example of $\mathfrak{so}(2n+1)_k$ that the gluing procedure of [24] generalizes also to the orthosymplectic case.

2 Primary bootstrap

Let us summarize and slightly extend the results of [16] where the authors used the OPE bootstrap to construct even spin $\mathcal{W}_\infty$ in the primary basis.¹ We assume to have one generating field of each even spin. The most general form of the operator product expansions between fields up to sum of spins 14 is

$$\begin{aligned}
 W_4 W_4 &\sim C_{44}^0 \mathbb{1} + C_{44}^4 W_4 + C_{44}^6 W_6 \\
 W_4 W_6 &\sim C_{46}^4 W_4 + C_{46}^6 W_6 + C_{46}^{[44]} [W_4 W_4] + C_{46}^8 W_8 \\
 W_4 W_8 &\sim C_{48}^4 W_4 + C_{48}^6 W_6 + C_{48}^8 W_8 + C_{48}^{[44]} [W_4 W_4] \\
 &\quad + C_{48}^{10} W_{10} + C_{48}^{[44]^{(2)}} [W_4 W_4]^{(2)} + C_{48}^{[46]} [W_4 W_6] + C_{48}^{[46]^{(1)}} [W_4 W_6]^{(1)} \\
 W_6 W_6 &\sim C_{66}^0 \mathbb{1} + C_{66}^4 W_4 + C_{66}^6 W_6 + C_{66}^8 W_8 + C_{66}^{[44]} [W_4 W_4] \\
 &\quad + C_{66}^{10} W_{10} + C_{66}^{[44]^{(2)}} [W_4 W_4]^{(2)} + C_{66}^{[46]} [W_4 W_6] + C_{66}^{[46]^{(1)}} [W_4 W_6]^{(1)} \\
 W_4 W_{10} &\sim C_{4,10}^4 W_4 + C_{4,10}^6 W_6 + C_{4,10}^8 W_8 + C_{4,10}^{[44]} [W_4 W_4] \\
 &\quad + C_{4,10}^{10} W_{10} + C_{4,10}^{[44]^{(2)}} [W_4 W_4]^{(2)} + C_{4,10}^{[46]} [W_4 W_6] + C_{4,10}^{[46]^{(1)}} [W_4 W_6]^{(1)} \\
 &\quad + C_{4,10}^{12} W_{12} + C_{4,10}^{[48]} [W_4 W_8] + C_{4,10}^{[66]} [W_6 W_6] + C_{4,10}^{[444]} [W_4 W_4 W_4] \\
 &\quad + C_{4,10}^{[44]^{(4)}} [W_4 W_4]^{(4)} + C_{4,10}^{[46]^{(2)}} [W_4 W_6]^{(2)} + C_{4,10}^{[48]^{(1)}} [W_4 W_8]^{(1)} + C_{4,10}^{[46]^{(3)}} [W_4 W_6]^{(3)} \\
 W_6 W_8 &\sim C_{68}^4 W_4 + C_{68}^6 W_6 + C_{68}^8 W_8 + C_{68}^{[44]} [W_4 W_4] \\
 &\quad + C_{68}^{10} W_{10} + C_{68}^{[44]^{(2)}} [W_4 W_4]^{(2)} + C_{68}^{[46]} [W_4 W_6] + C_{68}^{[46]^{(1)}} [W_4 W_6]^{(1)} \\
 &\quad + C_{68}^{12} W_{12} + C_{68}^{[48]} [W_4 W_8] + C_{68}^{[66]} [W_6 W_6] + C_{68}^{[444]} [W_4 W_4 W_4] \\
 &\quad + C_{68}^{[44]^{(4)}} [W_4 W_4]^{(4)} + C_{68}^{[46]^{(2)}} [W_4 W_6]^{(2)} + C_{68}^{[48]^{(1)}} [W_4 W_8]^{(1)} + C_{68}^{[46]^{(3)}} [W_4 W_6]^{(3)}
 \end{aligned} \tag{2.1}$$

We listed all possible primary fields that can appear in the singular part of the OPE consistent with spin assignments. The descendants also appear in the OPE but their coefficients are uniquely determined by the Virasoro subalgebra. We used the brackets to denote the primary projection of the normal ordered product of fields. We did not the negative powers of $(z - w)$ because they are determined by the dimensions of the fields. Assuming this form of the operator product expansions, the Jacobi identities impose many algebraic relations between the structure constants. The equations up to sum of spins 12 are given in more detail in appendix A. From these equations we can see the following: apart from the central charge c there is essentially one undetermined structure constant. We choose it to be the scale-invariant ratio

$$x \equiv \frac{C_{46}^6}{C_{44}^4}. \tag{2.2}$$

All other undetermined parameters can be fixed to arbitrary non-zero values by rescaling the generating fields. We can choose C_{44}^4 to take any non-zero value by rescaling W_4 .

¹The bootstrap procedure is mathematically formalized in [25] and reviewed in [11, 18].

Afterwards the constant C_{44}^6 (assuming it to be generically non-vanishing) can be chosen to take any non-zero value if we appropriately rescale W_6 . At dimension 8 we have apart from W_8 a new composite primary $[W_4W_4]$ so now we may freely choose C_{46}^8 and $C_{46}^{[44]}$ by redefinition of W_8 (W_8 can be shifted by any multiple of $[W_4W_4]$) etc. The conjecture of [16] proven in [17] is that this continues to hold in all higher OPEs, i.e. that there exists a two-parametric family of algebras parametrized by the central charge and the parameter x .

Note that unlike in [16] where the independent structure constant was chosen to be

$$\frac{C_{44}^0}{(C_{44}^4)^2} = \frac{c(5c+22)}{72(c+24)} - \frac{7c(c-1)(5c+22)}{72(2c-1)(7c+68)}x + \frac{c(c-1)(c+24)(5c+22)}{12(2c-1)(7c+68)^2}x^2, \quad (2.3)$$

here we work instead with x defined by (2.2). The advantage of working with x is that we avoid the ambiguity of choosing a square root when solving for other structure constants. We will see that the group of duality symmetries of the algebra will have order 6 just like the triality in $\mathcal{W}_\infty$. The 6 other candidates for dualities found in [16] are not symmetries of the algebra because they don't preserve the invariant ratio (2.2) and thus not all of the structure constants of even spin $\mathcal{W}_\infty$.

2.1 Minimal representations

In order to study the dualities of the algebra, we need to parametrize the algebra in terms of rank-like parameter [11, 16]. This was done in [16] by applying the method of [26] and expressing the conformal dimensions of the minimal representations of the algebra in terms of rank of the B , C and D series of simple Lie algebras entering the Drinfeld-Sokolov reduction. The minimal representations of even spin $\mathcal{W}_\infty$ are those representations whose character is (for generic values of parameters)

$$\chi_{\min} = \frac{q^{h_m}}{1-q} \chi_{vac}. \quad (2.4)$$

This character has a simple interpretation: the local fields associated to the minimal representation are normal ordered products of fields in even spin $\mathcal{W}_\infty$ with an arbitrary derivative of the minimal primary. Following [16], we assume OPE of the minimal primary ϕ_m with $\mathcal{W}$ -algebra generators to be of the form

$$W_4\phi_m \sim w_{m4}\phi_m \quad (2.5)$$

$$W_6\phi_m \sim w_{m6}\phi_m + C_{6m}^{[4m]}[W_4\phi_m] + C_{6m}^{[4m]^{(1)}}[W_4\phi_m]^{(1)} \quad (2.6)$$

where the notation is like in the previous section. The Jacobi identity ($W_4W_4\phi_m$) imposes the following constraints on the conformal dimension and higher spin charges:

$$\begin{aligned} \frac{w_{m4}}{C_{44}^4} &= -\frac{h_m(ch_m+c+3h_m^2-7h_m+2)(2ch_m+c+16h_m^2-10h_m)}{12(2c^2h_m^2-2c^2h_m-9c^2+36ch_m^3-147ch_m^2+120ch_m-6c+24h_m^3+10h_m^2-28h_m)} \\ \frac{w_{m6}C_{44}^6}{(C_{44}^4)^2} &= \frac{(c-1)(5c+22)^2h_m(ch_m+c+3h_m^2-7h_m+2)}{54(c+24)(2c-1)(7c+68)} \\ &\quad \times \frac{(ch_m+2c+15h_m^2-26h_m+8)(2ch_m+c+16h_m^2-10h_m)(2ch_m+3c+48h_m^2-28h_m)}{(2c^2h_m^2-2c^2h_m-9c^2+36ch_m^3-147ch_m^2+120ch_m-6c+24h_m^3+10h_m^2-28h_m)^2} \end{aligned} \quad (2.7)$$

together with the following algebraic equation restricting h_m :

$$\begin{aligned}
 & (4c^3h_m^2x - 28c^3h_m^2 - 4c^3h_mx + 28c^3h_m - 18c^3x + 91c^3 + 72c^2h_m^3x \\
 & - 364c^2h_m^3 - 198c^2h_m^2x + 988c^2h_m^2 + 144c^2h_mx - 715c^2h_m - 444c^2x + 1010c^2 \\
 & + 1776ch_m^3x - 4040ch_m^3 - 7036ch_m^2x + 12646ch_m^2 + 5704ch_mx - 9616ch_m - 288cx \\
 & + 1224c + 1152h_m^3x - 4896h_m^3 + 480h_m^2x + 3944h_m^2 - 1344h_mx - 272h_m) \times \\
 & \times (12c^3h_m^2x - 14c^3h_m^2 - 12c^3h_mx + 14c^3h_m - 54c^3x + 168c^3 + 216c^2h_m^3x \\
 & - 672c^2h_m^3 - 594c^2h_m^2x + 3287c^2h_m^2 + 432c^2h_mx - 2783c^2h_m - 1332c^2x + 1548c^2 \\
 & + 5328ch_m^3x - 6192ch_m^3 - 21108ch_m^2x + 31544ch_m^2 + 17112ch_mx - 26900ch_m - 864cx \\
 & - 816c + 3456h_m^3x + 3264h_m^3 + 1440h_m^2x - 16592h_m^2 - 4032h_mx + 14144h_m) = 0. \quad (2.8)
 \end{aligned}$$

Given h_m and c , there are thus two possible values for the structure constant x of the algebra,

$$x = \frac{(2c-1)(7c+68)(h_m-4)(ch_m+3c+48h_m^2-52h_m)}{6(c+24)(2c^2h_m^2-2c^2h_m-9c^2+36ch_m^3-147ch_m^2+120ch_m-6c+24h_m^3+10h_m^2-28h_m)} \quad (2.9)$$

$$x = \frac{(7c+68)(4c^2h_m^2-4c^2h_m-13c^2+52ch_m^3-180ch_m^2+141ch_m-18c+72h_m^3-58h_m^2+4h_m)}{2(c+24)(2c^2h_m^2-2c^2h_m-9c^2+36ch_m^3-147ch_m^2+120ch_m-6c+24h_m^3+10h_m^2-28h_m)}, \quad (2.10)$$

so knowing c and h_m still does not determine the algebra uniquely. Solving the next Jacobi identity ($W_4W_6\phi_m$) determines the charge w_{m8} but more importantly also picks the solution (2.9) rather than (2.10). This means that the structure constant x of the algebra is uniquely determined once we know the dimension of the minimal representation (and the central charge).

Given c and x there are generically three solutions of the equation (2.9) for the conformal dimension of the minimal representation. This means that there are three minimal representations of even spin $\mathcal{W}_\infty$ and as we will see they are all permuted by the triality symmetry of the algebra. This is slightly different than the situation in the case of $\mathcal{W}_\infty$ where we have six minimal representations, but there we have to use the triality symmetry together with the charge conjugation symmetry to find all of these representations. The difference between $\mathcal{W}_\infty$ and even spin $\mathcal{W}_\infty$ comes from the fact that while the minimal representations of $\mathcal{W}_\infty$ are charged (i.e. transform under charge conjugation symmetry which changes sign of odd spin fields), the even spin $\mathcal{W}_\infty$ has no conjugation symmetry.

2.2 Parametrizations

Analogously to [18] we can introduce a redundant but triality covariant parametrization of the structure constants using three parameters permuted by the triality symmetry. Let us introduce three parameters (μ_1, μ_2, μ_3) such that the conformal dimensions of the minimal representations are

$$h_{m1} = \frac{1+\mu_1}{2}, \quad h_{m2} = \frac{1+\mu_2}{2}, \quad h_{m3} = \frac{1+\mu_3}{2}. \quad (2.11)$$

These parameters are not independent but satisfy a constraint

$$\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} = 0 \quad (2.12)$$

analogous to a similar constraint in $\mathcal{W}_\infty$. The central charge expressed in terms of these parameters is

$$c = \frac{(\mu_1 + 1)(\mu_2 + 1)(\mu_3 + 1)}{2}. \quad (2.13)$$

We can also introduce a parameter ψ ,

$$\mu_1 = \eta, \quad \mu_2 = -\frac{\eta}{\psi}, \quad \mu_3 = \frac{\eta}{\psi - 1}. \quad (2.14)$$

The parameter ψ is natural parameter from point of view of Drinfeld-Sokolov reductions — is the DS level shifted such that the critical level is at $\psi = 0$. It also agrees with the Kapustin-Witten parameter Ψ in Gaiotto-Rapčák construction [7]. Under triality transformations it transforms by fractional linear transformations permuting $(0, 1, \infty)$. The other parameter η measures the overall scale of μ_j (and is equal to one of μ_j depending on the triality frame). The condition (2.12) is identically satisfied. The central charge takes a simple form

$$c = \frac{(\eta + 1)(\psi - \eta)(\eta + \psi - 1)}{2(\psi - 1)\psi}. \quad (2.15)$$

There is one more parametrization of the algebra used in [17]. The parameter λ used there is related to x by

$$x = \frac{(7c + 68)(1 + 49\lambda - 49\lambda c)}{84\lambda(1 - c)(24 + c)}, \quad (2.16)$$

which is a fractional linear transformation so at given generic c the correspondence between x and λ is one-to-one.

2.3 Orthogonal cosets

We may now identify the parameters of even spin $\mathcal{W}_\infty$ with parameters of orthogonal cosets

$$\frac{\mathfrak{so}(n)_k \times \mathfrak{so}(n)_1}{\mathfrak{so}(n)_{k+1}} \quad (2.17)$$

which are known to have even spin $\mathcal{W}_\infty$ symmetry [14, 16]. Note that the actual symmetry algebra of the cosets is a certain truncation of the even spin $\mathcal{W}_\infty$. First of all, usually we take the rank n to be a positive integer. We can also choose the level k to be one of the rational for which additional null states appear in the algebra. In the following, especially when comparing different cosets, we usually mean the full even spin $\mathcal{W}_\infty$ before the null vectors are quotiented out. But at given values of parameters one can also compare the corresponding simple quotients.

For basics of affine Lie algebras and cosets see [27, 28]. Using the formula for the central charge of affine Lie algebra associated to a simple Lie algebra $\mathfrak{g}$

$$\frac{k \dim \mathfrak{g}}{k + h^\vee} \quad (2.18)$$

where k is the level of affine Lie algebra $\hat{\mathfrak{g}}$ and $h^\vee$ is the dual Coxeter number, we can calculate the central charge of the coset (2.17) and we find

$$c = \frac{kn(2n+k-3)}{2(n+k-1)(n+k-2)} \quad (2.19)$$

(which is uniform for both B and D series of cosets). The conformal dimension of the minimal representation is [14, 16]

$$h_2 = \frac{2n+k-3}{2(n+k-2)} \quad (2.20)$$

for $(\square, \square; \bullet)$ and

$$h_3 = \frac{k}{2(n+k-1)} \quad (2.21)$$

for $(\bullet, \square; \square)$. Expressing x in terms of c and one of h_j , we can find the dimensions of the other two minimal representations. In this way we find that the third minimal representation has conformal dimension

$$h_1 = \frac{n}{2}. \quad (2.22)$$

This is nice because in the even orthogonal case this is exactly the dimension of the additional Pfaffian generator that we might add to the truncation of even spin $\mathcal{W}_\infty$ to get the algebra $\mathcal{WD}_{\frac{n}{2}}$.

To identify the even spin $\mathcal{W}_\infty$ corresponding to these cosets, we can use the central charge together with one of the minimal dimensions in formula (2.9) and this determines the parameter x uniquely. It is also possible to verify explicitly that (2.9) is the correct branch (and not the one given by (2.10)) by direct evaluation of the $k \rightarrow \infty$ limit of these orthogonal cosets. In that case the coset simplifies to

$$\frac{\mathfrak{so}(n)_1}{\mathfrak{so}(n)} \quad (2.23)$$

which is well-known to be realized by the singlet part of VOA of n free real fermions with OPE [27]

$$\psi_j(z)\psi_k(w) \sim \frac{\delta_{jk}}{z-w}. \quad (2.24)$$

The central charge of this VOA is $c = \frac{n}{2}$ while the invariant ratio of structure constants x is

$$x = \frac{C_{46}^6}{C_{44}^4} = \frac{49(n-8)(7n+136)}{6(n+48)(19n-68)}. \quad (2.25)$$

This exactly agrees with the first branch (2.9). The expression for the parameter x in terms of n and k is thus

$$x = \frac{(n-8)(7k+6n-13)(7k+8n-8)(k^2+2kn-3k-n+2)}{6(k^2n+48k^2+2kn^2+93kn-144k+48n^2-144n+96)} \quad (2.26)$$

$$\times \frac{(7k^2n+136k^2+14kn^2+251kn-408k+136n^2-408n+272)}{(19k^4n-68k^4+76k^3n^2-386k^3n+408k^3+94k^2n^3-599k^2n^2+1267k^2n-860k^2+36kn^4-252kn^3+857kn^2-1336kn+744k+24n^4-52n^3-124n^2+376n-224)}.$$

This completely determines the map from (n, k) parameters to (c, x) . The parameters ψ and η are

$$\psi = 2 - n - k, \quad \eta = n - 1. \quad (2.27)$$

In fact, ψ is determined only up to a S_3 subgroup of Möbius transformations permuting $(0, 1, \infty)$. The other five choices of ψ correspond to 5 other embeddings related by the triality symmetry. In terms of coset parameters (n, k) , the following 6 values correspond to the triality-equivalent even spin $\mathcal{W}_\infty$:

$$\begin{aligned} (n, k), \quad & (n, 3 - 2n - k), \quad \left(\frac{k}{n+k-1}, \frac{n}{n+k-1} \right), \\ \left(\frac{k}{n+k-1}, \frac{2n+k-3}{n+k-1} \right), \quad & \left(\frac{2n+k-3}{n+k-2}, \frac{k}{n+k-2} \right), \quad \left(\frac{2n+k-3}{n+k-2}, -\frac{n}{n+k-2} \right). \end{aligned} \quad (2.28)$$

This is one example where the discussion below (2.17) applies. Starting for example with the coset (2.17) with both n and k positive integers, we see that the triality maps parameters (n, k) to values of parameters where typically neither n nor k is an integer. In this situation the triality symmetry implies that for these rational non-integer values of n and k the even spin $\mathcal{W}_\infty$ develops an ideal and the associated simple quotient is isomorphic to (2.17) with n and k positive integers that we started with.

Symplectic quotients. We can use the duality between the orthogonal and the symplectic algebras to study the corresponding symplectic quotients. In general, the Grassmannian cosets of the type²

$$\frac{\mathfrak{so}(n)_k \times \mathfrak{so}(n)_l}{\mathfrak{so}(n)_{k+l}} \simeq \frac{\mathfrak{so}(k+l)_n}{\mathfrak{so}(k)_n \times \mathfrak{so}(l)_n} \quad (2.29)$$

with central charge

$$\frac{kl n(n-1)(2n+k+l-4)}{(n+k-2)(n+l-2)(n+k+l-2)} \quad (2.30)$$

have a triality symmetry acting by permuting parameters $k_1 = k, k_2 = l, k_3 = 4 - 2n - k - l$. This is analogous to the situation in unitary cosets [31]. The unitary cosets have an additional $\mathbb{Z}_2$ symmetry which changes signs of all k_j parameters. In the case of orthogonal cosets this $\mathbb{Z}_2$ symmetry instead maps the orthogonal cosets to the symplectic ones,³ i.e.

$$\frac{\mathfrak{sp}(2n)_{\frac{k}{2}} \times \mathfrak{sp}(2n)_{\frac{l}{2}}}{\mathfrak{sp}(2n)_{\frac{k+l}{2}}} \simeq \frac{\mathfrak{so}(-2n)_{-k} \times \mathfrak{so}(-2n)_{-l}}{\mathfrak{so}(-2n)_{-k-l}}. \quad (2.31)$$

Note that if we want to get a symplectic quotient of positive rank, we would have to start with an orthogonal coset of negative rank. Again, one should interpret these isomorphisms

²The equivalence of the simple quotients is the statement of the level-rank duality [29, 30].

³The reason that we have half-integer levels is a consequence of the usual convention for normalization of Killing form such that the length squared of long roots is 2. In the C_n case that we are considering this leads to dual Coxeter number $n+1$ which is half of we would get if we worked in more symmetric conventions where the length squared of short roots in C_n would be 2.

at the level of the even spin $\mathcal{W}_\infty$ algebra (which is defined for arbitrary values of parameters) and of its truncations.

The symplectic analogue of the cosets (2.17) are therefore the cosets

$$\frac{\mathfrak{sp}(2n)_k \times \mathfrak{sp}(2n)_{-\frac{1}{2}}}{\mathfrak{sp}(2n)_{k-\frac{1}{2}}} \quad (2.32)$$

with the central charge

$$- \frac{kn(4n+2k+3)}{(n+k+1)(2n+2k+1)}. \quad (2.33)$$

The dimensions of the minimal representations are

$$h_1 = -n, \quad h_2 = \frac{4n+2k+3}{4(n+k+1)}, \quad h_3 = \frac{k}{2n+2k+1}. \quad (2.34)$$

The simplest level $-\frac{1}{2}$ representation may be realized as a singlet part of VOA of $2n$ free symplectic bosons with OPE

$$\xi_j(z)\xi_k(w) \sim \frac{\omega_{jk}}{z-w} \quad (2.35)$$

where ω_{jk} is a symplectic form. The structure constants of this symplectic quotient algebra exactly agree with those of $2n$ free fermions (i.e. level 1 even orthonormal coset) if we change the sign of n everywhere.

2.4 Drinfeld-Sokolov reductions

Let us summarize the central charges and dimensions of minimal representations in principal Drinfeld-Sokolov reduction of B_n , C_n and D_n type algebras following [16, 28]. The first main formula that we will use is the expression for the central charge

$$c = \ell - 12|\alpha_+\rho + \alpha_-\rho^\vee|^2. \quad (2.36)$$

Here ℓ is the rank of the Lie algebra, ρ is the Weyl vector and $\rho^\vee$ the dual Weyl vector. The parameters $\alpha_\pm$ are defined as

$$\alpha_+ = \frac{1}{\sqrt{k+h^\vee}}, \quad \alpha_- = -\sqrt{k+h^\vee} \quad (2.37)$$

where k is the level of the affine Lie algebra entering the Drinfeld-Sokolov reduction and $h^\vee$ is the dual Coxeter number. The second useful formula is the formula for the dimension of maximally degenerate representation parametrized by the pair of highest (co)weights (Λ^+, Λ^-) where Λ^+ is integral dominant weight and Λ^- integral dominant co-weight:

$$h = \frac{1}{2} \langle \alpha_+ \Lambda_+ + \alpha_- \Lambda_-, \alpha_+ (\Lambda_+ + 2\rho) + \alpha_- (\Lambda_- + 2\rho^\vee) \rangle. \quad (2.38)$$

Odd orthogonal case — $\mathfrak{so}(2n_B + 1)$. Turning now to Lie algebra B_{n_B} , from (2.36) we find the central charge

$$c_B = -\frac{n_B(4n_B^2 + 2n_B k_B + 2k_B - 3)(4n_B^2 + 2n_B k_B - 2n_B + k_B)}{2n_B + k_B - 1} \quad (2.39)$$

and from (2.38) the minimal conformal dimension

$$h_{B2} = -\frac{n_B(2n_B + k_B - 2)}{2n_B + k_B - 1} \quad (2.40)$$

(corresponding to $\Lambda_+ = \omega_1$) and

$$h_{B1} = \frac{1}{2} (4n_B^2 + 2n_B k_B - 2n_B + k_B). \quad (2.41)$$

(corresponding to $\Lambda_- = \omega_1^\vee$ which here agrees with ω_1). These two values are compatible with third minimal dimension

$$h_{B3} = \frac{4n_B^2 + 2n_B k_B + 2k_B - 3}{2(2n_B + k_B - 2)}. \quad (2.42)$$

The shifted level ψ_B is

$$\psi_B = 2n_B - 1 + k_B \quad (2.43)$$

and the scale parameter η_B is

$$\eta_B = 2n_B \psi_B - 2n_B + \psi_B. \quad (2.44)$$

Symplectic case — $\mathfrak{sp}(2n_C)$. For Lie algebra C_{n_C} we find the central charge

$$c_C = -\frac{n_C(2n_C^2 + 2n_C k_C + 2n_C + k_C)(4n_C^2 + 4n_C k_C - 2k_C - 3)}{n_C + k_C + 1} \quad (2.45)$$

and dimensions of minimal representations

$$h_{C2} = -\frac{4n_C^2 + 4n_C k_C - 2k_C - 3}{4(n_C + k_C + 1)} \quad (2.46)$$

(corresponding to $\Lambda_+ = \omega_1$) and

$$h_{C1} = n_C(2n_C + 2k_C + 1). \quad (2.47)$$

(corresponding to $\Lambda_- = \omega_1^\vee$ which is twice as long as ω_1). The third minimal dimension compatible with these is

$$h_{C3} = \frac{2n_C^2 + 2n_C k_C + 2n_C + k_C}{2n_C + 2k_C + 1}. \quad (2.48)$$

The shifted level ψ_C is

$$\psi_C = 2n_C + 2 + 2k_C \quad (2.49)$$

and the scale parameter η_C

$$\eta_C = 2n_C \psi_C - 2n_C - 1. \quad (2.50)$$

Even orthogonal case — $\mathfrak{so}(2n_D)$. In the case of D_{n_D} the calculation is slightly simpler because the Lie algebra is simply laced. We find

$$c_D = -\frac{n_D(4n_D^2 + 2n_D k_D - 10n_D - 2k_D + 5)(4n_D^2 + 2n_D k_D - 8n_D - k_D + 4)}{2n_D + k_D - 2} \quad (2.51)$$

and the minimal dimensions are

$$h_{D2} = -\frac{4n_D^2 + 2n_D k_D - 10n_D - 2k_D + 5}{4n_D + 2k_D - 4} \quad (2.52)$$

(for $\Lambda_+ = \omega_1$) and

$$h_{D1} = \frac{1}{2} (4n_D^2 + 2n_D k_D - 8n_D - k_D + 4) \quad (2.53)$$

(for $\Lambda_- = \omega_1$ since now the weights and co-weights agree). The third minimal dimension compatible with these is simply

$$h_{D3} = n_D \quad (2.54)$$

(just like in the case of orthogonal coset, this is compatible with assumption that the Pfaffian generator transforms in the minimal representation of the algebra). Finally we define the shifted level ψ_D to be

$$\psi_D = 2n_D - 2 + k_D \quad (2.55)$$

and the parameter η_D is

$$\eta_D = 2n_D \psi_D - 2n_D - \psi_D + 1 = (2n_D - 1)(\psi_D - 1). \quad (2.56)$$

Note on Pfaffian generator. Let us briefly discuss the Pfaffian generator of dimension n_D . There is no corresponding field in even spin $\mathcal{W}_\infty$, although we have just seen that one of the minimal primaries has exactly the correct conformal dimension. The reason why this field is not present in even spin $\mathcal{W}_\infty$ is that it is unstable as we vary n . In this sense the Drinfel'd-Sokolov $\mathcal{W}$ -algebra of type $\mathcal{WD}$ is not a truncation of even spin $\mathcal{W}_\infty$, only its $\mathbb{Z}_2$ projection which removes the Pfaffian generator is [17, 32, 33]. On the other hand, we will see in the next section when we discuss the Miura transformation that the Pfaffian generator can be naturally embedded into $\mathfrak{u}(1) \times \mathcal{W}_N$ truncation of $\mathcal{W}_{1+\infty}$ and actually this operator plays a crucial role in construction of the embedding of even spin $\mathcal{W}_\infty$ into $\mathcal{W}_{1+\infty}$.

2.5 Gaiotto-Rapčák

In [7] the authors found an interesting realization of $\mathcal{W}$ -algebras in gauge theory setting. The theory they considered was four-dimensional twisted $\mathcal{N} = 4$ super Yang-Mills theory with three semi-infinite co-dimension one defects meeting at co-dimension two subspace. The degrees of freedom living at this co-dimension 2 subspace were found to be organized by a certain truncation of $\mathcal{W}_{1+\infty}$ algebra determined by the ranks of the gauge groups in three subsectors of the full four-dimensional space cut out by the co-dimension 1 defects [7, 24]. This setup can be modified by introducing an orientifold plane. The unitary gauge groups are then projected to orthosymplectic groups and one expects the degrees of freedom at

Figure 1. Gaiotto-Rapčák orthosymplectic Y -algebras.

co-dimension 2 subspace to be reduced to even spin $\mathcal{W}_\infty$. Here we verify that the central charge formula derived in [7] is compatible with the form of the central charge in even spin $\mathcal{W}_\infty$ and later that the orthosymplectic Y -algebras can be identified with the truncations of even spin $\mathcal{W}_\infty$.

As discussed in [7] there are actually four different ways how to introduce an orientifold plane in the theory leading to four different families of Y -algebras. They are shown in figure 1. Although we expect that the orthosymplectic algebras constructed in [7] should be truncations of even spin $\mathcal{W}_\infty$, to identify the parameters one would need to know the central charge and one of the structure constants. Unfortunately only the central charge was calculated in [7]. On the other hand, the orthosymplectic Y -algebras transform nicely under triality transformations and the Kapustin-Witten parameter Ψ has exactly the properties of the parameter ψ introduced in (2.14) so one can try to identify this Ψ with ψ . The fact that we find rational expressions for the minimal dimensions and the compatibility of various truncations and restrictions are a non-trivial verification of the correctness of the proposed identification.

Algebra Y_{N_1,N_2,N_3}^- . Starting with the first algebra of the figure 1, Y^- , the central charge is given by (2.15) with

$$\eta^- = 1 + 2(N_1 - N_3) - (1 + 2(N_2 - N_3))\psi. \quad (2.57)$$

From this we can immediately find the three μ parameters of the algebra using (2.14). The next algebra is $\tilde{Y}^-$. The central charge calculated in [7] is of the form (2.15) with

$$\tilde{\eta}^- = 2(N_1 - N_3) + (1 - 2(N_2 - N_3))\psi. \quad (2.58)$$

The third algebra, Y^+ has parameter η equal to

$$\eta^+ = -1 + 2(N_1 - N_3) - 2(N_2 - N_3)\psi. \quad (2.59)$$

The last algebra of figure 1 is $\tilde{Y}^+$ with η parameter equal to

$$\tilde{\eta}^+ = 2(N_1 - N_3) - 2(N_2 - N_3)\psi. \quad (2.60)$$

In all four cases we get a nice polynomial expression for η which has the same structure as in the case of the cosets and the Drinfeld-Sokolov reductions. Let's summarize some of the properties of these algebras:

1. The parameters of even spin $\mathcal{W}_\infty$ don't change if we shift all three N_j parameters at the same time by a constant. This is analogous to what happens in $\mathcal{W}_\infty$ and is a consequence of (2.12). This doesn't mean though that the Y_{N_1, N_2, N_3} algebras are the same: only their simple quotient is expected to be the same. In the case of $\mathcal{W}_\infty$ this is discussed in [24] and in particular in [34] in connection with free field representations — Gaiotto-Rapčák algebras are not simple when all three N_j parameters are non-zero.
2. The transformation $\psi \leftrightarrow 1 - \psi$ in parametrization (2.14) exchanges $\mu_1 \leftrightarrow \mu_3$ and transforms η only by its action on ψ . The effect on orthosymplectic Y -algebras is

$$\begin{aligned} Y^-(N_1, N_2, N_3) &\leftrightarrow \tilde{Y}^-(N_1, N_3, N_2), & Y^+(N_1, N_2, N_3) &\leftrightarrow Y^+(N_1, N_3, N_2) \\ \tilde{Y}^+(N_1, N_2, N_3) &\leftrightarrow \tilde{Y}^+(N_1, N_3, N_2) \end{aligned} \quad (2.61)$$

which is exactly the claim of [7]. Note that pictorially it exchanges the upper right and lower right gauge groups in figure 1.

3. To see the effect of the transformation $\psi \rightarrow \frac{1}{\psi}$ on Y -algebras it's better to work directly with μ_j parameters. We find

$$\begin{aligned} Y^-(N_1, N_2, N_3) &\leftrightarrow Y^-(N_2, N_1, N_3), & \tilde{Y}^-(N_1, N_2, N_3) &\leftrightarrow Y^+(N_2, N_1, N_3) \\ \tilde{Y}^+(N_1, N_2, N_3) &\leftrightarrow \tilde{Y}^+(N_2, N_1, N_3) \end{aligned} \quad (2.62)$$

again in agreement with [7].

4. The third operation of exchanging two gauge groups corresponds to $\psi \rightarrow \frac{\psi}{\psi-1}$. The action on Y -algebras is

$$\begin{aligned} Y^-(N_1, N_2, N_3) &\leftrightarrow Y^+(N_3, N_2, N_1), & \tilde{Y}^-(N_1, N_2, N_3) &\leftrightarrow \tilde{Y}^-(N_3, N_2, N_1) \\ \tilde{Y}^+(N_1, N_2, N_3) &\leftrightarrow \tilde{Y}^+(N_3, N_2, N_1). \end{aligned} \quad (2.63)$$

The fact that Y^- and Y^+ exchange their roles is again manifest in figure 1.

5. At the level of the parameters of the universal algebra, all four orthosymplectic Y -algebras are connected formally by half-integer shifts of rank parameters: apart from the shift

$$\tilde{\eta}^+(N_1, N_2, N_3) = \eta^+(N_1 + \frac{1}{2}, N_2, N_3) \quad (2.64)$$

used already in [7] and its generalization

$$\tilde{\eta}^-(N_1, N_2, N_3) = \eta^-(N_1, N_2 - \frac{1}{2}, N_3 + \frac{1}{2}) \quad (2.65)$$

which also involves the transformation $\psi \rightarrow 1 - \psi$ we have also formally

$$\begin{aligned} \tilde{\eta}^-(N_1, N_2, N_3) &= \eta^-(N_1 - \frac{1}{2}, N_2 - 1, N_3) \\ \eta^+(N_1, N_2, N_3) &= \eta^-(N_1 - 1, N_2 - \frac{1}{2}, N_3) \\ \tilde{\eta}^+(N_1, N_2, N_3) &= \eta^-(N_1 - \frac{1}{2}, N_2 - \frac{1}{2}, N_3). \end{aligned} \quad (2.66)$$

This shows that working in the universal even spin algebra, all the orthosymplectic Y -algebras are connected by analytic continuation of their parameters.

3 Miura transformation and quadratic basis

In this section we use the results of [19] to find a free field representation of even spin $\mathcal{W}_\infty$ and embed it in $\mathcal{W}_{1+\infty}$. First of all, recall that given N free fields with currents satisfying OPE

$$J_j(z)J_k(w) \sim \frac{\delta_{jk}}{(z-w)^2} \quad (3.1)$$

we can construct the Miura operator

$$(\alpha_0 \partial + J_1(z)) \cdots (\alpha_0 \partial + J_N(z)) = \sum_{k=0}^N U_k(z) (\alpha_0 \partial)^{N-k}. \quad (3.2)$$

The currents $U_k(z)$ defined in this way represent algebra $\widehat{u(1)} \times \mathcal{W}_N$ [35, 36] and moreover the operator product expansions in this basis [18, 36] are quadratic.

An important observation of [19] is that the fields appearing in the OPE of the generating field of the highest spin W_N in $\widehat{u(1)} \times \mathcal{W}_N$ generate an even spin subalgebra. Following [28], we can define fields $V_j(z)$ by

$$U_N(z)U_N(w) = \frac{a(N-1)}{(z-w)^{2N}} + \sum_{k=1}^{N-1} \frac{(-1)^k a(N-1-k)}{(z-w)^{2N-2k}} [V_{2k}(z) + V_{2k}(w)] \quad (3.3)$$

where we choose the normalization factors as

$$a(j) = \prod_{r=1}^j (1 - (2j)(2j+1)\alpha_0^2). \quad (3.4)$$

These are not so easy to calculate explicitly at larger values of N , because even if we are interested in fields V_{2j} with j small, we still need to know the OPE of U_N with itself. Fortunately, we can use the result that the OPE can be written in the form [18]

$$U_N(z)U_N(w) = \sum_{l+m \leq 2N} C_{NN}^{lm}(\alpha_0, N) \frac{U_{lm}(z, w)}{(z-w)^{2N-l-m}} \quad (3.5)$$

where $U_{lm}(z, w)$ are certain bi-local fields of the form $(U_l U_m)(w) + \text{derivatives}$ and can be explicitly written in terms of fields $U_j(z)U_k(w)$ with $j + k \leq l + m$. More concretely they are equal to

$$U_{lm}(z, w) = \sum_{j+k \leq l+m} \frac{D_{lm}^{jk} U_j(z) U_k(w)}{(z-w)^{l+m-j-k}} \quad (3.6)$$

and the matrix of constants D_{lm}^{jk} is the inverse of the matrix of structure constants C_{lm}^{jk} (considering (j, k) and (l, m) as bi-indices as explained in [18]). The structure constants for OPE of U_N with itself in our situation simplify to

$$C_{NN}^{jk}(\alpha_0, N) = (-1)^{\frac{j-k}{2}} \prod_{r=1}^{\frac{2N-j-k-2}{2}} (1 - 2r(2r+1)\alpha_0^2) = (-1)^{\frac{j-k}{2}} a \left(\frac{2N-j-k-2}{2} \right) \quad (3.7)$$

for $j + k$ even and to

$$\begin{aligned} C_{NN}^{jk}(\alpha_0, N) &= (-1)^{\frac{j-k-1}{2}} (2n-j-k-1)\alpha_0 \prod_{r=1}^{\frac{2N-j-k-3}{2}} (1 - 2r(2r+1)\alpha_0^2) \\ &= (-1)^{\frac{j-k-1}{2}} (2n-j-k-1)\alpha_0 a \left(\frac{2N-j-k-3}{2} \right) \end{aligned} \quad (3.8)$$

for $j + k$ odd. In both cases, these depend only on the sum $j + k$ (except for an overall sign). Now the problem of extracting lower spin fields V_{2j} at larger values of N is solved, because we can use the expression (3.5) to directly extract V_{2s} fields, a calculation which involves knowledge of OPE of fields of spin $\leq 2s$ only.

Let us write a formula that we can use to extract the generators of even spin $\mathcal{W}_\infty$ in terms of those of $\mathcal{W}_{1+\infty}$. For that, we Taylor expand (3.3) at $z = w$ obtaining an ordinary OPE. The coefficient of pole of order $2N - 2s$ is

$$2(-1)^s a(N-1-s)V_{2s}(w) + \sum_{r=1}^{s-1} \frac{(-1)^r a(N-1-r)}{(2(s-r))!} V_{2r}^{(2s-2r)}(w). \quad (3.9)$$

On the other hand, the coefficient of the same pole in the expansion of the form (3.5) is equal to⁴

$$\sum_{l+m \leq j+k \leq 2s} C_{NN}^{jk} D_{jk}^{lm} (U_l, U_m)_{l+m-2s} \quad (3.10)$$

which is a convenient expression involving only OPE of fields with spins $\leq 2s$. Here $(U_l, U_m)_{l+m-2s}$ is the operator appearing as the coefficient of the pole of order $l+m-2s$ in OPE of $U_l(z)$ and $U_m(w)$. In the notation of `OPEdefs` this is `OPEPole`[$l+m-2s$][U_l, U_m].

Equating these last two expressions, we can recurrently calculate expressions for V_{2s} fields in terms of U_j fields. The first few fields are given in the next section. It is a non-trivial check of our calculations that the OPEs of V_j fields close as we verified up to sum of spins 20.

⁴Note that the multiplication of the matrix components with the components of the inverse matrix does not give the identity matrix because of the restriction on the range of the indices.

3.1 Map between parameters of even spin $\mathcal{W}_\infty$ and $\mathcal{W}_{1+\infty}$

Using the OPE of the Pfaffian field as described in the previous section we can extract first two V_j fields:

$$\begin{aligned}
 V_2 &= U_2 - \frac{1}{2}(U_1 U_1) - (N-1)\alpha_0 U_1' \\
 V_4 &= U_4 - (U_1 U_3) + \frac{1}{2}(U_2 U_2) - (N-2)\alpha_0 U_3' + (N-2)\alpha_0 (U_1 U_2') - (N-2)\alpha_0 (U_1' U_2) \\
 &\quad + \frac{1}{4}U_2'' - \frac{N-1}{4}(U_1'' U_1) + \frac{4N^2\alpha_0^2 - 14N\alpha_0^2 + 12\alpha_0^2 - 1}{4}(U_1' U_1') \\
 &\quad + \frac{(N-1)\alpha_0(4N^2\alpha_0^2 - 14N\alpha_0^2 + 12\alpha_0^2 - N - 1)}{12}U_1'''.
 \end{aligned} \tag{3.11}$$

The third field, V_6 , is given in the appendix. Since V_2 and V_4 generate the even spin $\mathcal{W}_\infty$ subalgebra [17, 19], we can identify the parameters of even spin $\mathcal{W}_\infty$ in terms of those of $\mathcal{W}_{1+\infty}$ (N and α_0). We need to find the stress-energy tensor which is simply $-V_2$ and the primary combination of spin 4 fields and spin 6 fields to extract the central charge and the parameter x . The result is

$$c = N(1 - 2(N-1)(2N-1)\alpha_0^2) \tag{3.12}$$

and

$$\begin{aligned}
 x &= \left[(N-4)(-6\alpha_0^2 + 4\alpha_0^2 N^2 + 10\alpha_0^2 N - 49)(28\alpha_0^2 N^3 - 42\alpha_0^2 N^2 + 14\alpha_0^2 N - 7N - 68) \right. \\
 &\quad \times (4\alpha_0^2 N^2 - 4\alpha_0^2 N - 1) \Big] / \left[6(4\alpha_0^2 N^3 - 6\alpha_0^2 N^2 + 2\alpha_0^2 N - N - 24) \right. \\
 &\quad \times (12\alpha_0^2 + 16\alpha_0^4 N^5 - 40\alpha_0^4 N^4 - 40\alpha_0^4 N^3 - 80\alpha_0^2 N^3 + 100\alpha_0^4 N^2 \\
 &\quad \left. + 302\alpha_0^2 N^2 - 36\alpha_0^4 N - 204\alpha_0^2 N + 19N - 34) \right]
 \end{aligned} \tag{3.13}$$

Expressing N and α_0^2 in terms of parameters λ of $\mathcal{W}_\infty$ defined by [18]

$$c_\infty = (\lambda_1 - 1)(\lambda_2 - 1)(\lambda_3 - 1), \quad \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0 \quad \lambda_3 = N \tag{3.14}$$

we can write even spin $\mathcal{W}_\infty$ parameters as

$$\begin{aligned}
 h_1 &= 1 + \lambda_1 + \frac{\lambda_1}{2\lambda_2} = \frac{1 + \mu_1}{2} \\
 h_2 &= 1 + \lambda_2 + \frac{\lambda_1}{2\lambda_2} = \frac{1 + \mu_2}{2} \\
 h_3 &= \lambda_3 = \frac{1 + \mu_3}{2}.
 \end{aligned} \tag{3.15}$$

In terms of parameters μ we have

$$\begin{aligned}
 \mu_1 &= \frac{\lambda_1 + \lambda_2 + 2\lambda_1\lambda_2}{\lambda_2} \\
 \mu_2 &= \frac{\lambda_1 + \lambda_2 + 2\lambda_1\lambda_2}{\lambda_1} \\
 \mu_3 &= -\frac{\lambda_1 + \lambda_2 + 2\lambda_1\lambda_2}{\lambda_1 + \lambda_2}.
 \end{aligned} \tag{3.16}$$

For reference, the map between parameters (N, α_0) and μ_j is

$$N = \frac{\mu_1 + \mu_2 - \mu_1\mu_2}{2(\mu_1 + \mu_2)} = \frac{\mu_3}{2} \left(1 - \frac{1}{\mu_1} - \frac{1}{\mu_2} \right) \quad (3.17)$$

$$\alpha_0^2 = -\frac{(\mu_1 + \mu_2)^2}{\mu_1\mu_2} = -\frac{\mu_1\mu_2}{\mu_3^2} = \frac{(1-\psi)^2}{\psi}. \quad (3.18)$$

We see that the embedding of even spin $\mathcal{W}_\infty$ in $\mathcal{W}_{1+\infty}$ breaks the triality symmetry of $\mathcal{W}_{1+\infty}$ to a $\mathbb{Z}_2$ exchanging $\lambda_1 \leftrightarrow \lambda_2$ or $\mu_1 \leftrightarrow \mu_2$. This is related to the fact that the Miura transformation depends on a choice of a preferred direction. So although both algebras have a triality symmetry, the triality in $\mathcal{W}_{1+\infty}$ does not restrict to the triality in even spin $\mathcal{W}_\infty$. The choice of even spin $\mathcal{W}_\infty$ subalgebra in $\mathcal{W}_{1+\infty}$ breaks the triality symmetry to $\mathbb{Z}_2$ but when restricted to this subalgebra, the duality is enhanced to a triality of the even spin subalgebra. This is analogous to enhancement of duality to triality in unitary Grassmannian cosets when one of the levels is one. We also see that there are at least six ways of embedding even spin $\mathcal{W}_\infty$ in $\mathcal{W}_{1+\infty}$, each associated to different asymptotic direction (times two because of the complex conjugation in $\mathcal{W}_\infty$).

3.2 Operator product expansions in V_j basis

As a result of our definition of V_j fields, they are quadratic composites of the U_j fields. Since $\mathcal{W}_{1+\infty}$ is filtered with filtration degree given by the number of U_j fields in each term and since the operator product expansions preserve the degree, we can also expect the operator product expansions of V_j field to have quadratic operator product expansions.

To find these operator product expansions, we can first calculate the OPE of V_2 with V_{2s} ,

$$\begin{aligned} V_2(z)V_{2j}(w) \sim & \frac{4N(-1)^j [2^{2j+2}-1] a(N-1)B_{2j+2}}{(2j+2)a(N-j-1)} \frac{1}{(z-w)^{2j+2}} \\ & + \sum_{k=1}^{j-1} \frac{8(N-k)(-1)^{j-k} [2^{2j-2k+2}-1] a(N-k-1)B_{2j-2k+2}}{(2j-2k+2)a(N-j-1)} \frac{V_{2k}(w)}{(z-w)^{2j-2k+2}} \\ & + \frac{(\text{derivatives})}{(z-w)^{\geq 3}} - \frac{2jV_{2j}(w)}{(z-w)^2} - \frac{\partial V_{2j}(w)}{z-w}, \end{aligned} \quad (3.19)$$

where B_n are the Benoulli numbers. The Jacobi identity ($V_2V_2V_{2j}$) fixes all the derivative terms in V_2V_{2j} OPE. The OPE of V_4 with itself is given in the appendix B. With this input (actually only the coefficient of the identity and of V_4 in V_4V_4 OPE is necessary) the Jacobi identities determine all the other operator product expansions. The resulting OPEs have the following properties

1. the operator product expansions are purely quadratic, i.e. all the operators appearing in the singular part of the OPE are normal ordered products of (at most) two V_j fields and their derivatives. This is analogous to the case of $\mathcal{W}_{1+\infty}$ [18, 36].
2. All the structure constants are polynomial functions of N and α_0 . This is again analogous to [18, 23].

3. Unlike in $\mathcal{W}_{1+\infty}$, the derivatives do not seem to be simply summable into bi-local fields (this seems to be the case even after a simple linear redefinition of the fields). This is probably related to different form of the Miura operator which is ‘folded’. As a consequence of this, the calculation of commutation relations between mode operators is more involved because one needs to consider the terms with derivatives.
4. We verified these claims for OPE of fields V_j and V_k with $j + k \leq 20$. At every step the determination of the OPE reduces to solution of a linear system of equations.

For later purposes, it is useful to determine at each even spin the primary field W_{2j} whose pole of order $4j$ with all dimension $2j$ fields not involving V_{2j} vanishes, i.e. field which is orthogonal to all lower dimension fields and their derivatives and composites. Actually we don’t even need to require this field to be primary, the field orthogonal to all composites of lower dimension fields is automatically primary. A useful property of this field is that it is the field whose two-point function vanishes for truncations of the algebra. We can thus avoid searching for zeros of Kac determinant to find the truncations of the algebra. It is enough to identify these primaries and find zeros of their two point functions. We choose the normalization such that $W_{2j} = V_{2j} + \dots$. With this choice, the two-point functions are

$$\begin{aligned}
 \langle W_2 W_2 \rangle &= -\frac{1}{2}n(2\alpha_0^2 + 4\alpha_0^2 n^2 - 6\alpha_0^2 n - 1) \\
 \langle W_4 W_4 \rangle &= -\frac{1}{2(20\alpha_0^2 n^3 - 30\alpha_0^2 n^2 + 10\alpha_0^2 n - 5n - 22)} \times n(2n-1)(\alpha_0^2 n^2 + \alpha_0^2 n - 1) \\
 &\quad \times (12\alpha_0^2 + 4\alpha_0^2 n^2 - 14\alpha_0^2 n - 1)(2\alpha_0^2 + 4\alpha_0^2 n^2 - 6\alpha_0^2 n - 1)(-2\alpha_0^2 + 4\alpha_0^2 n^2 - 2\alpha_0^2 n - 9) \\
 \langle W_6 W_6 \rangle &\sim (n-1)n(2n-1)(\alpha_0^2 n^2 + \alpha_0^2 n - 1)(\alpha_0^2 n^2 + 2\alpha_0^2 n - 4)(30\alpha_0^2 + 4\alpha_0^2 n^2 - 22\alpha_0^2 n - 1) \\
 &\quad \times (12\alpha_0^2 + 4\alpha_0^2 n^2 - 14\alpha_0^2 n - 1)(2\alpha_0^2 + 4\alpha_0^2 n^2 - 6\alpha_0^2 n - 1)(4\alpha_0^2 n^2 - 2\alpha_0^2 n - 1) \\
 &\quad \times (-2\alpha_0^2 + 4\alpha_0^2 n^2 - 2\alpha_0^2 n - 9)(-6\alpha_0^2 + 4\alpha_0^2 n^2 + 2\alpha_0^2 n - 25) \\
 \langle W_8 W_8 \rangle &\sim (n-1)n(n+1)(2n-3)(2n-1)(n^2\alpha_0^2 + n\alpha_0^2 - 1)(n^2\alpha_0^2 + 2n\alpha_0^2 - 4) \\
 &\quad \times (n^2\alpha_0^2 + 3n\alpha_0^2 - 9)(4n^2\alpha_0^2 - \alpha_0^2 - 4)(4n^2\alpha_0^2 - 30n\alpha_0^2 + 56\alpha_0^2 - 1) \\
 &\quad \times (4n^2\alpha_0^2 - 22n\alpha_0^2 + 30\alpha_0^2 - 1)(4n^2\alpha_0^2 - 14n\alpha_0^2 + 12\alpha_0^2 - 1) \\
 &\quad \times (4n^2\alpha_0^2 - 10n\alpha_0^2 + 4\alpha_0^2 - 9)(4n^2\alpha_0^2 - 6n\alpha_0^2 + 2\alpha_0^2 - 1)(4n^2\alpha_0^2 - 2n\alpha_0^2 - 1) \\
 &\quad \times (4n^2\alpha_0^2 - 2n\alpha_0^2 - 2\alpha_0^2 - 9)(4n^2\alpha_0^2 + 2n\alpha_0^2 - 6\alpha_0^2 - 25)(4n^2\alpha_0^2 + 6n\alpha_0^2 - 10\alpha_0^2 - 49)
 \end{aligned} \tag{3.20}$$

where $\sim$ means that we didn’t write the denominator (because we are mainly interested in zeros of these two-point functions). One can actually find higher order two-point function by the following trick: we have

$$C_{46}^8 C_{88}^0 = C_{48}^6 C_{66}^0 \tag{3.21}$$

if the field W_8 is chosen to be orthogonal to $W_{[44]}$ which is equivalent to condition

$$C_{48}^4 = 0. \tag{3.22}$$

Similarly at the next level

$$C_{48}^{10} C_{10,10}^0 = C_{4,10}^8 C_{88}^0 \tag{3.23}$$

if we choose W_{10} to be orthogonal to $W_{[46]}$ and $W_{[44]^{(2)}}$ which means

$$C_{4,10}^4 = 0 \quad \text{and} \quad C_{4,10}^6 = 0. \quad (3.24)$$

In this way, we were able to find the zeros of Kac determinant up to level 12 which would otherwise require to knowing 24th order pole of W_{12} with itself. The fact that the numerator of $C_{10,10}^0$ and $C_{12,12}^0$ obtained in this way factorizes into factors of the form of (3.20) is a nice consistency check of this procedure.

4 Truncations

We will now collect all the results about various truncations discussed so far. All the truncation curves will have formally the same form as in $\mathcal{W}_\infty$,

$$\frac{N_1}{\mu_1} + \frac{N_2}{\mu_2} + \frac{N_3}{\mu_3} = 1 \quad (4.1)$$

with non-negative integers N_1, N_2 and N_3 . Due to redundancy in parametrization (2.12) shifting all N_j by a constant does not change the truncation curve, but we can use these triples of integers differing by a constant to describe different truncations of the algebra with the same truncation curve.

4.1 Truncations from Gaiotto-Rapčák algebras

The expressions for η for Y -algebras discussed in section 2.5 can be immediately translated into truncation curves, i.e. curves in μ -parameter space where the universal even spin $\mathcal{W}_\infty$ develops a singular vector and can be truncated to a smaller algebra. These curves have the form

$$\begin{aligned} Y^- : \quad & \frac{2N_1 + 1}{\mu_1} + \frac{2N_2 + 1}{\mu_2} + \frac{2N_3}{\mu_3} = 1 \\ \tilde{Y}^- : \quad & \frac{2N_1 + 1}{\mu_1} + \frac{2N_2}{\mu_2} + \frac{2N_3 + 1}{\mu_3} = 1 \\ Y^+ : \quad & \frac{2N_1}{\mu_1} + \frac{2N_2 + 1}{\mu_2} + \frac{2N_3 + 1}{\mu_3} = 1 \\ \tilde{Y}^+ : \quad & \frac{2N_1}{\mu_1} + \frac{2N_2}{\mu_2} + \frac{2N_3}{\mu_3} = 1 \end{aligned} \quad (4.2)$$

The pattern is clear: if the gauge group associated to face μ_j is $\text{Sp}(2N)$, the coefficient of μ_j^{-1} is $2N$, while if the gauge group is $\text{SO}(N)$, the coefficient is $N - 1$ (both for even and odd N). Whenever the parameters (μ_1, μ_2, μ_3) of the even spin $\mathcal{W}_\infty$ satisfy one of these equations for non-negative integer values of (N_1, N_2, N_3) , the algebra develops an ideal so can be truncated to a smaller subalgebra. Note that in general for a fixed truncation curve there might be various choices of this ideal corresponding to the fact that the map from triples (N_1, N_2, N_3) to truncation curves is not one-to-one (in particular an overall shift of all three ranks by a constant leads to the same truncation curve). In the case of $\mathcal{W}_\infty$ there is always a maximal ideal corresponding to a truncation where (at least) one of the integers N_j was vanishing [7, 34]. One can see the differences between the different truncations sharing the same truncation curve by studying the level at which the first singular vector appears.

4.2 Truncations from cosets and DS reductions

We can similarly translate the value of η for cosets and Drinfel'd-Sokolov reductions to truncation curves. The orthogonal cosets (2.17) have simple truncation curves

$$\frac{n-1}{\mu_1} = 1 \quad (4.3)$$

while the symplectic ones

$$\frac{2n+1}{\mu_2} + \frac{2n+1}{\mu_3} = 1. \quad (4.4)$$

The Drinfel'd-Sokolov reductions lead to curves

$$\begin{aligned} B_n : \quad & \frac{1}{\mu_1} + \frac{2n+1}{\mu_3} = 1 \\ C_n : \quad & \frac{1}{\mu_2} + \frac{2n+1}{\mu_3} = 1 \\ D_n : \quad & \frac{2n-1}{\mu_3} = 1. \end{aligned} \quad (4.5)$$

Unitary minimal models. These of course don't exhaust all the possible truncations that we may get by studying cosets and Drinfel'd-Sokolov reductions. For example cosets (2.17) can be studied at fixed non-negative integer value of level k and generic n . The central charges of these models are $c = 0$ ($k = 0$, only the vacuum state), $c = 1$ ($k = 1$), $c = \frac{4n-1}{2n+1}$ ($k = 2$), $\dots$. For n positive integer these are the unitary minimal models of the corresponding truncated algebras. For each fixed k (and generic n), the parameters of the associated even spin $\mathcal{W}_\infty$ lie on a curve

$$\frac{k}{\mu_2} + \frac{k-1}{\mu_3} = 1 \quad (4.6)$$

so we can think of this curve as cutting out the unitary minimal models in the parameter space. This is again very similar to the situation in $\mathcal{W}_{1+\infty}$ and in fact even the form of these curves is the same.

Non-unitary minimal models. The class of all minimal models is larger than the one of the unitary minimal models. Consider following [19] the minimal models of $\mathcal{W}$ -algebra associated to D_n via Drinfel'd-Sokolov reduction parametrized by coprime integers (p', p) such that the central charge is

$$c = n \left[1 - 2(n-1)(2n-1) \frac{(p'-p)^2}{p'p} \right]. \quad (4.7)$$

Choosing $p' = p + 1$ and $p = 2n - 2 + k$ we get for $k = 0, 1, \dots$ the sequence of unitary minimal models discussed in the previous paragraph. For $|p' - p| \neq 1$ we still get minimal models but no longer unitary. The level of Drinfel'd-Sokolov reduction can be chosen either

$$k_D = \frac{p' - 2(n-1)p}{p} \quad \text{or} \quad \frac{p - 2(n-1)p'}{p'}. \quad (4.8)$$

Choosing the first one, we can identify the parameters of even spin $\mathcal{W}_\infty$ as

$$\begin{aligned}\mu_1 &= \frac{p}{(2n-1)(p'-p)} \\ \mu_2 &= -\frac{p'}{(2n-1)(p'-p)} \\ \mu_3 &= \frac{1}{2n-1}.\end{aligned}\tag{4.9}$$

These lie on truncation curve

$$\frac{p'-2n+1}{\mu_1} + \frac{p-2n+1}{\mu_2} = 1.\tag{4.10}$$

Choosing $p'-p=1$ we reduce to the truncation curve of minimal models discussed in the previous paragraph (although in another triality frame).

4.3 Truncations from explicit OPE calculations

Let us summarize truncation curves that we see from the explicit calculation of the operator product expansions. This is easier to see in the quadratic basis because we have a natural normalization of fields such that the OPEs have only polynomial coefficients in this basis.

Truncation to vacuum. Just like the $c=0$ truncation of Virasoro algebra where the vacuum representation is one-dimensional, in even spin $\mathcal{W}_\infty$ for

$$\frac{1}{\mu_1} + \frac{1}{\mu_2} = 1\tag{4.11}$$

(and permutations of μ) the dimension two field V_2 is singular so the theory reduces to a single state. This happens in the zeroth unitary minimal model $k=0$ where there is just the vacuum state and the central charge vanishes.

Truncation to $W[2]$ (Virasoro). The Virasoro algebra generated by $T = -V_2$ is always a subalgebra of even spin $\mathcal{W}_\infty$. It is a quotient of even spin $\mathcal{W}_\infty$ if the dimension 4 field is singular. This happens if

$$\frac{1}{\mu_1} + \frac{3}{\mu_2} = 1\tag{4.12}$$

(or a triality equivalent equation) is satisfied. In this case the singular vector is at level 4. These truncations of even spin $\mathcal{W}_\infty$ admit free field representation in terms of only one free boson.

Truncation to $W[2,4]$. Working in the primary basis, a truncation to $\mathcal{W}$ -algebra with additional spin 4 field is a little bit irregular because our parameter x is in this case not defined. The only condition coming from associativity of the algebra is

$$\frac{C_{44}^0}{(C_{44}^4)^2} = \frac{c(2c-1)(5c+22)(7c+68)}{216(c+24)(c^2-172c+196)}\tag{4.13}$$

Translated to truncation curves, we find three curves of the form

$$\frac{1}{\mu_1} = 1 \quad (4.14)$$

(which corresponds to first unitary minimal models) and we also have an orbit of six curves of the form (these are associated to WB_2 or WC_2 truncations)

$$\frac{1}{\mu_1} + \frac{5}{\mu_2} = 1. \quad (4.15)$$

All these algebras have level 6 singular vector.

Truncation to $W[2, 4, 6]$ The bootstrap for algebras of type $W[2, 4, 6]$ is consistent if x takes one of the values

$$\frac{5(2c-1)(3c+20)(7c+68)}{6(c+24)(10c^2+47c-82)}, \quad \frac{5(c+50)(2c-1)(7c+68)}{3(c+24)(5c^2+309c-14)} \quad (4.16)$$

as well as one of two roots of the quadratic equation

$$a_2x^2 + a_1x + a_0 = 0 \quad (4.17)$$

with

$$\begin{aligned} a_2 &= 18(c+24)^2(85c^4 + 5275c^3 + 101736c^2 + 1806268c - 2633664) \\ a_1 &= 3(c+24)(7c+68)(65c^4 + 2409c^3 - 161760c^2 - 11131676c + 17536992) \\ a_0 &= 98(c+50)(2c-1)(7c+68)^2(13c+1320) \end{aligned} \quad (4.18)$$

The first solution for x corresponds to truncations

$$\frac{2}{\mu_1} = 1 \quad (4.19)$$

(and triality images of this), the second solution to

$$\frac{3}{\mu_1} + \frac{3}{\mu_2} = 1 \quad (4.20)$$

and the pair of algebraic solutions satisfying the quadratic equation for x correspond to six truncation curves of the form

$$\frac{1}{\mu_1} + \frac{7}{\mu_2} = 1 \quad (4.21)$$

(these are the WB_3 or WC_3 truncations). There are also some spurious co-dimension two specializations of parameters where the algebra truncates (for example $(c = -\frac{22}{5}, x = \frac{31}{28})$ or $(c = -\frac{68}{7}, x = 0)$ but we are interested in co-dimension 1 specializations so we don't discuss these here.

Truncation to $W[2, 4, 6, 8]$. Here the truncation curves are of the form

$$\frac{3}{\mu_1} = 1 \quad (4.22)$$

and

$$\frac{1}{\mu_1} + \frac{9}{\mu_2} = 1. \quad (4.23)$$

Truncation to $W[2, 4, 6, 8, 10]$. At level 12 there are four different types of truncations,

$$\frac{4}{\mu_1} = 1, \quad \frac{2}{\mu_1} + \frac{1}{\mu_2} = 1, \quad \frac{5}{\mu_1} + \frac{3}{\mu_2} = 1, \quad \frac{11}{\mu_1} + \frac{1}{\mu_2} = 1. \quad (4.24)$$

4.4 Summary up to level 12 and a conjecture

Let's summarize the truncations discussed so far. The following table lists all the truncations with singular vector up to level 12:

(N_1, N_2, N_3)	level of singular vector	construction of truncation
(1, 1, 0)	2	vacuum
(3, 1, 0)	4	Virasoro
(1, 0, 0)	6	first unitary minimal models
(5, 1, 0)	6	$WB_2 \simeq WC_2$
(2, 0, 0)	8	$\mathfrak{so}(3)$ coset
(7, 1, 0)	8	$WB_3 \simeq WC_3$
(3, 3, 0)	8	$\mathfrak{sp}(2)$ coset
(3, 0, 0)	10	$WD_2, \mathfrak{so}(4)$ coset
(9, 1, 0)	10	$WB_4 \simeq WC_4$
(4, 0, 0)	12	$\mathfrak{so}(5)$ coset
(2, 1, 0)	12	second unitary minimal models
(5, 3, 0)	12	
(11, 1, 0)	12	$WB_5 \simeq WC_5$

To write a general conjecture for a level of a given truncation we need to distinguish three cases depending on the parity of the parameters N_j in (4.1):

1. For the truncation curves of the form

$$\frac{2N_1 + 1}{\mu_1} + \frac{2N_2 + 1}{\mu_2} = 1 \quad (4.25)$$

the truncation has first singular vector at level

$$\frac{1}{2}(2N_1 + 2) \times (2N_2 + 2) \times 1. \quad (4.26)$$

In Gaiotto-Rapčák picture this corresponds to one of algebras $Y_{N_1, N_2, 0}^-$, $Y_{N_2, N_1, 0}^-$, $\tilde{Y}_{N_1, 0, N_2}^-$, $\tilde{Y}_{N_2, 0, N_1}^-$, Y_{0, N_1, N_2}^+ or Y_{0, N_2, N_1}^+ . In each of these cases we have gauge groups $\text{Sp}(2N_1)$ or $\text{SO}(2N_1 + 1)$ and $\text{Sp}(2N_2)$ or $\text{SO}(N_2 + 1)$. The third gauge group is formally $\text{SO}(0)$.

2. Second type of truncation curves are those of the form

$$\frac{2N_1 + 1}{\mu_1} + \frac{2N_2}{\mu_2} = 1. \quad (4.27)$$

These truncations have their first singular vector at level

$$\frac{1}{2}(2N_1 + 3) \times (2N_2 + 2) \times 2 \quad (4.28)$$

and the Gaiotto-Rapčák algebras are now Y_{0,N_2,N_1+1}^- , $Y_{N_2,0,N_1+1}^-$, $\tilde{Y}_{0,N_1+1,N_2}^-$, $\tilde{Y}_{N_2,N_1+1,0}^-$, $Y_{N_1+1,0,N_2}^+$ or $Y_{N_1+1,N_2,0}^+$. The associated gauge groups are $\text{SO}(2N_1+2)$, either $\text{Sp}(2N_2)$ or $\text{SO}(2N_2+1)$ and formally $\text{Sp}(0)$.

3. The last type of truncation curves are those of the form

$$\frac{2N_1}{\mu_1} + \frac{2N_2}{\mu_2} = 1. \quad (4.29)$$

The Y -algebras are of the form $\tilde{Y}_{N_1,N_2,0}^+$ and permutations and the gauge groups are either $\text{Sp}(2N_1)$ or $\text{SO}(2N_1+1)$ and either $\text{Sp}(2N_2)$ or $\text{SO}(2N_2+1)$. The third gauge group is formally $\text{Sp}(0)$. The level of such truncations is

$$\frac{1}{2}(2N_1+2) \times (2N_2+2) \times 2 \quad (4.30)$$

The level of the truncation can be written uniformly as

$$\frac{1}{2}\rho(G_1) \times \rho(G_2) \times \rho(G_3) \quad (4.31)$$

where $\rho(G)$ is a factor associated to each gauge group,

$$\rho(G) = \begin{cases} 2n+2, & \text{Sp}(2n) \\ 2n+2, & \text{SO}(2n+1) \\ 2n+1, & \text{SO}(2n) \end{cases} \quad (4.32)$$

or in other words twice the (Cartan) rank plus the lacity (1 for simply laced D_n and 2 for doubly laced algebras B_n and C_n).

We explicitly verified the existence these truncation curves only by studying the first appearance of the singular vector in the universal even spin algebra. In this way we can only see the truncations where one of the N_j parameters vanishes. This corresponds to simple quotients of the algebra. The class of Y -algebras introduced in [7] however includes also algebras which are not simple. These are still interesting for example when one considers the gluing construction [24] because a simple algebra can contain a subalgebra which is not simple. In the unitary case the free field representations of these non-simple quotients were found in [34]. Since in the unitary case which is better understood and also in all examples discussed here the level of the first singular vector follows a simple uniform factorized formula (4.31) where the individual gauge groups don't interact and which makes good sense even if all parameters parametrizing the truncation curve are non-zero, we conjecture that this correctly describes the truncations of even spin $\mathcal{W}_\infty$ to Y -algebras in the non-simple situation as well.

Comparison of truncation curves of even spin $\mathcal{W}_\infty$ and $\mathcal{W}_\infty$. In general each truncation curve

$$\frac{N_1}{\mu_1} + \frac{N_2}{\mu_2} + \frac{N_3}{\mu_3} = 1 \quad (4.33)$$

Figure 2. Part of gluing diagram for $\mathfrak{u}(N)_k$.

in even spin $\mathcal{W}_\infty$ lies on a curve

$$\frac{N_1}{2\lambda_1} + \frac{N_2}{2\lambda_2} + \frac{N_3 + 1}{2\lambda_3} = 1, \quad (4.34)$$

in the parameter space of $\mathcal{W}_\infty$. Due to factor of 2 in the denominator there are curves in the parameter space of $\mathcal{W}_\infty$ where the full algebra does not truncate but the even spin subalgebra still does. Truncations to algebras WB_n, WC_n and WD_n are examples of truncations which lie on truncation curves in $\mathcal{W}_{1+\infty}$, actually they lie on curves corresponding to truncations to W_n algebras (this is also true for exceptional algebras of E and G type where there are known embeddings in $\hat{\mathfrak{u}}(1) \times \mathcal{W}_n$ ⁵).

5 Gluing

In the last section we illustrate how the gluing procedure discussed in [24] applies to orthogonal affine Lie algebras. Let us first review the case of unitary affine Lie algebras. The gluing diagram of $\mathfrak{u}(N)_k$ is based on the decomposition

$$\mathfrak{u}(N)_k \supset \frac{\mathfrak{u}(N)_k}{\mathfrak{u}(N-1)_k} \times \frac{\mathfrak{u}(N-1)_k}{\mathfrak{u}(N-2)_k} \times \cdots \times \frac{\mathfrak{u}(2)_k}{\mathfrak{u}(1)} \times \mathfrak{u}(1). \quad (5.1)$$

Each of the factors on the right hand side is a truncation of $\mathcal{W}_{1+\infty}$. Identifying parameters as in figure 2 we can calculate the λ -parameters of the correspond $\mathcal{W}_{1+\infty}$ algebras sitting at the vertices [24]. We have

$$\lambda_3 = \frac{(N-1)\epsilon_2 + N\epsilon_3}{\epsilon_3} = \frac{N(\psi-1) - (N-1)\psi}{\psi-1}, \quad (5.2)$$

$$\lambda'_2 = \frac{(N-2)\epsilon'_2 + (N-1)\epsilon'_3}{\epsilon'_2} = -\frac{(N-1)(\psi'-1) - (N-2)\psi'}{\psi'} \quad (5.3)$$

⁵In all these cases deleting one node from the Dynkin diagram results in Dynkin diagram of A_{n-1} and using the additional $\hat{\mathfrak{u}}(1)$ current one can define the remaining screening charge [28]. We learnt this argument from T. Creutzig.

Figure 3. Part of gluing diagram for $\mathfrak{so}(2N+1)_k$.

From the (p, q) charges of the five-branes we see that $\epsilon'_1 = \epsilon_1$ and $\epsilon'_2 = \epsilon_1 + \epsilon_2$ so

$$\psi' \equiv -\frac{\epsilon'_2}{\epsilon'_1} = \psi - 1 \quad (5.4)$$

which guarantees that the dimension of the fundamental gluing fields is

$$h = h_3 + h'_2 = \frac{1 + \lambda_3}{2} + \frac{1 + \lambda'_2}{2} = 1 \quad (5.5)$$

This is exactly what we need in order to find dimension 1 fields charged under Cartan $\mathfrak{u}(1)$ currents coming from the vertices. These correspond in the language of affine Lie algebras to currents associated to positive and negative simple roots. The other generators associated to roots that are not simple correspond to line operators stretched between vertices which are not neighbouring.

Let's now consider the orthogonal Lie algebra $\mathfrak{so}(2n+1)_k$. We have a similar decomposition

$$\mathfrak{so}(2N+1)_k \supset \frac{\mathfrak{so}(2N+1)_k}{\mathfrak{so}(2N)_k} \times \frac{\mathfrak{so}(2N)_k}{\mathfrak{so}(2N-1)_k} \times \cdots \times \frac{\mathfrak{so}(3)_k}{\mathfrak{so}(2)} \times \mathfrak{so}(2). \quad (5.6)$$

The first coset on the right hand side has parameters compatible with $\tilde{Y}_{0,N,N}^-$ with the truncation curve

$$\frac{2N}{\mu_2} + \frac{2N+1}{\mu_3} = 1 \quad (5.7)$$

while the second term can be identified with $Y_{0,N-1,N}^-$ with truncation curve

$$\frac{2N-2}{\mu'_2} + \frac{2N-1}{\mu'_3} = 1. \quad (5.8)$$

The first part of the gluing diagram looks like figure 3. Let us verify that the gluing fields have compatible dimensions. The upper vertex has parameter

$$\mu_3 = \frac{\psi - 2N}{\psi - 1} \quad (5.9)$$

while the corresponding parameter of the lower vertex is

$$\mu'_2 = -\frac{1 - 2N + \psi'}{\psi'}. \quad (5.10)$$

The relative orientation of the two vertices is just like in the unitary case so we still have (5.4). Now we can calculate the conformal dimension of gluing fields and find

$$h_3 + h'_2 = 1 \quad (5.11)$$

which is exactly what we want in order to find dimension 1 currents.

Note that the way the currents appear is slightly different than in the unitary situation. In the unitary case each vertex represented a truncation of $\mathcal{W}_{1+\infty}$ algebra which by definition carried an affine $\mathfrak{u}(1)$ current. There are as many of these as is the rank of the algebra and all these currents give the Cartan subalgebra of $\mathfrak{u}(N)_k$. As already discussed the elementary gluing fields give rise to simple positive and negative roots. In the orthogonal case the truncations of even spin $\mathcal{W}_\infty$ algebras at vertices do not have any spin 1 fields so we do not find any Cartan fields in this way. On the other hand, we have an alternating sequence of Y^- and $\tilde{Y}^-$ algebras and associated to each neighbouring pair of these there is an elementary dimension 1 gluing field (which now do not appear in complex conjugate pairs because the minimal representations of even spin $\mathcal{W}_\infty$ are real). For example for rank 2 algebra the first Cartan generator can be chosen to correspond to line operator stretched from $\tilde{Y}_{022}^-$ to Y_{012}^- and the second generator to line operator between $\tilde{Y}_{011}^-$ to Y_{001}^- .

6 Discussion

The understanding of the universal orthosymplectic $\mathcal{W}_\infty$ algebra is still much more limited than that of $\mathcal{W}_{1+\infty}$. In particular

1. All the truncations that we found are associated to truncation curves of the form (4.1) and also each of these can be associated to a certain Y -algebra. We conjecture that this happens at level (4.31), but our calculations give no proof that this is what actually happens. From the from explicit coset and Drinfel'd-Sokolov reduction description of Y -algebras it should be possible to verify this.
2. The free field representations of truncations of $\mathcal{W}_{1+\infty}$ are reasonably well understood [34]. Since the even spin algebra is a subalgebra of $\mathcal{W}_{1+\infty}$, we can find many free field representations of truncations of even spin algebra from the free field representations of $\mathcal{W}_{1+\infty}$. But one should understand if there are any other representations and how are these related to truncations of the algebra, i.e. if we have a correspondence between free field representations and co-dimension 1 truncations like in the case of $\mathcal{W}_{1+\infty}$ [34].

3. The combinatorial box counting interpretation [37] of characters of even spin $\mathcal{W}_\infty$ is not known. One cannot simply restrict to subset of box configurations in $\mathcal{W}_{1+\infty}$ because the canonical Virasoro generators do not agree and the higher spin generators of $\mathcal{W}_{1+\infty}$ do not seem to preserve the even spin subalgebra.
4. No analogue of Tsybaliuk presentation of $\mathcal{W}_{1+\infty}$ as affine Yangian is known. One could try to repeat the steps of [38] to find the ladder operators in Yangian but first the folding of $GL(N)$ Miura operator should be understood. It is very reminiscent to spin chains with boundary where the boundary reflection operator is ∂ or ∂^{-1} [20]. This surely deserves a deeper study. Once this is understood one can try to apply the techniques of quantum inverse scattering method or algebraic Bethe ansatz to construct Yangian operators for even spin $\mathcal{W}_{1+\infty}$.
5. Although the algebra admits a quadratic basis, the derivatives of fields don't seem to follow the same simple pattern as in the case of $\mathcal{W}_{1+\infty}$. If one understands this, one might hope to be able to write a closed-form formulas for OPEs and commutators in even spin $\mathcal{W}_\infty$ just like those in [18, 23].
6. In $\mathcal{W}_{1+\infty}$ and its matrix extension the fusion and its associated coproduct were extremely efficient tools for construction of free field representations or representations in terms of affine Lie algebras. Also the space of co-dimension 1 truncations can be seen as a cone generated by elementary Miura transformations [23, 39]. The Miura operator immediately allows us to extract the coproduct. On the other hand, because of the folding of the Miura operator in the orthosymplectic case, it is not obvious if the orthosymplectic version of $\mathcal{W}_\infty$ admits this structure.

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A Structure constants in primary basis

Here is the list of the structure constants in the primary basis for sum of spins up to 12:

$$C_{44}^0 = \frac{c(5c+22)}{72(c+24)} (C_{44}^4)^2 - \frac{7c(c-1)(5c+22)}{72(2c-1)(7c+68)} C_{44}^4 C_{46}^6 + \frac{c(c-1)(c+24)(5c+22)}{12(2c-1)(7c+68)^2} (C_{46}^6)^2$$

$$C_{46}^4 = \frac{4(5c+22)}{9(c+24)} \frac{(C_{44}^4)^2}{C_{44}^6} - \frac{96(c^2-172c+196)}{c(2c-1)(7c+68)} \frac{C_{44}^0}{C_{44}^6}$$

$$\begin{aligned}
 C_{48}^4 = & -\frac{32(c-151)(c-1)(c+24)(5c+22)(c^2-172c+196)}{(2c-1)^2(7c+68)^3(13c+516)} \frac{(C_{46}^6)^3}{C_{44}^6 C_{46}^8} \\
 & -\frac{56(c-1)(5c+22)(c^2-172c+196)(20c^3+24807c^2+765640c-185172)}{3(c+31)(2c-1)^2(7c+68)^2(13c+516)(55c-6)} \frac{C_{44}^4 (C_{46}^6)^2}{C_{44}^6 C_{46}^8} \\
 & +\frac{4(c-1)(5c+22)(5605c^4-408494c^3-70820464c^2-1703657536c+1312613664)}{9(c+24)(c+31)(2c-1)(7c+68)(13c+516)(55c-6)} \frac{(C_{44}^4)^2 C_{46}^6}{C_{44}^6 C_{46}^8} \\
 & -\frac{5(c-1)(c+50)(5c+22)(715c^4+90933c^3+2851076c^2+21154896c+6967008)}{12(c+24)(c+31)(3c+46)(5c+3)(13c+516)(55c-6)} \frac{C_{46}^{[44]} (C_{44}^4)^2}{C_{46}^8} \\
 & +\frac{7(c-1)(5c+22)(65c^4+8637c^3+364470c^2+2897944c+36384)}{12(2c-1)(3c+46)(5c+3)(7c+68)(13c+516)} \frac{C_{46}^{[44]} C_{44}^4 C_{46}^6}{C_{46}^8} \\
 & -\frac{(c-1)(c+24)(5c+22)(65c^4+8637c^3+364470c^2+2897944c+36384)}{2(2c-1)(3c+46)(5c+3)(7c+68)^2(13c+516)} \frac{C_{46}^{[44]} (C_{46}^6)^2}{C_{46}^8} \\
 & +\frac{140(c-1)(c+50)(5c+22)(11c+656)}{27(c+24)^2(c+31)(55c-6)} \frac{(C_{44}^4)^3}{C_{44}^6 C_{46}^8} \\
 C_{48}^6 = & \frac{8(25c^3+615c^2-88272c+102332)}{3(c+24)(c+31)(55c-6)} \frac{C_{44}^4 C_{46}^6}{C_{46}^8} - \frac{4(c-151)}{(13c+516)} \frac{C_{46}^{[44]} C_{44}^4 C_{46}^6}{C_{46}^8} \\
 & +\frac{16(425c^4+15145c^3+233766c^2+6507708c-7565544)}{(c+31)(7c+68)(13c+516)(55c-6)} \frac{(C_{46}^6)^2}{C_{46}^8} \\
 & +\frac{7840(c+50)(2c-1)(7c+68)}{9(c+24)^2(c+31)(55c-6)} \frac{(C_{44}^4)^2}{C_{46}^8} - \frac{35(c+50)(2c-1)(7c+68)}{(3c+24)(c+31)(55c-6)} \frac{C_{46}^{[44]} C_{44}^4 C_{46}^4}{C_{46}^8} \\
 C_{48}^8 = & \frac{8(33c^2+1087c+11760)}{(7c+68)(13c+516)} C_{46}^6 - \frac{(31c-192)}{2(13c+516)} C_{44}^6 C_{46}^{[44]} - 2C_{44}^4 \\
 C_{48}^{[44]} = & -\frac{896(3c+46)(5c+3)(c^2-172c+196)}{(c+31)(2c-1)(7c+68)^2(55c-6)} \frac{(C_{46}^6)^2}{C_{44}^6 C_{46}^8} + \frac{8(33c^2+1087c+11760)}{(7c+68)(13c+516)} \frac{C_{46}^6 C_{46}^{[44]}}{C_{46}^8} \\
 & +\frac{3136(3c+46)(5c+3)(c^2-172c+196)}{3(c+24)(c+31)(2c-1)(7c+68)(55c-6)} \frac{C_{44}^4 C_{46}^6}{C_{44}^6 C_{46}^8} - \frac{(31c-192)}{2(13c+516)} \frac{C_{44}^6 (C_{46}^{[44]})^2}{C_{46}^8} \\
 & -\frac{4(165c^3+10763c^2+140036c+38568)}{3(c+24)(c+31)(55c-6)} \frac{C_{44}^4 C_{46}^{[44]}}{C_{46}^8} + \frac{448(3c+46)(5c+3)(11c+656)}{9(c+24)^2(c+31)(55c-6)} \frac{(C_{44}^4)^2}{C_{44}^6 C_{46}^8} \\
 C_{48}^{[46]^{(1)}} = & -\frac{112}{3(c+24)} \frac{C_{44}^4}{C_{46}^8} + \frac{(13c+918)}{(13c+516)} \frac{C_{44}^6 C_{46}^{[44]}}{C_{46}^8} - \frac{16(113c-1338)}{(7c+68)(13c+516)} \frac{C_{46}^6}{C_{46}^8} \\
 C_{66}^0 = & -\frac{7(c-1)c(5c+22)^2(8c^2+1161c-1244)}{162(c+24)(2c-1)^2(7c+68)^2} \frac{(C_{44}^4)^3 C_{46}^6}{(C_{44}^6)^2} \\
 & +\frac{14(c-1)^2 c(c+24)(5c+22)^2(c^2-172c+196)}{9(2c-1)^3(7c+68)^4} \frac{C_{44}^4 (C_{46}^6)^3}{(C_{44}^6)^2} \\
 & -\frac{2(c-1)^2 c(c+24)^2(5c+22)^2(c^2-172c+196)}{3(2c-1)^3(7c+68)^5} \frac{(C_{46}^6)^4}{(C_{44}^6)^2} \\
 & -\frac{(c-1)c(5c+22)^2(17c^3-13105c^2+25330c-12092)}{54(2c-1)^3(7c+68)^3} \frac{(C_{44}^4)^2 (C_{46}^6)^2}{(C_{44}^6)^2} \\
 & +\frac{(c-1)c(5c+22)^2(11c+656)}{162(c+24)^2(2c-1)(7c+68)} \frac{(C_{44}^4)^4}{(C_{44}^6)^2} \\
 C_{66}^4 = & \frac{28(c-1)(5c+22)(c^2-172c+196)}{3(2c-1)^2(7c+68)^2} \frac{C_{44}^4 (C_{46}^6)^2}{(C_{44}^6)^2} - \frac{8(c-1)(c+24)(5c+22)(c^2-172c+196)}{(2c-1)^2(7c+68)^3} \frac{(C_{46}^6)^3}{(C_{44}^6)^2} \\
 & +\frac{4(c-1)(5c+22)(11c+656)}{9(c+24)(2c-1)(7c+68)} \frac{(C_{44}^4)^2 C_{46}^6}{(C_{44}^6)^2}
 \end{aligned}$$

$$\begin{aligned}
 C_{66}^6 &= -\frac{20(13c^4 - 1637c^3 - 113622c^2 + 32168c + 859328)}{27(c+24)^2(2c-1)(7c+68)} \frac{(C_{44}^4)^2}{C_{44}^6} \\
 &\quad + \frac{10(28c^5 - 5425c^4 - 525974c^3 + 387728c^2 + 3726976c - 3870208)}{9(c+24)(2c-1)^2(7c+68)^2} \frac{C_{44}^4 C_{46}^6}{C_{44}^6} \\
 &\quad + \frac{20(92c^5 + 2389c^4 + 39632c^3 + 4060c^2 - 212032c + 193984)}{(2c-1)^2(7c+68)^3} \frac{(C_{46}^6)^2}{C_{44}^6} \\
 C_{66}^8 &= \frac{4(4c+61)}{(7c+68)} \frac{C_{46}^6 C_{46}^8}{C_{44}^6} - \frac{(11c+656)}{6(c+24)} \frac{C_{44}^4 C_{46}^8}{C_{44}^6} \\
 C_{66}^{[44]} &= \frac{784(c^2 - 172c + 196)}{3(c+24)(2c-1)(7c+68)} \frac{(C_{44}^4) C_{46}^6}{(C_{44}^6)^2} - \frac{(11c+656)}{6(c+24)} \frac{(C_{44}^4) C_{46}^{[44]}}{C_{44}^6} \\
 &\quad - \frac{224(c^2 - 172c + 196)}{(2c-1)(7c+68)^2} \frac{(C_{46}^6)^2}{(C_{44}^6)^2} + \frac{112(11c+656)}{9(c+24)^2} \frac{(C_{44}^4)^2}{(C_{44}^6)^2} + \frac{4(4c+61)}{(7c+68)} \frac{C_{46}^6 C_{46}^{[44]}}{C_{44}^6} \\
 C_{66}^{[44]^{(2)}} &= \frac{1960(47c-614)(c^2 - 172c + 196)}{3(c+24)(c+31)(2c-1)(7c+68)(55c-6)} \frac{C_{44}^4 C_{46}^6}{(C_{44}^6)^2} + \frac{3}{4} \frac{C_{46}^8 C_{46}^{[44]^{(2)}}}{C_{44}^6} \\
 &\quad + \frac{5(c+76)(5c+22)(11c+232)}{12(c+24)(c+31)(55c-6)} \frac{C_{44}^4 C_{46}^{[44]}}{C_{44}^6} \\
 &\quad - \frac{560(47c-614)(c^2 - 172c + 196)}{(c+31)(2c-1)(7c+68)^2(55c-6)} \frac{(C_{46}^6)^2}{(C_{44}^6)^2} + \frac{280(11c+656)(47c-614)}{9(c+24)^2(c+31)(55c-6)} \frac{(C_{44}^4)^2}{(C_{44}^6)^2} \\
 C_{66}^{10} &= \frac{3}{4} \frac{C_{46}^8 C_{48}^{10}}{C_{44}^6} \\
 C_{66}^{[46]} &= -\frac{56}{(c+24)} \frac{C_{44}^4}{C_{44}^6} + \frac{48(81c+1274)}{(7c+68)(13c+516)} \frac{C_{46}^6}{C_{44}^6} + \frac{3(13c+248)}{2(13c+516)} C_{46}^{[44]} + \frac{3}{4} \frac{C_{46}^8 C_{48}^{[46]}}{C_{44}^6} \\
 C_{46}^{[46]^{(1)}} &= -\frac{140}{3(c+24)} \frac{C_{44}^4}{C_{44}^6} - \frac{20(113c-1338)}{(7c+68)(13c+516)} \frac{C_{46}^6}{C_{44}^6} + \frac{5(13c+918)}{4(13c+516)} C_{46}^{[44]} - \frac{5}{4} \frac{C_{46}^8 C_{46}^{[44]^{(1)}}}{C_{44}^6}
 \end{aligned}$$

B Quadratic basis in even spin $\mathcal{W}_\infty$

The field V_6 expressed in terms of U_j fields is

$$\begin{aligned}
 V_6 &= U_6 - (U_1 U_5) + (U_2 U_4) - \frac{1}{2}(U_3 U_3) + \frac{1}{4}(5 - 2N)(U_1'' U_3) - \alpha_0(N - 3)(U_1' U_4) \\
 &\quad + \alpha_0(N - 3)(U_1 U_4') + \frac{1}{4}(N - 2)(U_2'' U_2) + \alpha_0(N - 3)(U_2' U_3) - \alpha_0(N - 3)(U_2 U_3') \\
 &\quad - \frac{1}{4}(U_1 U_3'') + \frac{1}{96}(N - 1)(72\alpha_0^2 + 8\alpha_0^2 N^2 - 50\alpha_0^2 N - N)(U_1^{(4)} U_1) \\
 &\quad + \frac{1}{12}\alpha_0(N - 2)(30\alpha_0^2 + 4\alpha_0^2 N^2 - 22\alpha_0^2 N - N)(U_1^{(3)} U_2) \\
 &\quad - \frac{1}{16}(30\alpha_0^2 + 4\alpha_0^2 N^2 - 22\alpha_0^2 N - 1)(12\alpha_0^2 + 4\alpha_0^2 N^2 - 14\alpha_0^2 N - 1)(U_1'' U_1'') \\
 &\quad + \frac{1}{4}\alpha_0(N - 2)(30\alpha_0^2 + 4\alpha_0^2 N^2 - 22\alpha_0^2 N + 2N - 7)(U_1'' U_2') \\
 &\quad - \frac{1}{4}\alpha_0(N - 2)(30\alpha_0^2 + 4\alpha_0^2 N^2 - 22\alpha_0^2 N - 1)(U_1' U_2'') \\
 &\quad + \frac{1}{2}(30\alpha_0^2 + 4\alpha_0^2 N^2 - 22\alpha_0^2 N - 1)(U_1' U_3') \\
 &\quad - \frac{1}{12}\alpha_0(N - 2)(30\alpha_0^2 + 4\alpha_0^2 N^2 - 22\alpha_0^2 N - N)(U_1 U_2^{(3)})
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{4}(-30\alpha_0^2 - 4\alpha_0^2 N^2 + 22\alpha_0^2 N + 1)(U_2' U_2') \\
 & + \frac{1}{24}[-360\alpha_0^4 - 12\alpha_0^2 - 16\alpha_0^4 N^4 + 144\alpha_0^4 N^3 + 8\alpha_0^2 N^3 - 476\alpha_0^4 N^2 \\
 & - 42\alpha_0^2 N^2 + 684\alpha_0^4 N + 60\alpha_0^2 N - N](U_1^{(3)} U_1') \\
 & + \frac{1}{12}\alpha_0(N-2)(30\alpha_0^2 + 4\alpha_0^2 N^2 - 22\alpha_0^2 N - 3)U_3^{(3)} \\
 & + \frac{1}{48}(-30\alpha_0^2 + \alpha_0^2 N^3 - 7\alpha_0^2 N^2 + 24\alpha_0^2 N + 1)U_2^{(4)} \\
 & - \frac{1}{480}\alpha_0(N-1)[1440\alpha_0^4 - 168\alpha_0^2 + 64\alpha_0^4 N^4 - 576\alpha_0^4 N^3 - 14\alpha_0^2 N^3 \\
 & + 1904\alpha_0^4 N^2 + 52\alpha_0^2 N^2 - 2736\alpha_0^4 N + 26\alpha_0^2 N + 5N]U_1^{(5)} \\
 & - \alpha_0(N-3)U_5'
 \end{aligned}$$

OPE of field V_4 with itself is

$$\begin{aligned}
 V_4(z)V_4(w) \sim & -\frac{n}{2}(12\alpha_0^2 + 4\alpha_0^2 n^2 - 14\alpha_0^2 n - 1)(2\alpha_0^2 + 4\alpha_0^2 n^2 - 6\alpha_0^2 n - 1) \times \\
 & \times (101\alpha_0^2 + 34\alpha_0^2 n^2 - 117\alpha_0^2 n - n - 8) \frac{1}{(z-w)^8} \\
 & + 4\alpha_0^2(n-1)(2n-5)(2n-1)(12\alpha_0^2 + 4\alpha_0^2 n^2 - 14\alpha_0^2 n - 1) \frac{V_2}{(z-w)^6} \\
 & + 2\alpha_0^2(n-1)(2n-5)(2n-1)(12\alpha_0^2 + 4\alpha_0^2 n^2 - 14\alpha_0^2 n - 1) \frac{V_2'}{(z-w)^5} \\
 & - 2(n-1)(12\alpha_0^2 + 4\alpha_0^2 n^2 - 14\alpha_0^2 n - 1) \frac{(V_2 V_2)}{(z-w)^4} \\
 & + (n-1)(-3\alpha_0^2 + 4\alpha_0^2 n^2 - 8\alpha_0^2 n - 1)(12\alpha_0^2 + 4\alpha_0^2 n^2 - 14\alpha_0^2 n - 1) \frac{V_2''}{(z-w)^4} \\
 & + 6(6\alpha_0^2 + 4\alpha_0^2 n^2 - 12\alpha_0^2 n + 1) \frac{V_4}{(z-w)^4} \\
 & - 2(n-1)(12\alpha_0^2 + 4\alpha_0^2 n^2 - 14\alpha_0^2 n - 1) \frac{(V_2' V_2)}{(z-w)^3} \\
 & + \frac{1}{3}(n-1)(-7\alpha_0^2 + 4\alpha_0^2 n^2 - 6\alpha_0^2 n - 1)(12\alpha_0^2 + 4\alpha_0^2 n^2 - 14\alpha_0^2 n - 1) \frac{V_2^{(3)}}{(z-w)^3} \\
 & + 3(6\alpha_0^2 + 4\alpha_0^2 n^2 - 12\alpha_0^2 n + 1) \frac{V_4'}{(z-w)^3} \\
 & - (n-1)(12\alpha_0^2 + 4\alpha_0^2 n^2 - 14\alpha_0^2 n - 1) \frac{(V_2'' V_2)}{(z-w)^2} \\
 & + \frac{1}{2}(-12\alpha_0^2 - 4\alpha_0^2 n^2 + 14\alpha_0^2 n + 1) \frac{(V_2' V_2')}{(z-w)^2} \\
 & - \frac{4(V_2 V_4)}{(z-w)^2} - \frac{6V_6}{(z-w)^2} \\
 & + \frac{1}{2}(-22\alpha_0^2 + 4\alpha_0^2 n^2 + 1) \frac{V_4''}{(z-w)^2} \\
 & + \frac{1}{24}(12\alpha_0^2 + 4\alpha_0^2 n^2 - 14\alpha_0^2 n - 1) \times
 \end{aligned}$$

$$\begin{aligned}
& \times (22\alpha_0^2 + 8\alpha_0^2 n^3 - 16\alpha_0^2 n^2 - 14\alpha_0^2 n - 2n + 1) \frac{V_2^{(4)}}{(z-w)^2} \\
& - \frac{1}{3}(n-1)(12\alpha_0^2 + 4\alpha_0^2 n^2 - 14\alpha_0^2 n - 1) \frac{(V_2^{(3)} V_2)}{z-w} \\
& + \frac{1}{2}(-12\alpha_0^2 - 4\alpha_0^2 n^2 + 14\alpha_0^2 n + 1) \frac{(V_2'' V_2')}{z-w} \\
& + \frac{1}{60}(12\alpha_0^2 + 4\alpha_0^2 n^2 - 14\alpha_0^2 n - 1)(15\alpha_0^2 + 4\alpha_0^2 n^3 - 6\alpha_0^2 n^2 - 13\alpha_0^2 n - n) \frac{V_2^{(5)}}{z-w} \\
& - \frac{2(V_2' V_4)}{z-w} - \frac{2(V_2 V_4')}{z-w} + \alpha_0^2(3n-7) \frac{V_4^{(3)}}{z-w} - \frac{3V_6'}{z-w}.
\end{aligned}$$

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