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Modular forms as classification invariants of 4D $\mathcal{N} = 2$ Heterotic-IIA dual vacua

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ABSTRACT: We focus on 4D $\mathcal{N} = 2$ string vacua described both by perturbative Heterotic theory and by Type IIA theory; a Calabi-Yau three-fold X_{IIA} in the Type IIA language is further assumed to have a regular K3-fibration. It is well-known that one can assign a modular form Φ to such a vacuum by counting perturbative BPS states in Heterotic theory or collecting Noether-Lefschetz numbers associated with the K3-fibration of X_{IIA} . In this article, we expand the observations and ideas (using gauge threshold correction) in the literature and formulate a modular form Ψ with full generality for the class of vacua above, which can be used along with Φ for the purpose of classification of those vacua. Topological invariants of X_{IIA} can be extracted from Φ and Ψ , and even a pair of diffeomorphic Calabi-Yau's with different Kähler cones may be distinguished by introducing the notion of “the set of Ψ 's for Higgs cascades/for curve classes”. We illustrated these ideas by simple examples.

KEYWORDS: String Duality, Superstrings and Heterotic Strings, Differential and Algebraic Geometry, Topological Strings

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1 Introduction

The duality between the Heterotic string theory and the Type IIA string theory has been known for a long time. The duality with $SO(5,1)$ symmetry and 16 supersymmetry charges — the duality at 6D — comes with just one piece of moduli space $[\text{Isom}(\text{II}_{4,20}) \setminus SO(4,20)/SO(4) \times SO(20)] \times \mathbb{R}_{>0}$, and its various aspects are understood very well [1–5]. The duality with $SO(3,1)$ symmetry and 8 supersymmetry charges [6] — the duality at 4D — is less understood. The moduli space of Heterotic-IIA dual 4D vacua forms a complicated network of branches. It is desirable that those individual branches are characterized both in the languages of Heterotic string and Type IIA string, and the dictionary between the branch-characterizing data on both sides are understood. At the moment, we do not have one for the 4D Heterotic-IIA duality¹ as clear as Batyrev’s dual polyhedra for mirror symmetry.

For a systematic approach, we need to find invariants characterizing the branches of the moduli space. A lattice pair $\tilde{\Lambda}_S \oplus \Lambda_T$ that fits into $\text{II}_{4,20}$ is assigned for those branches [7–10], and a modular form Φ of certain type that depends on Λ_S is also assigned [11, 12]. It is known, however, that there are physically distinct branches of vacua that cannot be distinguished by the triple of invariants $(\Lambda_S, \Lambda_T, \Phi)$. The primary purpose of this article is to introduce more invariants by using modular forms to improve the state of affairs.

From the perspective of pure mathematics, this task is equivalent to classification of Calabi-Yau three-folds with a K3-fibration. The modular forms introduced in this article can be used therefore for study of such a geometry classification. It should be mentioned, however, that we consider only regular K3-fibrations in this article.

The organization of this article is as follows. We begin in section 2.1 with a short review on the Heterotic-Type IIA duality and a summary of technical limitations on the class of vacua to be considered in this article. The modular form Φ is parametrized by low-energy BPS indices, which are bounded from below as we argue in section 2.2; in the Heterotic language, the bounds come from the quantization of the level of current algebra and the spin under the $SU(2)$ action. In sections 2.3 and 2.4, those bounds and the modular property are combined to constrain the possibilities of Φ and the Euler number of a Calabi-Yau three-fold X_{IIA} that compactifies the Type IIA theory.

We formulate in section 3 a modular form Ψ for X_{IIA} and a little more data, and use it to define new invariants for X_{IIA} in addition to $(\Lambda_S, \Lambda_T, \Phi)$. Section 3.1.1 includes the basic definition of Ψ , which follows from the idea of [11] and subsequent works. This modular form appears in the integrand of the 1-loop gauge threshold correction in the Heterotic string. In section 3.1.2 we comment on some restrictions on the degree of freedom of Ψ , that comes from its modular property and some physical constraints. We introduce a map $(\text{diff}_{\text{coarse}}, \text{diff}_{\text{fine}})$ in section 3.1.3, which extracts from (Φ, Ψ) the full information in specifying the diffeomorphism class of a Calabi-Yau three-fold X_{IIA} . Combining this map with the degree-of-freedom study of Φ and Ψ in sections 2 and 3.1.2, we can use modular property of Φ and Ψ to obtain non-trivial results in the diffeomorphism classes of real

¹An exception is for the case $\Lambda_S = U$ (see main text for what it is), when the duality lifts to the Heterotic-F-theory duality at 6D.

six-dimensional manifolds realized by Calabi-Yau three-folds. We also propose to use the notion of “the set of Ψ ’s for Higgs cascades” or of “the set of Ψ ’s for curve classes” as an invariant that may resolve a diffeomorphic pair of Calabi-Yau three-folds with different cone of curves. In section 3.2, all those ideas are illustrated by simple examples.

In section 4 we comment on some open questions.

The appendix A contains basics about (vector-valued) modular forms and explicit Fourier expansions of modular forms in the main text. In appendix B we review the lattice unfolding method and the embedding trick of Borchers’s [13], presented in a form we need for threshold calculations in the Heterotic theory. The embedding trick is used in explicitly evaluating integrals, for example in the case of $\Lambda_S = \langle +2 \rangle$ in appendix B.3.1.

2 Coarse classification

2.1 A brief review

Let us first review what is known in the literature about the classification using the new supersymmetric index / the generating function of the Noether-Lefschetz number.

2.1.1 Heterotic description: the new supersymmetric index

A Heterotic string compactification to 3+1-dimensions has an unbroken $\mathcal{N} = 2$ supersymmetry (8 supersymmetry charges), if and only if the right-mover of the internal worldsheet CFT contains an $N = 4$ superconformal algebra (SCA) with central charge $\tilde{c} = 6$ and an $N = 2$ free SCA with $\tilde{c} = 3$ corresponding to a flat space of one complex dimension [14–16]. We restrict our attention in this article only to compactifications without an NS5-brane or its generalizations discussed in section 5 of [17].

Let ρ be the number of free chiral bosons in the left-mover in such a compactification. There are vertex operators of the form $e^{ip_L \cdot X_L + ip_R \cdot X_R}$ in the $(c, \tilde{c}) = (22, 9)$ CFT, where X_L and X_R are the $\rho + 2$ chiral bosons in the left-mover and right-mover; the set of $U(1)$ charges $\{(p_L, p_R)\} = \tilde{\Lambda}_S$ forms a lattice with the quadratic form given² by $p_R^2/2 - p_L^2/2$, so its signature is $(+, -) = (2, \rho)$. This lattice $\tilde{\Lambda}_S$ should be even, $p_R^2/2 - p_L^2/2 \in \mathbb{Z}$ for any element of $\tilde{\Lambda}_S$, since the contribution of the state $e^{ip_L \cdot X_L + ip_R \cdot X_R}$ to the partition function should be invariant under $T : \tau \rightarrow \tau + 1$. The $U(1)$ charge of any worldsheet operator should lie in³ $\tilde{\Lambda}_S^\vee$, the dual lattice of $\tilde{\Lambda}_S$. Note that we deal with cases where $\tilde{\Lambda}_S$ is not necessarily unimodular, so that the discriminant group $G_S := \tilde{\Lambda}_S^\vee / \tilde{\Lambda}_S$ may be non-trivial. We assume, however, that $\tilde{\Lambda}_S$ is a primitive sublattice of⁴ $\Pi_{4,20} = U^{\oplus 4} \oplus E_8[-1]^{\oplus 2}$; the orthogonal complement $[\tilde{\Lambda}_S^\perp \subset \Pi_{4,20}]$ is denoted by Λ_T .

For example, when we compactify the Heterotic theory on $K3 \times T^2$ with instantons in $\mathfrak{g} \subset E_8 \times E_8$, the lattice $\tilde{\Lambda}_S$ is equal to $U^{\oplus 2} \oplus W$, where $W = [\mathfrak{g}^\perp \subset E_8[-1]^{\oplus 2}]$. The

²We use the convention $\alpha' = 2$.

³We assume any charge $v \in \tilde{\Lambda}_S^\vee$ is realized by some state. The Type IIA counter part of this assumption is that the pairing $(H^2(X; \mathbb{Z}) / \mathbb{Z}D_s) \times [H_2(X; \mathbb{Z})]^{\text{vert}} \rightarrow \mathbb{Z}$ is represented by the unit matrix; see page 8 for notations.

⁴The even unimodular lattice of signature $(1,1)$ is denoted by U . We use the same notation $\mathcal{R}$ in this article for one of ADE types, its Lie algebra, and its root lattice with positive definite signature.

classical STU model corresponds to the case $W = \{0\}$. One may turn on a Wilson line to add one degree of freedom to the Coulomb branch moduli space (2.2), where the rank-1 lattice W can be $A_1[-1] = \langle -2 \rangle$ for example (see e.g. [50, 51]), and the 24 instantons on K3 stay away from W . The ST model⁵ (discussed in the context of Het-IIA duality in [6]), on the other hand, fits into the form $\Lambda_S = \langle +2 \rangle$; the fact that this Heterotic compactification does not allow purely geometric description using $K3 \times T^2$ is reflected in the choice of Λ_S not in the form of $U \oplus W$ for some negative definite lattice W . In this paper we choose to use abstract CFT language to describe a general class of Heterotic compactifications all at once, including (but not limited at all to) the classical $K3 \times T^2$ compactifications and the ST model.

The Hilbert space of the internal CFT can be decomposed using the action of the free boson algebra and the $N = 4$ right-mover SCA [11, 18]:

$$\mathcal{H}^{\text{int}} = \bigoplus_{(\gamma, \tilde{h}, \tilde{I})} \mathcal{H}_{\gamma, (\tilde{h}, \tilde{I})}^{(22-\rho, 0)} \otimes \mathcal{H}_{\gamma}^{(\rho, 3)} \otimes \mathcal{H}_{\tilde{h}, \tilde{I}}^{(0, 6)}, \quad (2.1)$$

where superscripts indicate the central charge $(c, \tilde{c})$. The rank- $(\rho + 2)$ $U(1)$ charge in $\tilde{\Lambda}_S$ is denoted by v , and γ is its corresponding element $v + \tilde{\Lambda}_S$ in the discriminant group G_S . The vector space $\mathcal{H}_{\gamma}^{(\rho, 3)}$ consists of states of charge $v \in \gamma$ with any free bosonic/fermionic oscillator excitations. A pair $(\tilde{h}, \tilde{I})$ of conformal weight and $SU(2)$ spin labels a unitary irreducible representation $\mathcal{H}_{\tilde{h}, \tilde{I}}^{(0, 6)}$ of the $N = 4$ SCA. The spectrum of the free sector $\mathcal{H}_w^{(\rho, 3)}$ is specified by the central charge $p_R^{\mathbb{C}} : \tilde{\Lambda}_S^{\vee} \rightarrow \mathbb{C}$, which appears in the 4D $\mathcal{N} = 2$ supersymmetry algebra. At the Heterotic string perturbative level, $p_R^{\mathbb{C}}$ is governed by the Coulomb branch moduli space⁶

$$D(\tilde{\Lambda}_S) := \mathbb{P} \left\{ \mathcal{U} \in \tilde{\Lambda}_S \otimes \mathbb{C} \mid (\mathcal{U}, \mathcal{U})_{\tilde{\Lambda}_S} = 0, (\mathcal{U}, \bar{\mathcal{U}})_{\tilde{\Lambda}_S} > 0 \right\}, \quad (2.2)$$

which constitutes the special geometry along with the dilaton complex scalar $s := 4\pi i S$; the weak coupling limit⁷ is $s_2 \gg 1$. The rest of spectrum information (i.e. the spectrum of $\mathcal{H}_{w, (\tilde{h}, \tilde{I})}^{(22-\rho, 0)}$ and $\mathcal{H}_{\tilde{h}, \tilde{I}}^{(0, 6)}$ for each possible $(w; \tilde{h}, \tilde{I})$) depends also on the hypermultiplet moduli space.

The new supersymmetric index [11, 19] of the internal CFT is defined by⁸

$$Z_{\text{new}}(\tau, \bar{\tau}) := \frac{-i}{\eta(\tau)^2} \text{Tr}_{\text{R-sector}}^{(22, 9)} \left[e^{\pi i F_R} F_R q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} \right], \quad (2.3)$$

⁵Best effort for a geometric description of this model is as follows: put $10 + 10$ instantons in $E_8[-1] \oplus E_8[-1]$ to break them all and put 4 instantons to break $\langle -2 \rangle$ in $\langle -2 \rangle \oplus \langle +2 \rangle \subset U$, to obtain the unbroken $\langle +2 \rangle$. We need to tune the radius of T^2 at a stringy regime so an $SU(2) = \langle -2 \rangle$ gauge group is enhanced for that. So, this model does not allow us to describe the T^2 part with purely classical geometry, or to use geometric intuitions or a sugra description associated with a direct product of K3 and T^2 .

⁶This space is a generalization of the Narain moduli space for the torus compactifications. For example, in the case of $K3 \times T^2$ compactifications, it contains complex/Kähler structure of T^2 as well as the degree of freedom for the Wilson lines. See also the footnote 96.

⁷The real and imaginary components of s are denoted by $s_{1,2}$. Similar notations $(t_2, \tau_{1,2}, \rho_2$ etc.) are used throughout this article.

⁸To be more precise, only the trace part is called as the new supersymmetric index of the internal CFT. The $1/\eta^2$ factor is included within Z_{new} here because the trace part appears in Δ_{grav} in the combination (2.3); the $1/\eta^2$ factor is from the 4D Minkowski part in the light-cone gauge.

where $F_R := 2(\tilde{J}_3^{\tilde{c}=6})_0 + (\tilde{J}^{\tilde{c}=3})_0$ is the zero mode of the total $U(1)$ current in the right-mover; $\tilde{J}_3^{\tilde{c}=6}$ is the $SU(2)$ Cartan current in the $N = 4$ SCA and $\tilde{J}^{\tilde{c}=3}$ the $U(1)$ current in the $N = 2$ SCA. An important point is that this index does not depend on any continuous deformations of the hypermultiplet moduli, although it does on that of vector multiplet moduli. In addition, it is used in computing the 1-loop correction Δ_{grav} to the gravitational coupling $\sqrt{-g}R^2$ in the 4D effective theory [20–22];

$$\Delta_{\text{grav}} = \int \frac{d\tau_1 d\tau_2}{\tau_2} (\mathcal{B}_{\text{grav}} - b_{\text{grav}}), \quad \mathcal{B}_{\text{grav}} = Z_{\text{new}} \hat{E}_2, \quad (2.4)$$

where the integration is over the fundamental region of $SL(2; \mathbb{Z})$ in the upper complex half plane (of the torus world sheet complex structure τ), and $\hat{E}_2 := E_2 - \frac{3}{\pi\tau_2}$ is a non-holomorphic modular form of weight $(2, 0)$. The constant b_{grav} is set to the q^0 coefficient of $\mathcal{B}_{\text{grav}}$, to cut off the IR divergent 1-loop contributions and brings the massless degrees of freedom back into the path integration in low energy effective theory. The index Z_{new} must have weight $(-1, 1)$ due to the modular invariance of the integrand.

The action of the free boson algebra on $\mathcal{H}^{\text{int}}$ leads to the following decomposition

$$Z_{\text{new}}(\tau, \bar{\tau}) = \sum_{\gamma \in G_S} \theta_{\tilde{\Lambda}_S[-1]+\gamma}(\tau, \bar{\tau}) \frac{\Phi_\gamma(\tau)}{\eta(\tau)^{24}}, \quad (2.5)$$

$$\theta_{\tilde{\Lambda}_S[-1]+\gamma}(\tau, \bar{\tau}) := \sum_{v \in \tilde{\Lambda}_S + \gamma} q^{p_L^2(v)/2} \bar{q}^{p_R^2(v)/2}, \quad (2.6)$$

where $\theta_{\tilde{\Lambda}_S[-1]+\gamma}$ is the Siegel theta function, which describes the whole continuous dependence of the index Z_{new} on the vector multiplet moduli, while Φ_γ allows only discrete choices as seen below. This Siegel theta function is a vector-valued modular form of weight $(\rho/2, 1)$ and type $\rho_{\tilde{\Lambda}_S[-1]}$. Similarly, $\Phi = \sum_{\gamma \in G_S} e_\gamma \Phi_\gamma$ lies in $\text{Mod}(11 - \rho/2, \rho_{\tilde{\Lambda}_S})$.⁹ In particular, Φ is holomorphic at cusps, i.e. has no negative power of q in its expansion. Transformation law under $T : \tau \rightarrow \tau + 1$ fixes the fractional part of power of q , so Φ_γ/η^{24} can be expanded as¹⁰

$$\frac{\Phi_\gamma(\tau)}{\eta^{24}} = \sum_{\nu \in \gamma^2/2 + \mathbb{Z}} c_\gamma(\nu) q^\nu, \quad c_\gamma(\nu) = 0 \quad \text{for } \nu < -1. \quad (2.7)$$

Non-zero contributions to Z_{new} comes only from those states whose right-movers are given by the (Ramond) ground states of the $\tilde{c} = 3$ sector and the short representations of the $N = 4$ SCA in the $\tilde{c} = 6$ sector. There are two types of short representations of the $N = 4$ SCA: vector-type and hyper-type. Their quantum numbers $(\tilde{h}, \tilde{I})$ are given by the following table.

	R-sector	NS-sector
vector-type:	$(1/4, 1/2)$	$(0, 0)$
hyper-type:	$(1/4, 0)$	$(1/2, 1/2)$

⁹In fact, we impose a stronger restriction on Φ . See (2.14).

¹⁰Note that $c_\gamma(\nu) = c_{-\gamma}(\nu)$ follows from properties of the Weil representation. See appendix A.1.

Their contributions can be written as [11]

$$\frac{\Phi_\gamma}{\eta^{24}} = -2 \times \frac{1}{\eta^2} \text{Tr}_{\mathcal{H}_{\gamma,V}^{(22,0)}} \left[q^{L_0 - \frac{c}{24}} \right] + 1 \times \frac{1}{\eta^2} \text{Tr}_{\mathcal{H}_{\gamma,H}^{(22,0)}} \left[q^{L_0 - \frac{c}{24}} \right], \quad (2.8)$$

where $\mathcal{H}_{\gamma,(\tilde{h},\tilde{I})}^{(22,0)} := \mathcal{H}_{\gamma,(\tilde{h},\tilde{I})}^{(22-\rho,0)} \otimes \mathcal{H}^{(\rho,0)}$, with $\mathcal{H}^{(\rho,0)}$ being the space of the (neutral) left-moving oscillations in the $c = \rho$ sector, and

$$\mathcal{H}_{\gamma,V}^{(22,0)} := \mathcal{H}_{\gamma,(1/4,1/2)}^{(22,0)} = \mathcal{H}_{\gamma,(0,0)}^{(22,0)}, \quad (2.9)$$

$$\mathcal{H}_{\gamma,H}^{(22,0)} := \mathcal{H}_{\gamma,(1/4,0)}^{(22,0)} = \mathcal{H}_{\gamma,(1/2,1/2)}^{(22,0)}. \quad (2.10)$$

The second equality in each line comes from the spectral flow of the $N = 4$ SCA that brings the representations in the R-sector to those in the NS-sector. The coefficients -2 and $+1$ in (2.8) correspond to the Witten index of the representations $(1/4, 1/2)$ and $(1/4, 0)$, respectively [23, 24]:

$$\text{Tr}_{(1/4,1/2)}(-1)^{2(\tilde{J}_3^{\tilde{c}=6})_0} = -2, \quad \text{Tr}_{(1/4,0)}(-1)^{2(\tilde{J}_3^{\tilde{c}=6})_0} = +1. \quad (2.11)$$

It follows from (2.8), in particular, that all the Fourier coefficients $c_\gamma(\nu)$ are integers, so we have only discrete choices of Φ . In fact, since Φ/η^{24} has a negative weight as a modular form, it can be uniquely specified¹¹ by the coefficients $c_\gamma(\nu)$ with $\nu < 0$, i.e.

$$n_\gamma := -2n_\gamma^V + n_\gamma^H = c_\gamma([\gamma^2/2]_{\text{frac}} - 1), \quad \gamma \in G_S. \quad (2.12)$$

Here the fractional part $[x]_{\text{frac}}$ of $x \in \mathbb{R}$ is defined by $[x]_{\text{frac}} \equiv x \bmod \mathbb{Z}$ and $0 \leq [x]_{\text{frac}} < 1$. $n_\gamma^{V/H}$ is the number of states with conformal weight $[\gamma^2/2]_{\text{frac}}$ in $\mathcal{H}_{\gamma,V/H}^{(22,0)}$.

We can deduce $n_{\gamma=0} = -2$ from the supersymmetry constraints. First, the uniqueness of the ground state forces $n_0^V = 1$; n_0^V is the number of states in $\mathcal{H}_{\gamma=0,V}^{(22,0)}$ with conformal weight 0. Such state, tensored with the right-moving highest weight state of $(\tilde{h}, \tilde{I}) = (0, 0)$, gives the ground state in the Hilbert space of the internal CFT. Second,¹² $\partial X^{\mu=2,3}$ tensored with the right-moving Ramond ground states (in the $\tilde{c} = 3$ sector) and the highest weight states of $(\tilde{h}, \tilde{I}) = (1/4, 1/2)$ (in the $\tilde{c} = 6$ sector) gives the exactly required number of gravitino states for 4D $\mathcal{N} = 2$ supersymmetry. If $n_0^H > 0$, the 4D effective theory would have $\mathcal{N} = 2 + n_0^H$ supersymmetry in a similar way. Since we focus on the case with only 8 supercharges, n_0^H should be zero, so $n_0 = -2$. For the explanation of $n_0 = -2$ from the Type IIA perspective, see section 2.1.2.

The BPS index n_γ for $\gamma \neq 0$ also has a simple interpretation in terms of spacetime effective theory: for each $v \in \gamma \subset \tilde{\Lambda}_S^\vee$ such that $-2 \leq v^2 < 0$,¹³ there exist n_γ^V BPS vector multiplets and n_γ^H BPS half-hypermultiplets,¹⁴ both of which have $U(1)$ -charge v and BPS mass-square $p_R^2(v)$. $v^2 < 0$ implies that these states¹⁵ will be massless at some points in

¹¹In fact, we assume some n_γ to be zero. See the comments around (2.14).

¹² $X^{\mu=2,3}$ are the scalar operators representing Minkowski coordinates under the light cone gauge.

¹³The condition $-2 \leq v^2 < 0$ comes from the left-right matching of conformal weights.

¹⁴If $2\gamma \neq 0 \in G_S$, these $n_\gamma^H + n_{-\gamma}^H = 2n_\gamma^H$ half-hypermultiplets are combined to become n_γ^H full hypermultiplets.

¹⁵In section 2.2, we show that there are non-trivial constraints about γ for BPS massless vector multiplet states to exist, coming from charge and level quantization conditions.

the (weak coupling) Coulomb branch moduli space $D(\tilde{\Lambda}_S)$. For this reason, we call n_γ 's the low-energy BPS indices in this article.

In this article, we impose some technical constraints on $\tilde{\Lambda}_S$ and Φ for simplicity. First, we only consider the case

$$\tilde{\Lambda}_S = U[-1] \oplus \Lambda_S \quad (2.13)$$

for some even lattice Λ_S of signature $(1, \rho - 1)$. Note that the direct summand $U[-1]$ does not contribute to the discriminant group: $G_S = \tilde{\Lambda}_S^\vee / \tilde{\Lambda}_S = \Lambda_S^\vee / \Lambda_S$. The factor $U[-1]$ corresponds to $H^0(\text{K3}; \mathbb{Z}) \oplus H^4(\text{K3}; \mathbb{Z})$ of the generic fibre of a K3-fibred Calabi-Yau manifold that compactifies the Type IIA theory. See also section 2.1.2.¹⁶

Second, we assume that if $\gamma^2/2 \equiv 0 \pmod{\mathbb{Z}}$ and $\gamma \neq 0$ then $n_\gamma = c_\gamma(-1) = 0$. In other words, Φ has an expansion of the form $-2q^0 e_0 + (\text{strictly higher power of } q)$. We denote this condition^{17,18} as

$$\Phi \in \text{Mod}_0(11 - \rho/2, \rho_{\tilde{\Lambda}_S}). \quad (2.14)$$

We define $h_{\min}(\gamma)$ so that the Fourier expansion of Φ_γ begins with $\mathcal{O}(q^{h_{\min}(\gamma)})$ (or higher):

$$h_{\min}(\gamma) \in \gamma^2/2 + \mathbb{Z}, \quad h_{\min}(0) = 0, \quad 0 < h_{\min}(\gamma) \leq 1 \text{ for } \gamma \neq 0. \quad (2.15)$$

We denote

$$G_S^< = \{\gamma \in G_S \mid h_{\min}(\gamma) < 1\}, \quad d^< = |G_S^< / \pm|. \quad (2.16)$$

After all, the modular form Φ is specified by $n_0 = -2$ and at most $(d^< - 1)$ integers

$$n_{|\gamma|} = n_{\pm\gamma} \geq -2, \quad |\gamma| \in G_S^< / \pm, \quad \gamma \neq 0. \quad (2.17)$$

Sometimes, the modular properties predict linear relations among the low-energy BPS indices $\{n_{|\gamma|}\}$; see section 2.3.2.

2.1.2 Type IIA description: the generating function of the Noether-Lefschetz numbers

In this article, we consider the Type IIA string theory compactified on a non-singular Calabi-Yau three-fold $X = X_{\text{IIA}}$ that has a K3-fibration $\pi : X \rightarrow \mathbb{P}^1 = \mathbb{P}_{\text{IIA}}^1$; complexified Kähler parameters may be analytically continued out of a geometric phase, but otherwise we remain in a geometric phase.¹⁹ This restriction means, in particular, that we do not

¹⁶A choice of $\tilde{\Lambda}_S$ that does not have $U[-1]$ as a direct summand may correspond to some non-geometric background of in the Type IIA language, such as mirror-folds, etc.

¹⁷This condition corresponds to the claim (2.24) in the IIA side. Even when there is a Heterotic construction that is not subject to (2.14), there would probably be no Type IIA dual with a geometric phase. See also section 2.1.2.

¹⁸This is an extra non-trivial condition only in a lattice Λ_S with a non-zero isotropic element γ . Examples of such lattices include $\Lambda_S = \langle 2n^2 m \rangle$ for any $n, m \in \mathbb{N}$. Indeed, let e be a generator of the rank-1 abelian group $\mathbb{Z}$ underlying Λ_S , so $(e, e)_{\Lambda_S} = 2n^2 m$. Elements such as $\gamma = e/n + \Lambda_S \in G_S$ give $\gamma^2/2 = m \equiv 0 \pmod{\mathbb{Z}}$.

¹⁹Both $R^4 \pi_* \mathbb{Z}$ and $R^0 \pi_* \mathbb{Z}$ are trivial rank-1 local systems on $\mathbb{P}^1$.

treat T-folds or mirror folds [25–27]. This restriction corresponds to (2.13) in the Heterotic side. We also assume that

$$h^{1,0}(X) = h^{2,0}(X) = 0. \quad (2.18)$$

The effective theory on $3 + 1$ -dimensions has strictly $\mathcal{N} = 2$ supersymmetry, not more, not less.

K3-fibres in $\pi : X \rightarrow \mathbb{P}^1$ may degenerate at isolated points on the base $\mathbb{P}^1$. Degenerations of a K3-fibration are classified (by allowing a base change locally) into Type I, Type II, and Type III [28]. When a K3-fibration $\pi : X \rightarrow \mathbb{P}^1$ only has degenerations classified as Type I,²⁰ such a K3-fibration is said to be *regular*. In this article, we only consider regular K3-fibrations, because that is when one can find Heterotic dual descriptions without (generalization of) NS5-branes [17].

Let $\pi : X \rightarrow \mathbb{P}^1$ be a regular K3-fibration. Then the cohomology groups of X have the following filtration:

$$H^2(X; \mathbb{Z}) \supset \mathbb{Z}D_s \supset \{0\}, \quad H^2(X; \mathbb{Z})/\mathbb{Z}D_s =: \Lambda_S, \quad (2.19)$$

$$H_2(X; \mathbb{Z}) \supset [H_2(X; \mathbb{Z})]^{\text{vert}} \supset \{0\}, \quad (2.20)$$

where D_s is the total K3-fibre divisor class, and $[H_2(X; \mathbb{Z})]^{\text{vert}}$ the subgroup generated by curves that are projected to points on $\mathbb{P}^1$. The free abelian group Λ_S is a subgroup of the Neron-Severi lattice L_S of X_p , the fibre K3 surface over a generic point $p \in \mathbb{P}^1$. So, an intersection form is introduced on Λ_S by restricting the intersection form of L_S ; Λ_S is now regarded as a lattice. The natural pairing between Λ_S and $[H_2(X; \mathbb{Z})]^{\text{vert}}$ is non-degenerate, and we have an isomorphism $[H_2(X; \mathbb{Z})]^{\text{vert}} \cong_{ab} \Lambda_S^\vee$ as abelian groups. We reserve ρ for the rank of Λ_S , not for L_S . Λ_S [resp. L_S] is a primitive sublattice of $\text{II}_{3,19} \cong H^2(K3; \mathbb{Z})$; the orthogonal complement lattice is denoted by Λ_T [resp. L_T].

For a regular K3-fibration $\pi : X \rightarrow \mathbb{P}^1$, one can think of a generating function Φ of the number of Noether-Lefschetz points on the base $\mathbb{P}_{\text{IIA}}^1$. When it is defined appropriately (see below until (2.24)), it is known to be a modular form [12]. First, there is a holomorphic map

$$\iota_\pi : \mathbb{P}_{\text{IIA}}^1 \longrightarrow D(\Lambda_T)/\Gamma_T, \quad (2.21)$$

where $\Gamma_T := \text{Ker}(\text{Isom}(\Lambda_T) \rightarrow \text{Isom}(G_T))$; this is because $D(\Lambda_T)/\Gamma_T$ is the coarse moduli space of Λ_S -polarized K3 surfaces. At points on the base $\mathbb{P}_{\text{IIA}}^1$ where the ι_π -image hits the Noether-Lefschetz divisor $D_{NL(F_\perp)}$ of $D(\Lambda_T)$ for $F_\perp \in \Lambda_T^\vee$, the transcendental cycles $(F_\perp + \Lambda_S^\vee) \cap \text{II}_{3,19}$ become algebraic.

Second, think of the Heegner divisor [12]

$$\Phi_{\text{pre}} := \sum_{F_\perp \in \Lambda_T^\vee} D_{NL(F_\perp)} q^{-F_\perp^2/2} e_{[F_\perp]}/\Gamma_T \in \text{Pic}(D(\Lambda_T)/\Gamma_T)[[q^{1/N}]] \otimes \mathbb{C}[G_T], \quad (2.22)$$

²⁰We avoid saying “only has Type I fibres” here, because such expressions as “Type I fibre” are reserved only for the central fibres after the local geometry of $\pi : X \rightarrow \mathbb{P}^1$ around a degeneration point $p \in \mathbb{P}^1$ is brought into a Kulikov model by a base change around p .

where $\{e_\gamma\}_{\gamma \in G_T}$ is the set of formal basis elements of the vector space $\mathbb{C}[G_T]$, $q = e^{2\pi i \tau}$ a formal variable, and N the level of the quadratic discriminant form of the lattice Λ_T . The sum over $F_\perp \in \Lambda_T^\vee$ does not include $F_\perp$'s with $(F_\perp)^2 > 0$ because the Noether-Lefschetz divisor is empty for them. Besides the terms of $F_\perp$'s with $(F_\perp)^2 < 0$, however, Φ_{pre} is defined to include an extra term $e_0 q^0 D_{NL(0)}/\Gamma_T$; we do not provide a description of the divisor “ $D_{NL(0)}/\Gamma_T$ ” (see [12] for details), but all of its necessary properties are provided later on. Note that Φ_{pre} in this definition does not have a term²¹ proportional to $e_\gamma q^0$ for a non-zero isotropic $\gamma \in G_T$. Note also that Φ_{pre} is defined for Λ_T without referring to a K3-fibration.

Finally, given a K3-fibration ι_π , we obtain a $\mathbb{C}[G_T]$ -valued function by pairing Φ_{pre} with the ι_π -image inside $D(\Lambda_T)/\Gamma_T$:²²

$$\Phi = \sum_{\gamma \in G_T} \Phi_\gamma(\tau) e_\gamma := \Phi_{\text{pre}} \cdot \iota_\pi(\mathbb{P}^1) = \sum_{\gamma \in G_T} \sum_{[F_\perp] \in (\gamma + \Lambda_T)/\Gamma_T} NL_{[F_\perp], \gamma} q^{-(F_\perp)^2/2} e_\gamma. \quad (2.23)$$

Define $NL_{\nu, \gamma} = \sum_{[F_\perp], -(F_\perp)^2/2 = \nu} NL_{[F_\perp], \gamma}$; it is the Fourier coefficients of $\Phi_\gamma = \sum_\nu NL_{\nu, \gamma} q^\nu$. Ref. [12] arrives at a statement (by using earlier math results in [29], but not relying on the duality with Heterotic string) that

$$\Phi \in \text{Mod}_0 \left(11 - \frac{\rho}{2}, \rho_{\Lambda_S} \right). \quad (2.24)$$

See footnote 21 for why Φ is in Mod_0 , not in Mod .

The coefficient $NL_{\nu, \gamma}$ with $\gamma = 0$ and $\nu = 0$ in Φ does not describe the number of Noether-Lefschetz points on $\mathbb{P}_{\text{IIA}}^1$. Following the definition of the divisor “ $D_{NL(0)}/\Gamma_T$ ” in [12], one arrives at

$$NL_{0,0} = \deg_{\mathbb{P}_{\text{IIA}}^1} (R^2 \pi_* (\mathcal{O}_X)) = -2, \quad (2.25)$$

where we used $h^{0,3}(X) = 1$ and $h^{0,2}(X) = 0$ at the last equality. So, as a consequence²³ of (2.18), we have the expansion $\Phi \sim -2q^0 e_0 + (\text{strictly higher power of } q)$.

2.1.3 Heterotic-Type IIA duality and effective theory

When branches of moduli space of Heterotic theories and Type IIA theories associated with a pair $(\tilde{\Lambda}_S, \Lambda_T)$ are identified under the duality, both descriptions should give rise to the same modular form:

$$\{\Phi_\gamma^{\text{Het}}(q)\}_{\gamma \in G_S} = \{\Phi_\gamma^{\text{IIA}}(q)\}_{\gamma \in G_T}. \quad (2.26)$$

²¹This leads to the subtle fact that Φ lies in Mod_0 rather than Mod in (2.24).

²²To be precise, this procedure is well-defined only for smooth fibrations. In the case of a Calabi-Yau three-fold X with finitely many nodal singular K3-fibres (these degenerations are classified as Type I), one has to take a double cover of X and resolve the conifold singularities, so that a smooth fibration is obtained. One can apply the procedure to this fibration and divide the modular form by 2. We denote as Φ the modular form defined in this way. See [30, 31] for details.

²³If $X = K3 \times T^2$, where $h^{0,2}(X) = 1$, then $R^2 \pi_* (\mathcal{O}_X) = \mathcal{O}_{T^2}$. So, $NL_{0,0} = \deg_{T^2} (\mathcal{O}_{T^2}) = 0$. This is consistent with the Heterotic string description ($n_0 = -2 + n_0^H = 0$ in the $\mathcal{N} = 4$ supersymmetry situation).

The isomorphism $G_S \cong_{ab} G_T$ as abelian groups comes from the mutually orthogonal embedding²⁴ $(\tilde{\Lambda}_S \oplus \Lambda_T) \subset \Pi_{4,20} \subset (\tilde{\Lambda}_S^\vee \oplus \Lambda_T^\vee)$.

That is because all the Fourier coefficients of Φ/η^{24} determine physical quantities in the $\mathcal{N} = 2$ supersymmetric effective theory on 3+1-dimensions; the helicity supertrace is defined by

$$\Omega(0, 0; (r, w), q_0) := -2\text{Tr}_{\mathcal{H}(0,0;(r,w),q_0)} [(-1)^{2J_3} (J_3)^2] \quad (2.27)$$

on the Hilbert space $\mathcal{H}(0, 0; (r, w), q_0)$ of particles on $\mathbb{R}^{3,1}$ with a given pure electric charge under the $(\rho + 2)$ $U(1)$ gauge fields; $r, q_0 \in \mathbb{Z}$ and $w \in \Lambda_S^\vee$. J_3 is the 3-component of the angular momentum of this space-time. In the Heterotic language [32]

$$\Omega(0, 0; (1, w), q_0) = c_{[v]}^{\text{Het}}(v^2/2), \quad v = e^0 + w + q_0 e^4 \in \tilde{\Lambda}_S^\vee, \quad (2.28)$$

where e^0 and e^4 are the generators of the $U[-1]$ factor in (2.13), and $w \in \Lambda_S^\vee$. In the Type IIA language, we apply the electromagnetic duality transformation in the effective theory for the one of the $(\rho + 1)$ $U(1)$ gauge fields originating from the Ramond-Ramond 3-form field, the one associated with the 2-form D_s ; then a D4-brane wrapped on the fibre class along with a 2-form $F \in H^2(K3; \mathbb{Z})$ and $N_{\geq 0}$ units of anti-D0-brane gives rise to a particle on $\mathbb{R}^{3,1}$ with a pure electric charge (Mukai vector on the K3 surface)

$$v = e^0 + F_{\parallel} + q_0(F, N)e^4 \in \tilde{\Lambda}_S^\vee, \quad q_0(F, N) = 1 - N + \frac{F^2}{2}. \quad (2.29)$$

Here, $\tilde{\Lambda}_S = U[-1] \oplus \Lambda_S$ is regarded as the Mukai lattice of Λ_S -polarized K3 surface, and e^0 and e^4 generate the factor $U[-1] \cong_{ab} H^0(K3; \mathbb{Z}) \oplus H^4(K3; \mathbb{Z}) \cong_{ab} \mathbb{Z}e^0 \oplus \mathbb{Z}e^4$. $F_{\parallel}$ is the projection of $F \in H^2(K3; \mathbb{Z})$ to $\iota^*(H^2(X; \mathbb{Z})) \cong \Lambda_S$, and $\iota : (\text{a fibre K3}) \hookrightarrow X$ is an embedding. The helicity supertrace is [33–38] (and [39])

$$\Omega(0, 0; (1, F_{\parallel}), q_0) = c_{[v]}^{\text{IIA}}(v^2/2). \quad (2.30)$$

The values of $v^2/2$ in the set of states above exhaust²⁵ all $\nu \in h_{\min}(\gamma) + \mathbb{Z}_{\geq 0}$, so all the Fourier coefficients of $\Phi^{\text{Het}}/\eta^{24}$ and $\Phi^{\text{IIA}}/\eta^{24}$ should be the same. The Coulomb branch moduli space $D(\tilde{\Lambda}_S)$ in (2.2) is parametrized by $t \in \Lambda_S \otimes \mathbb{C}$ with $(t_2, t_2) > 0$, which is interpreted as Narain moduli parameters in the Heterotic description and as complexified Kähler parameters in the Type IIA description:

$$\mathcal{U}(t) = e^0 + e^4 \frac{(t, t)}{2} + t. \quad (2.31)$$

²⁴More generally, there is such a unique isomorphism $\phi_M : G_L \cong_{ab} G_{L'}$ that satisfies $\phi_M^*(-, -)_{L'} = -(-, -)_L$ for any mutually orthogonal pair L and L' of primitive sublattices of a unimodular lattice M . In this article, elements of G_L and $G_{L'}$ are identified without referring to ϕ_M (e.g., in (B.24)).

²⁵The Gopakumar-Vafa invariants of vertical curve classes $\beta \in [H_2(X; \mathbb{Z})]^{\text{vert}} \cong \Lambda_S^\vee$ are equal to $c_{[\beta]}(\beta^2/2)$, and are also physical in the effective field theory because of its appearance in the prepotential (3.28), (3.31); see [11, 12, 40]. Not all the coefficients $c_\gamma(\nu)$ of Φ correspond to those Gopakumar-Vafa invariants for some Λ_S (e.g., $\Lambda_S = \langle +2 \rangle$), however.

The central charge of a BPS state with the electric charge $v \in \tilde{\Lambda}_S^\vee$ is proportional to $(p_R^C(v) \propto (\mathcal{U}(t), v))$. We will focus only on the $s_2 \gg 1$ and $s_2 \gg |t_2|$ region of the special geometry throughout this article; that is the weak coupling region in the Heterotic description, and the large base $\mathbb{P}^1$ region in the Type IIA description.²⁶

To wrap up, the moduli space of the Heterotic-Type IIA dual vacua with 4D $\mathcal{N} = 2$ supersymmetry has a branch structure, and each branch is labeled by a pair of lattices $\tilde{\Lambda}_S$ and Λ_T , and a modular form $\Phi \in \text{Mod}_0(11 - \rho/2, \rho_{\Lambda_S})$. It is known [41] that one can find a $\mathbb{C}$ -basis $\{\phi_i\}$ of $\text{Mod}_0(11 - \rho/2, \rho_{\Lambda_S})$ so that all the Fourier coefficients of $\phi_{i,\gamma}(\tau)$ of $\phi_i = \sum_{\gamma \in G_S} e_\gamma \phi_{i,\gamma}(\tau)$ are integers. So, Φ must be in the free abelian group within $\text{Mod}_0(11 - \rho/2, \rho_{\Lambda_S})$ whose rank is the same as the dimension of $\text{Mod}_0(11 - \rho/2, \rho_{\Lambda_S})$. This free abelian group is denoted by $\text{Mod}_0^{\mathbb{Z}}(11 - \rho/2, \rho_{\Lambda_S})$.

Without relying on explicit constructions (such as toric complete intersection), we can therefore hope to use properties of the free abelian group $\text{Mod}_0^{\mathbb{Z}}(11 - \rho/2, \rho_{\Lambda_S})$ to derive some properties of Λ_S -polarized K3-fibred Calabi-Yau three-folds.

2.2 Conditions for $n_\gamma^V = 1$

As seen in section 2.1.1, n_γ^V is the number of BPS vector multiplets of fixed charge $v \in \gamma$ subject to $-2 \leq v^2 < 0$. Since any charged massless gauge boson²⁷ should be a part of non-abelian gauge bosons for some compact Lie group, its multiplicity must be one for each possible charge. This implies that $n_\gamma^V \in \{0, 1\}$ for any non-zero $\gamma \in G_S$, so that $n_\gamma \geq -2$. Let us consider what happens when $n_\gamma^V = 1$.

Suppose $n_{\gamma_0}^V = 1$ for a given non-zero $\gamma_0 \in G_S$. Fix $v_0 \in \gamma_0 \subset \tilde{\Lambda}_S$ such that $-2 \leq (v_0, v_0) < 0$. Let us consider the points in the Coulomb branch moduli space $D(\tilde{\Lambda}_S)$ where the states of charge v_0 become massless: $p_R(v_0) = 0$. At these points, the massless vector bosons of charge v_0 should be a part of non-abelian gauge bosons. In the language of the Heterotic worldsheet theory, this gauge symmetry is described by some current algebra carried by the left-mover. Therefore we have $\text{SU}(2)$ current algebra (that may be a sub-algebra of a larger current algebra) that consists of

$$J^\pm(z) := e^{\pm i p_L(w_0) \cdot X_L} \mathcal{O}^\pm, \quad J^3(z) := \frac{i}{-(v_0, v_0)} p_L(v_0) \cdot \partial X_L, \quad (2.32)$$

where X_L is the internal free chiral bosons in the $c = \rho$ sector, and $\mathcal{O}^\pm$ are some vertex operators of conformal weight $[(\gamma_0, \gamma_0)/2]_{\text{frac}}$ in $\mathcal{H}_{\gamma_0, V}^{(22,0)}$. Note that $J^\pm$ have conformal weight 1 as well as J^3 , because a charge v_0 with $-2 \leq (v_0, v_0) < 0$ is under consideration. Their OPEs are

$$J^3(z) J^\pm(0) \sim \frac{\pm 1}{z} J^\pm(0), \quad J^3(z) J^3(0) \sim \frac{-1/(w_0, w_0)}{z^2}. \quad (2.33)$$

This means that²⁸ the level of the current algebra is $k = -2/(v_0, v_0)$.

²⁶That is when we expect little contributions to low-energy physics from the NS5-branes in Heterotic string and D-branes wrapped on cycles that are mapped surjectively to $\mathbb{P}_{\text{IIA}}^1$ in the Type IIA string. BPS states of such origins are not used in defining the modular form Φ .

²⁷Heterotic string non-perturbative effects modify the infrared dynamics and the moduli space, as in Seiberg-Witten theory, but they are all known story [42].

²⁸In other words, $\mathcal{O}^\pm$ are from the coset $\text{SU}(2)_k / \text{U}(1)_k$. This coset model is known as parafermion theory with $\mathbb{Z}_k$ symmetry.

n	1	2	3	4	5	6	...
(j, k)	$\emptyset$	$(2, 2)$	$(3, 4)$	$\emptyset$	$(\pm 4, 5)$	$(\pm 4, 3), (\pm 6, 2)$	...

Table 1. list of $(\gamma_0, k) = ([\frac{j}{2n}e], k)$ satisfying the condition (2.35) when $\Lambda_S = \langle +2n \rangle$.

Let us consider some constraints from unitarity. First, the level k needs to be a positive integer. Second, any state (of U(1)-charge $v \in \tilde{\Lambda}_S^\vee$) should have half-integral spin in terms of this SU(2): this results in

$$\frac{(v_0, v)}{(v_0, v_0)} \in \frac{1}{2}\mathbb{Z} \quad \forall v \in \tilde{\Lambda}_S^\vee, \quad (2.34)$$

because that²⁹ is the charge of $e^{i(p_L(v) \cdot X_L + p_R(v) \cdot X_R)}$ under J_3 . These mean the following constraints³⁰ for $\gamma_0 \in G_S = \tilde{\Lambda}_S^\vee / \tilde{\Lambda}_S$:

$$\frac{(\gamma_0, \gamma_0)}{2} \equiv \frac{-1}{k} \pmod{\mathbb{Z}} \quad \text{and} \quad k\gamma_0 \equiv 0 \in G_S, \quad \text{for some } k \in \mathbb{Z}_{\geq 2}. \quad (2.35)$$

The possibility $k = 1$ does not have to be included here, because $k = 1$ would simply mean $\gamma_0 = 0$ in (2.35), where we know that $n_{\gamma_0=0}^V = 1$ from the beginning.

Not all $\gamma \in G_S$ satisfy the conditions (2.35), but those that satisfy (2.35) are not extremely rare. Table 1 shows the list of such γ 's for some of the $\rho = 1$ lattices, $\Lambda_S = \langle +2n \rangle \cong_{ab} \mathbb{Z}e$. In the case of lattices of the form $\Lambda_S = U \oplus \langle -2m \rangle =:_{ab} U \oplus \mathbb{Z}e$, at least $(\gamma_0 = [\frac{1}{2m}e], k = 4m)$ and $(\gamma_0 = [\frac{2}{2m}e], k = m)$ satisfy (2.35).

There are a couple of different behaviors in the Type IIA geometry X_{IIA} that correspond to appearance of a massless non-abelian gauge boson in the low-energy effective field theory. Let $v = re^0 + q_0e^4 + F_{\parallel} \in \tilde{\Lambda}_S^\vee$ be the U(1) charge of such a gauge boson (as in (2.29)).

Suppose $rq_0 = 0$. Then the Calabi-Yau X_{IIA} should have a curve class $F_{\parallel} \in \Lambda_S^\vee \cong [H_2(X_{\text{IIA}}; \mathbb{Z})]^\text{vert}$ realized algebraically over a generic point of the base $\mathbb{P}_{\text{IIA}}^1$ (B_∞ in section 3.2 of [12]); this is because

$$c_{\gamma_0}(F_{\parallel}^2/2) = NL_{1+(F_{\parallel})^2/2, \gamma_0} < 0, \quad -1 < \frac{F_{\parallel}^2}{2} < 0 \quad (2.36)$$

is possible only when $\iota_\pi(\mathbb{P}^1)$ stays within a Noether-Lefschetz divisor $(\sum_{F_\perp} D_{NL(F_\perp)})/\Gamma_T$ with the sum ranging over $F_\perp$'s with $F_{\parallel}^2 + F_\perp^2 = -2$. The algebraic curve is $\pm F = \pm(F_{\parallel} + F_\perp) \in \text{II}_{3,19}$, which must be a (-2) curve. The vector boson on the spacetime $\mathbb{R}^{3,1}$ is massless when this (-2) curve in the K3-fibres collapses to zero volume $((t_2, F_{\parallel}) = 0)$. $F_\perp$ must be non-zero since we think of a case $\gamma_0 \neq 0 \in G_S$. Then there must be non-trivial monodromy on $F_\perp \in \Lambda_T$ so that F is in L_S , but not in Λ_S . Λ_S is a proper subset of L_S then.

²⁹This condition is required for all $v \in \tilde{\Lambda}_S^\vee$, because we assumed in section 2.1 (footnote 3) that there exists at least one state of charge v for any $v \in \gamma \subset \tilde{\Lambda}_S^\vee$.

³⁰It follows from these constraints that, for any non-zero isotropic γ in G_S , we get $n_{\gamma_0}^V = 0$. The assumption $n_{\gamma_0} = 0$ in (2.14) — automatic in a Type IIA geometric phase (footnote 21) — further implies $n_{\gamma_0}^H = 0$ then.

In the case of $\Lambda_S = U \oplus \langle -2m \rangle$ and $\gamma_0 = \pm[e/m]$ (the level k is m), the following interpretation seems to work: in terms of lattices, $L_S \cong U \oplus A_1[-1]^{\oplus m}$ and the lattice $\langle -2m \rangle \subset \Lambda_S$ is embedded diagonally in $A_1[-1]^{\oplus m}$; in terms of geometry, each K3 fibre of $X \rightarrow \mathbb{P}^1$ has m points of A_1 singularity, and those singular points form a curve in X that is an m -fold cover over the base $\mathbb{P}^1$. The gauge kinetic term of the vector field is ms in Type IIA reasoning, which agrees nicely with the gauge kinetic function ks for gauge fields associated with the level- k current algebra in Heterotic constructions [43].

There should also be geometry / lattice interpretation along the line of $\Lambda_S \subsetneq L_S$ also for the case $\Lambda_S = U \oplus \langle -2m \rangle$ and $\gamma_0 = \pm[e/(2m)]$, but we have not been able to find a functioning interpretation yet.

Suppose instead that $rq_0 \neq 0$. The massless vector boson is then a D4-D2-D0 bound state, not just of D2- and D0-branes ($r \neq 0$). The Kähler parameter t must be of order unity for the vector boson to become massless (for moderate choices of r , q_0 , and $F_{\parallel}$), so the base $\mathbb{P}_{\text{IIA}}^1$ may be large ($s_2 \gg 1$), but the fibre K3 is not safely in the large radius geometric regime. In any case of $\Lambda_S = \langle +2n \rangle$ with $\gamma_0 \in G_S$ satisfying (2.35), the U(1) charge w_0 for such a massless vector boson should be the one of this category, because $F_{\parallel}^2/2$ is positive definite in $\Lambda_S = \langle +2n \rangle$.

2.3 Examples of Φ

One can list up modular forms $\Phi \in \text{Mod}_0^{\mathbb{Z}}(11 - \rho/2, \rho_{\Lambda_S})$ that satisfy the condition $n_0 = -2$ and the lower bounds $n_{\gamma} \geq -2$ or $n_{\gamma} \geq 0$ depending on $\gamma \in G_S$ as seen in section 2.2. The easiest and well-known case is when Λ_S is unimodular:

$$\begin{aligned} (\tilde{\Lambda}_S, \Lambda_T) = & (U^{\oplus 2}, U^{\oplus 2} \oplus E_8^{\oplus 2}[-1]), \quad (U^{\oplus 2} \oplus E_8[-1], U^{\oplus 2} \oplus E_8[-1]), \\ & (U^{\oplus 2} \oplus E_8[-1]^{\oplus 2}, U^{\oplus 2}). \end{aligned} \quad (2.37)$$

The rank ρ is 2, 10, 18 for each case and Φ should be a scalar-valued modular form of weight $(22 - \rho)/2$ with $n_0 = -2$. So $\Phi = -2E_4E_6$ and $-2E_6$ for the first and second case, respectively. There is no candidate of Φ for the third case; $\tilde{\Lambda}_S = U^{\oplus 2} \oplus E_8[-1]^{\oplus 2}$ (i.e., zero instantons in $E_8 \times E_8$ in the Heterotic string) cannot be realized at least in our setup reviewed in section 2.1.

2.3.1 Cases $\Lambda_S = \langle +2 \rangle$, $\langle +4 \rangle$, and $\langle +6 \rangle$

We attempt at assessing how well/poorly the combination of the modular invariance, integrality of the BPS indices, and their lower bounds explains possible topological choices of K3-fibrations of Calabi-Yau three-folds for non-unimodular Λ_S . In section 2.3.1, we first work on the cases $\Lambda_S = \langle +2 \rangle$, $\langle +4 \rangle$, and $\langle +6 \rangle$, where a set of independent generators of $\text{Mod}_0(21/2, \rho_{\Lambda_S})$ is known [12, 38, 44, 62], as summarized in appendix A.2.1. The list of Φ 's determined in this way is compared against a list of Calabi-Yau three-folds with those Λ_S -polarized K3-fibration explicitly constructed in [38].

Let us illustrate the procedure using the case $\Lambda_S = \langle +2 \rangle$ as an example. To prepare a notation, let e be the generator of the free abelian group $\langle +2n \rangle =: {}_{ab} \mathbb{Z}e$; the formal basis elements of $\mathbb{C}[G_S]$ [resp. the low-energy BPS indices] are denoted by $e_{\gamma} = e_{j/2n}$

Λ_S	$\chi(X)$
$\langle +2 \rangle$	$\chi = -252 + 56n_{1/2}$
$\langle +4 \rangle$	$\chi = -168 + 128n_{1/4} + 14n_{2/4}$
$\langle +6 \rangle$	$\chi = -148 + 108n_{1/6} + 54n_{2/6} + 2n_{3/6}$

Table 2. The Euler number χ in terms of the low-energy BPS indices n_γ . $n_{2/4}$ and $n_{3/6}$ are not less than -2 , and all the rest are non-negative.

[resp. $n_\gamma = n_{j/2n}$] for short, when $\gamma = (j/2n + \mathbb{Z})e \in G_S$ for some $j \in \{0, 1, \dots, 2n-1\}$. Now, one can³¹ choose $\phi_{(i)} = [\theta_{\langle +2 \rangle}, E_{10-2i}]_i$, $i = 0, 1$ as a $\mathbb{C}$ -basis of the vector space $\text{Mod}_0(21/2, \rho_{\langle -2 \rangle})$ [38, 44]. The modular form Φ is parametrized by

$$\Phi = -2(\phi_{(0)} + \phi_{(1)}/2) - \frac{n_{1/2}}{4}\phi_{(1)}, \quad (2.38)$$

$$= e_0 \left(-2 + (300 - 56n_{1/2})q + (217200 - 13680n_{1/2})q^2 + \dots \right) \\ + e_{1/2} \left(n_{1/2}q^{\frac{1}{4}} + (2496 + 360n_{1/2})q^{\frac{5}{4}} + (665600 + 30969n_{1/2})q^{\frac{9}{4}} + \dots \right); \quad (2.39)$$

this linear combination is chosen so that the coefficients of e_0 and $e_{1/2}q^{1/4}$ be $n_0 = -2$ and $n_{1/2}$ respectively.³² It follows from the discussion in section 2.2 that $n_{1/2}$ should be a non-negative integer. This condition indicates that the Euler number $\chi = -c_0(0) = -[\Phi_0/\eta^{24}]_{q^0}$ of X_{IIA} is quantized and bounded from below:

$$\chi(X_{\text{IIA}}) = -252 + 56n_{1/2}, \quad n_{1/2} \in \mathbb{Z}_{\geq 0}. \quad (2.40)$$

It is also bounded from above, because $h^{2,1}(X_{\text{IIA}}) \geq 0$ (and hence $\chi(X_{\text{IIA}}) \leq 2h^{1,1} = 4$) in the Type IIA language.³³ So,

$$n_{1/2} \in \{0, 1, 2, 3, 4\}. \quad (2.41)$$

As a result, we see that when $\Lambda_S = \langle +2 \rangle$, there are only finite possibilities for Φ, χ , and BPS indices $n_{|\gamma|}$.

Similar procedure can be carried out for $\Lambda_S = \langle +4 \rangle$ and $\langle +6 \rangle$. See appendix A.2.1 for details. Here we just quote the results on Euler number in table 2. The lattice $\Lambda_S = \langle +4 \rangle, \langle +6 \rangle$ also allows only finite possibilities for $\chi, \{n_\gamma\}$ and Φ .

The authors of [38] scanned combinatorial data in the toric complete intersection construction and produced some explicit examples of Calabi-Yau three-folds with a $\langle +2n \rangle$ -polarized regular K3 fibration: see table 1 in [38]. All of their examples satisfy the integrality of n_γ and the bounds of n_γ and χ as described above, of course, but do not exhaust

³¹Here, $[-, -]_i$ is the Rankin-Cohen bracket. See appendix A.1.

³²Because Φ was parametrized by $\Phi = -2(\phi_{(0)} + \phi_{(1)}/2) + n/4\phi_{(1)}$ in [38, 44], the parameter n should be interpreted as $-n_{1/2}$.

³³In the Heterotic language, $c_0(0) = (252 - 56n_{1/2})$ is $\dim_{\mathbb{C}}([\mathcal{H}_{0,(1/2,1/2)}]_{L_0=1}) - 2 \dim_{\mathbb{C}}([\mathcal{H}_{0,(0,0)}]_{L_0=1}) - 4$, where the last term -4 is from η^{-2} of the first term in (2.8). By definition, $\dim_{\mathbb{C}}([\mathcal{H}_{0,(0,0)}]_{L_0=1}) =: \rho$, and the assumption that the Heterotic construction in consideration has a Type IIA dual in the geometric phase implies that $\dim_{\mathbb{C}}([\mathcal{H}_{0,(1/2,1/2)}]_{L_0=1}) \geq 2$ because the Type IIA compactifications has at least one hypermultiplet containing the dilaton. So, $c_0(0) \geq -2(\rho + 1)$.

all possibilities under these conditions. For example, when $\Lambda_S = \langle +2 \rangle$, their list includes threefolds with $\chi = -252, -196, -140, -84$, which correspond to $n_{1/2} = 0, 1, 2, 3$ in the parametrization above, but it does not include any cases where $\chi = -28$ and $n_{1/2} = 4$. The discussion above indicates that there cannot exist Calabi-Yau three-folds with a $\langle +2 \rangle$ -polarized regular K3 fibration (whether toric complete intersection or not) with $\chi < -252$, or $-252 < \chi < -196$, etc. But, we did not find any theoretical reason to rule out the case $n_{1/2} = 4$ and $\chi = -28$.

Similar stories hold for $\Lambda_S = \langle +4 \rangle$ and $\langle +6 \rangle$; table 1 in [38] shows geometric realizations for $n_{1/4} = 0, n_{2/4} = -2, 0, 2, 4, 6$ and $n_{1/6} = n_{2/6} = 0, n_{3/6} = 0, 8, 10, 14, 16, 20$. But there exist other values of integers $(n_{1/4}, n_{2/4})$ or $(n_{1/6}, n_{2/6}, n_{3/6})$ satisfying the bounds $n_{1/2}, n_{1/4}, n_{1/6}, n_{2/6} \geq 0$ and $n_{2/4}, n_{3/6} \geq -2$.

The absence of examples with odd $n_{2/4}$ and $n_{3/6}$ in geometry constructions is presumably explained as follows. The BPS index $n_{2/4}$ [resp. $n_{3/6}$] in the case of $\Lambda_S = \langle +4 \rangle$ [resp. $\langle +6 \rangle$] is $NL_{4/8,2/4}$ [resp. $NL_{9/12,3/6}$]. To the Noether-Lefschetz number $NL_{4/8,2/4}$, for example, both $F_\perp = (2/4)e'$ and $F_\perp = -(2/4)e'$ give rise to separate contributions $NL_{[F_\perp],2/4}$ where $[F_\perp] = F_\perp \Gamma_T$; here, e' is the generator of $\langle -4 \rangle =: {}_{ab} \mathbb{Z}e'$ in $\Lambda_T \cong \langle -4 \rangle \oplus U^{\oplus 2} \oplus E_8[-1]^{\oplus 2}$. They are separate contributions, because the two $F_\perp$'s shown above are not in a common orbit³⁴ of Γ_T . Their Noether-Lefschetz divisors $D_{NL(F_\perp)}$ in $D(\Lambda_T)/\Gamma_T$ are the same, so the sum of the two contributions is twice the single contribution³⁵ $NL_{[F_\perp],2/4}$ (cf. [12]). This argument is adapted in an obvious way³⁶ to the case of $\Lambda_S = \langle +6 \rangle$ and $n_{3/6}$.

We could not rule out $n_{1/4}, n_{1/6}, n_{2/6}$ that are non-zero,³⁷ or $n_{3/6} = 2, 4, 6$. We may have missed some additional physical/mathematical constraints,³⁸ or it is possible that some of such Calabi-Yau three-folds may exist, either outside of the scanned range of the combinatorial data in [38], or in the form that do not allow their realization by complete intersections in toric varieties.

2.3.2 Linear relations on the spectrum of a local effective field theory

As stated already in section 2.1.1, the classification invariant Φ is in the free abelian group whose rank is the $\dim_{\mathbb{C}}(\text{Mod}_0(11 - \rho/2, \rho\Lambda_S))$, and is also completely determined by the low-energy BPS indices $\{n_{|\gamma|}\}_{\pm\gamma \in G_S^<}$ (because the weight $(-1 - \rho/2)$ of Φ/η^{24} is strictly

³⁴In the argument here, we discuss only the contributions $NL_{[F_\perp],2/4}$ only from the Γ_T -orbits $[e'/2]$ and $[-e'/2]$. Note in v.2: It turns out that the Γ_T -orbit decomposition of $\{F_\perp \in e'/2 + \Lambda_T \mid -(F_\perp)^2/2 = 1/2 = \nu\}$ consists of one more orbit besides $[e'/2]$ and $[-e'/2]$. So, unless we can prove that $NL_{[F_\perp],2/4}$ from this orbit is even, we cannot rule out an odd $n_{2/4}$. Also for $\Lambda_S = \langle +6 \rangle$, $\{F_\perp \in e'/2 + \Lambda_T \mid (F_\perp)^2 = -3/2\}$ consists of three Γ_T orbits.

³⁵The authors do not have confidence to say that the divisor-curve intersection number $NL_{[F_\perp],2/4}$ is definitely an integer, because $D(\Lambda_T)/\Gamma_T$ has orbifold singularity associated with K3-surfaces of a non-trivial group of purely non-symplectic automorphisms. If $NL_{[F_\perp],2/4} \in \mathbb{Z}$, then $2NL_{[F_\perp],2/4} \in 2\mathbb{Z}$.

³⁶This argument is not applicable to the case $\Lambda_S = \langle +2 \rangle$. That is because $F_\perp = (1/2)e$ and $F_\perp = -(1/2)e$ are in one orbit under Γ_T . That is consistent with the fact that ref. [38] found threefolds for $\Lambda_S = \langle +2 \rangle$ with odd $n_{1/2}$'s.

³⁷ $n_{1/4} \geq 2$ and $n_{1/6} \geq 2$ are ruled out because $\chi(X)$ would be larger than $2h^{1,1}$ then.

³⁸The procedure explained in section 3 and exemplified in appendix B.1.3 is one of the ways to find a constraint on n_γ 's.

negative for any $\rho \in \{1, 2, \dots, 20\}$). This implies immediately that there is a linear relation among n_γ 's when $\dim_{\mathbb{C}}(\text{Mod}_0(11 - \rho/2, \rho_{\Lambda_S}))$ is strictly less than $d^< = |G_S^< / \pm|$. As is clear from the Heterotic description,³⁹ the low-energy BPS indices n_γ 's with $\gamma \neq 0$ are the multiplicities of fields with purely electric charge under the $(\rho + 2)$ gauge bosons. So, such a linear relation is that of the spectrum of $\mathcal{N} = 2$ supersymmetric Lagrangian-based effective field theory on $\mathbb{R}^{3,1}$ with charge lattice Λ_S . It cannot be related to the 4D triangle anomaly because of non-chiral matter representations (possibly to 6D box anomaly if Λ_S contains U), but it originates from the modular invariance of Φ .

Examples of the lattice Λ_S with such a prediction are found by using the dimension formula (A.12), (A.13) if $\rho < 18$ (so $11 - \rho/2 > 2$). Within rank-1 Λ_S 's, the lattice $\langle +2n \rangle = \langle +14 \rangle$ is the first example, where $\dim_{\mathbb{C}}(\text{Mod}_0(21/2, \rho_{\langle +14 \rangle})) = 7$, less than $d^< = d = 8$. We found that

$$14n_{1/14} + 8n_{2/14} - 13n_{3/14} - 6n_{4/14} - n_{5/14} - 6n_{6/14} + n_{7/14} + 28 = 0; \quad (2.42)$$

see appendix A.2.2 for necessary details. In the series of $\rho = 3$ lattices $\Lambda_S = U \oplus \langle -2m \rangle$ with $m \in \mathbb{N}$, the $m = 2$ case already has a prediction, because $\dim_{\mathbb{C}}(\text{Mod}_0(19/2, \rho_{\langle -4 \rangle})) = 2$ and $d^< = d = 3$.

$$n_{1/4} = 8n_{2/4} + 96. \quad (2.43)$$

Details are found in appendix A.2.1. In both of the series $\Lambda_S = \langle +2n \rangle$ and $\Lambda_S = U \oplus \langle -2m \rangle$, we confirmed that the dimension of $\text{Mod}_0(11 - \rho/2, \rho_{\Lambda_S})$ lies strictly below $d^<$ and also above zero for large n 's and m 's, by evaluation of the dimension formula (A.12), (A.13). The swampland surely exists within the space of local effective field theories,⁴⁰ if we restrict our attention to the class of Heterotic-Type IIA dual vacua reviewed in section 2.1.

It is also found that the vector spaces $\text{Mod}_0(21/2, \rho_{\langle +2n \rangle})$ and $\text{Mod}_0(19/2, \rho_{\langle -2m \rangle})$ continue to have strictly positive dimensions⁴¹ for larger n and m (by numerically evaluating (A.12), (A.13)). So, the modular invariance of Φ and the integrality of its coefficients alone do not rule out existence of Calabi-Yau three-folds with a $\Lambda_S = \langle +2n \rangle$ -polarized K3-fibration for an arbitrary large n , or also with $\Lambda_S = U \oplus \langle -2m \rangle$ for an arbitrary large m .

³⁹Modular form Φ 's can be defined in a Type IIA compactification on a Calabi-Yau X that does not necessarily have a K3-fibration; for a divisor P in X , Φ is the generating function of the helicity supertrace for states originating from a D4-brane wrapped on P . It is then in $\text{Mod}(11 - r/2, \rho_\Lambda)$, where the lattice Λ is the sublattice of $H^2(P)$ corresponding to the image $\iota_P^*(H^2(X))$ of the embedding $\iota_P : P \hookrightarrow X$; $r := \text{rank}(\Lambda)$. See [45]. Here, the weight $(-1 - r/2)$ of Φ/η^{24} is always strictly negative, and the n_γ 's are the multiplicities of states whose central charge may vanish at a positive Kähler parameter, even in this more general set-up ([39], section 3). So, the same argument as in the main text also holds; whenever $\dim_{\mathbb{C}}(\text{Mod}(11 - r/2, \rho_\Lambda))$ is strictly less than $|G_\Lambda / \pm|$, there is a linear relation among the low-energy BPS indices.

For a general divisor P in X , however, such a linear relation is among the multiplicities of states whose $U(1)$ charges are not necessarily mutually local. So, it cannot be regarded as a prediction on a spectrum of a Lagrangian-based local effective field theory. In the set-up discussed in the main text, P is the total fibre class D_s , where $P \cdot P = 0$, and $r = h^{1,1}(X) - 1$, not $h^{1,1}(X)$. This property makes all the states from a D4-brane on P free from magnetic charge (obvious in the Heterotic description from the start).

⁴⁰The lattice Λ_S is characterized within the language of local effective field theory on $\mathbb{R}^{3,1}$; it appears in the prepotential (3.28).

⁴¹There are more modular forms of a fixed weight and for $\Gamma(4n)$ for large n . So, that is not surprising.

2.3.3 Cases with $\rho = 20$ and $\rho = 19$

Here, we have a look at the cases with $\rho = 20$ and 19 . In some of them, we will see that the vector space $\text{Mod}(11 - \rho/2, \rho_{\Lambda_S})$ is empty, and that there cannot be such a lattice-polarized regular K3-fibration in a Calabi-Yau three-fold, so studies from both sides agree nicely.

When $\rho = 20$, the lattice Λ_T is of rank-2, positive definite, and even. One can see that the vector space $\text{Mod}(1, \rho_{\Lambda_S})$ is empty in the following way. For any $\Phi \neq 0$ in this vector space, $\theta_{\Lambda_T} \cdot \Phi$ must be a scalar-valued weight-2 modular form starting with $-2 + \mathcal{O}(q)$. Because there is no such modular form, the vector-valued modular form Φ should have been zero.

One can also arrive at almost the same conclusion independently by using geometry available in the Type IIA language. If a Calabi-Yau three-fold X_{IIA} has a K3-fibration with a generic fibre having $\rho = 20$ Neron-Severi lattice, the fibre K3 surface has a fixed complex structure over the entire base $\mathbb{P}_{\text{IIA}}^1$, so X_{IIA} must be of the form $(\rho = 20 \text{ K3}) \times \mathbb{P}_{\text{IIA}}^1$. This is not a Calabi-Yau three-fold, so there should not be such a K3-fibred Calabi-Yau three-fold. It should be noted, however, that this second argument does not rule out non-geometric phase⁴² Type IIA constructions with $\rho = 20$, and hence the first argument is stronger.

Similar arguments also rule out a family of $\rho = 19$ cases

$$\tilde{\Lambda}_S = U^{\oplus 2} \oplus E_8[-1]^{\oplus 2} \oplus \langle -2n \rangle, \quad \Lambda_T = U \oplus \langle +2n \rangle, \quad (2.44)$$

where $n \in \mathbb{Z}_{>0}$. The first argument for the $\rho = 20$ cases can be repeated by replacing θ_{Λ_T} with $\theta_{\langle +2n \rangle}$, to see that the vector space $\text{Mod}(3/2, \rho_{\Lambda_S})$ is empty. So there is no suitable Φ in this case.

The absence of such a Φ (with $n_0 = -2$) is also understandable in terms of geometry. The Fourier coefficients of Φ are the intersection numbers of the Noether-Lefschetz divisors in $D(\Lambda_T)/\Gamma_T$ (they are points in the $\rho = 19$ cases) and the image $\iota_\pi(\mathbb{P}_{\text{IIA}}^1)$ of the base $\mathbb{P}_{\text{IIA}}^1$. The moduli space⁴³ $D(\Lambda_T)/\Gamma_T = \mathcal{H}/\Gamma_0(n)$ contains the large complex structure limit point (when $\mathcal{H}/\Gamma_0(n)$ is compactified); if ι_π is surjective, then there are points in the base $\mathbb{P}_{\text{IIA}}^1$ where the K3-fibration is not regular. All the $\rho = 19$ K3-fibrations studied in [47] are in this category. If ι_π were to be a constant map, then X_{IIA} would not be a Calabi-Yau three-fold (see the second argument for the $\rho = 20$ case).

This geometry-based argument for the absence of Calabi-Yau three-folds with a Λ_S -polarized regular K3-fibration holds true for all the $\rho = 19$ cases, not necessarily for the $\Lambda_T = U \oplus \langle +2n \rangle$ cases discussed above. The authors do not have a proof yet that $\text{Mod}_0(3/2, \rho_{\Lambda_S})$ is empty for such general Λ_T 's with $\rho = 19$, however.

The consequence that the dimension of $\text{Mod}_0(11 - \rho/2, \rho_{\Lambda_S})$ is smaller for larger ρ is understandable intuitively in itself. A K3 fibration is specified, after all, by specifying a map from the base $\mathbb{P}^1$ to the period domain $D(\Lambda_T)$; less complicated geometry $D(\Lambda_T)$ allows less variety in the map from $\mathbb{P}^1$ to $D(\Lambda_T)$.

⁴²Because we only think of Type IIA compactifications in the $s_2 \gg 1$ regime, this premise almost implies that the base $\mathbb{P}^1$ is in the geometric phase. But we still need to find an appropriate technical language to extend this argument to cover constructions without entirely geometric X_{IIA} .

⁴³In the case of $\Lambda_T = U \oplus \langle +2n \rangle$, $\text{Isom}(\Lambda_T) \cong \Gamma_0(n)_+$ (see [46]), but $\Gamma_T \cong \Gamma_0(n)$.

2.4 Lower bounds on Euler numbers

We have seen in table 2 that the Euler number $\chi(X_{\text{IIA}})$ of any Calabi-Yau three-fold X_{IIA} that admits Λ_S -polarized K3-fibration is given by a linear sum of the low-energy BPS indices $\{n_{|\gamma|}\}$, for a few choices of Λ_S . In table 2, all the coefficients of $\{n_{|\gamma|}\}$ are positive, from which it follows that $\chi(X_{\text{IIA}})$ is bounded from below. Actually, this is true for any choice of Λ_S , as we see below.

Suppose⁴⁴ $\phi \in \text{Mod}(3 + \rho/2, \rho_{\Lambda_S}^\vee)$. Then

$$\phi \cdot \frac{\Phi}{\eta^{24}} = -2[\phi]_{q^0} \frac{E_4^2 E_6}{\eta^{24}} \quad (2.45)$$

because $\phi \cdot \Phi$ must be a scalar-valued $\text{SL}(2; \mathbb{Z})$ modular form of weight 14 with the leading coefficient $[\phi \cdot \Phi]_{q^0} = -2[\phi]_{q^0}$. By comparing the coefficients of the q^0 term on both sides,⁴⁵ we obtain one linear relation of $\{n_\gamma\}$ and $c_0(0) = -\chi(X_{\text{IIA}})$ for one $\phi \in \text{Mod}(3 + \rho/2, \rho_{\Lambda_S}^\vee)$.

A non-trivial ϕ can be constructed by a theta function for a suitable lattice. The $(1, 7 + \rho)$ lattice $\Lambda_S \oplus E_8[-1]$ has a primitive sublattice⁴⁶ isometric to U . Since U is unimodular, the orthogonal complement $L := [U^\perp \subset \Lambda_S \oplus E_8[-1]]$ satisfies

$$\Lambda_S \oplus E_8[-1] \cong U \oplus L. \quad (2.46)$$

This lattice L has signature $(0, 6 + \rho)$ and shares the same discriminant form $(G_S, (-, -)_{G_S})$ with Λ_S , so the lattice theta function⁴⁷ $\theta_{L[-1]}$ is in $\text{Mod}(3 + \rho/2, \rho_{\Lambda_S}^\vee)$. The linear relation for $\phi = \theta_{L[-1]}$,

$$\sum_{b \in L^\vee} c_{[b]}(b^2/2) = 0, \quad (2.47)$$

leads to

$$\chi(X) = -c_0(0) = \sum_{b \in L^\vee}^{(-2 < b^2 < 0)} n_{[b]} + \sum_{b^2 = -2} c_{[b]}(-1) = \sum_{b \in L^\vee}^{(-2 < b^2 < 0)} n_{[b]} - 2 \cdot \#(\text{roots of } L). \quad (2.48)$$

The sum over $b \in L^\vee$ is a finite sum since $L[-1]$ is positive definite and $c_\gamma(\nu) = 0$ for $\nu < -1$. The last equality follows from $c_\gamma(-1) = 0$ for $\gamma \neq 0 \in G_S$ (see (2.14)). For example, when $\Lambda_S = U \oplus W$ for an even lattice W , the relation for $\phi = \theta_{W[-1]} E_4$ yields

$$\chi(X_{\text{IIA}}) = \left(\sum_{b \in W^\vee}^{0 < b^2 < -2} n_{[b]} \right) + (-2) \times |\{b \in W \mid b^2 = -2\}| - 480. \quad (2.49)$$

In particular, $\chi = -960$ when $W = E_8[-1]$.

⁴⁴There is a more general version of this argument. A linear relation among Fourier coefficients of arbitrary $\Phi' \in \text{Mod}(2k + 11 - \rho/2, \rho_{\Lambda_S})$ may be obtained by using $\phi \in \text{Mod}(2k' + 1 + \rho/2, \rho_{\Lambda_S}^\vee)$; the combination $\phi \cdot \Phi'$ must be a scalar-valued $\text{SL}(2; \mathbb{Z})$ modular form of weight $(2k + 2k' + 12)$.

⁴⁵The same relation is obtained also from $0 = \oint_{\tau \sim i\infty} d\tau \phi \cdot (\Phi/\eta^{24}) = [\phi \cdot (\Phi/\eta^{24})]_{q^0}$, because $\phi \cdot \Phi/\eta^{24}$ is of weight 2 (e.g., [48]).

⁴⁶To see this, choose an element $x \in \Lambda_S$ of positive norm $2n$; there exists such x for some $n > 0$, since Λ_S is indefinite. One can choose a primitive element $y \in E_8[-1]$ of norm $-2n$, and also an element $z \in E_8[-1]$ such that $(y, z) = 1$. Now, the sublattice $\text{span}_{\mathbb{Z}}\{x + y, z\} \subset \Lambda_S \oplus E_8$ is isometric to U .

⁴⁷For example, when $\Lambda_S = U \oplus W$ for some even lattice W of signature $(0, \rho - 2)$, there is always an obvious embedding of U into $\Lambda_S \oplus E_8[-1] = U \oplus W \oplus E_8[-1]$. $L = W \oplus E_8[-1]$, so $\theta_{L[-1]} = \theta_{W[-1]} E_4$.

$2n$	2	4	6	8	10	12	14	14
$\chi \geq$	-252	-196	-152	-112	-124	-124	-144	-92

Table 3. Lower bounds on $\chi(X)$ for X 's that have a regular $\Lambda_S = \langle +2n \rangle$ -polarized K3-fibration. Two lower bounds are listed for $\langle 2n \rangle = \langle 14 \rangle$ because we can construct two inequivalent lattices L .

The linear coefficient of $n_{|\gamma|}$ on the right hand side of (2.48) is positive for any $|\gamma|_{\neq 0} \in G_S/\pm$, because we have used a lattice theta function for ϕ . Then the lower bounds on n_γ 's lead to a lower bound on $\chi(X)$:

$$\chi \geq -2 \cdot \#\{b \in L^\vee; -2 \leq b^2 < 0\}. \quad (2.50)$$

If there are no $\gamma \in G_S$ that satisfy the condition (2.35), then $n_\gamma \geq 0$ for $\gamma_{\neq 0} \in G_S$, and

$$\chi \geq -2 \cdot \#(\text{roots of } L). \quad (2.51)$$

The relation (2.48) reproduces those in table 2 by using $\phi = \theta_{L[-1]}$ for an L determined as in footnote 46. We applied the same procedure for some $\Lambda_S = \langle +2n \rangle$ not covered in section 2.3, and obtained the result summarized in table 3; calculations that led to table 3 are found in appendix A.2.2. Note that it is not necessary to work out a basis of $\text{Mod}_0(11 - \rho/2, \rho_{\Lambda_S})$ (where Φ lies in) in deriving these results.

All the lower bounds of $\chi(X_{\text{IIA}})$ for individual Λ_S in table 3 (and $\chi = -960$ for $\Lambda_S = U \oplus E_8[-1]$) are safely above the absolute lower bound for all Calabi-Yau three-folds X_{IIA} [49],

$$\chi \gtrsim -\frac{5}{3}e^{4\pi} \sim -5 \times 10^5. \quad (2.52)$$

The bound (2.52) was derived [49] by exploiting the modular invariance of the fundamental string partition function of Type II compactification (with X_{IIA} as the target space), while the bound (2.50) is due to the modular property of Φ , the generating function of the helicity supertraces of vertical D4-D2-D0 BPS bound states on X_{IIA} . Recall the latter modular property can be easily derived from that of the new supersymmetric index Z_{new} of the fundamental string in the Heterotic description. (See section 2.1.1.)

One also finds from the relation (2.48) that the low-energy BPS indices also have an upper bound. This is because

$$\chi(X_{\text{IIA}}) = 2h^{11} - 2h^{21} \leq 2h^{11} = 2(\rho + 1). \quad (2.53)$$

This is a generalization of the same observation made already in section 2.3.1. Note, however, that all of the coefficients n_γ do not necessarily appear in the equation (2.48); in the case of $\Lambda_S = U \oplus \langle -2n \rangle =:_{ab} U \oplus \mathbb{Z}e$, for example, the linear relation (2.48) for $\phi = \theta_{\langle +2n \rangle} E_4$ has contributions only from $\gamma = [\frac{x}{2n}e] \in G_S$ with $0 \leq x \leq \sqrt{4n}$. These are only $\mathcal{O}(\sqrt{n})$ coefficients among the $\mathcal{O}(n)$ low-energy BPS indices. So, we cannot use this argument to claim that only finite choices of the low-energy BPS indices $\{n_\gamma\}$ correspond to geometric phases in the Type IIA description.

3 Finer classifications

The modular form Φ (new supersymmetric index (Het)/Noether-Lefschetz number generating function (IIA)) is not enough discrete data for classification of branches of moduli space of the Het-IIA dual vacua. Let us take the $\Lambda_S = U$ case as an example. In the Heterotic language, all the $K3 \times T^2$ compactifications with the 24 instantons on K3 distributed by $(12 - n, 12 + n)$ to the two weakly coupled E_8 gauge groups share the same $\Phi = (-2E_4E_6)$ for all $0 \leq n \leq 2$, but they form three distinct branches of moduli space. In the Type IIA language, the modular form Φ determines the Gopakumar-Vafa invariants of all the vertical curve classes of an elliptic-K3 fibred Calabi-Yau three-fold X_{IIA} , but the classical trilinear intersection numbers of the divisors are not completely determined [11] (for precise statements, see section 3.1.3). X_{IIA} can be any one of the elliptic fibrations over F_n with $n = 0, 1, 2$. Presence of such multiple branches of moduli space sharing Φ has been reported also for the case of $\Lambda_S = \langle +2 \rangle$ and $\langle +4 \rangle$ [38] (see also [17]).

We introduce invariants of branches of moduli space of the Het-IIA dual vacua that can distinguish those sharing a common Φ . That is done by developing observations and ideas that are found in the literatures. Those invariants do not rely on supergravity approximation or explicit construction of geometries, but use modular forms.

3.1 The idea

Consider a branch of moduli space of the Het-IIA dual vacua, where we have special geometry and hypermultiplet moduli space of fixed dimensions ($h^{1,1}(X_{\text{IIA}})$ - and $(h^{2,1}(X_{\text{IIA}}) + 1)$ -dimensions, respectively, if the branch contains a geometric phase in the Type IIA language). We call it the original branch. It often comes with special loci in the hypermultiplet moduli space where non-abelian gauge symmetry $\mathcal{R}$ is enhanced in the effective theory on $\mathbb{R}^{3,1}$; one ventures into other branches of moduli space by turning on non-zero Coulomb vacuum expectation values in $\mathcal{R}$. Modular forms denoted by Ψ and $\underline{\Phi}$ are assigned to such a symmetry-enhanced branch (see below for details); the idea is to use the set of such modular forms as an invariant of the original branch. We will see in this section 3 that the set of Ψ 's or the set of $\underline{\Phi}$'s distinguish multiple branches sharing the same $\tilde{\Lambda}_S$, Λ_T , and Φ ; moreover, the modular form Ψ or $\underline{\Phi}$ of even just one symmetry-enhanced branch attached to the original branch already improves the classification by the modular form Φ alone.

3.1.1 Higgs cascades and modular forms

Let us first assign two modular forms $\underline{\Phi}$ and Ψ for a symmetry-enhanced branch. We will discuss in section 3.1.3 the information of the target-space geometry in the original branch that we can extract from such a modular form Ψ or from the set of Ψ 's associated with all the symmetry-enhanced branches.

We restrict our attention to the case where $\mathcal{R}$ is one of ADE types and its non-abelian gauge bosons are given by left-moving level $k = 1$ current algebra in the Heterotic

language.⁴⁸ The lattice $\mathcal{R}[-1]$ is chosen within Λ_T ; now we introduce lattices

$$\underline{\Lambda}_T := \left[\mathcal{R}[-1]^\perp \subset \Lambda_T \right], \quad \tilde{\Lambda}_S := \left[\underline{\Lambda}_T^\perp \subset \Pi_{4,20} \right] \quad (3.1)$$

for the symmetry-enhanced branch. The lattice $\tilde{\Lambda}_S$ is $\tilde{\Lambda}_S \oplus \mathcal{R}[-1]$ or its extension.

It is assumed here that the symmetry-enhanced branch is also realized without NS5-branes and the likes in the Heterotic description, or without a degeneration of K3 fibre classified as Type II or III in the Type IIA description. A related discussion is found at the end of section 3.2.2.

Under this assumption, there must be a modular form

$$\underline{\Phi} \in \text{Mod}_0(11 - \underline{\rho}/2, \rho_{\underline{\Lambda}_S}); \quad (3.2)$$

it describes the BPS indices of the Heterotic description and the Noether-Lefschetz numbers of the K3-fibre in the Type IIA description in the symmetry-enhanced branch, just like Φ does for the original branch. Here $\underline{\rho} := \rho + \text{rank}(\mathcal{R})$, and $\rho_{\underline{\Lambda}_S}$ the representation of $\text{Mp}(2; \mathbb{Z})$ associated with the lattice $\underline{\Lambda}_S$.

The modular form $\underline{\Phi}$ of the symmetry-enhanced branch should be related to Φ of the original branch in the following way. At the entrance of the symmetry-enhanced branch (so the Coulomb branch moduli still stays within the subset $D(\tilde{\Lambda}_S)$ of $D(\tilde{\Lambda}_S)$), the non-abelian symmetry $\mathcal{R}$ remains unbroken. Since this vacuum belongs to both of the original and symmetry-enhanced branches, the new supersymmetric index Z_{new} of both branches should be equal at this point:

$$\sum_{\underline{\gamma} \in \underline{G}_S} \theta_{\tilde{\Lambda}_S[-1] + \underline{\gamma}} \underline{\Phi}_{\underline{\gamma}} = \sum_{\gamma \in G_S} \theta_{\tilde{\Lambda}_S[-1] + \gamma} \Phi_{\gamma}. \quad (3.3)$$

Here $\underline{G}_S = \tilde{\Lambda}_S^\vee / \tilde{\Lambda}_S$. Define $G_0 := \tilde{\Lambda}_S^\vee / (\tilde{\Lambda}_S \oplus \mathcal{R}[-1]) \subset G_S \times G_{\mathcal{R}}$ and $[\gamma, \delta] := (\gamma, \delta) + \tilde{\Lambda}_S \in \underline{G}_S$ for $(\gamma, \delta) \in G_0$. Then we can rewrite the left hand side of the above as

$$\sum_{(\gamma, \delta) \in G_0} \theta_{\tilde{\Lambda}_S[-1] + \gamma} \theta_{\mathcal{R} + \delta} \underline{\Phi}_{[\gamma, \delta]}. \quad (3.4)$$

Therefore, the modular form $\underline{\Phi}$ of the symmetry-enhanced branch should reproduce Φ of the original branch through

$$\Phi_{\gamma} = \sum_{(\gamma, \delta) \in G_0} \theta_{\mathcal{R} + \delta} \underline{\Phi}_{[\gamma, \delta]}. \quad (3.5)$$

In the case of $\tilde{\Lambda}_S = \tilde{\Lambda}_S \oplus \mathcal{R}$, this simplifies to [50–52]

$$\Phi_{\gamma} = \sum_{\delta \in G_{\mathcal{R}}} \theta_{\mathcal{R} + \delta} \underline{\Phi}_{(\gamma, \delta)}. \quad (3.6)$$

⁴⁸It is possible for a gauge group with level $k > 1$ to enhance, although we do not use it as a probe in this article.

The modular form Ψ is introduced in association with 1-loop⁴⁹ threshold correction $\Delta_{\mathcal{R}}$ to the coupling constant of the enhanced non-abelian gauge group $\mathcal{R}$, which is given by [21, 53]

$$\Delta_{\mathcal{R}} \delta^{IJ} = \int_{\mathcal{H}/\mathrm{SL}_2\mathbb{Z}} \frac{d\tau_1 d\tau_2}{\tau_2} (\mathcal{B}^{IJ} - b_{\mathcal{R}} \delta^{IJ}), \quad (3.7)$$

$$\mathcal{B}^{IJ} = \frac{-i}{\eta^2} \mathrm{Tr}_{\mathrm{R-sector}}^{(c,\tilde{c})=(22,9)} \left[e^{\pi i F_R} F_R q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} \left(\frac{Q^I Q^J}{2} - \frac{\delta^{IJ}}{8\pi\tau_2} \right) \right],$$

where the Heterotic string is used as a language. Here, $I, J \in \{1, \dots, \mathrm{rank}(\mathcal{R})\}$ label $\mathrm{rank}(\mathcal{R})$ left-moving free bosons⁵⁰ X_L^I , and Q^I is the zero-mode momentum in the expansion $X_L^I(z) = x^I + Q^I \ln z + \text{oscillators}$. Q^I works as the Cartan charge operator of $\mathcal{R}$. $b_{\mathcal{R}} = (\sum_{\text{hyper}} 2T_{\text{rep}}) - 2T_{\mathcal{R}}$ is the 1-loop beta function of the probe gauge group $\mathcal{R}$. Contracting the indices I, J , we arrive at [52]

$$\Delta_{\mathcal{R}} = \int_{\mathcal{H}/\mathrm{SL}_2\mathbb{Z}} \frac{d\tau_1 d\tau_2}{\tau_2} (\mathcal{B}_{\mathcal{R}} - b_{\mathcal{R}}), \quad (3.8)$$

$$\mathcal{B}_{\mathcal{R}} = \frac{1}{24} \sum_{\gamma \in G_S} \theta_{\tilde{\Lambda}_S[-1]+\gamma} \frac{\Phi_{\gamma} \hat{E}_2 - \Psi_{\gamma}}{\eta^{24}}, \quad (3.9)$$

$$\Psi_{\gamma} = -\frac{24}{\mathrm{rank}(\mathcal{R})} \sum_{(\gamma, \delta) \in G_0} (\partial^S \theta_{\mathcal{R}+\delta}) \Phi_{[\gamma, \delta]}. \quad (3.10)$$

Here ∂^S is the Ramanujan-Serre derivative (see appendix A.1). Immediately from (3.2), (3.10),

$$\Psi \in \mathrm{Mod}_0(13 - \rho/2, \rho_{\Lambda_S}). \quad (3.11)$$

Obviously the modular form Ψ/η^{24} (for not necessarily unimodular Λ_S) is the generalization of $\Psi/\eta^{24} \in \mathrm{span}_{\mathbb{C}}\{E_4^3, E_6^2\}/\eta^{24}$ for $\Lambda_S = U$ and $\Psi/\eta^{24} = (-2E_4^2)/\eta^{24}$ for $\Lambda_S = U \oplus E_8[-1]$ (e.g. [11, 50–52]).

The modular form Ψ of the symmetry-enhanced branch captures only a part of information in $\underline{\Phi}$, because Ψ can be determined from $\underline{\Phi}$ as in (3.10). In fact, when there is a chain of symmetry enhancements $\mathcal{R}_1 \subsetneq \mathcal{R}_2 \subsetneq \dots$ accompanied by a chain of tunings in the hypermultiplet moduli space, the chain of the invariants $(\tilde{\Lambda}_S, \underline{\Lambda}_T, \underline{\Phi})_{\mathcal{R}=\mathcal{R}_i}$ all reproduce one common modular form Ψ through (3.10). This is because the corrections to the gauge coupling constants remain unchanged by continuous change in the hypermultiplet vacuum expectation values in $\mathcal{N} = 2$ supersymmetric gauge theories on $\mathbb{R}^{3,1}$, an observation implicit already in [11]. For this reason, the modular form $\underline{\Phi}$ is assigned to each symmetry-enhanced branch, but Ψ to a chain of symmetry-enhanced branches attached to the original branch (such a chain is called a Higgs cascade).

⁴⁹The corrections to special geometry that are regarded as 1-loop contributions in the Heterotic string language are those in the Type IIA language that neither diverge (tree in Het) nor vanish (non-perturbative in Het) in the large base limit.

⁵⁰Normalized so $X_L^I(z)X_L^J(w) \sim -\delta^{IJ} \ln(z-w)$.

We will see in section 3.1.3 that the modular form Ψ for one Higgs cascade attached to the original branch — an arbitrarily chosen cascade is fine — specifies the diffeomorphism class of X_{IIA} of the original branch (of the individual geometric phase chambers of the original branch, to be more precise); the modular form Φ alone does not have enough information for this purpose in general. Furthermore, the set of Ψ 's for the set of Higgs cascades attached to the original branch can also be used as a classification invariant of the original branch. This viewpoint is sometimes useful for distinguishing two different branches of moduli space with the same diffeomorphism class of Λ_S -polarized K3-fibred Calabi-Yau three-folds. The modular forms $\underline{\Phi}$ are more useful in capturing the network of symmetry-enhanced branches.

3.1.2 The space of the modular form Ψ 's

For a given branch of the Het-IIA dual moduli space characterized by $(\tilde{\Lambda}_S, \Lambda_T, \Phi)$, the modular form Ψ of a Higgs cascade of the original branch is not a completely arbitrary element of the vector space (3.11). We will derive a few constraints on Ψ in the following.

General constraints. First, recall the definition:

$$\frac{\theta_{\tilde{\Lambda}_S[-1]} \cdot (\Phi E_2 - \Psi)}{\eta^{24}} \delta^{IJ} = 12 \cdot \frac{-i}{\eta^2} \text{Tr}_{\text{R-sector}}^{(c, \tilde{c})=(22,9)} \left[e^{\pi i F_R} F_R q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} Q^I Q^J \right] \quad (3.12)$$

for $1 \leq I, J \leq \text{rank}(\mathcal{R})$. Choose a basis of the left-moving free bosons X_L^I so that roots of a fixed subalgebra $\mathfrak{su}(2) \subset \mathcal{R}$ have charges only in $I = 1$; $Q^I = \pm \sqrt{2} \delta_1^I$. Now set $I = J = 1$ in the above equation. Since all the states in the Hilbert space with a definite charge under $\mathcal{R}$ have $Q^{I=1} Q^{J=1} \in \frac{1}{2}\mathbb{Z}$, contributions from states with a charge (w, Q^1) under $\tilde{\Lambda}_S \oplus \mathfrak{su}(2)$ and those with a charge $(w, -Q^1)$ add up to be an integer. This implies that

$$d_\gamma(\nu) \in 12\mathbb{Z}, \quad \frac{\Phi_\gamma E_2 - \Psi_\gamma}{\eta^{24}} =: \sum_{\nu \in h_{\min}(\gamma) + \mathbb{Z}} d_\gamma(\nu) q^\nu. \quad (3.13)$$

As an immediate consequence, all the Fourier coefficients $c_\gamma^\Psi(\nu)$ in

$$\frac{\Psi}{\eta^{24}} = \sum_{\gamma \in G_S} e_\gamma \sum_{\nu \in \mathbb{Q}} c_\gamma^\Psi(\nu) q^\nu \quad (3.14)$$

are all integers.

In other words, this comes just from the properties of lattice theta functions of simple Lie algebra: Using the relations (3.10), (3.5), we see that

$$\frac{\Phi_\gamma E_2 - \Psi_\gamma}{\eta^{24}} = 12 \frac{2}{\text{rank}(\mathcal{R})} \sum_{(\gamma, \delta) \in G_0} q \frac{\partial \theta_{\mathcal{R}+\delta}}{\partial q} \cdot \frac{\Phi_{[\gamma, \delta]}}{\eta^{24}}. \quad (3.15)$$

Defining $a_\delta^{(\mathcal{R})}(\nu)$ by

$$\sum_{\nu \in \delta^2/2 + \mathbb{Z}} a_\delta(\nu) q^\nu := \frac{2}{\text{rank}(\mathcal{R})} q \frac{\partial}{\partial q} \theta_{\mathcal{R}+\delta}, \quad (3.16)$$

we have

$$d_\gamma(\nu) = 12 \sum_{(\gamma, \delta) \in G_0} \sum_{\nu' + \nu'' = \nu} a_\delta(\nu') \underline{c}_{[\gamma, \delta]}(\nu''). \quad (3.17)$$

The integrality of $a_\delta(\nu)$ can be seen in essentially the same way as the discussion above (see also appendix A.2.3). So we have $d_\gamma(\nu) \in 12\mathbb{Z}$, because the BPS indices $\underline{c}_{[\gamma, \delta]}(\nu)$ are also integers.

As we see later, consistency in the low-energy effective field theory on $\mathbb{R}^{3,1}$ implies that $d_0(0) \in 24\mathbb{Z}_{\geq 0}$. We have not tried much to think whether this condition can be derived directly from consistency of string theory.

Not all the $d_\gamma(\nu)$'s are arbitrary integers divisible by 12 (or 24). From the fact $a_0^{(\mathcal{R})}(0) = 0$ and that $\nu_\delta := \min\{x^2/2 \mid x \in \delta \subset \mathcal{R}^\vee\}$ is strictly positive for $\delta \neq 0$, it follows immediately that

$$d_\gamma(-1) = 0, \quad \forall \gamma \in G_S, \quad \text{s.t.} \quad (\gamma, \gamma)/2 = 0 \in \mathbb{Q}/\mathbb{Z}. \quad (3.18)$$

We can say a little more. Define $m_\gamma := d_\gamma([\gamma^2/2]_{\text{frac}} - 1)$ for $(\Phi E_2 - \Psi)$, like we defined n_γ for Φ :

$$m_\gamma = 12 \sum_{(\gamma, \delta) \in G_0}^{([\gamma, \gamma]/2]_{\text{fr}} > \nu_\delta} a_\delta(\nu_\delta) \underline{n}_{[\gamma, \delta]}. \quad (3.19)$$

Here only $\delta \in G_{\mathcal{R}}$ such that $[(\gamma, \gamma)/2]_{\text{frac}} > \nu_\delta$ can contribute to the sum. In particular,

$$m_\gamma = 0 \quad \forall \gamma \in G_S, \quad \text{s.t.} \quad [\gamma^2/2]_{\text{frac}} \leq 1/4, \quad (3.20)$$

regardless of $\mathcal{R}$, because $(\delta = e_1, \mathcal{R} = A_1)$ gives the smallest⁵¹ ν_δ among all possibilities for $(\delta, \mathcal{R})$. This supersedes the condition (3.18). See appendix A.2.3 for a list of values of ν_δ for various $\mathcal{R}$'s.

Ref. [11] sets a constraint that Heterotic string tachyon states should have zero contribution to the gauge threshold correction $\Delta_{\mathcal{R}}$ (because they are not charged under $\mathcal{R}$) in relating Ψ to Φ , and this reasoning was enough to determine Ψ completely in the case of $\Lambda_S = U \oplus E_8[-1]$. The relation (3.18) and (3.20) are kinds of generalizations of this argument. On the other hand, m_γ for γ satisfying $[\gamma^2/2]_{\text{frac}} \geq 1/4$ can be non-trivial in many cases. An easiest example is for $\Lambda_S = U + A_1[-1]$ with an enhancement of symmetry $\mathcal{R}$ so that $A_1 + \mathcal{R}$ within contained in one weakly coupled E_8 of the Heterotic string theory.

Let us introduce a space denoted by $\text{Mod}_0^\Phi(13 - \rho/2, \rho_{\Lambda_S})$ for a given Φ , which is the set of modular forms Ψ satisfying $d_\gamma(\nu) \in 12\mathbb{Z}$, $d_0(0) \in 24\mathbb{Z}_{\geq 0}$, and (3.20). When the modular form Ψ is for a Higgs cascade attached to the original branch with Φ , then Ψ must be in this set. For example, in the case of $\Lambda_S = U$ (where $\Phi = -2E_4E_6$), the space $\text{Mod}_0^\Phi(12, \rho_U)$ consists of $\Psi = -E_4^3 - E_6^2 + (288 - d(0))\eta^{24}$ with $d(0) \in 24\mathbb{Z}_{\geq 0}$.

⁵¹This is because A_1 is a Lie subalgebra of any $\mathcal{R}$.

Field-theory argument for $d_0(0) \in 24\mathbb{Z}_{\geq 0}$. It follows that $d_0(0) \in 24\mathbb{Z}$, not just in $12\mathbb{Z}$, from its relation to the 1-loop beta function $b_{\mathcal{R}}$ through $b_{\mathcal{R}} = d_0(0)/24$ (e.g. [11]); this relation itself is obvious also from the expression (3.17):

$$\frac{d_0(0)}{24} = -a_0^{(\mathcal{R})}(1) + \sum_{(0,\delta) \neq 0 \in G_0}^{(0 < \nu_\delta < 1)} \frac{a_\delta^{(\mathcal{R})}(\nu_\delta)}{2} \underline{n}_{[0,\delta]}(-\nu_\delta) \rightarrow b_{\mathcal{R}} = -2T_{\mathcal{R}} + \sum_{\text{halfhyp}} T_\delta, \quad (3.21)$$

where we made replacements $a_0^{(\mathcal{R})}(1) \rightarrow 2T_{\mathcal{R}}$ and $a_\delta^{(\mathcal{R})}(\nu_\delta) \rightarrow 2T_\delta$ for $\nu_\delta < 1$ (see appendix A.2.3), and interpreted $\underline{n}_{[0,\delta]} = \underline{n}_{[0,-\delta]}$ as the (effective) number of half-hyper multiplets in the corresponding representation. Now, we see that the fact $d_0(0) \in 12\mathbb{Z}$ implies $b_{\mathcal{R}} \in \mathbb{Z}/2$. But $b_{\mathcal{R}}$ can be in $1/2 + \mathbb{Z}$ only when $\mathcal{R} = A_1$ and there are odd number of hypers in the fundamental representation, which is not allowed because it would cause the $SU(2)$ global anomaly. So $b_{\mathcal{R}} \in \mathbb{Z}$, and $d_0(0) \in 24\mathbb{Z}$.

In addition, $b_{\mathcal{R}}$ should be non-negative because there must be plenty of matter fields to Higgs the gauge symmetry $\mathcal{R}$ completely; the Higgsing brings the symmetry-enhanced branch back to the original branch. In the case of $\Lambda_S = U$ with the probe symmetry set in one of the two weakly coupled E_8 's of the Heterotic string, for example, it is known that there are $I := 10 + 12^{-2}d(0)$ instantons in the rest of the E_8 . The condition that $d(0) \geq 0$ corresponds to the fact that at least $I = 10$ instantons on K3 are necessary to break the E_8 symmetry completely.

As a side remark, one notices (see appendix A.2.3) that the coefficients $a_\delta^{(\mathcal{R})}(\nu)$ for smaller values of ν are divisible by 2 in $D_{r \geq 4}$, by $6 = 2T_{27}$ in $\mathcal{R} = E_6$, $12 = 2T_{56}$ in $\mathcal{R} = E_7$, and by $60 = 2T_{248}$ in $\mathcal{R} = E_8$, although the authors do not know if this property persists for arbitrary large values of ν . So, a modular form Ψ is such that $d_\gamma(\nu)$'s for small values of ν are divisible not just by 12, but by 24 [resp. 12×6 , 12×12 , or 12×60] if the Higgs cascade to which Ψ is assigned has an enhanced symmetry $\mathcal{R}$ as large as⁵² $D_{r \geq 4}$ [resp. E_6 , E_7 , or E_8].

Extra degrees of freedom. The set $\text{Mod}_0^\Phi(13 - \rho/2, \rho_{\Lambda_S})$ is parametrized by a finite number of $d_\gamma(\nu)$'s in $12\mathbb{Z}$. Those with $\nu < 1$ — the m_γ 's — are enough in the case of $\rho > 2$, and those with $\nu < 2$ are enough if $\rho = 1, 2$, because Ψ/η^{24} [resp. Ψ/η^{48}] has a negative weight when $\rho > 2$ [resp. $\rho = 1, 2$]. Those $d_\gamma(\nu)$'s (or equivalently the Fourier coefficients $c_\gamma^\Psi(\nu)$'s) may be subject to some linear constraints, just like we discussed for Φ in section 2.3.2.

In the case $\rho = 2$: the remaining freedom in the space $\text{Mod}_0^\Phi(0, \rho_{\Lambda_S})$ not specified by the m_γ 's is in the free abelian group $\text{Mod}^\mathbb{Z}(k = 0, \rho_{\Lambda_S})$. This abelian group is equivalent to that of a (τ -independent⁵³) vector $\phi = \{\phi_\gamma = \Delta d_\gamma(0)\} \in 12\mathbb{Z}[G_S/\pm]$ invariant under $\rho_{\Lambda_S}(g)$ for any $g \in \text{Mp}(2; \mathbb{Z})$.

⁵²Remember that we pose here the question how large an enhanced symmetry can be within Heterotic compactifications without 5-branes, or within Type IIA compactifications where the K3-fibration remains regular.

⁵³Consider lifting $\phi \in \text{Mod}(k = 0, \rho_{\Lambda_S})$ to a modular curve $\overline{\mathcal{H}/\Gamma(N)}$ where $\Gamma(N)$ is in the kernel of the representation ρ_{Λ_S} . The lift ϕ should be $\mathbb{C}$ -valued functions on the compact curve, so it must be τ -independent.

It is enough to make sure that $\{\phi_\gamma\}$ is invariant under $\rho_{\Lambda_S}(T)$ and $\rho_{\Lambda_S}(S)$. The invariance under $\rho_{\Lambda_S}(T)$ implies that ϕ_γ can be non-zero only if $(\gamma, \gamma)/2 = 0 \in \mathbb{Q}/\mathbb{Z}$. From the invariance under the $\rho_{\Lambda_S}(S)$, it follows that a non-zero ϕ is possible only when G_S contains a non-zero isotropic element $\gamma_{\neq 0} \in G_S$ or $G_S = \{0\}$ (so $\Lambda_S = U$). For most of rank-2 Λ_S 's, therefore, there is no extra degree of freedom for Ψ/η^{24} .

Suppose that G_S contains an isotropic⁵⁴ subgroup H and that $|H| = \sqrt{|G_S|}$. Whenever there is such a subgroup H , the vector $\phi = \{\phi_\gamma = 1 \text{ if } \gamma \in H, \phi_\gamma = 0 \text{ otherwise}\}$ is invariant under $\rho_{\Lambda_S}(S)$. So, there is one independent extra degree of freedom in the form of $\Delta\Psi = x\phi\eta^{24}$, with a parameter x . For example, in the case of⁵⁵ $\Lambda_S = U[N]$ for some integer $N > 1$, there are two isotropic subgroups $H \cong \mathbb{Z}_N$ in $G_S = \mathbb{Z}_N \times \mathbb{Z}_N$, so there are at least two extra degrees of freedom for Ψ/η^{24} besides $n_{|\gamma|}$'s and m_γ 's. We have neither been able to prove that all the invariant vectors of a Weil representation are written as linear combinations of vectors associated with isotropic subgroups H with $|H| = \sqrt{|G_S|}$, nor to find a counter example.

In the case of $\rho = 1$: namely, $\Lambda_S = \langle +2n \rangle$ for some $n \in \mathbb{N}_{>0}$, the $2(n+1)$ integers $\{\Delta m_\gamma\}$ and $\{\Delta d_\gamma(0 \leq \nu < 1)\}$ must be enough to parametrize Ψ for a given Φ . They are often redundant, however.⁵⁶ We do not have a general theory about how many linear constraints exist within them without relying on a case-by-case analysis. About $\{\Delta d_\gamma(0 \leq \nu < 1)\}$, at least we know that there is one degree freedom not captured by $\{m_\gamma\}$; we stay within the set $\text{Mod}_0^\Phi(25/2, \rho_{\langle +2n \rangle})$ under a change by

$$\frac{\Delta\Psi}{\eta^{24}} \propto -(\Delta d_0(0))\theta_{\langle 2n \rangle}. \quad (3.22)$$

With a case-by-case analysis, it is possible to find out which of those $d_\gamma(\nu < 1)$'s are linearly independent, when an explicit basis of the vector space $\text{Mod}(13 - \rho/2, \rho_{\langle +2n \rangle})$ is available. An alternative is to find a basis of $\text{Mod}(k', \rho_{\langle -2n \rangle})$ with $k' \equiv \rho/2 + 1 \pmod{2}$, and find linear constraints on $\Delta d_\gamma(\nu < 1)$ as in the discussion in footnote 44. A basis of $\text{Mod}(k', \rho_{\langle -2n \rangle})$ can be worked out by using the vector space of $\langle +2n \rangle$ -polarized Jacobi forms of weight $(k' + 1/2)$ (see [65] or appendix A.1). For any $\phi \in \text{Mod}(7/2, \rho_{\langle -2 \rangle})$ and $\phi' \in \text{Mod}(9/2, \rho_{\langle -2 \rangle})$, for example, the coefficients a, b and a', b' in

$$\phi \cdot \frac{\Psi}{\eta^{24}} = \frac{aE_4^4 + bE_4E_6^2}{\eta^{24}}, \quad \phi' \cdot \frac{\Psi}{\eta^{24}} = \frac{a'E_4^3E_6 + b'E_6^3}{\eta^{24}} \quad (3.23)$$

are determined by ϕ and a small number of $\Delta d_\gamma(\nu < 1)$'s by comparing the coefficients of the q^{-1} and q^0 terms; comparison of the coefficients of the q^1 term yields a linear constraint on $\{d_\gamma(\nu < 1)\}$ and $d_\gamma(\nu = 1)$ for isotropic γ 's.

⁵⁴A subgroup H of a discriminant group G is *isotropic*, if the restriction of the discriminant quadratic form on H is trivial.

⁵⁵The case of $U[N]$ seems to appear in the context of CHL orbifold using order N automorphism of K3 (see e.g. [70, 71]).

⁵⁶Because the dimension of $\text{Mod}_0(25/2, \rho_{\langle +2n \rangle})$ does not grow as fast as $\sim (2n)$.

3.1.3 Modular forms and topological invariants

Let X [resp. $\underline{X}$] be a Calabi-Yau three-fold with a regular Λ_S -polarized [resp. $\underline{\Lambda}_S$ -polarized] K3 fibration, and suppose that $\underline{X}$ with some cycles collapsed is regarded as a limit of complex structure of X in a way a complex codimension-2 singularity of type $\mathcal{R}$ emerges along a curve $C_{\mathcal{R}}$; $\lim_{\text{cpx str}} X = \lim_{\text{Kahler}} \underline{X}$. The modular forms Φ and $\underline{\Phi}$ assigned to X and $\underline{X}$, respectively, determine such information as Noether-Lefschetz numbers of X and $\underline{X}$, but there are also some topological invariants of X and $C_{\mathcal{R}}$ that can be determined from them [11].

We start off with quickly reviewing the matching relation between the data (Φ, Ψ) and the low-energy effective theory, and proceed to discuss how we can use such modular forms for classification of regular Λ_S -polarized K3-fibrations.

A quick summary of the matching. The low-energy effective theory on $\mathbb{R}^{3,1}$ has a prepotential $\mathcal{F}$, the gravitational coupling F_1 , and the gauge kinetic function $f_{\mathcal{R}}$ of the enhanced symmetry $\mathcal{R}$, when the Type IIA string is compactified on $\lim_{\text{cpx str}} X = \lim_{\text{Kahler}} \underline{X}$. Those functions of the effective theory in the $s_2 \gg 1$ limit⁵⁷

$$\mathcal{F}^{\text{pert}} = \frac{s}{2}(t, t) + f^{(1)}(t), \quad F_1^{\text{pert}}, \quad f_{\mathcal{R}}^{\text{pert}} = s + 4\pi i h^{(1)}(t) \quad (3.24)$$

are determined⁵⁸ from the microscopic data $\Delta_{\mathcal{R}}$ and Δ_{grav} through the relation⁵⁹ [53–55]

$$4\pi \text{Re}(F_1^{\text{pert}}) = 24s_2 + \frac{1}{4\pi} \left(24V_{\text{GS}} + \Delta_{\text{grav}} - b_{\text{grav}} \hat{K} \right), \quad (3.25)$$

$$4\pi \text{Re}(h^{(1)}) = \frac{1}{4\pi} \left(V_{\text{GS}} + \Delta_{\mathcal{R}} - b_{\mathcal{R}} \hat{K} \right), \quad (3.26)$$

$$V_{\text{GS}} = \frac{4\pi}{(t_2, t_2)} \text{Im}[(1 - it_2^a \partial_{t^a}) f^{(1)}]. \quad (3.27)$$

Those low-energy functions are of the following form,⁶⁰ because we already assume a geometric phase Type IIA compactification:

$$\mathcal{F}^{\text{pert}} = \frac{1}{2} s(t, t)_{\Lambda_S} + \frac{d_{abc}}{3!} t^a t^b t^c - \frac{\zeta(3)}{(2\pi i)^3} \frac{\chi}{2} + \frac{1}{(2\pi i)^3} \sum_{\beta_{\text{eff}}} n_{\beta_{\text{eff}}}^0 \text{Li}_3(e^{2\pi i(\beta_{\text{eff}}, t)}), \quad (3.28)$$

$$4\pi i F_1^{\text{pert}} = 24s + (c_2)_a t^a - \frac{2}{2\pi i} \sum_{\beta_{\text{eff}}} (n_{\beta_{\text{eff}}}^0 + 12n_{\beta_{\text{eff}}}^1) \text{Li}_1(e^{2\pi i(\beta_{\text{eff}}, t)}), \quad (3.29)$$

$$s + 4\pi i h^{(1)} = s + d'_a t^a - \sum_{\beta_{\text{eff}}} \frac{A_{\beta_{\text{eff}}}}{2\pi i} \text{Li}_1(e^{2\pi i(\beta_{\text{eff}}, t)}). \quad (3.30)$$

⁵⁷We consider only the cases where the curve $C_{\mathcal{R}}$ covers the base $\mathbb{P}^1$ just once. (See footnote 48.)

⁵⁸See footnote 49.

⁵⁹ $\hat{K} = -\ln(t_2, t_2) + \text{const.}$ is the (Heterotic string) tree-level Kähler potential of the non-dilaton vector-multiplet scalars.

⁶⁰In an $\mathcal{N} = 2$ field theory on $\mathbb{R}^{3,1}$, the prepotential itself is not physical (e.g., [18, 54]); different choices of $(2n_V + 2)$ -tuple of symplectic sections (X^I, F_I) related by a $\text{Sp}(2n_V + 2; \mathbb{Z})$ duality transformation may have different prepotentials (a prepotential does not exist for some frames). The prepotential here is for a frame where a D2-brane wrapped on any real 2-dimensional cycle (in the Type IIA language) is treated as an electrically charged particle in $\mathbb{R}^{3,1}$. cf. section 2.1.3.

Here, a component description $\{t^{a=1,\dots,\rho}\}$ is given to $t \in \Lambda_S \otimes \mathbb{C}$ by choosing an integral basis $\{D_s, D_{a=1,\dots,\rho}\}$ of $H^2(X; \mathbb{Z})$ consistent with the filtration structure⁶¹ in (2.19); the divisors $\{D_{a=1,\dots,\rho}\}$ modulo $\mathbb{Z}D_s$ may be regarded as a basis of Λ_S . The complexified Kähler class of X is $t_{CY} = sD_s + t^a D_a = sD_s + t$ when $e^{2\pi i s}$ corrections are ignored (as we will everywhere in this article).

The sums of exponential terms run over effective vertical curve classes β_{eff} , because we retain only the terms that remain non-zero in the large base ($s_2 \gg 1$) region of the moduli space. $n_{\beta_{\text{eff}}}^r$ is a Gopakumar-Vafa invariant of X_{IIA} . $A_{\beta_{\text{eff}}}$ is related $r = 0$ Gopakumar-Vafa invariants of $\underline{X}_{\text{IIA}}$. It is well-known that the matching relations (3.25), (3.26), (3.27) determine those parameters in terms of the coefficients of Φ and Ψ as (e.g. [11])

$$\chi = -c_0(0), \quad n_w^0 = c_{[w]}(w^2/2), \quad A_w = \frac{d_{[w]}(w^2/2)}{12}, \quad n_w^1 = \frac{\tilde{c}_{[w]}(w^2/2) - c_{[w]}(w^2/2)}{12} \quad (3.31)$$

for any $w \in [H_2(X; \mathbb{Z})]^{\text{vert}} \subset \Lambda_S^\vee$ that is effective; $\tilde{c}_\gamma(\nu)$ is the Fourier coefficient $[E_2 \Phi_\gamma / \eta^{24}]_{q^\nu}$. We have nothing to add or discuss about them in this article, however.

The non-exponential part of the low-energy functions capture topological invariants of X_{IIA} and the curve $C_{\mathcal{R}}$ in $\lim_{\text{cpx str}} X = \lim_{\text{Kahler}} \underline{X}$; $\chi = \chi(X)$ is the Euler number, and

$$d_{abc} t^a t^b t^c = \int_{X_{\text{IIA}}} t \wedge t \wedge t, \quad t = t^a D_a, \quad (3.32)$$

$$24s + (c_2)_a t^a = \int_{X_{\text{IIA}}} c_2(X_{\text{IIA}}) \wedge t_{CY} = 24s + t^a \int_X c_2(TX) D_a, \quad (3.33)$$

$$s + d'_a t^a = \langle t_{CY}, C_{\mathcal{R}} \rangle = s + \langle t, C_{\mathcal{R}} \rangle. \quad (3.34)$$

The coefficients d'_a can also be regarded as trilinear intersection numbers in $\underline{X}$ among D_a and a pair of exceptional divisors that emerge after resolving the singularity of type $\mathcal{R}$.

Those invariants are determined by⁶² the matching conditions (3.25), (3.26), (3.27) as⁶³

$$\mathcal{F}_{\text{cub}} := \frac{1}{2}s(t, t) + \frac{d_{abc}}{3!} t^a t^b t^c = \frac{1}{2}\tilde{s}(t, t) + \frac{1}{3!}P_3(t), \quad (3.35)$$

$$4\pi i(F_1)_{\text{nonexp}} = 24s + (c_2)_a t^a = 24\tilde{s} + P_1(t), \quad (3.36)$$

$$s + 4\pi i(h^{(1)})_{\text{nonexp}} = s + d'_a t^a =: \tilde{s}, \quad (3.37)$$

⁶¹For a Calabi-Yau three-fold, the structure of K3-fibration $\pi : X_{\text{IIA}} \rightarrow \mathbb{P}_{\text{IIA}}^1$ is in one-to-one with a divisor class D_s of X_{IIA} satisfying $D_s^2 = 0$ and $\int_{X_{\text{IIA}}} D_s \cdot c_2(TX_{\text{IIA}}) = 24$. The divisor class characterized in this way is the topological class of the K3 fibre over a generic point in the base $\mathbb{P}^1$ [10]. Choice of a divisor D_a has ambiguity $D_a \rightarrow D_a + \delta n'_a D_s$ with $\delta n'_a \in \mathbb{Z}$. In terms of the Kähler parameters, this corresponds to $s \rightarrow s - \delta n'_a t^a$ and t^a unchanged.

⁶²Here we have used $\tilde{s} = s + d'_a t^a$ for convenience. Since d'_a is an integer, using $(\tilde{s}, t^a)$ instead of (s, t^a) corresponds to the integral basis change $D_a \rightarrow D_a - d'_a D_s$.

⁶³The 1-loop threshold Δ_{grav} and $\Delta_{\mathcal{R}}$ in (3.25), (3.26)—see also (B.3)—are expanded as in (B.5); the last two lines of (B.5) are used to determine the exponential part of the low energy functions and $\chi = -c_0(0)$. The non-exponential part are determined by the first term in (B.5), or equivalently (B.9).

where $P_3(t)$ and $P_1(t)$ are polynomials of t given by the integrals over the fundamental region $\text{SL}(2; \mathbb{Z}) \backslash \mathcal{H}$:

$$P_3(t_2) := \frac{-t_2^2 |t_2|}{32\pi \sqrt{2}} \int \frac{d\tau_1 d\tau_2}{\tau_2^{3/2}} \overline{\theta_{\Lambda_S}(\tau, \bar{\tau}; t_2)} \frac{\Phi(\tau) \hat{E}_2(\tau, \bar{\tau}) - \Psi(\tau)}{\eta^{24}}, \quad (3.38)$$

$$P_1(t_2) := \frac{1}{4\pi \sqrt{2}} \int \frac{d\tau_1 d\tau_2}{\tau_2^{3/2}} \overline{\theta_{\Lambda_S}(\tau, \bar{\tau}; t_2)} \frac{\Psi(\tau)}{\eta^{24}}. \quad (3.39)$$

See appendix B for details of the integrals; an evaluation method for the case Λ_S has a non-trivial null element is reviewed in appendix B.1. Any Λ_S with $\rho \geq 5$ is known to have such a non-zero null element, and the same is also true for any lattice $\Lambda_S = U \oplus W'$ for some even lattice W' of signature $(0, \rho - 2)$. Appendix B.3 explains how to reduce a case of Λ_S without such an element to cases with such an element.

Discussion 1. For a given X , choose any Higgs cascade attached to the branch of moduli space⁶⁴ of the Type IIA compactification of X . We see in the following that the pair of modular forms Φ and Ψ contains complete information in specifying the diffeomorphism class of X .

Let us recall Wall's theorem [56], which states that the set of diffeomorphism classes of real six-dimensional, simply-connected, spin, and oriented manifolds with torsion-free cohomology groups and a given set of Betti numbers b_2 and b_3 are in one-to-one with the set

$$\{(\mu, p_1) \mid \mu \in \text{Hom}^{\text{Sym}}(H^2 \times H^2 \times H^2, \mathbb{Z}), \ p_1 \in \text{Hom}(H^2, \mathbb{Z}), \ (a), (b)\} / \sim, \quad (3.40)$$

where $H^2 \cong \mathbb{Z}^{\oplus b_2}$ and

$$(a) \quad \mu(x, x, y) + \mu(x, y, y) \equiv 0 \pmod{2} \text{ for any } x, y \in H^2,$$

$$(b) \quad 4\mu(x, x, x) - p_1(x) \equiv 0 \pmod{24} \text{ for any } x \in H^2;$$

the equivalence relation is such that $(\mu, p_1) \sim (\mu', p'_1)$ if and only if there is an isomorphism $\phi : H^2 \rightarrow H^2$ so that $(\mu', p'_1) = (\mu, p_1) \cdot \phi$. For a manifold X , the trilinear symmetric form μ is the wedge product of $H^2(X; \mathbb{Z})$, and p_1 the linear form $\int_X p_1(TX) \wedge x$ for $x \in H^2(X; \mathbb{Z})$.

Its subset of interest in this article is those where H^2 contains an element D_s of the property described in footnote 61. It is given by

$$\begin{aligned} \text{Diff}_{\Lambda_S} := \{&(d, c_2^{\Lambda_S}, \chi) \mid d \in \text{Hom}^{\text{Sym}}(\Lambda_S \times \Lambda_S \times \Lambda_S, \mathbb{Z}), \ c_2^{\Lambda_S} \in \text{Hom}(\Lambda_S, \mathbb{Z}), \ \chi \in \mathbb{Z} \\ &\text{subject to } (a', b')\} / \sim_{\Lambda_S}, \end{aligned} \quad (3.41)$$

where Λ_S is an even lattice of signature $(1, \rho - 1)$ with $\rho = b_2 - 1$ and

$$(a') \quad d_{aab} + d_{abb} \equiv 0 \pmod{2} \text{ for any } a, b \in \{1, \dots, \rho\},$$

$$(b') \quad 4d_{aaa} + 2(c_2)_a \equiv 0 \pmod{24} \text{ for any } a \in \{1, \dots, \rho\};$$

⁶⁴Mathematically, the relation (3.10) can be regarded as the definition of the modular form Ψ of a Higgs cascade; Φ is defined as in [12] (and reviewed in section 2.1.2).

d_{abc} and $(c_2)_a$ for $a, b, c = 1, \dots, \rho$ are the component description of d and $c_2^{\Lambda_S}$ for some basis $\{D_{a=1, \dots, \rho}\}$ of Λ_S ; the equivalence relation is⁶⁵ given by setting $(d, c_2^{\Lambda_S}) \sim_{\Lambda_S} (d', c_2^{\Lambda_S'})$ if and only if they become identical for some combination of isometries of Λ_S and the basis changes in footnote 61.

The modular form Φ of X determines the combinations

$$P_3(t_2) - \frac{(t_2, t_2)}{8} P_1(t_2) = \frac{-|t_2|^3}{32\pi\sqrt{2}} \int \frac{d\tau_1 d\tau_2}{\tau_2^{3/2}} \overline{\theta_{\Lambda_S}} \frac{\Phi \hat{E}_2}{\eta^{24}}, \quad (3.42)$$

and hence the combinations

$$d'_{abc} := d_{abc} - \frac{[(c_2)_a C_{bc}^{\Lambda_S} + \text{cycl.}]}{24}, \quad \text{where } C_{ab}^{\Lambda_S} := (D_a, D_b)_{\Lambda_S}. \quad (3.43)$$

They remain invariant under the shifts $D_a \rightarrow D_a + (\delta n'_a) D_s$ with $\delta n'_a \in \mathbb{Q}$ for a basis $\{D_s, D_{a=1, \dots, \rho}\} \in H^2 \otimes \mathbb{Q}$. So the modular form Φ of a Calabi-Yau three-fold X determines an element of

$$\text{Diff}_{\Lambda_S}(\mathbb{Q}) := \left\{ d'_{abc} := (d_{abc} - [(c_2)_a C_{bc}^{\Lambda_S} + \text{cycl.}]/24) \mid (a'), (b') \right\} / \text{Isom}(\Lambda_S) \times \{\chi \in \mathbb{Z}\}. \quad (3.44)$$

There may be a pair of three-folds X and X' sharing the same modular form Φ that are not diffeomorphic to each other. They must have the same combination $d_{abc} - [(c_2)_a C_{bc}^{\Lambda_S} + \dots]/24$, but $(d_{abc}, (c_2)_a)$ of X may be converted to that of X' only by allowing the shifts with $(\delta n'_a + \mathbb{Z}) \neq 0 \in \mathbb{Q}/\mathbb{Z}$.

$$\text{Diff}_{\Lambda_S}^{d'} := \{(d_{abc} \in \mathbb{Z}, (c_2)_a \in \mathbb{Z}) \mid (a'), (b'), \text{ fixed } d'_{abc}, \} / \sim_{\Lambda_S}. \quad (3.45)$$

With just the modular form Ψ of one arbitrary chosen Higgs cascade of X (along with Φ), however, the dictionary (3.35), (3.36), (3.37) determines $(d_{abc}, (c_2)_a)$ precisely with the relation $\sim_{\Lambda_S}$, because the integrality of d'_a allows only the shifts $s \rightarrow s - \delta n'_a t^a$ with $\delta n'_a \in \mathbb{Z}$.

To summarize, the modular forms Φ and Ψ may be seen as information of the spectrum of BPS states of string theory, or that of Noether-Lefschetz numbers and curve counting invariants, but they also carry full information of the diffeomorphism class of the original manifold X . The way to extract the information has already been described.

$$\begin{array}{ccccc}
 & & \{X's\}_{\Lambda_S} & & \\
 & \swarrow & & \searrow & \\
 [\text{Mod}_0^{\mathbb{Z}}(11 - \frac{\rho}{2}, \rho_{\Lambda_S})]^{\S 2} \supset [\text{Mod}_0^{\mathbb{Z}}(11 - \frac{\rho}{2}, \rho_{\Lambda_S})]^{\text{r.mfd}} & \xrightarrow{\quad\quad\quad} & \text{Diff}_{\Lambda_S}(\mathbb{Q}) & \xleftarrow{\quad\quad\quad} & \text{Diff}_{\Lambda_S} \\
 & \swarrow & & \searrow & \\
 \text{Mod}_0^{\Phi}(13 - \frac{\rho}{2}, \rho_{\Lambda_S}) \supset [\text{Mod}_0^{\Phi}(13 - \frac{\rho}{2}, \rho_{\Lambda_S})]^{\text{r.mfd}} & \xrightarrow{\quad\quad\quad} & \text{Diff}_{\Lambda_S}^{d'} & \xrightarrow{\quad\quad\quad} & \text{Diff}_{\Lambda_S}
 \end{array} \quad (3.46)$$

The following discussions explain why some of the maps are drawn in double lines.

⁶⁵Note that an element $D_s \in H^2$ with the property in footnote 61 is mapped by $\phi : H^2 \cong H^2$ for the relation $(\mu, p_1) \sim (\mu', p'_1)$ to an element $\phi(D_s) \in H^2$ that also has the same property.

For a given three-fold X , there may be multiple Higgs cascades attached to the original branch of the moduli space, and hence multiple modular form Ψ 's. The arrow from $\{X\text{'s}\}$ to $\text{Mod}_0^\Phi(12 - \rho/2, \rho_{\Lambda_S})$ in (3.46) is shown in a double line because of that. Those Ψ 's should yield the same element in $\text{Diff}_{\Lambda_S}^{d'}$. It follows that the difference $\Delta\Psi$ must be such that the resulting $\Delta P_1(t)$ is of the form $24(\Delta d'_a)t^a$ with $(\Delta d'_a) \in \mathbb{Z}$.

One may also ask which subset of the diffeomorphism classes Diff_{Λ_S} of real six-dimensional manifolds are realized under the restriction that X is a Calabi-Yau three-fold (with Λ_S -polarized regular K3-fibrations). Because we do not know well the set of such Calabi-Yau three-folds, $\{X\text{'s}\}$, one may think of applying the procedure of assigning (d'_{abc}, χ) to an abstract general element $\Phi \in \text{Mod}_0^\mathbb{Z}(11 - \rho/2, \rho_{\Lambda_S})$. First, it is not true that the resulting d'_{abc} can be always interpreted as (3.43) for some $d_{abc} \in \mathbb{Z}$ and $(c_2)_a \in \mathbb{Z}$, satisfying (a') and (b'), if we just require that⁶⁶ $\Phi \in \text{Mod}_0^\mathbb{Z}(11 - \rho/2, \rho_{\Lambda_S})$ is subject to the inequalities we derived in section 2; see an example in appendix B.1.3. So, the subset of the Φ 's whose d'_{abc} backed by integer $(d_{abc}, (c_2)_a)$ subject to (a', b') is denoted by $[\text{Mod}_0^\mathbb{Z}(11 - \rho/2, \rho_{\Lambda_S})]^{\text{r.mfd}}$. Similarly, not a general element of $\Psi \in \text{Mod}_0^\Phi(13 - \rho/2, \rho_{\Lambda_S})$ yields an element of $\text{Diff}_{\Lambda_S}^{d'}$ (see section 3.2), so those that fall into $\text{Diff}_{\Lambda_S}^{d'}$ forms a subset denoted by $[\text{Mod}_0^\Phi(13 - \rho/2, \rho_{\Lambda_S})]^{\text{r.mfd}}$. We have the map

$$\text{diff}_{\text{coarse}} : [\text{Mod}_0^\mathbb{Z}(11 - \rho/2, \rho_{\Lambda_S})]^{\text{r.mfd}} \longrightarrow \text{Diff}_{\Lambda_S}(\mathbb{Q}), \quad (3.47)$$

$$\text{diff}_{\text{fine}} : [\text{Mod}_0^\Phi(13 - \rho/2, \rho_{\Lambda_S})]^{\text{r.mfd}} \longrightarrow \text{Diff}_{\Lambda_S}^{d'}; \quad (3.48)$$

see (3.46). The set of diffeomorphism classes represented by Calabi-Yau three-folds must be⁶⁷ within the image of the map $(\text{diff}_{\text{coarse}}, \text{diff}_{\text{fine}})$ in Diff_{Λ_S} .

The map $(\text{diff}_{\text{coarse}}, \text{diff}_{\text{fine}})$ can be worked out by dealing with purely mathematical objects. In setting up the relation between the modular forms and diffeomorphism classes, however, we have combined two physics observations under the Heterotic-Type IIA string duality; the parameters d_{abc} and $(c_2)_a$ in the low-energy effective theory is determined i) by the topology of the target space X in a Type IIA string compactification, and ii) also by 1-loop integrals in the Heterotic string where the integrands are modular forms. The $\mathcal{R}$ -independence of Ψ in a given Higgs cascade may well be proved purely in math, although physics reasonings are enough; the claim that the difference among the Ψ 's from different Higgs cascades of a given original branch disappears in the image of $\text{diff}_{\text{fine}}$ also relies on physics reasonings.

Discussion 2. For a given X with its modular form Φ , one may specify a curve class $C \in H_2(X; \mathbb{Z})$ and ask whether complex structure of X can be tuned to have singularity of some type $\mathcal{R}$ along C (and its resolution is still a regular K3-fibration). In general, there is no guarantee that a modular form Ψ exists in $\text{Mod}_0^\Phi(13 - \rho/2, \rho_{\Lambda_S})$ such that the resulting $P_1(t)$ and $P_3(t)$, hence the right hand side of (3.35), (3.36), (3.37), reproduce all the input data d_{abc} , $(c_2)_a$ (from X) and d'_a (from C) on the left hand side. If there is no

⁶⁶The subset of the Φ 's that meet these requirements is denoted by $[\text{Mod}_0^\mathbb{Z}(11 - \rho/2, \rho_{\Lambda_S})]^{\S 2}$ in (3.46).

⁶⁷A cautionary remark is that the diffeomorphism class of X may not be contained in the image of $\text{diff}_{\text{fine}}$, if X does not have any symmetry-enhanced branch of $\underline{X}$ with a regular K3-fibration.

such modular form Ψ , we learn that complex structure of X cannot be tuned in that way. See section 3.2.1 for an example.

Discussion 3. Suppose that a pair of Calabi-Yau three-folds X and X' have a diffeomorphism between them, but not a holomorphic one-to-one map. The Type IIA string compactifications over X and X' form two different branches of moduli space then. Such a pair of branches of moduli space cannot be distinguished by the invariants $\text{diff}_{\text{coarse}}(\Phi)$ and $\text{diff}_{\text{fine}}(\Psi)$.

Instead of finding the modular form Ψ for one Higgs cascade of the branch of X and specify $\text{diff}_{\text{fine}}(\Psi)$, we can specify the subset of $[\text{Mod}_0^\Phi(13 - \rho/2, \rho_{\Lambda_S})]^{\text{r.mfd}}$ of all the Ψ 's of the Higgs cascades attached to the original branch of X . In some cases, this subset of Ψ 's is different for X and X' , so that it works as an invariant for distinguishing such a pair of Calabi-Yau three-folds. An example of such a case is discussed in section 3.2.1.

That idea of extracting an invariant of a branch of moduli space is faithful to the way we analyze the moduli space by using the low-energy effective field theory. In practice, however, it is not easy to work out all the possible ways to tune complex structure of a manifold to obtain singularity. A close alternative to the idea of using the set of Ψ 's of all the Higgs cascades is i) to think of all the holomorphic curves in $H_2(X; \mathbb{Z})$, ii) apply the reasoning in Discussion 2 to eliminate some of those curves, and finally, iii) to extract the set of Ψ 's for those remaining curves. The latter set of Ψ 's contains the former set of Ψ 's. Those two set of Ψ 's may be called as the set of Ψ 's for curve classes (the latter) and the set of Ψ 's for Higgs cascades (the former).

The latter idea detects difference in the Kähler cone, or in the cone of curves. Let $f : X \rightarrow X'$ be a diffeomorphism; if $C \in H_2(X; \mathbb{Z})$ is in the cone of curves of X , but $f_*(C)$ is not in that of X' , then the modular form Ψ for C may be in the set of Ψ 's for X , but not in the the set for X' . In the example of section 3.2.2, we discuss this latter set of Ψ 's.

Discussion 4. The positive cone⁶⁸ of $\Lambda_S \otimes \mathbb{R}$ may contain multiple chambers separated by walls orthogonal to some elements in $\Lambda_S^\vee$; some of those chambers correspond to three-folds with different topology [57, 58]. The modular form Φ remains the same on both sides of the wall, but the integral in (3.42) does not necessarily yield the same polynomial on both sides of the walls (see appendix B.2), and hence not necessarily the same d'_{abc} ; the arrow from $[\text{Mod}_0^\mathbb{Z}(11 - \rho/2, \rho_{\Lambda_S})]^{\text{r.mfd}}$ to $\text{Diff}_{\Lambda_S}(\mathbb{Q})$ in (3.46) is shown in a double line because of that.

There are not many things we can say with confidence about whether a symmetry-enhanced branch available on one side of the wall continues to exist on the other side of the wall. We think it is likely, however, if $\lim_{\text{cpx.str}} X$ has singularity of type $\mathcal{R}$ along a curve $C \subset X$, then a flop transition on X along a curve disjoint from C yields a three-fold X' that continues to have a symmetry-enhanced phase with the singularity along C . Even in such cases, the map $\text{diff}_{\text{fine}}$ depends on the choice of a chamber, because the integrals P_1 and P_3 have singularity along the walls.

⁶⁸Just one piece of the two connected components of the cone $(t_1, t_2) > 0$ is (more than) enough in parametrizing the Coulomb branch moduli space $D(\tilde{\Lambda}_S)/\text{Ker}[\text{Isom}(\tilde{\Lambda}_S) \rightarrow \text{Isom}(G_S, q_s)]$.

Such invariants as $\text{diff}_{\text{coarse}}(\Phi)$ and $\text{diff}_{\text{fine}}(\Psi)$ are not assigned to branches of Coulomb-and-hyper moduli space, but for branches of individual chambers-and-hyper moduli space.

Because the integrals (polynomials) P_3 for $\Phi \in [\text{Mod}_0^{\mathbb{Z}}(11 - \rho/2; \rho_{\Lambda_S})]^{\S 2}$ and P_1 for $\Psi \in \text{Mod}_0^{\Phi}(13 - \rho/2; \rho_{\Lambda_S})$ vary from one chamber to another, their subsets $[\text{Mod}_0^{\mathbb{Z}}(11 - \rho/2; \rho_{\Lambda_S})]^{\text{r.mfd}}$ and $[\text{Mod}_0^{\Phi}(13 - \rho/2; \rho_{\Lambda_S})]^{\text{r.mfd}}$ may also vary.⁶⁹ In fact, we have confirmed that those subsets remain the same for all the chambers in the positive cone; for a given choice of $\Phi \in [\text{Mod}_0^{\mathbb{Z}}(11 - \rho/2; \rho_{\Lambda_S})]^{\S 2}$ and $\Psi \in \text{Mod}_0^{\Phi}(13 - \rho/2; \rho_{\Lambda_S})$, suppose that d_{abc} and $(c_2)_a$ are all integral and the conditions (a') and (b') are both satisfied when P_3 and P_1 in one chamber is used; the wall crossing formula (B.19), (B.20) can be used to prove that d_{abc} and $(c_2)_a$ are still integral and the conditions (a', b') are still satisfied for P_3 and P_1 in other chambers. The $\text{diff}_{\text{coarse}}$ -image of $[\text{Mod}_0^{\mathbb{Z}}(11 - \rho/2; \rho_{\Lambda_S})]^{\text{r.mfd}}$ and the $\text{diff}_{\text{fine}}$ -image of $[\text{Mod}_0^{\Phi}(13 - \rho/2; \rho_{\Lambda_S})]^{\text{r.mfd}}$ can be different sets for different chambers when seen in the data of d_{abc} and $(c_2)_a$'s; when the images for all the chambers in the positive cone are joined, the symmetry $\text{Isom}(\Lambda_S)$ is restored, and we can take a quotient $\sim_{\Lambda_S}$.

Discussion 5. For Heterotic string compactifications reviewed in section 2.1.1, the invariants such as the set of Ψ 's and the set of Φ 's are assigned for (individual chambers of) the branches of moduli space.⁷⁰ Those invariants are given in terms of the CFT of the fundamental string, and are well-defined, without relying on Heterotic supergravity approximation, or a geometric phase.

Those invariants beyond $(\Lambda_S, \Lambda_T, \Phi)$ generalize the integer n of the $(12 + n, 12 - n)$ instanton number distribution in the case of $\Lambda_S = U$, and detect difference among branches of Heterotic string moduli space already present at perturbative level (with corrections of order $(e^{2\pi i s})$ ignored); see also [17, 38].

Remark. This is a small side remark before closing this section 3.1. The modular form Φ 's for a chain of symmetry enhancement $\mathcal{R}_1 \subset \mathcal{R}_2 \subset \dots$ also contain how many hypermultiplet moduli need to be tuned to have the enhanced symmetries in the chain:

$$\begin{aligned} \Delta h_{\mathcal{R}}^{2,1} &:= h^{2,1}(X) - h^{2,1}(\underline{X}^{(\mathcal{R})}) = h^{1,1}(X) - h^{1,1}(\underline{X}^{(\mathcal{R})}) - (\chi(X) - \chi(\underline{X}^{(\mathcal{R})}))/2, \\ &= -\text{rank}(\mathcal{R}) + [c_0(0) - c_0^{(\mathcal{R})}(0)]/2. \end{aligned} \quad (3.49)$$

This $\Delta h_{\mathcal{R}}^{2,1}$ was used in [17] to distinguish four different branches $X = M_{\langle +2 \rangle}^{(n=2,1,0,-1)}$ of moduli space that share the same $\Lambda_S = \langle +2 \rangle$ and Φ (those that are discussed in section 3.2.2). But it is enough to have $\Phi^{(\mathcal{R})}$ without $\Delta h_{\mathcal{R}}^{2,1}$ as an invariant of branches of the moduli space.

3.2 Examples

Let us see in simple examples how the invariants Φ , Ψ and “the set of possible Ψ 's” work in distinguishing different branches of the moduli space.

⁶⁹This paragraph was added in v2.

⁷⁰Their Type IIA dual do not necessarily have a geometric phase, so we may think of Φ in $\text{Mod}^{\mathbb{Z}}(11 - \rho/2, \rho_{\Lambda_S})$ rather than in $[\text{Mod}_0^{\mathbb{Z}}(11 - \rho/2, \rho_{\Lambda_S})]^{\text{r.mfd}}$.

3.2.1 $\Lambda_S = U$

Let us begin with a traditional example, $\Lambda_S = U$. The maps ($\text{diff}_{\text{coarse}}, \text{diff}_{\text{fine}}$) are worked out first.

We know that $\Phi = -2E_4E_6$ and a general element of $\text{Mod}_0^\Phi(12, \rho_U)$ is $\Psi = -(E_4^3 + E_6^2) + (288 - 24b_{\mathcal{R}})\eta^{24}$, where we use $b_{\mathcal{R}} = d(0)/24 \in \mathbb{Z}_{\geq 0}$ as a parameter. It is known that the integrals $P_3(t)$ and $P_1(t)$ for those Φ and Ψ are given by [11, 50, 51, 68]

$$\frac{1}{3!}P_3(t) = \frac{2}{3!}\rho^3 + \frac{n' - 2}{2}\rho^2u + \frac{n'}{2}\rho u^2, \quad (3.50)$$

$$P_1(t) = -4\rho + 12(2 + n')(\rho + u), \quad (3.51)$$

where $n' := 2 - 12^{-2}d(0) = 2 - b_{\mathcal{R}}/6$. Here, the component description of $t \in \Lambda_S \otimes \mathbb{C} = U \otimes \mathbb{C}$ is that of (B.6) associated with an obvious null element $z \in U$. The expressions above are for the chamber $0 \leq \rho_2 \leq u_2$; those for the other chamber $0 \leq u_2 \leq \rho_2$ are obtained by exchanging ρ and u . As we will take a quotient by $\text{Isom}(U)$, which includes the $\rho \leftrightarrow u$ exchange, it is enough to focus on the $0 \leq \rho_2 \leq u_2$ chamber in the following.

One can see that

$$[\text{Mod}_0^\mathbb{Z}(10, \rho_U)]^{\text{r.mfd}} = [\text{Mod}_0^\mathbb{Z}(10, \rho_U)]^{\S 2} = \{\Phi = -2E_4E_6\}, \quad (3.52)$$

$$[\text{Mod}_0^\Phi(12, \rho_U)]^{\text{r.mfd}} = \{b_{\mathcal{R}} = 0, 6, \dots\} \subsetneq \{b_{\mathcal{R}} \in \mathbb{Z}_{\geq 0}\} = \text{Mod}_0^\Phi(12, \rho_U), \quad (3.53)$$

after working out details by using the expressions of P_3 and P_1 above. The restriction on the value of $b_{\mathcal{R}}$ is from the integrality of d_{abc} 's; once the condition $n' \in \mathbb{Z}$ is imposed, then (a') and (b') are automatically satisfied in this $\rho_{\Lambda_S} = U$ case. Corresponding to the shift $\tilde{s} \rightarrow \tilde{s} + (\delta n'_a)t^a$ with $\delta n'_a \in \mathbb{Z}$ is $\Delta n' = 2$, $\Delta b_{\mathcal{R}} = -12$, which mods out $[\text{Mod}_0^\Phi(12, \rho_U)]^{\text{r.mfd}}$ in passing to $\text{Diff}_{\Lambda_S}^{d'}$ by the map $\text{diff}_{\text{fine}}$.

The image of the map ($\text{diff}_{\text{coarse}}, \text{diff}_{\text{fine}}$) must be in

$$\text{Diff}_{\Lambda_S=U} = \{(d_{\rho\rho\rho}, d_{uuu}, N_\rho, N_u) \in \mathbb{Z}^{\oplus 4}\} / (\rho \leftrightarrow u) \times \{\nu = 0, 1\} \times \{\chi \in \mathbb{Z}\}, \quad (3.54)$$

$$(c_2)_\rho = -2d_{\rho\rho\rho} + 12N_\rho + 24\delta n'_\rho, \quad d_{\rho\rho u} = \nu + 2\delta n'_\rho, \quad (3.55)$$

$$(c_2)_u = -2d_{uuu} + 12N_u + 24\delta n'_u, \quad d_{\rho uu} = \nu + 2\delta n'_u. \quad (3.56)$$

Both $\nu = 0, 1$ of $\text{Diff}_{\Lambda_S}^{d'}$ are realized by the images of even n' and odd n' . Only just one element of $\text{Diff}_{\Lambda_S}(\mathbb{Q}) = \mathbb{Z}^{\oplus 4} / (\rho \leftrightarrow u) \times \{\chi \in \mathbb{Z}\}$ is in the image of $\text{diff}_{\text{coarse}}$, however. It is the element represented by $d_{\rho\rho\rho} = 2$, $d_{uuu} = 0$, $N_\rho = 4 + \nu$, $N_u = 4 + \nu$, and $\chi = -480$. The modular property of Φ 's behind the scene indicates that the diffeomorphism classes realized in the form of Calabi-Yau three-folds are significantly less.⁷¹

As is well-known, there are Calabi-Yau three-folds for both of even n' and odd n' . Think of a Weierstrass-model elliptic fibration over the Hirzebruch surface F_n that is Calabi-Yau, and denote it by $X^{(n)}$; we denote by D_7 the zero-section divisor of the Weierstrass-model $X^{(n)}$. The base surface F_n is a $\mathbb{P}^1$ -fibration over $\mathbb{P}_{\text{IIA}}^1$, where D_f is the $\mathbb{P}^1$ -fibre class, and the two sections denoted by D_+ and D_- have self-intersection $+n$ and $-n$, respectively.

⁷¹In the case of $\Lambda_S = U$, this is not surprising, because a simple argument [59, 60] shows that $X^{(n)}$ with $n = 0, 1, 2$ (explained shortly in the main text) are all the possibilities.

The pull-back of the divisors D_f , D_+ , and D_- to $X^{(n)}$ are denoted by D_s , D_3 , and D_4 , respectively. $D_3 \sim D_4 + nD_s$. Some of the triple intersection numbers are

$$D_7 \cdot \begin{pmatrix} D_3 \cdot D_3 & D_3 \cdot D_s \\ D_s \cdot D_3 & D_s \cdot D_s \end{pmatrix} = \begin{pmatrix} +n & 1 \\ 1 & 0 \end{pmatrix}, \quad (3.57)$$

$$D_s \cdot \begin{pmatrix} (D_7 + D_3)^2 & (D_7 + D_3) \cdot D_3 \\ D_3 \cdot (D_7 + D_3) & D_3^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (3.58)$$

which mean that we can use $\{D_s, (D_7 + D_3), D_3\}$ as a basis of $H^2(X^{(n)}; \mathbb{Z})$ in a way that $\{(D_7 + D_3) + \mathbb{Z}D_s, (D_3) + \mathbb{Z}D_s\}$ becomes a basis of $\Lambda_S = U$. We use the parametrization $t_{CY} = sD_s + \rho(D_7 + D_3) + uD_3$.

$$\int_X t_{CY}^3 = s\rho u + \frac{2}{3!}\rho^3 + \frac{n-2}{2}\rho^2 u + \frac{n}{2}\rho u^2, \quad (3.59)$$

$$\int_X c_2(TX) \wedge t_{CY} = 24s - 4\rho + 12(2+n)(\rho + u). \quad (3.60)$$

Now, think of a symmetry-enhanced limit $\lim_{\text{cpx str}} X^{(n)}$ of this $X^{(n)}$ so a singularity of type $\mathcal{R}$ emerges in the fibre⁷² of $D_- \subset F_n$. Then $(f_{\mathcal{R}})_{\text{nonexp}} = s = \tilde{s}$. The modular form Ψ for this Higgs cascade must be the one for $n' = 2 - 12^{-2}d(0) = n$, because $\tilde{s}\rho u + P_3(t; n' = n)/3!$ and $24\tilde{s} + P_1(t; n' = n)$ reproduces the topological invariants (3.59), (3.60) of $X^{(n)}$. In particular, the set $\text{Diff}_{\Lambda_S}^{d'} \simeq \mathbb{Z}/2\mathbb{Z}$ is realized indeed by the diffeomorphism classes of $X^{(n=0)} \sim X^{(n=2)}$ and $X^{(n=1)}$.

It is possible to find a broader class of symmetry-enhanced limits of $X^{(n)}$ by using F-theory. Let $C' \in H^2(F_n; \mathbb{Z})$ be represented by an irreducible curve (we call it an irreducible curve class). Now, choose f and g of the Weierstrass model $y^2 = x^3 + fx + g$ so

$$f = -3h^2 + \sigma a + \mathcal{O}(\sigma^2), \quad g = 2h^3 - h\sigma a + \mathcal{O}(\sigma^2), \quad (3.61)$$

where $h \in \Gamma(F_n; \mathcal{O}(-2K_{F_n}))$, $a \in \Gamma(F_n; \mathcal{O}_{F_n}(-4K_{F_n} - C'))$, and $\sigma \in \Gamma(F_n; \mathcal{O}_{F_n}(C'))$ [61]; in this limit, the three-fold $X^{(n)}$ has a singularity of type $\mathcal{R} = A_1$ along a curve C in the fibre of the $\sigma = 0$ curve in F_n . Think of C' of the form⁷³ $C' \sim D_- + mD_f$ labeled by $m \in \mathbb{Z}$, so that $C' \cdot D_f = 1$ in F_n , and $C \cdot D_s = +1$ in $X^{(n)}$. For this type of symmetry-enhanced limit (Higgs cascade), we have⁷⁴ $(f_{\mathcal{R}})_{\text{nonexp}} = s + m(\rho + u) = \tilde{s}$. Therefore, we find that the modular form Ψ of this Higgs cascade must be that of $n' = n - 2m$ ($b_{\mathcal{R}}/6 = 2m + 2 - n$), which we find by requiring that the topological invariants (3.59), (3.60) of $X^{(n)}$ must be reproduced by $\tilde{s}\rho u + P_3(t)$ and $24\tilde{s} + P_1(t)$, respectively.

⁷²In the Type IIA-Heterotic duality dictionary, this class of singularities corresponds to choosing the probe gauge group $\mathcal{R}$ in one E_8 in the Heterotic language; Heterotic string is compactified on $K3 \times T^2$, and the $E_8 \times E_8$ instanton configuration on $K3$ remains to have value within $[\mathcal{R}^\perp \subset E_8 \oplus E_8]$. Non-zero Kähler parameters that resolve the singularity of type $\mathcal{R}$ correspond to non-zero Wilson lines in $\mathcal{R}[-1] \subset \underline{\Lambda}_S = U \oplus \mathcal{R}[-1]$. See also the footnote 73.

⁷³We have discussed the invariant Ψ for the Higgs cascade with $C' = D_-$ ($m = 0$); the choice $C' = D_+$ ($m = n$) corresponds to placing the probe gauge group $\mathcal{R}$ in the other weakly coupled E_8 in the Heterotic language, but there are more varieties (m) in the symmetry-enhancement limits.

⁷⁴In this construction of $\lim_{\text{cpx str}} X^{(n)}$, the curve C of A_1 singularity is along $(x, y) = (c, 0)$ in the elliptic fibre, so it does not touch the zero section divisor D_7 . So $D_7 \cdot C = 0$.

The class C' is represented by a curve when $m \geq 0$; the curve is not irreducible for the choice $m = 1$ in the case of $n = 2$, however, because the divisor $D_- + D_f$ has D_- as a base locus then. There is also an upper bound, $m \leq 8 + 4n$; when the divisor class $-4K_{F_n} - C'$ is not effective, $X^{(n)}$ has singular fibre over any point on F_n because $f = -3h^2$ and $g = 2h^3$. The effectiveness of $-4K_{F_n} - C'$ is translated into the upper bound. So, we have found a class of Higgs cascades attached to the branch of the moduli space of $X^{(n)}$ whose invariants are

$$X^{(n=2)} : \{ \Psi_{b_{\mathcal{R}}/6=2m+2-2} \mid m = 0, 2, 3, \dots, 16 \} \subset [\text{Mod}_0^\Phi(12, \rho_U)]^{\text{r.mfd}}, \quad (3.62)$$

$$X^{(n=0)} : \{ \Psi_{b_{\mathcal{R}}/6=2m+2-0} \mid m = 0, 1, \dots, 8 \} \subset [\text{Mod}_0^\Phi(12, \rho_U)]^{\text{r.mfd}}, \quad (3.63)$$

$$X^{(n=1)} : \{ \Psi_{b_{\mathcal{R}}/6=2m+2-1} \mid m = 0, 1, \dots, 12 \} \subset [\text{Mod}_0^\Phi(12, \rho_U)]^{\text{r.mfd}}. \quad (3.64)$$

The difference among Ψ 's for one given $X^{(n)}$ is precisely of the form we expected in Discussion 1. The set of Ψ 's of $X^{(n=2)}$ and $X^{(n=0)}$ are not the same, however, reflecting the fact that this pair of three-folds have a diffeomorphism but not a holomorphic one-to-one map between them, and the Kähler cones are not identical when $H^2(X^{(2)}; \mathbb{R})$ and $H^2(X^{(0)}; \mathbb{R})$ are identified by using the diffeomorphism between them.

One will wonder if there are other symmetry-enhancement limits of $X^{(n)}$. At least we can rule out cases where singularity of type $\mathcal{R}$ emerges along a curve C satisfying $C \cdot D_s = 1$ and $C \cdot D_7 \neq 0$ (Discussion 2). To see this, suppose that there is such a limit. Then $(f_{\mathcal{R}})_{\text{nonexp}} = s + m_\rho \rho + m_u u = \tilde{s}$ with $m_\rho \neq m_u$; no choice of Ψ from $[\text{Mod}_0^\Phi(12, \rho_U)]^{\text{r.mfd}}$ for the polynomials $P_3(t)$ and $P_1(t)$ can reproduce $\mathcal{F}_{\text{cub}}$ and F_1 appropriate for $X^{(n)}$, and hence the assumption must be wrong.⁷⁵ We do not have an argument to rule out the possibility that there are other limits of $X^{(n)}$ for symmetry-enhancements with $C \cdot D_s = C \cdot D_7 = 0$ that cannot be obtained in the form (3.61).

3.2.2 $\Lambda_S = \langle +2 \rangle$

Consider the case $\Lambda_S = \langle +2 \rangle \cong_{ab} \mathbb{Z}e$ now, where the modular form Φ is of the form (2.38). The modular form $\Psi \in \text{Mod}_0^\Phi(25/2, \rho_{\langle +2 \rangle})$ is parametrized by $m_0 = d_0(-1)$, $m_{1/2} = d_{1/2}(-3/4)$, and one more, because $\dim_{\mathbb{C}} \text{Mod}_0(25/2, \rho_{\langle +2 \rangle}) = 3$. We can use the $d_0(0) = 24b_{\mathcal{R}}$ as the third parameter.⁷⁶ Just one among them, $d_0(0) \in 24\mathbb{Z}_{\geq 0}$, is the free parameter, while $m_0 = m_{1/2} = 0$ because of (3.20). So, a general element Ψ in $\text{Mod}_0^\Phi(25/2, \rho_{\langle +2 \rangle})$ is given by⁷⁷

$$\begin{aligned} \Psi &= \frac{E_6}{E_4} \Phi - (d_0(0) + 1440) \theta_{\langle +2 \rangle} \eta^{24} \\ &= e_0 \left[-2 + (348 - d_0(0) - 56n_{1/2})q + (-280656 + 22d_0(0) + 27984n_{1/2})q^2 + \dots \right] \\ &\quad + e_{1/2} \left[n_{1/2} q^{\frac{1}{4}} + (-384 - 2d_0(0) - 384n_{1/2})q^{\frac{5}{4}} \right. \\ &\quad \left. + (-1122304 + 48d_0(0) - 77103n_{1/2})q^{\frac{9}{4}} \dots \right]. \end{aligned} \quad (3.65)$$

⁷⁵This argument still allows a limit of $X^{(n)}$ whose singularity resolution $\underline{X}^{(n)}$ does not have a *regular* K3-fibration over $\mathbb{P}_{\text{IIA}}^1$.

⁷⁶ $d_{1/2}(1/4)$ must be linearly dependent with the other three.

⁷⁷The modular form J of [62] corresponds to $\Psi = -J\eta^{24}$ with $n_{1/2} = 0$ and $d_0(0) = 300$.

We computed the integrals P_3 and P_1 in (3.38), (3.39) for those Φ in (2.38) and Ψ in (3.65) parametrized by $n_{1/2}, b_{\mathcal{R}} \in \mathbb{Z}_{\geq 0}$; details⁷⁸ are left to appendix B.3, and only the result is shown here:

$$\frac{1}{3!}P_3(t) = \frac{(4 - b_{\mathcal{R}} - n_{1/2})}{3!}(t^{a=1})^3, \quad P_1(t) = (52 - 4b_{\mathcal{R}} - 10n_{1/2})(t^{a=1}), \quad (3.66)$$

where $t \in \Lambda_S \otimes \mathbb{C}$ is parametrized by $t = (t^{a=1})e$ with $t^{a=1} \in \mathbb{C}$. So, the dictionary (3.35), (3.36), (3.37) yields

$$d_{111} + 6\delta n'_a = 4 - b_{\mathcal{R}} - n_{1/2}, \quad (c_2)_1 + 24\delta n'_1 = (52 - 4b_{\mathcal{R}} - 10n_{1/2}) \quad (3.67)$$

for some $\delta n'_1 \in \mathbb{Z}$. By comparing this with

$$\text{Diff}_{\Lambda_S=\langle +2 \rangle} = \{N \in \mathbb{Z}\} \times \{\nu \in \{0, 1, 2, \dots, 5\}\}, \quad (3.68)$$

$$d_{111} = \nu + 6(\delta n'_1), \quad (c_2)_1 = 12N - 2\nu + 24(\delta n'_1), \quad (3.69)$$

we find that

$$[\text{Mod}_0^{\mathbb{Z}}(21/2, \rho_{\langle +2 \rangle})]^{\text{r.mfd}} = [\text{Mod}_0^{\mathbb{Z}}(21/2, \rho_{\langle +2 \rangle})]^{\S^2} = \{n_{1/2} = 0, 1, 2, 3, 4\}, \quad (3.70)$$

$$[\text{Mod}_0^{\Phi}(25/2, \rho_{\langle +2 \rangle})]^{\text{r.mfd}} = \{b_{\mathcal{R}} = 0, 2, 4, \dots\} \subsetneq \{b_{\mathcal{R}} = 0, 1, 2, \dots\} = \text{Mod}_0^{\Phi}(25/2, \rho_{\langle +2 \rangle}). \quad (3.71)$$

Within the set $\text{Diff}_{\Lambda_S}(\mathbb{Q}) = \{(2/3)(d_{111} - (c_2))/4 = (\nu - 2N) \in \mathbb{Z}\} \times \{\chi \in \mathbb{Z}\}$, the image of $\text{diff}_{\text{coarse}}$ consists of $\{(\nu - 2N, \chi) = (n_{1/2} - 6, -252 + 56n_{1/2}) \mid n_{1/2} = 0, 1, 2, 3, 4\}$. For a given d'_{111} , all the three elements of $\text{Diff}_{\Lambda_S}^{d'} \simeq \mathbb{Z}/3\mathbb{Z} = \{\Delta\nu/2 = \Delta N = 0_{+3\mathbb{Z}}, 1_{+3\mathbb{Z}}, 2_{+3\mathbb{Z}}\}$ are realized by the image of the map $\text{diff}_{\text{fine}}$ of $\{b_{\mathcal{R}} = 0, 2, 4, \dots\}/\{\Delta b_{\mathcal{R}} = 6\}$.

Some diffeomorphism classes of three-folds with a $\langle +2 \rangle$ -polarized regular K3-fibration are constructed by using toric technique, and are listed up in ([38], table 1).⁷⁹ Those in the list must have the invariants Φ and $\Psi + \{\Delta\Psi\}$ specified in table 4. Such choices as $(n_{1/2}, b_{\mathcal{R}} + 6\mathbb{Z}) = (1, 4_{+6\mathbb{Z}}), (2, 4_{+6\mathbb{Z}}), (3, 2_{+6\mathbb{Z}}), (3, 4_{+6\mathbb{Z}})$, and all of $(4, 2\mathbb{Z}/6\mathbb{Z})$ are in the images of the map $(\text{diff}_{\text{coarse}}, \text{diff}_{\text{fine}})$ but are not found in the table; we have not made an effort to search in a larger Calabi-Yau topology database.

There is at least one pair of three-folds in the same diffeomorphism class but a holomorphic one-to-one map between them may or may not exist, also in the $\Lambda_S = \langle +2 \rangle$ case. The three-folds denoted by $M_{\langle +2 \rangle}^{(n)}$ with $n = 2, 1, 0, -1$ in [17] are in the diffeomorphism classes with $n_{1/2} = 0$ for all of them, and $b_{\mathcal{R}} + 6\mathbb{Z} = 0_{+6\mathbb{Z}}, 2_{+6\mathbb{Z}}, 4_{+6\mathbb{Z}}$, and $0_{+6\mathbb{Z}}$, respectively; the pair $M_{\langle +2 \rangle}^{(n=2)}$ and $M_{\langle +2 \rangle}^{(n=-1)}$ are in the same class. Unlike the diffeomorphic pair $X^{(n=2)}$ and $X^{(n=0)}$ in the $\Lambda_S = U$ case, however, the Kähler cones of $M_{\langle +2 \rangle}^{(n=2)}$ and $M_{\langle +2 \rangle}^{(n=-1)}$ are identical. To see this, let $\{D_s^{(n)}, D_{a=1}^{(n)}\}$ be the basis of $H^2(M_{\langle +2 \rangle}^{(n)}; \mathbb{Z})$ characterized by $(D_{a=1}^{(n)})^3 = 2n$, and $\{\Sigma_B^{(n)}, \Sigma_F^{(n)}\}$ its dual basis of $H_2(M_{\langle +2 \rangle}^{(n)}; \mathbb{Z})$. Toric techniques are used to find that $\Sigma_B^{(n)}$ and $\Sigma_F^{(n)}$ generate the cone of curves for $n = 0, 1, 2$, while the generators

⁷⁸One can employ the “embedding trick” to apply the lattice unfolding method.

⁷⁹Here we cited only the case $\Lambda_S = \langle +2 \rangle$. They treat also the case $\Lambda_S = \langle +4 \rangle$ and $\langle +6 \rangle$.

[38], table 1			discrete parameters	
χ	d_{111}	$(c_2)_1$	$n_{1/2}$	$b_{\mathcal{R}}$ modulo $6\mathbb{Z}$
-252	4	52	0	0
	2	44		2
	0	36		4
-196	3	42	1	0
	1	34		2
-140	2	32	2	0
	0	24		2
-84	1	22	3	0

Table 4. Topological invariants of Calabi-Yau three-folds with a regular $\langle +2 \rangle$ -polarized K3-fibration quoted from [38]. Corresponding parameters $(n_{1/2}, b_{\mathcal{R}} + 6\mathbb{Z})$ of the modular forms Φ and Ψ are shown on the right.

should be $(\Sigma_B^{(n)} + n\Sigma_F^{(n)})$ and $\Sigma_F^{(n)}$ for $n = 0, -1$. It is not hard to find that the diffeomorphism $f : M_{\langle +2 \rangle}^{(n=2)} \rightarrow M_{\langle +2 \rangle}^{(n=-1)}$ maps $\Sigma_F^{(2)}$ to $\Sigma_F^{(-1)}$ and $\Sigma_B^{(2)}$ to $\Sigma_B^{(-1)} - \Sigma_F^{(-1)}$, so the cone of curves are identical indeed.

For the three-folds $M_{\langle +2 \rangle}^{(n)}$ with $n = 2, 1, 0, -1$, the Discussion 2 cannot rule out a possibility for any C in the cone of curves that complex structure of $M_{\langle +2 \rangle}^{(n)}$ can be tuned to have singularity of type $\mathcal{R}$ along C . For a curve class $C = \Sigma_B^{(n)} + m\Sigma_F^{(n)}$ (with $m \in \mathbb{Z}_{\geq 0}$ for $n \geq 0$), we have $f_{\mathcal{R}} = s + m(t^{a=1})$. The dictionary (3.67) with $n_{1/2} = 0$ and $-\delta n'_{a=1} = m$ reads $2n = 4 - b_{\mathcal{R}} + 6m$. So, the modular form Ψ is that of $b_{\mathcal{R}} = 6m + 4 - 2n$, if it is possible to tune complex structure and the corresponding Higgs cascade exists for this class $C = \Sigma_B^{(n)} + m\Sigma_F^{(n)}$. The set of Ψ 's for curve classes is

$$M_{\langle +2 \rangle}^{(n)} : \quad \{\Psi_{b_{\mathcal{R}}=6m+4-2n} \mid m \geq 0\} \subset [\text{Mod}_0^{\Phi}(25/2, \rho_{\langle +2 \rangle})]^{\text{r.mfd}}. \quad (3.72)$$

The three-folds $M_{\langle +2 \rangle}^{(n)}$ with different $n + 3\mathbb{Z}$ are also distinguished in this way. For further attempt at finding difference between $M_{\langle +2 \rangle}^{(2)}$ and $M_{\langle +2 \rangle}^{(-1)}$, see discussion at the end of this section.

A look at $\mathcal{R}$ -dependence. The analysis up to this point relied on Ψ of a Higgs cascade as a whole, so it is independent of the choice of the symmetry $\mathcal{R}$. Now let us use Φ and look into the information which type of singularity may develop in a given manifold.

Suppose that a three-fold X is in the diffeomorphism class characterized by $n_{1/2}$ and $b_{\mathcal{R}} + 6\mathbb{Z}$. Unless $(b_{\mathcal{R}} + 6\mathbb{Z}) = 0 \in \mathbb{Z}/6\mathbb{Z}$, any Higgs cascade attached to the branch of the Type IIA compactification on X does not lead to an enhancement of singularity of type $\mathcal{R} = E_6$ (or higher) whose resolution $\underline{X}$ has a regular K3-fibration.

As a test for whether an enhancement of singularity of a given type $\mathcal{R}$ is possible, one may ask whether an appropriate modular form $\Phi^{(\mathcal{R})}$ can be found. For example, take $\underline{\Lambda}_S = \langle +2 \rangle \oplus A_3[-1]$. Then the modular form Φ of the hypothetical branch of Type IIA

on $\underline{X}$ is in the form of⁸⁰

$$\begin{aligned} \underline{\Phi} = & \frac{-68 + 6\underline{n}_1 + \underline{n}_2 + \underline{n}_3}{72} E_6 \theta_{D_5} \otimes \theta_{\langle +2 \rangle} + \frac{-52 - \underline{n}_2 - \underline{n}_3}{72} E_4 \tilde{\partial}^S \theta_{D_5} \otimes \theta_{\langle +2 \rangle} \\ & + \frac{8 - 4\underline{n}_1 - 3\underline{n}_2 + 6\underline{n}_3}{48} E_4 \theta_{D_5} \otimes \tilde{\partial}^S \theta_{\langle +2 \rangle} + \frac{-8 + \underline{n}_2 - 2\underline{n}_3}{16} (\tilde{\partial}^S)^2 \theta_{D_5} \otimes \tilde{\partial}^S \theta_{\langle +2 \rangle} \end{aligned} \quad (3.73)$$

parametrized⁸¹ by $\underline{n}_{1,2,3} \in \mathbb{Z}$ and $\underline{n}_0 = -2$. This parametrization is for an obvious reason:

$$\begin{aligned} \underline{\Phi} = & e_0 \otimes e_0 (-2 + (324 - 56\underline{n}_1 - 8\underline{n}_2 - 6\underline{n}_3)q + \dots) \\ & + e_0 \otimes e_1 (\underline{n}_1 q^{1/4} + \dots) + (e_1 + e_3) \otimes e_0 (\underline{n}_2 q^{5/8} + \dots) \\ & + (e_1 + e_3) \otimes e_1 ((16\underline{n}_1 - 2\underline{n}_2 + 8\underline{n}_3 + 96)q^{7/8} + \dots) \\ & + e_2 \otimes e_0 (\underline{n}_3 q^{1/2} + \dots) + e_2 \otimes e_1 ((8 + 10\underline{n}_1 + 4\underline{n}_2 - 6\underline{n}_3)q^{3/4} + \dots). \end{aligned} \quad (3.74)$$

By the discussion in section 2.2, the coefficients $\underline{n}_{1,2,3} \in \mathbb{Z}$ need to satisfy

$$\underline{n}_1, \underline{n}_2 \geq 0, \quad \text{and} \quad \underline{n}_3, 16\underline{n}_1 - 2\underline{n}_2 + 8\underline{n}_3 + 96, 8 + 10\underline{n}_1 + 4\underline{n}_2 - 6\underline{n}_3 \geq -2. \quad (3.75)$$

Because Φ and Ψ is given by $\Phi = \theta_{A_3} \cdot \underline{\Phi}$ and $\Psi = \tilde{\partial}^S \theta_{A_3} \cdot \underline{\Phi}$, we have

$$n_{1/2} = \underline{n}_1, \quad b_{\mathcal{R}} = -8 + \underline{n}_2 + \underline{n}_3. \quad (3.76)$$

For $n_{1/2} = 0$ and $b_{\mathcal{R}} = 0, 2, 4$, for example, there are solutions $(\underline{n}_1, \underline{n}_2, \underline{n}_3)$ to (3.75). Therefore, we do not have to rule out a possibility that there exists tuning of complex structure of X so it develops singularity of type A_3 and its resolution $\underline{X}$ has a regular K3-fibration. Indeed, in the case of $X = M_{\langle +2 \rangle}^{(n)}$, such an enhancement can be realized as we see below.⁸²

The Calabi-Yau three-folds $M_{\langle +2 \rangle}^{(n=2,1,0,-1)}$ are given by a hypersurface surface equation

$$X_1^2 + F^{(6)}(X_{2,3,4}) = 0. \quad (3.77)$$

$$\begin{aligned} F^{(6)} = & a_0 X_4^6 + b_0 X_4^4 (X_2 X_3) + c_0 X_4^2 (X_2 X_3)^2 + d_0 (X_2 X_3)^3 \\ & + (a_1 X_4^5 X_2 + a_2 X_4^4 X_2^2 + \dots + a_6 X_2^6) \\ & + (b_1 X_4^3 X_2^2 + b_2 X_4^2 X_2^3 + b_3 X_4 X_2^4 + b_4 X_2^5) X_3 + (c_1 X_4 X_2^3 + c_2' X_2^4) X_3^2 \\ & + (a_1' X_4^5 X_3 + a_2' X_4^4 X_3^2 + \dots + a_6' X_3^6) \\ & + (b_1' X_4^3 X_3^2 + b_2' X_4^2 X_3^3 + b_3' X_4 X_3^4 + b_4' X_3^5) X_2 + (c_1' X_4 X_3^3 + c_2' X_3^4) X_2^2. \end{aligned} \quad (3.78)$$

Homogeneous coordinates X_1 and $X_{2,3,4}$ of the toric ambient space are subject to the $\mathbb{C}^\times$ action $X_1 \rightarrow X_1 \lambda^3$ and $X_{2,3,4} \rightarrow X_{2,3,4} \lambda$ ($\lambda \in \mathbb{C}^\times$) for projectivization. The coefficients $a_{i=1,\dots,6}$, $b_{i=1,\dots,4}$, $c_{i=1,2}$, $a'_{i=1,\dots,6}$, $b'_{i=1,\dots,4}$, $c'_{i=1,2}$, and a_0 , b_0 , c_0 and d_0 are regarded as sections of appropriate line bundles of the base $\mathbb{P}^1$; those line bundles should have the degree specified in table 5 for construction of $M_{\langle +2 \rangle}^{(n)}$. More details are found in [17].

⁸⁰ $\tilde{\partial}^S$ is a shorthand: for modular form F of weight k , $\tilde{\partial}^S F := (\partial^S F)/(-\frac{k}{12}) = (\frac{1}{2\pi i} \frac{\partial}{\partial \tau} - \frac{k}{12} E_2) F / (-\frac{k}{12})$.

⁸¹Remembering that $SU(4) \times SO(10)$ fits into E_8 , it is a reasonable idea to try to construct a basis by using $\theta_{\langle +2 \rangle} \otimes \theta_{D_5}$, derivatives, and the Eisenstein series E_4 and E_6 (see appendix A.1). The dimension formula (A.12), (A.13) indicates that the vector space is of 4-dimensions for the weight and type for this $\underline{\Phi}$.

⁸²Of course, there is no guarantee in general that there exists an actual enhancement $\underline{X}$ when we can construct a candidate for modular form $\underline{\Phi}$.

	a_i, a'_i	b_i, b'_i	c_i, c'_i	d_0
deg	$12 - 2i + (i - 4)(2 - n)$	$8 - 2i + (i - 2)(2 - n)$	$4 - 2i + i(2 - n)$	$2(2 - n)$

Table 5. The “coefficients” a_i , b_i , etc. in (3.78) are sections of $\mathcal{O}_{\mathbb{P}^1}(\text{deg})$, with the degree “deg” specified in this table.

Singularity of type $\mathcal{R} = E_7$ develops in $M_{\langle +2 \rangle}^{(n)}$ for any one of $n = 2, 1, 0, -1$, when all of the sections $a_{2,\dots,6}$, $b_{2,\dots,4}$, and $c_{1,2}$ are set to zero. The singularity is along the curve⁸³ $(x_1, x_4, x_3) = (0, 0, 0)$, which is in the class $C_{\mathcal{R}} = \Sigma_B^{(n)}$; it is of type E_7 because $x_1^2 + b_1 x_4^3 x_3 + d_0 x_3^3 \simeq 0$ is in the direction of the K3 fibre.

It is only in the case of $n = 2$, however, that the K3-fibration in the resolution $\underline{M}$ of the singularity remains regular; for $n \neq 2$, the coefficient d_0 is a section of a line bundle of the base $\mathbb{P}^1$ of positive degree. The section d_0 vanishes at some points in the base, and the three-fold $\underline{M}$ has reducible fibres at those points. So, this failure in finding a tuning of complex structure for $M_{\langle +2 \rangle}^{(n=1,0)}$ is consistent with $b_{\mathcal{R}} + 6\mathbb{Z} = 2_{+6\mathbb{Z}}, 4_{+6\mathbb{Z}}$ that does not allow interpretation as a 1-loop beta function of $\mathcal{R} = E_7$ gauge group. On the other hand, singularity of type $\mathcal{R} = A_3$ enhances along $C_{\mathcal{R}} = \Sigma_B^{(n)}$ by setting $a_{3,\dots,6}$ and $b_{3,4}$ to zero, and we can see by using toric data (just like in [17]) that their $\underline{M}$ ’s have a regular K3-fibration, as announced earlier.

Given the diffeomorphism $f : M_{\langle +2 \rangle}^{(2)} \rightarrow M_{\langle +2 \rangle}^{(-1)}$, the symmetry-enhancement branch of $M_{\langle +2 \rangle}^{(2)}$ with a singularity along $C_{\mathcal{R}} = \Sigma_B^{(2)}$ should be compared with the symmetry-enhancement branch of $M_{\langle +2 \rangle}^{(-1)}$ with a singularity along $\Sigma_B^{(-1)} - \Sigma_F^{(-1)}$. For a singularity of type $\mathcal{R} = E_7$ for the latter, we can tune $a'_{4,\dots,0}$, b'_1 , b_0 , a_1 , b_1 , and a_2 to zero, for example, because the hypersurface equation is $\xi_1^2 + b'_2 \xi_3^3 \xi_2 + a_3 \xi_2^3 \simeq 0$ near the curve⁸⁴ $\xi_1 = \xi_3 = \xi_2 = 0$. So, this Higgs cascade for $M_{\langle +2 \rangle}^{(-1)}$ and $C_{\mathcal{R}} = \Sigma_B^{(-1)} - \Sigma_F^{(-1)}$ must have the same modular form Ψ as the Higgs cascade for $M_{\langle +2 \rangle}^{(2)}$ and $C_{\mathcal{R}} = \Sigma_B^{(2)}$. Those two symmetry-enhancement branches are still different, because a_3 is a section of $\mathcal{O}(3)$ on $\mathbb{P}_{\text{IIA}}^1$ in the former, whereas d_0 that of $\mathcal{O}(0)$ on $\mathbb{P}_{\text{IIA}}^1$ in the latter, and the three-fold $\underline{M}$ for the former does not have a regular K3-fibration. One possibility is that the branches of $\text{IIA}/M_{\langle +2 \rangle}^{(2)}$ and $\text{IIA}/M_{\langle +2 \rangle}^{(-1)}$ are still identical one branch of moduli space, where we have just yet to find tuning of complex structure of $M_{\langle +2 \rangle}^{(-1)}$ so there is an E_7 singularity along $\Sigma_B^{(-1)} - \Sigma_F^{(-1)}$ whose resolution remains to have a regular K3-fibration (put differently, there may be multiple Higgs cascades sharing one common Ψ). The other possibility is that they are two physically distinct branches of moduli space of Heterotic-IIA dual vacua, and that the invariants Λ_S , Λ_T , Φ , and the set of Ψ ’s are not enough to distinguish the two branches.

⁸³We used inhomogeneous coordinates $x_1 = X_1/X_2^3$, $x_4 = X_4/X_2$, and $x_3 = X_3/X_2$.

⁸⁴Now, the inhomogeneous coordinates are $\xi_1 = X_1/X_4^3$, $\xi_{2,3} = X_{2,3}/X_4$.

4 Open questions

Practical questions remain: how small are the subspaces $[\text{Mod}_0^{\mathbb{Z}}(11 - \rho/2, \rho_{\Lambda_S})]^{\text{r.mfd}}$ and $[\text{Mod}_0^{\Phi}(13 - \rho/2, \rho_{\Lambda_S})]^{\text{r.mfd}}$ within $[\text{Mod}_0^{\mathbb{Z}}(11 - \rho/2, \rho_{\Lambda_S})]^{\S 2}$ and $\text{Mod}_0^{\Phi}(13 - \rho/2, \rho_{\Lambda_S})$, respectively. We worked on this question for $\Lambda_S = U$, $\langle +2 \rangle$, and $U \oplus \langle -2 \rangle$ in this article, but not for a general (Λ_S, Λ_T) that fits into $\text{II}_{3,19}$. For example, it is possible (at least in theory) to study whether the subspaces remain non-empty for the series $\Lambda_S = \langle +2n \rangle$ with large n .

The image of the map $(\text{diff}_{\text{coarse}}, \text{diff}_{\text{fine}})$ restricts possible diffeomorphism classes of real six-dimensional manifolds that can be realized by Calabi-Yau three-folds with Λ_S -polarized regular K3-fibrations. This method has been applied only for $\Lambda_S = U$ and $\langle +2 \rangle$; we found that the image of $\text{diff}_{\text{coarse}}$ is much smaller than the set $\text{Diff}_{\Lambda_S}(\mathbb{Q})$ for both Λ_S 's, and the image of $\text{diff}_{\text{fine}}$ is all of $\text{Diff}_{\Lambda_S}^{d', \chi}$ for (d', χ) in the image of $\text{diff}_{\text{coarse}}$. One can find out whether that remains to be true for various different Λ_S 's, by working out the images of $[\text{Mod}_0^{\mathbb{Z}}(11 - \rho/2, \rho_{\Lambda_S})]^{\text{r.mfd}}$ and $[\text{Mod}_0^{\Phi}(13 - \rho/2, \rho_{\Lambda_S})]^{\text{r.mfd}}$.

A few theoretical questions can also be put down. There are two possibilities for a pair of modular forms $\Phi \in [\text{Mod}_0^{\mathbb{Z}}(11 - \rho/2, \rho_{\Lambda_S})]^{\S 2}$ and $\Psi \in \text{Mod}_0^{\Phi}(13 - \rho/2, \rho_{\Lambda_S})$ that are not in the subsets $[\text{Mod}_0^{\mathbb{Z}}(11 - \rho/2, \rho_{\Lambda_S})]^{\text{r.mfd}}$ and $[\text{Mod}_0^{\Phi}(13 - \rho/2, \rho_{\Lambda_S})]^{\text{r.mfd}}$. One is that there are more theoretical constraints of string theory that we failed to capture in sections 2 and 3 and such a (Φ, Ψ) is in conflict with those constraints. The other is that such a pair (Φ, Ψ) is for a branch of moduli space whose Type IIA description does not involve a geometric phase. It remains to be an open question how to determine the boundary between those two possibilities in the space of (Φ, Ψ) .

We have already seen that the image of the map $(\text{diff}_{\text{coarse}}, \text{diff}_{\text{fine}})$ is small compared with the set Diff_{Λ_S} for some Λ_S 's. But not all the diffeomorphism classes of real six-dimensional manifold in the image are guaranteed to be realized as a Calabi-Yau three-fold. By taking advantage of large database of topology of Calabi-Yau three-folds, one may try to get the feeling how much fraction of the diffeomorphism classes in the image of $(\text{diff}_{\text{coarse}}, \text{diff}_{\text{fine}})$ are indeed guaranteed to be realized by Calabi-Yau three-folds. Such a study may provide hints in considering the “determining the boundary” issue above.

In pure mathematics literatures, some inequalities on topological invariants of Calabi-Yau three-folds have been derived (e.g., [63], section 2). It is beyond the scope of this article to study how those inequalities are related to the bounds that we discussed in this article. Also in physics approach, various integer parameters are likely not just bounded from below, but also from above (for maintaining strictly⁸⁵ a given lattice Λ_S). But we have not given enough thoughts on how this intuition is related or unrelated to the bounds and classifications discussed in this article.

A few ideas are also available in improving the effort of introducing invariants for classification of the branches of moduli space of the Het-IIA dual vacua. We just introduced the idea of using the set of Ψ 's of all the Higgs cascade as an invariant of a branch (than

⁸⁵In the case of $\Lambda_S = U$, for example, we know that the three-fold $X^{(n)}$ with $n > 2$ has Λ_S strictly larger than U (non-Higgsable phenomenon [60]). This phenomenon leaves its trace in the fact that the set $\{b_{\mathcal{R}} = 2m + 2 - n \mid m \in \mathbb{Z}_{\geq 0}\}$ contains $b_{\mathcal{R}} < 0$ (hence non-Higgsable) when $n > 2$.

just using its image by $\text{diff}_{\text{fine}}$) in this article; more knowledge in the cone of curves (and tuning of complex structure to have certain singularity along a curve class) would make it possible to compute the set of Ψ 's for general Λ_S 's, not just for $\Lambda_S = U$. Also, a part of the idea (using $\Delta h_{\mathcal{R}}^{2,1}$) for invariants in the case $\Lambda_S = \langle +2 \rangle$ in [17] has been incorporated as a part of $\Phi^{(\mathcal{R})}$ for general Λ_S 's in this article, but a bit more idea beyond $\Delta h_{\mathcal{R}}^{2,1}$ in [17] has not been generalized to other Λ_S 's, or brought into the language of world-sheet CFT in this article.

Finally, as a reminder, K3-fibrations of a Calabi-Yau three-fold were assumed to be regular in this article. Classifications of Calabi-Yau three-folds with a non-regular K3-fibration (and their Heterotic duals) should be considered separately from this article. Note also that we set some other technical limitations in section 2.1 on the class of Heterotic-IIA dual vacua to study in this article. Structure of branches of the whole vacua is yet to be figured out.

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A Modular forms

A.1 Notations and basic facts

In this subsection we explain our notations and some basic facts about modular forms.

Metaplectic group. The *metaplectic group* $\text{Mp}(2; \mathbb{Z})$ is defined by

$$\text{Mp}(2; \mathbb{Z}) = \left\{ (A, f(\tau)) \left| A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2; \mathbb{Z}), f(\tau)^2 = c\tau + d \right. \right\}. \quad (\text{A.1})$$

f is a holomorphic function in the upper half plane $\mathcal{H} \subset \mathbb{C}$. f specifies the choice of sign $\pm\sqrt{c\tau + d}$, so $\text{Mp}(2; \mathbb{Z})$ is a double covering of $\text{SL}(2; \mathbb{Z})$. The multiplication of two elements in the group $\text{Mp}(2; \mathbb{Z})$ is defined by

$$(A, f(\tau))(B, g(\tau)) = (AB, f(B \cdot \tau)g(\tau)). \quad (\text{A.2})$$

The group $\text{SL}(2; \mathbb{Z})$ is generated by two elements

$$S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}, \quad (\text{A.3})$$

which satisfy $S^2 = (ST)^3 = -1$. Similarly, $\text{Mp}(2; \mathbb{Z})$ is generated by

$$S = \left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \sqrt{\tau} \right), \quad T = \left(\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}, 1 \right), \quad (\text{A.4})$$

which satisfy $S^2 = (ST)^3 = Z$, where

$$Z = \left(\left(\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, i \right), \quad Z^4 = 1. \quad (\text{A.5})$$

Vector-valued modular form. Let $k^\pm \in \frac{1}{2}\mathbb{Z}$, and $\rho : \text{Mp}(2; \mathbb{Z}) \rightarrow \text{GL}(V)$ be a representation on a vector space V . A real analytic (but not necessarily holomorphic) function $F : \mathcal{H} \rightarrow V$ is called a *(vector-valued) modular form of weight (k^+, k^-) and type ρ* if F satisfies the modular transformation laws

$$F(A \cdot \tau) = f(\tau)^{2k^+} \overline{f(\tau)}^{2k^-} \rho(A, f) F(\tau), \quad (A, f) \in \text{Mp}(2; \mathbb{Z}). \quad (\text{A.6})$$

A modular form $F(\tau, \bar{\tau})$ is said to be *almost holomorphic*, if it is a polynomial of $(1/\tau_2)$ with τ -dependent coefficients and has finite values at all the cusp points. The vector space of almost holomorphic modular forms of weight (k^+, k^-) and type ρ is denoted by $\text{Mod}((k_+, k_-), \rho)$. The vector space of truly holomorphic modular forms — no $\bar{\tau}$ dependence — in $\text{Mod}((k^+, 0), \rho)$ is denoted by $\text{Mod}(k^+, \rho)$. We mainly consider the case where ρ is a Weil representation (explained shortly in the following). A subspace $\text{Mod}_0(k, \rho)$ of $\text{Mod}(k, \rho)$ is also defined below.

Weil representation. Let M be an even lattice of signature (b^+, b^-) , and $G_M = M^\vee/M$ the discriminant group. Define $\mathbb{C}[G_M] := \text{span}_{\mathbb{C}}\{e_\gamma \mid \gamma \in G_M\}$ where e_γ is a formal symbol. The *Weil representation* $\rho_M : \text{Mp}(2; \mathbb{Z}) \rightarrow \text{GL}(\mathbb{C}[G_M])$ is defined by

$$\rho_M(T)e_\gamma = e_\gamma \mathbb{E}\left(\frac{(\gamma, \gamma)}{2}\right), \quad (\text{A.7})$$

$$\rho_M(S)e_\gamma = \sum_{\delta \in G_M} e_\delta \frac{1}{\sqrt{|G_M|}} \mathbb{E}\left(-\frac{\text{sgn}(M)}{8} - (\delta, \gamma)\right), \quad (\text{A.8})$$

where $\mathbb{E}(x) = e^{2\pi i x}$. The element Z acts as $\rho_M(Z)e_\gamma = e_{-\gamma} \mathbb{E}(-\text{sgn}(M)/4)$. The relation $S^2 = (ST)^3$ holds because of Milgram's formula:

$$\sum_{\gamma \in G_M} \mathbb{E}((\gamma, \gamma)/2) = \sqrt{|G_M|} \mathbb{E}(\text{sgn}(M)/8). \quad (\text{A.9})$$

If two even lattices M_1, M_2 are primitive sublattices of a certain unimodular lattice L and are orthogonal to each other inside $L \otimes \mathbb{R}$, then ρ_{M_1} and ρ_{M_2} are the dual (contragredient) representation of $\text{Mp}(2; \mathbb{Z})$ of each other. For example ρ_{Λ_S} and ρ_{Λ_T} in the main text are dual.

Any modular form $F \in \text{Mod}(k, \rho_M)$ takes value only in the vector subspace $\mathbb{C}[G_M]^+ := \text{span}_{\mathbb{C}}\{e_\gamma + e_{-\gamma} \mid \gamma \in G_M\} \subset \mathbb{C}[G_M]$, when $k/2 \equiv \text{sgn}(M)/4 \pmod{\mathbb{Z}}$.

Subspaces of $\text{Mod}(k, \rho)$ of interest. A modular form $\Phi \in \text{Mod}(k, \rho_M)$ has a Fourier expansion

$$\Phi(\tau) = \sum_{\gamma \in G_M} e_\gamma \Phi_\gamma = \sum_{\gamma \in G_M} e_\gamma \sum_{\nu \in \gamma^2/2 + \mathbb{Z}} x_\gamma(\nu) q^\nu, \quad x_\gamma(\nu) = 0 \text{ for } \nu < 0. \quad (\text{A.10})$$

Here $q = e^{2\pi i\tau}$. $\gamma^2/2$ denotes the quadratic form on G_M . We define $\text{Mod}_0(k, \rho)$ by imposing the cusp condition on the components Φ_γ for isotropic (but non-zero) γ 's:

$$\text{Mod}_0(k, \rho) = \{\Phi \in \text{Mod}(k, \rho) \mid x_\gamma(0) = 0 \text{ if } \gamma \neq 0 \text{ and } \gamma^2/2 = 0\}. \quad (\text{A.11})$$

In this article, the Fourier coefficients $x_\gamma(\nu)$ are denoted by $[\Phi_\gamma]_{q^\nu}$.

Dimensional formula. The dimension of the vector space $\text{Mod}(k, \rho_M)$ is determined by the following formula in the case of $k > 2$ and $k/2 \equiv \text{sgn}(M)/4 \pmod{\mathbb{Z}}$ [29]:

$$\begin{aligned} \dim_{\mathbb{C}}(\text{Mod}(k, \rho_M)) &= d + \frac{dk}{12} \\ &\quad - \alpha\left(\mathbb{E}\left(\frac{k}{4}\right)\rho_M(S)\right) - \alpha\left(\left(\mathbb{E}\left(\frac{k}{6}\right)\rho_M(ST)\right)^{-1}\right) - \alpha(\rho_M(T)), \end{aligned} \quad (\text{A.12})$$

where $d := \dim_{\mathbb{C}} \mathbb{C}[G_M]^+$, and $\alpha(X) := \sum_{i=1}^d \beta_j$ when $\beta_j \in [0, 1)$ is the complex phase (divided by 2π) of the each eigenvalue of the representation matrix $X|_{\mathbb{C}[G_M]^+} \sim \mathbb{E}(\text{diag}(\beta_j))$. The restriction $k > 2$ of this formula is due to the fact that $\dim_{\mathbb{C}} \text{Mod}(k, \rho_M^\vee) = \dim_{\mathbb{C}} \text{Mod}(2-k, \rho_M)$ is not necessarily zero for $k \leq 2$.

The dimension of the subspace $\text{Mod}_0(k, \rho_M)$ is ([48], chapter 1.2.3)

$$\dim_{\mathbb{C}}(\text{Mod}_0(k, \rho_M)) = \dim_{\mathbb{C}}(\text{Mod}(k, \rho_M)) - \#\{\pm\gamma \in G_M/\pm \mid \gamma \neq 0, \gamma^2/2 = 0\}. \quad (\text{A.13})$$

Siegel theta function. Let M be an even lattice of signature (b^+, b^-) . Fix a point v in the Grassmannian $Gr(M) = Gr(M \otimes \mathbb{R}; b^+)$, i.e. a pair of positive/negative definite $b^\pm$ -dimensional subspaces of $M \otimes \mathbb{R}$, orthogonal to each other, and define $v_\pm : M \otimes \mathbb{R} \rightarrow M \otimes \mathbb{R}$ as the orthogonal projections. The *Siegel theta function* is defined by

$$\theta_M(\tau, \bar{\tau}; v) = \sum_{\gamma \in M^\vee/M} e_\gamma \sum_{\lambda \in M+\gamma} q^{v_+^2(\lambda)/2} \bar{q}^{v_-^2(\lambda)/2}, \quad q := e^{2\pi i\tau}. \quad (\text{A.14})$$

Here $v_\pm^2(\lambda) := |(v_\pm(\lambda), v_\pm(\lambda))|$ are both non-negative. $\theta_M(\tau, \bar{\tau}; v)$ is a $\mathbb{C}[G_M]$ -valued modular form of weight $(b^+/2, b^-/2)$ and type ρ_M .

Eisenstein series.

$$E_2 = 1 - 24(q + 3q^2 + \cdots) = 1 - 24 \sum_{n=1}^{\infty} q^n \sigma_1(n) = 1 - 24 \sum_{m=1}^{\infty} \frac{mq^m}{1-q^m}, \quad (\text{A.15})$$

$$E_4 = 1 + 240(q + 9q^2 + \cdots) = 1 + 240 \sum_{n=1}^{\infty} q^n \sigma_3(n), = 1 + 240 \sum_{m=1}^{\infty} \frac{m^3 q^m}{1-q^m}, \quad (\text{A.16})$$

$$E_6 = 1 - 504(q + 33q^2 + \cdots) = 1 - 504 \sum_{n=1}^{\infty} q^n \sigma_5(n) = 1 - 504 \sum_{m=1}^{\infty} \frac{m^5 q^m}{1-q^m}. \quad (\text{A.17})$$

$E_4(\tau)$ and $E_6(\tau)$ are modular forms of weight 4 and 6, respectively, for $\text{SL}(2; \mathbb{Z})$, but $E_2(\tau)$ is not modular (it is Mock modular). The space of scalar-valued modular forms can be identified with the polynomial ring $\mathbb{C}[E_4, E_6]$.

The q^0 -term vanishes in the combination

$$E_4^3 - E_6^2 = (1 + 3 \cdot 240q + \dots) - (1 - 2 \cdot 504q - \dots) = 1728q + \dots = 1728\eta^{24}, \quad (\text{A.18})$$

where

$$\eta = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \quad (\text{A.19})$$

Ramanujan-Serre derivative and Rankin-Cohen bracket. For a modular form $F \in \text{Mod}(k, \rho)$, the *Ramanujan-Serre derivative* $\partial^S : \text{Mod}(k, \rho) \rightarrow \text{Mod}(k+2, \rho)$ is defined by

$$\partial^S F := \left(\frac{1}{2\pi i} \frac{\partial}{\partial \tau} - \frac{k}{12} E_2 \right) F = \eta^{2k} q \partial_q \left(\frac{F}{\eta^{2k}} \right). \quad (\text{A.20})$$

Vector-valued modular forms and their Ramanujan-Serre derivatives can be multiplied to produce yet another vector-valued modular form; the *Rankin-Cohen bracket* $[F, G]_n$ of a pair of modular forms $F \in \text{Mod}(w_F, \rho_F)$ and $G \in \text{Mod}(w_G, \rho_G)$ and $n \in \mathbb{N}$ is

$$[F, G]_n := \frac{1}{(2\pi i)^n} \sum_{r+s=n} (-1)^r \binom{w_F + n - 1}{s} \binom{w_G + n - 1}{r} \frac{\partial^r F}{\partial \tau^r} \frac{\partial^s G}{\partial \tau^s}. \quad (\text{A.21})$$

For example, $[F, G]_0 = FG$, and $[F, G]_1 = w_F F q(\partial_q G) - w_G q(\partial_q F) G$. It is known that $[F, G]_n \in \text{Mod}(w_F + w_G + 2n, \rho_F \otimes \rho_G)$.

Lattice-polarized Jacobi forms and vector-valued modular forms. For a positive definite even lattice M , and $k \in \frac{1}{2}\mathbb{Z}$, a holomorphic function $\phi : \mathcal{H} \times (M \otimes_{\mathbb{Z}} \mathbb{C}) \ni (\tau, z) \mapsto \mathbb{C}$ is said to be a *Jacobi form of weight k and index M* , if it satisfies

$$\phi \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = (c\tau + d)^k \mathbb{E} \left(\frac{c(z, z)}{2(c\tau + d)} \right) \phi(\tau, z), \quad (\text{A.22})$$

$$\phi(\tau, z + \mu + \lambda\tau) = \mathbb{E} \left(-(\lambda, z) - \tau \frac{(\lambda, \lambda)}{2} \right) \phi(\tau, z), \quad (\text{A.23})$$

where the bilinear form $(-, -)$ of M has been extended linearly to $M \otimes_{\mathbb{Z}} \mathbb{C}$. The classical definition of a Jacobi form of weight k and index m is regarded as that of a Jacobi form of weight k and index M with the lattice $M = \langle +2m \rangle$.

To any Jacobi form $\phi(\tau, z)$ of weight k and index M , one can assign a vector-valued modular form of weight $(k - 1/2)$ and type $\rho_M^{\vee}$. That is through

$$\phi(\tau, z) = \sum_{x \in G_M} \left(\sum_{\lambda \in x + M} q^{\frac{(\lambda, \lambda)}{2}} e^{2\pi i(\lambda, z)} \right) f_x(\tau) \iff \sum_{x \in G_M} e_x f_x(\tau), \quad (\text{A.24})$$

and this is one-to-one. The modular form $\sum_x e_x f_x(\tau)$ is holomorphic at the cusps if and only if the expansion $\phi(\tau, z) = \sum_{n \in \mathbb{Z}} \sum_{\lambda \in M^{\vee}} c(n, \lambda) q^n e^{2\pi i(\lambda, z)}$ has support in $n \geq (\lambda, \lambda)/2$. See [64, 65] for more information.

A.2 Explicit examples

Some of the modular forms used in the main text are written down explicitly here.

A.2.1 Explicit basis of $\text{Mod}_0(11 - \rho/2, \rho_{\Lambda_S})$

The case $\Lambda_S = \langle +2 \rangle$: (cf. [12, 38, 44]) The vector space $\text{Mod}_0(21/2, \rho_{\langle +2 \rangle})$ is 2-dimensional over $\mathbb{C}$. One can use $\phi_{(i)} := [\theta_{\langle +2 \rangle}, E_{10-2i}]_i$ with $i = 0, 1$ as a basis. Their Fourier expansions are

$$\phi_{(0)} = e_0 (1 - 262q - 135960q^2 + \cdots) + e_{1/2} \left(2q^{\frac{1}{4}} - 528q^{\frac{5}{4}} - 270862q^{\frac{9}{4}} + \cdots \right), \quad (\text{A.25})$$

$$\phi_{(1)} = e_0 (0 + 224q + 54720q^2 + \cdots) + e_{1/2} \left(-4q^{\frac{1}{4}} - 1440q^{\frac{5}{4}} - 123876q^{\frac{9}{4}} + \cdots \right), \quad (\text{A.26})$$

where $\{e_0, e_{1/2}\}$ is the basis of $\mathbb{C}[G_S]$.

The case $\langle +4 \rangle$: (cf. [38, 44]) The vector space $\text{Mod}_0(21/2, \rho_{\langle +4 \rangle})$ is of 3-dimensions, and is generated by $\phi_{(i)} := [\theta_{\langle +4 \rangle}, E_{10-2i}]_i$ with $i = 0, 1, 2$. Their Fourier expansions are

$$\phi_{(0)} = 1e_0 + q^{1/8}(e_{1/4} + e_{3/4}) + 2q^{1/2}e_{2/4} - 264qe_0 - 263q^{9/8}(e_{1/4} + e_{3/4}) + \cdots, \quad (\text{A.27})$$

$$\phi_{(1)} = -q^{1/8}(e_{1/4} + e_{3/4}) - 8q^{1/2}e_{2/4} + 240qe_0 - 249q^{9/8}(e_{1/4} + e_{3/4}) + \cdots, \quad (\text{A.28})$$

$$\frac{64}{21}\phi_{(2)} = +q^{1/8}(e_{1/4} + e_{3/4}) + 32q^{1/2}e_{2/4} - 576qe_0 + 1017q^{9/8}(e_{1/4} + e_{3/4}) + \cdots. \quad (\text{A.29})$$

The modular form Φ for a Heterotic-IIA dual vacuum is parametrized by the low-energy BPS indices $n_0 = -2$, $n_{1/4}$, and $n_{1/2}$. It must be

$$\Phi = -2\phi_{(0)} + \frac{(n_{1/2} - 32n_{1/4} - 60)}{24}\phi_{(1)} + \frac{(n_{1/2} - 8n_{1/4} - 12)}{24}\frac{64}{21}\phi_{(2)}, \quad (\text{A.30})$$

$$\begin{aligned} &= e_0 \left(-2 + (216 - 14n_{1/2} - 128n_{1/4})q + (153900 - 57344n_{1/4} - 568n_{2/4})q^2 + \cdots \right) \\ &\quad + (e_{1/4} + e_{3/4}) \left(n_{1/4}q^{\frac{1}{8}} + (640 - 7n_{1/4} + 32n_{1/2})q^{\frac{9}{8}} \right. \\ &\quad \left. + (273028 - 272n_{1/4} + 544n_{2/4})q^{\frac{17}{8}} \cdots \right) \\ &\quad + e_{2/4} \left(n_{1/2}q^{1/2} + (10032 + 4864n_{1/4} - 188n_{1/2})q^{\frac{3}{2}} + \cdots \right). \end{aligned} \quad (\text{A.31})$$

It follows that

$$\chi(X_{\text{IIA}}) = 48 - NL_{1,0} = -168 + 128n_{1/4} + 14n_{2/4}. \quad (\text{A.32})$$

The case $\langle +6 \rangle$: the vector space $\text{Mod}_0(21/2, \rho_{\langle +6 \rangle})$ is of 4-dimensions, and is generated by $\phi_{(i)} = [\theta_{\langle +6 \rangle}, E_{10-2i}]_i$ with $i = 0, 1, 2, 3$. The modular form Φ for a Heterotic-IIA vacuum is parametrized by the low-energy BPS indices $n_0 = -2$, and n_γ with $\gamma = 1/6, 2/6, 3/6$:

$$\frac{\Phi}{\eta^{24}} = F^{-1} + n_{1/6}F^{-11/12} + n_{2/6}F^{-8/12} + n_{3/6}F^{-3/12}, \quad (\text{A.33})$$

where we can use the following [44]

$$\begin{aligned}
 F^{-1} &= \frac{-2}{q}e_0 + 148e_0 + 336q^{\frac{1}{12}}(e_{1/6} + e_{5/6}) + 2730q^{\frac{4}{12}}(e_{2/4} + e_{4/6}) + 35360q^{\frac{9}{12}}e_{3/6} + \dots, \\
 F^{-11/12} &= \frac{q^{\frac{1}{12}}}{q}(e_{\frac{1}{6}} + e_{\frac{5}{6}}) - 108e_0 - 134q^{\frac{1}{12}}(e_{\frac{1}{6}} + e_{\frac{5}{6}}) + 924q^{\frac{4}{12}}(e_{\frac{2}{6}} + e_{\frac{4}{6}}) + 20196q^{\frac{9}{12}}e_{\frac{3}{6}} + \dots, \\
 F^{-8/12} &= \frac{q^{\frac{4}{12}}}{q}(e_{\frac{2}{6}} + e_{\frac{4}{6}}) - 54e_0 + 56q^{\frac{1}{12}}(e_{\frac{1}{6}} + e_{\frac{5}{6}}) + 214q^{\frac{4}{12}}(e_{\frac{2}{6}} + e_{\frac{4}{6}}) - 3024q^{\frac{9}{12}}e_{\frac{3}{6}} + \dots, \\
 F^{-3/12} &= \frac{q^{\frac{9}{12}}}{q}e_{3/6} - 2e_0 + 3q^{\frac{1}{12}}(e_{1/6} + e_{5/6}) - 6q^{\frac{4}{12}}(e_{2/6} + e_{4/6}) + 14q^{\frac{9}{12}}e_{3/6} + \dots.
 \end{aligned} \tag{A.34}$$

It follows that

$$\chi(X_{\text{IIA}}) = -c_0(0) = -148 + (108n_{1/6} + 54n_{2/6} + 2n_{3/6}). \tag{A.35}$$

All the details so far in this appendix A.2.1 are used in section 2.3.1.

The case $\Lambda_S = U \oplus \langle -4 \rangle$: there are two linearly independent holomorphic Jacobi forms of weight 10 and index 2 (see [65]). One can work out a basis of the 2-dimensional vector space $\text{Mod}_0(19/2, \rho_{\langle -4 \rangle})$ by using those holomorphic Jacobi forms. The modular form Φ in this vector space is parametrized by $n_0 = -2$ and $n_{2/4}$, as

$$\begin{aligned}
 \Phi &= e_0(-2 + (336 - 18n_{2/4})q + (116340 - 16n_{2/4})q^2 + \dots) \\
 &\quad + (e_{1/4} + e_{3/4})\left((96 + 8n_{2/4})q^{\frac{7}{8}} + (66976 + 120n_{2/4})q^{\frac{15}{8}} + (2539488 - 1368n_{2/4})q^{\frac{23}{8}} + \dots\right) \\
 &\quad + e_{2/4}\left(n_{2/4}q^{\frac{1}{2}} + (10192 - 120n_{2/4})q^{\frac{3}{2}} + (771456 + 900n_{2/4})q^{\frac{5}{2}} + \dots\right).
 \end{aligned} \tag{A.36}$$

We can read out $n_{1/4} = 8n_{2/4} + 96$ from Φ above; this is used in section 2.3.2.

A.2.2 Some lower bounds on $\chi(X)$ for $\Lambda_S = \langle +2n \rangle$

In section 2.4, we derive an expression for $\chi(X)$ for Calabi-Yau three-folds X that have a regular Λ_S -polarized K3-fibration, by using $\phi \in \text{Mod}(3 + \rho/2, \rho_{\Lambda_S}^\vee)$ whose Fourier coefficients at lower powers of q 's are all positive. In this appendix A.2.2, we give details of $\phi = \theta_{L[-1]}$ in some of the $\rho = 1$ cases,⁸⁶ and the expression for $\chi(X)$ that follows.

In the case of $\Lambda_S = \langle +2n \rangle =: \mathbb{Z}e$, choose a primitive element $y' \in E_8$ of norm $2n$ and think of a decomposition (2.46) as in footnote 46. One can then show that the positive definite $L[-1]$ is isometric to the sublattice $[(y')^\perp \subset E_8]$. In the $n = 1$ case, $L[-1] \cong E_7$; the relation (2.48) for $\phi = \theta_{E_7}$ reproduces $\chi(X) = -252 + 56n_{1/2}$. Similarly, the expressions of $\chi(X)$ for $n = 2, 3$ in section 2.3 (table 2) are also reproduced from (2.48) by choosing $L[-1]$ as above. Here, we write $n_{x/2n} = n_{\frac{x}{2n}e}$ for $[(x/2n)e] \in G_{L[-1]}$.

⁸⁶In the $\rho = 1$ cases ($\Lambda_S = \langle +2n \rangle$), we can use knowledge on the vector space of holomorphic Jacobi forms of weight 4 and index n to construct a basis of $\text{Mod}(7/2, \rho_{\langle +2n \rangle}^\vee)$. So, it is also possible to find ϕ 's with positive Fourier coefficients from this vector space, instead of finding lattices L and using $\phi = \theta_{L[-1]}$.

We apply the same procedure to the $n = 4, 5, 6$ cases. It turns out that we can use the lattice theta functions $\phi = \theta_{L[-1]}$ shown in the following to obtain an expression for $\chi(X)$:

$$\begin{aligned} \theta_{L[-1]}^{n=4}(\tau) = & e_0(1 + 56q) + (e_{\frac{1}{8}} + e_{\frac{7}{8}})(56q^{15/16}) + (e_{\frac{2}{8}} + e_{\frac{6}{8}})(28q^{3/4}) + (e_{\frac{3}{8}} + e_{\frac{5}{8}})(8q^{7/16}) \\ & + e_{\frac{4}{8}}(0q^0 + 70q) + \mathcal{O}(q^{1+\epsilon}), \end{aligned} \quad (\text{A.37})$$

$$\begin{aligned} \theta_{L[-1]}^{n=5}(\tau) = & e_0(1 + 60q) + (e_{\frac{1}{10}} + e_{\frac{9}{10}})(44q^{19/20}) + (e_{\frac{2}{10}} + e_{\frac{8}{10}})(33q^{4/5}) + (e_{\frac{3}{10}} + e_{\frac{7}{10}})(12q^{11/20}) \\ & + (e_{\frac{4}{10}} + e_{\frac{6}{10}})(q^{1/5}) + e_{\frac{5}{10}}(32q^{3/4}) + \mathcal{O}(q^{1+\epsilon}), \end{aligned} \quad (\text{A.38})$$

$$\begin{aligned} \theta_{L[-1]}^{n=6}(\tau) = & e_0(1 + 46q) + (e_{\frac{1}{12}} + e_{\frac{11}{12}})(48q^{23/24}) + (e_{\frac{2}{12}} + e_{\frac{10}{12}})(30q^{5/6}) + (e_{\frac{3}{12}} + e_{\frac{9}{12}})(16q^{5/8}) \\ & + (e_{\frac{4}{12}} + e_{\frac{8}{12}})(3q^{1/3}) + (e_{\frac{5}{12}} + e_{\frac{7}{12}})(48q^{23/24}) + e_{\frac{6}{12}}(10q^{1/2}) + \mathcal{O}(q^{1+\epsilon}). \end{aligned} \quad (\text{A.39})$$

The relation (2.48) for those $\phi = \theta_{L[-1]}$ are shown in table 2. For all of $\Lambda_S = \langle +2n \rangle$ with $n = 1, 2, 3, 4, 5, 6$, we know from the dimension formula (A.12), (A.13) that all the $(n + 1)$ Fourier coefficients $\{n_{|\gamma|}\}_{|\gamma| \in G_S/\pm}$ of Φ/η^{24} are linearly independent for $\Phi \in \text{Mod}_0(21/2, \rho_{\langle +2n \rangle})$; so there cannot be any other linear expressions of $\chi(X) = -c_0(0)$ in terms of those independent $n_{|\gamma|}$'s. So, it is enough just to find one $L[-1]$.

In the $n = 7$ case, there must be one linear relation among the 8 coefficients $n_{|\gamma|}$ with $|\gamma| \in G_S/\pm$, because the dimension formula indicates that the vector space $\text{Mod}_0(21/2, \rho_{\langle +14 \rangle})$ is 7-dimensional. So, an expression of the form $\chi(X) = -c_0(0) = \sum_{|\gamma|} \kappa_{|\gamma|} n_{|\gamma|}$ is not expected to be unique. Indeed, we can think of two choices of

$$y' = 3e'_1 + 2e'_2 + e'_3 \in E_8, \quad y' = 3e'_1 + (e'_2 + \cdots + e'_6) \in E_8; \quad (\text{A.40})$$

here, the lattice E_8 is expressed as the abelian group $\mathbb{Z}^{\oplus 8} \oplus ((1/2, \dots, 1/2) + \mathbb{Z}^{\oplus 8})$, and the intersection form on this is given by setting $(e'_i, e'_j) = +\delta_{ij}$ for the generators e'_i of the i -th factor of $\mathbb{Z}^{\oplus 8}$. The sublattices of E_8 isometric to $L[-1]$ are worked out for both choices, and it turns out that the corresponding lattice theta functions

$$\begin{aligned} \theta_{L[-1]}(\tau) = & e_0(1 + 72q) + (e_{\frac{1}{14}} + e_{\frac{13}{14}})(28q^{27/28}) + (e_{\frac{2}{14}} + e_{\frac{12}{14}})(27q^{6/7}) \\ & + (e_{\frac{3}{14}} + e_{\frac{11}{14}})(27q^{19/28}) + (e_{\frac{4}{14}} + e_{\frac{10}{14}})(q^{3/7}) + (e_{\frac{5}{14}} + e_{\frac{9}{14}})(q^{3/28}) \\ & + (e_{\frac{6}{14}} + e_{\frac{8}{14}})(27q^{5/7}) + e_{\frac{7}{14}}(0q^{1/4}) + \mathcal{O}(q^{1+\epsilon}) \end{aligned}$$

and

$$\begin{aligned} \theta_{L[-1]}(\tau) = & e_0(1 + 44q) + (e_{\frac{1}{14}} + e_{\frac{13}{14}})(42q^{27/28}) + (e_{\frac{2}{14}} + e_{\frac{12}{14}})(35q^{6/7}) \\ & + (e_{\frac{3}{14}} + e_{\frac{11}{14}})(14q^{19/28}) + (e_{\frac{4}{14}} + e_{\frac{10}{14}})(7q^{3/7}) + (e_{\frac{5}{14}} + e_{\frac{9}{14}})(0q^{3/28}) \\ & + (e_{\frac{6}{14}} + e_{\frac{8}{14}})(21q^{5/7}) + e_{\frac{7}{14}}(2q^{1/4}) + \mathcal{O}(q^{1+\epsilon}), \end{aligned}$$

are not the same. Two expressions for χ are obtained from the relation (2.48), and are shown in table 6.

Multiple choices of $\phi \in \text{Mod}_0(3 + \rho/2, \rho_{\Lambda_S}^\vee)$ result in multiple expressions for $\chi(X)$ in terms of $\{n_{|\gamma|}\}_{|\gamma| \in G_S/\pm}$, as we have seen above in the $\Lambda_S = \langle +14 \rangle$ case. When we form a linear combinations of such ϕ 's so that the leading Fourier coefficient vanishes (i.e., a cusp

Λ_S	$\chi(X_{\text{IIA}})$
$\langle +2 \rangle$	$\chi = -252 + 56n_{1/2}$
$\langle +4 \rangle$	$\chi = -196 + 128n_{1/4} + 14(n_{2/4} + 2)$
$\langle +6 \rangle$	$\chi = -152 + 108n_{1/6} + 54n_{2/6} + 2(n_{3/6} + 2)$
$\langle +8 \rangle$	$\chi = -112 + 112n_{1/8} + 56n_{2/8} + 16n_{3/8}$
$\langle +10 \rangle$	$\chi = -124 + 88n_{1/10} + 66n_{2/10} + 24n_{3/10} + 2(n_{4/10} + 2) + 32n_{5/10}$
$\langle +12 \rangle$	$\chi = -124 + 96n_{1/12} + 60n_{2/12} + 32n_{3/12} + 6(n_{4/12} + 2) + 96n_{5/12} + 10(n_{6/12} + 2)$
$\langle +14 \rangle$	$\chi = -144 + 56n_{1/14} + 54n_{2/14} + 54n_{3/14} + 2n_{4/14} + 2n_{5/14} + 54n_{6/12}$
$\langle +14 \rangle$	$\chi = -92 + 84n_{1/14} + 70n_{2/14} + 28n_{3/14} + 14n_{4/14} + 0n_{5/14} + 42n_{6/12} + 2(n_{7/14} + 2)$

Table 6. Euler number χ in terms of BPS indices n_γ . n_γ or $(n_\gamma + 2)$ above is non-negative. For example, $\chi \geq -196$ when $\Lambda_S = \langle +4 \rangle$. The last two lines show an example where Weyl-inequivalent sub-lattices of E_8 for $L[-1]$ give different expressions of χ in terms of $\{n_\gamma\}$.

form), then we obtain linear relations among $\{n_{|\gamma|}\}$ discussed in section 2.3.2. Multiple expressions for $\chi(X)$ are consistent because of the linear relations among those low-energy BPS indices.

A.2.3 The first derivative of some lattice theta functions

For any $\mathcal{R}$ of ADE type, the coefficients $a_\delta^{(\mathcal{R})}(\nu)$ in

$$\frac{2}{\text{rank}(\mathcal{R})} q \partial_q \theta_{\mathcal{R}} =: \sum_{\delta \in G_{\mathcal{R}}} e_\delta a_\delta^{(\mathcal{R})}(\nu) q^\nu \quad (\text{A.41})$$

are all integers. To see this, we use Lemma 6.1 and 6.2 of [66] for the lattice $\mathcal{R}$. We have a formula

$$a_\delta^{(\mathcal{R})}(\nu) = \frac{\sum_{\mathbf{b} \in \mathcal{R}^\vee} ((\mathbf{b}, \mathbf{b})_{\mathcal{R}/2=\nu}) ((\mathbf{b}, \mathbf{a})_{\mathcal{R}})^2}{(\mathbf{a}, \mathbf{a})_{\mathcal{R}}} \quad (\text{A.42})$$

for any $\mathbf{a} \in \mathcal{R} \otimes \mathbb{R}$. By using any root in $\mathcal{R}$ as the vector $\mathbf{a}$, we see that any pair $\mathbf{b}$ and $-\mathbf{b}$ in the sum⁸⁷ contributes by

$$\frac{((\mathbf{b}, \mathbf{a})_{\mathcal{R}})^2 + ((-\mathbf{b}, \mathbf{a})_{\mathcal{R}})^2}{(\mathbf{a}, \mathbf{a})_{\mathcal{R}}} \in \mathbb{Z}. \quad (\text{A.43})$$

The values of $a_\delta^{(\mathcal{R})}(\nu)$ for $\nu < 1$ are recorded here.

$$\mathcal{R} \quad \sum_{\delta \in G_{\mathcal{R}}} \sum_{\nu}^{\nu < 1} e_\delta a_\delta^{(\mathcal{R})}(\nu) q^\nu \quad (\text{A.44})$$

$$A_1 : \quad = e_1 q^{1/4}, \quad (\text{A.45})$$

$$A_2 : \quad = (e_1 + e_2) q^{1/3}, \quad (\text{A.46})$$

$$A_3 : \quad = (e_1 + e_3) q^{3/8} + e_2 2q^{1/2}, \quad (\text{A.47})$$

$$A_4 : \quad = (e_1 + e_4) q^{2/5} + (e_2 + e_3) 3q^{3/5}, \quad (\text{A.48})$$

⁸⁷ $\mathbf{b} = 0$ does not form a pair, but $a_0(0) = 0$ obviously.

$$A_5 : \quad = (e_1 + e_5) q^{5/12} + (e_2 + e_4) 4q^{2/3} + e_3 6q^{3/4}, \quad (\text{A.49})$$

$$A_6 : \quad = (e_1 + e_6) q^{3/7} + (e_2 + e_5) 5q^{5/7} + (e_3 + e_4) 10q^{6/7}, \quad (\text{A.50})$$

$$A_7 : \quad = (e_1 + e_7) q^{7/16} + (e_2 + e_6) 6q^{3/4} + (e_3 + e_5) 15q^{15/16} \quad (\text{A.51})$$

$$A_8 : \quad = (e_1 + e_8) q^{4/9} + (e_2 + e_7) 7q^{7/9} \quad (\text{A.52})$$

$$A_{r \geq 8} : \quad = (e_1 + e_r) q^{\frac{r}{2(r+1)}} + (e_2 + e_{r-1}) (r-1) q^{\frac{2(r-1)}{2(r+1)}} \quad (\text{A.53})$$

$$D_{r=2m} : \quad = (e_{01} + e_{10}) 2^{r-3} q^{r/8} + e_{11} 2q^{1/2} \quad \text{ignore the } q^{r/8} \text{ term for } r \geq 8, \quad (\text{A.54})$$

$$D_{r=2m+1} : \quad = (e_1 + e_3) 2^{r-3} q^{r/8} + e_2 2q^{1/2} \quad \text{ignore the } q^{r/8} \text{ term for } r > 8, \quad (\text{A.55})$$

$$E_6 : \quad = (e_1 + e_2) 6q^{2/3}, \quad (\text{A.56})$$

$$E_7 : \quad = e_1 12q^{3/4}. \quad (\text{A.57})$$

All the $a_\delta(\nu)$'s recorded here are equal to $2T_R$, where T_R is the Dynkin index of the fundamental representation R of the algebra $\mathcal{R}$ associated with $\delta \in G_{\mathcal{R}}$.

B Evaluation of integrals by lattice unfolding

We need to evaluate the integrals Δ_{grav} in (2.4), (2.3) and $\Delta_{\mathcal{R}}$ in (3.8), just like in refs. [11, 68]. The evaluation method in [68] (with extension by [11]), however, is applicable immediately only for lattices Λ_S of the form $\Lambda_S = U \oplus W$ with a lattice W of signature $(0, \rho - 2)$. Instead, we rely on the evaluation method presented in [13]; here, we describe the outline of the evaluation method in [13], and quote results relevant to this article (a review is also found in [67], section 3). The method is applied to cases of our interest in appendices B.1.3 and B.3.1; some of those results are used in the main text. The embedding trick is presented in a more general form in appendix B.3 than in the original ([13], section 8); we use this general form in the calculation in B.3.1.

A class of integrals considered in [13] was⁸⁸

$$I_M(v, F) := \int_{\text{SL}(2; \mathbb{Z}) \backslash \mathcal{H}} \frac{d\tau_1 d\tau_2}{\tau_2^2} \tau_2^{b^+/2} \overline{\theta_M(\tau, \bar{\tau}; v)} F(\tau, \bar{\tau}), \quad (\text{B.1})$$

where M is an even lattice of signature (b_+, b_-) , and v denotes a point in the Grassmannian $Gr(M) := Gr(M \otimes \mathbb{R}; b_+)$. One point $v \in Gr(M)$ has the same information as isometries $M \rightarrow \mathbb{R}^{b_+, b_-}$ modulo $\text{SO}(b_+) \times \text{SO}(b_-)$, when the lattice M is indefinite. In the case M is negative definite ($b_+ = 0$), v has empty information, and $\overline{\theta_M(\tau, \bar{\tau}; v)}$ is the theta function $\theta_{M[-1]}(\tau)$. The other factor $F(\tau, \bar{\tau})$ in the integrand is a $\mathbb{C}[M^\vee/M]$ -valued almost holomorphic modular form of weight $(\frac{b^+ - b^-}{2}, 0)$ and type ρ_M ;

$$F(\tau, \bar{\tau}) = \sum_{\gamma \in M^\vee/M} e_\gamma \sum_{\nu \in \mathbb{Q}, \nu \geq \nu_{\min}} \sum_{k=0}^{k_{\max}} c_\gamma^{(k)}(\nu) q^\nu \left(\frac{-3}{\pi \tau_2} \right)^k, \quad c_\gamma^{(k)}(\nu) \in \mathbb{C}. \quad (\text{B.2})$$

⁸⁸When the integral shows some divergence, we understand the integral as regularized by subtracting the integrand by $\text{const} \times \tau_2^{b^+/2-2}$ (equivalent of integrating in IR degrees of freedom) [68] or replacing $d\tau_2/\tau_2^2 \rightarrow d\tau_2/\tau_2^{2+s}$ as in [13].

Those integrals with $M = \tilde{\Lambda}_S$ are of immediate relevance, because

$$\Delta_{\text{grav}} = I_{\tilde{\Lambda}_S} \left(p, \frac{\Phi \hat{E}_2}{\eta^{24}} \right), \quad \Delta_{\text{gauge}} = \frac{1}{24} I_{\tilde{\Lambda}_S} \left(p, \frac{\Phi \hat{E}_2 - \Psi}{\eta^{24}} \right), \quad (\text{B.3})$$

where we used the right-mover and left-mover momenta $p = (p_R, p_L)$ in the Heterotic description to refer to a choice of $v \in Gr(\tilde{\Lambda}_S)$. We can just take $\nu_{\min} = -1$ and $k_{\max} = 1$, and both Δ_{grav} and Δ_{gauge} are within the class of integrals (B.1), (B.2) introduced above.

In evaluating the integral $I_M(v, F)$, ref. [13] relates it to an integral in the same class, but with a lattice M' of $b_+(M') = b_+(M) - 1$, $v' \in Gr(M')$, and F' of the form⁸⁹ (B.2) with M replaced by M' , as we review in appendices B.1 and B.3. This procedure is called the lattice unfolding method. At the end, we are left with evaluating integrals of the form (B.1), (B.2) for a negative definite lattice M'' . Since $\overline{\theta_{M''}}(\tau)$ is holomorphic, the integral $I_{M''}(F'')$ can be regarded as that for 0-dimensional lattice: $I_{M''}(F'') = I_{\{0\}}(\overline{\theta_{M''}}(\tau)F'')$. This type of integrals can be evaluated by simple partial integrals. When $F(\tau) = \phi(\tau)\hat{E}_2^m$ with ϕ some scalar-valued modular form of weight $-2m$, the formula is [68, 69]

$$I_{\{0\}}(\phi \hat{E}_2^m) = \frac{\pi}{3(m+1)} [\phi E_2^{m+1}]_{q^0}. \quad (\text{B.4})$$

B.1 Lattice unfolding formula

When the lattice M of signature (b_+, b_-) has a non-zero element z of norm $z^2 = 0$, a lattice $M' := [z^\perp \subset M]/\mathbb{Z}z$ has signature $(b_+ - 1, b_- - 1)$. For $v \in Gr(M)$, $v' \in Gr(M')$ is determined by the $(b_+ - 1)$ -dimensional vector subspace of the b_+ -dimensional positive definite subspace corresponding to v , orthogonal to z . Discussions in [13] rewrite $I_M(v; F)$ as a sum of $I_{M'}(v', F')$ for an appropriately chosen F' and additional terms that are completely determined in terms of v and F .

Since $U[-1]$ has a non-zero null vector, evaluation of $I_{\tilde{\Lambda}_S}$ with $\tilde{\Lambda}_S = U[-1] \oplus \Lambda_S$ can be reduced to that of I_{Λ_S} (see appendix B.1.1). When the lattice Λ_S of signature $(1, \rho - 1)$ also has a non-zero null vector, we can again reduce I_{Λ_S} to an integral for a smaller lattice $[z^\perp \subset \Lambda_S]/\mathbb{Z}z$ of signature $(0, \rho - 2)$; see appendix B.1.2. Since this lattice is negative definite, we can apply the formula (B.4).

When the lattice Λ_S does not have a non-zero null vector, we can use the embedding trick (explained in appendix B.3) to think of $I_{\Lambda_S}(v', F')$ as $I_{\tilde{M}}(\tilde{v}, \tilde{F})$ for some lattice $\tilde{M}$ and some modular form $\tilde{F}$. Here $\tilde{M}$ contains Λ_S as a sublattice and $b_+(\tilde{M}) = b_+(\Lambda_S)$, and it can be chosen so that it has a non-zero null element. In this way, the integrals for all Λ_S can be reduced to integrals with some negative definite lattice.

B.1.1 From $\tilde{\Lambda}_S$ to Λ_S

In the lattice unfolding process from the lattice $M = \tilde{\Lambda}_S$ to $M' = \Lambda_S$, the positive definite $(b_+ = 2)$ -plane $v \in Gr(M)$ is specified by the real and imaginary parts $\mathcal{U}$ in (2.31).

⁸⁹The value of $k_{\max}$ does not change under this reduction, but the value of $\nu_{\min}$ may not be the same as before if one applies the embedding trick (see appendix B.3).

Let $\{e^0, e^4\}$ be a basis of $U[-1]$ with the intersection form $(e^0, e^0) = (e^4, e^4) = 0$ and $(e^0, e^4) = -1$; they are the generators of $H^0(K3; \mathbb{Z})$ and $H^4(K3; \mathbb{Z})$, as in section 2.1.3. When we choose $z = -e^4$, M' is Λ_S and the positive definite $(b_+(M') = 1)$ -plane v' in $M' \otimes \mathbb{R}$ is the imaginary part of $\mathcal{U}$, namely $\mathbb{R}t_2$. Following ([13], Theorem 7.1), one finds that⁹⁰

$$\begin{aligned} I_{\tilde{\Lambda}_S}(v, F) &= \frac{|t_2|}{\sqrt{2}} I_{\Lambda_S}(v', F) \\ &+ c_0^{(0)}(0) [-\log(2\pi t_2^2) + \text{const.}] - c_0^{(1)}(0) \frac{6\zeta(3)}{\pi^2 t_2^2} \\ &+ \sum_{0 \neq w \in \Lambda_S^\vee} \left[2c_{[w]}^{(0)}(w^2/2) \text{Li}_1(e^{2\pi i(w \hat{\cdot} t)}) - \frac{24}{t_2^2} c_{[w]}^{(1)}(w^2/2) \text{Li}_3(e^{2\pi i(w \hat{\cdot} t)}) \right]. \end{aligned} \quad (\text{B.5})$$

Here $(w \hat{\cdot} t) = (w, t_1) + i|(w, t_2)|$ and $\text{Li}_3(e^{iz}) = \text{Li}_3(e^{iz}) + \text{Im}(z) \text{Li}_2(e^{iz})$. The modular form F is used as F' in the first term on the right hand side. The details of the second and the third line are necessary when working out the matching calculation (3.31), but are not relevant to the matching (3.35), (3.36), (3.37) that we need to discuss in this article.

The integral $I_{\tilde{\Lambda}_S}(v, F)$ as a function of $v \in Gr(\tilde{\Lambda}_S)$ has singularity only along real codimension- $(b_+(\tilde{\Lambda}_S) = 2)$ subspace, whereas I_{Λ_S} as a function of $v' \in Gr(\Lambda_S)$ has singularity (“wall-crossing”) along real codimension- $(b_+(\Lambda_S) = 1)$ walls; $I_{\tilde{\Lambda}_S}(v, F)$ has logarithmic singularity at the locus $\mathcal{U}(w) = 0$ for some $w \in \tilde{\Lambda}_S^\vee$, while $I_{\Lambda_S}(v', F)$ has conical singularity at $(w, t_2) = 0$ for some $w \in \Lambda_S^\vee$. This implies that there is (partial) cancellation of singularity between $I_{\Lambda_S}(v', F')$ and the third line of (B.5) so the sum of them remains non-singular at the codimension-1 walls (except the codimension-2 points).⁹¹ See appendix B.2 for the wall-crossing formula of I_{Λ_S} . For more information, see ([13], section 6).

B.1.2 I_{Λ_S} for Λ_S with a null element

In the rest of appendix B.1, we discuss evaluation of $I_{\Lambda_S}(v', F)$ for Λ_S that has a non-zero null element z . Let z be a non-zero primitive null element of Λ_S , and N be the GCD of the values of $(z, \lambda) \in \mathbb{Z}$ for $\lambda \in \Lambda_S$. We then choose $\tilde{z} \in \Lambda_S$ so that $(z, \tilde{z}) = N$, and a free abelian subgroup $W \subset \Lambda_S$ such that $z \perp W$ and $\Lambda_S \cong_{ab} \mathbb{Z}z \oplus \mathbb{Z}\tilde{z} \oplus W$; here $\cong_{ab}$ stands for an isomorphism between abelian groups. The same W also denotes the lattice $[z^\perp \subset \Lambda_S]/\mathbb{Z}z$. The complexified Kähler parameter $t \in \Lambda_S \otimes \mathbb{C}$ can be parametrized by

$$t = uz + \rho\tilde{z} + a, \quad u, \rho \in \mathbb{C}, \quad a \in W \otimes \mathbb{C}. \quad (\text{B.6})$$

Here is additional preparation. The pairing $(-, z) : \Lambda_S \rightarrow \mathbb{Z}$ induces a homomorphism $(-, z) : G_S \rightarrow \mathbb{Z}_N$, where $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$. The kernel of this homomorphism $\ker(-, z)$ includes the subgroup $\langle z/N \rangle \cong \mathbb{Z}_N$ of G_S generated by $z/N + \Lambda_S$. We have a natural

⁹⁰Here, $k_{\max} \leq 1$ is assumed. The value of “const” in the expression is regularization dependent.

⁹¹The equation (B.5) holds even when t_2 does not have a large norm (although ([13], Theorem 7.1) proves the corresponding claim only when the norm (t_2, t_2) is sufficiently large), because both sides of (B.5) are well-defined even for small t_2 and they remain (real-)analytic inside any single chamber, so they must be the same.

isomorphism $j : \ker(-, z) / \langle z/N \rangle \cong G_W$ of finite abelian groups, which preserves the quadratic form. The subset $j^{-1}(\lambda) \subset \ker(-, z)$ for $\lambda \in G_W$ is of the form $\delta_0 + \langle z/N \rangle$ for some δ_0 that depends on λ . The linear map

$$\mathbb{C}[G_S] \rightarrow \mathbb{C}[G_W], \quad \begin{cases} e_\delta \mapsto e_{j(\delta)} & (\delta \in \ker(-, z)), \\ e_\delta \mapsto 0 & (\delta \notin \ker(-, z)), \end{cases} \quad (\text{B.7})$$

is compatible with the Weil representation ([13], Theorem 5.3). For a modular form F of type ρ_{Λ_S} , we denote by F_W the modular form of type ρ_W obtained by composing $F : \mathcal{H} \rightarrow \mathbb{C}[G_S]$ and this linear map.

Now Theorem 7.1, Lemma 7.3, and Theorem 10.2 of ref. [13] can be used to rewrite the integrals $I_{\Lambda_S}(v', F)$ in terms of u , ρ , a , and the coefficients $c_\gamma^{(k)}(\nu)$ of F . We just quote the results for the case $k_{\max} \leq 1$: when t_2 lies in a fundamental chamber^{92,93} $\mathcal{C}_z$,

$$\begin{aligned} \frac{|t_2|}{\sqrt{2}} I_{\Lambda_S}(v', F) &= \frac{t_2^2}{2N\rho_2} I_W(F_W) \\ &+ 2\pi \sum_{b \in W^\vee} \sum_{\delta \in j^{-1}([b])} \left(N\rho_2 c_\delta^{(0)}(b^2/2) \mathbb{B}_2(\cdots) + \frac{2N^3\rho_2^3}{t_2^2} c_\delta^{(1)}(b^2/2) \mathbb{B}_4(\cdots) \right), \end{aligned} \quad (\text{B.8})$$

where the argument of the functions⁹⁴ $\mathbb{B}_2$ and $\mathbb{B}_4$ are $(b, a_2/(N\rho_2)) + N^{-1}(\delta, \tilde{z})$. The first term $I_W(F_W)$ can be evaluated by the formula (B.4).

When one expands the right hand side, there appear some terms with ρ_2 in the denominator, but they should cancel out because of consistency with wall-crossing behavior of I_{Λ_S} (see [13], Theorem 10.3).⁹⁵ Due to this cancellation, the integral $I_{\Lambda_S}(v', F) \times |t_2|^{1+2k_{\max}}$ turns out to be a chamber-wise homogeneous function of the components of t_2 of degree $(1 + 2k_{\max})$.

The integrals of our interest are those in (B.3), and we find it useful to assign notations for the following combinations (the same as (3.39), (3.38)):

$$P_1(t_2) := \frac{1}{4\pi} \frac{|t_2|}{\sqrt{2}} I_{\Lambda_S}(v', \Psi \eta^{-24}), \quad P_3(t_2) := \frac{-t_2^2}{32\pi} \frac{|t_2|}{\sqrt{2}} I_{\Lambda_S}(v', (\Phi \hat{E}_2 - \Psi) \eta^{-24}); \quad (\text{B.9})$$

P_1 is a chamber-wise polynomial of degree 1 and P_3 of degree 3. They are used to determine some of the functions of the low-energy effective theory in (3.35), (3.36).

⁹²The positive cone, one of two components of $\{t_2 \in \Lambda_S \otimes \mathbb{R} \mid (t_2, t_2) > 0\}$, is divided into *chambers* based on which subset of $\Pi := \{w \in \Lambda_S^\vee \mid 2\nu_{\min} \leq w^2 < 0\}$ is characterized as $\{w \in \Pi \mid (w, t_2) > 0\}$. We call a chamber $\mathcal{C}$ a *fundamental chamber*, if it contains a region $|z, t_2| \ll |t_2|$ (equivalently $N|\rho_2| \ll |t_2|$). There may be multiple fundamental chambers in general.

⁹³This formula is guaranteed to be valid in fundamental chambers. For evaluating the integral in other chambers, one needs to use the wall-crossing formula (B.18). See appendix B.2 for details.

⁹⁴The function $\mathbb{B}_m(x)$ is defined (for $m > 0$) by

$$\mathbb{B}_m(x) = -m! \sum_{n \neq 0} \frac{\mathbb{E}(nx)}{(2\pi i n)^m} = -\frac{m!}{(2\pi i)^m} (\text{Li}_m(\mathbb{E}(x)) + (-1)^m \text{Li}_m(\mathbb{E}(-x))).$$

It satisfies $\mathbb{B}_m(x+1) = \mathbb{B}_m(x)$ and $\mathbb{B}_m(-x) = (-1)^m \mathbb{B}_m(x)$ by definition. Its value for $0 < x < 1$ is given by the usual Bernoulli polynomial $B_m(x)$; e.g. $B_2(x) = x^2 - x + 1/6$ and $B_4(x) = x^4 - 2x^3 + x^2 - 1/30$.

⁹⁵Roughly, the claim is proved by observing that different choices of z put different variable (e.g. u_2) in the denominator and may cause different type of singularity. But the wall-crossing is just a polynomial so the fractional terms must vanish.

B.1.3 Examples: $\Lambda_S = U \oplus W$

The polynomials P_3 and P_1 have simpler expressions when Λ_S is of the form⁹⁶ $U \oplus W$ for some even and negative definite lattice W . In the fundamental chambers,⁹⁷ they are

$$\begin{aligned} \frac{1}{3!}P_3(t) = & \frac{2}{3!}\rho^3 + \frac{n'-2}{2}\rho^2u + \frac{n'}{2}\rho u^2 + \frac{n'}{2}(u+\rho)(a,a) \\ & - \frac{u+\rho}{2} \sum_{b \in W^\vee} \frac{d_{[b]}(b^2/2)}{12}(b,a)^2 + \frac{2u\rho + (a,a)}{4} \sum_{(b,a_2)>0} \frac{d_{[b]}(b^2/2)}{12}(b,a) \\ & + \frac{1}{3!} \frac{1}{2} \sum_{\substack{b \in W^\vee \\ (b,a_2)>0}} c_{[b]}(b^2/2)(b,a)^3, \end{aligned} \quad (\text{B.10})$$

$$P_1(t) = -4\rho + 12(2+n')(u+\rho) - \sum_{\substack{b \in W^\vee \\ (b,a_2)>0}} c_{[b]}^\Psi(b^2/2)(b,a), \quad (\text{B.11})$$

where we have taken care of cancellation referred below the formula (B.8). The parameter n' extracts⁹⁸ a combination of $d_\gamma(\nu)$'s through $(2-n') := 144^{-1} \sum_b d_{[b]}(b^2/2)$.

An element $\Phi \in [\text{Mod}_0^\mathbb{Z}(11-\rho/2, \rho_{\Lambda_S})]^{\S 2}$ and $\Psi \in \text{Mod}_0^\Phi(13-\rho/2, \rho_{\Lambda_S})$ is in $[\text{Mod}_0^\mathbb{Z}(11-\rho/2, \rho_{\Lambda_S})]^{\text{r.mfd}}$ and $[\text{Mod}_0^\Phi(13-\rho/2, \rho_{\Lambda_S})]^{\text{r.mfd}}$, respectively, if the following conditions are satisfied. For integrality of the coefficients d_{abc} of the $\rho^2u/2$ and $\rho u^2/2$ terms in P_3 ,

$$\left(2 - \frac{1}{12} \sum_{b \in W^\vee} \frac{d_{[b]}(b^2/2)}{12} \right) =: n' \in \mathbb{Z}. \quad (\text{B.12})$$

The upa term and $aaa/3!$ term have integer coefficients, only if

$$\sum_{\substack{(b,a_2)>0 \\ b \in W^\vee}} \frac{d_{[b]}(b^2/2)}{12} b \in 2W^\vee \quad (\text{B.13})$$

and

$$\sum_{\substack{(b,a_2)>0 \\ b \in W^\vee}} c_{[b]}(b^2/2) (b, r_1)(b, r_2)(b, r_3) \in 2\mathbb{Z} \quad \text{for } r_{1,2,3} \in W, \quad (\text{B.14})$$

respectively. The condition (a') is translated into

$$\sum_{\substack{(b,a_2)>0 \\ b \in W^\vee}} c_{[b]}(b^2/2) (b, r_1)(b, r_2)(b, r_1 + r_2) \equiv 0 \pmod{4} \quad \text{for } r_{1,2} \in W, \quad (\text{B.15})$$

⁹⁶Heterotic compactifications on $K3 \times T^2$ with instantons in $\mathfrak{g} \subset E_8[-1] \oplus E_8[-1]$ fits into this case with $W = [\mathfrak{g}^\perp \subset E_8[-1] \oplus E_8[-1]]$, where u and ρ describe complex/Kähler structures of T^2 and a is the Wilson line in W .

⁹⁷This expression is for fundamental chambers that contain the region with $(b, a_2) < \rho_2$ for b 's appearing in the sum.

⁹⁸Suppose that $W \subset U^{\oplus 2} \oplus E_8^{(1)}$, the probe gauge group $\mathcal{R}$ is in $E_8^{(2)}$, and there are $I = 12 - n'$ instantons on $K3$ (of the $K3 \times T^2$ internal space) in $[\mathcal{R}^\perp \subset E_8^{(2)}]$ in the Heterotic language. Then Ψ should be the one with $d_\gamma(\nu)$'s that are related to n' in this way.

and the condition (b') to

$$\sum_{b \in W^\vee}^{(b, a_2) > 0} \left(c_b(b^2/2)(b, r)^3 - c_{[b]}^\Psi(b^2/2)(b, r) \right) \equiv 0 \pmod{12} \quad \text{for } r \in W, \quad (\text{B.16})$$

or equivalently, to

$$\sum_{b \in W^\vee}^{(b, a_2) > 0} c_{[b]}(b^2/2)(b, r)((b, r) + 1)((b, r) - 1) \equiv 0 \pmod{12} \quad \text{for } r \in W. \quad (\text{B.17})$$

The special case $W = \{0\}$ has been treated in section 3.2.1. The $W = A_1[-1]$ case yields an example where $[\text{Mod}_0^\mathbb{Z}(11 - \rho/2, \rho_{\Lambda_S})]^{\text{r.mfd}}$ is a proper subset of $[\text{Mod}_0^\mathbb{Z}(11 - \rho/2, \rho_{\Lambda_S})]^{\S 2}$. To see this, note that the modular forms Φ and Ψ are parametrized by $n_{1/2} \in \mathbb{Z}_{\geq 0}$ and $m_{1/2} \in 12\mathbb{Z}$ (after using $\dim_{\mathbb{C}}(\text{Mod}_0(23/2, \rho_{\langle -2 \rangle})) = 2$ and imposing $n_0 = -2$ and $m_0 = 0$). The integrality of $n' = 2 - [56 - n_{1/2} + m_{1/2}]/12$ is translated into $n_{1/2} \in 8 + 12\mathbb{Z}$, so $[\text{Mod}_0^\mathbb{Z}(11 - \rho/2, \rho_{\Lambda_S})]^{\text{r.mfd}}$ is strictly smaller. The other four conditions above follow automatically for $n' \in \mathbb{Z}$. One may also find from this that the image of $\text{diff}_{\text{fine}}$ in the set $\text{Diff}_{\Lambda_S}^{d'}$ is $\mathbb{Z}/2\mathbb{Z}$ for $\Lambda_S = U \oplus W$ with $W = A_1[-1]$.

B.2 Wall-crossing behavior

We give some comments on the wall-crossing behavior. See section 6 of [13] for details. $I_{\Lambda_S}(v, F)$ as a function of $v = \mathbb{R}t_2 \in Gr(\Lambda_S)$ shows conical singularities along the walls $(w, t_2) = 0$ for some⁹⁹ w 's in $\Lambda_S^\vee$. These real-codimension-1 walls separate $Gr(\Lambda_S)$ into many chambers; I_{Λ_S} is analytic in each chamber but shows jump from its analytic continuation when t_2 crosses a wall. Let $I_{\Lambda_S}(v, F; \mathcal{C})$ be the analytic continuation of the restriction of $I_{\Lambda_S}(v, F)$ to a chamber $\mathcal{C}$. The difference for two different chambers $\mathcal{C}_1$ and $\mathcal{C}_2$ is given by

$$\begin{aligned} & I_{\Lambda_S}(v; \mathcal{C}_1) - I_{\Lambda_S}(v; \mathcal{C}_2) \\ &= \sum_{\substack{(w, \mathcal{C}_1) > 0 \\ (w, \mathcal{C}_2) < 0}} \left\{ c_w^{(0)}(w^2/2)(-8\sqrt{2}\pi)(w, v) + c_w^{(1)}(w^2/2)(-32\sqrt{2}\pi)(w, v)^3 \right\}. \end{aligned} \quad (\text{B.18})$$

Here $(w, \mathcal{C}) > 0$ means $(w, t_2) > 0$ for any $t_2 \in \mathcal{C}$. Note that the difference as shown above is a polynomial of $v = t_2/|t_2|$. Using this fact, ([13], Theorem 10.3) shows that $I_{\Lambda_S}(v, F)$ gives chamber-wise polynomial of v with degree at most 3 in our case (in fact, without even degree terms).

As a corollary, we obtain wall-crossing formulas for $P_1(t)$ and $P_3(t)$, which are directly relevant to the topological invariants of X_{IIA} :

$$\frac{1}{3!}P_3(t; \mathcal{C}_1) - \frac{1}{3!}P_3(t; \mathcal{C}_2) = \sum_w \left\{ \frac{c_w(w^2/2)}{3!}(w, t)^3 + \frac{(t, t)}{2} \frac{d_w(w^2/2)}{12}(w, t) \right\}, \quad (\text{B.19})$$

$$P_1(t; \mathcal{C}_1) - P_1(t; \mathcal{C}_2) = - \sum_w 2c_w^\Psi(w^2/2)(w, t). \quad (\text{B.20})$$

The summation over w is the same as in (B.18).

⁹⁹ w that gives singularity satisfies $-2 \leq w^2 < 0$; the condition $w^2 < 0$ [resp. $w^2/2 \geq \nu_{\min} = -1$] follows from $t_2^2 > 0$ [resp. $c_\lambda(\nu) = 0$ for $\nu < \nu_{\min}$].

B.3 Embedding trick

Even when Λ_S has no non-zero null elements, one can evaluate the integral $I_{\Lambda_S}(v, F)$ by embedding Λ_S into a larger lattice(s) $\widetilde{M}$ with a non-zero null element so that $I_{\Lambda_S}(v, F)$ is equal to (linear combination of) $I_{\widetilde{M}}(v, G)$ for a suitable modular form(s) G . One can then apply the lattice unfolding method to compute it. This method is called the “embedding trick” in [13]. In this section, we explain the original embedding trick and its slight modifications. We also treat a concrete example (the case $\Lambda_S = \langle +2 \rangle$).

The original embedding trick is as follows ([13], Theorem 8.1). To begin with, choose a pair of (negative definite) Niemeier lattices M_1, M_2 with different numbers of roots:

$$r_1 - r_2 \neq 0, \quad r_i := \text{the number of roots in } M_i. \quad (\text{B.21})$$

Then we obtain

$$1 = (\bar{\theta}_{M_1}(\tau) - \bar{\theta}_{M_2}(\tau)) \frac{1}{(r_1 - r_2)\eta(\tau)^{24}}; \quad (\text{B.22})$$

this is because the right hand side is a scalar-valued modular form of weight 0 with the term q^{-1} vanishing and the coefficient of q^0 normalized. By inserting the expression on both sides to the integrand of $I_{\Lambda_S}(v, F)$, we get

$$I_{\Lambda_S}(v, F) = I_{\Lambda_S \oplus M_1}(v, G) - I_{\Lambda_S \oplus M_2}(v, G), \quad (\text{B.23})$$

where $G = F[(r_1 - r_2)\eta^{24}]^{-1}$ and $v \in Gr(\Lambda_S)$ is regarded as the same positive definite ($b_+ = 1$)-dimensional subspace $v \in Gr(\Lambda_S) \subset Gr(\Lambda_S \oplus M_i)$. Since $\Lambda_S \oplus M_i$ has U as direct summand (see [13], section 8), we can apply the lattice unfolding formula to evaluate the right hand side.

There are a few points to keep in mind. First, G has a pole of higher order at cusps than F (i.e. $\nu_{\min}(G) = \nu_{\min}(F) - 1 = -2$). Second, in many cases (e.g., in appendix B.3.1), we need to care about wall-crossings of $I_{\Lambda_S \oplus M_i}$ for walls between v and the fundamental region in $Gr(\Lambda_S \oplus M_i)$ where the unfolding formula (B.8) is valid.

This embedding trick adds Niemeier lattice to the original lattice and increases the rank by as many as $\text{rank } M_i = 24$; sometimes it is troublesome to handle such a big lattice and consider all the relevant wall-crossings. But actually, it suffices to add a smaller lattice: choose sublattices $N_i \subset M_i$ such that $N_1 \cong N_2 (\cong: N)$. Then by decomposing the theta function of M_i as

$$\bar{\theta}_{M_i}(\tau) = \sum_{\delta \in N_i^\vee / N_i} \bar{\theta}_{N_i + \delta} \bar{\theta}_{N_i^\perp + \delta}(\tau), \quad (\text{B.24})$$

we can rewrite (B.22) to

$$1 = \sum_{\delta \in N^\vee / N} \bar{\theta}_{N+\delta}(\tau) h_\delta(\tau), \quad h_\delta(\tau) = \frac{\bar{\theta}_{N_1^\perp + \delta}(\tau) - \bar{\theta}_{N_2^\perp + \delta}(\tau)}{(r_1 - r_2)\eta(\tau)^{24}}. \quad (\text{B.25})$$

$\{h_\delta\}$ is a modular form of type ρ_N . Inserting this expression to the integrand of $I_{\Lambda_S}(v, F)$, we get

$$I_{\Lambda_S}(v, F) = I_{\Lambda_S \oplus N}(v, G), \quad G_{\gamma, \delta} = F_\gamma h_\delta. \quad (\text{B.26})$$

Here G is a modular form of type $\rho_{\Lambda_S \oplus N}$. Note that also in this case $\nu_{\min}(G)$ is less than $\nu_{\min}(F) = -1$ (but still $\nu_{\min}(G) \geq -2$).

After all, an important point for the embedding trick is to find a decomposition

$$1 = \sum_{\delta \in N^\vee / N} \bar{\theta}_{N+\delta}(\tau) h_\delta(\tau) \quad (\text{B.27})$$

for some lattice N and a modular form h , not necessarily associated with Niemeier lattices. There are many choices. For example, let¹⁰⁰ $N = \langle -2m \rangle$, and $\varphi(\tau, z)$ be a weak Jacobi form of weight 0 and index m , satisfying $\varphi(\tau, z=0) \neq 0$. We can theta-expand φ using a modular form h :

$$\varphi(\tau, z) = \sum_{\delta \in \mathbb{Z}_{2m}} \bar{\theta}_{\langle -2m \rangle + \delta}(\tau, z) h_\delta(\tau). \quad (\text{B.28})$$

Since a modular form of weight 0 and holomorphic at cusps is necessarily just a constant, setting $z = 0$ in the above equation leads to the required decomposition (B.27) up to some normalization. If $m \geq 2$, there are multiple choices for such φ .

So, even for a given lattice Λ_S , there are multiple different ways to use the embedding trick, $I_{\Lambda_S}(v, F) = \sum_i I_{\widetilde{M}_i}(v, G_i)$. There is no unique choice for $\widetilde{M} = \Lambda_S \oplus N$; even for a given N , the choice of G is not necessarily unique. One can use just any version of the embedding trick, so practical calculations of one's interest are easier.

B.3.1 Example: $\Lambda_S = \langle +2 \rangle$

Let us compute $I_{\Lambda_S = \langle +2 \rangle}(F)$ for $F = \Psi \eta^{-24}$, $(\Phi \hat{E}_2 - \Psi) \eta^{-24}$ using the embedding trick.¹⁰¹ They contribute to (B.3) through (B.5); as a reminder, Φ and Ψ to be used here (for $\Lambda_S = \langle +2 \rangle$) are those in (2.38), (3.65).

As a practical implementation of the embedding trick, we choose $N = \langle -2 \rangle$, so $\widetilde{M} = \langle +2 \rangle \oplus \langle -2 \rangle$, and $G_{\widetilde{M}} = \mathbb{Z}_2 \times \mathbb{Z}_2$. We will use the notation $\mathbb{Z}_2 \cong \{0, 1\}$ (instead of $\{0, 1/2\}$) in this appendix B.3.1. Let us use the weak Jacobi form $\varphi(\tau, z)$ of weight 0 and index 1 (i.e., index $\langle +2 \rangle$) normalized so that $\varphi(\tau, z=0) = 1$, and determine a vector-valued modular form h through (A.24), (B.28). Now

$$I_{\langle +2 \rangle}(v, \Psi \eta^{-24}) = I_{\widetilde{M}}(v, G), \quad I_{\langle +2 \rangle}(v, (\Phi \hat{E}_2 - \Psi) \eta^{-24}) = I_{\widetilde{M}}(v, \tilde{G}), \quad (\text{B.29})$$

where $G = \sum_{\delta, \gamma \in \mathbb{Z}_2} e_{\delta, \gamma} h_\delta \Psi_\gamma \eta^{-24}$ and $\tilde{G} = \sum_{\delta, \gamma \in \mathbb{Z}_2} e_{\delta, \gamma} h_\delta (\Phi_\gamma \hat{E}_2 - \Psi_\gamma) \eta^{-24}$. Because

$$h(\tau) = \frac{e_0}{12} \left(10 + 108q^{4/4} + \mathcal{O}(q^{8/4}) \right) + \frac{e_1}{12} \left(\frac{1}{q^{1/4}} - 64q^{3/4} + \mathcal{O}(q^{7/4}) \right), \quad (\text{B.30})$$

$\nu_{\min}(G)$ and $\nu_{\min}(\tilde{G})$ are both $-5/4$, rather than -1 .

¹⁰⁰We can choose N to be rank-1 for any Λ_S . This is because Λ_S has necessarily an element of norm $2m$ for some positive integer m ; the lattice $\Lambda_S \oplus N$ with $N = \langle -2m \rangle$ has a non-zero null element then.

¹⁰¹When F is holomorphic on the upper half-plane (e.g. $\Psi \eta^{-24}$), $I_{\langle +2n \rangle}(F)$ can be also calculated by using ([13], Corollary 9.6), which uses Zagier's modular form and Stokes' theorem.

(n, m)	$\lambda^2/2$	$(\lambda, t_2 = u_2 z + \rho_2 \tilde{z})$	wall at $u_2/\rho_2 =$
$\pm(2, -3)$	$-5/4$	$\pm(5u_2 - 3\rho_2)$	$3/5$
$\pm(1, -2)$	$-3/4$	$\pm(3u_2 - 2\rho_2)$	$2/3$
$\pm(0, 1)$	$-1/4$	$\mp(u_2 - \rho_2)$	1
$\pm(0, 2)$	-1	$\mp 2(u_2 - \rho_2)$	
$\pm(1, 2)$	$-3/4$	$\mp(u_2 - 2\rho_2)$	2
$\pm(2, 3)$	$-5/4$	$\mp(u_2 - 3\rho_2)$	3

Table 7. The list of walls where $I_{\widetilde{M}}(\mathbb{R}\tilde{t}_2, G)$ and $I_{\widetilde{M}}(\mathbb{R}\tilde{t}_2, \tilde{G})$ are singular. They are sorted in the order of their slopes in the ρ_2 - u_2 plane. They are all within the positive cone $(2u_2 - \rho_2)\rho_2 > 0$ (and $\rho_2 > 0$) of the lattice $\widetilde{M}$.

Let us describe the chamber structure in $\widetilde{M} \otimes \mathbb{R}$. To start, we introduce a parametrization of the space $\{\tilde{t}_2 \in \widetilde{M} \otimes \mathbb{R}\}$ as in (B.6). We denote a generator of $\Lambda_S = \langle +2 \rangle$ and $N = \langle -2 \rangle$ by v_{+2} and v_{-2} , respectively. In a basis $(z, \tilde{z}) = (v_{+2} + v_{-2}, -v_{-2})$, the intersection form of $\widetilde{M}$ is given by

$$\begin{pmatrix} 0 & 2 \\ 2 & -2 \end{pmatrix}. \quad (\text{B.31})$$

So we can use z as the null element for the lattice unfolding method; now $\tilde{t}_2 = u_2 z + \rho_2 \tilde{z}$. Next, the walls of interest are of the form $(\tilde{t}_2, \lambda) = 0$ for some $\lambda \in \widetilde{M}^\vee$, with $\nu_{\min} \leq \lambda^2/2 < 0$. This condition is equivalent to $0 < m^2 - n^2 \leq 5$ for $\lambda = nv_{+2}/2 + mv_{-2}/2 \in \widetilde{M}^\vee$, $n, m \in \mathbb{Z}$. So, there are only finite number of solutions (n, m) :

$$(n, m) = (0, \pm 1) \text{ and } (0, \pm 2), (\pm 1, \pm 2), (\pm 2, \pm 3). \quad (\text{B.32})$$

See table 7 for the list of those walls in $\widetilde{M} \otimes \mathbb{R}$.

The integrals (B.29) are evaluated, first, in the fundamental chamber $\mathcal{C}_z$ by using the lattice unfolding formula (B.8). The fundamental chamber is the region $0 < 3\rho_2 < u_2$. The result is

$$P_1(\tilde{t}_2; \mathcal{C}_0) = (2u_2 - \rho_2) \left(10 - b_{\mathcal{R}} - \frac{7}{2}n_{1/2} \right) + \rho_2 \frac{(248 - 18b_{\mathcal{R}} - 29n_{1/2})}{6}, \quad (\text{B.33})$$

$$\begin{aligned} P_3(\tilde{t}_2; \mathcal{C}_0) &= (2u_2 - \rho_2)^2 \rho_2 \frac{(-b_{\mathcal{R}} - 3n_{1/2})}{4} \\ &\quad + (2u_2 - \rho_2) \rho_2^2 \frac{(20 - 3b_{\mathcal{R}} + 5n_{1/2})}{4} + (\rho_2)^3 \frac{1 - 7n_{1/2}}{3}; \end{aligned} \quad (\text{B.34})$$

in using the formula (B.8) for the lattice $\widetilde{M}$ and the null element $z = (v_{2+} + v_{2-})$, the lattice W is $\{0\}$, and $N = 2$; the subgroup $\ker(-, z)$ is $\{0, (z/2)_{+\widetilde{M}}\} = \{(0, 0), (1, 1)\} \subset G_{\widetilde{M}}$, which is also equal to $j^{-1}(0)$ for $0 \in G_W$; relevant Fourier coefficients of G and $\tilde{G}$ are computed from (B.29) and (2.38), (3.65).

The fundamental region $\mathcal{C}_z$ is separated from the $\Lambda_S \otimes \mathbb{R} \subset \widetilde{M} \otimes \mathbb{R}$ locus by the two walls $u_2 = 3\rho_2$ and $u_2 = 2\rho_2$. The integrals in the chamber $\rho_2 \leq u_2 \leq 2\rho_2$ are obtained by

adding the following terms to (B.33), (B.34),

$$\Delta P_1 = \frac{-2}{6}(u_2 - 3\rho_2) + \frac{10n}{6}(u_2 - 2\rho_2), \quad (\text{B.35})$$

$$\Delta P_3 = \frac{2}{12}(u_2 - 3\rho_2)^3 - \frac{10n_{1/2}}{12}(u_2 - 2\rho_2)^3, \quad (\text{B.36})$$

because of the wall crossing formula (B.18). The first and second terms are associated with the walls at $u_2 = 3\rho_2$ and $u_2 = 2\rho_2$, respectively. Relevant Fourier coefficients are $[G_{10}]_{q^{-5/4}} = -2/12$, $[G_{01}]_{q^{-3/4}} = 10n/12$, $[h_1\Phi_0\eta^{-24}]_{q^{-5/4}} = -2/12$, and $[h_0\Phi_1\eta^{-24}]_{q^{-3/4}} = 10/12$.

The integrals $P_1(\tilde{t}_2)$ and $P_3(\tilde{t}_2)$ for the argument of real interest, $\tilde{t}_2 = (t_2^{a=1})v_{2+} \in (\Lambda_S \otimes \mathbb{R}) \subset (\tilde{M} \otimes \mathbb{R})$, is obtained by taking the limit $u_2 \rightarrow t_2^{a=1}$, $\rho_2 \rightarrow t_2^{a=1}$ in their expressions in the chamber $\rho_2 \leq u_2 \leq 2\rho_2$. Therefore,

$$\begin{aligned} P_1(t_2^{a=1}) &= \left[\left(10 - b_{\mathcal{R}} - \frac{7}{2}n_{1/2} + \frac{(248 - 18b_{\mathcal{R}} - 29n_{1/2})}{6} \right) + \frac{4}{6} - \frac{10n_{1/2}}{6} \right] (t_2^{a=1}), \\ &= (52 - 4b_{\mathcal{R}} - 10n_{1/2})(t_2^{a=1}), \end{aligned} \quad (\text{B.37})$$

$$\begin{aligned} P_3(t_2^{a=1}) &= \left[\left(\frac{20 - 4b_{\mathcal{R}} + 2n_{1/2}}{4} + \frac{1 - 7n_{1/2}}{3} \right) + \frac{2(-8)}{12} - \frac{10}{12}n_{1/2} \right] (t_2^{a=1})^3, \\ &= (4 - b_{\mathcal{R}} - n_{1/2})(t_2^{a=1})^3. \end{aligned} \quad (\text{B.38})$$

Those two results are used in section 3.2.2.

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